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ABSTRACT

MATHCOUNTS is a national coaching and competition program designed to stimulate 7th- and 8th-grade students' interest and achievement in mathematics. Teachers and volunteers coach students beginning each fall and continuing throughout the year. Participating schools can select individuals or four-person teams to compete in on-site competitions. Students compete in local meets in February, winners progress to state contests in March, and the top four scorers and top coach at the state competition represent their state at the national finals in May. Each year the MATHCOUNTS question writing committee selects and announces a special topic designed to stimulate student interest in mathematics. The special topic for the 1998-99 MATHCOUNTS program is Investigation and Exploration. This book provides detailed information about the competition, activities, and suggested workouts that can be completed in the coaching program. Contains 21 references. (ASK)

98-99 MATHCOUNTS School Handbook

Math is not a Spectator Sport

MATHCOUNTS

MATHCOUNTS is a cooperative project of CNA, the National Society of Professional Engineers, The Dow Chemical Company Foundation, General Motors Foundation, Phillips Pe[illegible] Company, Texas Instruments Incorporated, 3M Foundation, the National Council of [illegible] of Mathematics, and the National Aeronautics and Space Administration.

1998–1999
MATHCOUNTS
SCHOOL HANDBOOK

In order to develop
a technically literate population
and to improve quality of life,
the national MATHCOUNTS
coaching and competition program
promotes math interest and achievement
among intermediate school students.

MATHCOUNTS Critical Dates

- ☐ IMMEDIATELY — For easy reference, write your chapter coordinator's address and phone number here. If you do not know who your chapter coordinator is, contact your state coordinator (see page 4).

  ____

  ____

  ____

  ____

- ☐ SEPTEMBER — If you are ordering coaching and/or novelty items, send the correct form with payment or purchase order to the address below. Please allow 3-4 weeks for delivery.

  Sports Awards
  4351 N. Milwaukee Avenue
  Chicago, IL 60641

  Questions? Call toll-free 800-621-5830 or 773-282-8060 within Illinois.

- ☐ OCTOBER 30 — Have you sent your school's registration invoice along with check, money order or purchase order? Don't delay! Deadline is in two weeks! Send to:

  MATHCOUNTS Registration
  P.O. Box 441
  Annapolis Junction, MD 20701

  Questions? Call 301-498-6141 or e-mail: math@pmds.com
  Or confirm your registration on-line at http://mathcounts.org

- ☐ **NOVEMBER 13** — **REGISTRATION DEADLINE:**
  Your school competition will be mailed in early December, if your registration form is *received* by November 13.

- ☐ JANUARY 8 — LATE REGISTRATION:
  Your school competition will be mailed in late January, if your registration form is *received* between November 14 and January 8.

- ☐ AFTER JANUARY 8 — RESTRICTED LOCAL REGISTRATION:
  Local registration is at the discretion of chapter coordinators. Contact your chapter coordinator or state coordinator (see pg. 4) as soon as possible to determine local registration policy. *Registration forms received after January 8 will be returned.*

- ☐ MID-JANUARY — If you have not been contacted by your chapter coordinator with competition details, call your chapter or state coordinator!
  Call the MATHCOUNTS Registration Office (301-498-6141) with questions regarding the school competition.

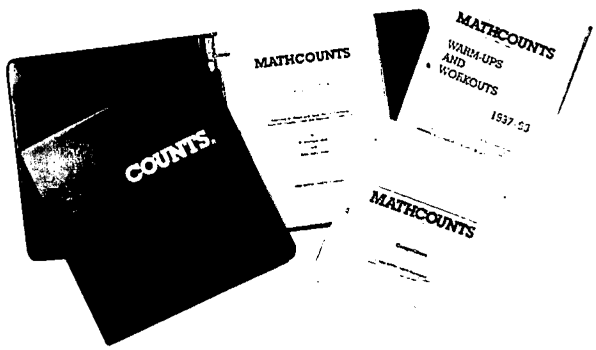
- ☐ FEBRUARY 6–28 — Chapter Competitions

- ☐ MARCH 6–28 — State Competitions

- ☐ MAY 13–16 — National Competition in Washington, DC

MATHCOUNTS

COACHING MATERIALS



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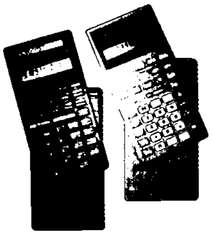

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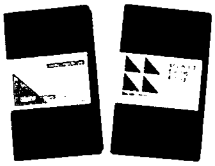

MATHCOUNTS Video

Hear coaches and mathletes describe the benefits of MATHCOUNTS in this fast-paced video. Includes scenes from coaching sessions and the National Competition. A great overview of MATHCOUNTS to show at coaching sessions, PTA meetings, open houses and teacher gatherings. 13 minutes.

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TEACHER'S NAME ______________________________

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Prices are good through July 1999.

Send completed order form along with purchase order, check or money order to:

SPORTS AWARDS COMPANY
4351 N. MILWAUKEE AVE.
CHICAGO, IL 60641

In Illinois: 773-282-8060
FAX 773-282-3019
Outside Illinois call:

1-800-621-5803

Item #	Description	Each	Qty.	Total
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MCC98	1997-98 Competition Set	$18.95		
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MCS84-88	5-yr Solution Set (1984-88)	$34.95		
MCS93	1992-93 Individual Solution Set	$ 9.95		
MCWW98-99	1998-99 Warm-ups and Workouts	$ 7.50		
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MCWW94-98	5-yr Warm-ups and Workouts set (1984-98)	$29.95		
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MCS97	1996-97 School Handbook Solutions	$13.95		
MCS96	1995-96 School Handbook Solutions	$11.95		
MCS98A	Solutions: 1997-98 and 1996-97	$24.95		
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MCCR	Problem of the Week	$18.50		
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  ☐ VISA ☐ MasterCard

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Order Total $

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GRAND TOTAL $

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Thanks to the writers for creating the 1998–99 MATHCOUNTS coaching materials and competitions:

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MATHCOUNTS National Judges

Thanks to the judges for reviewing competition materials and for developing Masters Round questions:

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Reviewers

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Editor: G. Patrick Vennebush, Alexandria, VA

This handbook was designed for use by participating schools, teams and coaches. It is also intended for use by chapter and state coordinators since some information contained in this handbook is not duplicated in the *Chapter & State Coordinators Handbook*.

Coaches and coordinators should pay special attention to the following new information and program changes described in this handbook:

- Competition registration deadlines, page 7
- School registration fee, page 7
- Chapter and State competition dates, page 9
- Extended activities, pages 34–39
- Stretches, pages 40–44
- Warm-Ups and Workouts, pages 45–99
- Solutions, pages 101–126
- Problem Index, page 128

MATHCOUNTS Mission

The mission of MATHCOUNTS is to increase interest and involvement in mathematics among all intermediate school students in order to assist in developing a technically literate population essential to US global competitiveness and the quality of life.

Preface

MATHCOUNTS is a national coaching and competition program designed to stimulate 7th- and 8th-grade students' interest and achievement in mathematics. Teachers and volunteers coach student mathletes beginning each fall and continuing through the year. Participating schools can select four students to compete as teams and individuals in on-site competitions. Students compete in local meets in February. Winners progress to state contests in March. The top four scorers and top coach at the state competition represent their state at the national finals in May.

At the national level, MATHCOUNTS is sponsored by:

- CNA
- National Society of Professional Engineers (NSPE)
- General Motors Foundation (GM)
- Phillips Petroleum Company
- The Dow Chemical Company Foundation
- Texas Instruments Incorporated (TI)
- 3M Foundation
- National Council of Teachers of Mathematics (NCTM)
- National Aeronautics and Space Administration (NASA)

The national MATHCOUNTS program also receives contributions from the following corporations:

- Alcoa Foundation
- AlliedSignal Inc.
- American Association of University Women Educational Foundation
- ANR Pipeline
- Champion International Corporation
- Exxon Educational Foundation
- Fluor Daniel
- Ford Motor Company Fund
- GEICO
- GTE Foundation
- Lockheed Martin
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- Northrop Grumman Corporation
- Rockwell
- TRW Foundation

The National Association of Secondary School Principals has placed this program on the NASSP Advisory List of National Contests and Activities for 1998–99.

MATHCOUNTS Information

Questions concerning competition registration? Contact your chapter coordinator, state coordinator (listed below) or the MATHCOUNTS Registration Office at:

MATHCOUNTS Registration
P.O. Box 441
Annapolis Junction, MD 20701
Phone: (301) 498-6141
E-mail: math@pmds.com

To confirm your school's registration, check MATHCOUNTS's on-line registration information at http://mathcounts.org/registration/schools. Other questions regarding the MATHCOUNTS program should be addressed to:

MATHCOUNTS Foundation
1420 King Street
Alexandria, VA 22314
Phone: (703) 684-2828
E-mail: mathcounts@nspe.org

Or, reference the MATHCOUNTS homepage on the World Wide Web at http://mathcounts.org/.

1998–99 State Coordinators

(*Note: DD=Defense Department; MP=Northern Mariana Islands; SS=State Department)

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CA	Scott Vorhees	916-422-7786
CO	Jennifer Vrynios, P.E.	303-796-4608
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WY	Dr. Sally Steadman, E.I.T.	307-766-6105

For your coordinator's e-mail address, reference the back of the competition registration form, or visit the Program area of the MATHCOUNTS web site at http://mathcounts.org/Program/coord.html.

Building a MATHCOUNTS Program

Recruiting Mathletes

Ideally the materials in this handbook will be incorporated into the regular classroom curriculum so that all students are exposed to problem-solving techniques and develop critical thinking skills. When a school MATHCOUNTS program is limited to extracurricular sessions, all interested students should be invited to participate, regardless of their academic standing. Because the greatest benefits of the MATHCOUNTS program are realized at the school level, the more mathletes involved, the better.

Students should view their experience with MATHCOUNTS as fun as well as challenging, so let them know from the very first organizational meeting that one goal is to have a good time. The following are suggestions from successful MATHCOUNTS coaches on how to stimulate interest at the beginning of the school year:

- Build a display case using MATHCOUNTS T-shirts, posters and balloons. Include trophies and photos from previous years' coaching sessions or competitions.
- Post intriguing math questions (involving specific school activities and situations) in hallways, the library, and the cafeteria, and refer students to the first meeting for answers.
- Make a presentation at the first pep rally or student assembly.
- Approach students through other extracurricular clubs (science club, computer club, chess club).
- Inform parents of the benefits of MATHCOUNTS participation via the school newsletter or parent-teacher organization.
- Create a MATHCOUNTS display for the first open house of the school year.
- Call on former mathletes to speak about the rewards of the program.

The Coaching Session

In order for students to reap the full benefits of MATHCOUNTS (and be prepared to compete in February), it is important to begin coaching early in the school year. The Stretches, Warm-Ups and Workouts in this handbook should carry a coaching program from October through January. If your school is competing, postpone selection of the official school team until just before the chapter competition.

On average, MATHCOUNTS coaches meet with mathletes one or two times a week at the beginning of the year and with increasing frequency as the competitions approach. Experienced coaches suggest holding sessions before school to eliminate scheduling conflicts with other activities. The following are suggestions for getting the most out of the WARM-UPS and WORKOUTS at the coaching sessions:

- Encourage discussion of the problems so that students learn from each other.
- Encourage a variety of methods for solving problems.
- Have students write problems for each other (see information about the F.L.I.P. contest on page 45).
- Use the MATHCOUNTS Problem of the Week (POW), which is posted every Monday on the MATHCOUNTS web site at **http://mathcounts.org**.
- Practice working in groups to develop teamwork (and to prepare for the Team Round).
- Practice oral presentations to reinforce understanding (and to prepare for the Masters Round).
- Use previous years' competition materials to provide an extra challenge (or to practice for the chapter competition). See page 6 for additional coaching materials that are available.
- Provide refreshments and vary the location of your meetings to create a relaxing, fun atmosphere.
- Invite the school principal to a session to offer words of support.

MAINTAINING A STRONG PROGRAM

Keep the school program strong by soliciting local support and focusing attention on the rewards of MATHCOUNTS. Publicize success stories. Let the rest of the student body see how much fun mathletes have. Remember, the more this year's students get from the experience, the easier recruiting will be next year. Here are some suggestions:

- Use the school newspaper and local media to publicize MATHCOUNTS events. Let individual mathletes tell their success stories.
- Inform parents of events through the PTA, open houses and the school newsletter.
- Schedule a special pep rally for the mathletes.
- Recognize the achievements of the mathletes at a school awards program.
- Have a students versus teachers mock Countdown Round and invite the student body to watch.
- Solicit donations from local businesses to be used as prizes in practice competitions.
- Plan retreats or field trips for the mathletes, perhaps to area college campuses.
- Take photos at coaching sessions and competition events and keep a scrapbook.
- Distribute MATHCOUNTS shirts to participating students.
- Form an alumni association and hold an annual reunion. Publish an alumni newsletter.
- Start a MATHCOUNTS summer school program.
- Encourage teachers of lower grades to participate in mathematics enrichment programs.

CALLING ON VOLUNTEERS

Volunteer assistance can be used to enrich the program and to expand the program to more students. Fellow teachers can serve as an assistant coaches. Individuals outside the school can also be helpful:

- **MATHCOUNTS alumni and senior high students** are wonderful role models. Each can work with a small group of mathletes. Their involvement also reinforces their own education.
- **Parents** may be willing to provide snacks for the group or to help drive mathletes to competitions.
- **Community professionals** (including retirees) can be called on to speak to the mathletes about careers in mathematics and to assist with coaching.

ADDITIONAL COACHING MATERIALS

To supplement materials in this handbook (which may be duplicated for use by US schools), MATHCOUNTS makes a variety of coaching products available to stimulate interest in the program and to enhance the educational experience.

Materials include special editions of current and past Warm-Ups and Workouts, problems from previous years' competitions, and videotapes which can be used to promote the MATHCOUNTS program throughout your school. A wide selection of novelty items (t-shirts, hats, calculators, etc.) is also available.

Coaching materials and novelty items can be ordered for a nominal charge using the coaching materials order form or the novelty items order form. **Orders should be sent with payment to Sports Awards at the address on the order forms—sending orders to the national office or the registration office will significantly delay your order.** Orders are filled on a first-come, first-served basis and are filled year round. Please allow three to four weeks for delivery.

MATHCOUNTS is on the World Wide Web at **http://mathcounts.org**. At this site, coaches, students and coordinators have access to the Problem of the Week, various sections of this handbook, *MATHCOUNTS News*, *The Circular*, school registration information, discussion forums, and other program information.

MATHCOUNTS Competition Rules

The following rules and regulations govern all MATHCOUNTS competitions. The national MATHCOUNTS Foundation reserves sole authority to alter these rules at any time with discretion. **Coaches are responsible for being familiar with the rules and procedures outlined in this handbook.** Coaches should bring any difficulty in procedures or in student conduct to the immediate attention of the appropriate chapter, state or national official on-site at any competition. Students violating any rules will be subject to immediate disqualification.

COMPETITION REGISTRATION

The non-refundable competition registration fee is $50 per school.

To register your school for MATHCOUNTS competition, complete and return the School Registration Form (separate attachment), along with a check, money order or purchase order to MATHCOUNTS Registration, P.O. Box 441, Annapolis Junction, MD 20701. All registered schools will receive a subscription to *MATHCOUNTS News* published three times a year.

One copy of the school competition (including instructions and an answer key), four school team ribbons, four team alternate ribbons, and fifteen student participation certificates will be mailed to all registered schools according to the following schedule:

<u>Registration Deadline:</u> **November 13, 1998**	School competition kits will be mailed in early December to schools whose registration forms are received by November 13.
<u>Late Registration:</u> **January 8, 1999**	School competition kits will be mailed in late January to schools whose registration forms are received between November 13 and January 8.
<u>Restricted Local Registration:</u> **After January 8, 1999**	*Registration forms received after January 9 will be returned.* Local registration is at the discretion of chapter coordinators. Schools deciding to compete after January 8 should contact their chapter coordinator to determine the local registration policy.

Upon receipt of a school registration form, the MATHCOUNTS registration office will send an invoice to the registered school indicating payment received or payment due. Invoices will be sent to the school address provided on the registration form. If a purchase order accompanies a registration form, the invoice will instead be sent to the address on the purchase order. A school's invoice will serve as confirmation of registration. Confirmation can also be available via the MATHCOUNTS on-line registration information at **http://mathcounts.org/registration/schools/**.

Your MATHCOUNTS chapter and state coordinator will be notified of your registration. They will then inform you of dates and locations of the chapter and state competitions. **If you are not contacted by your chapter coordinator by mid-January, it is your responsibility to contact your state coordinator**, listed on page 4, to confirm that your registration has been properly routed. Contact your state coordinator for the name and telephone number of your chapter coordinator, who has chapter competition details.

PARTICIPANTS

Students officially enrolled in the 7th or 8th grade are eligible to participate in MATHCOUNTS competitions. Students taking 7th- or 8th-grade mathematics classes who are not full-time 7th or 8th graders are not eligible. Participation in the competition phase is limited to a maximum of two years for each student, though there is no limit to the number of years a student may participate in the coaching phase. A school is eligible to send only one team of four students from among the 7th- and 8th-graders enrolled in the school. **A school team must consist of four students**. Teams of fewer than four will not be permitted to compete in any part of the chapter competition.

In the event that an individual is prevented from competition due to special needs, the chapter and state coordinator, in conjunction with the MATHCOUNTS Foundation, will review the needs of the student. If at all possible, reasonalbe accommodations may be made to allow the student to compete.

A student enrolled part-time at a magnet school may not compete in MATHCOUNTS as a member of that magnet school's team. A student may compete as a member of a magnet school's MATHCOUNTS team only if enrolled as a full-time student at the magnet school.

Schools are not permitted to combine students from two or more schools to form a team unless each school has nine or fewer students enrolled in the 7th and 8th grade combined.

Homeschools in compliance with the homeschool laws of the state in which they are located are eligible to participate in MATHCOUNTS competitions. Homeschools that register to compete must comply with competition rules and must compete in the chapter and state in which the homeschool is located. Homeschool coaches must complete an affidavit verifying that students from the homeschool are in either the 7th or 8th grade and that the homeschool complies with applicable state laws. Completed affidavits must be submitted to the chapter coordinator prior to competition.

At all levels of competition, student substitutions are not permitted after on-site competition registration has been completed.

If one or more members of a school team are observing religious beliefs on the same day as a MATHCOUNTS competition, the team may, at the discretion of the chapter and state coordinators, take the written portion of the competition up to one week in advance of the scheduled competition under MATHCOUNTS proctored conditions. An individual competitor, though, may not take the competition in advance, except at the state level in the case where he or she is the only competitor from a school.

Only students taking the competition on the regularly scheduled date are eligible to participate in the Countdown Round. In chapters and states where the Countdown Round counts as an official part of an individual score, students taking the competition in advance will automatically forfeit one place standing in the Countdown Round.

COMPETITION LEVELS

MATHCOUNTS competitions are organized at four levels: school, chapter, state and national.

Coordinators may release a blank copy of the exam and answer key for the chapter, state and national competitions to participating coaches after all competitions at that level have been completed. In addition, coaches should expect to receive the scores of their students, anonymous rankings of all scores, and a list of the top 25% of students and top 40% of teams from their coordinator. **Student competition papers become the confidential property of the MATHCOUNTS Foundation and will not be viewed by nor distributed to coaches, parents, competitors or other individuals.**

School Competition

The real success of MATHCOUNTS is measured by the coaching component at the school level that comprises the longest period of time and demands the greatest involvement. These school coaching sessions spur students to view mathematics as exciting, challenging, rewarding and fun.

In January, after several months of coaching using this handbook, schools registered for the competition phase of the program should administer the school competition to all interested 7th- and 8th-grade students. Coaches should prepare copies of the school competition for all participating students. The school competition is intended to be an aid to the coach in determining team members for the chapter competition. Selection of team members is entirely at the discretion of coaches.

It is important that the coach look upon coaching sessions during the academic year as opportunities to develop better math skills in all 7th- and 8th-grade students. Therefore, **it is suggested that the coach postpone selection of the official school team until just prior to the chapter competition in February**. At that time, the coach should select four mathletes from among the 7th- and 8th-graders to represent the school in the chapter round of competition. Coaches should also select four mathletes (also from among the 7th- and 8th-graders) to serve as alternates in case of illness or other unavoidable circumstances. Unless the school coach is otherwise notified by the chapter or state coordinator, alternates should not plan to attend competitions.

Chapter Competition

Chapter competitions will be held between February 6 and February 28 on a date selected by the chapter coordinator. A competition using problems supplied by National MATHCOUNTS will be conducted for schools within their chapter areas at the discretion of each chapter. (All participating chapters administer the same exam.) The chapter competition will consist of the Sprint, Target, Team and Countdown rounds. One team of four students from each school within the chapter's area will be eligible to participate in the chapter competition.

Chapters will select a winning team and team place-winners. In addition, the chapter will select a winning individual and individual place-winners.

State Competition

State competitions will be held between March 6 and March 28 on a date selected by the state coordinator. A competition using problems supplied by National MATHCOUNTS will be conducted at the discretion of each state. (All participating states administer the same exam.) The state competition will consist of the Sprint, Target, Team and Countdown rounds, and may also include an optional Masters round. One or more school teams (consisting of four mathletes) from each participating chapter within the state is eligible to participate in the state competition.

Substitutions of team members for the state competition may only take place when a student voluntarily releases his/her position on the team. Such a request must be submitted in writing to the state coordinator. Additional rules for substitutions of team members are at the discretion of the state coordinator, although student substitutions will not be permitted after on-site registration is completed.

In addition to the winning team(s), a chapter may select from one to four of the next highest-ranking individuals (not necessarily in the top four) at the chapter level not on the winning school team(s) will be eligible to participate in state competition. (For example, if each chapter sends one team to the state competition, and members of the winning team placed 1st, 2nd, 3rd and 4th, then the participants who placed 5th, 6th, 7th and 8th are also eligible to compete at the state competition as individuals. These mathletes may compete in all rounds that yield individual scores and are eligible to compete for a place on the state team advancing to the national competition.) The number of teams and individual mathletes progressing to the state competition from each chapter will be determined by the state coordinator according to the above guidelines, but will be consistent for all chapters within the state.

States will select a team winner, team place-winners, an individual winner and individual place-winners. Special category winners and individual place-winners in oral competition may also be recognized.

National Competition

The MATHCOUNTS national competition will be held on Friday, May 14, 1999. The national competition will consist of the Sprint, Target, Team, Countdown and Masters rounds. One team from each state will be eligible to compete at the national competition. The state team will be composed of the top four individual scorers in the state competition plus the coach of the winning team.

Expenses of the state team and coach to travel to the national competition will be paid by MATHCOUNTS. In addition, one team and one coach from the Department of Defense Dependent Schools, Department of State School System, the District of Columbia, Guam, the Commonwealth of the Northern Mariana Islands, Puerto Rico, and the Virgin Islands will be eligible to compete in the national competition. No provisions will be made for alternate students or alternate coaches.

The top ten individuals and the top three teams, as well as special category winners, will be recognized.

COMPETITION COMPONENTS

MATHCOUNTS competitions are designed to be completed in approximately three hours. Competitions consists of the following rounds:

Sprint Round

The Sprint Round (40 minutes) consists of 30 problems that are distributed to individual competitors. This round is intended to test accuracy, with time being such that only the most capable students will complete all of the problems. Calculators are not permitted during this round.

Target Round

The Target Round (approximately 30 minutes) consists of eight problems presented to individual competitors in four pairs (6 minutes per pair). This round features multi-step problems that engage mathletes in mathematical reasoning and problem-solving processes. Calculators are permitted during this round; problems assume calculator use. See page 12 for rules on acceptable calculators.

Team Round

The Team Round (20 minutes) consists of ten problems on which the team works together to solve. Team member interaction is permitted and encouraged during this portion. Calculators are permitted during this round; problems assume calculator use. See page 12 for rules on acceptable calculators.

Countdown Round

At chapter and state competitions, a Countdown Round may be conducted officially, may be conducted unofficially for fun, or may be omitted. The Countdown Round is a fast-paced, oral competition for top-scoring individuals (based on Sprint and Target round scores). Pairs of mathletes challenge each other in head-to-head competition. Calculators are not permitted during this round.

The use of the Countdown Round will be consistent for all chapters within a state. That is, *all* chapters within a state must use the round officially in order for *any* chapter within a state to use it officially. An official Countdown Round is defined as one which is included in determining an individual's final overall rank in the competition. If the Countdown Round is used officially, chapters and states must use the official procedures as established by the MATHCOUNTS Foundation, described below. These procedures are used to conduct the Countdown Round at the national competition.

If a Countdown Round is conducted unofficially, chapters and states are not required to follow the official procedures. Chapters and states choosing not to conduct the round officially must determine individual winners (who will progress to higher levels of competition) on the sole basis of students' scores in the Sprint and Target rounds of the competition.

<u>Countdown Round Procedures</u>

a. Top individuals are ranked as a result of their total score on the written portion of the competition.

b. Participants in the Countdown Round consist of the top 25% of students up to a maximum of 10 students.

c. The two lowest-ranked Countdown Round participants are paired.

d. A problem is simultaneously projected on a screen in view of the mathletes and read aloud. Once the problem is projected, mathletes have a maximum of 45 seconds to solve it. A ten-second warning is given to the students, as appropriate.

e. A mathlete indicates readiness to answer using a method determined by the competition organizers. The mathlete is called on to respond to the problem orally. Once called, the mathlete has 3 seconds to begin an answer. The other mathlete may continue working while the first is responding to the problem.

f. If the first mathlete to respond answers correctly, he/she scores one point in the round. If the first to respond answers incorrectly, the second mathlete is given the remainder of the 45 seconds to respond correctly and score a point in the round.

g. If neither mathlete responds within 45 seconds, the next problem is posed to the two mathletes.

h. The mathlete who answers the most of three questions correctly wins the round, captures the higher place, and progresses to the next round to challenge the next highest placeholder. (Please note that this procedure does not necessarily require a mathlete to answer two out of three questions correctly. For instance, a mathlete answering only one question of three will progress to the next round if his/her opponent has not responded correctly to any problems in the round.)

   If the mathletes are tied after three problems (1-1 or 0-0), a sudden victory situation is declared. Problems will continue to be read until one of them successfully answers a problem. The first to correctly answer a problem wins the round.

i. Procedure continues until the fourth-ranked mathlete and his/her opponent are paired. From this point on, the first in each of the last four rounds to answer three questions correctly wins the round. (At the national competition, this new procedure begins when students 3 and 4 are paired.)

j. Procedure continues until the first-place individual is identified.

k. Protests are be accepted during the Countdown Round. Decisions of the moderator and judges are final and may not be appealed.

Masters Round

The Masters Round (not included at chapter level; optional at state level) is a special competition in which top individual scorers are asked to respond orally to questions on a specific topic that are written and presented by a panel of judges. At the state level, the state coordinator will determine the number of mathletes that participate in the round. At the national level, four mathletes participate. (Participation in the Masters Round is optional. A mathlete declining to participate will not be penalized.)

Each mathlete is given 30 minutes to prepare a presentation for the Masters Round. The presentation will be 15 minutes in total length—up to 11 minutes can be used for the mathlete's oral response to the problem, and the remaining time can be used as a period of questioning by the judges. This competition values creativity and oral expression as well as mathematical accuracy. Judging of Masters Round presentations is based on the following criteria:

a. Knowledge
   1. Are the written questions answered?
   2. Is an appropriate approach to the solution of the problem used?
   3. Are the facts correct?
   4. Is the logic correct?
   5. Is the use of vocabulary and notation correct?
   6. Are relevant theorems, axioms, etc., used and mentioned when they contribute to the result?

b. Presentation
   1. Is time used effectively?
   2. Are important ideas emphasized (outlined, summarized)?
   3. Are ideas explained clearly?
   4. Is sufficient detail used?
   5. Are oral and visual techniques used effectively?

c. Response to Judges' Questions — Are the judges' questions answered correctly and adequately?

ADDITIONAL RULES

1. Use of notes or other reference materials (including dictionaries) is not permitted in the competition.
2. **The Target and Team Rounds assume the use of a calculator**. These rounds include problems that would be extremely time-consuming without the use of a calculator. Mathletes are not permitted to use calculators during the Sprint, Countdown or Masters Rounds.

   Where calculators are permitted, mathletes may use any type of calculator except programmable or graphing instruments. Calculators that lose memory when turned off are considered non-programmable.

   > NOTE: The National Council of Teachers of Mathematics' *Curriculum and Evaluation Standards for School Mathematics* recommends that all students in the 7th and 8th grades have calculators. Calculators should include the following features: algebraic logic including order of operations; computation in decimal and common fraction form; constant function for addition, subtraction, multiplication and division; and memory, percent, square root, exponent, reciprocal and +/– keys.

   Coaches are responsible for ensuring that their students use acceptable calculators, and mathletes are responsible for providing their own calculators. **Coordinators are not responsible for providing mathletes with calculators, AC outlets or batteries before or during MATHCOUNTS competitions**.

   Coaches are strongly advised to bring back-up calculators and spare batteries to the competition for their team members in case of a malfunctioning calculator or weak/dead batteries. **Neither the MATHCOUNTS Foundation nor MATHCOUNTS coordinators shall be responsible for the consequences of a calculator's malfunctioning**.

   Texas Instruments is the official calculator company of the 1998–99 MATHCOUNTS program.
3. Talking and signals are permitted only during the Team Round. Interaction with coaches is prohibited during all of the rounds but permitted during breaks. All other communication between guests and mathletes is prohibited until completion of the written portion of the competition. Only the judges and individuals making presentations may speak during the Masters Round.
4. Pencils and paper will be provided for mathletes. However, mathletes may bring their own pencils, pens and erasers if they wish.
5. All answers must be legible.
6. Specific instructions stated in a given problem take precedence over any general rule or procedure.
7. Protests concerning the correctness of an answer on the written portion of the competition must be registered with the room supervisor in writing by the team coach within 30 minutes after questions and answers are provided for each round. Rulings on protests are final and may not be appealed.
8. Protests will not be accepted during the Countdown or Masters Rounds. Decisions of the moderator and judges are final and may not be appealed.

FORMS OF ANSWERS

The following list explains acceptable forms for answers. Coaches should ensure that mathletes are familiar with these rules prior to participating at any level of competition.

Judges will score competition answers in compliance with these rules for forms of answers.

1. All answers must be expressed in simplest form. A "common fraction" is to be considered a fraction in the form $\pm\frac{a}{b}$, where a and b are natural numbers, and GCF$(a,b) = 1$. In some cases the term "common fraction" is to be considered a fraction in the form $\frac{A}{B}$, where A and B are algebraic expressions, and A and B do not share a common factor. A simplified "mixed number" ("mixed numeral," "mixed fraction") is to be considered a fraction in the form $\pm N\frac{a}{b}$, where N, a, and b are natural numbers, $a < b$, and GCF$(a,b) = 1$. Examples:

   Problem: Express 8 divided by 12 as a common fraction.
   Answer: $\frac{2}{3}$ Unacceptable Answers: $\frac{4}{6}$, $\frac{8}{12}$

Problem: Express 12 divided by 8 as a common fraction.

Answer: $\frac{3}{2}$ Unacceptable Answers: $\frac{6}{4}$, $\frac{12}{8}$

Problem: Express the sum of the lengths of the radius and the circumference of a circle with a diameter of $\frac{1}{4}$ as a common fraction in terms of p.

Answer: $\frac{1 + 2\pi}{8}$

Problem: Express 20 divided by 12 as a mixed number.

Answer: $1\frac{2}{3}$ Unacceptable Answers: $1\frac{8}{12}$, $\frac{5}{3}$

2. Unless otherwise specified in the problem, ratios should be expressed as simplified common fractions. Examples:

   Simplified, Acceptable Forms: $\frac{7}{2}$, $\frac{3}{\pi}$, $\frac{4 - \pi}{6}$

   Unacceptable Forms: $3\frac{1}{2}$, $\frac{\frac{1}{4}}{3}$, 3.5, 2:1

3. Radicals must be simplified. A simplified radical must satisfy the following conditions: 1) no radicands have perfect square factors other than one; 2) no radicands contain fractions; and 3) no radicals appear in the denominator of a fraction. For example:

   Problem: Evaluate $\sqrt{15} \times \sqrt{5}$.

   Answer: $5\sqrt{3}$ Unacceptable Answer: $\sqrt{75}$

   Numbers with fractional exponents are *not* considered to be in radical form.

4. Answers to problems asking for a response in the form of a dollar amount or an unspecified monetary unit (e.g. "How many dollars...," "What is the amount of interest...") should be expressed in the form (\$) *a.bc*, where *a* is an integer and *b* and *c* are digits. The *only* exceptions to this rule are when *a* is zero, in which case it may be omitted, or when *b* and *c* are both zero, in which case they may both be omitted.

   Acceptable Forms: 2.35, 0.38, .38, 5.00, 5

   Unacceptable Forms: 4.9, 8.0

5. Units of measurement are generally not required in answers, but must be correct if given in an answer. When a problem asks for an answer expressed in a specific unit of measure, equivalent answers expressed in other units are not acceptable. For example, if a problem asks for the number of ounces and 36 oz is the correct answer, 2 lbs 4 oz will not be accepted. Similarly, if a problem asks for the number of cents and 25 cents is the correct answer, \$0.25 will not be accepted.

6. In general, do not make approximations for numbers (e.g., p, $\frac{2}{3}$, $\sqrt{5}$) in the data given or in solutions unless the problem says to do so.

7. Do not do any intermediate rounding (other than the "rounding" a calculator performs) when calculating solutions to problems. All rounding should be done at the end of the calculation process.

8. When answers are to be expressed in scientific notation, they should be in the form $a \times 10^n$ where a is a decimal, $1 \leq |a| < 10$, and n is an integer. For example:

   Problem: Write 6,895 in scientific notation.

   Answer: 6.895×10^3

9. An answer expressed to a greater or lesser degree of accuracy than called for in the problem will not be accepted. Whole number answers should be expressed in their whole number form. Thus, 25.0 will not be accepted for 25, nor vice versa.

SCORING

All MATHCOUNTS competitions are written with the expectation that mathletes will be coached for a full season using MATHCOUNTS materials (see syllabus, page 31). Scores on the competition do not conform to traditional grading scales. Students and coaches should view an individual written competition score of 23 (out of a possible 46) as highly commendable.

Individual and team scores for the written portion of the competition are calculated as follows (perfect scores for each round are indicated in parentheses):

Round	Individual Score	Team Score
Sprint	Number Correct (30)	Avg. Number Correct (30)
Target	Number Correct x 2 (16)	Avg. Number Correct x 2 (16)
Team	(0)	Number Correct x 2 (20)
Final Written Score	Sum (46)	Sum (66)

Scores are kept for both individuals and teams throughout all MATHCOUNTS competitions. The Sprint and Target rounds yield scores for both individual and team standings. The Team Round yields a score for only the team standings. If used officially (see page 9), the Countdown Round yields final individual standings. The Masters Round is a competition for the top-scoring individuals that yields a separate special-category winner (see page 11).

At all levels of competition, ties among students or teams are resolved using a tiebreaker mechanism. The tiebreaker mechanism compares the number of problems answered correctly in a series of subsets of competition problems. Ties are resolved in favor of the tied competitor or team having the greatest number of problems correct in the first subset of problems. If ties remain, problems from the second subset of problems are compared, and so forth.

If ties still remain after the tiebreaking mechanism is implemented, a tiebreaker round will be held in which tied competitors or teams are given tiebreaker problems. Ties will be resolved in the order in which the students or teams provide correct solutions to each tiebreaker round problem.

AWARDS

At all levels of the competition, appropriate recognition or awards will be given to winning individuals, teams, coaches and schools. The top four individuals in each state competition will receive an all-expense-paid trip to Washington, DC, to compete in the national competition.

At the national competition, the first-, second- and third-place individuals will receive $8000, $6000 and $4000 college scholarships, respectively. The student scoring the highest in the written portion of the competition will be awarded a $2000 college scholarship. Each member of the winning team will receive a $2000 college scholarship.

COMPETITION QUESTIONS

MATHCOUNTS questions fit into 7th- and 8th-grade curricula throughout the country. However, many problems are designed to accelerate students in mathematics accomplishments, and tests become progressively harder as mathletes advance to higher levels of competition.

Problems used in all levels of competition as well as in the School Handbook are developed by the Question Writing Committee, which is composed of experienced mathematics teachers appointed by the National Council of Teachers of Mathematics. The following are topics that may be included in the handbook or in competitions:

- Estimation/approximation
- Computation — wholes, integers, rationals, roots, percentages, averages, exponents, proportions
- Equivalent Expressions — simplify fractions, decimals, percents, exponential expressions
- Statistics — means, medians, modes, min/max, range, statistical graphs

- Probability — simple events, combinations, permutations, counting properties
- Measurement — linear, area, volume, conversions, temperature, time
- Equations/Inequalities — linear equations, inequalities, percent sentences
- Geometry — parallel and perpendicular lines, circles, polygons, areas, volumes, Pythagorean property
- Charts, Graphs, Tables — interpreting and applying Cartesian coordinates
- Number Theory — series, patterns, primes and composites, LCM, GCF, modular arithmetic, divisibility
- Scientific Notation
- Consumer Math — mixtures, discounts, mark-ups, other word problems
- Algebra Topics — order of operations, simplifying variable expressions
- Logic — boolean algebra, Venn diagrams, symbolic logic

A problem index is included on page 128 of the *1998–99 MATHCOUNTS School Handbook* for the Warm-Up and Workout problems.

Special Topic

Investigation & Exploration

Each year, the MATHCOUNTS Question Writing Committee selects and announces a special topic designed to stimulate students' interest in mathematics. The topic is explored in the MATHCOUNTS coaching materials. The Special Topic for the 1998–99 MATHCOUNTS program is *Investigation & Exploration.*

Background

As America enters the technology age, teaching students to formulate, understand and solve problems on their own is vital to our nation's future. For that reason, both the National Research Council and the National Council of Teachers of Mathematics have criticized the traditional curriculum as well as the methods used to teach that curriculum. The mathematical knowledge and skills traditionally taught, which was typically facts coupled with computational efficiency, are no longer required. Instead, students need to learn problem solving, critical thinking and mathematical confidence. They need to be able to apply learned knowledge in various situations, and they must be able to acquire new knowledge on their own.

The development of classroom exploration and learning through discovery is motivated by a belief that inquiry has advantages unavailable to expository teaching. The scholars Spencer and Seneca believed that the value of discovery lay in improved retention. Moreover, Rousseau and Kant believed that discovery deepened understanding, and Wyse thought that discovery provided intrinsic motivation. Modern mathematical educators view knowledge as being constructed through a process of investigation where uncertainty, conflict and doubt provide the motivation which leads to a continuous search for resolution and contributes to a more refined understanding of the world.

Process

In the early part of the twentieth century, philosopher and educator John Dewey wrote that "reflective thought" involved first a state of doubt and mental difficulty in which thinking begins and, second, an act of searching to find a resolution to that doubt. Inquiry learning, which is the technical term ascribed to investigative and exploratory educational activities, facilitates both states. Initially, students formulate a problem to be solved. Then, students seek to find a solution, either individually or in groups. During this process of exploration, students learn mathematical knowledge, problem solving techniques, fundamental concepts and self-confidence.

In inquiry learning, students actively engage in constructing mathematical knowledge. Students construct this knowledge not by themselves, but as a community of investigators. Through discussion with other students, anomalies, ambiguities and controversies provide potential stimuli for further investigation, and through repeated explorations students begin to make sense of the concepts, rules and problems that they encounter.

Characteristics

There are four basic characteristics of a good inquiry problem:

1. The problem must be *realistic*. Students must see the problem encountered as having meaning in their lives as well as having applications to the real world. Although a scenario which involves charting the number of CDs purchased at a record store provides interest because the item is familiar to students, and a scenario regarding sales of stock relates real-world mathematics, a scenario which has students investigate potential sales of CDs at their school and using mathematics to forecast the best possible sale price is probably more realistic—and engaging—to students.
2. The exploration needs to be *complete*; that is, all of the information necessary to find at least one

solution is available to the students. Knowing that the land speed record has increased by 10% for each of the last 10 years is not enough if the students' task is to predict the land speed record 50 years from now. Additional necessary information might be the percent increase for the past 100 years as well as the exact increase each year.

3. The problem situation and requirements are *unambiguous*. Students need to clearly understand the concept(s) that they are to identify. To ask students to find relationships among the parts of a triangle may provide interesting results. For example, students may learn that the measure of an exterior angle is equal to the sum of the measures of the opposite interior angles, or they may learn that the sum of the lengths of any two sides must always be greater than the third side. However, if the point of the exploration is to teach students that the sum of the interior angles is 180 degrees, the lesson is, at least in one way, a failure.
4. A problem solving exploration is *solvable*. The resources available to students allow for a solution. Providing square tiles, access to a computer with Geometer's Sketchpad, or even just graph paper and a ruler is required if a student is to investigate the relationship between the area and perimeter of a polygon.

MODES OF EXPLORATION

There are three modes of exploration when using inquiry learning:

- **Guided Investigation:** In guided investigation, although the students solve the problem, the teacher provides the source of the problem as well as the source of the solution. Guided discovery is most effective when used to teach new concepts.
- **Modified Investigation:** Unlike guided investigation, students generate the source of the solution in modified investigation. However, the teacher still provides the question to be answered. To provide students with exposure to particular phenomena or to develop problem-solving skills, modified investigation can be very successful.
- **Free Investigation:** In free investigation, students provide both the question to be solved as well as the source of the solution. For topics and concepts not required in the curriculum, free investigation provides an opportunity for students to construct knowledge on their own as well as to hone their individual problem-solving skills.

An Example of a Guided Investigation

Guided investigations help students answer the question, "What can you say about _____?"

Mr. Moreno believes that students need to construct their own mathematical knowledge by investigating and exploring various areas of mathematics. On the first day of a unit which will focus on basic number theory and pattern recognition, Mr. Moreno gives the following problem to his students:

> What is the sum of the first several odd positive integers?
> That is, what is $1 + 3 + 5 + \ldots + n$?

Knowing that his students may be able to get a specific answer if a value of n is given, Mr. Moreno continues the lesson by asking for the value of the sum if $n = 9$; that is, he asks for the value of $1 + 3 + 5 + 7 + 9$. Mr. Moreno knows that Ron, who is not one of his better students, should have no trouble answering this question. (Mr. Moreno also knows that asking Ron a question that he can surely solve will help to increase Ron's confidence.) So, he asks Ron for the value of that expression. Not surprisingly, in a few seconds Ron responds, "25?"

"Correct!" replies Mr. Moreno, which causes both he and Ron to smile. Mr. Moreno then asks the class for the value of the sum when n takes on a few other values, and various students answer his questions. Knowing that his students may not see a pattern from random examples, he asks the class how they might investigate the pattern created by these sums. Carol, one of the top students in the class, suggests that they create a chart which shows the value of the sum for several different values of n. Mr. Moreno then allows a few other students to comment, and they generally agree with Carol, so he lets the students work in small groups to create individual charts and to look for patterns.

Mr. Moreno then walks around the room and he notices that most students are creating charts which show that, for values of $n = 1, 3, 5, 7, \ldots$, the corresponding values of the sum are 1, 4, 9, 16, … . Mr. Moreno then startles some students as he asks from the back of the class, "What are you finding?"

Rachel excitedly raises her hand. "These look like the numbers you showed us last week! The square numbers!"

"Very good, Rachel! But is that always the case? Will the sum always be a square number?"

"Well," says Ron, "I tried it up to 35, and I got 324. And 324 was on our list of square numbers."

Mr. Moreno then agreed that it looked like the sum would always be a square number. And to prove it, he demonstrated a geometric proof which showed that each time an odd number of blocks was added to a square, a larger square is formed. (See the Investigation & Exploration of Warm-Up 10.)

Mr. Moreno provided the question for the students to investigate. And although students may feel that they have ownership of the solution (Carol suggested how to solve the problem), it was Mr. Moreno's intent to lead them to this source for a solution.

An Example of a Modified Investigation

Modified investigations give opportunities to students to develop their investigative skills. Modified investigation helps students answer the question, "What would happen if we did this instead of that?"

"Wow! That's pretty neat, Mr. Moreno!" exclaimed Bill as he saw the geometric proof of odd numbers always having a perfect square number sum.

"I think so," said Mr. Moreno. Looking toward the other side of the classroom, he asks, "Judy, did you have a question?"

"I was just wondering if there were always tricks like this to add numbers."

"What do you think, class?" asked Mr. Moreno.

Most of the class agreed that there probably was, so Mr. Moreno gave the groups some time to work on two problems generated by a further discussion: "What is the sum of the first several consecutive positive integers?" and "What is the sum of the first several even integers?" As he made his way around the room, he stopped and listened to the discussions, and, when necessary, interjected some advice for how they might proceed, although he was careful to offer suggestions but never a solution—finding a solution was the task of the students.

An Example of a Free Investigation

As in modified discovery, students use free investigation to answer the question, "What would happen if we did this instead of that?" The major difference, however, is that students form their own questions.

As most students were concluding, Mr. Moreno noticed that Carol's group was working furiously as well as holding a heated debate. He made his way to their group to observe, fearing that something might be awry.

"Is there a problem here?" Mr. Moreno asked.

"Yes!" cried Julie. "Carol thinks that there's a rule for the square numbers. And we think so, too, but we can't find one."

"I don't understand, Julie," replied Mr. Moreno. "The square numbers, by definition, are just some smaller number squared."

"We know *that*," retorted Julie, indignant. "But we think there's a rule when for when you *add up* the square numbers."

"Oh! Well, that is an interesting question!" Mr. Moreno agreed. And once again, he smiled.

The process of investigation in Mr. Moreno's class led Carol and Julie to investigate a pattern he hadn't even considered. Identifying a pattern within the sums of the square numbers is not an easy task, but Mr. Moreno is delighted that some of his students are willing to search on their own. In the end, he knows that these students will develop exemplary problem-solving skills, teamwork ability, the capacity to teach themselves, and a love for learning.

Problem Solving

The NCTM *Curriculum and Evaluation Standards for School Mathematics* recommends that the mathematics curriculum "include numerous and varied experiences with problem solving as a method of inquiry and application." There are many problems within the MATHCOUNTS program that may be considered difficult if attacked by setting up a series of equations, but quite simple when attacked by problem-solving strategies such as looking for a pattern, drawing a diagram, making an organized list, and so on.

The problem-solving method that will be used in the following discussion consists of four basic steps:

FIND OUT	Look at the problem. Have you seen a similar problem before? If so, how is this problem similar? How is it different? What facts do you have? What do you know that is not stated in the problem?
CHOOSE A STRATEGY	How have you solved similar problems in the past? What strategies do you know? Try a strategy that seems as if it will work. If it doesn't, it may lead you to one that will.
SOLVE IT	Use the strategy you selected and work the problem.
LOOK BACK	Reread the question. Did you answer the question asked? Is your answer in the correct units? Does your answer seem reasonable?

Specific strategies may vary in name. Most, however, fall into these basic categories:

- Compute or Simplify (C)
- Use a Formula (F)
- Make a Model or Diagram (M)
- Make a Table, Chart or List (T)
- Guess, Check and Revise (G)
- Consider a Simpler Case (S)
- Eliminate (E)
- Look for Patterns (P)

To assist in using these problem-solving strategies, the answers to the STRETCHES, WARM-UPS and WORKOUTS have been coded to indicate possible strategies. The single-letter codes above for each strategy appear in parentheses after each answer.

In the next section, the strategies above are applied to previously-published MATHCOUNTS problems. The following examples model possible approaches to teaching problem solving.

Compute or Simplify (C)

Many problems are straightforward and require nothing more than the application of arithmetic rules. When solving problems, simply apply the rules and remember the order of operations.

Given $(6^3)(5^4) = (N)(900)$, find N.

FIND OUT — What are we asked? The value of N that satisfies an equation.

CHOOSE A STRATEGY — Will any particular strategy help here? Yes, factor each term in the equation into primes. Then, solve the equation noting common factors on both sides of the equation.

SOLVE IT — Break down the equation into each term's prime factors.

$$6^3 = 6 \times 6 \times 6 = 2 \times 2 \times 2 \times 3 \times 3 \times 3$$

$$5^4 = 5 \times 5 \times 5 \times 5$$

$$900 = 2 \times 2 \times 3 \times 3 \times 5 \times 5$$

Two 2's and two 3's from the factorization of 6^3 and two 5's from the factorization of 5^4 cancel the factors of 900. The equation reduces to $2 \times 3 \times 5 \times 5 = N$, so $N = 150$.

LOOK BACK — Did we answer the question asked? Yes.

Does our answer make sense? Yes—since $900 = 30^2 = (2 \times 3 \times 5)^2$, we could have eliminated two powers of 2, 3 and 5 to obtain the same answer.

Example — The conversion from heartbeats to days in problem #9 of Warm-Up 11 is a good example of a problem which can be solved with computation.

Use a Formula (F)

Formulas are one of the most powerful mathematical tools at our disposal. Often, the solution to a problem involves substituting values into a formula or selecting the proper formula to use. Some of the formulas that will be useful for students to know are listed on page 39. However, other formulas will be useful to mathletes, too. If the strategy code for a problem is (F), then the problem can be solved with a formula. When students encounter problems for which they don't know an appropriate formula, they should be encouraged to discover the formula for themselves.

The formula $F = 1.8C + 32$ can be used to convert temperatures between degrees Fahrenheit (F) and degrees Celsius (C). How many degrees are in the Celsius equivalent of -22°F?

FIND OUT — What are we trying to find? We want to know a temperature in degrees Celsius instead of degrees Fahrenheit.

CHOOSE A STRATEGY — Since we have a formula which relates Celsius and Fahrenheit temperatures, let's replace F in the formula with the value given for degrees Fahrenheit.

SOLVE IT — The formula we're given is $F = 1.8C + 32$. Substituting -22 for F in the equation leads to the following solution:

$$-22 = 1.8C + 32$$

$$-22 - 32 = 1.8C$$

$$-30 = C$$

The answer is -30°C.

LOOK BACK — Is our answer reasonable? Yes.

Example Problem #2 of Warm-Up 4 can be solved by applying the distance formula to the coordinates given.

MAKE A MODEL (M)

Mathematics is a way of modeling the real world. A mathematical model has traditionally been a form of an equation. The use of physical models is often useful in solving problems. There may be several models appropriate for a given problem. The choice of a particular model is often related to the mathlete's previous knowledge and problem-solving experience. Objects and drawings can help to visualize problem situations. Acting out is also a way to visualize the problem. Writing an equation is an abstract way of modeling a problem situation. The use of modeling provides a method for organizing information that could lead to the selection of another problem-solving strategy.

Use Physical Models

Four holes are drilled in a straight line in a rectangular steel plate. The distance between hole 1 and hole 4 is 35mm. The distance between hole 2 and hole 3 is twice the distance between hole 1 and hole 2. The distance between hole 3 and hole 4 is the same as the distance between hole 2 and hole 3. What is the distance, in millimeters, between the center of hole 1 and the center of hole 3?

FIND OUT We want to know the distance between hole 1 and hole 3.

What is the distance from hole 1 to hole 4? 35 mm.

What is the distance from hole 1 to hole 2? Half the distance from hole 2 to hole 3.

What is the distance from hole 3 to hole 4? The same as from hole 2 to hole 3.

CHOOSE A STRATEGY Make a model of the problem to determine the distances involved.

SOLVE IT Mark off a distance of 35 mm.

Place a marker labeled #1 at the zero point and one labeled #4 at the 35-mm point.

Place markers #2 and #3 between #1 and #4.

1) Move #2 and #3 until the distances between #2 & #3 and #3 & #4 are equal.
2) Is the distance between #1 & #2 equal to half the distance between #2 & #3?

Adjust the markers until both of these conditions are met.

Measure the distances to double check. The distance between #1 and #3 is 21 mm.

LOOK BACK Does our answer seem reasonable? Yes, the answer must be less than 35.

Example The cone described in problem #2 of Workout 5 is one that can easily be constructed. By creating a cone and then filling it with a material of known volume, a good approximation of the volume can be found. Further, constructing the model will lend insight to how a specific answer to the problem can be found.

Act Out the Problem

There may be times when you experience difficulty in visualizing a problem or the procedure necessary for its solution. In such cases you may find it helpful to physically act out the problem situation. You might use people or objects exactly as described in the problem, or you might use items that represent the people or objects. Acting out the problem may itself lead you to the answer, or it may lead you to find another strategy that will help you find the answer. Acting out the problem is a strategy that is very effective for young children.

There are five people in a room and each person shakes every other person's hand exactly one time. How many handshakes will there be?

FIND OUT We are asked to determine the total number of handshakes.

How many people are there? Five.

How many times does each person shake another's hand? Only once.

CHOOSE A STRATEGY Would it be possible to model this situation in some way? Yes, pick five friends and ask them to act out the problem.

Should we do anything else? Keep track of the handshakes with a list.

SOLVE IT Get five friends to help with this problem.

Make a list with each person's name at the top of a column.

Have the first person shake everyone's hand. How many handshakes were there? Four.

Repeat this four more times with the rest of the friends. Write down who each person shook hands with. Our table should look something like this:

Rhonda	Jagraj	Rosario	Kiran	Margot
Jagraj	Rosario	Kiran	Margot	Rhonda
Rosario	Kiran	Margot	Rhonda	Jagraj
Kiran	Margot	Rhonda	Jagraj	Rosario
Margot	Rhonda	Jagraj	Rosario	Kiran

There were a total of twenty handshakes. But notice that each person actually shook everyone else's hand twice. (For example, Rhonda shook Jagraj's hand, and Jagraj shook Rhonda's hand.) Divide the total number of handshakes by two to find out the total number if each person had shaken every other person's hand only once. There were ten handshakes.

LOOK BACK Did we answer the question? Yes.

Does our answer seem reasonable? Yes.

Example The science fair problem given in Workout 18, problem #4, is a good problem that can be acted out. Using twelve students in six pairs and then enacting the situation for various possibilities will eventually lead to a solution.

Use Drawings or Sketches

If an eight-inch square cake serves four people, how many twelve-inch square cakes are needed to provide equivalent servings to eighteen people?

FIND OUT We are to find how many 12 x 12 cakes are needed.

How big is the original cake? 8 x 8.

How many people did it feed? 4.

How big are the other cakes? 12 x 12.

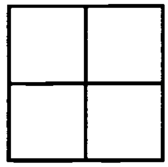
How many people must they feed? 18.

CHOOSE A STRATEGY — How should we approach this problem? Diagram the cakes to understand the size of the portions.

SOLVE IT — Draw an 8 x 8 cake and cut it into 4 equal pieces. Since each piece is a square with side length of 4, the area of each piece is 4 x 4 = 16 square inches.

4

4



So each person gets 16 square inches of cake.

18 people times 16 square inches per person equals 288 total square inches of cake needed.

We know that a 12 x 12 cake contains 144 square inches of cake.

288 divided by 144 equals 2, so two 12 x 12 cakes are required to feed 18 people.

LOOK BACK — Did we answer the correct question, and does our answer seem reasonable? Yes.

Example — Problem #10 of Warm-Up 11 is best represented with a drawing—perhaps on graph paper—of the floor described. Then, the number of tiles the diagonal crosses can be explored.

Use Equations

Lindsey has a total of $82.00, consisting of an equal number of pennies, nickels, dimes and quarters. How many coins does she have in all?

FIND OUT — We want to know how many coins Lindsey has.

How much money does she have total? $82.00.

How many of each coin does she have? We don't know exactly, but we know that she has an equal number of each coin.

CHOOSE A STRATEGY — We know how much each coin is worth, and we know how much all of her coins are worth total, so we can write an equation that models the situation.

SOLVE IT — Let p be the number of pennies, n the number of nickels, d the number of dimes, and q the number of quarters.

We then have the equation $p + 5n + 10d + 25q = 8200$.

We know that she has an equal number of each coin, so $p = n = d = q$. Substituting p for the other variables gives an equation in just one variable. The equation above becomes $p + 5p + 10p + 25p = 41p = 8200$, so $p = 200$.

Lindsey has 200 pennies. Since she has an equal number of each coin, she also has 200 nickels, 200 dimes and 200 quarters. Therefore, she has 800 coins.

LOOK BACK — Did we answer the question asked? Yes.

Does our answer seem reasonable? Yes, we know the answer must be less than 8200 (the number of coins if they were all pennies) and greater than 328 (the number of coins if they were all quarters).

Example — The number of pets in the store in problem #7 of Warm-Up 9 can be represented with a variable. Then, each of the quantities can be represented in terms of that variable, and an equation can be used to solve the problem.

MAKE A TABLE, CHART OR LIST (T)

Making a table, chart, graph, or list is a way to organize data presented in a problem. This problem-solving strategy allows the problem solver to discover relationships and patterns among data.

Use Tree Diagrams or Organized Lists

Customers at a particular yogurt shop may select one of three flavors of yogurt. They may choose one of four toppings. How many one-flavor, one-topping combinations are possible?

FIND OUT — What question do we have to answer? How many flavor-topping combinations are possible?

How many flavors are available? Three.

How many toppings are available? Four.

Are you allowed to have more than one flavor or topping? No, the combinations must have only one flavor and one topping.

CHOOSE A STRATEGY — How could we organize the possible combinations help? With letters and numbers in a list.

SOLVE IT — Make an organized list. Use F and T to denote either flavor or topping. Use the numbers 1 – 3 and 1 – 4 to mark different flavors and toppings.

F1T1, F1T2, F1T3, F1T4
F2T1, F2T2, F2T3, F2T4
F3T1, F3T2, F3T3, F3T4

Now count the number of combinations. There are 12 combinations possible.

LOOK BACK — Did we answer the question asked? Yes.

Does our answer seem reasonable? Yes.

Example — Problem #9 in Workout 7 can and should be solved with a tree diagram or organized list. Although there are not a large number of possibilities, some acceptable routes can be easily overlooked, and a systematic method for keeping track of them will ensure that none are missed.

Make a Chart

How many hours will a car travelling at 45 miles per hour take to catch up with a car travelling at 30 miles per hour if the slower car starts one hour before the faster car?

FIND OUT — What is the question we have to answer? How long does it take for the faster car to catch the slower car.

What is the speed of the slower car? 30 miles per hour.

What is the speed of the faster car? 45 miles per hour.

CHOOSE A STRATEGY — What strategy will help here? We could model this on paper, but accuracy would suffer. We could also use equations. But let's make a table with the time and distance traveled since that will explicitly show what's happening here.

SOLVE IT — Make a table with two rows and four columns.

The rows will identify the cars, and the columns will mark the hours.

Where the rows and columns intersect will indicate distance traveled, since distance equals the speed times the amount of time traveled.

Car \ Hour	1	2	3	4
Slow Car	30	60	90	120
Fast Car	0	45	90	135

At the end of the first hour, the faster car was just starting. At the end of the second hour, the faster car had gone 45 miles. At the end of the third hour, the faster car had gone 90 miles. This equals the distance traveled by the slower car in three hours. So, the faster car only traveled for two hours.

LOOK BACK Did we answer the question asked? Yes.

Does our answer seem reasonable? Yes.

Example The problem of how long it would take one person to paint a room—problem #9 of Warm-Up 5—can be easily understood and solved if the information is organized into a chart.

Guess, Check and Revise (G)

The guess-check-and-revise strategy for problem solving can be helpful for many types of problems. When using this strategy, mathletes are encouraged to make a reasonable guess, check the guess, and revise the guess if necessary. By repeating this process mathletes can arrive at a correct answer that has been checked. Using this strategy does not always yield a correct solution immediately but provides information that can be used to better understand the problem and may suggest the use of another strategy. Students have a natural affinity for this strategy and should be encouraged to use it when appropriate.

To use the guess-check-and-revise strategy, follow these steps:

1. Make a guess at the answer.
2. Check your guess. Does it satisfy the problem?
3. Use the information obtained in checking to help you make a new guess.
4. Continue the procedure until you get the correct answer.

Leah has \$4.05 in dimes and quarters. If she has 5 more quarters than dimes, how many of each does she have?

FIND OUT What are we asked to determine? We need to find how many dimes and how many quarters Leah has.

What is the total amount of money? \$4.05.

What else do we know? There are five more quarters than dimes.

CHOOSE A STRATEGY Will listing combinations help? Yes, but creating an extended list of possible combinations of dimes and quarters could be cumbersome to create.

What other strategy would work? Pick a number, try it, and adjust the estimate.

SOLVE IT Try 5 dimes. That would mean 10 quarters.

5 x $0.10 + 10 x $0.25 = $3.00

Increase the number of dimes to 7.

7 x $0.10 + 12 x $0.25 = $3.70

Try again. This time use 8 dimes.

8 x $0.10 + 13 x $0.25 = $4.05

Leah has 8 dimes and 13 quarters.

LOOK BACK Did we answer the question asked, and does our answer seem reasonable? Yes.

Trevor had 60 markers he could turn in at the end of the year for extra-credit points he had earned during the year. Some markers were worth one point and others were worth two points. If he was entitled to a total of 83 extra-credit points, how many one point markers did he have?

FIND OUT What question are we trying to answer? The question is how many one-point markers did Trevor have.

What is the total number of markers he had? 60.

What were their possible values? One or two points.

What was the total value of all the markers? The markers totalled 83 points.

CHOOSE A STRATEGY How can we approach this problem? Make a table of the possible number of markers and their total value.

SOLVE IT Make a guess as to the first value. We can adjust our guess as we get closer to the desired answer.

Pick 10 as the number of one-point markers. This means he has 50 two-point markers since we know he has 60 markers total. The value of this combination is 110 points.

We can keep track of our guesses in a table by listing the number of one-point markers, the number of two-point markers, and the total number of points various combinations would give.

# of 1-point Markers	# of 2-point Markers	Total Value
10	50	110
50	10	70
40	20	80
38	22	82
37	23	83

Trevor had 37 one-point markers.

LOOK BACK Did we answer the question? Yes.

Does our answer seem reasonable? Yes, we know the answer has to be less than 60. Also, 23 points more than 60 implies that 23 markers were worth 2 points.

Example Problem #2 of Warm-Up 15 can be solved by first choosing two consecutive odd integers, calculating the amount by which their squares differ, and then revising your guess.

CONSIDER A SIMPLER CASE (S)

The problem-solving strategy of simplifying is most often used in conjunction with other strategies. Writing a simpler problem is one way of simplifying the problem-solving process. Rewording the problem, using smaller numbers, or using a more familiar problem setting may lead to an understanding of the solution strategy to be used. Many problems may be divided into simpler problems to be combined to yield a solution. Some problems can be made simpler by working backwards.

Sometimes a problem is too complex to solve in one step. When this happens, it is often useful to simplify the problem by dividing it into cases and solving each one separately.

Divide into smaller problems

Three shapes—a circle, a rectangle, and a square—have the same area. Which shape has the smallest perimeter?

FIND OUT	We want to know which of three shapes has the smallest perimeter.
CHOOSE A STRATEGY	Will any particular strategy help here? Yes, we can compare the perimeters of the shapes pairwise. This will be easier than calculating the area of each since numbers are not given.
SOLVE IT	First, compare the circumference of the circle to the perimeter of the square. They have equal area, so the area of the circle, πr^2, equals the area of the square, s^2. Consequently, the perimeter of the square will be slightly greater than the circumference of the circle. Next, compare the perimeter of the square to the perimeter of the rectangle. A square is the quadrilateral which has minimum perimeter, so the perimeter of the square must be less than the perimeter of the rectangle. By the transitive property, then, the perimeter of the rectangle will be greater than the circumference of the circle. Hence, the circle has the smallest perimeter.
LOOK BACK	Did we answer the question asked? Yes. Does our answer make sense? Yes. If we arbitrarily choose 100 units^2 as the area of each shape, the circumference of the circle is roughly 35.5 units, the perimeter of the square is 40 units, and the perimeter of the rectangle could be any amount greater than 40 units and less than 100 units.
Example	By first seeing what numbers leave remainder 1 when divided by 3 and comparing those with the numbers that leave remainder 1 when divided by 4, the solution to problem #2 in Warm-Up 14 is a step closer. Then comparing the results with the numbers that leave remainder 1 when divided by 5 gives the answer.

Work Backwards

A student needs at least a 95% average to receive a grade of A. On the first three tests the student averaged 92%. What is the minimum a student must average on the last two tests to receive a grade of A?

FIND OUT	We are asked to find what a student must average on her last two tests to get an A. What average is required for an A? 95%. How many tests will be figured into the average? Five. How many test has she taken so far? Three. What is her average on the first three tests? 92%.

CHOOSE A STRATEGY — What strategy would work well in this situation? Work backwards from the minimum required average needed for an A to find the scores needed on the last two tests.

SOLVE IT — Work backwards from the required average on all five tests.

The average of the tests must be 95%. There are five tests so the total number of points scored on the five tests must be, at least, 5 x 95 = 475.

So far the student averages 92% on three tests. While we don't know all of the individual scores, the total number of points scored on the three tests must be 3 x 92 = 276.

475 points required minus 276 scored so far equals 199 required on the next two tests.

199 divided by 2 equals 99.5.

The student must average 99.5% on her next two tests if she is to get an A.

LOOK BACK — Did we answer the question asked? Yes.

Does our answer seem reasonable? Yes, we knew we were looking for a number between 95 and 100.

Example — The cube in problem #1 of Warm-Up 10 is known to have surface area 600 square inches. By working backwards, it becomes obvious that the area of each face is 100 square inches, the length of an edge is 10 inches, and the volume is 1000 cubic inches.

ELIMINATE (E)

The strategy of elimination is commonly used by people in everyday life. In a problem-solving context, mathletes must list and then eliminate possible solutions based upon information presented in the problem. The act of selecting a problem-solving strategy is an example of the elimination process. Logical reasoning is a problem-solving strategy that is used in all problem-solving situations. It can result in the elimination of incorrect answers, particularly in "if-then" situations and in problems with a listable number of possible solutions.

What is the largest two-digit number that is divisible by 3 whose digits differ by 2?

FIND OUT — What are we asked to find? A certain number.

What do we know about the number? The number is less than 100. It is divisible by 3. The digits of the number differ by 2.

CHOOSE A STRATEGY — What strategy will help here? Working backwards from 99, list numbers and eliminate those that do not satisfy the conditions given. (Notice that we have already eliminated numbers greater than 99.)

SOLVE IT — 99, 98, 97, 96, 95, 94, 93, 92, 91, 90,
89, 88, 87, 86, 85, 84, 83, 82, 81, 80,
79, 78, 77, 76, 75, 74, 73, 72, 71, 70, . . .

Eliminate those numbers that are not divisible by 3:

99, ~~98~~, ~~97~~, **96**, ~~95~~, ~~94~~, **93**, ~~92~~, ~~91~~, **90**,
~~89~~, ~~88~~, **87**, ~~86~~, ~~85~~, **84**, ~~83~~, ~~82~~, **81**, ~~80~~,
~~79~~, **78**, ~~77~~, ~~76~~, **75**, ~~74~~, ~~73~~, **72**, ~~71~~, ~~70~~, . . .

From these, eliminate all numbers whose digits do not differ by 2:

~~99~~, ~~96~~, ~~93~~, ~~90~~, ~~87~~, ~~84~~, ~~81~~, ~~78~~, **75**, ~~72~~, . . .

75 is the largest number that remains.

LOOK BACK Did we answer the question asked? Yes.

Do we have a two-digit number divisible by 3 whose digits differ by 2? Yes.

Example Two-digit numbers that are not prime can be eliminated when solving problem #5 of Warm-Up 1. Then, the two largest numbers remaining can be multiplied to find an answer.

LOOK FOR PATTERNS (P)

When mathletes use this problem-solving strategy, they are required to analyze patterns in data and make predictions and generalizations based on their analysis. Mathletes then must check the generalization against the information in the problem and possibly make a prediction from, or extension of, the given information. A pattern is a regular, systematic repetition. A pattern may be numerical, visual or behavioral. By identifying the pattern, you can predict what will come next and what will happen again and again in the same way. Looking for patterns is a very important strategy for problem solving, and is used to solve many different kinds of problems. Sometimes you can solve a problem just by recognizing a pattern, but often you will have to extend a pattern to find a solution. Making a number table often reveals patterns, and for this reason it is frequently used in conjunction with this strategy.

Laura was given an ant farm by her grandparents for her 13th birthday. The farm could hold a total of 100,000 ants. Laura's farm had 1500 ants when it was given to her. If the number of ants in the farm on the day after her birthday was 3000 and the number of ants the day after that was 6000, in how many days will the farm be full?

FIND OUT We need to know when the ant farm will be full.

How many ants will the farm hold? 100,000.

How many ants are in the farm the first day? 1500.

How many ants are in the farm the second day? 3000.

How many ants are in the farm the third day? 6000.

CHOOSE A STRATEGY Is a pattern developing? Yes, each day twice as many ants are in the farm as the day before. Make a table to count the ants systematically.

SOLVE IT Draw a table with two lines for numbers.

The top line is the number of days after Laura's birthday, and the bottom line is the number of ants in the farm on that day.

# days	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
# ants	1500	3000	6000	12,000	24,000	48,000	96,000	192,000

The ant farm will be full seven days after her birthday.

LOOK BACK Read the question again. Did we answer all of the question? Yes.

Does our answer seem reasonable? Yes.

What assumption are we making? We are assuming that the pattern—the number of ants doubles each day—continues indefinitely.

Example Problem #5 of Warm-Up 3 can be solved by identifying a pattern. A cycle of four digits repeats when powers of 13 are divided by 5, and recognizing that cycle is certainly easier than calculating an exact value.

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Teacher's Syllabus

The problems for the MATHCOUNTS school coaching program are separated into three sections: Stretches, Warm-Ups and Workouts. The Stretches are collections of problems centered around a specific topic. Each Warm-Up contains problems that generally survey the 7th- and 8th-grade mathematics curriculum. And each Workout contains multi-step problems that challenge students to apply their mathematical skills.

The problems are designed to provide mathletes with a large variety of challenges. These materials may be used as the basis for an exciting extra-curricular mathematics club or may simply supplement the normal junior high mathematics curriculum.

STRETCHES

The Stretches are included so that students may have in-depth practice on specific mathematical topics. The Stretches can be used as prerequisites to the Warm-Up and Workout problems, because they focus on concepts that are prevalent throughout the Handbook. The Stretches can also be used after the Warm-Ups and Workouts to reinforce concepts that students learn while solving those problems.

The Stretches can easily be incorporated into a unit which focuses on the concept covered.

The answers to the Stretches can be found on page 44.

WARM-UPS AND WORKOUTS

The Warm-Ups and Workouts can be used to provide students with practice in a variety of problem-solving situations. All of the problems may be used to diagnose skill levels, to practice and apply skills, or to evaluate growth in skills.

The Warm-Up and Workout problems can be used to prepare for MATHCOUNTS competitions. The wording and content of these handbook problems is consistent with the contest problems. The Warm-Ups may be used as practice for the Sprint Round, and the Workouts may be used as practice for the Target and Team Rounds. Along with discussion and review of the solutions, it is recommended that mathletes be provided with opportunities to present solutions to problems from the Workouts as preparation for the Masters Round.

The answers to each Warm-Up and Workout are given on a preceding answer page. All answers have been coded as to applicable problem-solving techniques. A discussion of these problem-solving techniques begins on page 19.

SCHEDULE

The Stretches should be used as necessary throughout the coaching phase. The following chart is the recommended schedule for using the Warm-Ups and Workouts:

October	WARM-UPS 1–2	WORKOUT 1	EXTENDED ACTIVITY 1
November	WARM-UPS 3–6	WORKOUTS 2–3	
December	WARM-UPS 7–12	WORKOUTS 4–6	EXTENDED ACTIVITY 2
January	WARM-UPS 13–16	WORKOUTS 7–8	
	School Level MATHCOUNTS Competition		
	WARM-UPS 17–18	WORKOUT 9	
February	Selection of School Team		
	Chapter Level MATHCOUNTS Competition		

If you have questions or comments concerning the Stretches, Warm-Ups or Workouts, please contact:

MATHCOUNTS Curriculum Coordinator
1420 King Street
Alexandria, VA 22314-2794
(703) 684-2859
E-mail: mathcounts@nspe.org

Vocabulary and Formulas

The following list is representative of terminology used in the problems but should not be viewed as all-inclusive. It is recommended that coaches review this list with their mathletes.

absolute value
acute angle
additive inverse (opposite)
adjacent angles
algorithm
alternate interior angles
alternate exterior angles
altitude (height)
area
arithmetic mean
arithmetic sequence
base 10
binary
bisect
center
chord
circle
circumscribe
circumference
coefficient
collinear
combination
common divisor
common denominator
common factor
common fraction
common multiple
complementary angles
composite number
compound interest
concentric
cone
congruent
convex
coordinate plane/system
coordinates of a point
corresponding angles
counting numbers
cube
cylinder
data
decimal
degree measure
denominator
diagonal of a polygon
diagonal of a polyhedron
diameter
difference
digit
direct variation
dividend
divisible
divisor
edge
endpoint
equation
equiangular
equidistant
equilateral
evaluate
exponent
expression
exterior angle of a polygon
factor
factorial
Fibonacci sequence
finite
formula
frequency distribution
frustum
function
geometric sequence
height (altitude)
hemisphere
hexagon
hypotenuse
image(s) of a point(s) (under a transformation)
improper fraction
inequality
infinite series
inscribe
integer
interior angle of a polygon
intersection
inverse variation
irrational number
isosceles
lateral surface area
lateral edge
lattice point(s)
linear equation
median of a set of data
median of a triangle
midpoint
mixed number
mode(s) of a set of data
multiple
multiplicative inverse (reciprocal)
natural number
numerator
obtuse angle
octagon
octahedron
opposite of a number (additive inverse)
ordered pair
origin
palindrome
parallel
parallelogram
Pascal's triangle
pentagon
percent increase/decrease
perimeter
permutation
perpendicular
planar
polygon
polyhedron
prime factorization
prime number
prism
probability
product

proper divisor
proper factor
proper fraction
proportion
pyramid
Pythagorean triple
quadrant
quadrilateral
quotient
radius
random
range of a data set
rate
ratio
rational number
ray
real number
reciprocal (multiplicative inverse)
rectangle
reflection
regular polygon
relatively prime
remainder
repeating decimal
revolution
rhombus
right angle
right circular cone
right circular cylinder
right polyhedron
right triangle
rotation
scalene triangle
scientific notation
segment of a line
semicircle
sequence
set
similar figures
simple interest
slope
slope-intercept form
solution set
sphere
square
square root
stem-and-leaf plot
sum
supplementary angles
system of equations/ inequalities
tangent figures
tangent line
term
terminating decimal
tetrahedron
total surface area
transformation
translation
trapezoid
triangle
trisect
union
unit fraction
variable
vertical angles
vertex
volume
whole number
x-axis
x-coordinate
x-intercept
y-axis
y-coordinate
y-intercept

The list of formulas below is representative of those needed to solve MATHCOUNTS problems but should not be viewed as the only formulas which may be used. Many other formulas which are useful in problem solving should be discovered and derived by mathletes.

CIRCUMFERENCE

Circle	$c = 2 \cdot \pi \cdot r$
	$c = \pi \cdot d$

AREA

Square	$A = s^2$
Rectangle	$A = l \cdot w$
Parallelogram	$A = b \cdot h$
Trapezoid	$A = \frac{1}{2} \cdot (b_1 + b_2) \cdot h$
Circle	$A = \pi \cdot r^2$
Triangle	$A = \frac{1}{2} \cdot b \cdot h$

SURFACE AREA & VOLUME

Sphere	$SA = 4 \cdot \pi \cdot r^2$
Sphere	$V = \frac{4}{3} \cdot \pi \cdot r^3$
Rectangular prism	$V = l \cdot w \cdot h$
Circular cylinder	$V = \pi \cdot r^2 \cdot h$
Circular cone	$V = \frac{1}{3} \cdot B \cdot h$
Pyramid	$V = \frac{1}{3} \cdot B \cdot h$

PYTHAGOREAN THEOREM

Right triangle	$a^2 + b^2 = c^2$

Extended Activities

The following pages offer extended, open-ended activities that develop mathematics skills within real-world contexts. These activities are designed to supplement classroom curricula or as extensions to a MATHCOUNTS coaching program.

Extended Activity 1: Eat Right!

The first activity has students evaluate their eating habits. Through body mass index and ideal weight calculations, students will determine the number of calories that a person of their height and frame size should consume to maintain an ideal weight.

Students will then analyze one day of their own diet by recording all of the foods eaten in a 24-hour period. Based on percentages of calories from fat, carbohydrate and protein, comparison with the guidelines given by the food pyramid, and from the calculations of ideal weight, students will analyze their diets.

The final result of the project is a written analysis of the student's diet. They are to include a chart listing foods consumed as well as all calculations. In this analysis, students are to suggest one change in their diet to make it more aligned with recommendations or provide an explanation as to why their diet needs no modification.

Extended Activity 2: Leavin' on a Jet Plane

The second activity has students devise a flight plan for the NASA 8 turbo-prop plane. Working as assistants to the administrator, students are to design a plan that will allow the administrator to visit each of seven NASA field centers while traveling the least distance in the least time and consuming a minimum amount of fuel. Students are to determine an acceptable route for the NASA 8 so that each of the centers is visited exactly once.

Many algorithms have been devised to find the optimal route when considering problems of this nature. Although several algorithms will find good routes, no algorithm has been found which will necessarily find the optimal route. The number of possible routes to be checked can be very large (for visiting just ten destinations, there are 9! = 362 880 possible routes). Consequently, mathematicians often use an algorithm which gives a good—though not necessarily optimal—route, because these algorithms can produce results very quickly. While completing this activity, students should be encouraged to find algorithms which generate acceptable results while finding a route fairly quickly.

The final result of this project is a presentation of a designed flight plan. This presentation should include a graphic representation (such as a chart or spanning tree) indicating how the flight plan was designed and possibly a supplemental written report.

Usage

Integrate this material into your mathematics classroom or coaching sessions. The activities were designed to be used by students working individually or in small groups. The activities can be used as day-long lessons or can be expanded to become in-depth student projects.

Each activity contains a teacher's guide which presents an overview of the activity; a student guide which explains the activity; and information and details required to adequately complete each project.

Teachers may wish to alter these activities to meet your own classroom needs; in fact, you are encouraged to modify these activities to make them appropriate for your students. You are also encouraged to extend these activities where appropriate. Each activity may be extended by having the students apply the mathematics involved to a real situation that they discover or by requiring that students give a presentation to other members of the class. Other possible extensions for each activity are listed on the accompanying teacher's guide.

Finally, students should be encouraged to find other sources which supplement the information provided. The student guide and information sheet which accompany each activity provide the minimum information that the students will need to successfully complete the activity. Possible alternative resources include reference materials in the school and public libraries, access to the world wide web, or local professionals who might be familiar with the demands of the activity.

Extended Activity 1:

Eat Right!

Teacher's Guide

In this activity, students will explore their personal eating habits and use mathematics to determine how healthy (or unhealthy) their habits are. Students will compute the number of daily calories they should consume; calculate the percent of their daily caloric intake that is fat, carbohydrate and protein; and compare their habits with the suggested servings detailed in the US Department of Agriculture's food pyramid. In addition to using computation skills to calculate percents, they will also use data analysis to critique their diets; algebraic reasoning to compute fat, carbohydrate and protein content and to calculate their ideal weight; and logic to determine how to eat healthier.

Materials and Resources

In order for students to complete this project effectively, a nutritional booklet which lists the caloric, fat, carbohydrate and protein content of various foods should be available. One such book, *The Diabetes Carbohydrate and Fat Gram Guide*, is available from the American Diabetes Association. It can be ordered from their web site, **http://merchant.diabetes.org/adabooks**, or by calling 1-800-DIABETES. Similar books are available at most bookstores. A calorie counter available on the web which gives the caloric and fat content of many foods is at **http://www.caloriecontrol.org**, the home page of the Calorie Control Council.

Objectives:

1. Students will calculate the number of calories they should consume daily using the formula $c = kw$, where c is the number of calories that should be consumed daily, k is a coefficient determined by level of daily activity, and w is the preferred weight for a person.
2. Students will calculate their BMI (body mass index) and determine an estimate of their desired weight based on the BMI.
3. Using information provided regarding ideal weight, students will be able to generate the ideal weight formula for men, $I = 106 + 6(h - 60)$, where I is ideal weight in pounds and h is height in inches. For women, the formula is $I = 100 + 5(h - 60)$.
4. Students will record for one day (a 24-hour period) the foods they consume and calculate the caloric percentage of fat, carbohydrate and protein of that daily intake.
5. Students will analyze their diet by comparing their daily intake to guidelines from the US Department of Agriculture by percentage of fat, carbohydrate and protein as well as through a comparison of the food pyramid.

Eat Right!

A healthy diet consists of two parts: eating the right foods, and eating the proper amounts of the right foods. Based on suggested percentages of fat, carbohydrate and protein, a diet consisting of only pasta may seem appropriate, but such a diet would lack the balance necessary to ensure that all the vitamins and minerals required by the body are being consumed. Conversely, a diet composed of various foods may provide an adequate amount of nutrients but too many calories. Consideration of both aspects—a balance of foods, while getting the proper number of calories and adequate percentages of fat, carbohydrate and protein—is important when assessing your diet.

This project will analyze your diet by considering the food you eat for one day.

Over the course of one full day (a 24-hour period), record all of the foods you eat. Be sure to include the butter on a piece of bread (it's not enough to count just the bread) as well as the three M&M's that a friend gives you at lunch. Also include beverages you drink—although the caloric content of water is negligible, other drinks can contribute more than 100 calories! When possible, record the **number of calories** and the **number of grams of fat, carbohydrate and protein** for each item. Be complete; although you won't be able to get all of this information for every item, keep as thorough a record as possible. This information is contained on the nutrition label of most items.

After recording all of your foods, use a reference guide to determine the caloric, fat, carbohydrate and protein content of those items for which you don't already have information. Then compute the total number of calories you consumed, and determine the number of grams of fat, carbohydrate and protein. Finally, compute the percentage of fat calories, carbohydrate calories and protein calories in your daily intake.

Project Requirements

Upon completion of these calculations, write a report which analyzes your diet. Include a chart similar to the one shown below which indicates the foods you consumed, the total calories consumed, and the percentage of fat, carbohydrate and protein. Also include calculations of your ideal weight and the number of calories a day you should consume based on your ideal weight. Compare your diet with the recommended servings given in the food pyramid. Finally, conclude by suggesting possible changes in your diet that would make your eating more healthy and aligned with the guidelines, or explain why your diet needs no modification.

Things to Consider:

- What is your ideal weight? How many calories should you consume to maintain your ideal weight?
- What percent of your diet is fat, carbohydrate or protein? What should those percents be?
- How does the food you eat compare with the recommendations given in the food pyramid?

Food Item	Quantity Eaten	Calories	Fat (g)	Carbohydrate (g)	Protein (g)
________	________	________	________	________	________
________	________	________	________	________	________
________	________	________	________	________	________
________	________	________	________	________	________
________	________	________	________	________	________

Fact Sheet

Ever wonder how many candy bars a day is too many? Ever wonder how much a person your height ought to weigh? Wanna know how much and which foods to eat to look and feel your best?

Body Mass Index

The body mass index (BMI) is a comparison of a person's height and weight. The BMI is calculated as the ratio of weight (in kilograms) to height (in meters) squared. The BMI is one measure for determining ideal weight. Federal guidelines suggest that a person should have a BMI of 25. A BMI lower than 25 is inconsequential, unless the BMI is below 20, at which point it is an indication of malnutrition. A BMI of 27–28 indicates that a person is about 20% overweight, which greatly increases the risk for certain diseases. A BMI of 30 or above is an indication of obesity.

Ideal Weight

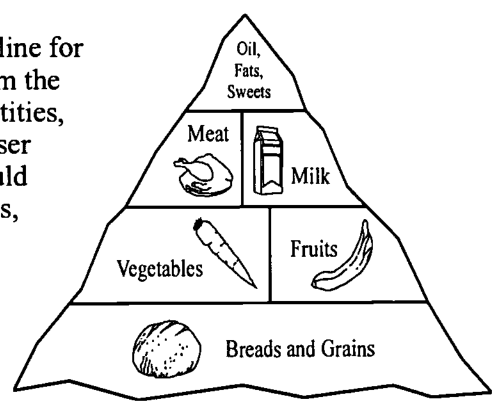
A simple formula used by some doctors to estimate ideal weight for women is 100 pounds for the first 5 feet of height plus 5 pounds for each additional inch. Similarly for men, ideal weight is 106 pounds for the first 5 feet and 6 pounds for each additional inch. A person with a large frame should then add 10%, and a person with a small frame should subtract 10%. (To determine your frame size, place your thumb and index finger around your wrist: if they overlap, you have a small frame; if they barely touch, you have a medium frame; and if they do not touch, you have a large frame.)

Daily Intake

According to Judy Fitzgibbons, M.S., R.D., L.D., on the "Fuel for Fitness" web site, the number of calories a person should consume per day is based on the person's ideal weight and the level of activity a person maintains. The number of calories a sedentary person should consume is equal to 13 times their weight in pounds. A moderately active person should consume 15 times their weight, while a very active person should consume 17 times their weight.

The Food Pyramid

In 1992, the US Department of Agriculture produced a guideline for dietary intake known as the Food Pyramid. The name comes from the fact that foods better for you should be consumed in greater quantities, and foods that are not as good for you should be consumed in lesser quantities. Based on these daily recommendations, a person should eat 6–11 servings of breads and grains, 3–5 servings of vegetables, 2–3 servings of fruit, 2–3 servings of meat and 2–3 servings of milk. Fats, oils and sweets should be consumed only in moderation.



A serving in the bread group is one slice of bread, one ounce of cold cereal, or one-half cup of rice or pasta. A serving of vegetables is one cup of raw greens or one-half cup of chopped vegetables. A fruit serving is an apple or an orange, a melon wedge, one-half cup of berries, or three-fourths cup of juice. A serving in the milk group is a cup of milk or 1.5 ounces of cheese. A meat serving is three ounces lean meat, poultry or fish, one egg, one-half cup of cooked beans or two tablespoons of seeds or nuts.

Calories

One gram of fat contains 9 calories, whereas one gram of carbohydrate or protein contains only 4 calories. The combined fat, carbohydrate and protein of any food constitutes its entire caloric content. For example, the nutritional label for vanilla ice cream says that one-half cup contains 270 calories, 18g fat and 21g carbohydrate. Consequently, it contains $270 - 18(9) - 21(4) = 24$ calories which come from protein, and since there are 4 calories per gram of protein, the ice cream contains 6g protein. According to many nutritionists, a healthy diet contains 58% carbohydrate calories, 12% protein calories, and 30% fat calories.

Extended Activity 2:

Leavin' on a Jet Plane

Teacher's Guide

Students will use this activity to explore mathematics in a realistic way. Students are to prepare a flight plan for the NASA 8 turbo-prop plane to visit each of seven NASA field centers. The mathematics students will use include the concepts of measurement, computation, discrete mathematical models (such as graphs and trees), and information arrangement, organization and communication.

NASA has enlisted your students to assist in designing an optimal flight plan for the NASA 8 which carries the agency's administrator to NASA's field centers. The flight plan must include a stop at each of seven centers. NASA 8 must stop at each of the field centers exactly once, and the flight plan must begin and end at the same location (Goddard Space Flight Center) to form a round trip.

Serving as assistants to NASA's administrator, your students will be asked to design a flight plan and create a diagram to identify the best route according to specified criteria, such as the least total distance, the minimum amount of time, and the least fuel consumption.

This activity is based on a famous problem in mathematics, the *Traveling Salesman Problem*. To find the shortest possible trip when visiting just three centers, A, B and C, there are only two possible routes to consider: A-B-C or A-C-B. But for seven centers, there are 6! = 720 possible routes to check. Because checking so many possibilities is cumbersome, mathematicians have devised several algorithms which will generate good—although not necessarily optimal—routes. One algorithm, called the *nearest neighbor* algorithm, considers the distances from the starting center to each of the other centers and chooses the shortest distance; then, the distances from that center to each of the centers not yet visited are considered, and the shortest distance is chosen; and so on, for each of the remaining centers. While working on this activity, students should attempt to create legitimate algorithms for determining good routes.

Objectives:

1. Students will design a flight plan for the NASA 8. The flight plan must include a visit to each of nine NASA field centers exactly once as well as originate and end at the same center.
2. Students will determine an acceptable algorithm for designing a short route to visit each field center.
3. Students will compute the time required for NASA 8 to complete the shortest route identified.
4. Students will find the amount of fuel required to complete the route identified. Additionally, students will determine impossible flights because the amount of fuel needed is more than the fuel tanks hold.
5. Students will create a graphical representation or model (such as a spanning tree) which presents their solution for a flight plan.

Possible Extensions

- Adapt the problem so that the flight plan needs to visit fewer centers. The optimal flight plan for four centers is significantly easier to find than the optimal flight plan for seven centers. Then, discuss possible algorithms for finding a good flight plan (although not necessarily optimal) when the number of destinations is large.
- Have students solve analogous problems using other locations and distances. Perhaps the locations could be stops on a dream vacation, or maybe the students can pretend to visit various world leaders.
- Students can attempt to improve the route taken by their school transportation system. Perhaps the best project can be presented at a meeting of the school board or parent-teacher association.
- Modify the problem so that the limiting factors are not distance but cost—for example, the costs of flights on commercial airlines.
- Add additional constraints and parameters to the problem. For example, consider the amount of time required to refuel at each location, the number of miles that can be traveled before refueling is necessary, and so on.

Leavin' on a Jet Plane

Background

NASA's administrator is required to visit each of seven NASA field centers. The administrator must submit a flight plan prior to each departure. Serving as the administrator's assistant, you have been asked to design a flight plan whose route requires the least amount of flying distance. While constructing this flight plan, consider also the traveling time and fuel consumption. The NASA 8 turbo-prop plane which carries the administrator must stop exactly once at each field center and must return to its point of origin. The field center at Goddard is closest to NASA Headquarters in Washington, DC, so the flight plan should use Goddard as its origin.

Project Requirements

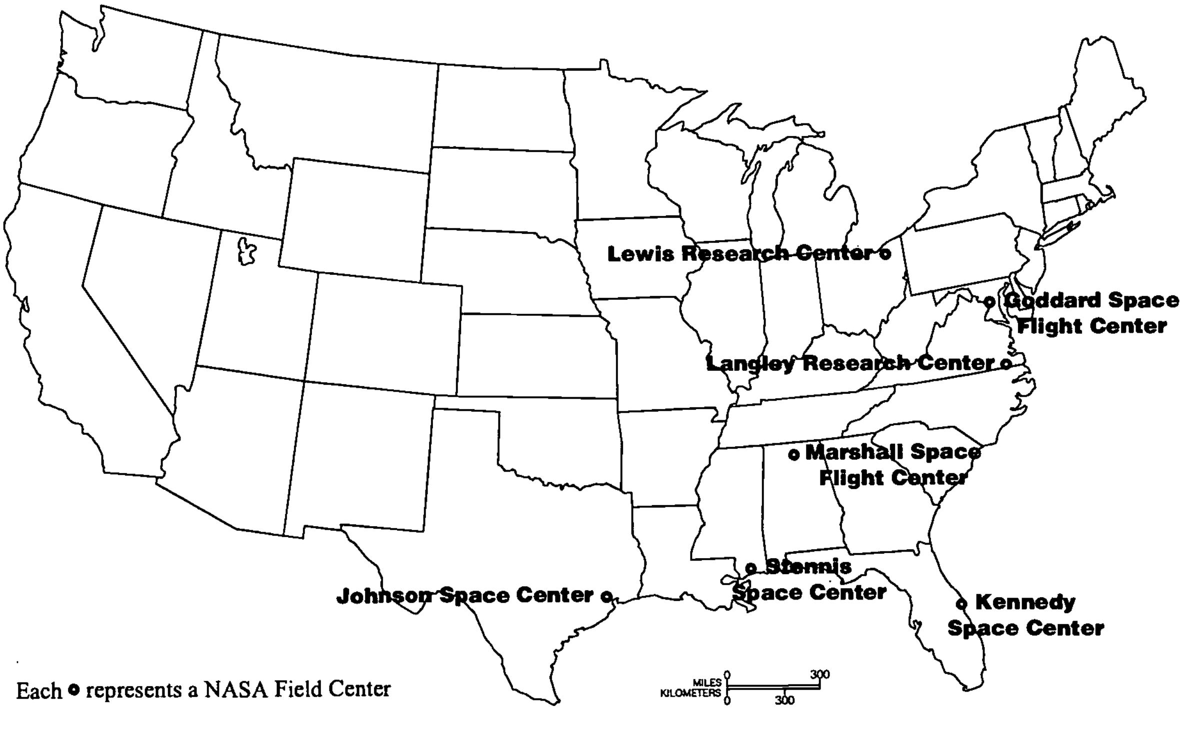
A flight plan needs to be created which indicates the best route for visiting the seven centers. A graphical representation or model (such as a spanning tree) may be designed to show the best route, although other graphic representations are certainly possible. The flight plan must include the total distance traveled, the amount of time to complete the trip, and the amount of fuel consumed.

Specifications

The NASA 8 flies at a speed of 270 knots (300 mph). Its fuel tanks can hold 544 gallons of JP4 fuel, which weighs 6.5 lbs/gal. Its fuel consumption is 800 lbs of fuel for the first hour of flight, 700 lbs for the second hour, and 600 lbs for each hour thereafter.

Things to Consider:

- Choose four centers and generate all possible flight plans; then, choose the best one. How is the problem similar for seven centers? How is it different?
- Generating all possible flight plans can be cumbersome. What would be a method to choose a good flight plan that requires less work but still gives a solution which is *close* to optimal?
- How are distance, time and fuel consumption related? What limit does fuel consumption impose on the maximum distance of a flight?
- What type of models might be helpful? Would a table or graph help? How about a tree diagram?



Each • represents a NASA Field Center

Stretches

The following pages contain three Stretches, segments created to give mathletes practice with specific subject areas. The Stretches for 1998–99 focus on the following areas:

Statistics	Data Analysis and Representation
Algebra	Symbolic Manipulation and Algebraic Thinking
Probability	Selections and Geometric Probability

The Stretches can be used in several different ways. As part of the regular curriculum, the Stretches can easily be included as part of regular classroom work during an appropriate unit. They can be used to teach problem-solving skills, to practice and apply skills, to diagnose skill levels, or to evaluate growth in skills.

As part of the MATHCOUNTS coaching phase, the Stretches can be used to prepare mathletes for more advanced problem-solving situations. They can be used prior to the Warm-Ups and Workouts to introduce mathematical topics, or they can be used to teach and reinforce concepts after mathletes have attempted the Warm-Ups and Workouts. Finally, they can be used when preparing for competition to aid mathletes with troublesome concepts.

Answers to the Stretches include one-letter codes indicating appropriate problem-solving strategies. These strategies are explained on pages 19–29. Some problems have a calculator icon indicating which problems might be more easily solved with a calculator.

MATHCOUNTS Symbols and Notation

Standard abbreviations have been used for units of measure. Complete words or symbols are also acceptable. Square units or cube units may be expressed as $units^2$ or $units^3$.

Statistics Stretch

1. What is the positive difference between the mean of set $A = \{12, 13, 23, 34, 143\}$ and the mean of set $B = \{12, 13, 23, 34, 43\}$?

1. ________

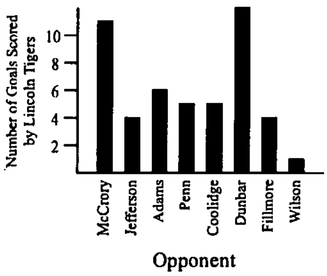
2. What is the positive difference between the range and the median of the set $\{1, 3, 3, 5, 5, 5, 7, 7, 7, 7\}$?

2. ________

3. In the town of Olney, a survey was conducted. Of all people surveyed, 241 750 were identified as employed, and 3.3% of were identified as unemployed. How many people were surveyed?

3. ________

4. According to the chart shown, in what percent of the games did the Lincoln Tigers field hockey team score more than 8 goals? Express your answer to the nearest whole percent.



4. ________

5. In Clark, $\frac{1}{2}$ of 1% of the town's 100 000 families have incomes over \$80 000 per year. How many families with incomes over \$80 000 per year are in Clark?

5. ________

6. A set of three numbers has an average of 8. What number should be added to the set to make the average of all four numbers 9?

6. ________

7. Kelsey scored better than Terry on 7 of the 9 holes of golf they played, and their scores on each hole are shown below. If compared by average number of strokes, by how many strokes per hole is Terry's average better than Kelsey's? Express your answer as a decimal to the nearest hundredth.

7. ________

Hole #	1	2	3	4	5	6	7	8	9
Kelsey	3	4	4	3	11	10	4	3	3
Terry	4	6	5	4	3	4	5	5	6

8. A man walks down a street and happens across a ten-dollar bill. Having \$40 in his pocket, he thinks, "Cool, I just increased my money by 25%." Later, he discovers that he lost a \$5-dollar bill. He thinks, "That's OK. I increased my money by 25% earlier, and I decreased my money by 10% now, so I'm still 15% ahead." His logic is wrong—what is the percent by which the amount of money in his pocket actually increased? Express your answer to the nearest tenth.

8. ________

9. A baseball player's batting average is computed as the ratio of hits to at-bats. Hammerin' Hank had 160 hits in 452 at-bats. Joltin' Joe had 148 hits in 421 at-bats. Whose average was higher?

9. ________

10. How many modes does the set $\{1, 2, 2, 3, 3, 4, 4, 5, 5, 6\}$ have?

10. ________

Algebra Stretch

1. Solve for p: $p + 17 = 3p + 9$. 1. ______

2. Bobbi has nine stamps, some of which are 32¢ stamps and some of which are 23¢ stamps. If the total value of all the stamps is \$2.52, how many of her stamps are worth 32¢? 2. ______

3. What is the value of the following expression when $k = 4$ and $\ell = 6$? 3. ______

$$k^3 + k - k\ell + 3\ell - 7$$

4. What is the sum of all values of q for which $q^2 + 7q = 120$? 4. ______

5. In the following equation, what is the product of all possible values for r? 5. ______

$$\frac{3}{5r} = \frac{r}{15}$$

6. Given the following simultaneous equations, what is the value of the sum $a + b$? 6. ______

$$3a + 4b = 59$$
$$4a + 3b = 53$$

7. Shelley has three pigs. She unfortunately doesn't have a scale, but the farm on which she lives has a large balance. Using the balance, Shelley found that Pinky weighs as much as Oinky and Binky combined, that Oinky and Binky weigh the same amount, and that she weighs as much as Pinky and Oinky combined. Shelley also knows that she weighs 120 lb. What is the number of pounds in Pinky's weight? 7. ______

8. The *Perfect Turkey* cookbook offers the following formula: 8. ______

$$t = 16w + 15$$

A turkey weighing w pounds should be cooked t minutes. Using this formula, how many minutes should a turkey weighing 15 pounds be cooked?

9. For what positive value of m will the following equation be true if $n = 5$? 9. ______

$$(m + n)^2 + (m - n)^2 + m^2 + n^2 = 318$$

10. *Twistee* is 40 percent cranberry juice. Write a formula for the percent of cranberry juice in a mixture of t quarts Twistee and j quarts orange juice. Express your answer as a common fraction in terms of t and j. 10. ______

Probability Stretch

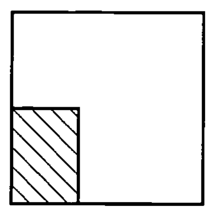
1. A bag contains 3 red and 5 blue marbles. If one marble is randomly drawn from the bag, what is the probability that it is blue? Express your answer as a common fraction. **1.** ____________

2. A pocket full of change consists of 5 nickels, 2 dimes and 1 quarter. Two coins are randomly chosen from the bag without replacement. What is the probability that their combined value is 15¢? Express your answer as a common fraction. **2.** ____________

3. Two fair coins are flipped. What is the probability that both show heads? Express your answer as a common fraction. **3.** ____________

4. Two of the digits 1–5 are selected at random without replacement. What is the probability that the positive difference between the two numbers is 3? Express your answer as a common fraction. **4.** ____________

5. The height of the shaded rectangle shown is one-half the height of the square, and the width of the rectangle is one-third the width of the square. If a point is chosen at random inside the square, what is the probability that it is also inside the rectangle? Express your answer as a common fraction. **5.** ____________



6. Chi-Bin selected a positive multiple of 7 less than 70. Chi-Kai selected a positive multiple of 11 less than 70. What is the probability that they selected the same integer? **6.** ____________

7. Alex has a $\frac{3}{10}$ chance of making a free throw. What is the probability that she will make both of her next two free throws? Express your answer as a common fraction. **7.** ____________

8. Tiles numbered 1–6 are each placed randomly into one of three different boxes. What is the probability that each box contains 2 tiles? Express your answer as a common fraction. **8.** ____________

9. Matt is given a jar with eight marbles, six of which are red and two of which are green. Matt then randomly draws marbles without replacement. If he selects a red marble, he puts it in the first box; similarly, he continues to place all red marbles he selects into the first box until a green marble is chosen. When a green marble is chosen, he puts the first box aside. He then selects and places red marbles in the second box until the second green marble is chosen. When the second green marble is selected, he will put the second box aside and place all remaining red marbles in a third box. After all marbles are drawn, what is the probability that each box contains two red marbles? Express your answer as a common fraction. **9.** ____________

10. One letter is randomly selected from the word *MATH* and one letter is randomly selected from the word *COUNTS*. What is the probability that both letters selected are consonants? Express your answer as a common fraction. **10.** ____________

ANSWER KEY – Statistics Stretch

1.	20	(CF)	**2.**	1	(C)	**3.**	250 000	(C)
4.	25	(FM)	**5.**	500	(C)	**6.**	12	(F)
7.	0.33	(C)	**8.**	12.5	(CE)	**9.**	Hank's	(CF)
10.	4	(C)						

ANSWER KEY – Algebra Stretch

1.	4	(FG)	**2.**	5	(MG)	**3.**	55	(C)
4.	-7	(CM)	**5.**	-9	(MG)	**6.**	16	(CTG)
7.	80	(FMG)	**8.**	255	(F)	**9.**	9	(CG)
10.	$\frac{40t}{t+j}$	(F)						

ANSWER KEY – Probability Stretch

1.	$\frac{5}{8}$	(MT)	**2.**	$\frac{5}{14}$	(MT)	**3.**	$\frac{1}{4}$	(MT)
4.	$\frac{1}{5}$	(T)	**5.**	$\frac{1}{6}$	(FM)	**6.**	0	(TEP)
7.	$\frac{9}{100}$	(FM)	**8.**	$\frac{10}{81}$	(FM)	**9.**	$\frac{1}{28}$	(MT)
10.	$\frac{1}{2}$	(CT)						

Warm-Ups and Workouts

Answers to the Warm-Ups and Workouts include one-letter codes, in parentheses, indicating appropriate problem solving strategies for that specific problem. A discussion of these stragtegies appears on pages 19–29. It should be noted that the strategies indicated may not be the only applicable strategies. Additionally, some problems have a calculator icon indicating possible calculator use.

The following codes will be used in the answer keys:

(C) Compute or Simplify
(F) Use a Formula
(M) Make a Model or Diagram
(T) Make a Table, Chart or List
(G) Guess, Check and Revise
(S) Consider a Simpler Case
(E) Eliminate
(P) Look for Patterns

As appropriate, a mathematical connection to a problem or an investigation and exploration activity has been noted on the answer page accompanying each Warm-Up and Workout.

MATHCOUNTS Symbols and Notation

Standard abbreviations have been used for units of measure. Complete words or symbols are also acceptable. Square units or cube units may be expressed as $units^2$ or $units^3$.

F.L.I.P. Contest

Problems submitted to MATHCOUNTS F.L.I.P. (Formulate Logical, Interesting Problems) contest will be considered for inclusion in future editions of the School Handbook. If you have an original, creative problem, send it to F.L.I.P., c/o MATHCOUNTS Curriculum Coordinator, 1420 King Street, Alexandria, VA 22314. Entrants who send a self-addressed, stamped envelope with their problem(s) will receive an "I F.L.I.P. for MATHCOUNTS" button.

ANSWER KEY - WARM-UP 1

1.	12	(MT)	**2.**	$-11\frac{1}{3}$	(C)	**3.**	3	(MTE)
4.	8000	(FM)	**5.**	8633	(CE)	**6.**	C	(P)
7.	$\frac{2}{3}$	(FMP)	**8.**	5	(MT)	**9.**	8	(MGS)
10.	22.40	(CF)						

SOLUTION **Problem #1**

FIND OUT What are we asked to find? The number of games Sasha won.

CHOOSE A STRATEGY We can use the variable s to represent the number of times Sasha won. Then, the number of pieces of candy that Sasha and Trudy received can be represented in terms of s.

SOLVE IT Assuming there are no ties, let s represent the number of times Sasha won. Then, $30 - s$ represents the number of times that Trudy won. After 30 games, Trudy had $2(30 - s)$ pieces of candy and Sasha had $3s$ pieces. Since they each had the same number of pieces, these two quantities must be equal.

$$2(30 - s) = 3s$$
$$60 - 2s = 3s$$
$$60 = 5s$$
$$12 = s$$

Consequently, Sasha must have won 12 games.

LOOK BACK Does this answer check? Yes. Sasha won 12 games, which means that Trudy gave him $3 \times 12 = 36$ pieces of candy. Consequently, Trudy won 18 games, which means Sasha gave her $2 \times 18 = 36$ pieces of candy. Since Sasha started and ended with the same number of pieces, our answer is correct.

MAKING CONNECTIONS... to Recreational Mathematics Problem #8

A famous math problem—which might be better described as a math joke—asks the following question: "Which would you rather have, a broken clock with minute and second hands that do not move, or a clock that gains one minute every hour?" The rationale behind the answer is that the clock which gains one minute every hour gains 24 minutes every day, which means that it will only be correct once every 60 days. Conversely, the clock with hands that are not able to move is correct twice a day.

The humor inherent in such mathematics is that the clock which gains speed regularly is at least predictable. The clock with broken hands, however, has no level of predictability—it will indicate the same time no matter what time of the day you check it.

INVESTIGATION & EXPLORATION Problem #9

Exploring with a *geoboard* (a board with lattice points indicated, like the array described in this problem) is a good way for students to understand area, perimeter, and the relationship between them. In this problem, students found the area of the largest possible triangle that could be formed by choosing three points. In how many different ways can three points be chosen to yield that same maximum area? What is the maximum possible area of a quadrilateral when four points are chosen? ...of a pentagon when five points are chosen? ...of a hexagon when six points are chosen? What is the minimum area that can be chosen in each of these cases?

WARM-UP 1

1. Sasha and Trudy are playing tic-tac-toe. Sasha agrees to give Trudy two pieces of candy for each game that Sasha loses. Trudy agrees to give Sasha three pieces of candy for each game that Sasha wins. After playing thirty games, Sasha has the same number of pieces of candy with which she started. If there were no ties, how many games did Sasha win?

2. Express the value of $\dfrac{0.\overline{13}+0.\overline{5}}{0.\overline{16}-0.\overline{2}}$ as a mixed number.

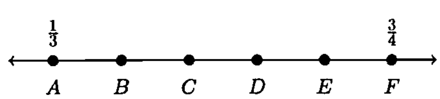
3. One day, 60 flights departed Washington National Airport for five different destinations. One-quarter of the flights went to Atlanta, one-third went to Chicago, one-fifth to Miami, and one-sixth to New York. The remaining flights went to Pittsburgh. How many flights went to Pittsburgh?

4. What is the maximum number of cubes measuring 5 cm on an edge that can be obtained by cutting a solid cube that measures 1 m on an edge?

5. Two distinct prime numbers, each less than 100, are multiplied. What is the greatest possible product of these two primes?

6. The pattern AABBBCCCCAABBBCCCC... continually repeats. What is the 369th letter in this pattern?

7. Points A–F are evenly spaced on a number line. What common fraction corresponds to point E?



8. A 12-hour clock stopped at 11:00 AM Friday morning. From noon Friday through midnight Sunday, how many times did this clock show the correct time?

9. Three points of a 5×5 square array are connected. How many square units are in the area of the largest possible triangle that can be formed?

10. A wholesaler purchased a pair of jeans from the manufacturer for $40. The wholesaler then tried to sell the jeans for 140% of the purchase price. When customers wouldn't pay that price, the jeans were placed on sale at a 20% discount off the selling price. At the end of the season, a final reduction of 50% off the lowered selling price was given. How many dollars was the final price of the jeans? Express your answer to the nearest cent.

ANSWER KEY – WARM-UP 2

1.	446.25	(FP)	**2.**	100	(CTP)	**3.**	40	(C)
4.	−3	(CG)	**5.**	36	(TP)	**6.**	3	(FP)
7.	120	(M)	**8.**	38	(CP)	**9.**	777	(GE)
10.	4	(TE)						

SOLUTION **Problem #10**

FIND OUT We want to know the number of pairs of emirps which have both numbers in the pair being two-digit prime numbers.

CHOOSE A STRATEGY Logic indicates that any number with an even digit cannot be a member of a pair of emirps, because if the last digit is even the number is divisible by 2 (and not prime), and if the first digit is even, the reversed number is divisible by 2 (and not prime). Similar logic holds with any number which has 5 as a digit. This rules out many possibilities. Consequently, a solution can be found by looking at the remaining numbers.

SOLVE IT Both digits of a number must be 1, 3, 7 or 9, based on the logic used above. Consider all 16 possible numbers: 11, 13, 17, 19, 31, 33, 37, 39, 71, 73, 77, 79, 91, 93, 97 and 99. Any pair of emirps must have both of its numbers on this list. A quick scan yields that there are five pairs, (11, 11), (13, 31), (17, 71), (37, 73) and (79, 97).

LOOK BACK Has the question been answered? No. The question stated that each number in the pair is greater than 11, and the first pair identified contains an 11. Hence, we must exclude that first pair, so there are four two-digit pairs of emirps.

MAKING CONNECTIONS... to Cryptography Problem #8

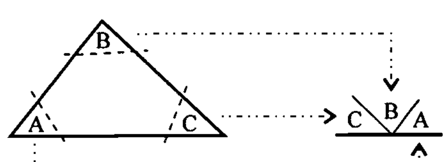
In 1977, Ronald Rivest, Adi Shamir and Leonard Adleman offered a $100 reward to any person who could determine the two prime factors of a 129-digit number. The number, which has come to be known as RSA-129, has just two prime factors. Obviously, factoring such a number is no easy task. In fact, using the factoring methods and technology that was available in 1977, Rivest, Shamir and Adleman estimated that factoring RSA-129 would have taken 40 quadrillion years!

Why would anyone want to know the factors of such a large number? Well, actually, they already knew the factors. What they wanted to know was whether anyone else could find those two factors quickly. You see, Rivest, Shamir and Adleman had developed a cryptographic system—a secret coding system which assigns values to words or letters, like the system used in problem #8—based on RSA-129 and its two factors. If this large number was difficult to factor, their system would be very hard to crack. However, if the number could be factored easily, their system would be useless. (For more information on RSA-129 and public key codes, see the book *For All Practical Purposes*, produced by COMAP, the Consortium for Math and its Applications.)

As it turns out, RSA-129 was finally factored in 1994—almost 20 years after the initial challenge was posed. The method used to find the factors is known as the *Multiple Polynomial Quadratic Sieve*, and the factoring process took about eight months to complete using approximately 750 ten-MIPS computers. Granted, that's significantly less than 40 quadrillion years—but if a secret code takes eight months to be broken, the sender can reasonably assume that the message will arrive safely.

INVESTIGATION & EXPLORATION Problem #7

In order to solve problem #7, students need to know that the sum of the measures of the three angles of a triangle have a combined sum of 180°. One way for students to learn this rule is to replicate it with paper. A triangle of any size can be cut from a piece of paper. Then, its three angles can be removed and rejoined, as shown, to form a straight angle. The measure of a straight angle is 180°, so the sum of the measures of the three angles of a triangle is also 180°.



WARM-UP 2

1. What is the 100th term of the arithmetic sequence $\frac{3}{4}, 5.25, \frac{39}{4}, 14.25, \ldots$? Express your answer as a decimal to the nearest hundredth. **1.** ________

2. When five brothers gathered to trade cards, only Jeb had some cards to trade. Jeb gave $\frac{1}{2}$ of his cards to Karl; then, Jeb gave Karl 2 more cards. Karl then gave $\frac{1}{2}$ plus 2 of the cards he received to Doug. Likewise, Doug gave $\frac{1}{2}$ plus 2 of the cards he received to Greg, and Greg gave $\frac{1}{2}$ plus 2 of the cards he received to Chris. Chris received 10 cards from Greg. How many cards did Jeb have before he gave any to Karl? **2.** ________

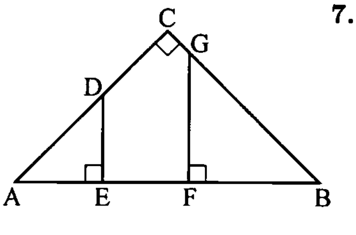
3. A magazine advertises its $12 annual subscription price as 70% off the newsstand price. What is the number of dollars per year a reader would spend if the magazine were purchased at the newsstand price? **3.** ________

4. Solve for x: $\left(\frac{1}{4}\right)^{2x+8} = (16)^{2x+5}$. **4.** ________

5. A five-digit positive integer is a "mountain number" if the first three digits are in ascending order and the last three digits are in descending order. For example, 35 761 is a mountain number, but 32 323 and 35 655 are not. How many five-digit numbers greater than 70 000 are mountain numbers? **5.** ________

6. What is the remainder when the sum of the first 102 counting numbers is divided by 5250? **6.** ________

7. In the figure, $m\angle ADE = 30°$. What is the number of degrees in $m\angle CGF$? **7.** ________



8. What is the positive difference in the sums of the letters in the words MATH and COUNTS, if each letter is assigned a value as follows: A = 1, B = −2, C = 3, D = −4, ..., Y = 25, Z = −26? **8.** ________

9. A three-digit number has the same hundreds, tens and units digit. The sum of the prime factors of the number is 47. What is the three-digit number? **9.** ________

10. A pair of *emirps* consists of two prime numbers such that reversing the digits of one number gives the other. How many pairs of two-digit emirps exist such that each number in the pair is greater than 11? **10.** ________

ANSWER KEY – WARM-UP 3

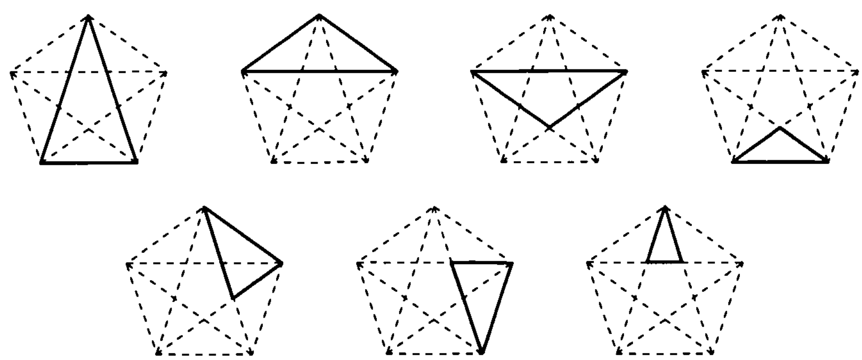
1.	35	(TSP)	**2.**	12.5	(CT)	**3.**	$\frac{8}{27}$	(C)
4.	100	(CFM)	**5.**	2	(TSP)	**6.**	11	(M)
7.	9 000 000	(FT)	**8.**	3	(TE)	**9.**	$\frac{3}{8}$	(MS)
10.	$\frac{6}{7}$	(T)						

SOLUTION **Problem #1**

FIND OUT We want to count the number of triangles that appear in a pentagon with its five diagonals drawn.

CHOOSE A STRATEGY A regular pentagon has five-way rotational symmetry. Therefore, each time a triangle is found, there are four additional congruent triangles. Hence, we need to identify the number of distinct triangles and multiply by 5. (Note that *distinct* here does not imply non-congruent; congruent triangles in different orientations may account for additional triangles.)

SOLVE IT We can count the triangles by identifying the number of smaller regions of which the triangles are composed. First, are there triangles of just one region? Yes, there are two of them (one at the end of each row below). Are there triangles composed of two regions? Yes, and there are two of these as well (the first two in the bottom row). There are also two triangles of three regions (the middle figures in the top row). Note that these triangles are congruent, but their orientation makes it so that there are five of each of them. There are no triangles formed from four regions, and, finally, there is one 5-region triangle (top left).



In total, seven distinct triangles have been identified. Since there is five-way symmetry, multiplying by 5 indicates that there are $5 \times 7 = 35$ triangles total in the diagram.

LOOK BACK As a check, we could list all triangles by indicating their regions. Doing this would again indicate that there are 35 triangles in the diagram, so we can have confidence in our answer.

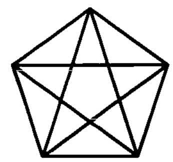
MAKING CONNECTIONS... to Measures of Central Tendency **Problem #2**

Collectively, the mean, median and mode(s) of a set of data are known as *measures of central tendency*, because they are meant to represent what is typical in a situation. However, if not interpreted correctly, they can be misleading. For example, imagine investing \$1000 in a stock that has an equal likelihood of gaining 40% or losing 30% each year. The mean gain is +5%, so after 20 years the expected value of the stock is $1000 \times (1.05)^{20} = \$2653$. But, in reality, the most likely outcome is that the stock will rise in value 10 years and decrease in value the other 10 years. Under those conditions, the stock will probably lose money and be worth only $1000 \times (1.4)^{10} \times (0.7)^{10} = \$817$.

One of the corollaries associated with expected values is the law of averages. If a coin is flipped many times, it is expected that half of the flips will be heads and half of the flips will be tails. However, this does not imply that the outcome of any individual flip can be predicted. Often called the *gambler's fallacy*, if a coin has shown tails many times in a row, there is no reason to believe that a successive flip will show heads. The probability of heads on each successive flip is still $\frac{1}{2}$.

WARM-UP 3

1. How many triangles of any size are in the picture? 1. ______



2. What is the number of centimeters in the median of the following set of measurements? Express your answer as a decimal to the nearest tenth. 2. ______

$$\{126\,\text{mm}, 11.8\,\text{cm}, 1.25\,\text{dm}, 0.13\,\text{m}, 10^{-4}\,\text{km}\}$$

3. Simplify and express your answer as a common fraction: $\dfrac{\frac{2}{3!}+1}{\frac{1}{2!}+4}$. 3. ______

4. What is the number of degrees in the measure of the larger of two supplementary angles if the ratio of their measures is $4:5$? 4. ______

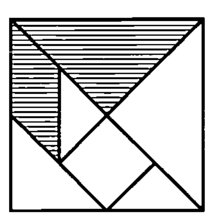
5. What is the remainder when 13^{51} is divided by 5? 5. ______

6. Kelsey was delivering lunch to a local office building. Kelsey's first delivery was to the middle floor of the building. She then took the elevator up 4 floors for the next delivery, down 6 floors for her third delivery, and up 7 floors for her final delivery. She was then on the top floor of the building. How many floors were in this office building? 6. ______

7. How many different seven-digit telephone numbers are available if the only restriction is that the first digit cannot be 0? 7. ______

8. For how many two-digit prime numbers is the sum of its digits 8? 8. ______

9. The square shown is constructed with the seven *tangram* pieces. Two of the seven tangram pieces are small isosceles right triangles. Three other tangram pieces—a square, a parallelogram, and another isosceles right triangle—are equal in area to two of the smaller triangles. The final two tangram pieces—the two large isosceles right triangles—are equal in area to four of the small triangles. What is the probability that a randomly chosen point inside the square will lie in the shaded region? Express your answer as a common fraction. 9. ______



10. Three quarters are tossed, and a tail appears on at least one of them. What is the probability that at least one head appears? Express your answer as a common fraction. 10. ______

ANSWER KEY – WARM-UP 4

1.	$\frac{25}{16}$	(C)	**2.**	10	(FM)	**3.**	103.25	(C)
4.	91	(MT)	**5.**	192	(FM)	**6.**	−11	(MG)
7.	4153	(CFT)	**8.**	51	(MP)	**9.**	$\frac{1}{5}$	(TP)
10.	142	(TG)						

SOLUTION **Problem #10**

FIND OUT What are we looking for? The sum of two 2-digit numbers which have a product of 4641.

CHOOSE A STRATEGY Since we are looking for two numbers which have a product of 4641, it makes sense to determine the factors of 4641. Further, we can determine the prime factors and then logically assemble the two-digit numbers.

SOLVE IT The number 4641 is difficult to factor, but we know that 3 is a factor because the digital sum $4 + 6 + 4 + 1 = 15$ is divisible by 3. We can then use a calculator to determine the other prime factors, and we find that there are three of them, 7, 13 and 17.

We now need to arrange these four factors into two numbers which have two digits each. The product $7 \times 13 = 91$, and $3 \times 17 = 51$. Hence, we have found two 2-digit numbers with a product of 4641. The question asks for the sum of these two numbers, which is $91 + 51 = 142$.

LOOK BACK Do both numbers contain only 2 digits? Yes. Is the product of the numbers 4641? Yes.

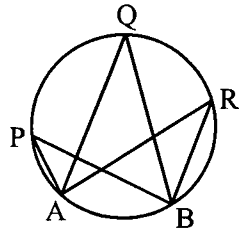
MAKING CONNECTIONS... to Exponential Growth **Problem #7**

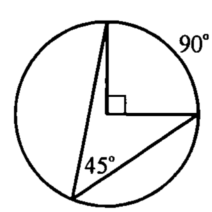
Exponential growth (or decline) is the growth attained by successive increases (or decreases) at a constant rate. The depreciation of a car like the one in problem #7 is an example of exponential decrease, because the car loses 25% each year. The car's value after n years will be $(0.75)^n$ of its original value. This implies that, though the loss the first year is large (25% of the original value), the loss in successive years will be less drastic; during the seventh year, for example, the loss will only be $(0.75)^6 - (0.75)^7 = 4.4\%$ of the original value. On the other hand, exponential growth has a more rapid increase as time passes. For instance, a \$1000 investment which earns 15% a year will earn \$150 the first year; during the seventh year, however, it will earn $1000\left((1.15)^7 - (1.15)^6\right) = \$347$. And during its 14th year, that investment will earn \$1061—which is more than the original amount invested!

INVESTIGATION & EXPLORATION **Problem #5**

Have students draw several blank circles. Then, choose two points (A and B) on the circumference of the circle. Finally, choose several points (P, Q, R, ...) along the major arc AB and construct $\angle APB$, $\angle AQB$, $\angle ARB$, and so on. By visual inspection—or, with a protractor, if skepticism rears its ugly head—the students can compare the measures of the angles created. Most will be quite astounded when they see that all of the angles created have the same measure.

As a follow-up to this first investigation, allow students to explore the relationship between the measures of these angles, the measures of corresponding central angles, and the measures of the arcs intercepted. (See the figure to the right below.)





WARM-UP 4

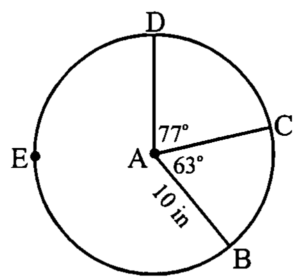
1. Compute: $(\frac{4}{5})^{-2}$. Express your answer as a common fraction. **1.** ________

2. What is the number of units in the distance from the origin to the point $(-8, 6)$ in a coordinate system? **2.** ________

3. Speedy Green Lawn Service will fertilize lawn at a rate of 59¢ per 100 square feet. What is the price Speedy Green will charge to fertilize a $125' \times 140'$ rectangular lawn? Express your answer to the nearest cent. **3.** ________

4. Fourteen students in Ms. Power's Algebra class were asked to shake hands with each other. Each student shook hands with each of the other students exactly once. How many handshakes occurred in Ms. Power's class? **4.** ________

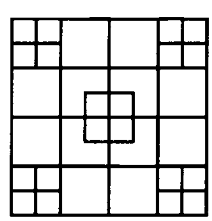
5. Points B, C and D lie on circle A. Circle A has radius 10 inches. What is the number of square inches in the area of the sector bounded by radii $\overline{AD}$ and $\overline{AB}$ and major arc BED? Express your answer to the nearest square inch. **5.** ________



6. What is the least integer value of x such that $|2x + 7| \leq 16$? **6.** ________

7. Each year, the value of a car decreases by 25% of its value the previous year. After 5 years, what is the value of a car originally worth \$17 500? Express your answer to the nearest dollar. **7.** ________

8. How many squares of any size are pictured in the diagram? **8.** ________



9. If the tens digit d is replaced with one of the digits 0–9, what is the probability that the four-digit positive integer $12d4$ is divisible by 12? Express your answer as a common fraction. **9.** ________

10. The number 4641 can be expressed as the product of two 2-digit whole numbers. What is the sum of these two numbers? **10.** ________

ANSWER KEY - WARM-UP 5

1.	210	(CTG)	**2.**	5	(TP)	**3.**	$\frac{1}{\pi}$	(CM)
4.	266	(M)	**5.**	47	(CF)	**6.**	867	(TP)
7.	18	(M)	**8.**	12.5	(C)	**9.**	$6\frac{1}{4}$	(MTG)
10.	90	(FM)						

SOLUTION **Problem #6**

FIND OUT What are we asked to find? The sum of the positive odd multiples of 3 less than 100.

CHOOSE A STRATEGY The numbers in this sum are multiples of 3, so we can factor a 3 out of the entire sum. Then, if we can identify a pattern among the sum that remains, our work will be easy.

SOLVE IT The sum of the positive odd multiples of 3 is $3 + 9 + 15 + 21 + \ldots + 99$. That sum can be factored to

$$3(1 + 3 + 5 + 7 + \ldots + 33)$$

which is 3 times the sum of the first 17 positive odd integers. The sum of the first n positive odd integers is equal to n^2, so the section of the expression in parentheses is equal to $17^2 = 289$. The entire sum, then, is 3 times that, or $3 \times 289 = 867$.

LOOK BACK Does this make sense? Yes. As a check, we can pair the first term, 3, with the last term, 99, which has a sum of 102. Likewise, the second term, 9, can be paired with the next-to-last term, 93, which also has a sum of 99. Further, 8 pairs with sum 102 can be created, and the middle term, 51, will be alone. Hence, the sum is equal to $8 \times 102 + 51 = 867$.

MAKING CONNECTIONS...to Continued Fractions **Problem #1**

The value in problem #1 can be expressed as a simplified radical: $\sqrt{200} = \sqrt{100 \times 2} = 10\sqrt{2}$. But what is this value expressed as a decimal? And how is that decimal value found?

A continued fraction is a complex fraction whose consecutive denominators repeat to infinity. Notice that the continued fraction

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$$

can be rewritten, simply, as $x = \frac{1}{2+x}$, because the repeated portion of the fraction is a replication of the entire fraction. This gives the following quadratic equation:

$$2x + x^2 = 1$$
$$x^2 + 2x - 1 = 0$$

This equation may seem harmless enough, but notice that the quadratic equation yields

$$x = \frac{-2 + \sqrt{2^2 - 4(1)(-1)}}{2(1)}$$
$$x = -1 + \sqrt{2}$$
$$x + 1 = \sqrt{2}.$$

Now, by finding the value of x (the continued fraction) and adding 1, the value of $\sqrt{2}$ can be calculated.

INVESTIGATION & EXPLORATION **Problem #2**

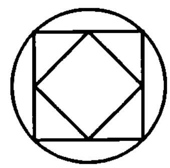
Students can use a calculator to learn about the repetend of decimals which repeat when converted from common fractions. Using a calculator to investigate $\frac{1}{3}$ and $\frac{2}{3}$ isn't all that interesting, because each has a repetend of only one digit. Investigating the repetend when dividing by 7, however, is a little more intriguing, as $\frac{1}{7} = 0.\overline{142857}$. In fact, each proper fraction with denominator 7 has the same 6 digits in its repetend, albeit in different order. Dividing by 11 doesn't offer much insight, but it is worth noting that the repetend always has two repeating digits. Dividing by 13 is engaging, as $\frac{1}{13} = 0.\overline{076923}$. The repetend has 6 digits, and it will have 6 digits for any fraction with denominator 13 when converted to a decimal. Can your students discover that the number of digits in the repetend is always 1 less than a factor of the denominator of the fraction?

WARM-UP 5

1. The numbers a and b are consecutive positive integers, and $a < \sqrt{200} < b$. What is the value of the product ab? **1.** ________

2. Which digit will appear in the 534th place after the decimal point in the decimal representation of $\frac{5}{13}$? **2.** ________

3. The radius of the circle is 10 centimeters, and the vertices of the smaller square are the midpoints of the larger square. What is the ratio of the area of the smaller square to the area of the circle? Express your answer as a common fraction in terms of π. **3.** ________



4. What is the minimum number of $8'' \times 8''$ square tiles needed to completely cover a $12'8'' \times 9'4''$ rectangular floor? **4.** ________

5. The Wildcats scored an average of 60 points per game during the 1998 season. They averaged 72 points for each of their 13 home games. How many points did they average for each of their 12 away games? **5.** ________

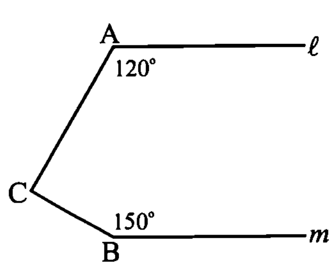
6. What is the sum of all positive odd multiples of 3 that are less than 100? **6.** ________

7. Regular pentagon ABCDE is inscribed in circle O. What is the number of degrees in the measure of $\angle$OCE? **7.** ________

8. What percent of 20 is 2.5? Express your answer to the nearest tenth. **8.** ________

9. Peter painted $\frac{1}{3}$ of a room while Richard painted $\frac{2}{5}$ of the same room. It then took Peter 1 hour, 40 minutes to finish painting the remainder of the room by himself. In how many hours could Peter have painted the entire room by himself? Express your answer as a mixed number. **9.** ________

10. Lines ℓ and m are parallel. $m\angle A = 120°$, and $m\angle B = 150°$. What is the number of degrees in $m\angle C$? **10.** ________



ANSWER KEY – WARM-UP 6

1.	24.92	(CF)	**2.**	16	(FMG)	**3.**	$32\pi - 64$	(FM)
4.	7	(TG)	**5.**	22 050	(TGE)	**6.**	$\frac{27\pi}{1024}$	(FM)
7.	$\frac{101}{180}$	(C)	**8.**	30	(M)	**9.**	4	(TGE)
10.	11	(T)						

SOLUTION **Problem #9**

FIND OUT For what are we searching? The largest even integer which can not be expressed as a sum of only 3's, 5's and 7's.

CHOOSE A STRATEGY List even numbers and determine if they can be expressed as a sum of 3's, 5's and 7's. When we can express three consecutive even integers as a sum of 3's, 5's and 7's, our work is complete, because we can add 6 $(3 + 3)$ or a multiple of 6 to one of these consecutive integers to make any greater even integer.

SOLVE IT The first even integer, 2, can not be obtained. Similarly, it is impossible to get 4. However, 6 can be made with the expression $3 + 3$. Likewise, 8 can be made with $3 + 5$, and 10 can be made with $3 + 7$. Because of our logic above, 4 must be the greatest even integer that can not be expressed as a sum of 3's, 5's and 7's.

LOOK BACK Can we prove that our answer is the greatest even integer which cannot be formed? Yes. Any multiple of 6 can be made by adding some multiple of 6 $(3 + 3)$ to 6. Any even integer congruent to 2 modulo 6 can be made by adding some multiple of 6 to 8, and any even integer congruent to 4 modulo 6 can be made by adding some multiple of 6 to 10.

MAKING CONNECTIONS... to Shipping Problem #2

Efficiency is a factor in transportation. Proper design of packages can maximize the amount a truck, ship, train or aircraft can carry. Cylindrical objects, like the popcorn container in this problem, result in wasted space when packed into a carrying compartment. The empty space could be eliminated by using packages which are rectangular prisms instead of cylinders.

When cylindrical containers are shipped, several of them are typically placed into larger, rectangular packages. The ratio of the cubic inches of empty space to the volume of the rectangular box is a measure of the wasted space. In a transportation vehicle, the number of additional units that could be carried as a result of more efficient packing could be calculated. A solution to the problem of wasted space might be to calculate the volume of the popcorn container and redesign the packaging so that the same amount of popcorn could fit in a rectangular container. Of course, there are other concerns. Strength and the ability to handle the object are important. Using packages that are bigger than the content is a marketing concern. And let's face it—it's just easier to drink beverages from a cylindrical can than from a rectangular one!

INVESTIGATION & EXPLORATION Problem #7

Using a calculator with the proper guidance, students can learn a lot about number theory. With the technology available on even the most simplistic calculators, students can investigate which fractions terminate, which ones repeat, and the length of repetends. Have students enter $1 \div 9$ into their calculators, and ask them how the result relates to problem #7. Ask them what expression needs to be entered to get a result of $0.0\overline{1}$. See if your students can determine a general rule for the result when dividing a two-digit number by 99, when dividing a three-digit number by 999, and so on. And if you really want to get their mathematical juices flowing, generate a discussion of whether $0.\overline{9} = 1$ by showing the following pattern:

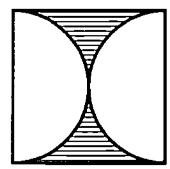
$$\frac{2}{9} = 0.\overline{2} \qquad \frac{6}{9} = 0.\overline{6}$$
$$\frac{3}{9} = 0.\overline{3} \qquad \frac{7}{9} = 0.\overline{7}$$
$$\frac{4}{9} = 0.\overline{4} \qquad \frac{8}{9} = 0.\overline{8}$$
$$\frac{5}{9} = 0.\overline{5} \qquad \frac{9}{9} = 0.\overline{9} = 1?$$

WARM-UP 6

1. Mr. Byrd can buy a 757-page Algebra book for \$14.99, or he can make copies of it for 5¢ a page. In either case, a 9% sales tax applies. What is the number of dollars in the positive difference between the costs for buying and copying the book? Express your answer to the nearest cent. (*Problem submitted by mathlete David Reid.*) **1.** ________

2. A popcorn company wants to create a circular cylindrical container with diameter 10 inches and volume 1256 cubic inches. How many inches should the height of the container be? Express your answer to the nearest inch. **2.** ________

3. In the figure, two tangent semicircles are drawn inside a square with side length 8 cm, as shown. The diameters of the semicircles are sides of the square. What is the number of square centimeters in the positive difference between the areas of the shaded and the unshaded regions? Express your answer in terms of π. **3.** ________



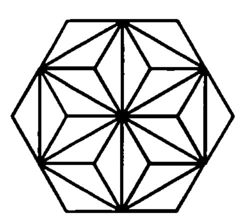
4. How many positive integer solutions exist for $3(x-5) \leq 7$? **4.** ________

5. A positive integer has all single-digit prime factors. None of its prime factors occurs more than twice in the prime factorization, but the number is not a perfect square. What is the greatest possible number with these properties? (*Problem submitted by alumnus Todd Stohs.*) **5.** ________

6. An antacid tablet is in the shape of a right circular cylinder. The diameter of the base is $\frac{3}{4}''$, and the tablet is $\frac{3}{16}''$ thick. How many cubic inches are in the volume of the tablet? Express your answer as a common fraction in terms of π. **6.** ________

7. Express the sum $0.25 + 0.3\overline{1}$ as a common fraction. **7.** ________

8. The tessellation shown consists of congruent obtuse triangles arranged in a regular hexagon. What is the number of isosceles triangles of any size in the figure? **8.** ________



9. The numbers 3, 5 and 7 are important in many aspects of Japanese life. Most positive integers can be expressed as a sum of only 3's, 5's and 7's. For example, 15 can be expressed as $5+5+5$ or $3+5+7$. What is the greatest even integer that cannot be expressed as a sum of 3's, 5's and 7's? (*Problem submitted by former question writer Diane Spresser.*) **9.** ________

10. For how many prime numbers between 10 and 100 is the last digit also a prime number? **10.** ________

ANSWER KEY – WARM-UP 7

1.	123	(C)	**2.**	7	(T)	**3.**	$4\sqrt[5]{10}$	(C)
4.	11.2	(M)	**5.**	32	(M)	**6.**	48	(MG)
7.	18π	(M)	**8.**	64	(MG)	**9.**	495	(C)
10.	15 000	(CM)						

SOLUTION **Problem #10**

FIND OUT We want to find the amount of money that Eve will receive.

CHOOSE A STRATEGY We can solve this problem by making a model or by using an equation. Let's use an equation and work backwards to represent the amount that each person receives.

SOLVE IT If we look at $145 000 as the whole, then each person receives a fraction of the whole. We can express the amount that each person receives in terms of the amount that John receives. Let's represent the amount that John receives by the variable x. Then,

Maya $\rightarrow \frac{x}{2}$

Ciro $\rightarrow \frac{1}{4}\cdot\frac{x}{2} = \frac{x}{8}$

Eve $\rightarrow \frac{1}{2}\cdot\frac{x}{2} - 5000 = \frac{x}{4} - 5000$

We can combine these amounts to total the whole:

$$x + \frac{x}{2} + \frac{x}{8} + \left(\frac{x}{4} - 5000\right) = 145\,000$$
$$8x + 4x + x + 2x - 40\,000 = 145\,000 \times 8$$
$$15x - 40\,000 = 1\,160\,000$$
$$x = 1\,200\,000 \div 15 = 80\,000$$

Therefore, Eve will receive $\frac{80\,000}{4} - 5000 = 15\,000$.

LOOK BACK Is the amount that Eve received $5000 less than 50% of the amount Maya received? Yes. Maya received $40 000, and Eve received $15 000.

MAKING CONNECTIONS... to Nature **Problem #5**

Symmetry is found in nature, in the petals of a daisy, in the bilateral form of the human body, and in the spirals of the scales on a pine cone. Natural symmetry has inspired art, and the result can be observed in the artistic patterns of many cultures. The Moors of Alhambra used symmetry in their decorative designs, and the Indians of the Southwest used symmetry in their weavings. The ancient Greeks used frieze patterns on their urns, repeatedly translating a pattern. And Jan Vansina even tells the story of how a Bakuba (Zaire) king was so enthralled by the pattern of motorcycle tires in the sand that he had it copied and gave the pattern his name.

Symmetry is defined by mathematicians as the transformations that leave an object—or a picture, or an equation—unchanged. Many species of animals have two sides, one of which appears to be a mirror image of the other. *Bilateral symmetry* implies that one side cannot be distinguished from the other when reflected, and biologists borrowed this term from mathematics to describe the symmetry of the human body.

WARM-UP 7

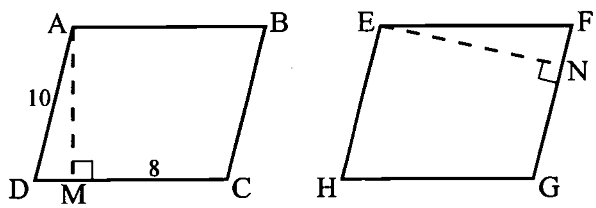
1. When gold sold for \$16 an ounce, Johnny found \$6 worth of gold at his claim. Gold presently sells for \$328 an ounce. How many dollars would Johnny's amount of gold be worth today? 1. ______

2. How many positive three-digit prime numbers less than 200 have two digits the same? 2. ______

3. Simplify and express your answer in simplest radical form: 3. ______

$$\sqrt[5]{32^2 + 16^3 + 8^4 + 4^5}$$

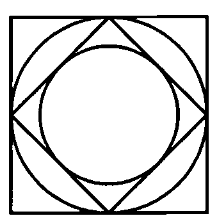
4. ABCD and EFGH are congruent parallelograms. AD = 10 cm, MC = 8 cm, and the area of ABCD is 112 cm^2. What is the number of centimeters in EN? Express your answer as a decimal to the nearest tenth. 4. ______



5. Two vertices of a rectangle are $(-3, 1)$ and $(-3, -3)$. The line with equation $x = 1$ is a line of symmetry for this rectangle. How many square units are in the area of the rectangle? 5. ______

6. A dog consists of a tail, a body, and a head. A dog's tail is half as long as its body, and its body is twice the length of its head. The dog's head is 12 in long. How many inches are in the total length of the dog? 6. ______

7. The length of each side of the larger square is 12 cm. The remaining figures are an inscribed square and two inscribed circles. What is the number of square centimeters in the area of the smaller circle? Express your answer in terms of π. 7. ______



8. The numbers of inches in Jenny's, Benny's, Penny's and Lenny's heights are four consecutive integers. The sum of their heights is 254 inches. Lenny is tallest, Jenny is shortest, and Penny is taller than Benny. What is the number of inches in Penny's height? 8. ______

9. Compute: $\dfrac{440 \times 0.45}{0.4}$ 9. ______

10. Four siblings are to divide a \$145 000 inheritance among them. The money will be divided using the following system: John will receive a certain sum; Maya will receive one-half as much as John; Ciro will receive one-fourth as much as Maya; and Eve will receive \$5000 less than half of the amount given to Maya. How many dollars will Eve receive? 10. ______

ANSWER KEY - WARM-UP 8

1.	$-\frac{43}{11}$	(C)	**2.**	6	(MG)	**3.**	13	(MG)
4.	55	(M)	**5.**	15 871 867	(CP)	**6.**	1.25×10^{37}	(C)
7.	$\frac{1}{5}$	(CM)	**8.**	$54\sqrt{6}$	(FM)	**9.**	23	(M)
10.	13	(C)						

SOLUTION **Problem #8**

FIND OUT We need to compute the area of a triangle, given only the lengths of its sides.

CHOOSE A STRATEGY Heron's Formula for finding the area of a triangle can be used to solve this problem. Heron's formula is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are the lengths of the sides of the triangle and s is the *semiperimeter* (one-half of the perimeter) of the triangle.

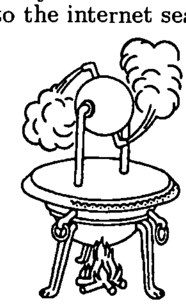
SOLVE IT The semiperimeter s is one-half the perimeter, which means that $s = \frac{1}{2}(15 + 18 + 21) = 27$. Using Heron's formula with this value, we can find the area of the triangle.

$$A = \sqrt{27(27-15)(27-18)(27-21)}$$
$$A = \sqrt{27(12)(9)(6)}$$
$$A = 54\sqrt{6}$$

LOOK BACK This problem is rather difficult to solve without the knowledge of Heron's formula. Who was the mathematician who discovered this formula? See below.

MAKING CONNECTIONS... to the History of Mathematics **Problem #8**

The internet offers a wealth of information about mathematics and related topics. At the site **http://www-groups.dcs.st-and.ac.uk/~history/Mathematician/Heron.html**, the following information can be obtained regarding Heron. "Born about 65 in (possibly) Alexandria, Egypt. Sometimes called Hero, Heron was an important geometer and worker in mechanics. Book I of his treatise *Metrica* deals with areas of triangles, quadrilaterals, regular polygons (of between 3 and 12 sides), surfaces of cones, cylinders, prisms, pyramids and spheres. A method, known to Babylonians 2000 years before, is also given for approximating the square root of a number. Heron's proof of his famous formula—if A is the area of a triangle with sides a, b and c and $s = \frac{a+b+c}{2}$, then $A^2 = s(s-a)(s-b)(s-c)$—is also given." The article goes on to describe some of Heron's inventions, including a description of a steam-powered engine called an *aeolipile*, which has much in common with a jet engine. Heron's aeolipile is described as follows: "The aeolipile was a hollow sphere mounted so that it could turn on a pair of hollow tubes that provided steam to the sphere from a cauldron. The steam escaped from the sphere from one or more bent tubes projecting from its equator, causing the sphere to revolve. The aeolipile is the first known device to transform steam into rotary motion." The Internet has a lot of information about Heron as well as about other mathematicians. A trip to the internet searching for mathematics is a voyage worth taking.



Hero's aeolipile

INVESTIGATION & EXPLORATION **Problem #10**

The traditional approach to teaching order of operations often involves explaining the acronym PEMDAS. With the help of a calculator, though, students can discover the order of operations on their own. For instance, students' inquisitiveness will make them wonder why the calculator gives an answer of 7 when the expression $1 + 2 \times 3$ is entered but gives 5 for the expression $1 \times 2 + 3$. The precedence of parentheses over both multiplication and addition will be apparent when they try the expression $1 \times (2 + 3) \times 4$. And a few other expressions will make the order of operations for exponents, division and subtraction obvious, too.

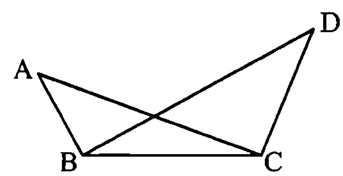
WARM-UP 8

1. Simplify and express your answer as a common fraction: 1. ______

$$\frac{2\cdot(3)^{-2}+3\cdot(2)^{-3}}{2\cdot(3)^{-2}-3\cdot(2)^{-3}}$$

2. The sum of two positive integers is 30 and their product is 216. Find the positive difference between the two numbers. 2. ______

3. $\triangle$ ABC and $\triangle$ DBC share $\overline{BC}$. AB = 5 cm, AC = 12 cm, DC = 8 cm, and BD = 20 cm. What is the least possible integral number of centimeters in BC? 3. ______



4. Each dimension of a parallelogram is increased to four times its original size to form a similar parallelogram. If the new parallelogram has an area of 880 square units, what is the number of square units in the area of the original parallelogram? 4. ______

5. The population of a midwestern city is increasing at an annual rate of 13%. The present population is 11 000 000 people. If this increase continues at the same rate, what will the population be in 3 years? 5. ______

6. Express the product of 2^{35} and 5^{38} in scientific notation. 6. ______

7. In *Robinson's Progressive Intellectual Arithmetic*, copyright 1863, a problem similar to the following is found. "If $\frac{2}{3}$ of a ton of hay costs $\frac{4}{5}$ of an eagle, how many eagles will $\frac{1}{6}$ of a ton cost?" What is the number of eagles in the answer to this problem? Express your answer as a common fraction. 7. ______

8. What is the number of square centimeters in the area of a triangle whose sides measure 15 cm, 18 cm, and 21 cm? Express your answer in simplest radical form. 8. ______

9. The coordinates of the vertices of $\triangle$ ABC are A$(-5,1)$, B$(-2,3)$ and C$(-4,7)$. The triangle is reflected over the line $x=2$. What is the sum of the x-coordinates of the images of points A, B and C? 9. ______

10. What is the positive difference between $(42+7-6\times6+3\times(-1))\times0$ and $42+7-6\times6+3\times(-1)\times0$? (*Problem submitted by alumna Debbie Berg.*) 10. ______

ANSWER KEY – WARM-UP 9

1. $\frac{551}{9212}$	(MTS)	**2.** 1	(FM)	**3.** $\frac{2}{195}$	(FT)
4. $\frac{3}{8}$	(M)	**5.** 14	(MT)	**6.** 0.0081	(C)
7. 30	(MT)	**8.** B	(T)	**9.** 1	(C)
10. 32π	(FM)				

SOLUTION **Problem #6**

FIND OUT We need to compute the value of $\left(\frac{2}{15}\right)^4 \times \left(\frac{3}{2}\right)^8$.

CHOOSE A STRATEGY We can use the laws of exponents to simplify the problem.

SOLVE IT In general, for $b \neq 0$, $\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$, so the problem can be rewritten as follows:

$$\frac{2^4 \times 3^8}{15^4 \times 2^8}.$$

Much cancellation occurs, because $15^4 = 3^4 \times 5^4$. Four powers of 3 in the numerator cancel with 3^4 in the denominator, and four powers of 2 in the denominator cancel with 2^4 in the numerator. The result is

$$\frac{1 \times 3^4}{1 \times 5^4 \times 2^4} = \frac{81}{10\,000} = 0.0081$$

LOOK BACK This problem does not require a calculator to be solved, but we can check our answer with a calculator. Did we give the answer in the form requested? Yes, the answer is expressed as a decimal.

MAKING CONNECTIONS... to Coincidence **Problem #3**

The probability of choosing four points at random from this array and having them be the vertices of a square is fairly low; in fact, as solving this problem showed, the probability is a mere $\frac{2}{195}$. However, the probability of getting four vertices of a square jumps to 50% if we select just 15 different sets of points. And, if we select just 26 sets, the probability is over 80%. Although it is unlikely to get the vertices of a square when choosing just one set, it is very likely to get the vertices of a square when multiple sets are chosen.

Similarly, the coincidences that happen in everyday life are not nearly as extraordinary as often believed. For instance, most people are quite shocked when they meet someone who has the same birthday. But in a room of just 23 people, there is a 50% chance that two of the people will share a common birthday. That probability is calculated as $1 - \left(\frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \cdots \frac{343}{365}\right) \approx 0.507$. What is more rare is the probability that two people have a *specific* birthday—say, for example, June 4th. Each person has a $\frac{364}{365}$ chance of not having a June 4th birthday. The probability of finding two people with a June 4th birthday, then, doesn't reach 50% until you meet 253 people, because $1 - \left(\frac{364}{365}\right)^{253} \approx 0.500$.

INVESTIGATION & EXPLORATION **Problem #5**

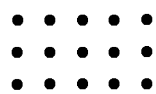
The triangle inequality is a relationship among the sides of a triangle. It stipulates that the sum of the lengths of any two sides of a triangle must be greater than the length of the third side. A great exploration for students involves using pieces of plastic straw to create triangles. Have students cut straws into 1-inch, 2-inch, 3-inch, 4-inch and 5-inch pieces. Then, have them randomly select three pieces. Which groups of three can be used to form a triangle? Students will soon realize that the sum of the lengths of the two smaller pieces must be greater than the length of the longer piece for a triangle to be formed—that is, students will discover the triangle inequality on their own.

WARM-UP 9

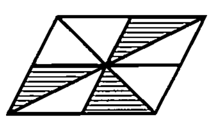
1. A bag contains 20 black balls and 30 white balls. Four balls are randomly drawn without replacement. What is the probability of selecting a white ball followed by a black ball, and then selecting a black ball followed by a white ball? Express your answer as a common fraction.

2. If the volume of a sphere inscribed in a cube is $\frac{\pi}{6}$ cubic inches, what is the number of cubic inches in the volume of the cube?

3. What is the probability that four different points chosen at random from the fifteen equally-spaced points shown are the vertices of a square? Express your answer as a common fraction.



4. In the parallelogram shown, the midpoints of opposite sides are connected with line segments. Likewise, the opposite vertices are connected. What is the probability that a point randomly selected inside the parallelogram will lie inside one of the shaded regions? Express your answer as a common fraction.



5. A triangle has two sides of length 36 cm and 8 cm. What is the number of centimeters in the positive difference between the greatest possible whole number length and the least possible whole number length of the third side?

6. What is the terminating decimal value of $\left(\frac{2}{15}\right)^4 \times \left(\frac{3}{2}\right)^8$?

7. In a pet store, $\frac{1}{2}$ of the animals are fish, $\frac{1}{10}$ are dogs, $\frac{1}{5}$ are birds and the remaining six are cats. How many animals are in the store?

8. In the accompanying numeration system, $B * C = B$. What is the value of $(B * B) * (C * C)$?

*	A	B	C
A	B	A	C
B	A	C	B
C	B	C	A

9. If $f(x) = x^2 - 2x + 1$ and $g(x) = \sqrt{2x+1}$, what is the value of $f(g(4)) - g(f(3))$?

10. A square is inscribed in a circle. The area of the square is 64 square inches. How many square inches are in the area of the circle? Express your answer in terms of π.

ANSWER KEY – WARM-UP 10

1. 1000	(FM)	**2.** 155	(TP)	**3.** 120	(C)
4. 112.5	(C)	**5.** 60	(FM)	**6.** 37	(TE)
7. 0.17	(M)	**8.** 24	(FT)	**9.** 1 000 000	(FP)
10. $\frac{1}{2}$	(TP)				

SOLUTION **Problem #9**

FIND OUT What question do we wish to answer? The value of the sum of the consecutive odd integers 1 through 1999.

CHOOSE A STRATEGY There are two quick methods that can be used to solve this problem. The first is the method used by Gauss, which is to pair the first and last terms to find their sum, and then multiply by the number of pairs. The second method relies on the fact that the sum of the first n consecutive odd integers is equal to n^2. Let's use the first method to solve this problem, and then check our answer with the second method.

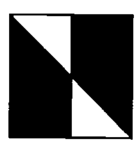
SOLVE IT Pairing the first term, 1, with the last term, 1999, gives a sum of 2000. Likewise, pairing 3 with 1997 also gives a sum of 2000. Continuing, many such pairs with a sum of 2000 can be generated. How many pairs are there?

Each positive odd integer is of the form $2n - 1$. When $n = 1$, we get the first odd number, $2(1) - 1 = 1$. Similarly, if $2n - 1 = 1999$, $n = 1000$, so 1999 must be the 1000th positive odd integer. Consequently, there are $\frac{1000}{2} = 500$ pairs generated by the method above. The sum, then, is equal to $2000 \times 500 = 1\,000\,000$.

LOOK BACK Does this answer check when using the second method? Yes. We determined that there are 1000 numbers in this sum, and the sum of the first n odd consecutive integers is equal to n^2. In this case, the sum is $1000^2 = 1\,000\,000$, which agrees with the first answer attained.

MAKING CONNECTIONS...to Logos **Problem #7**

The logo in problem #7 is the one currently used by the Central Broadcasting System. A painfully simple design—two concentric circles and two arcs. But CBS uses this mathematical gem to represent an eye for newscasts, and to represent a baseball (the small circle), a football (the arcs) and a basketball (the large circle) during sports broadcasts.

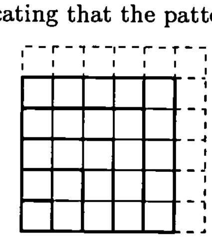


Mathematics plays a significant role in logo design. The figure above consists of two trapezoids and two triangles. Seemingly a simple, worthless design, right? Well, not exactly. This symbol used to be the logo for the National Broadcasting Company. Several years after a designer created this symbol for them, the University of Nebraska independently created the same logo for their school. And as the story goes, NBC paid a substantial amount of money to the University of Nebraska so that they wouldn't use the same logo.

INVESTIGATION & EXPLORATION **Problem #9**

Students should investigate sums of consecutive positive odd integers and try to identify a pattern. The first odd integer is 1. The sum of the first two odd integers is $1 + 3 = 4$. The sum of the first three is 9; of the first four is 16; and so on. The pattern of sums is $1, 4, 9, 16, \ldots$, which appears to be the sequence of square numbers.

A proof of this theorem can be constructed in several different ways. Using induction, it can be shown that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$. Using geometry, though, we can see below that adding $(2n - 1)$ squares increases the figure from $(n - 1)^2$ to n^2 total squares, indicating that the pattern is the sequence of square numbers.



WARM-UP 10

1. What is the number of cubic inches in the volume of a cube with surface area 600 square inches?

1. ______________

2. The triangular array of positive integers shown continues indefinitely, with each row containing one number more than the row above it. The number directly above 50 is added to the number directly below 100. What is the sum?

2. ______________

1				
2	3			
4	5	6		
7	8	9	10	
11	12	13	14	15

3. Given that $a = 4$ and $b = 3$, evaluate $(a^b + b^a) - (a^2 + b^2)$.

3. ______________

4. What percent of 88 is 99? Express your answer as a decimal to the nearest tenth.

4. ______________

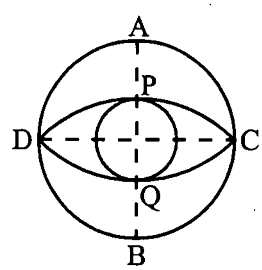
5. The length of each edge of the base of a regular square pyramid is 6 centimeters. The height of the pyramid is 4 centimeters. What is the number of square centimeters in the lateral surface area of the pyramid?

5. ______________

6. What is the greatest prime factor of 1998?

6. ______________

7. The logo of a national broadcasting company is shown. In the figure, AB and CD are perpendicular diameters of the large circle. $\overset{\frown}{DPC}$ has center B and radius BD; likewise, $\overset{\frown}{DQC}$ has center A and radius AD. The inner circle is tangent to the arcs at P and Q. What is the ratio of the area of the smaller circle to the area of the larger circle? Express your answer as a decimal to the nearest hundredth.



7. ______________

8. In how many distinct ways can five children be seated around a circular merry-go-round which has five identical seats?

8. ______________

9. What is the value of the sum $1 + 3 + 5 + 7 + \ldots + 1997 + 1999$?

9. ______________

10. A pair of two distinct points is selected at random from the set P. What is the probability that the length of the segment formed by joining the chosen points is an integer? Express your answer as a common fraction.

$$P = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

10. ______________

ANSWER KEY – WARM-UP 11

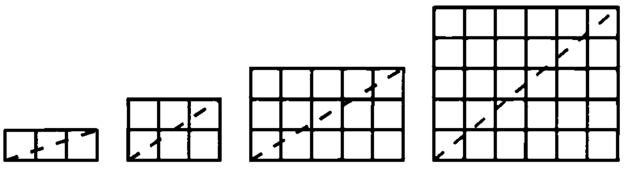
1.	42	(MTP)	**2.**	6	(MG)	**3.**	9	(TEP)
4.	136	(MP)	**5.**	384	(MG)	**6.**	$\frac{4}{7}$	(T)
7.	−30	(TP)	**8.**	20	(M)	**9.**	10	(C)
10.	20	(M)						

SOLUTION **Problem #10**

FIND OUT What do we wish to know? How many tiles the diagonal of a $10' \times 12'$ floor crosses.

CHOOSE A STRATEGY Let's consider simple examples to see if a pattern exists.

SOLVE IT Let's first consider a few simple cases.



In each case, notice that the diagonal crosses $w + \ell - 1$ tiles, where w is the width and ℓ is the length of the rectangle. The floor we are to consider is a 10×12 rectangle. Two 5×6 rectangles joined at their corners simulates the situation. Hence, the diagonal of a 10×12 rectangle will cross the same number of tiles as the diagonals of two 5×6 rectangles, which is $2(10) = 20$.

LOOK BACK Does the rule work for all rectangles? Yes, it does, provided that the dimensions w and ℓ are relatively prime. If the dimensions are not relatively prime, as is the case here with a 10×12 rectangle, the rectangle must be divided into smaller parts.

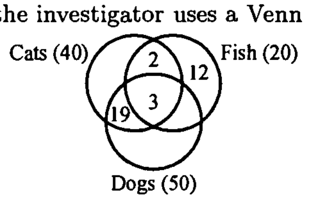
MAKING CONNECTIONS...to Large Numbers Problem #9

One million is a large number. But would you have thought that one million heart beats could happen in just ten days? Probably not. You might be more surprised to learn that biologists have determined that most animals have a life span of about one billion heartbeats, although humans live longer than that—about three billion heartbeats—because of developments in health care.

Large numbers can be hard for the human mind to comprehend, because we rarely encounter a billion of anything. Think of some simple examples that represent a billion. How many homes could be purchased with a billion dollars? How long is a billion seconds? How far away is a billion inches? Try to name a few things which occur in billions. Establishing points of reference makes comprehending a billion much easier.

INVESTIGATION & EXPLORATION Problem #8

Problem #8 is not a difficult problem—if the investigator uses a Venn diagram like the one shown below.



Venn diagrams help to organize the information presented in the problem, and they are especially helpful in a situation where some of the information overlaps (like having both a dog and a cat). Only one section of the fish circle remains open, and we know that 20 students own a fish, so the dog-fish section must contain $20 - 12 - 2 - 3 = 3$ students. Similar logic reveals that 16 students own only a cat and 25 students own only a dog. Since there are 100 students total in the survey, $100 - (16 + 12 + 25 + 2 + 3 + 19 + 3) = 20$ students have no pet at all.

WARM-UP 11

1. A magic square is an array of numbers in which the sum of the numbers in each row, column and diagonal is the same. In a 5×5 magic square which uses the integers 1–25, the numbers are arranged so that 4, 6 and 13 are in the same row. What is the sum of the other two numbers in that row?

1. ________

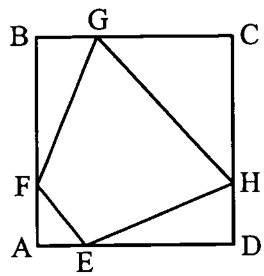
2. Arwin needs 100 lbs of bird seed, but the seed is only sold in bags containing 16, 17, 23, 24, 39 or 40 lbs. How many bags will he need to buy in order to purchase exactly 100 lbs?

2. ________

3. In how many ways can two number cubes, each with faces numbered 1–6, be rolled so that the sum of the numbers on the top faces is divisible by 4?

3. ________

4. In rectangle ABCD, AF = BG = 5 cm, FB = HC = DE = 12 cm, and AE = 4 cm. How many square centimeters are in the area of quadrilateral EFGH?



4. ________

5. The ratio of the sum and difference of two positive integers is $\frac{11}{5}$. If the difference between the sum and difference is 24, what is the product of the two integers?

5. ________

6. What common fraction is the median of the following set?

$$\left\{\frac{2}{5}, \frac{1}{3}, \frac{5}{8}, \frac{1}{4}, \frac{3}{4}, \frac{2}{3}, \frac{4}{7}\right\}$$

6. ________

7. Given that f is a linear function where $f(0) = 20$ and $f(4) = 0$, what is the value of $f(10)$?

7. ________

8. A poll of 100 eighth-grade students was conducted to determine the number of students who had a dog, a cat or a fish. The data showed that 50 students had a dog, 40 students had a cat, and 20 students had a fish. Further, 19 students had both a dog and cat, 2 students had a cat and a fish, 3 students had a dog and a fish, and 12 students had only a fish. How many students had none of these pets?

8. ________

9. The average adult's heart beats about 70 times each minute. At that rate, how many days would it take for an adult's heart to beat 1 000 000 times? Express your answer to the nearest whole number.

9. ________

10. A rectangular floor measuring 10 feet by 12 feet is tiled with one-foot square tiles. Through how many tiles would the diagonal of this rectangle pass?

10. ________

ANSWER KEY - WARM-UP 12

1. 1	(C)	**2.** $\frac{1}{64}$	(T)	**3.** 130π	(FM)
4. $\frac{7}{2}$	(FM)	**5.** $\frac{5}{17}$	(FT)	**6.** $24\sqrt{3}$	(FM)
7. 810	(TG)	**8.** 8	(CP)	**9.** $\frac{2}{11}$	(TP)
10. 14	(TG)				

SOLUTION **Problem #10**

FIND OUT What do we want to identify? The smallest sum for a set of distinct integers which have a product of 84.

CHOOSE A STRATEGY Since we are concerned with a product, we should begin by factoring 84 into its prime factors. Then, we can combine these prime factors to find the lowest possible sum.

SOLVE IT The integer 84 has four prime factors and can be written as $2 \times 2 \times 3 \times 7$. Now, let's combine the factors in all possible ways, and choose the combination with the lowest total sum.

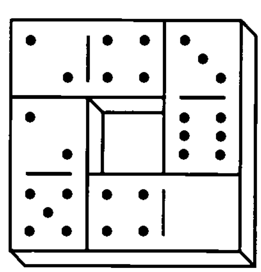
Combination	Sum
1, 84	85
2, 42	44
3, 28	31
7, 12	19
2, 3, 14	19
2, 6, 7	15
3, 4, 7	14

The last combination shown in the chart indicates that 14 is the smallest possible sum when the factors are 3, 4 and 7.

LOOK BACK Did we answer the question? Yes. Does the answer seem reasonable? Yes, because the sum of all the prime factors is $2 + 2 + 3 + 7 = 14$. Although we should expect the answer to be close, or even equal, to that sum, the answer cannot be lower than 14.

INVESTIGATION & EXPLORATION **Problem #9**

A basic set of *double-six dominoes* has 28 rectangular tiles. (Zero is paired with the seven digits 0–6; one is paired with the six digits 1–6; two, likewise, is paired with the five digits 2–6; and so on. The number of dominoes in this set is equal to $7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$, which happens to be the 7th triangular number. Several mathematical concepts can be investigated using dominoes. Have students arrange sets of four dominoes in such a way that each domino touches two—and only two—other dominoes and such that the number of dots in each row and column is the same. (One example is shown below.) There are several different arrangements which have a sum of 9 in each row and each column. For what other sums do arrangements exist? (3, 6, 8, 10, 11, 16.) Which dominoes need to be used for each of these sums? Can your students construct the arrangements?

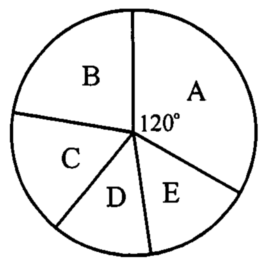


WARM-UP 12

1. What number is $33\frac{1}{3}\%$ of the positive difference between 25% of 60 and 75% of 16? 1. ________

2. The faces of a regular tetrahedron are numbered 1–4, the faces of another tetrahedron are lettered A–D, and the faces of a third tetrahedron are colored red, blue, yellow and green. If these tetrahedrons are rolled, what is the probability that the bottom faces are 1, A and red, respectively? Express your answer as a common fraction. 2. ________

3. A circle has sectors A, B, C, D and E, as shown. The circumference of the circle is 60π centimeters. The relationships of the areas of the sectors are $B = \frac{2}{3}A$, $C = \frac{3}{4}B$ and $D = \frac{4}{5}C$. What is the number of square centimeters in the area of sector E? Express your answer in terms of π. 3. ________



4. The equation of a line is $y = -\frac{2}{3}x + 5$. What is the y-coordinate of the y-intercept of a line that contains the point $(-1, 2)$ and is perpendicular to the given line? Express your answer as a common fraction. 4. ________

5. A class of 21 students wants to select a committee to manage its annual fund drive. They are going to form a committee with either four or five students. What is the ratio of the number of possible four-member committees to the number of possible five-member committees? Express your answer as a common fraction. 5. ________

6. A new solid is formed by gluing a face of one regular tetrahedron to a face of an identical tetrahedron. The length of one edge of a tetrahedron is 4 inches. How many square inches are in the surface area of the new solid? Express your answer in simplest radical form. 6. ________

7. The ratio of the sum and difference between two positive integers a and b is $\frac{7}{3}$. What is the greatest possible product of the two integers such that the product is less than 1000? 7. ________

8. Simplify: $\dfrac{(37\,037 \times 15) + (37\,037 \times 9)}{37\,037 \times 3}$ 8. ________

9. A domino is a rectangular tile composed of two squares. An integer is represented on both squares, and each integer 0–9 is paired with every integer 0–9 exactly once to form a complete set. A *double* is a domino which has the same integer on both of its squares. What is the probability that a domino randomly selected from a set will be a double? Express your answer as a common fraction. 9. ________

10. The product of a set of distinct positive integers is 84. What is the least possible sum of these integers? 10. ________

ANSWER KEY – WARM-UP 13

1.	56	(FM)	**2.**	36	(T)	**3.**	42	(CM)
4.	56.25	(MTG)	**5.**	60	(FMS)	**6.**	57π	(FM)
7.	1192	(MG)	**8.**	7	(MG)	**9.**	24	(CG)
10.	455	(MP)						

SOLUTION **Problem #8**

FIND OUT What are we asked to determine? The sum of two relatively prime integers, given some Pythagorean relationships among them.

CHOOSE A STRATEGY We are given three expressions and three values, but we are not told how they are related. Therefore, let's use logic to match the expressions to the values and then work from there.

SOLVE IT The three values given are 20, 21 and 29, only one of which is even. Further, one of the three expressions, $2mn$, must also be even. Therefore, $2mn = 20$. Dividing both sides by 2 yields that $mn = 10$, and if both m and n are integers, either $m = 10$ and $n = 1$, or $m = 5$ and $n = 2$. A simple check using the other expressions shows that 10 and 1 do not satisfy the requirements ($10^2 + 1^2 = 101$), while 5 and 2 do ($5^2 + 2^2 = 29$).

LOOK BACK Have we answered the question? No. The question asks for the sum of the numbers, not the numbers themselves. The sum is $5 + 2 = 7$.

MAKING CONNECTIONS...to Engineering **Problem #5**

The honeycomb structure shown in problem #5 is an example of natural engineering. The honeycomb, with its hexagonal structure and tessellating cells, is the home of the honeybee. The bee uses the honeycomb to store food and raise its young. Honeybees create honeycombs with two sizes of hexagonal cells. The larger cells are used for raising worker bees or for storing pollen; the smaller cells are used for raising drone bees or for storing honey.

This structure is widely used in engineering because it is light-weight yet structurally solid. Additionally, this pattern has good energy absorbing capabilities, which is why it is used in the solar liners that cover and heat pools.

INVESTIGATION & EXPLORATION **Problem #10**

Pascal's triangle is rich with patterns. Students should be encouraged to investigate this mathematical treasure. Can your students find the powers of 2 in Pascal's triangle? What about the powers of 11? The consecutive integers? Or the triangular numbers? Have your students find other patterns in Pascal's triangle—there are many of them.

WARM-UP 13

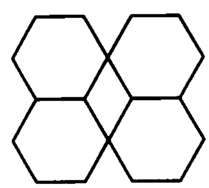
1. How many distinct triangles can be formed by connecting three different vertices of a cube? — 1. ________

2. How many times in a 12-hour period does the sum of the digits on a digital clock equal 6? — 2. ________

3. The dimensions of a box are $5'' \times 3'' \times 10''$. Two dimensions are increased by 20%, and a third dimension is decreased by 50%. What is the positive difference between the number of cubic inches in the original volume and in the new volume? — 3. ________

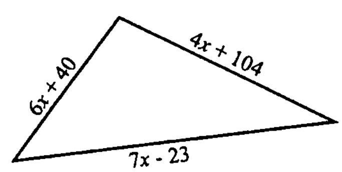
4. What is the number of square centimeters in the maximum area possible for a rectangle with perimeter 30 centimeters? Express your answer to the nearest hundredth. — 4. ________

5. A tessellation is composed of four regular hexagons and a rhombus as shown. How many degrees are in an acute angle of the rhombus? — 5. ________



6. Rectangular labels are produced to cover the lateral surfaces of two cylindrical cans. One of the cans is 10 inches tall and has a radius of 3 inches. The other can is 13 inches tall and has a radius of $4\frac{1}{2}$ inches. How many more square inches of paper must be used to label the larger can? Express your answer in terms of π. — 6. ________

7. The triangle shown is isosceles. What is the greatest possible perimeter? — 7. ________



8. One formula for generating a Pythagorean Triple is to choose two relatively prime integers m and n such that $m > n$ and calculate the values of $m^2 + n^2$, $m^2 - n^2$, and $2mn$. These values correspond to the lengths of the sides of a right triangle. What is the value of $m + n$ if the sides of a right triangle are 20, 21 and 29? — 8. ________

9. Given that $2^x + 2^x + 2^x + 2^x = 128$, what is the value of $(x+1)(x-1)$? — 9. ________

10. The pattern of Pascal's triangle is illustrated in the diagram shown. What is the fourth element in Row 15 of Pascal's triangle? — 10. ________

Row 0:					1				
Row 1:				1		1			
Row 2:			1		2		1		
Row 3:		1		3		3		1	
Row 4:	1		4		6		4		1

ANSWER KEY - WARM-UP 14

1.	70	(MTG)	**2.**	61	(MTG)	**3.**	63	(MG)
4.	4	(TP)	**5.**	9.5	(CF)	**6.**	35	(CG)
7.	16	(MS)	**8.**	18	(GSP)	**9.**	1260	(MTG)
10.	3	(CT)						

SOLUTION **Problem #7**

FIND OUT What are we asked to find? The number of inches in the length of the segment that extends from a vertex of a triangle to the point at which the medians meet.

CHOOSE A STRATEGY We can begin with the fact that △BCD, △ADC and △ABD have equal areas, and then use the formula for the area of a triangle.

SOLVE IT Because each of the three smaller triangles, △BCD, △ADC and △ABD, have equal area, the area of each of them must be equal to one-third the area of △ABC. Although it is impossible to compute the exact area of △ABC, we do know that its area is equal to one-half its base ($\overline{AC}$) times its height. Similarly, although it is not possible to compute the exact area of △ADC, we do know that its area is equal to one-half its base (also $\overline{AC}$) times its height. Further, we know that its area is one-third the area of △ABC. Hence,

$$\frac{1}{3}A_{\text{ABC}} = \frac{1}{3}\cdot\frac{1}{2}(\text{AC})\cdot h_{\text{ABC}} = \frac{1}{2}(\text{AC})\cdot h_{\text{ADC}} = A_{\text{ADC}}$$

$$\frac{1}{3}h_{\text{ABC}} = h_{\text{ADC}}$$

This result is important, because it establishes that the height of △ABC is three times the height of △ADC; likewise, then, the length of $\overline{BG}$ is three times the length of $\overline{DG}$. Hence, DG = 8 in, so BD = 16 in.

LOOK BACK To check this result, apply it to the specific case when △ABC is an equilateral triangle with side length $16\sqrt{3}$ in. In that case, the area of △ABC is $192\sqrt{3}$ sq in, the area of △ADC is $64\sqrt{3}$ sq in, the ratio DG : BG = 1 : 3, and our answer checks.

INVESTIGATION & EXPLORATION **Problem #6**

While this problem can be solved using prime factorization or divisibility rules, it can also be solved geometrically. Suppose we want to find the greatest common denominator (GCD) of 12 and 30. On paper, draw a rectangle with the smaller dimension representing the width and the greater dimension representing the length. Cut off the largest square that can be contained in the rectangle, which is a 12 × 12 square (although it is not necessary to physically remove the square, separate it from the rest with a vertical line). The remaining rectangle is 12 × 18. Again, separate off the largest square that can be drawn within this rectangle, which is again 12 × 12. The remaining rectangle is 6 × 12. Separate off the largest square that can be contained in this rectangle, which is 6 × 6. Now, there is no area left to remove, since the piece that is left is a 6 × 6 square—to remove another square would be to eliminate the last piece entirely! The 6 × 6 square is the largest square that will completely cover the original rectangle. Consequently, the GCD of 12 and 30 is 6.

Analogously, students can use their calculators for the same process. First, subtract 12 from 30 repeatedly until the result is less than or equal to 12. After two subtractions, the difference 30 − 2(12) = 6 is obtained. Then, repeatedly subtract 6 from 12 until the result is less than or equal to 6; after just one subtraction, the difference is 6. Hence, the GCD of 12 and 30 is 6.

Both of these methods—the geometric and the technologic—demonstrate the theory which underlies the Euclidean algorithm.

WARM-UP 14

1. From any letter in the diagram, a move can only be made to a letter diagonally adjacent and below. In how many different ways can a path that spells MATHCOUNTS be taken? **1.** ______

```
      M
     A A
    T T T
   H H H H
  C C C C C
   O O O O
    U U U
     N N
      T
       S
```

2. Natasha has more than $1 but less than $10 worth of dimes. When she puts her dimes in stacks of 3, she has 1 left over. When she puts them in stacks of 4, she has 1 left over. When she puts them in stacks of 5, she also has 1 left over. How many dimes does Natasha have? **2.** ______

3. There once was a woman from Dundee,
whose age had last digit three.
    If the square of the first (digit)
    is her whole age reversed,
then what must the woman's age be? **3.** ______

4. In how many zeroes does 20! end? **4.** ______

5. Elizabeth ran the Governor's Cup, a 10-kilometer road race, in 58 minutes, 54 seconds. What was the number of minutes per mile in Elizabeth's pace? Express your answer as a decimal to the nearest tenth. **5.** ______

6. What is the greatest common factor of the numbers 2835 and 8960? **6.** ______

7. In $\triangle ABC$, the medians meet at point D. Points E, F and G are the midpoints of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$, respectively. $BG = 24$ in. What is the number of inches in the length of $\overline{BD}$? **7.** ______

8. Each letter d, n and a in the addition below represents a different digit. What is the sum $d + n + a$? **8.** ______

$$\begin{array}{r} dna \\ +\ dan \\ \hline and \end{array}$$

9. Mackenzie bought 142 feet of fencing with which to enclose her rectangular garden. If the numbers of feet in the lengths of the garden's sides are natural numbers, what is the maximum number of square feet that can be enclosed by the fencing? **9.** ______

10. How many integers are greater than $\sqrt{27}$ and less than $\sqrt{72}$? **10.** ______

ANSWER KEY – WARM-UP 15

1.	18	(M)	**2.**	1023	(TG)	**3.**	567	(M)
4.	3	(TP)	**5.**	3	(MG)	**6.**	8	(FM)
7.	8	(CG)	**8.**	$\frac{1}{4}$	(C)	**9.**	6	(M)
10.	12	(GP)						

SOLUTION **Problem #5**

FIND OUT We are asked to find the number of minutes it will take to fill the tub.

CHOOSE A STRATEGY An equation for this problem can be created if the various times for filling and emptying the tub are considered.

SOLVE IT Using the fill and empty times given in the problem, and letting x represent the total number of minutes needed to fill the tub, expressions for each aspect can be formulated. Since the hot water tap can fill the tub in 6 minutes, this tap will fill $\frac{x}{6}$ of the tub. Likewise, the cold water tap will fill $\frac{x}{4}$, and the drain will empty $\frac{x}{12}$. Finally, since the tub will be filled in x minutes, the tub fills $\frac{x}{x} = 1$ in x minutes. Hence, a solution results from the following equation:

$$\frac{x}{6} + \frac{x}{4} - \frac{x}{12} = 1$$
$$2x + 3x - x = 12$$
$$4x = 12$$
$$x = 3$$

It will take three minutes to fill the tub.

LOOK BACK Does the answer make sense? In three minutes, the hot water tap will have filled $\frac{1}{2}$ the tub, and the cold water tap will have filled $\frac{3}{4}$ of the tub. Fortunately, during the same three minutes, $\frac{1}{4}$ of the tub will have drained. In total, that's $\frac{1}{2} + \frac{3}{4} - \frac{1}{4} = 1$, so the tub will be perfectly full after three minutes.

MAKING CONNECTIONS. . . to the Pythagorean theorem Problem #9

Although the Pythagorean theorem is named for the Greek mathematician Pythagoras (circa 540 BC), the theorem was known many years before by the Babylonians and Egyptians. Some of the basic ideas of geometry were developed in measuring land, and, while measuring right triangles, the Babylonians observed the important relationship between the sides. According to legend, when Pythagoras discovered this theorem, he was so overjoyed that he offered a sacrifice of oxen. The first deductive proof, however, was not given until nearly 200 years later by Euclid. But that's not to say that it's difficult to find a proof for the Pythagorean theorem—the *Pythagorean Proposition* by E. S. Loomis contains 370 proofs of this theorem, one by former president James Garfield!

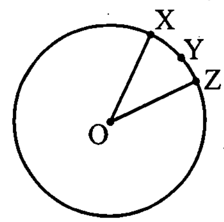
INVESTIGATION & EXPLORATION Problem #6

If an unlimited number of copies of the polygon in this problem were arranged so that they were adjacent but not overlapping, they would not be able to completely cover a plane. There would be a square "hole" in the middle of four adjacent octagons.

However, many shapes will completely cover the plane, and such a complete covering is called a *tessellation.* Make several copies of an equilateral triangle. Can these be arranged to tessellate? What about several copies of a square? . . .a pentagon? . . .a hexagon? Explore with polygons that are not regular, too. Will all triangles tessellate? Will all quadrilaterals tessellate? What other shapes tessellate the plane? Are there combinations of polygons which will tessellate?

WARM-UP 15

1. The circumference of circle O is 56 centimeters, and the length of $\widehat{XYZ}$ is 2.8 centimeters. What is the number of degrees in the measure of acute $\angle XOZ$?



2. The positive difference between the squares of two consecutive odd integers is 128. What is the product of the two integers?

3. An octagon is inscribed in a square so that the vertices of the octagon trisect the sides of the square. The perimeter of the square is 108 centimeters. What is the number of square centimeters in the area of the octagon?

4. Janelle cleans her aquarium by replacing $\frac{2}{3}$ of the water with new water, but that doesn't clean the aquarium to her satisfaction. She decides to repeat the process, again replacing $\frac{2}{3}$ of the water with new water. How many times will Janelle have to do this so that at least 95% of the water is new water?

5. Steve's bathtub, when full, will drain in 12 minutes. The hot water tap takes 6 minutes to fill the tub, and the cold water tap takes 4 minutes to fill the tub. If Steve opens both taps but forgets to put in the plug, in how many minutes will the tub be filled?

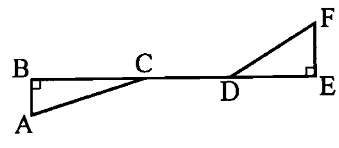
6. Two congruent regular polygons are placed side-by-side so they share a common side. The angle formed between the polygons is a right angle. How many sides does each polygon have?

7. For what value of n is the following equation true?

$$5\frac{4}{5} = 1 + \frac{n}{1 + \frac{1}{1+\frac{1}{1+1}}}$$

8. One acre equals 43 560 square feet. What is the ratio of 160 acres to 1 square mile? Express your answer as a common fraction.

9. In the diagram, AB = 2 cm, AC = $2\sqrt{17}$ cm, DF = $3\sqrt{5}$ cm and EF = 3 cm. If the distance from A to F is $5\sqrt{17}$ cm, what is the number of centimeters in CD?



10. Each asterisk in the diagram indicates a missing digit in the division. What is the value of the two-digit divisor?

```
       8**
  ** )*****
      **
      ***
      ***
        **
        **
```

ANSWER KEY - WARM-UP 16

1.	13	(FM)	**2.**	$-\frac{2}{3}$	(FM)	**3.**	50	(FM)
4.	2:50	(T)	**5.**	15	(FMT)	**6.**	4	(FTE)
7.	$\frac{3}{4}$	(MT)	**8.**	6π	(FM)	**9.**	36	(MTG)
10.	44	(TG)						

SOLUTION **Problem #9**

FIND OUT We want to know the difference between the greatest and least possible perimeters of a rectangle when the areas of some regions are known.

CHOOSE A STRATEGY We can represent the dimensions of the region with area $12\,\text{in}^2$ using the variable m, and we can represent the dimensions of the region with area $48\,\text{in}^2$ using n. Then, we can determine possible values for m, n and the perimeter.

SOLVE IT If the height of the lower-left rectangle is m, then its length is $\frac{12}{m}$. Similarly, if the height of the upper-right rectangle is n, then its length is $\frac{48}{n}$. More importantly, the area of the lower-right region is $\frac{48m}{n}$, and the area of the upper-left region is $\frac{12n}{m}$. Since these two areas are equal, we have

$$\frac{48m}{n} = \frac{12n}{m}$$
$$48m^2 = 12n^2$$
$$2m = n$$

The negative possibilities were not considered in the last step, as the length can never be negative. Regardless, the result implies that the value of n is twice the value of m.

Because the dimensions of each region are whole numbers, m must be one of the integer factors of 12. Further, the perimeter p of the entire rectangle is given by the expression $2\left(m + n + \frac{12}{m} + \frac{48}{n}\right)$. This information allows us to generate the following table:

If m is...	1	2	3	4	6	12
Then n is...	2	4	6	8	12	24
And p is...	78	48	42	42	48	78

The greatest perimeter is 78 and the least perimeter is 42, so the difference is 36.

LOOK BACK We can check by calculating the areas of the two unknown regions for the values of m and n used in the table. For $m = 1$ and $n = 2$, the lower-right region has dimensions 1×24 with area $24\,\text{in}^2$, and the upper-left region has dimensions 2×12 with area $24\,\text{in}^2$, also. For other values of m and n, the results check, too, so we can be confident that 36 is the correct answer.

MAKING CONNECTIONS... to Traffic **Problem #5**

Highway traffic can only go where there is space to go. Consider an ideal traffic condition in which there is one lane of cars all traveling at a speed of v ft/s. Further, all vehicles are of the same length ℓ ft, and a distance of d ft is between all cars. If you were standing on the side of the road, it would take $\frac{\ell+d}{v}$ seconds for a vehicle to completely pass you. Thus, the number of vehicles passing in an hour would be $3600 \div \frac{\ell+d}{v}$. For example, if $\ell = 20$ ft, $d = 80$ ft, and $v = 60$ mph $= 88$ ft/sec, the number of vehicles passing in one hour would be $3600 \div \frac{20+80}{88} = 3128$. On a four-lane highway, under the ideal conditions given, 12 672 cars could be accommodated each hour.

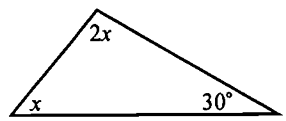
Transportation engineers deal with the issues that surround highway traffic. They determine how the capacity changes if the speed limit changes from 60 mph to 65 mph, or if the highway narrows to just three lanes. They also study how the traffic patterns change because of individual driver differences. For instance, what happens if one vehicle slows down to 55 mph? How does a slower driver affect a one-lane highway differently than a three-lane highway? What if one-tenth of all drivers travel at 55 mph, and they are randomly distributed among the traffic flow?

WARM-UP 16

1. A plane passes through a sphere, and the area of the circle formed by that intersection is 144π square centimeters. The perpendicular distance from the center of the sphere to the plane is 5 centimeters. What is the number of centimeters in the length of the radius of the sphere?

2. What is the slope of the line that is tangent to a circle at point $(5,5)$ if the center of the circle is $(3,2)$? Express your answer as a common fraction.

3. What is the number of degrees in x?



4. The three afternoon classes at Euclid Middle School were shortened one day so that students would have extra time to prepare for the math competition. Each of the last three periods were shortened to 35 minutes, with a 5-minute passing time between classes. What time would the final class end if afternoon classes started at 12:55?

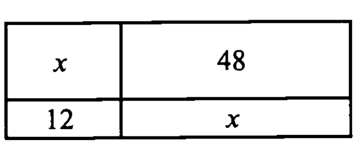
5. Joanne rode her bike 30 miles. If she had increased her average speed by 3 mph, she would have decreased her travel time by 20 minutes. How many miles per hour was her original speed?

6. For what digit n is the five-digit number $3n85n$ divisible by 6?

7. What is the probability of tossing exactly two heads or exactly two tails when three fair coins are tossed? Express your answer as a common fraction.

8. A cable is tied around the Earth at the equator. A longer second cable is placed so that it is always exactly 3 feet directly above the Earth's equator. What is the positive difference in the number of feet in the lengths of the two cables? Express your answer in terms of π.

9. Each rectangular region in the figure has the area shown: the largest region has area $48\,\text{in}^2$; another region has area $12\,\text{in}^2$; and the other two regions have the same unknown area $x\,\text{in}^2$. All dimensions are whole numbers. What is the number of inches in the difference between the greatest and least possible perimeters for the entire rectangle?



10. The product of two positive integers is 315. One of them is a single-digit number, and the other is a two-digit number. What is the least possible sum of the two numbers? (*Problem submitted by former question writer Diane Spresser.*)

ANSWER KEY – WARM-UP 17

1.	$\frac{1}{2}$	(MT)	**2.**	24	(M)	**3.**	576	(MT)
4.	−6	(FM)	**5.**	$28\frac{4}{5}$	(M)	**6.**	3	(CM)
7.	39	(MT)	**8.**	12	(TG)	**9.**	2	(FM)
10.	28	(M)						

SOLUTION **Problem #8**

FIND OUT What are we asked to determine? The number whose positive integer factors have a sum of 28.

CHOOSE A STRATEGY First, let's determine the factors of the number 28. Then, using some knowledge about the sums of factors, we can deduce the answer.

SOLVE IT For a prime number p raised to the nth power, the sum of the factors of p^n is $\frac{p^{n+1}-1}{p-1}$. For instance, the fourth power of 3 is $3^3 = 27$—the sum of its factors is $1 + 3 + 9 + 27 = 40$, and $\frac{3^4-1}{3-1} = 40$.

Using this rule, we can create a table which shows the sums of the factors of powers of prime numbers:

Powers of 2	2^0	2^1	2^2	2^3	2^4
Sum of Factors	1	3	7	15	31
Powers of 3	3^0	3^1	3^2	3^3	3^4
Sum of Factors	1	4	13	40	121

For the problem at hand, we are concerned with a number whose factors have a sum of 28. One factorization of 28 is 4×7. This is helpful, because both 4 and 7 appear in the chart above. The number 4 appears as the factor sum of 3^1, and 7 appears as the factor sum of 2^2. Consequently, the number for which we are searching has a prime factorization of $3^1 \times 2^2$. The number is 12.

LOOK BACK Does our answer check? Yes. The factors of 12 are 1, 2, 3, 4, 6 and 12, which have a sum of $1 + 2 + 3 + 4 + 6 + 12 = 28$.

MAKING CONNECTIONS...to Art Problem #4

In 1514, German artist Albrecht Dürer (1471–1528) made the copperplate engraving *Melancholia*, in which the following grid of numbers appeared:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

There is no consensus as to how this square of numbers relates to the theme of the painting. But Dürer was taken by this square because the sum of the numbers in each row, each column, and even along each of the long diagonals, was the same. This square of numbers is a magic square with magic sum 34. Further, in the middle left part of this painting, Dürer placed a geometric solid which has triangular and pentagonal faces. (A web site with this painting can be found at **http://familiar.sph.umich.edu/cjackson/durer/p-durer23.htm.**) Dürer was concerned with mathematics in other realms, too, as is evident by his treatise *Four Books on Human Proportions*. Commonly, artists use mathematical ideas—and geometric patterns in particular—to enhance a painting or to influence a design.

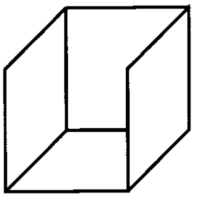
INVESTIGATION & EXPLORATION Problem #1

This problem can be solved in many ways. A tree diagram could be helpful in organizing the information. Logic would dictate that the chance of replacing a red or white ball is equally likely, so the probability of choosing red on the second draw is still $\frac{1}{2}$. But a hands-on method that would not only familiarize students with probability but also lead them into data analysis is sampling. By running this experiment a large number of times, students will begin to see that the probability of choosing a red ball is close to the probability of choosing a white ball. And if the class compiles their results, the probability will likely tend even closer to $\frac{1}{2}$.

WARM-UP 17

1. There are two red balls and two white balls in a jar. One ball is drawn and replaced with a ball of the other color. The jar is then shaken and one ball is chosen. What is the probability that this ball is red? Express your answer as a common fraction. **1.** ______

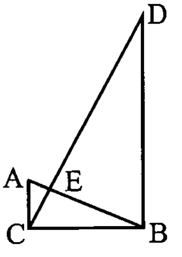
2. Each side of a cube is painted a different color. The cube is then inserted into the container shown which only allows the top and front faces to show. In how many different ways can the cube be inserted into the box? **2.** ______



3. What three-digit positive integer is exactly 32 times the sum of its digits? **3.** ______

4. In a magic square, numbers are arranged in an $n \times n$ array so that the sum of the numbers listed in each row and column is the same. This sum is called the "magic sum". Use the nine numbers $-10, -8, -6, -4, -2, 0, 2, 4$ and 6 to form a 3×3 magic square. What is the magic sum for this magic square? **4.** ______

5. In the diagram, right $\triangle ACB$ and right $\triangle DBC$ share base BC as shown. $\overline{CD} \perp \overline{AB}$ at E, $\overline{AC} \parallel \overline{BD}$, AC = 5 cm, and BC = 12 cm. What is the number of centimeters in BD? Express your answer as a mixed number. **5.** ______



6. It took Jim $1\frac{1}{3}$ hours to walk $\frac{4}{9}$ of the way home. At the same rate, how many hours would it take Jim to walk all the way home? **6.** ______

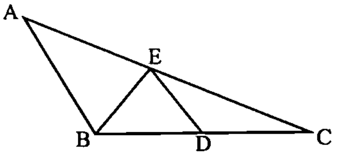
7. The lattice shown continues indefinitely. What is the sum of the row number and column number in which the integer 254 appears? **7.** ______

	Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7
Row 1	1	2	3	4	5	6	7
Row 2	8	9	10	11	12	13	14
Row 3	15	16	17	18	19	20	21
⋮	⋮			…			⋮

8. The sum of all positive integer factors of a number is 28. What is the number? **8.** ______

9. Point $(6, 3)$ is reflected over line ℓ to $(2, 5)$. What is the slope of line ℓ? **9.** ______

10. In $\triangle ABC$, BE = DE = DC, and $m\angle AEB = 84°$. What is the number of degrees in $m\angle C$? **10.** ______



ANSWER KEY – WARM-UP 18

1.	20	(M)	**2.**	$\frac{7}{10}$	(M)	**3.**	46.8	(FMP)
4.	5	(MTG)	**5.**	250	(TS)	**6.**	9	(C)
7.	12	(MG)	**8.**	$\frac{1}{2}$	(FM)	**9.**	0	(CM)
10.	288	(FM)						

SOLUTION **Problem #7**

FIND OUT What are we asked to find? How many of each item Marva could afford to purchase.

CHOOSE A STRATEGY We can use a modified guess-and-check strategy. We can assign values for the prices of the pencils, cards and pens that satisfy the initial conditions and then use that information to answer the question.

SOLVE IT Marva can afford to purchase 21 pencils and 21 cards (42 items total), or 28 pens. Consequently, the ratio of the number of pencils and cards (total) to the number of pens is 3:2. Hence, the ratio of the price of a pencil or a card to a pen must be $2:3$. Therefore, let's assign a value of 2 to the price of a pencil, 2 to the price of a card, and 3 to the price of a pen. This works, because the price of the pencils was $21 \times 2 = 42$, and the price of the cards was $21 \times 2 = 42$. Their combined price was 84. The price of the pens was also $28 \times 3 = 84$.

Now that prices have been assigned, we can determine how many of each item Marva could afford. The cost of one pencil, one card and one pen is $2+2+3=7$. From above, we know that she had money totalling 84. Consequently, she could afford to buy $84 \div 7 = 12$ of each item.

LOOK BACK Does our logic work in general? Yes. Using variables to represent the prices, we see that $21x + 21y = 28z$, or $3(x+y) = 4z$. Hence, the cost of one of each item is $x+y+z = (x+y)+z = \frac{4}{3}z + z = \frac{7}{3}z$. Since she was able to afford the cost of the pens which was $28z$, then she could afford to purchase $28z \div \frac{7}{3}z = 12$ of each item.

MAKING CONNECTIONS... to Braille, Credit Cards and Combinatorics Problem #5

The multiplication principle, which says that the number of ways for two events to happen is equal to the product of the number of ways for each event to happen independently, underlies many of the situations we encounter everyday. For instance, Braille, the typesetting system used to produce books for blind people, represents each symbol, letter or number with six dots, and each dot can be raised or unraised. Since there are two possibilities for each dot, the multiplication principle tells us that there are $2^6 = 64$ possible representations. (Actually, one of these representations is six unraised dots, which is just a blank space, so there are only 63 representations which are useful.)

The account numbers assigned to credit cards are another example of where the multiplication principle rears its head. A typical credit card account number has 16 digits. The first number is usually predetermined by the company offering the credit card, and the last number is a "check digit" so its value is preset, too. But the other 14 numbers can be any digit 0–9, so there are at least 10^{14} possible account numbers. But 10^{14} is 20 000 times the population of the world. Why are so many account numbers needed? To protect cardholders from having their account number correctly guessed by would-be frauds.

INVESTIGATION & EXPLORATION Problem #4

Acting out a problem is a way to solidify a method of solution in a student's mind. The underlying principle is the age-old adage, "Seeing is believing." The traditional handshake problem, "When n people shake hands with each other exactly once, how many handshakes occur?" is a perfect example of a problem that can be modeled. For several values, n students can act out the situation and determine an answer. More importantly, watching the problem being acted out allows a student to recognize the underlying pattern required to solve this problem and provides the necessary information for generalizing a solution.

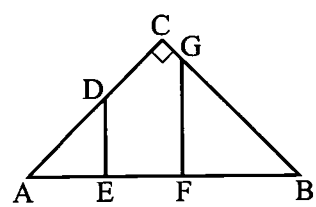
The problem here can be acted out similarly. An arbitrary value can be assigned to Hannah's partner, and then the situation can be acted out to see if a contradiction occurs. When a value is given to Hannah's partner that doesn't result in a contradiction, the answer has been determined.

WARM-UP 18

1. The width of a rectangle is increased by 6 inches while the length is decreased by 2 inches. The resulting rectangle is a square with perimeter 28 inches. What was the number of inches in the perimeter of the original rectangle? 1. ________

2. A rectangle has vertices with coordinates $(-2,3)$, $(8,3)$, $(8,11)$ and $(-2,11)$. What is the probability that a point randomly chosen inside the rectangle will be to the right of the line $y = 2x + 5$? Express your answer as a common fraction. 2. ________

3. In right $\triangle$ ACB, $\overline{DE} \perp \overline{AB}$ and $\overline{GF} \perp \overline{AB}$. AE = 5 cm, DE = 12 cm, and FG = 18 cm. What is the number of centimeters in GB? Express your answer as a decimal to the nearest tenth. 3. ________



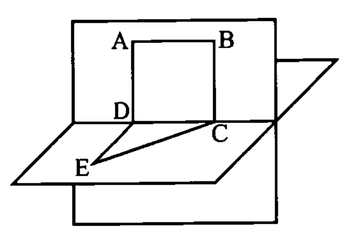
4. In one room of the science fair were six projects, and each project was entered by a pair of students. During the fair, some of the students visited other projects in the room and shook hands with some of the other project entrants. No students shook hands with their own partners. After the fair, the students discussed the projects they had visited. Hannah learned that no two of the other eleven students had shaken the same number of hands. How many hands did Hannah's partner shake? 4. ________

5. How many even four-digit counting numbers can be formed by choosing digits from the set $\{1,2,3,4,5\}$ if digits can be repeated? 5. ________

6. Given that $f(x) = \frac{x^2-1}{x+1}$ and $g(x) = x^2 - 2x + 1$, what is $g(f(5))$? 6. ________

7. When shopping for presents, Marva had exactly enough money to buy either 21 pencils and 21 cards, or 28 pens. Instead, she decided to buy the same number of pens, pencils and cards. How many of each item could she afford to purchase? 7. ________

8. Square ABCD and isosceles right $\triangle$ CDE lie in perpendicular planes. $\angle$ CDE is a right angle, and EC = 6 cm. What is the ratio of the area of $\triangle$ CDE to the area of square ABCD? Express your answer as a common fraction. 8. ________



9. Point (a, b) is reflected over the x-axis and then is reflected over the y-axis. After these reflections, the new coordinates of the point are (c, d). What is the value of $ab - cd$? 9. ________

10. A space diagonal of a cube is 12 inches long. How many square inches are in the surface area of the cube? 10. ________

ANSWER KEY - WORKOUT 1

1.	Saturday	(FT)	**2.**	6.3	(FM)	**3.**	17.46	(CF)
4.	27	(FM)	**5.**	64	(TS)	**6.**	2	(TP)
7.	310	(TE)	**8.**	12	(MG)	**9.**	4.5	(C)
10.	34	(FM)						

SOLUTION **Problem #5**

FIND OUT What do we want to know? The number of distinct values that can be obtained by placing parentheses in an expression.

CHOOSE A STRATEGY By considering a simpler case, we can identify a pattern. From that, we can generalize to find a solution.

SOLVE IT Let's first consider the expression $1 \div 2 \div 3$, which is much less cumbersome. One possible placement of parentheses gives $(1 \div 2) \div 3 = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$; another placement gives $1 \div (2 \div 3) = \frac{1}{1} \cdot \frac{3}{2} = \frac{3}{2}$. Of course, we could use fewer or more parentheses, but any other result would be equivalent to one of these two results.

Similarly, when the expression contains four numbers—$1 \div 2 \div 3 \div 5$—there are four possible results: $\frac{1}{30}, \frac{3}{10}, \frac{5}{6}, \frac{15}{2}$. When 7 is added to the expression, there are eight possible results. It is probably becoming apparent that the number of possibilities doubles each time a number is added to the expression. The reason for this is the movement of each number because of parentheses. The 1 and the 2 are fixed; that is, in the final result, the 1 will be a factor of the numerator and the 2 will be a factor of the denominator. However, the other numbers may move to either the numerator or the denominator. For each number, there are two possibilities. Because there are six numbers beyond the 1 and 2 in the expression, then, there are $2^6 = 64$ possible values.

LOOK BACK Have we accounted for all possibilities? Yes. Notice that all the numbers in the expression are prime numbers. Consequently, each number is relatively prime to every other number in the expression. If the numbers shared a common factor, our method of counting would allow for overlap. But since all numbers are relatively prime, there is no overlap, and we can be certain that our method counted all possible values.

MAKING CONNECTIONS...to the Stock Market Problem #3

Stocks on the NYSE (New York Stock Exchange), AMEX (American Stock Exchange) and NASDAQ (National Association of Securities Dealers Automated Quotations) stock exchanges are traded in fractions, not decimals, but this trend looks like it may soon change. In December, 1996, the NASDAQ opted to quote stock prices in increments of $\frac{1}{16}$ of a dollar. Prior to that, quoting was done only in increments of $\frac{1}{8}$. The move toward *decimalization*, however, would mean that quotes could be given in increments of $0.01 = \frac{1}{100}$ rather than $0.0625 = \frac{1}{16}$.

The little-known fact about the NASDAQ is that, although quotes must be given in increments of $\frac{1}{8}$ for stocks above \$10 and in increments of $\frac{1}{32}$ for stocks below \$10, stocks may be traded in increments of $\frac{1}{256}$, which is roughly equal to 0.4¢—a higher level of precision than decimalization provides. And if the trade price is determined using decimals, trading currently may be done with precision up to eight decimal places of accuracy. Typically, however, there is no need for that level of precision, so investors rarely use such small increments.

INVESTIGATION & EXPLORATION Problem #4

The Geometer's Sketchpad™ or any of the computer geometry applications now available are very handy tools in an exploratory classroom. Barring that, though, simple tools like a straightedge, protractor and compass will assist the teacher who uses discovery learning.

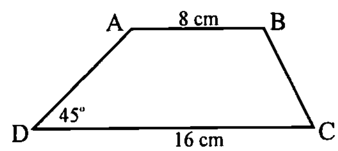
With the figure given in this problem, students can measure the angles to see what relationships exist. The trivial finding is learning that two angles which lie on a straight line are supplementary. A less trivial but fairly obvious discovery is that vertical angles have equal measure. And what students may have trouble noticing on their own, but what a good teacher will guide them toward, is the realization that alternate interior angles are congruent.

WORKOUT 1

1. What day of the week will it be one million seconds after 1:00 AM on Tuesday?

1. ______________

2. If the area of trapezoid ABCD is 72 sq cm, what is the number of centimeters in the length of $\overline{BC}$? Express your answer as a decimal to the nearest tenth.



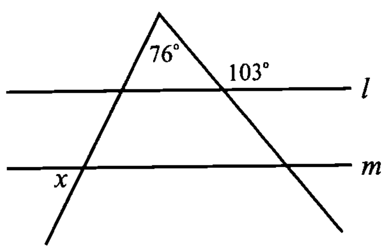
2. ______________

3. Five stocks had prices of $7\frac{7}{8}$, $7\frac{15}{16}$, $23\frac{5}{8}$, $19\frac{1}{2}$ and $28\frac{3}{8}$ dollars. What is the number of dollars in the average price for these five stocks? Express your answer to the nearest cent.

3. ______________

4. In the figure, $\ell \parallel m$. What is the number of degrees in x?

4. ______________



5. How many distinct values can be obtained for the expression $1 \div 2 \div 3 \div 5 \div 7 \div 11 \div 13 \div 17$ if an unlimited number of parentheses may be placed in the expression?

5. ______________

6. What is the remainder when 2^{133} is divided by 5?

6. ______________

7. There are two prime numbers between 100 and 199 such that the tens digit is a prime number, the units digit is a prime number, and the tens and units digit taken together are a two-digit prime number. Find the sum of these two prime numbers.

7. ______________

8. As Jenna goes from home to school, she passes Mary's house and then Jack's house. The distance from Jenna's house to Mary's is ten times the distance from Mary's house to Jack's. The distance from Mary's house to Jack's is ten times the distance from Jack's to school. The total distance from Jenna's house to school is 13.32 km. How many kilometers is it from Jenna's house to Mary's, assuming the houses and school are collinear?

8. ______________

9. Aida bought a jacket for $146.30 which included $6.30 in sales tax. What was the percent of tax that she paid? Express your answer to the nearest tenth.

9. ______________

10. $\angle C$ is the complement of $\angle A$ and the supplement of $\angle B$. Additionally, $m\angle A + m\angle B = 202°$. What is the number of degrees in $m\angle C$?

10. ______________

ANSWER KEY – WORKOUT 2

1.	47	(CP)	**2.**	72 239	(CT)	**3.**	36	(FM)
4.	369	(C)	**5.**	106	(CFP)	**6.**	400	(CP)
7.	9.16	(CG)	**8.**	36	(TG)	**9.**	30	(FM)
10.	3	(MG)						

SOLUTION **Problem #3**

FIND OUT What are we asked to determine? The area of a rectangle inscribed in a larger rectangle.

CHOOSE A STRATEGY Using knowledge of geometry, we can determine the lengths of many segments in the figure. From these measurements, we can determine the dimensions of the smaller rectangle and calculate its area.

SOLVE IT Near point D there are three angles, one of which is a right angle, and the other two measure 45°. Consequently, the triangle in the upper-left corner of the figure is a 45-45-90 triangle—its side lengths are 3″, and the hypotenuse $\overline{AD}$ has length $3\sqrt{2}''$.

The large triangle in the lower-left corner is also a 45-45-90 triangle. As a result, both angles in the lower-right corner of the figure measure 45°, and the triangle in the lower-right corner is a 45-45-90 triangle. Because ABCD is a rectangle and $AD = 3\sqrt{2}''$, $BC = 3\sqrt{2}''$, also. Hence, the hypotenuse of the lower-right triangle has length 6″. Because B is the midpoint of a side of the larger rectangle, both segments above and below B have length 6″. The triangle in the upper-right corner is a 45-45-90 triangle as well, and the length of its hypotenuse $\overline{AB}$ is, then, $6\sqrt{2}''$.

The area of rectangle ABCD is therefore $6\sqrt{2} \times 3\sqrt{2} = 36\,\text{in}^2$.

LOOK BACK Does this answer seem reasonable? Yes. Based on our deductions, the height of the larger rectangle is 12″, and its width is 9″. It has an area of $108\,\text{in}^2$. By inspection, it seems reasonable that the area of rectangle ABCD is roughly one-third the area of the larger rectangle. Further, the areas of the four triangles are $40\frac{1}{2}\,\text{in}^2$, $18\,\text{in}^2$, $9\,\text{in}^2$ and $4\frac{1}{2}\,\text{in}^2$, which means the area of rectangle ABCD is $108 - (40\frac{1}{2} + 18 + 9 + 4\frac{1}{2}) = 36\,\text{in}^2$.

MAKING CONNECTIONS... to Statistics Problem #1

Winston Churchill once said that there are three types of lies—lies, damn lies, and statistics. And Darrell Huff wrote a book titled, "How to Lie with Statistics". These sentiments stem from the belief that statistics often mislead, even if the intent is not to intentionally deceive. A real estate agent, for instance, may mention that the average value of the homes in a neighborhood is $200 000. Depending on the data underlying this statistic, however, there could be several scenarios. It could be that all of the homes are of the same general value. Or, it could mean that there are a few very expensive houses surrounded by many inexpensive tract homes. Like all statistics, the data underlying this realtor's statement needs to be considered before its meaning can be ascertained.

INVESTIGATION & EXPLORATION Problem #8

The solution to problem #8 in Warm-Up 17 shows the theory that will be helpful in solving problem #8 in this workout. However, that solution uses a theorem about the sum of the factors of a prime number—but that theorem is never proven. A nice exploration for students is to discover this formula on their own, and verify to themselves that it is correct.

The theorem states that the sum of the factors of p^n, where p is a prime number, is $\frac{p^{n+1}-1}{p-1}$. It is very easy to determine the sum of the factors of powers of 2, 3 or 5 fairly quickly. Using that information, students should attempt to discover the rule for determining the sum of the factors of a power of a prime number. Once students have made an intuitive guess at the formula, the following proof (which is given for the sum S of the factors of 3^n, but which can be extended for powers of other prime numbers) can be shown:

$$S = 3^0 + 3^1 + 3^2 + \ldots + 3^n$$

$$3S = 3^1 + 3^2 + \ldots + 3^n + 3^{n+1}$$

$$3S - S = 2S = 3^{n+1} - 3^0 = 3^{n+1} - 1$$

$$S = \frac{3^{n+1} - 1}{2}$$

WORKOUT 2

1. A, B, and C are three arithmetic sequences. What is the median of the means of the three sequences?

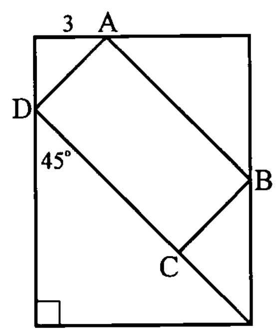
$$A: 1, 3, 5, \ldots, 91$$
$$B: 2, 4, 6, \ldots, 92$$
$$C: 3, 6, 9, \ldots, 93$$

1. ______________

2. What is the least common multiple of 1537 and 1363?

2. ______________

3. Point B is a vertex of rectangle ABCD and the midpoint of a side of the larger rectangle. What is the number of square inches in the area of rectangle ABCD?



3. ______________

4. The operation $a \star b$ is defined to be $(a+b)(a-b)$ and $a \nabla b$ is defined to be $a^2 + b^2$. What is the value of $(5 \star 4)(5 \nabla 4)$?

4. ______________

5. Compute: $2 - 4 + 6 - 8 + 10 - 12 + 14 - \ldots + 210$

5. ______________

6. A used car salesperson has a vehicle she can't sell. The original price was \$1024. She dropped the price initially to \$640. When the car still wouldn't sell, she made an equivalent percent decrease. How many dollars are in the next reduced price of the car?

6. ______________

7. In order to qualify for her age group in the Boston Marathon, Lisa must run a 26.2-mile marathon in a time of four hours or less. What is the slowest average pace in minutes per mile that Lisa can run and still qualify for Boston? Express your answer as a decimal to the nearest hundredth.

7. ______________

8. The sum of all positive integer factors of a number is 91. What is the number?

8. ______________

9. An isosceles trapezoid may be divided into three congruent non-overlapping equilateral triangles. The area of each triangle is $9\sqrt{3}$ square inches. What is the number of inches in the perimeter of the trapezoid?

9. ______________

10. What is the value of k if $2x + 1$ is a factor of $2x^2 + 7x + k$?

10. ______________

ANSWER KEY - WORKOUT 3

1.	22	(TP)	**2.**	45	(MTP)	**3.**	10	(MTG)
4.	123	(MT)	**5.**	123 456	(CP)	**6.**	8	(FM)
7.	7	(FP)	**8.**	292	(M)	**9.**	20	(MG)
10.	121	(TGP)						

SOLUTION **Problem #3**

FIND OUT What are we asked to find? The number of blue marbles in a jar.

CHOOSE A STRATEGY We can generate fractions equivalent to $\frac{1}{3}$. The numerator will represent the number of red marbles, and the denominator will represent the number of total marbles. Then, we can add 1 to the numerator and denominator to simulate adding one red marble until we find a ratio equivalent to $\frac{3}{8}$.

SOLVE IT A table showing equivalent fractions and the resulting fractions when the numerator and denominator are increased by 1 is shown below.

Fractions Equivalent to $\frac{1}{3}$	Fraction when Numerator is Increased by 1
$\frac{1}{3}$	$\frac{2}{3}$
$\frac{2}{6}$	$\frac{3}{6} = \frac{1}{2}$
$\frac{3}{9}$	$\frac{4}{9}$
$\frac{4}{12}$	$\frac{5}{12}$
$\frac{5}{15}$	$\frac{6}{16} = \frac{3}{8}$

The ratio will be $\frac{3}{8}$ when there are initially 5 red marbles and 15 marbles total. Hence, there must be 10 blue marbles in the jar.

LOOK BACK Does this answer check? Yes. There are 5 red marbles and 15 marbles total, so the ratio is $\frac{1}{3}$. If a red marble were added, there would be 6 red marbles and 16 marbles total, which is a ratio of $\frac{3}{8}$.

MAKING CONNECTIONS... to Construction Problem #8

The information given in problem #8 refers to section 4.8.2 of the *ADA Handbook*. That section states that the "...maximum slope for a ramp in new construction shall be 1:12. The maximum rise for any run shall be 30 in." An earlier section of the handbook states that any route with a slope greater than 1:20 shall be considered a ramp. These pieces lead to some interesting questions. For a ramp that must rise 30 in, what is the minimum run? What is the maximum run?

There is also one obvious question which must be asked. What happens if a ramp must have a rise of more than 30 in? The handbook requires that any ramp must have a landing at the top and bottom with a minimum length of 60 in. Two successive ramps, then, each with a rise of 30 in, must be separated by a landing. If a landing occurs between two ramps which change directions, the landing area must be 60 in × 60 in. And any landing must be at least twice the width of the ramp which leads to it. Why do you think such requirements are necessary?

INVESTIGATION & EXPLORATION Problem #2

Before tackling the problem here, it is beneficial for students to consider simpler cases, such as what happens with 1, 2, 3 or more lines in a plane. With only 1 line, there are no points of intersection. With 2 lines, there is 1 intersection point, and with 3 lines, there are 3 intersection points. The pattern of the maximum number of intersection points increases in a familiar way: $0, 1, 3, 6, 10, 15, 21, \ldots$—the triangular numbers. Hence, the maximum number of intersection points possible for 10 lines in a plane is equal to the 9th triangular number, 45.

An interesting extension question is to ask what possible numbers of intersection points are possible for n lines. For 2 lines, there are two possibilities: if the lines cross, there is 1 point of intersection, and if the lines are parallel, there are none. Have students search for the numbers possible for other values of n. The research they do will lead to some intriguing results as well as expose them to some interesting math.

WORKOUT 3

1. How many consecutive zeroes are at the end of the product $15 \times 16 \times 17 \times 18 \times 19 \times \cdots \times 100$? 1. ________

2. What is the maximum number of intersection points possible for ten non-collinear lines in the same plane? 2. ________

3. There are red and blue marbles in a jar. One-third of the marbles are red. If another red marble is added to the jar, then $\frac{3}{8}$ of the marbles will be red. How many blue marbles are in the jar? 3. ________

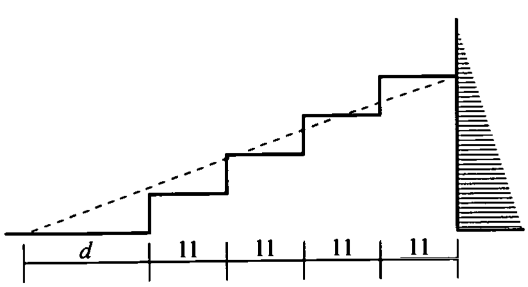
4. In a school with 2460 students, 65% of the students are enrolled in algebra, 45% are enrolled in French, and 55% are enrolled in history. One-fourth of the students are enrolled in algebra and French, two-fifths are enrolled in algebra and history, and one-fourth are enrolled in French and history. One-fifth of the students are enrolled in all three subjects. How many students in the school are not enrolled in any of the courses? 4. ________

5. Simplify: $\dfrac{123\,456}{(123\,457)^2 - (123\,456)(123\,458)}$ 5. ________

6. If exactly five diagonals can be drawn from one vertex of a convex polygon, how many sides does the polygon have? 6. ________

7. The third term of an arithmetic sequence is 15 and the fifth term is 23. What is the first term? 7. ________

8. The entrance to a museum consists of four steps. The tread of each step is 11″ long, and the riser of each step is 7″ high. To comply with section 4.8.2 of the *Americans with Disabilities Act Handbook*, the steps are being replaced by a ramp with a slope of 1:12 (indicated by the dotted line). How many inches d past the base of the lowest step must the ramp extend? 8. ________



9. At her bike factory, Eleanor manufactures two-wheel and three-wheel bikes. She has 96 seats and 212 wheels in inventory. She wants to use all the seats and wheels. How many three-wheelers should she manufacture? 9. ________

10. What is the least natural number that when divided by 11 gives a remainder of 0, but when divided by 2, 3, 4, 5 or 6 gives a remainder of 1? 10. ________

ANSWER KEY – WORKOUT 4

1.	6.5	(FM)	**2.**	40	(M)	**3.**	36	(FTP)
4.	3828	(FM)	**5.**	193	(TG)	**6.**	4096	(TP)
7.	36	(FT)	**8.**	$\frac{1}{27}$	(CG)	**9.**	88.7	(FM)
10.	−15	(M)						

SOLUTION **Problem #3**

FIND OUT What are we trying to determine? The number of seven-digit numbers that exist such that each digit is greater than the digit to its left.

CHOOSE A STRATEGY By considering the first digit and the last digit possibilities, we can use combinatorics to determine the number of possible arrangements of the digits in between. Then, we can calculate the sum of the numbers of arrangements to find the answer.

SOLVE IT If the first digit is 1 and the seventh digit is 9, five of the digits 2–8 must be chosen to fit between. (Note that because the five digits chosen will be placed in ascending order, each selection of five digits results in a unique solution.) Hence, there are $\binom{7}{5} = \frac{7!}{5!2!} = \frac{7\cdot 6}{2} = 21$ phone numbers with first digit 1 and last digit 9 for which the digits are in ascending order.

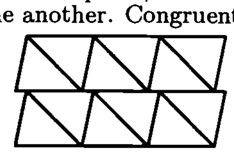
Similarly, if the first digit is 1 and the seventh digit is 8, five of the digits 2–7 must be chosen to fit between. That gives an additional $\binom{6}{5} = 6$ numbers with first digit 1 and last digit 8. Likewise, there are $\binom{5}{0} = 1$ number with first digit 1 and last digit 7, $\binom{6}{1} = 6$ numbers with first digit 2 and last digit 9, $\binom{5}{0} = 1$ number with first digit 2 and last digit 8, and $\binom{5}{0} = 1$ number with first digit 3 and last digit 9.

In total, there are $21 + 6 + 1 + 6 + 1 + 1 = 36$ phone numbers with digits in ascending order.

LOOK BACK It may be surprising that there are $9 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 9\,000\,000$ possible phone numbers, but only 36 of them have their digits in ascending order. But because we worked logically, we can feel confident that our result is correct.

MAKING CONNECTIONS... to Packaging **Problem #4 (& #9)**

Two-dimensional *tessellations* are arrangements of congruent polygons which cover the plane. Squares are one example of a congruent polygon which tessellates the plane, but congruent parallelograms will also tessellate when sides of the same length are placed next to one another. Congruent triangles, as shown below, tessellate, too.



Three-dimensional tessellations are arrangements of congruent polyhedrons which cover space. The three-dimensional analog to the square is the cube, and it is fairly easy to visualize how congruent cubes could completely fill a space. The soda can in problem #9 does not tessellate because of its round base. But when placed as a group in rectangular boxes, the boxes will tessellate. Similarly, most vacuum cleaners are irregularly-shaped and would probably not tessellate space, but the trapezoidal box in which they can be shipped, like the one in problem #4, would.

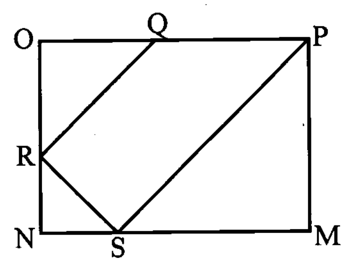
It is the concern of product designers to design packages which can be shipped easily and compactly. Moreover, when boxes can be packed tightly into the trailers of trucks, cargo bays of airplanes, or whatever space in which they are carried, the possibility of damage resulting from the packages moving during shipment is minimal.

WORKOUT 4

1. What is the number of square units in the area of a triangle with vertices A(2,3), B(7,4), and C(4,6)? Express your answer as a decimal to the nearest tenth.

1. ______________

2. Trapezoid PQRS has vertices which lie on rectangle MNOP such that $RN = 2\sqrt{2}$ and RO = OQ = QP. △RNS is a 45-45-90 triangle. What is the number of square inches in the area of trapezoid PQRS?

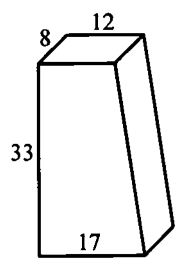


2. ______________

3. The first digit of a seven-digit telephone number can never be 0. How many telephone numbers exist such that each digit is greater than the digit to its left? For example, 124-5689 is one such number. (*Problem submitted by Stanley Levinson, P.E.*)

3. ______________

4. For shipment, an upright vacuum cleaner is packaged in a box shaped like the right trapezoidal prism shown. Using the measurements in inches given, calculate the number of cubic inches in the volume of the box.



4. ______________

5. The digits 1, 3, 4, 6, 8 and 9 are each used exactly once to form three two-digit primes. What is the sum of these prime numbers?

5. ______________

6. One section of a history test has 12 true/false questions. How many different combinations of answers are possible for this section if all twelve questions are answered?

6. ______________

7. Mara has 9 compact discs: 2 rap; 4 rock; and 3 country. How many different combinations of 5 CDs can she choose if she selects 1 rap, 2 rock and 2 country discs?

7. ______________

8. According to *Augustine's Fifth Law*, one-tenth of a committee's members produce one-third of the output, and the other members produce the rest. On average, what fraction of the output does each member of the other 90% produce if there are 20 members on the committee? Express your answer as a common fraction.

8. ______________

9. A 12-pack of soda is packaged in a rectangular box that is $10\frac{1}{2}$ inches long, $7\frac{7}{8}$ inches wide and 5 inches tall. Each can is a cylinder 5 inches tall with a diameter of $2\frac{5}{8}$ inches. How many cubic inches of empty space exist in the box after it is packed and sealed? Express your answer as a decimal to the nearest tenth.

9. ______________

10. Quadrilateral ABCD with vertices A(7,5), B(3,6), C(2,2) and D(5,3) is reflected across the y-axis, translated 3 units to the left, and then reflected across the x-axis. What will be the sum of the coordinates of vertex A after these transformations?

10. ______________

ANSWER KEY – WORKOUT 5

1.	$\frac{5}{4}$	(FM)	**2.**	24	(FM)	**3.**	$\frac{11}{25}$	(TS)
4.	6	(FTG)	**5.**	24.6	(FM)	**6.**	17	(MG)
7.	6.25	(MP)	**8.**	10	(FG)	**9.**	12	(MT)
10.	25 000	(C)						

SOLUTION **Problem #6**

FIND OUT What are we asked to find? The value of k such that $a_k = 5814$ and $a_n = n(n+1)(n+2)$.

CHOOSE A STRATEGY First, note that this question is actually asking for the value of k when $5814 = k(k+1)(k+2)$. The value of the expression for a_k is close to the value for k^3. Hence, determining the cube root of 5814 will identify a good first guess as to the value of k, and we can modify our guess from there.

SOLVE IT Using a calculator, we see that the cube root of 5814 is approximately 17.98. Taking $k = 18$ then gives $a_k = 18 \times 19 \times 20 = 6840$. This is too high, so try a modified guess with $k = 17$. This gives $a_k = 17 \times 18 \times 19 = 5814$. And, *voila!*, an answer is found.

LOOK BACK Algebraic manipulation is a useful tool, but so is algebraic reasoning. We could have multiplied the three factors of a_n to get $n^3 + 3n^2 + 2n$, but this would not have been very helpful. Instead, it was enough to notice that this expression would be close to n^3. If we had been a bit more savvy, we might have noticed that the product of the first and third factors, n and $n+2$, is $n^2 + 2n$, which is very close to the value of $(n+1)^2$. Then, we would have realized that the cube root of 5814 would have given us a better estimate of $n+1$ than of n.

MAKING CONNECTIONS... to Language **Problem #6**

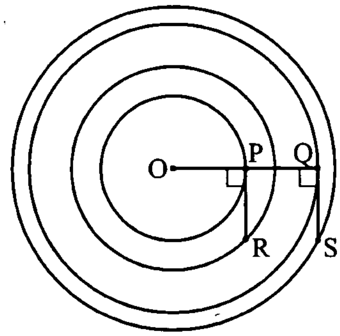
John Allen Paulos, in his book *Beyond Numeracy: Ruminations of a Numbers Man*, draws an analogy between pronouns and variables. The statement "John gave a diamond ring to his wife" can be rewritten as "He gave it to her" with pronouns referring to the people and items involved. Similarly, the mathematical statement "the general term of a sequence is the product of a first number, one more than the first number and two more than the first number" can be rewritten as $a_n = n(n+1)(n+2)$. Just as "her" is used to refer to "John's wife," n is used to mean "the first number." The use of variables is simply mathematical shorthand to say things that would require too many words to describe.

INVESTIGATION & EXPLORATION **Problem #10**

What happens when $2^5 = 32$ is divided by $2^3 = 8$? The result is $2^2 = 4$. Students who see this as well as several other examples will soon realize that $2^m \div 2^n = 2^{m-n}$. That realization lays the foundation for exploring negative exponents. When 2^5 is divided by 2^7, the rule says that the result should be $2^{5-7} = 2^{-2}$, but that may appear a little odd to most students. But coupling that with the display $\frac{2^5}{2^7} = \frac{1}{2^2} = \frac{1}{4}$ will cement the idea that a negative exponent simply implies that the result is the reciprocal of the positive exponent; that is, $2^{-n} = \frac{1}{2^n}$.

WORKOUT 5

1. In the figure, OP = PQ = PR = QS, and point O is the center of the four circles shown. What is the least ratio greater than 1 of the areas of any two concentric circles? Express your answer as a common fraction.



1. ______

2. Don inscribes a 120° central angle in circle Q with radius 6″. This central angle intersects the circle at A and B. Don cuts out sector AQB and rolls it to form a cone such that A meets B and there is no overlap. What is the number of cubic inches in the volume of the cone? Express your answer to the nearest cubic inch.

2. ______

3. The numerator of a fraction is randomly selected from the set $\{1, 3, 5, 7, 9\}$, and the denominator is randomly selected from the set $\{1, 3, 5, 7, 9\}$. What is the probability that the decimal representation of the resulting fraction is not a terminating decimal? Express your answer as a common fraction.

3. ______

4. The six-digit number $3730n5$, with tens digit n, is divisible by 21. What is the value of the digit n?

4. ______

5. What is the number of square inches in the total surface area of a regular square pyramid with a base area of $9\,\text{in}^2$ and with equilateral faces? Express your answer as a decimal to the nearest tenth.

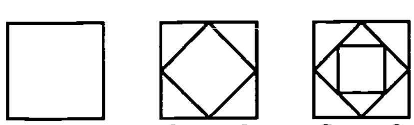
5. ______

6. Given that $a_n = n(n+1)(n+2)$, for what value of k does $a_k = 5814$?

6. ______

7. In the figure, each new stage is obtained by connecting the midpoints of each side of the smallest square at the previous stage. If the largest square has a perimeter of 40 centimeters, find the area in square centimeters of the smallest square in Stage 5. Express your answer as a decimal to the nearest hundredth.

7. ______



8. The equation $x! = 30y!$ holds for more than one set of values for x and y. What is the least possible sum of integer values for x and y?

8. ______

9. How many different arrangements are possible for 5 keys on a circular key ring?

9. ______

10. What is the value of $(900)^{\frac{1}{2}} \times (0.04)^{-2} \times (0.75)^{-1}$?

10. ______

ANSWER KEY – WORKOUT 6

1.	58	(CF)	**2.**	6	(TEP)	**3.**	0	(FM)
4.	92	(TE)	**5.**	58	(TEP)	**6.**	$\frac{3}{14}$	(FT)
7.	4	(MG)	**8.**	21.5	(M)	**9.**	9.52	(C)
10.	1	(FM)						

SOLUTION **Problem #10**

FIND OUT What do we want to find? The distance the dog traveled in the time it took for Donnie and Marie combined to walk one mile.

CHOOSE A STRATEGY Algebra and the distance-rate-time formula will yield the amount of time it took Donnie and Marie to meet. Then, multiplying the dog's speed by the time will give the dog's distance.

SOLVE IT We can use the variable x to indicate the distance Donnie traveled; then, the distance Marie traveled is $1 - x$. We also know that Donnie's speed was 3 mph, and Marie's speed was 4 mph. Because the time each of them walked was the same, we can use the formula $t = \frac{d}{r}$ to determine the time:

$$t_{\text{Donnie}} = t_{\text{Marie}}$$
$$\frac{x}{3} = \frac{1-x}{4}$$
$$4x = 3 - 3x$$
$$x = \frac{3}{7}$$

Since the distance Donnie traveled was $\frac{3}{7}$ of a mile, it must have taken him $\frac{3}{7} \div 3 = \frac{1}{7}$ of an hour to travel that far. The dog was traveling at a rate of 7 mph, so the dog traveled $7 \times \frac{1}{7} = 1$ mile.

LOOK BACK Does this answer make sense? Clearly. Because the combined speed of Donnie and Marie is equal to the speed of the dog, it makes sense that the combined distance traveled by Donnie and Marie equals the distance traveled by the dog.

MAKING CONNECTIONS...to the Golden Ratio Problem #7

Initially, if consecutive terms of this sequence are compared, the ratio among them is rather uninteresting: $\frac{8}{-4} = -2$; $\frac{-4}{4} = -1$; $\frac{4}{0} =$ undefined; $\frac{0}{4} = 0$; $\frac{4}{4} = 1$;.... But if we consider the ratio of the 10th term to the 9th term, the ratio is $\frac{32}{20} = 1.6$. And the ratio of the 15th term to the 14th term is $\frac{356}{220} = 1.6\overline{18}$. Eventually, as the numbers in this sequence grow larger, the ratio of consecutive terms approaches the golden ratio, $\frac{1+\sqrt{5}}{2}$. This is not too surprising, as this sequence is similar to the Fibonacci sequence in that each term is equal to the sum of the previous two, and the ratio of terms in the Fibonacci sequence approaches the golden ratio.

The ancient Greeks believed that the golden rectangle—a rectangle with sides in the golden ratio—was the most aesthetically pleasing of all rectangles. They used the golden rectangle extensively in their designs, especially in the architecture of the Parthenon. The golden ratio has been used more recently, by Stradivarius in the design of his violins and by artists Piet Mondrian and Georges Seurat in their paintings. Today, perhaps inadvertently—or perhaps not—credit cards are roughly in the shape of the golden rectangle.

INVESTIGATION & EXPLORATION Problem #7

The "Making Connections" section above talks about the golden ratio and how it is hidden in the sequence of problem #7. Using a calculator, students can approximate the golden ratio, represented by the Greek letter ϕ, in two different ways.

Method 1. Enter any number. Take the reciprocal of that number, and then add 1. Then, take the reciprocal of that result and add 1. Continue this process several times, and the result will eventually approach the value of $\phi = 1.618033\ldots$. This happens because one definition of the golden ratio is that $\phi = \frac{1}{\phi} + 1$.

Method 2. Enter any number. Add 1, and take the square root of the sum. Then, add 1 to the result and take the square root again. Continue this process, and again the result will approach the golden ratio. A second definition of the golden ratio is that $\phi^2 - \phi - 1 = 0$; written another way, $\phi = \sqrt{\phi + 1}$, which describes the process here.

WORKOUT 6

1. Mowry's department store discounted their sportswear by 40%. Before the end of the season, they reduced the same items by 30% of the discounted price. What was the total percent discount on sportswear items? 1. ________

2. A palindrome is a number that reads the same forwards and backwards, such as 1331 or 747. For how many integer values of n between 6 and 6 000 000 is it true that both $n+1$ and $n-1$ are palindromes? (*Problem submitted by coach Frank Kelly.*) 2. ________

3. The coordinates of right $\triangle$ABE are A(2, 4), B(8, 4), and E(8, 7). Point F is the midpoint of $\overline{AE}$. What is the difference in the number of square units in the area of $\triangle$ABF and the number of square units in the area of $\triangle$FBE? 3. ________

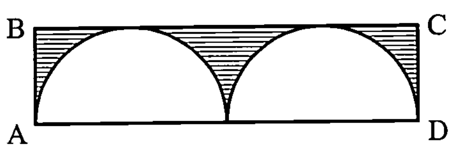
4. What is the greatest two-digit integer whose square is composed entirely of even digits? 4. ________

5. H is the set of all positive integers such that every element of H is not divisible by 3 or divisible by 8. Find the 34th smallest element in H. 5. ________

6. Four dimes and four pennies are randomly placed in a row. What is the probability that the coins at the ends are both pennies? Express your answer as a common fraction. 6. ________

7. The first and seventh terms of a sequence are both 8. Further, each term of the sequence is the sum of the previous two terms. What is the fifth term? 7. ________

8. ABCD is a rectangle such that BC = 20 cm. Two semicircles are tangent at the midpoint of AD, and each is tangent to BC. What is the number of square centimeters in the shaded area? Express your answer as a decimal to the nearest tenth. 8. ________



9. Simplify and express as a decimal to the nearest hundredth. 9. ________

$$4+\frac{8}{1+\frac{1}{2+\frac{1}{4+\frac{1}{2}}}}$$

10. Donnie and Marie were one mile apart when they began walking toward each other. Donnie traveled at 3 mph, and Marie traveled at 4 mph. When they began walking, Donnie's dog, who ran at 7 mph, ran toward Marie; when the dog reached Marie, it turned back and headed toward Donnie. The dog continued to run back and forth until Donnie and Marie met. When Donnie and Marie met, how many miles had the dog traveled? 10. ________

ANSWER KEY – WORKOUT 7

1.	14	(MG)	**2.**	28	(P)	**3.**	50	(CS)
4.	43	(MG)	**5.**	$\frac{2}{17}$	(FS)	**6.**	55.4	(FM)
7.	4	(FM)	**8.**	84 378 922	(TGS)	**9.**	15	(TP)
10.	8	(FM)						

SOLUTION **Problem #1**

FIND OUT What do we need to find? The smallest number in a 3×3 block that appears on the calendar page given.

CHOOSE A STRATEGY This problem can be solved in two very different ways. The first method uses algebra and assigns a variable to the number of the smallest date in the 3×3 block. The second method recognizes that the number in the center of the block will equal the average of all nine numbers. Hence, if we divide the sum of all nine numbers by 9, we get the middle number and can deduce the smallest number. Let's use the second method to get an answer, and then use the first method to verify that our answer is correct.

SOLVE IT The sum of the nine numbers is 198. Dividing by 9 gives $198 \div 9 = 22$, so 22 must be the date in the center of the 3×3 block. Consequently, 21, 22 and 23 are the numbers in the middle row; 14, 15 and 16 are the numbers in the top row; and 28, 29 and 30 are the numbers in the bottom row. Of these, 14 is the smallest number.

LOOK BACK We can use the algebraic method to check. Assigning the variable n to the smallest number, the numbers in the top row are n, $n+1$ and $n+2$. Similarly, the numbers in the middle row are $n+7$, $n+8$ and $n+9$, and the numbers in the bottom row are $n+14$, $n+15$, and $n+16$. Their sum is $9n+72$, and we know that sum is equal to 198. Solving for n yields that $n = 14$, so our solution checks.

MAKING CONNECTIONS...to Transportation **Problem #9**

The mathematics of paths is critical to the success of the transportation industry. Alternate highway or subway paths can be used to avoid delays when portions of a system are congested or closed. On radios across America every morning, the phrase, "Use an alternate route," is heard when the local traffic report is given and the announcer mentions heavy traffic, slow conditions, or an accident.

Consumer airline prices may be affected by alternate routes. Fares may be cheaper using an airline that may not fly directly to a destination but stops at intermediate locations. Similarly, considering several possible routes when scheduling deliveries to various cities is important to shippers and carriers. Determining different routes and assessing each based on the condition of the roadway, possible human exposure, and cost are factors in the transport of hazardous materials. And let's not forget one of the best reasons for choosing alternate routes—exploring new areas, and sightseeing!

INVESTIGATION & EXPLORATION **Problem #8**

Students should randomly select four digits 1–9 and attempt to form the largest sum possible from 2 two-digit numbers using those digits. Then, they should select five digits to determine the largest sum of a two-digit and a three-digit number. From that, they should extrapolate what the two-digit and three-digit numbers should look like in general, given any set of five digits. If necessary or desired, more or less digits could be investigated. Then, with a calculator, they should attempt to find the largest possible differences, products and quotients of a two-digit and three-digit number. Some patterns are sure to result, the students will develop their number sense in the process, and the knowledge needed to tackle this problem will be gained.

WORKOUT 7

1. On the calendar page shown, there is a 3×3 block of numbers which has a sum of 198. What is the least number in this block?

JUNE						
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30			

1. ______________

2. For what value of x does $\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdots\frac{55}{56} = \frac{1}{2x}$?

2. ______________

3. A speed limit sign reads 4400 feet per minute. How fast is that in miles per hour?

3. ______________

4. $\triangle$ ABC is equilateral and can be divided into three congruent isosceles triangles. The area of one of the isosceles triangles is $30\,\text{cm}^2$. How many centimeters are in the perimeter of $\triangle$ ABC? Express your answer to the nearest centimeter.

4. ______________

5. A box of golf tees contains 3 red tees, 5 green tees, 6 blue tees and 4 white tees. If four tees are selected without replacement, what is the probability that all four are different colors? Express your answer as a common fraction.

5. ______________

6. An octahedron is formed by gluing the square face of one regular square pyramid to the square face of a congruent square pyramid. The length of each edge of the octahedron is 4 inches. How many square inches are in the surface area of the new solid? Express your answer as a decimal to the nearest tenth.

6. ______________

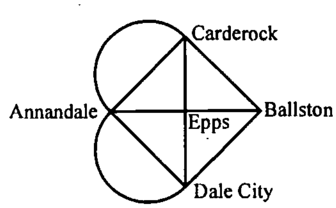
7. A six-digit perfect square has the same non-zero digit in the hundreds, tens, and units places. What is the tens digit?

7. ______________

8. Use each digit 0–9, the four basic operators $+$, $-$, $\times$ and $\div$, and one decimal point, each exactly once, to create a numerical expression. For example, $154 \times 876 + 90 \div .3 - 2 = 135\,202$ is one possible expression. Following the order of operations, what is the greatest possible value for such an expression? (*Problem submitted by mathlete David Presberry.*)

8. ______________

9. How many different ways can Marco travel from Annandale to Ballston? He may not visit any city more than once, and he must stay on the roads.



9. ______________

10. The distance between points $(-2, 4)$ and $(3, p)$ is $5\sqrt{5}$. What is the sum of all possible values of p?

10. ______________

ANSWER KEY – WORKOUT 8

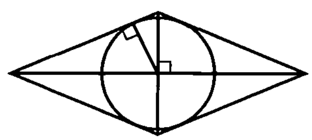
1.	1.73	(FM)	**2.**	19	(FM)	**3.**	66.9	(FM)
4.	25.5	(FM)	**5.**	5.4×10^8	(CM)	**6.**	63.7	(FM)
7.	$\frac{2}{15}$	(T)	**8.**	7	(MTP)	**9.**	$\frac{1}{5}$	(FM)
10.	112	(MTG)						

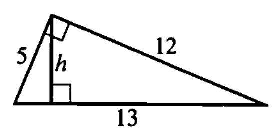
SOLUTION **Problem #3**

FIND OUT What do we want to know? The area of the circle inscribed in a rhombus, given the lengths of the diagonals.

CHOOSE A STRATEGY The diagonals of the rhombus divide the triangle into four right triangles. Further, the radius of the circle meets the rhombus at a right angle when drawn from the intersection of the diagonals to the point of tangency. This realization will allow us to form a proportion among corresponding pieces of two congruent right triangles.

SOLVE IT The diagonals of a rhombus meet at a right angle, and the radius of the circle is perpendicular to a side of the rhombus at the point of tangency, as shown in the figure to the left below.





The figure above to the right shows one-quarter of the rhombus, which contains two similar right triangles. One of them has a shorter leg of length 5 inches and hypotenuse of length 13 inches. The other has a shorter leg of unknown length h inches and hypotenuse of length 12 inches. By setting up a proportion among corresponding pieces of these congruent triangles, the length of the shorter leg h, which is also the radius of the circle, can be determined.

$$\frac{5}{13} = \frac{h}{12} \quad \longrightarrow \quad h = \frac{60}{13}$$

Because h is the radius of the circle, the area of the circle is $\pi\left(\frac{60}{13}\right)^2 = 66.9\,\text{in}^2$.

LOOK BACK The area of the rhombus is equal to one-half the product of the lengths of its diagonals, which is $\frac{1}{2} \times 10 \times 24 = 120\,\text{in}^2$. From the figure shown above, the area of the circle appears to be a little more than half the area of the rhombus, so our answer is reasonable.

MAKING CONNECTIONS... to Coin Production Problem #5

The US Mint, established 2 April 1792, is the organization responsible for producing the coins used daily, for minting commemorative coins, and for protecting the nation's gold and silver reserves. Two of the four mints, in Denver and Philadelphia, produce all of the daily-use coins—more than 18 billion of them a year, in fact. From January through June 1998, these two mints produced 2.4 billion pennies, 260 million nickels, 500 million dimes, 350 million quarters and 24 million half-dollars.

The mint is planning to release a new quarter dollar each year 1999–2008, each one commemorating five of the 50 states in the US. Additionally, when the supply of Susan B. Anthony dollar coins is depleted, a new dollar coin will be released. This new dollar will be gold, not silver; its diameter will be 26.5 mm, the same diameter as the Anthony dollar, which is much smaller than the $1\frac{1}{2}$-inch dollars that were minted prior to 1979; and its edge will have "tactile and visual features" that make it distinguishable from the quarter, which is of similar size.

INVESTIGATION & EXPLORATION Problem #9

If an analogous figure was constructed where the sides were trisected instead of bisected, what would the ratio of the area of one internal square to the area of the entire square be? And if the sides were divided into four parts? ...into five parts? Students should explore these relationships. In the process of searching, students will have an opportunity to learn that the sum of the first n odd positive integers is equal to n^2, to find that each figure can be divided into many congruent triangles, and to realize that the sequence of ratios follows a very predictable pattern.

WORKOUT 8

1. The volume of a cone is $27\,\text{cm}^3$ and its height is 5 cm. A smaller cone is cut from the first along a plane parallel to the base and at a perpendicular distance of 3 cm from the base. What is the number of cubic centimeters in the volume of the smaller cone? Express your answer as a decimal to the nearest hundredth.

1. ____________

2. A circular cylinder is formed by rolling an $8\frac{1}{2}'' \times 11''$ paper vertically and taping it with no overlap. A second cylinder is formed by rolling an $8\frac{1}{2}'' \times 11''$ paper horizontally and taping it with no overlap. What is the positive difference between the number of cubic inches in the volumes of the two cylinders? Express your answer to the nearest cubic inch.

2. ____________

3. The length of one diagonal of a rhombus is 10 inches and the other is 24 inches. A circle is inscribed in the rhombus. How many square inches are in the area of the circle? Express your answer as a decimal to the nearest tenth.

3. ____________

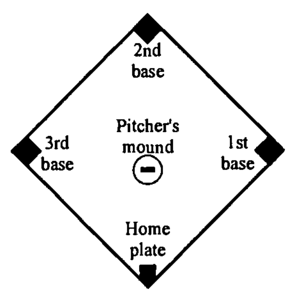
4. The area of each face of a regular tetrahedron is $15.6\,\text{in}^2$. What is the number of cubic inches in the volume of the tetrahedron? Express your answer as a decimal to the nearest tenth.

4. ____________

5. The United States Mint produced 13.5 billion pennies in 1995. If all of these pennies are redeemed for quarters, how many quarters would be needed? Express your answer in scientific notation.

5. ____________

6. A baseball diamond is a square with side length $90'$. The pitcher's mound is $60'6''$ from home plate and lies on the line between home plate and second base. How many feet must a player throw the ball from the pitcher's mound to first base? Express your answer as a decimal to the nearest tenth.



6. ____________

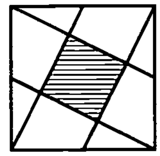
7. Given two different two-digit perfect squares, what is the probability that the sum of the digits of the two squares is the same? Express your answer as a common fraction.

7. ____________

8. Mini-puzzles come in 10 different varieties and are sold in packages of 15. There is at least one puzzle of each variety in every package. How many packages are needed to guarantee at least 10 puzzles of the same variety?

8. ____________

9. In the square, midpoints of sides are connected with vertices. What is the probability that a point inside the square will also be inside the shaded region? Express your answer as a common fraction.



9. ____________

10. What is the number of units in the positive difference between the maximum and minimum possible perimeters of a rectangle with whole number dimensions and area 72 square units?

10. ____________

ANSWER KEY – WORKOUT 9

1.	24	(TP)	**2.**	13	(TGP)	**3.**	7.3	(FM)
4.	16	(MG)	**5.**	$\frac{4}{9}$	(TP)	**6.**	144	(MP)
7.	19	(TG)	**8.**	1008	(FP)	**9.**	$\frac{3}{4}$	(M)
10.	180	(MP)						

SOLUTION **Problem #7**

FIND OUT What are we asked to find? The number from 0 through 20 which cannot be obtained by using the "4" key four times along with the other keys listed.

CHOOSE A STRATEGY Let's make a table giving one possible expression for each of the numbers.

SOLVE IT We must be careful to use parentheses when necessary and remember the order of operations. Since the problem implies that there is only one number for which an expression cannot be generated, the problem will be solved when only one number remains.

$0 \rightarrow 4+4-(4+4)$	$11 \rightarrow 44 \div 4 - \sqrt{4}$
$1 \rightarrow 4 \div 4 + 4 - 4$	$12 \rightarrow 4+4+\sqrt{4}+\sqrt{4}$
$2 \rightarrow 4 \div 4 + 4 \div 4$	$13 \rightarrow 44 \div 4 + \sqrt{4}$
$3 \rightarrow (4+4+4) \div 4$	$14 \rightarrow 4+4+4+\sqrt{4}$
$4 \rightarrow \sqrt{4+4+4+4}$	$15 \rightarrow 44 \div 4 + 4$
$5 \rightarrow \sqrt{4}+\sqrt{4}+(4 \div 4)$	$16 \rightarrow 4+4+4+4$
$6 \rightarrow (4+4) \div 4 + 4$	$17 \rightarrow 4 \times 4 + 4 \div 4$
$7 \rightarrow 44 \div 4 - 4$	$18 \rightarrow 4 \times 4 + 4 \div \sqrt{4}$
$8 \rightarrow 4+4+4-4$	$19 \rightarrow ??$
$9 \rightarrow 4+4+(4 \div 4)$	$20 \rightarrow 4 \times 4 + \sqrt{4} + \sqrt{4}$
$10 \rightarrow 4+4+4 \div \sqrt{4}$	

Since 19 is the only number for which an expression has not been generated, it must be the number that cannot be obtained.

LOOK BACK Expressions for all but one number have been found, and the question has been answered, so we can have confidence in our work. It is, however, interesting to note that 19 could be obtained if there is no restriction on the number of times a key may be used. Most calculators have a finite level of accuracy, so continually taking the square root of 4 will eventually give a value of 1 (due to calculator round-off error). Consequently,

$$19 = 4 \times 4 + 4 - \sqrt{\sqrt{\sqrt{\ldots \sqrt{4}}}}.$$

MAKING CONNECTIONS... to Photography **Problem #8**

The ASA rating of film indicates the film's emulsion speed, or its sensitivity to light. That number was previously determined by the American Standards Association, which is why the acronym ASA was used. Since 1982, however, the International Standards Organization has determined the rating, and the acronym ISO is now more common. Slow speed film has an ISO rating of 100 or less, and fast film has a rating of 400 or more. ISO 200 film is twice as sensitive to light as ISO 100, but only half as sensitive as ISO 400.

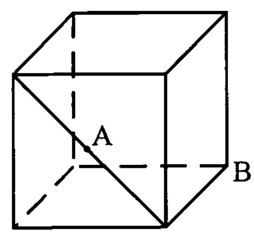
How does an automatic camera know what speed film is in the camera or how many exposures are on the roll? Much like the bar code on many products, the DX-code on the outside of a film container is a binary indicator. Contacts inside the film cartridge chamber relay data from this code to the camera's CPU. A 6-bit register indicates the film's speed, and a 3-bit register indicates the number of exposures on the roll.

WORKOUT 9

1. How many different five-digit positive integers are divisible by 4 and use each of the digits 1, 2, 3, 4 and 5 exactly once? **1.** __________

2. A five-digit perfect square in the form $33ab6$ has hundreds digit a and tens digit b. What is the sum $a + b$? **2.** __________

3. Each edge of a cube is 6 cm in length. If point A is the midpoint of a diagonal of a face, what is the distance from point A to point B? Express your answer as a decimal to the nearest tenth. **3.** __________



4. The geometric mean of two positive integers is defined as the positive square root of their product. The geometric mean of two positive integers is 6 and their arithmetic mean is 10. What is the positive difference between the two numbers? **4.** __________

5. What is the probability that the sum of two one-digit positive integers will be greater than 10? Express your answer as a common fraction. **5.** __________

6. Sixty-four unit cubes were glued together to make a $4 \times 4 \times 4$ cube. How many pairs of 1×1 unit faces were glued together to make the larger cube? **6.** __________

7. The only buttons on a calculator that work are the "4" key; the four basic operations $+$, $-$, $\times$, and $\div$; the parentheses (and); and the square root $\sqrt{\ }$. Most of the numbers 0–20 on can be obtained by using the "4" key exactly four times, together with the other keys that work. Which number from 0–20 cannot be obtained by using the "4" key exactly four times and using any of the other keys less than four times each? **7.** __________

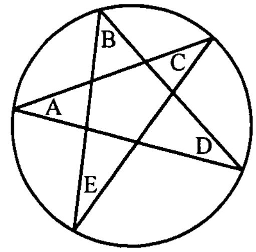
8. Every third ASA setting on a camera is labeled, as shown, and the settings increase geometrically. Rounded to the nearest integer, what is the value of the setting immediately after 800? **8.** __________

100 • • 200 • • 400 • • 800 • • 1600

(Problem submitted by alumnus Chris Smallwood.)

9. The midpoints of adjacent sides of a regular hexagon are connected to form an inscribed hexagon. What is the ratio of the area of the smaller hexagon to the larger hexagon? Express your answer as a common fraction. **9.** __________

10. A five-pointed star is inscribed in a circle as shown. What is the number of degrees in the sum of the measures of $\angle A$, $\angle B$, $\angle C$, $\angle D$ and $\angle E$ of the star? **10.** __________



Solutions to Problems from the Stretches, Warm-Ups and Workouts

The following pages contain selected solutions to the problems contained in the Stretches, Warm-Ups and Workouts. If an extended solution was written for a Warm-Up or Workout problem in the previous section, or if the Making Connections or Investigagion & Exploration section contains an explanation of the solution, a solution for that problem is not given here.

The solutions contained in this section should not be viewed as the only possible solutions to the problems contained in the *1998–99 MATHCOUNTS School Handbook*. At best, they represent one choice among many. Students should be encouraged to find alternate solutions in an attempt to extend their problem-solving skills, and they should be praised when they provide correct solutions which differ from those contained here.

Solutions to: **Begin on page:**

Statistics Stretch

Problem 1: The mean is found by summing the numbers in each set and dividing by 5. However, since four of the numbers are equal and the unequal numbers differ by 100, the sum of the numbers in A will be 100 more than the sum of the number in B. The mean, then, will differ by $100 \div 5 = 20$.

Problem 2: The numbers range from 1–7, so the set has a range of $7 - 1 = 6$.

To find the median of a set of numbers, first arrange the numbers in ascending order. Notice that these numbers are already arranged in ascending order. The median of a set with an odd number of terms is simply the middle term. For a set with an even number of terms, like this one, the median is the mean of the two middle numbers. The median for this set is the mean of the fifth and sixth terms. The fifth and sixth terms are both 5, so the median is also 5.

The difference between the range, 6, and the median, 5, is 1.

Problem 3: Because 3.3% were identified as unemployed, 96.7% must be employed. The number of people surveyed, p, can then be found with the following proportion:

$$\frac{p}{241\,750} = \frac{100}{96.7} \quad \longrightarrow \quad p = 250\,000$$

Problem 4: The chart shows that there were 8 games played. The Tigers scored more than 8 goals in 2 of the 8 games. The percentage is $\frac{2}{8} = 0.25 = 25\%$.

Problem 5: Change the fractions and percents to decimals. The fraction $\frac{1}{2}$ of 1% is represented by $0.5 \times 0.01 = 0.005$. There are 100 000 families, so there are $0.005 \times 100\,000 = 500$ families with incomes over \$80 000.

Problem 6: Let x, y and z be the three numbers. Their average is $\frac{x+y+z}{3} = 8$. This implies that $x + y + z = 24$. Now add m such that $\frac{(x+y+z)+m}{4} = 9$. Substituting gives $\frac{24+m}{4} = 9$. Hence, $m = 12$.

Problem 7: Find the average number of strokes per hole for each person; add all of Kelsey's strokes and divide by 9, and then do the same for Terry.

Kelsey's average: $45 \div 9 = 5$
Terry's average: $42 \div 9 \approx 4.67$

Kelsey's average score is $5 - 4.67 = 0.33$ better than Terry's.

Problem 8: The man started with \$40. He finds and loses money and ends up with \$45. He has increased his money by \$5. This is $\frac{45-5}{40} = \frac{1}{8} = 0.125$ of his original \$40, so he has increased his money by 12.5 percent.

Problem 9: Hank hit 160 out of 452, and Joe hit 148 out of 421. Joe had fewer hits, but he also had fewer at-bats. A calculator yields that $\frac{160}{452} \approx 0.354$ and $\frac{148}{421} \approx 0.352$. Hence, Hank's average is higher.

Problem 10: A mode is the most frequently appearing number(s) in a set of data, and there can be more than one mode. In this case, each of the numbers 2, 3, 4 and 5 appear twice. Thus, there are four modes for this set.

Algebra Stretch

Problem 1: Moving all terms containing a variable to the right side and all constants to the left side gives $-2p = -8$, so $p = 4$.

Problem 2: Although symbolic manipulation can solve this, algebraic reasoning probably attains a solution more quickly. Assume that 5 stamps are worth 32¢; then 4 stamps are worth 23¢, and the total value is $5 \times 32 + 4 \times 23 = 252$¢ = \$2.52. (This was a very lucky first guess. If it hadn't worked and our first guess was too low, we could have increased the number of 32¢ stamps. Conversely, if our first guess was too high, we could have increased the number of 23¢ stamps.

Problem 3: Substituting, $4^3 + 4 - (4)(6) + 3(6) - 7 = 64 + 4 - 24 + 18 - 7 = 55$.

Problem 4: Graphing this curve would yield a parabola which intersects the x-axis at -15 and 8, yielding the answer. Another method, factoring, instead shows that

$$\begin{aligned} q^2 + 7q &= 120 \\ q^2 + 7q - 120 &= 0 \\ (q + 15)(q - 8) &= 0 \\ q &= -15, 8. \end{aligned}$$

The sum of the values is $-15 + 8 = -7$.

Problem 5: The means-extremes theorem states that, in a proportion, the product of the means is equal to the product of the extremes. Colloquially, this is called cross-multiplication, and in this problem yields $5r^2 = 45$, or $r^2 = 9$. Hence, either $r = 3$ or $r = -3$, and the product of those values is -9.

Problem 6: There are several solutions to this problem, but the easiest results from adding the equations to give $7a + 7b = 112$. Rewritten, this is $7(a + b) = 112$, so $a + b = 112 \div 7 = 16$.

Problem 7: Using the first letter of each name as the variables for this problem, we have the following equations:

$$\begin{aligned} &(1) \quad p = o + b \\ &(2) \quad o = b \\ &(3) \quad s = p + o \\ &(4) \quad s = 120 \end{aligned}$$

From equations (1) and (2), we have $p = b + b = 2b$. Substituting into equation (3) yields that $s = p + b = p + \frac{1}{2}p = \frac{3p}{2}$. Equation (4) indicates that $s = 120$, so $120 = \frac{3p}{2}$, or $p = \frac{240}{3}$. Hence, Pinky's weight is 80 lb.

Problem 8: Substitute 15 for w gives $t = 16(15) + 15 = 255$.

Problem 9: Expanding the expressions before substituting for n gives

$$(m + n)^2 + (m - n)^2 + m^2 + n^2 = (m^2 + 2mn + n^2) + (m^2 - 2mn + n^2) + m^2 + n^2$$
$$318 = 3m^2 + 3n^2$$

Dividing both sides by 3 yields $m^2 + n^2 = 106$, and substituting gives $m^2 + 5^2 = 106$. Hence, $m^2 = 106 - 25 = 81$. The positive value of m is 9.

Problem 10: The percent of cranberry juice in t quarts of Twistee is $\frac{40t}{100t}$. Further, the percent of cranberry juice in j quarts of orange juice is $\frac{0}{100j}$ (that is, there is no cranberry juice in orange juice). Combining these gives $\frac{40t}{100(t+j)}$. To get the percent, multiply by 100, which gives $\frac{40t}{t+j}$.

Probability Stretch

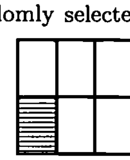
Problem 1: Because there are 3 red and 5 blue marbles, there are 8 ways to choose 1 marble and 5 ways to choose a blue marble. There is, therefore, a $\frac{5}{8}$ probability that a randomly chosen marble will be blue.

Problem 2: If the combined value of two coins is 15¢, then one of the coins is a nickel and the other is a dime. There are 8 coins to choose from, with 5 nickels and 2 dimes. There is a $\frac{5}{8}$ chance that a nickel is chosen first. After one coin is chosen, there are 7 coins remaining; thus, there is a $\frac{2}{7}$ probability that a dime is chosen second. The probability of choosing nickel and then dime is $\frac{5}{8} \times \frac{2}{7} = \frac{5}{28}$. However, remember that the dime could be chosen first and the nickel second. The probability of this is also $\frac{5}{28}$. The combined probability is therefore $\frac{5}{28} + \frac{5}{28} = \frac{5}{14}$.

Problem 3: Using logical reasoning, the probability of getting heads when a coin is flipped is $\frac{1}{2}$. Thus, the probability of getting two heads when flipping two coins is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. Another solution is to elaborate the sample space, which is HH, HT, TH and TT. There are four possible outcomes, and one of these outcomes is HH, two heads. In either case, the probability is $\frac{1}{4}$.

Problem 4: We are concerned with pairs that differ by 3. There are four such pairs: $(1,4), (4,1), (2,5), (5,2)$. Since there are $5 \times 4 = 20$ possible outcomes, and four that satisfy the requirements, there is a $\frac{4}{20} = \frac{1}{5}$ probability of success.

Problem 5: Because the shaded rectangle is $\frac{1}{2}$ the height and $\frac{1}{3}$ the width of the original square, the square can be partitioned into six identical rectangles, with one of the rectangles shaded. The shaded part is thus $\frac{1}{6}$ the area of the original square, and the probability that a randomly selected point lies with the shaded rectangle is also $\frac{1}{6}$.



Problem 6: The least positive integer that is both a multiple of 7 and 11 is 77, which is larger than 70. Therefore, the two events cannot happen together, so the probability is 0.

Problem 7: The probability that Alex makes both shots is $\frac{3}{10} \times \frac{3}{10} = \frac{9}{100}$.

Problem 8: There are $3^6 = 729$ possible arrangements of pogs into the boxes. Of those arrangements, there are $\binom{6}{2}$ ways to choose 2 pogs for the first box and $\binom{4}{2}$ ways to choose 2 of the remaining pogs for the second box. That leaves 2 pogs for the third box. The total number of successful ways, then is

$$\binom{6}{2} \times \binom{4}{2} \times \binom{2}{2} = 15 \times 6 \times 1 = 90.$$

The probability is $\frac{90}{729} = \frac{10}{81}$.

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Problem 9: There is only one way for the final result to be two marbles in each of the three boxes: if both the third and sixth marbles drawn are green. Because there are eight marbles initially, there are $\binom{8}{2} = 28$ ways to choose the positions of the green marbles. Hence, the probability is $\frac{1}{28}$.

Problem 10: There are $4 \times 6 = 24$ pairs in the sample space. There are 3 ways to choose a consonant from MATH and 4 ways to choose a consonant from COUNTS. There are, therefore, $3 \times 4 = 12$ pairs in the success space. The probability of success is $\frac{12}{24} = \frac{1}{2}$.

WARM-UP 1

Problem 2: The numerator of the fraction becomes $0.\overline{13} + 0.\overline{5} = 0.\overline{68}$, and the denominator becomes $0.\overline{16} - 0.\overline{2} = -0.\overline{06}$. The fraction is then $\frac{0.\overline{68}}{-0.\overline{06}}$, which can be rewritten as simply $-\frac{68}{6}$. This is equivalent to $-11\frac{1}{3}$.

Problem 3: There will be $\frac{1}{4}(60) = 15$ flights to Atlanta, $\frac{1}{3}(60) = 20$ flights to Chicago, $\frac{1}{5}(60) = 12$ flights to Miami, and $\frac{1}{6}(60) = 10$ flights to New York. These add up to $15 + 20 + 12 + 10 = 57$, so there will be 3 flights to Pittsburgh.

Problem 4: There are twenty 5-centimeter segments in 1 m, so there is a maximum of $20^3 = 8000$ cubes that can be cut from the 1-meter cube.

Problem 5: To solve this problem, find the two greatest prime numbers that are less than 100. These are 97 and 89, and $97 \times 89 = 8633$.

Problem 6: The period of the sequence AABBBCCCC is 9, and $369 \div 9 = 41$ with no remainder. Hence, the last letter of the sequence will be in the 369th position. That letter is C.

Problem 7: First, the distance between $\frac{1}{3}$ and $\frac{3}{4}$ is $\frac{3}{4} - \frac{1}{3} = \frac{9}{12} - \frac{4}{12} = \frac{5}{12}$.

Since there are five steps from A to F, each step must correspond to $\frac{1}{12}$. Thus, point E corresponds to

$$\frac{1}{3} + \frac{4}{12} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Problem 8: Every 12 hours, the clock will show the correct time, which happens $60 \div 12 = 5$ times.

Problem 9: The area of a triangle is one-half base times height, where the base and height are perpendicular. The largest triangle is constructed by choosing the three points at corners of the 5×5 grid. Since there are 4 one-unit segments along each edge, the total area enclosed is $\frac{1}{2}(4)(4) = 8$ square units.

Problem 10: We start with the \$40 wholesale price. Then, 140% of the purchase price simply multiplies \$40 by 1.4, the 20% discount multiplies this number by 0.8, and the final 50% reduction multiplies the previous result by 0.5. So, the final prices is $40 \times 1.4 \times 0.8 \times 0.5 = \$22.40$.

WARM-UP 2

Problem 1: The common difference is 4.5 and the initial term is $\frac{3}{4} = 0.75$, so the 100th term (a_{100}) is equal to $0.75 + 4.5(100 - 1) = 446.25$.

Problem 2: Work backwards. Start with the 10 cards Chris received from Greg. Of these cards, $10 - 2 = 8$ cards represent half of Greg's cards. That means that Greg had 16 cards before he gave some to Chris. Of these 16 cards, $16 - 2 = 14$ were half of Doug's cards. Hence, Doug had 28 cards before he gave some to Greg. Of these 28 cards, $28 - 2 = 26$ were half of Karl's cards. Karl had 52 cards. Of these, $52 - 2 = 50$ cards were half of Jeb's cards. Consequently, Jeb started with 100 cards.

Problem 3: All percent problems can be done by setting up a proportion with a data ratio and a percent ratio:

$$\frac{\text{data part}}{\text{data whole}} = \frac{\text{percent part}}{100}$$

where the percent whole, by definition, is always 100. Be careful that the parts and wholes are compatible. In this case, we do not know the data whole, but the percent whole is 100. We do know the data part is \$12. The percent we are given is 70% off the whole. This represents the percent not paid, but the \$12 represents the part paid. These are not comparable. However, if we know that 70% was not paid, then the percent paid is $100 - 70 = 30\%$. We can now set up the proportion $\frac{12}{x} = \frac{30}{100}$, so $30x = 1200$ and $x = 40$. The newsstand price is \$40.

Problem 4: The expression can be simplified with like bases to $(2^{-2})^{2x+8} = (2^4)^{2x+5}$, which can be further simplified to $2^{-4x-16} = 2^{8x+20}$. This equation can only be true if the exponents are equal; that is, $-4x - 16 = 8x + 20$, or $x = -3$.

Problem 5: We are looking for mountain numbers greater than 70 000. This means the first digit must be 7, 8 or 9. However, the first, second, and third digits must occur in ascending order, so 8 and 9 are not possible for the first slot. In fact, the first three digits must be 7, 8 and 9 respectively. Thus, we have choices for only the fourth and fifth digits of the number. We know the number is of the form 789ab, where a is greater than b and $a < 9$. Start with the greatest possible value for a, which is 8, and work down to the least possible value, which is 1.

For $a = 8$, the value of b can be any digit from 7 to 0, which is 8 choices.
For $a = 7$, the value of b can be any digit from 6 to 0, which is 7 choices.
For $a = 6$, the value of b can be any digit from 5 to 0, which is 6 choices.

⋮

For $a = 2$, the value of b can be any digit from 1 to 0, which is 2 choices.
For $a = 1$, the value of b can only be 0, which is 1 choice.

There are $8 + 7 + 6 + \ldots + 1 + 0 = 36$ possible numbers that fit the criteria.

Problem 6: To find the sum of the first 102 counting numbers, we will use a trick that is sometimes credited to the famous mathematician Gauss. Write the sum of the numbers twice, once forwards and once backwards, leaving out some of the middle terms, and then add vertically:

$$\begin{array}{r} 1 + 2 + 3 + \ldots + 100 + 101 + 102 \\ 102 + 101 + 100 + \ldots + 3 + 2 + 1 \\ \hline 103 + 103 + 103 + \ldots + 103 + 103 + 103 \end{array}$$

There are 102 sums of 103 in the addition. Thus the sum of all these numbers is $102 \times 103 = 10\,506$. This is double the sum we want, and half of 10 506 is 5253.

We want the remainder when 5253 is divided by 5250. The remainder is 3.

Problem 7: Given that $m\angle \text{ADE} = 30°$ and the triangle is a right triangle, $m\angle \text{DAE} = 60°$. This angle is also part of the large right triangle, $\triangle \text{ABC}$. That means that $m\angle \text{ABC} = 30°$, which implies that $m\angle \text{BGF} = 60°$. $\angle \text{CGF}$ is the supplement of $\angle \text{BGF}$, so its measure is $180 - 60 = 120°$.

Problem 8: Using the scheme given, the sum for the word MATH is $13 + 1 + (-20) + (-8) = -14$. The sum for the word COUNTS is $3 + 15 + 21 + (-14) + 19 + (-20) = 24$. The positive difference is $24 - (-14) = 38$.

Problem 9: The only three-digit numbers with the same units, tens and hundreds digits are multiples of 111. We know that the sum of the prime factors of the number is 47. The prime factors of 111 are 3 and 37, whose sum is 40. This leaves $47 - 40 = 7$ as the other prime factor. Hence, the number is $7 \times 111 = 777$.

WARM-UP 3

Problem 2: First convert each of the measurements to centimeters: 12.6, 11.8, 12.5, 13, 10. The median is the one in the middle when the numbers are put in order, and the order for these five measurements is 10, 11.8, 12.5, 12.6, 13. The number 12.5 is the third item in this five-item list, so it is the median.

Problem 3: Simplify the numerator by adding $\frac{2}{6} + 1 = \frac{8}{6}$. Simplify the denominator by adding $\frac{1}{2} + 4 = \frac{9}{2}$. Then $\frac{8}{6} \div \frac{9}{2} = \frac{8}{6} \times \frac{2}{9} = \frac{8}{27}$.

Problem 4: The ratio of the larger of two angles to the smaller is $\frac{5}{4}$. This means that the larger angle is 5 of the total $5 + 4 = 9$ parts. This is $\frac{5}{9}$ of the 180° in the supplementary angles, so the larger angle measures $\frac{5}{9} \times 180 = 100°$.

Problem 5: We can make a chart showing the remainders when powers of 13 are divided by 5:

Power of 13	13^1	13^2	13^3	13^4	13^5	13^6	13^7	...
Modulo 5	3	4	2	1	3	4	2	...

The chart shows that the remainders form a cycle of four. We need to find out where in the cycle the 51st power lies. We can write 13^{51} as $(13^4)^{12} \times 13^3$. Since 13^4 leaves remainder 1 when divided by 5, this indicates that the remainder of 13^{51} will be the same as the remainder of 13^3, which we found to be 2.

Problem 6: Because there is a middle floor, there must be an odd number of floors in the building. Kelsey starts at the middle floor and goes up 4, down 6, and up 7. If the middle floor is m, then $m + 4 - 6 + 7 = m + 5$. Since Kelsey is now on the top floor, there are 5 floors above the middle floor and there must also be 5 floors below the middle floor. Including the middle floor itself, there are 11 floors in the building.

Problem 7: There are nine digits that could be placed in the first spot, and there are ten digits that could be placed in each of the other six spots. So, the number of possible telephone numbers is

$$9 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 9 \times 10^6 = 9\,000\,000.$$

Problem 8: There are 22 two-digit prime numbers: 11, 13, 17, 19, 31, 37, 41, 43, 47, 53, 61, 67, 71, 73, 83, 89, 93, 97. Of these, there are only three—17, 53 and 71—whose digits have a sum of 8.

Problem 9: The two large isosceles right triangles cover half the area of the square, so one of these triangles is $\frac{1}{4}$ the area of the square. We know that each of these triangles is equal in area to four of the small triangles. Thus, each small triangle is equal to $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$ of the square. The parallelogram is formed from two of these small triangles, so it is equal in area to $2 \times \frac{1}{16} = \frac{1}{8}$ of the square. Together, the large isosceles right triangle and the parallelogram have area equal to $\frac{1}{4} + \frac{1}{8} = \frac{3}{8}$ of the square. Thus, the probability that a random point will be in the shaded region is $\frac{3}{8}$.

Problem 10: If three quarters are tossed, the eight possible results are HHH, HHT, HTH, THH, HTT, THT, TTH or TTT. There is at least one tail, so the first case may be eliminated. Of the other seven possibilities, only one incident does not include any heads. So, the probability of at least one head is $\frac{6}{7}$.

WARM-UP 4

Problem 1: Compute: $\left(\frac{4}{5}\right)^{-2} = \left(\frac{5}{4}\right)^{2} = \frac{25}{16}$

Problem 2: Use the Pythagorean theorem. The horizontal distance along the x-axis is $|x| = |-8-0| = 8$, and the vertical distance along the y-axis is $|y| = |6-0| = 6$. The distance from $(-8, 6)$ to the origin is just measured along the hypotenuse of this triangle, so it is $\sqrt{6^2+8^2} = \sqrt{36+64} = \sqrt{100} = 10$.

Problem 3: The total area of a $125\,\text{ft} \times 140\,\text{ft}$ lawn is $125 \times 140 = 17\,500\,\text{ft}^2$. Since the charge is 59¢ per $100\,\text{ft}^2$, we just need to calculate $59 \times 175 = 10\,325$. Hence, the Speedy Green Lawn Service will charge \$103.25.

Problem 4: The first student shakes hands with 13 other students. The second student (who has already had her hand shaken by the first student) shakes hands with 12 remaining students. The third student shakes hands with 11 students, the fourth student shakes hands with 10 students, and so on, until the thirteenth student shakes hands with just 1 student. So, the total number of handshakes is $13 + 12 + 11 + \ldots + 3 + 2 + 1 = \frac{13 \times 14}{2} = 91$.

Problem 5: The two small sectors have a total angle measurement of $63 + 77 = 140°$, so the major arc BED has angle measure $360° - 140° = 220°$. The area of this large sector is proportional to the area of the circle, so its area can be determined by

$$\begin{aligned}\frac{220}{360}\pi r^2 &\approx \frac{11}{18}(3.14)(10)^2\\ &\approx 191.89\end{aligned}$$

To the nearest square inch, the area of the major sector is $192\,\text{in}^2$.

Problem 6: Since absolute value measures distance from the origin, we can consider $|2x+7| \le 16$ as the points of the form $2x+7$ that lie within 16 units of the origin. This implies that $-16 \le 2x+7 \le 16$, so

$$\begin{aligned}-16 \le 2x+7 &\le 16\\ -23 \le 2x &\le 9\\ -11.5 \le x &\le 4.5\end{aligned}$$

The least integer value is -11.

Problem 7: Since the value of the car decreases by 25% the first year, the value after one year is 75% of its original value. For each subsequent year, we multiply by 0.75, so to find the value after five years we need to multiply by 0.75 five times.

$$(17\,500)(0.75)^5 \approx \$4153$$

Problem 8: There is one largest square (8×8), and twenty smallest squares (1×1). In between, there are seventeen 2×2 squares (don't miss the one in the middle), nine 4×4 squares and four 6×6 squares. This gives a total of $20 + 17 + 9 + 4 + 1 = 51$ squares.

Problem 9: The four-digit integer is of the form $12d4$; written algebraically, it can be expressed as $1200 + 10d + 4$. Obviously, 1200 is divisible by 12, so the question becomes, "For what value of d is $10d + 4$ divisible by 12?" Because $0 \le d \le 9$, we are only concerned with two-digit integers with 4 as the units digit. Only two values are divisible by 12—24 and 84—so the probability of $12d4$ being divisible by 12 is $\frac{2}{10} = \frac{1}{5}$.

WARM-UP 5

Problem 1: A calculator yields that $\sqrt{200} \approx 14.14$. Hence, we can say that $a < 14.14 < b$. Thus, we know that a is less than 14.14, and b is larger than 14.14. Since a and b are consecutive positive integers, $a = 14$ and $b = 15$. The product ab is $14 \times 15 = 210$.

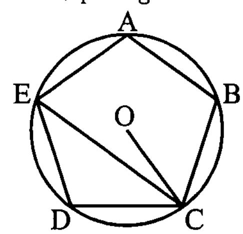
Problem 2: Dividing 5 by 13 gives a repeating decimal. Specifically, $\frac{5}{13} = 0.\overline{384615}$. Notice that there are six digits in the cycle. We need to know which digit will be in the 534th place. Because 6 divides 534 evenly, then the 534th digit is the last digit in the repeating cycle, which is 5.

Problem 3: The radius of the circle is 10 centimeters, so the area of the circle is $100\pi\,\text{cm}^2$. The diameter of the circle is 20 cm. The diameter of the circle is also the diagonal of the larger square. Hence, the side of the larger square may be found with the Pythagorean Theorem: $x^2+x^2=20^2$, or $x=10\sqrt{2}$. Then, the area of the larger square is $(10\sqrt{2})^2=200\,\text{cm}^2$. We could find the side of the smaller square, but if you imagine folding in the corners of the larger square, you can visualize the smaller square as half the area of the larger one. Thus, the area of the smaller square is $100\,\text{cm}^2$. The ratio of the area of the small square to the area of the circle is $\frac{100}{100\pi}=\frac{1}{\pi}$.

Problem 4: Converting all measurements to inches, the $12'8'' \times 9'4''$ floor has dimensions $152'' \times 112''$. The question then becomes, "How many sets of 8″ are in 152″, and how many sets of 8″ are in 112″?" The respective answers are 19 and 14. Hence, it will take $19 \times 14 = 266$ tiles to cover the floor.

Problem 5: The Wildcats played a total of 25 games. Since the average score per game was 60 points, the total number of points scored was $60 \times 25 = 1500$ points. Since there were 13 home games and the average score per home game was 72 points, the total number of points scored for the home games was $13 \times 72 = 936$ points. This means that $1500 - 936 = 564$ points must have been scored in the away games. Since there were 12 away games, $564 \div 12 = 47$ points is the average.

Problem 7: To find $m\angle\text{OCE}$, it helps to draw the pentagon inscribed in the circle.

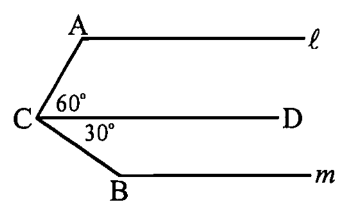


Since we know that ABCDE is a regular pentagon, $m\angle\text{EDC} = 108°$. We know that $\triangle\text{EDC}$ is isosceles, so $m\angle\text{DEC} = m\angle\text{DCE}$, and each is $\frac{180-108}{2} = 36°$. $m\angle\text{OCD} = \frac{1}{2}m\angle\text{BCD} = 54°$. Thus, $m\angle\text{OCE} = 54 - 36 = 18°$.

Problem 8: Solving the proportion $\frac{2.5}{20} = \frac{p}{100}$ gives $p = 12.5$, so the answer is 12.5%.

Problem 9: Combined, they painted $\frac{1}{3}+\frac{2}{5}=\frac{11}{15}$ of the room. Therefore, $\frac{4}{15}$ of the room was left to be painted. It took Peter 1 hr, 40 min = 100 minutes to paint the remainder. Thus, $\frac{4}{15}=\frac{100}{x}$ yields that it would have taken Peter 375 minutes, or $6\frac{1}{4}$ hours, to paint the room by himself.

Problem 10: Construct $\overline{CD}$ parallel to lines ℓ and m. Then, $m\angle\text{ACD} = 180 - 120 = 60°$, and $m\angle\text{DCB} = 180 - 150 = 30°$. Hence, $m\angle\text{ACB} = 60 + 30 = 90°$.



WARM-UP 6

Problem 1: The cost of copying a 757-page book at 5¢ per page is $757 \times 0.05 = \$37.85$. The 9% tax increases the price to $37.85 \times 1.09 = \$41.26$. The cost to buy the book is $14.99 \times 1.09 = \$16.34$. The difference is $41.26 - 16.34 = \$24.92$.

Problem 2: In the formula for the volume of a cylinder, $V = \pi r^2 h$, we know that $V = 1256$ and $r = 5$. We just solve for h:

$$\begin{aligned} V &= \pi r^2 h \\ 1256 &= \pi(5)^2 h \\ 1256 &= (3.14)(25)h \\ 1256 &= 78.5h \\ 16 &= h \end{aligned}$$

Problem 3: The two semicircles combine to make a circle with diameter 8 cm. The area of the circle is $\pi r^2 = \pi(4)^2 = 16\pi\,\text{cm}^2$. The area of the square is $8^2 = 64\,\text{cm}^2$. The area of the shaded region is the area of the entire square minus the area of the semicircles, which is $64 - 16\pi\,\text{cm}^2$. The positive difference between the unshaded and shaded regions is $16\pi - (64 - 16\pi) = 32\pi - 64\,\text{cm}^2$.

Problem 4: Algebraically,

$$\begin{aligned} 3(x-5) &\le 7 \\ 3x - 15 &\le 7 \\ 3x &\le 22 \\ x &\le 7\frac{1}{3} \end{aligned}$$

Since $7\frac{1}{3}$ is not an integer, the positive integer solutions are $1, 2, 3, 4, 5, 6, 7$. Hence, there are 7 solutions.

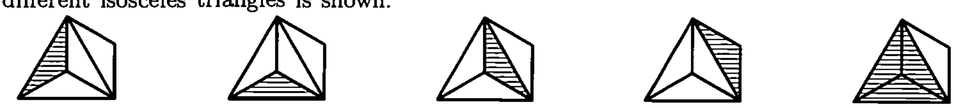
Problem 5: We know that 2, 3, 5 and 7 are the only possible prime numbers that we can use, and we know that none occurs more than twice. Thus, the greatest possible number we can make using each one twice is $2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 \times 7$. But this is a perfect square because each factor occurs twice. Thus, eliminate one of the least primes (2) to get $2 \times 3 \times 3 \times 5 \times 5 \times 7 \times 7 = 22\,050$.

Problem 6: The volume of a cylinder is $\pi r^2 h$. The radius is $\frac{3}{8}''$, the height is $\frac{3}{16}''$, so the volume is $\pi\left(\frac{3}{8}\right)^2\left(\frac{3}{16}\right) = \frac{27\pi}{1024}\,\text{in}^2$.

Problem 7: Convert each addend to common fractions: $0.25 = \frac{1}{4}$, and $0.3\overline{1} = \frac{1}{10}(3.\overline{1} = \frac{1}{10}\left(3 + \frac{1}{9}\right)$. Adding these fractions together gives

$$\begin{aligned} \frac{1}{4} + \frac{3}{10} + \frac{1}{90} &= \frac{5}{20} + \frac{6}{20} + \frac{1}{90} \\ &= \frac{11}{20} + \frac{1}{90} \\ &= \frac{99}{180} + \frac{2}{180} \\ &= \frac{101}{180} \end{aligned}$$

Problem 8: Each sixth of the hexagon contains five isosceles triangles (the equilateral triangle made up of three smaller triangles is also isosceles), so there are 30 isosceles triangles in the hexagon. One-sixth of the hexagon with the five different isosceles triangles is shown.



Problem 10: The prime numbers between 10 and 100 are 11, **13**, **17**, 19, **23**, 29, 31, **37**, 41, **43**, **47**, **53**, 59, 61, **67**, 71, **73**, 79, **83**, 89, 91, **97**. The 11 primes in bold are those whose ones digit is also a prime number.

WARM-UP 7

Problem 1: We simply solve the ratio $\frac{6}{16} = \frac{x}{328}$ to get $x = 123$.

Problem 2: Because we are dealing with three-digit prime numbers, the ones digit must be a 1, 3, 7 or 9, since numbers ending in 2, 4, 6 or 8 are divisible by 2 and those ending in 5 are divisible by 5. A list of the possible numbers is $101, 111, 113, 117, 119, 121, 131, 133, 141, 151, 161, 171, 177, 181, 191, 199$. We can eliminate 111, 117, 141 and 171 because the sum of their digits is a multiple of 3 which means that they are multiples of 3 (and not prime).

Next we look for factors of the remaining numbers. Because the numbers are all less than 199, we just need to check divisibility by primes less than or equal to $\sqrt{199} \approx 14.1$. Thus, we only need to check divisibility by 2, 3, 5, 7, 11 and 13. We have already eliminated numbers with factors of 2, 3 and 5. The only number divisible by 11 is 121, and the numbers divisible by 7 are 161, 133 and 119. Of the remaining numbers, none are divisible by 13. Thus, the only three-digit prime numbers less than 200 with 2 digits the same are 101, 131, 151, 181, 191, 113 and 199. There are 7 prime numbers that satisfy these conditions.

Problem 3: Without a calculator, we can perform the following simplification:

$$\begin{aligned} \sqrt[5]{32^2 + 16^3 + 8^4 + 4^5} &= \sqrt[5]{\left(2^5\right)^2 + \left(2^4\right)^3 + \left(2^3\right)^4 + 4\left(2^2\right)^5} \\ &= \sqrt[5]{2^{10} + 2^{12} + 2^{12} + 2^{10}} \\ &= \sqrt[5]{2^{10}\left(1 + 4 + 4 + 1\right)} \\ &= \sqrt[5]{2^{10}} \times \sqrt[5]{10} \\ &= 4\sqrt[5]{10}. \end{aligned}$$

Problem 4: Areas of congruent parallelograms are equal, so $A_{ABCD} = A_{EFGH} = 112$. The area of a parallelogram is the base times the height. $\overline{EN}$ is the perpendicular from C to side $\overline{AD}$ in parallelogram ABCD. Since, the area of ABCD is 112, $EN = \frac{112}{10} = 11.2$.

Problem 5: The two points have the same x-coordinate because the line of symmetry is a vertical line ($x = 1$). The area of the rectangle formed by the two points and the vertical line is

$$A = bh = \big(1 - (-3)\big)\big(1 - (-3)\big) = 16,$$

and the total area of the rectangle is twice this, or 32.

Problem 6: Since the body length is twice that of the head, we know that the body of the dog is $2 \times 12 = 24$ inches long. The tail is half the length of the body, so it must be $\frac{1}{2}(24) = 12$ inches. Hence, the total length of the dog is $12 + 24 + 12 = 48$ inches.

Problem 7: The inner square has area equal to one-half of the larger square, so its edge has length $6\sqrt{2}$. Therefore, the radius of the inner circle is $\frac{1}{2}(6\sqrt{2}) = 3\sqrt{2}$, and its area is $\pi(3\sqrt{2})^2 = 18\pi$.

Problem 8: The four consecutive heights are x, $x+1$, $x+2$ and $x+3$. Hence,

$$\begin{aligned} x + (x+1) + (x+2) + (x+3) &= 254 \\ 4x + 6 &= 254 \\ 4x &= 248 \\ x &= 62 \end{aligned}$$

So, Penny's height is $x + 2 = 64$ inches.

Problem 9: Simplify: $\frac{440 \times 0.45}{0.4} \times \frac{100}{100} = \frac{440 \times 45}{40} = 11 \times 45 = 495$

WARM-UP 8

Problem 1: First rewrite the problem without the negative exponents.

$$\frac{2 \cdot \frac{1}{3^2} + 3 \cdot \frac{1}{2^3}}{2 \cdot \frac{1}{3^2} - 3 \cdot \frac{1}{2^3}}$$

Next, multiply the numerator and denominator by the common denominator $3^2 \cdot 2^3$, which gives

$$\frac{2 \cdot 2^3 + 3 \cdot 3^2}{2 \cdot 2^3 - 3 \cdot 3^2} = \frac{2^4 + 3^3}{2^4 - 3^3} = \frac{16 + 27}{16 - 27} = -\frac{43}{11}.$$

Problem 2: Writing two equations in two unknowns and substituting yields the quadratic equation $x^2 - 30x + 216$. In order to factor this quadratic, we need to find two integers whose sum is 30 and whose product is 216—which is the original problem! Instead, consider pairs of numbers whose sum is 30. The pairs are $(1, 29), (2, 28), (3, 27) \ldots, (12, 18), \ldots, (15, 15)$. Checking the products of these pairs reveals that $12 \times 18 = 216$, so the positive difference is $18 - 12 = 6$.

Problem 3: From the diagram, we can see that $\overline{BC}$ is the third side of two triangles. From $\triangle ABC$, we see that BC must be greater than $12 - 5 = 7$ cm and less than $12 + 5 = 17$ cm. Likewise, from $\triangle DBC$, we see that BC must be greater than $20 - 8 = 12$ and less than $20 + 8 = 28$. Combining these facts, BC must be greater than 12 and less than 17. The least integral value is 13.

Problem 4: The original area is base times height, or bh. Both dimensions were multiplied by 4, so the area is $(4b)(4h) = 16bh = 880$. This means $bh = \frac{880}{16} = 55$, so the area of the original parallelogram was 55 square units.

Problem 5: In three years, the population will be $11\,000\,000 \times (1.13)^3 = 15\,871\,867$.

Problem 6: To express a number in scientific notation, we must write it as a number greater than or equal to 1 but less than 10, times a power of 10. Combining powers of 2 and 5 will give powers of 10. Thus, $2^{35} \cdot 5^{38} = 5^3 \times (2 \cdot 5)^{35} = 5^3 \times 10^{35} = 125 \times 10^{35} = (1.25 \cdot 10^2) \times 10^{35} = 1.25 \times 10^{37}$.

Problem 7: Algebraically, we are given that $\frac{2}{3}h = \frac{4}{5}e$. Because $\frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$, we can solve this immediately:

$$\begin{aligned} \frac{1}{4} \times \frac{2}{3}h &= \frac{1}{4} \times \frac{4}{5}e \\ \frac{1}{6}h &= \frac{1}{5}e \end{aligned}$$

Hence, $\frac{1}{6}$ of a ton will cost $\frac{1}{5}$ eagles.

Problem 9: Each of the points is reflected over the line $x = 2$, which is a vertical line 2 units to the right of the origin. Point A has x-coordinate -5. When -5 is reflected over the line $x = 2$, it travels $(2 - (-5)) = 7$ units to get to $x = 2$, and then travels another 7 units to the right for the reflection. Its new x-coordinate is $2 + 7 = 9$. Similarly, point B travels 4 units to get to $x = 2$ and then travels another 4 units to the right for the reflection. Its new x-coordinate is $2 + 4 = 6$. And point C travels 6 units to get to $x = 2$ and then another 6 units to the right. Its new x-coordinate is $2 + 6 = 8$. The sum of the new x-coordinates is $9 + 6 + 8 = 23$.

Problem 10: The first expression is equal to 0 because everything within the parentheses is multiplied by 0. The second expression, according to the order of operations, is equal to

$$\begin{aligned} 42 + 7 - 6 \times 6 + 3 \times (-1) \times 0 &= 42 + 7 - 36 + (-3) \times 0 \\ &= 49 - 36 + 0 \\ &= 13. \end{aligned}$$

The positive difference between the two expressions is 13.

WARM-UP 9

Problem 1: The probability of getting a white ball first is $\frac{30}{50}$ (30 white balls out of 50 total). There are now 20 black balls and 29 white balls remaining, so the probability of pulling a black second is $\frac{20}{49}$. For pulling a black ball again, we have 19 black balls and 48 balls total, so the probability is $\frac{19}{48}$. The last one we want to pull is a white ball, and the probability of that is $\frac{29}{47}$. Multiplying these probabilities together gives the probability of drawing four balls in the order stated, $\frac{30 \cdot 20 \cdot 19 \cdot 29}{50 \cdot 49 \cdot 48 \cdot 47} = \frac{551}{9212}$.

Problem 2: The volume of a sphere is $\frac{4}{3}\pi r^3$. The volume of this sphere is $\frac{\pi}{6}$. Thus, we can create the following proportion: $\frac{\pi}{6} = \frac{4}{3}\pi r^3$. Multiplying both sides by 6 and dividing both sides by π gives $1 = 8r^3$, which simplifies to $r^3 = \frac{1}{8}$, or $r = \frac{1}{2}$.

The radius of the sphere is $\frac{1}{2}$ in, and the length of the side of the inscribed cube is the length of diameter, or twice the radius, which is $2(\frac{1}{2}) = 1$ in. Thus, the volume of the cube is $1^3 = 1\,\text{in}^3$.

Problem 3: First, identify the different-sized squares possible. There are three types, as shown below:

There are eight 1×1 squares, three 2×2 squares, and three $\sqrt{2} \times \sqrt{2}$ squares. That is a total of 14 possible squares.

There are $\binom{15}{4} = 1365$ ways to choose 4 points from the 15 points on this lattice. We found that 14 of these choices will yield a square, so the probability is $\frac{14}{1365} = \frac{2}{195}$.

Problem 4: The parallelogram is divided into eight equal parts, three of which are shaded, so the probability that a randomly selected point will be in a shaded region is $\frac{3}{8}$.

Problem 5: If two sides of a triangle are 36 cm and 8 cm, the length of the third side must be less than the sum of the sides $(36 + 8 = 44)$, and must be greater than their difference $(36 - 8 = 28)$ because of the triangle inequality. Thus, the longest whole number length allowed is 43 cm, and the shortest is 29 cm. The difference between them is 14 cm.

Problem 7: The fish, dogs and birds account for $\frac{1}{2} + \frac{1}{10} + \frac{1}{5} = \frac{4}{5}$ of the animals in the store. We are told that the other 6 animals were cats, and the remaining part is $1 - \frac{4}{5} = \frac{1}{5}$. If 6 cats represent $\frac{1}{5}$ of the animals in the store, there were $5 \times 6 = 30$ animals.

Problem 8: To calculate $(B * B) * (C * C)$, we find that $B * B = C$ and $C * C = A$. From the chart, we see that $C * A = B$.

Problem 9: From the given functions, $f(3) = 3^2 - 2(3) + 1 = 4$ and $g(4) = \sqrt{2(4)+1} = 3$. Further, $f(g(4)) = f(3) = 4$ and $g(f(3)) = g(4) = 3$. So, $f(g(4)) - g(f(3)) = 4 - 3 = 1$.

Problem 10: Since the area of the square is $64\,\text{in}^2$, the side of the square is 8 in. The diagonal of the square is $8\sqrt{2}$, so the diameter of the circle is also $8\sqrt{2}$ in. The area of the circle is $\pi r^2 = \pi(4\sqrt{2})^2 = 32\pi\,\text{in}^2$.

WARM-UP 10

Problem 1: The surface area of a cube is six times the area of a face, or $6x^2$, where x is the length of an edge. Therefore, $6x^2 = 600$ and $x = 10$, so the volume is $10^3 = 1000$.

Problem 2: The portion of the array below shows the numbers directly above 50 and directly below 100. Their sum is $41 + 114 = 155$.

1														
2	3													
4	5	6												
7	8	9	10											
11	12	13	14	15										
16	17	18	19	20	21									
22	23	24	25	26	27	28								
29	30	31	32	33	34	35	36							
37	38	39	40	**41**	42	43	44	45						
46	47	48	49	50	51	52	53	54	55					
56	57	58	59	60	61	62	63	64	65	66				
67	68	69	70	71	72	73	74	75	76	77	78			
79	80	81	82	83	84	85	86	87	88	89	90	91		
92	93	94	95	96	97	98	99	100	101	102	103	104	105	
106	107	108	109	110	111	112	113	**114**	115	116	117	118	119	120

Problem 3: Simplify:

$$\begin{aligned}(a^b + b^a) - (a^2 + b^2) &= (4^3 + 3^4) - (4^2 + 3^2)\\ &= 4^3 - 4^2 + 3^4 - 3^2\\ &= 4^2(4-1) + 3^2(3^2-1)\\ &= 16 \times 3 + 9 \times 8\\ &= 48 + 72 = 120\end{aligned}$$

Problem 4: Solve the equation $\frac{99}{88} = \frac{x}{100}$ for x, giving $100 \times \frac{99}{88} = x$, or $x = 112.5\%$.

Problem 5: The slant height of the pyramid is 5 (since the height and half the length of the base are two legs of a 3-4-5 triangle). Therefore, the area of the four triangular faces is $A = 4 \times \frac{1}{2}(6)(5) = 60$.

Problem 6: The number 1998 has prime factorization $1998 = 2 \times 3^3 \times 37$, so the largest prime factor is 37.

Problem 7: Let O be the center of the circles, and let r be the radius of the large circle. Then $\text{BD} = r\sqrt{2} = \text{BP}$, which implies that $\text{PO} = r\sqrt{2} - r = r(\sqrt{2} - 1)$. Then the ratio of the area of the smaller circle to the larger circle is

$$\frac{\pi\left(r(\sqrt{2}-1)^2\right)}{\pi r^2} = \left(\sqrt{2}-1\right)^2 \approx 0.1681.$$

To the nearest hundredth, the answer is 0.17.

Problem 8: In the first seat, we can place any of the five children; in the second, four; in the third, three; in the fourth, two; and in the last, only one. But, since the children are in a circle, the arrangement ABCDE is equivalent to BCDEA, CDEAB, DEABC and EABCD. We have counted these five arrangements as different, but they are actually the same. In fact, we have counted every arrangement five times. Consequently, there are $\frac{5\times4\times3\times2\times1}{5} = 24$ possibilities.

Problem 10: The nine points form a 3×3 lattice so that the only integer-length segments are either horizontal or vertical segments. There are $9 \times 8 = 72$ different ways to pick two distinct points from the set. Once the first point has been picked, there are only four other points from which to choose that will guarantee an integer-length segment (those on the same horizontal or vertical line). Thus, there are $9 \times 4 = 36$ possible choices that guarantee an integer-length segment. So, the probability is $\frac{9\times4}{9\times8} = \frac{1}{2}$.

WARM-UP 11

Problem 1: The magic square contains five rows with five numbers in each row. If all the row sums are the same, then that sum must be $\frac{1}{5}$ of the total of all numbers in the square. The magic square contains the numbers 1–25. The sum of these numbers is $1 + 2 + 3 + \ldots + 25 = \frac{25 \cdot 26}{2} = 325$. One-fifth of this sum is $\frac{325}{5} = 65$. Thus, the sum of the numbers in each row is 65. Three of the numbers in the row are 4, 6 and 13, which add to 23. The other two numbers in the row will add to $65 - 23 = 42$.

Problem 2: We can relate all of the numbers here to 20.

$$16 = 20 - 4$$
$$17 = 20 - 3$$
$$23 = 20 + 3$$
$$24 = 20 + 4$$
$$39 = 2(20) - 1$$
$$40 = 2(20)$$

Let's begin with the fact that $100 = 20 + 20 + 20 + 20 + 20$. If we replace one of these 20's with a 16, we have 4 left over. Similarly, if we replace one of them with a 23, we have 3 too many. Hence, we need to find a combination of 4, 3, -3, -4 and 1 that will yield an amount equal to one of the bags. Notice that $4 + 4 + 3 + 3 + 3 = 17$. Hence, we can replace two 20's with 16's, three 20's with 17's, and then we need to add one more 17. This gives $(20 - 4) + (20 - 4) + (20 - 3) + (20 - 3) + (20 - 3) + (4 + 4 + 3 + 3 + 3) = 16 + 16 + 17 + 17 + 17 + 17 = 100$. This requires two 16-lb bags and four 17-lb bags, or six bags total.

Problem 3: The only results from rolling two dice that are divisible by 4 are 4, 8 and 12. The result of 4 can be obtained by rolling a pair of 2's (one way), or a 1 and a 3 (two ways). So 4 can be produced in three ways. The result of 8 can be obtained by rolling a pair of 4's (one way), a 5 and a 3 (two ways), or a 6 and a 2 (two ways). So 8 can be produced in five ways. The result of 12 can only be produced by a pair of 6's (one way). Thus, there are nine ways to produce a multiple of 4 with two dice.

Problem 4: Because $AD = 4 + 12 = 16\,\text{cm}$, $GC = 16 - 5 = 11\,\text{cm}$. Similarly, $AB = 12 + 5 = 17\,\text{cm}$, so $HD = 17 - 12 = 5\,\text{cm}$. With that information, we can now find the area of the four triangles: $\triangle BGF$ has area $\frac{1}{2}(5)(12) = 30\,\text{cm}^2$, $\triangle HCG$ has area $\frac{1}{2}(11)(12) = 66\,\text{cm}^2$, $\triangle AEF$ has area $\frac{1}{2}(5)(4) = 10\,\text{cm}^2$, and $\triangle EDH$ has area $\frac{1}{2}(12)(5) = 30\,\text{cm}^2$. The area of quadrilateral EFGH equals the area of the entire rectangle ABCD minus the areas of the four triangles. That is, $17 \times 16 - (30 + 66 + 10 + 30) = 136\,\text{cm}^2$.

Problem 5: If the integers are x and y, it must be that $\frac{x+y}{x-y} = \frac{11}{5}$. The difference between the numerator and denominator of this fraction is $11 - 5 = 6$. We need an equivalent fraction for which the difference is 24. Try $\frac{4}{4} \times \frac{11}{5} = \frac{44}{20}$. The difference here is 24, and the fraction is equivalent to the original. We now know that $x + y = 44$ and $x - y = 20$. Solving these two equations gives $x = 32$ and $y = 12$, and their product is $32 \times 12 = 384$.

Problem 6: There are several ways to compare fractions. Many of these ways look at benchmarks and do not require computation of common denominators or converting to decimal form.

Benchmark A—same denominators: If the denominators are the same, compare the numerators. The fractions will be in the same order as the numerators. For example, $\frac{1}{3} < \frac{2}{3}$ and $\frac{1}{4} < \frac{3}{4}$.

Benchmark B—same numerators: If the numerators are the same, compare the denominators. The fractions will be in the reverse order of the denominators. For instance, $\frac{1}{4} < \frac{1}{3}$ and $\frac{2}{5} < \frac{2}{3}$.

Benchmark C—compare numerators and denominators: If the numerator is 1 less than the denominator for two fractions, the fractions will be in the same order as the denominators. (Think of each as being a pie with one piece missing. The greater the denominator, the smaller the missing piece, thus the greater the amount remaining.) For example, $\frac{2}{3} < \frac{3}{4}$.

Benchmark D—compare numerators and denominators: If the numerator is x less than the denominator for two fractions, fractions will be in the same order as the denominators. (Think of each as being a pie with x pieces missing. The greater the denominator, the smaller the missing pieces, thus the greater the amount remaining.) That is, $\frac{2}{5} < \frac{4}{7} < \frac{5}{8}$ (three pieces missing from each pie).

Benchmark E—equivalent fractions: Find fractions equivalent to those being compared, and then use one of the above rules.

In this case, $\frac{5}{8} < \frac{3}{4}$ since $\frac{5}{8} < \frac{6}{8}$ (Rule A). Further, $\frac{5}{8} < \frac{2}{3}$ since $\frac{5}{8} < \frac{6}{9}$ (Rule C—three pieces missing in each); $\frac{1}{3} < \frac{2}{5}$ since $\frac{2}{6} < \frac{2}{5}$ (Rule B). Putting this information together, the order is $\frac{1}{4} < \frac{1}{3} < \frac{2}{5} < \frac{4}{7} < \frac{5}{8} < \frac{2}{3} < \frac{3}{4}$, and the median is $\frac{4}{7}$.

Problem 7: The function is linear. The values for x go from 0 to 4, so the change is $4 - 0 = 4$. The values for y go from 20 to 0, so the change is $0 - 20 = -20$. We want to find the change in y when x goes from 0 to 10, a change of 10. To find this change, use the proportion $\frac{4}{-20} = \frac{10}{c}$. Thus, $4c = -200$, and $c = -50$. Then y changes -50 from its value when $x = 0$, and $y = 20 + (-50) = -30$.

Problem 9: The heart beats 70 times in 1 minute. We must first find out how many beats occur in 1 hour and then in 1 day.

$$\frac{70\,\text{beats}}{1\,\text{min}} \times \frac{60\,\text{min}}{1\,\text{hr}} \times \frac{24\,\text{hr}}{1\,\text{day}} = \frac{100\,800\,\text{beats}}{1\,\text{day}}$$

At this rate, it would take more than 9 but fewer than 10 days for the heart to beat 1 000 000 times.

WARM-UP 12

Problem 1: Find 25% of 60, which is $0.25 \times 60 = 15$, and 75% of 16, which is $0.75 \times 16 = 12$. The positive difference between the two numbers is $15 - 12 = 3$. Then, $33\frac{1}{3}\%$ of 3 is 1.

Problem 2: Each tetrahedron has four sides, so there is a $\frac{1}{4}$ probability of the first having a 1 on the bottom, a $\frac{1}{4}$ probability of the second having an A on the bottom, and a $\frac{1}{4}$ probability of the third having red on the bottom. Combined, the probability is $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$.

Problem 3: We know the measure of the angle of sector A, and we know some relationships about the areas of the other sectors. Since $B = \frac{2}{3}A$, then the measure of the central angle in sector B is $\frac{2}{3}(120) = 80°$. Then, $C = \frac{3}{4}B$, so the central angle of sector C is $\frac{3}{4}(80) = 60°$. Finally, $D = \frac{4}{5}C$, so the measure of the central angle of sector D is $\frac{4}{5}(60) = 48°$. Thus, the measure of the central angle of sector E is $360 - (120 + 80 + 60 + 48) = 360 - 308 = 52°$.

Since we know that the circumference of the circle is 60π cm, we can find the length of the radius. The circumference is $2\pi r = 60\pi$, so the radius is 30 cm. Then the area of the circle is $\pi(30)^2 = 900\pi\,\text{cm}^2$. The area of sector E is $\frac{52}{360}(900\pi) = 130\pi\,\text{cm}^2$.

Problem 4: If a line has slope m, the slope of a perpendicular is $-\frac{1}{m}$. The slope of the line $y = -\frac{2}{3}x + 5$ is $-\frac{2}{3}$. Hence, the slope of the perpendicular line is $\frac{3}{2}$. We know that the perpendicular line contains the point $(-1, 2)$, so the equation of the line is $(y - 2) = \frac{3}{2}(x - (-1))$. This simplifies to $y = \frac{3}{2}x + \frac{7}{2}$. Consequently, the y-coordinate of the y-intercept is $\frac{7}{2}$.

Problem 5: The number of ways to choose a 4-person committee from 21 students is $\binom{21}{4} = \frac{21\cdot20\cdot19\cdot18}{4\cdot3\cdot2\cdot1}$. Similarly, the number of ways to choose a 5-person committee is $\binom{21}{5} = \frac{21\cdot20\cdot19\cdot18\cdot17}{5\cdot4\cdot3\cdot2\cdot1}$. The ratio is

$$\frac{(21\cdot 20\cdot 19\cdot 18)(5\cdot 4\cdot 3\cdot 2\cdot 1)}{(4\cdot 3\cdot 2\cdot 1)(21\cdot 20\cdot 19\cdot 18\cdot 17)} = \frac{5}{17}.$$

Problem 6: The total surface area is equal to six times the area of one face. The area of one face is $\frac{1}{2}(4)(2\sqrt{3}) = 4\sqrt{3}$. So, the total surface area is $6 \times 4\sqrt{3} = 24\sqrt{3}$ square inches.

Problem 7: We start with $\frac{a+b}{a-b} = \frac{7}{3}$. Consequently,

$$3a + 3b = 7a - 7b$$
$$10b = 4a$$

and $a = \frac{5b}{2}$. For the product to be less than 1000, $a \times b < 1000$. Then, by substitution,

$$\frac{5}{2}b \times b < 1000$$
$$\frac{5}{2}b^2 < 1000$$
$$b^2 < 400$$
$$b < 20.$$

By substituting for a, we get $a < \frac{5}{2}(20) = 50$. The next smallest number that is less than 50 and which has a factor of 5 is 45, so $a = 45$, $b = 18$, and the product is $45 \times 18 = 810$.

Problem 8: Simplify:

$$\frac{37\,037\cdot 15 + 37\,037\cdot 9}{37037\cdot 3} = \frac{37\,037(15+9)}{37\,037(3)} = \frac{24}{3} = 8$$

Problem 9: We must find the number of dominoes in a set and the number of doubles. The number 0 can be paired with any of ten digits 0–9; the number 1 can be paired with any of nine digits 1–9; the number 2 can be paired with any of eight digits 2–9; and so on, each digit can be paired with one digit fewer than the previous number, through the number 9 which can only be paired with just one digit, 9. In total, there are $10 + 9 + 8 + \ldots + 1 = 55$ dominoes. When each digit is paired with itself, a double is formed, and since there are 10 digits, there are 10 doubles. So the probability of selecting a double is $\frac{10}{55} = \frac{2}{11}$.

WARM-UP 13

Problem 1: Since choosing any three vertices will always create a triangle, the question can be rephrased as, "How many ways can we choose 3 of 8 vertices?" That is simply $\binom{8}{3} = \frac{8!}{3!5!} = \frac{8\times7\times6}{3\times2} = 56$.

Problem 2: The hours digit cannot be 7, 8 or 9 because the sum would be greater than 6. The range for the tens digit of the minutes depends on the hour. Once the hour and tens digit of the minutes are chosen, the ones digit for the minutes is fixed, so only the first two digits need to be considered. For example, if the hour digit is 1, then the tens digit could be any digit 0–5. There are, then, six possibilities for times between 1 o'clock and 2 o'clock: 1:05, 1:14, 1:23, 1:32, 1:41 and 1:50.

The following list shows the numbers for each possible hour choice.

Hour	Possible Tens Digit	Total Combinations
1	0–5	6
2	0–4	5
3	0–3	4
4	0–2	3
5	0–1	2
6	0	1
10	0–5	6
11	0–4	5
12	0–3	4

Adding the numbers in the last column, there are 36 times in a 12-hour period when the digits on the clock total 6.

Problem 3: The original volume is $5 \times 3 \times 10 = 150$. The new volume is $5 \cdot \frac{6}{5} \times 3 \cdot \frac{6}{5} \times 10 \cdot \frac{1}{2} = 150 \times \frac{36}{50} = 108$. The positive difference is 42.

Problem 4: Let ℓ and w be the length and width of the rectangle. Since the perimeter is 30 cm, we have $\ell + w = 15$. The maximum area of the rectangle occurs when $\ell = w$; that is, when the rectangle is a square. Hence, the length of each side should be $15 \div 2 = 7.5$, and the largest area is $(7.5)^2 = 56.25\,\text{cm}^2$.

Problem 5: There are 120° in an interior angle of a regular hexagon, so at each vertex where two hexagons meet there are $360 - 240 = 120°$ for which we have not yet accounted. Thus, each obtuse angle in the rhombus measures 120°, while the acute angle is half of this, or 60°.

Problem 6: The surface area of the face of a cylindrical can is $2\pi rh$. So the two areas are $2\pi(4.5)(13) = 117\pi$ and $2\pi(3)(10) = 60\pi$, and their difference is 57π.

Problem 7: The perimeter of the triangle is $p = 17x + 121$. There are three different ways this triangle can be isosceles: (a) $6x + 40 = 4x + 104$; (b) $6x + 40 = 7x - 23$; or (c) $4x + 104 = 7x - 23$. These three possibilities yield $x = 32$, $x = 63$ and $x = 42\frac{1}{3}$, respectively. Since case (b) produces the greatest value of x, the greatest perimeter is $p = 17(63) + 121 = 1192$.

Problem 9: The given equation $2^x + 2^x + 2^x + 2^x = 128$ can be simplified to $4 \times 2^x = 128$, or, even better, to $2^{2+x} = 2^7$. Thus, $x = 5$ and $(x-1)(x+1) = 6 \times 4 = 24$.

Problem 10: An examination of Pascal's triangle shows many patterns. For this problem, we might notice that the numbers in each row represent the numbers of ways to choose some numbers of items from a set containing n items. Thus, in Row 0 (the top row), the 1 represents the number of ways to choose 0 items from a set containing 0 items. In mathematical notation, this is written as $\binom{0}{0}$. Then in Row 1, the two numbers 1 and 1 represent the number of ways to choose 1 item from a set containing 1 item and the number of ways to choose 0 items from a set containing 1 item. These calculations are written as $\binom{1}{1}$ and $\binom{1}{0}$ and are read as "one choose one" and "one choose zero," respectively. Similarly, the numbers in Row n show the numbers of ways to choose $n, n-1, n-2, \ldots, 2, 1, 0$ items from a set containing n items. In general, the numbers of ways to choose r items from a set of n items is represented and calculated as follows: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

In Row 15, the first number represents 15 choose 15, the second number 15 choose 14, the third 15 choose 13, and the fourth 15 choose 12. The last of these equals $\binom{15}{12} = \frac{15!}{12!(15-12)!} = 455$.

Note that, because of symmetry, we could also have computed the number of ways to choose 3 items from 15 items. The answers are the same.

WARM-UP 14

Problem 1: The easiest way to solve this problem is to find a pattern. We know that we must start at M, so let's begin with the A. There is one way to reach the first A (down to the left from M) and one way to reach the second A (down to the right from the M). Let's next consider the T. There is one way to reach the first T (down to the left from the first A) and one way to reach the last T (down to the right from the last A). There are, however two ways to choose the middle T (down to the right from the first A, or down to the left from the second A).

If this analysis is continued, you will notice that the numbers are those of Pascal's Triangle, as shown in the diagram. Notice that starting with Row 5, not all numbers in the triangle are present, but we can derive each of the numbers by adding the two adjacent numbers in the preceding row. Notice that the S does not act as part of the triangle, since there is no choice but to go directly from the T to the S.

M	···					1				
A	···				1		1			
T	···			1		2		1		
H	···		1		3		3		1	
C	···	1		4		6		4		1
O	···		5		10		10		5	
U	···			15		20		15		
N	···				35		35			
T	···					70				
S	···						70			

Thus, there are 70 different ways that a path spelling MATHCOUNTS can be taken.

Problem 2: Because Natasha has more than 10¢ and less than \$10 worth of dimes, she has more than 1 and fewer than 100 dimes. Let d be the total number of dimes. Because d is 1 more than a multiple of 3, 4 and 5, it can be written as $3m+1$, $4n+1$ and $5p+1$, where m, n and p are integers. If $3m+1=4n+1$, then $3m=4n$. The simplest whole number solution is $m=4$ and $n=3$. This makes $3m=4n=12$. However, any solution such that n is a multiple of 3 and m is a multiple of 4 will work. Thus all numbers that satisfy the first two conditions are of the form $12k+1$ for integer values of k. Now we have $d=12k+1=5p+1$. Thus, $12k=5p$. The simplest solution here is $k=5$ and $p=12$, making $12k=5p=60$. The solutions that satisfy all three conditions are of the form $60r+1$. The simplest choice here is $r=1$, so $60r+1=61$. Hence, Natasha has 61 dimes.

Problem 3: The last digit of the woman's age is 3. When that numeral is reversed, we get a number in the 30's. This number is a perfect square. The only number in the 30's that is a perfect square is 36. The square of the digit 6 is 36, so the answer checks. Consequently, the woman from Dundee must be 63.

Problem 4: $20! = 20\times19\times18\times17\times16\times15\times14\times13\times12\times11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1$. The zeroes at the end of the product of 20! come from the multiplication of 2's and 5's. There are many factors of 2 in this product, but there are only four factors of 5 (found in 20, 15, 10 and 5). Thus, there are 4 zeroes at the end of 20!.

Problem 5: One kilometer is approximately 0.62 miles. Elizabeth ran 10 km in 58 minutes, 54 seconds. First, convert 58 minutes, 54 seconds to a decimal number of minutes: 54 seconds is $\frac{54}{60}=0.9$ minutes. Thus, the time is 58.9 minutes. Converting her time to minutes per mile gives:

$$\frac{58.9\,\text{min}}{10\,\text{km}}\times\frac{1\,\text{km}}{0.62\,\text{mi}}=9.5\,\text{min/mi}$$

Problem 6: One way to find the GCF of two numbers is to obtain their prime factorizations and determine which prime factors they share. In this case, $2835=3^4\times5\times7$ and $8960=2^8\times5\times7$. The common factors are 5 and 7, so their GCF is $5\times7=35$.

Problem 8: There are several clues here. Notice that the digits added in the ones and tens columns are the same, but their sums are not. How could this happen? The sum of the digits must be more than 10 so that the sum in the tens column is 1 more than the sum in the ones column. This tells us that n, the sum in the tens column, is 1 more than d, the sum in the ones column. Also note that, in the tens column, the sum of n and a gives a sum ending in n. How is this possible? Only if $a=0$ or $a=9$, but from the ones column we can see that $a\neq0$, since the sum is not n. Thus, $a=9$.

Because the sum in the tens column is more than 9, and the sum in the hundreds column is 9, it must be that $1+d+d=9$. Hence, $2d=8$, giving $d=4$. Further, $n=d+1$, so $n=5$. The sum $d+n+a=4+5+9=18$.

Problem 9: The amount of fencing dictates the perimeter of the garden. With 142 feet of fencing, the sum of the dimensions of the rectangle must be 71 feet, since the perimeter is found by doubling the sum of the dimensions. Since we know that the dimensions are natural numbers, the possible dimensions are $1\times70=70, 2\times69=138, 3\times68=204, \ldots, 35\times36=1260$.

The rectangle with the greatest area is that in which the two dimensions are closest together, making either a square or the rectangle closest to a square. In this case, the best choice is 35×36 giving an area of $1260\,\text{ft}^2$.

Problem 10: We know that $5<\sqrt{27}<6$ and $8<\sqrt{72}<9$. Thus, all integers greater than 5 will satisfy the first restriction, and all integers less than 9 will satisfy the second restriction. The integers 6, 7 and 8 are the only ones that satisfy both criteria. Thus, there are 3 integers.

WARM-UP 15

Problem 1: The degree measure for the central angle is going to be in the same ratio to 360° (the entire circle) as the arc is to the circumference. Thus,

$$\frac{m\angle \text{XOZ}}{360} = \frac{2.8}{56} = \frac{1}{20}$$
$$m\angle \text{XOZ} = 18°.$$

Problem 2: The difference between consecutive odd integers ($2n-1$ and $2n+1$) is

$$\left((2n)^2-(2n-1)^2\right)+\left((2n+1)^2-(2n)^2\right) = (4n-1)+(4n+1) = 8n.$$

In this problem, the difference is 128, so $8n = 128$, or $n = 16$. Therefore, the consecutive odd integers are $2n-1 = 31$ and $2n+1 = 33$, and their product is $31 \times 33 = 1023$.

Problem 3: Since the perimeter of the square is 108 cm, then each side is $108 \div 4 = 27$ cm. Since the sides are trisected, triangles formed at the corner of the squares are isosceles right triangles with legs measuring 9 cm. Each of the triangles has area $\frac{1}{2}(9 \times 9) = 40.5$. To find the area of the octagon, compute the area of the square and subtract the areas of the four triangles. The area of the octagon, then, is $27^2 - 4(40.5) = 567\,\text{cm}^2$.

Problem 4: Think of this problem with the volume of the tank equal to one cubic unit. The first time Janelle changes $\frac{2}{3}$ of the tank water, $\frac{1}{3} = 0.\overline{3}$ of the water will be old. The next time she changes $\frac{2}{3}$, $\frac{1}{3}$ of the water in the tank is old, so the old water remaining after the second change will be $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9} = 0.\overline{1}$. Continue this until we have less than 0.05 old water. Replacing $\frac{2}{3}$ again, $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27} \approx 0.04$ old water remains. Janelle is done, and the process only took three changes of water.

Problem 6: There is a right angle, and the two interior angles of the regular polygon share a vertex. Consequently, $2x + 90 = 360$, where x is the angle of the regular polygon. Then, $x = 135$. The measure of each interior angle of a polygon with n sides is $\frac{180(n-2)}{n}$ degrees. In this case, the measure is 135°, so $n = 8$. Hence, the polygons are octagons, which have 8 sides.

Problem 7: The most efficient way to solve this problem is to simplify the complex fraction working from the bottom up.

$$1+\cfrac{n}{1+\cfrac{1}{1+\cfrac{1}{1+1}}} = 1+\cfrac{n}{1+\cfrac{1}{1+\frac{1}{2}}} = 1+\cfrac{n}{1+\frac{1}{\frac{3}{2}}} = 1+\cfrac{n}{1+\frac{2}{3}} = 1+\frac{n}{\frac{5}{3}} = 1+\frac{3n}{5} = \frac{5+3n}{5}$$

Because this fraction is equal to $5\frac{4}{5} = \frac{29}{5}$, the numerators must be equal. Hence, $5 + 3n = 29$, so $n = 8$.

Problem 8: One acre is $43\,560\,\text{ft}^2$, and one square mile is $5280^2\,\text{ft}^2$. The ratio of 160 acres to 1 square mile is

$$\frac{160 \times 43\,560}{5280^2} = \frac{1}{4}.$$

Problem 9: We are given AB = 2 and EF = 3. If a new right triangle is constructed with hypotenuse AF, the other legs are AF′ (which is AB extended) and FF′ (which is parallel to EB).

Now we can use the Pythagorean theorem with leg AF′ and hypotenuse AF. Then FF′ is $\sqrt{(5\sqrt{17})^2 - 5^2} = 20$ cm. FF′ = BE, so BE = 20 cm, too. Again using the Pythagorean theorem, we find that BC = 8 cm and DE = 6 cm. Now, BE = BC + DE + CD = 8 + 6 + CD = 20, so CD = 6.

Problem 10: From the first partial product, we see that 8 times the two-digit divisor gives a two-digit number. Since $8 \times 10 = 80$ and $8 \times 13 = 104$, we know that the divisor is 10, 11 or 12. From the second partial product, we know that some one-digit number times the two-digit divisor gives a three-digit product. This one-digit number must be 9, since it is larger than 8 from the previous partial product. Thus, 9 times the divisor must be a three-digit number. Since $9 \times 10 = 90$, $9 \times 11 = 99$ and $9 \times 12 = 108$, the only possible divisor is 12.

WARM-UP 16

Problem 1: Since the area of the circle is 144π, the radius of the circle is $r = \sqrt{\frac{A}{\pi}} = 12$ cm. The plane that created this circle is 5 cm from the center of the sphere. These two values represent two legs of a right triangle whose hypotenuse is the radius of the sphere. The right triangle with legs 5 and 12 has hypotenuse 13 (a Pythagorean triple).

Problem 2: The slope of the radius line is $\frac{5-2}{5-3} = \frac{3}{2}$. Tangent lines to circles are perpendicular to radius lines, so the slope of the tangent line is equal to the negative reciprocal of the radius line, which is $-\frac{2}{3}$.

Problem 3: The three angles in a triangle have a sum of 180°, so $x + 2x + 30 = 180$, which means that $x = 50°$.

Problem 4: Add 35 minutes for each of three periods, plus 5 minutes for two passing times between them That's a total of 1:55. One hour, 55 minutes from 12:55 is 2:50.

Problem 5: One of the basic algebra formulas is rate times time equals distance. The distance is 30. Let x be the original rate and t be the original time. Since the original rate is x, the new rate is $x + 3$. And since the original time is t, the new time is $t - \frac{1}{3}$ (because 20 minutes is $\frac{1}{3}$ hour). The original rate times the original time gives the equation $xt = 30$, or $x = \frac{30}{t}$. The new rate times the new time gives the equation $(x+3)(t-\frac{1}{3}) = 30$. Substituting $x = \frac{30}{t}$ gives

$$\left(\frac{30}{t}+3\right)\left(t-\frac{1}{3}\right) = 30$$
$$30 + 3t - \frac{10}{t} - 1 = 30$$
$$3t^2 - t - 10 = 0$$
$$(3t+5)(t-2) = 0$$
$$t = -\frac{5}{3}, 2$$

The only value for t that makes sense is $t = 2$. Since we know that $x = \frac{30}{t}$, then $x = 15$, and her original speed was 15 mph.

Problem 6: In order for a number to be divisible by 6, it must be divisible by both 2 and 3. To be divisible by 2, the units digit n must be 0, 2, 4, 6 or 8. To be divisible by 3, the sum of the digits must be divisible by 3. Thus, the sum $3 + n + 8 + 5 + n = 2n + 16$ must be divisible by 3. Checking even values, we see that $2n + 16$ is divisible by 3 when $n = 4$.

Problem 7: There are eight outcomes when a coin is flipped three times. Six of those eight ways contain either two heads or two tails. Hence, the probability is $\frac{6}{8} = \frac{3}{4}$.

Problem 8: The length of the original cable is $2\pi r$, where r is the radius of the Earth. The length of the new cable is $2\pi(r+3) = 2\pi r + 6\pi$, so the difference in length is just 6π feet.

Problem 10: The possible products are 9×35, 7×45, and 5×63. The least sum is produced by the first pairing, $9 + 35 = 44$.

WARM-UP 17

Problem 1: The probability of drawing a red ball first, replacing it with a white ball, and then drawing a red ball is $\frac{2}{4} \times \frac{1}{4} = \frac{1}{8}$. The probability of drawing a white ball first, replacing it with a red ball, and then drawing a red ball is $\frac{2}{4} \times \frac{3}{4} = \frac{3}{8}$. The combined probability that one of these two scenarios will occur is $\frac{1}{8} + \frac{3}{8} = \frac{1}{2}$.

Problem 2: There are six faces on a cube, and two of them are to be chosen. These faces can be chosen in $\binom{6}{2} = 15$ ways. However, three of these pairs are not acceptable—when the two faces chosen are parallel faces of the cube, the cube cannot be inserted to show those two faces. Hence, there are only 12 pairs of adjacent faces which can be chosen. And the cube can be inserted in two different ways for each of these 12 pairs, so there are $2 \times 12 = 24$ ways to insert the cube into this box.

Problem 3: The maximum value of each digit is 9, so the maximum possible sum of the digits in a three-digit number is 27. Hence, we only need to check the first 27 multiples of 32. A list of the multiples of 32 gives $32, 64, 96, \ldots, 576, \ldots, 832, 864$. By inspection, the only number that works is 576. The sum of its digits is $5 + 7 + 6 = 18$, and $32 \times 18 = 576$.

Problem 4: The sum of all the numbers in the magic square is $-10+(-8)+(-6)+(-4)+(-2)+0+2+4+6=-18$. Since the sums of all rows must be the same and there are three rows, the sum for each row must be $\frac{1}{3}$ the total sum. Thus, the magic sum is $\frac{-18}{3} = -6$.

Problem 5: $\triangle DEB \sim \triangle DBC$, because they are both right triangles and they share $\angle EDB$. Because the triangles are similar, $m\angle EBC = m\angle EDB$. Hence, $\triangle BEC \sim \triangle BCA \sim \triangle DEB$. Since these triangles are similar, a proportion will yield BD. $\overline{BC}$ is the shorter leg of $\triangle DBC$ and the longer leg of $\triangle BCA$. The ratio of the shorter to longer leg of $\triangle BCA$ will be equal to the ratio of the shorter to longer leg of $\triangle DBC$, which gives $\frac{5}{12} = \frac{12}{BD}$. Solving that proportion yields the length of the segment, and $BD = 28\frac{4}{5}$ cm.

Problem 6: Jim took $\frac{4}{3}$ hours to walk $\frac{4}{9}$ of the way, thus he would take $\frac{1}{3}$ hour to walk $\frac{1}{9}$ of the way. Hence, to walk $\frac{9}{9}$ of the way, Jim will take $9\left(\frac{1}{3}\right) = 3$ hours.

Problem 7: There are seven numbers in each row. The first number in each row leaves remainder 1 when divided by 7, the second number leaves remainder 2, the third number leaves remainder 3, and so on. As $254 = 36(7) + 2$, the number 254 will appear in Column 2.

Notice that the last number in Row 1 is 1×7, the last number in Row 2 is 2×7, and so forth. We know that $254 = 36(7) + 2$. The last number in Row 36 is $36 \times 7 = 252$. Hence, 254 will appear in Row 37.

The sum of the row and column numbers for 254 is $37 + 2 = 39$.

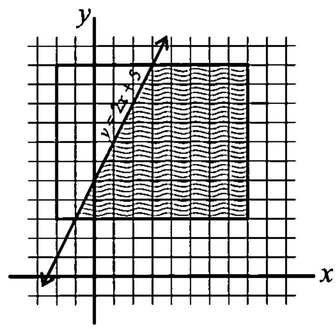
Problem 9: When the point $(6,3)$ is reflected over a line to obtain the image $(2,5)$, the line of reflection is the perpendicular bisector of the line segment connecting the point with its image. Since all that is needed is the slope of the line of reflection, we only need to find the slope of the line connecting the points and calculate the negative reciprocal. The y-coordinate goes from 3 to 5, a change of 2 units, and the x-coordinate goes from 6 to 2, a change of -4 units. The slope of this segment is then $\frac{2}{-4} = -\frac{1}{2}$. The slope of a line perpendicular to this is 2.

Problem 10: $\triangle$CDE is an isosceles triangle, and $\angle$C is one of the equal angles. Label the measure of this angle x. Then, $m\angle \text{CED} = x$, also, and $m\angle \text{EDC} = 180 - 2x$. This means that $\angle$EDB, which is supplementary to $\angle$EDC, will have measure $2x$. $\angle$EDB is one of the two equal angles in isosceles $\triangle$EBD. Thus, $m\angle \text{EBD} = 2x$. Then, $\angle \text{DEB} = 180 - 4x$. We can now show that $84 + (180 - 4x) + x = 180$, since the three angles $\angle$AEB, $\angle$BED and $\angle$DEC together comprise a straight angle. Thus, $x = m\angle \text{C} = 28°$.

WARM-UP 18

Problem 1: Let w represent the original length—the new width is $w + 6$. Let ℓ be the original length—then $\ell - 2$ is the new length. The new rectangle is a square with perimeter 28 in, so each side is 7 in. Hence, $\ell - 2 = 7$, so $\ell = 9$. Similarly, $w + 6 = 7$, so $w = 1$. The original perimeter was $1 + 1 + 9 + 9 = 20$ inches.

Problem 2: To solve a problem like this, it is best to draw the graph.



We are asked how much of the rectangle lies to the right of the line. Thus we need the ratio of the area of the shaded region to the area of the entire rectangle. The rectangle is 10 units wide and 8 units tall, so its area is 80 square units. The area of the shaded region can be found by combining a 5×8 unit rectangle and a right triangle with legs of 4 and 8 units. Thus the area of the shaded region is $40 + 16 = 56$ square units.

The ratio of the areas is $\frac{56}{80} = \frac{7}{10}$.

Problem 3: Using similar triangles, we can set up a proportion. $\triangle$ADE $\sim$ $\triangle$ABC because they are right triangles which share $\angle$A. Likewise, $\triangle$GBF $\sim$ $\triangle$ABC because they are right triangles which share $\angle$B. Consequently, $\triangle$ADE $\sim$ $\triangle$GBF. Using the Pythagorean theorem, $\text{AD} = \sqrt{12^2 + 5^2} = 13$. This gives

$$\frac{\text{AE}}{\text{AD}} = \frac{\text{GF}}{\text{GB}} \quad \rightarrow \quad \frac{5}{13} = \frac{18}{\text{GB}} \quad \rightarrow \quad \text{GB} = \frac{13 \times 18}{5} = 46.8\,\text{cm}$$

Problem 4: There are six pairs of partners: AB, CD, EF, GH, IJ and KL. Since eleven of them shook a different number of hands, and no one shook hands with their own partner, we know the number of hands shaken ranges from 0 to 10. Assume that A shook 10 hands; consequently, everyone except A's partner has shaken one hand (at least). Therefore, the only person who could have shaken 0 hands is A's partner, who is B. Similarly, assume that C shook 9 hands. Then D is the only person who could have shaken 1 hand. Continuing this logic, we can list all possible pairs: $(10,0), (9,1), (8,2), (7,3), (6,4), (5,5)$. Notice that there is an extra 5 in the list. Since none of the people that Hannah asked had shaken the same number of hands as anyone else that she asked, Hannah must be one of the people who shook 5 hands. Consequently, Hannah's partner also shook 5 hands.

Problem 5: The number must be even, so either 2 or 4 must be the units digit. For the other three spots, there are five possibilities. The total number of choices is $5 \times 5 \times 5 \times 2 = 250$.

Problem 6: Simplify:

$$g(f(5)) = g\left(\frac{5^2 - 1}{5 + 1}\right) = g\left(\frac{24}{6}\right) = g(4) = 4^2 - 2(4) + 1 = 9$$

Problem 8: Since $\triangle$DCE is isosceles, CD = 6 cm, and the area of the square is $36\,\text{cm}^2$. The area of the triangle is $\frac{1}{2}(6)(6) = 18\,\text{cm}^2$. The ratio of the area of the triangle to the area of the square is $\frac{18}{36} = \frac{1}{2}$.

Problem 9: When the point (a,b) is reflected over the x-axis, the new point has coordinates $(a,-b)$. When that point is reflected over the y-axis, another new point is created $(-a,-b)$, which was then named (c,d). So, $ab - cd = ab - (-a)(-b) = ab - ab = 0$.

Problem 10: The diagonal of a cube is the hypotenuse of a triangle whose legs are a side of the cube and a diagonal of an adjoining face. Call the side of the cube s. Then the diagonal of an adjoining face is $s\sqrt{2}$. Hence,

$s^2 + (s\sqrt{2})^2 = 12^2 = 144$. This gives $s^2 + 2s^2 = 3s^2 = 144$, and $s^2 = 48$. The area of one of the sides of the cube is $s^2 = 48$. The area of all six faces is $6s^2 = 6 \times 48 = 288\,\text{cm}^2$.

WORKOUT 1

Problem 1: Converting 1 000 000 seconds to days gives $1\,000\,000 \times \frac{1}{60} \times \frac{1}{60} \times \frac{1}{24} = 11.57$ days. And 11.57 days from Tuesday, it will be Saturday.

Problem 2: The area of a trapezoid is equal to the average of the lengths of the bases times the perpendicular height. Therefore, to find the height of the trapezoid, realize that the height is a leg of an isosceles right triangle.

Using the formula for area described above and substituting values for the two bases gives:

$$\frac{8+16}{2} \times h = 72$$
$$12h = 72$$
$$h = 6$$

Since a leg of the isosceles right triangle is 6 and DC = 16, then $16 - 6 - 8 = 2\,\text{cm}$ is the length of the remaining part of $\overline{\text{DC}}$. Finally, by the Pythagorean theorem,

$$6^2 + 2^2 = \text{BC}^2$$
$$36 + 4 = \text{BC}^2$$
$$40 = \text{BC}^2.$$

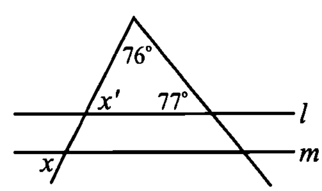
So, $\text{BC} = \sqrt{40} \approx 6.3$.

Problem 3: First, find the sum of all the stock prices.

$$7\frac{7}{8} + 7\frac{15}{16} + 23\frac{5}{8} + 19\frac{1}{2} + 28\frac{3}{8} = 87\frac{5}{16}$$

Since there are five stocks, the average price is $\frac{87}{5} + \frac{1}{16} \approx 17.46$.

Problem 4: Since $\ell \parallel m$, the number of degrees in the angle labeled x is the same as the number of degrees in the angle labeled x'.



The supplement of 103° is 77°. The three angles in the triangle must add to 180°, so

$$77 + 76 + x' = 180$$
$$x' = 180 - 77 - 76$$
$$x' = 27$$

and thus $x = x' = 27°$.

Problem 6: We can write 2^{133} as $2^{132+1} = 2^{132} \times 2^1 = (2^4)^{33} \times 2^1$. The remainder when $2^4 = 16$ is divided by 5 is 1, so the remainder from $(2^4)^{33} \times 2^1 = 1^{33} \times 2 = 2$.

Problem 7: The tens digits and the units digits can only come from the set $\{2, 3, 5, 7\}$. When paired, the only two-digit prime numbers from this set are 23, 37, 53 and 73. Hence, the only numbers between 100 and 199 that need to be considered are 123, 137, 153 and 173. But $123 = 41 \times 3$ and $153 = 9 \times 17$, so we must exclude those two, and the sum of the remaining numbers is $137 + 173 = 310$.

Problem 8: If we let x be the distance from school to Jack's house, then the distance from Jack's house to Mary's house is $10x$, and the distance from Mary's house to Jenna's house is $100x$. Therefore, the distance from Jack's house to the school can be determined from

$$100x + 10x + x = 13320$$
$$111x = 13320$$
$$x = 120$$

So this distance is 120 m. The distance from Jenna's house to Mary's house is $100x$, so it is $100(120) = 12\,000$ m or 12 km.

Problem 9: The pre-tax price is $146.30 - 6.30 = 140.00$. Thus, the percent tax that Aida paid was $\frac{6.30}{140.00} = 0.045$, or 4.5%.

Problem 10: Let $m\angle C = x$. Then, $m\angle A = 90 - x$ and $m\angle B = 180 - x$. Plugging this information into the equation for $m\angle A$ and $m\angle B$ gives

$$\begin{aligned} m\angle A + m\angle B &= 202 \\ (90 - x) + (180 - x) &= 202 \\ 270 - 2x &= 202 \\ -2x &= -68 \\ x &= 34. \end{aligned}$$

So, $m\angle C = 34°$.

WORKOUT 2

Problem 1: Since each of these is an arithmetic sequence, the mean for each sequence is the mean of the first and last numbers in the sequence. The means can be found very quickly:

Mean for A: $\frac{91+1}{2} = 46$
Mean for B: $\frac{92+2}{2} = 47$
Mean for C: $\frac{93+3}{2} = 48$

The median of these three numbers is 47.

Problem 2: Divisibility rules for 2, 3, 5 and 7 show that neither of these numbers has a one-digit factor. Hence, the Euclidean Algorithm probably ought to be used. (See the Investigation & Exploration section for Warm-Up 14.) Once the factors are identified, the LCM can be found easily.

$$\begin{aligned} 1537 &= 1363 \cdot 1 + 174 \\ 1363 &= 174 \cdot 7 + 145 \\ 174 &= 145 \cdot 1 + 29 \\ 145 &= 29 \cdot 1 + 0 \end{aligned}$$

Thus, the greatest common factor is 29. A calculator reveals that $1537 = 29 \times 53$ and $1363 = 29 \times 47$. The least common multiple, then, is $29 \times 47 \times 53 = 72\,239$.

Problem 4: First, note that the operation $a \star b$ can be slightly simplified to $a \star b = (a+b)(a-b) = a^2 - b^2$. Then,

$$\begin{aligned} (5 \star 4)(5 \nabla 4) &= (5^2 - 4^2)(5^2 + 4^2) \\ &= (5^2)^2 - (4^2)^2 \\ &= 5^4 - 4^4 \\ &= 369 \end{aligned}$$

Problem 5: Group the numbers as shown, notice that each group simplifies to -2, and compute the partial sums. (There are 52 groups with sum -2, plus 210 more.)

$$(2-4) + (6-8) + (10-12) + \ldots + (206-208) + 210 = (-2 \times 52) + 210 = 106$$

Problem 6: The percent decrease can be found by dividing the reduced price by the original price. $\frac{640}{1024} = 0.625 = 62.5\%$. The sale price is 62.5% of the original price. Hence, the next price will be $640 \times 0.625 = \$400$.

Problem 7: There are $4 \times 60 = 240$ minutes in 4 hours.

$$\frac{240\,\text{min}}{26.2\,\text{mi}} =\approx 9.2\,\text{min/mi}$$

Problem 8: We can factor 91 as 7×13. The sum of the factors of $2^2 = 4$ is 7, and the sum of the factors of $3^2 = 9$ is 13. Hence, the number we want is $2^2 \times 3^2 = 36$. As a check, we see that the sum of the factors of 36 is $1+2+3+4+6+9+12+18+36 = 91$. (See the solution to problem #8 from Warm-Up 7 for more information.)

Problem 9: The area of each of the three equilateral triangles is $9\sqrt{3}\,\text{in}^2$. Thus, the height of the triangle is $3\sqrt{3}$ in and the base is 6 in. The perimeter of the trapezoid is five times the base of one triangle, so the perimeter is $5 \times 6 = 30$ in.

Problem 10: Because $2x^2 + 7x + k$ is a quadratic expression, it will have two factors. One of factors is $2x + 1$, so the other factor will be of the form $x + q$ for some integer q. Hence,

$$\begin{aligned} (2x+1)(x+q) &= 2x^2 + 7x + k \\ 2x^2 + x + 2xq + q &= 2x^2 + 7x + k \\ 2x^2 + (2q+1)x + q &= 2x^2 + 7x + k \end{aligned}$$

Comparing the coefficients of x, $2q + 1 = 7$. Hence, $q = 3$. By comparing the constant terms, we see that $q = k$, so $k = 3$.

WORKOUT 3

Problem 1: Trailing zeros indicate powers of 10, and powers of 10 come from products of powers of 2 and 5. To find the trailing zeroes in this product, there is no need to count the number of factors of 2, because there are far fewer factors of 5. Factors of 5 occur at each multiple of 5, and because this sequence starts with 3×5 and ends with 20×5, there are 18 numbers in the list that contain a factor of 5. Among these 18 numbers, however, are 25, 50, 75 and 100, each of which contains two factors of 5 (because 5^2 is a factor of each of them). Hence, there are 22 factors of 5 in this product, so the product ends in 22 consecutive zeroes.

Problem 4: Of the 2460 students, 20% are taking all three subjects. Next, notice that 25% of the students take both algebra and French. Of them, 20% who take all three subjects have already been accounted for, so there are $25-20=5\%$ of students who take both algebra and French but not history. Likewise, $40-20=20\%$ of the students take algebra and history but not French, and $25-20=5\%$ of the students take history and French but not algebra. A total of $5+20+20=45\%$ of the students taking algebra have been accounted for, thus $65-45=20\%$ take algebra alone. There are $45-(20+20+5)=0\%$, or no students, taking history alone. Finally, we have $55-(5+20+5)=25\%$ taking French alone.

In total, we have accounted for $20+(5+20+5)+(20+0+25)=95\%$ of all students. Hence, $0.05 \times 2460 = 123$ students take none of these subjects.

Problem 5: Letting $x = 123\,456$, the fraction can be rewritten in terms of x. Using symbolic manipulation to simplify, we can solve the problem.

$$\frac{123\,456}{(123\,457)^2-(123\,456)(123\,458)} = \frac{x}{(x+1)^2-(x)(x+2)} = \frac{x}{x^2+2x+1-(x^2+2x)} = \frac{x}{1} = x$$

The fraction reduces to just x, so the answer is $x = 123\,456$.

Problem 6: The five diagonals hit all but three vertices of the polygon—the vertex from which the diagonals are being drawn and the two adjacent vertices. Thus, there are eight vertices, so there must be eight sides.

Problem 7: The difference between the fifth term and third term is 8. Since this is an arithmetic sequence, the difference between the fifth and third terms will be the same as the difference between the third and first terms. Thus, the first term is $15-8=7$.

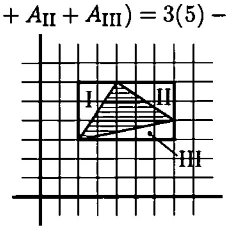
Problem 8: The total height of the steps' risers is $4 \times 7 = 28''$. Because the slope is required to be 1:12, the ramp must be $28 \times 12 = 336''$ long. The difference between the length of the ramp and the length of the steps is $336 - 4(11) = 292''$.

Problem 9: If all 96 seats were used to make two-wheel bikes, then 192 wheels would be needed. Since there are 212 wheels, 20 wheels would be left. Hence, 20 of the bicycles must have a third wheel.

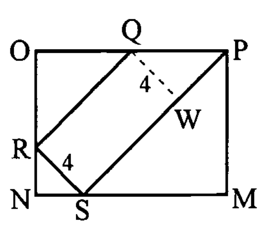
Problem 10: The number required must be 1 more than a multiple of the least common multiple of 2, 3, 4, 5 and 6. The least common multiple of these five numbers is $2 \times 2 \times 3 \times 5 = 60$ since the prime factorization of all five numbers occurs in that number. One more than 60 is 61, which is not a multiple of 11. But one more than $2 \times 60 = 120$ is 121, which is a multiple of 11.

WORKOUT 4

Problem 1: From the figure shown, the area of the triangle is equal to the area of the entire rectangle minus the areas of regions I, II and III. Hence, $A_{\text{rect}} - (A_{\text{I}} + A_{\text{II}} + A_{\text{III}}) = 3(5) - \left(\frac{1}{2}(3)(2) + \frac{1}{2}(3)(2) + \frac{1}{2}(5)(1)\right) = 6.5$ square units.



Problem 2: $\triangle$RNS is a 45-45-90 triangle, so $RN = NS = 2\sqrt{2}$. Therefore, $RS = \sqrt{2} \times 2\sqrt{2} = 4$. Likewise, QW = 4 also, since we draw parallelogram RQWS.



Because MNOP is a rectangle, $\angle$ROQ is a right angle. Further, RO = OQ. Consequently, $\triangle$ROQ is a 45-45-90 triangle.

Because the angles above and below it each measure 45°, $\angle$QRS is a right angle. The opposite sides ($\overline{RQ}$ and $\overline{PS}$) of trapezoid PQRS are parallel. Hence, $\angle$PSR is also a right angle.

With all the information gathered thus far, we can conclude that $\triangle$SMP is a 45-45-90 triangle. And from that, we can conclude that $m\angle$QPS $= 45°$. This is crucial, because it implies that $\triangle$PWQ is a 45-45-90 triangle. PW = RS = 4 in, so PQ $= 4\sqrt{2}$ in.

Because RO = OQ = QP, all three segments have length $4\sqrt{2}$ in. The rectangle MNOP, then, has length $8\sqrt{2}$ in and width $6\sqrt{2}$ in. Its area is 96 in^2. Further, all of the information acquired tells us that the area of $\triangle$RNS is 4 in^2, the area of $\triangle$SMP is 36 in^2, and the area of $\triangle$ROQ is 16 in^2. The area of trapezoid PQRS, then, is $96-4-36-16=40$ in^2.

Problem 4: The volume of a three-dimensional object is equal to the area of its base times its height. The area of the trapezoid, which serves as the base, is $\frac{12+17}{2} \times 33 = 478\frac{1}{2}$ in^2. The height is 8 in. The volume of the box is $8 \times 478\frac{1}{2} = 3828$ in^3.

Problem 5: To start, we know that 4, 6, and 8 must be the tens digits (since each of them is even, none can be the units digit of a prime number). So the possible numbers are 41, 43, 49, 61, 63, 69, 81, 83 and 89. But $49 = 7^2$, $63 = 7 \times 9$, $81 = 9^2$, and $69 = 3 \times 23$. Then, since each digit can only be used once, we are forced to throw out 41 and 83, leaving $43 + 61 + 89 = 193$.

Problem 6: There are two possibilities for each question. The number of different combinations, then, is $2^{12} = 4096$.

Problem 7: Since the choices are sequential, we multiply the combinations, $\binom{2}{1}\binom{4}{2}\binom{3}{2} = 2 \times 6 \times 3 = 36$.

Problem 8: One-tenth of the 20 members produce one-third of the output. Hence, the other 18 members produce $1 - \frac{1}{3} = \frac{2}{3}$ of the output. On average, then, each member of the 90% produces $\frac{2}{3} \div 18 = \frac{2}{54} = \frac{1}{27}$ of the output.

Problem 9: The volume of the cardboard box is $10.5 \times 7.875 \times 5 = 413.4375$. The volume of one can is approximately $\pi \times (1.3125)^2 \times 5 = 27.0594$. Therefore, the amount of empty space is $413.4375 - 12 \times 27.0594 \approx 88.7$.

Problem 10: The only point that we need to keep track of is $A = (7, 5)$. Its coordinates after reflection over the y-axis are $(-7, 5)$. The translation of 3 units to the left gives coordinates $(-10, 5)$, and the final reflection across the x-axis produces the final coordinates $(-10, -5)$. The sum of the abscissa and the ordinate of the transformed point is $-10 + (-5) = -15$.

WORKOUT 5

Problem 1: The area of each circle is πr^2, where r is the length of the radius. If R is the length of a longer radius and r is the length of a shorter radius, then the ratio is $\frac{\pi R^2}{\pi r^2} = \frac{R^2}{r^2}$. Thus, calculate the ratio of the squares of the radii for any two adjacent concentric circles, since the least ratio greater than 1 is desired.

Assume that the radius OP = 1 unit. Similarly, OQ = 2 units. Using the Pythagorean theorem, we find that OR $= \sqrt{2}$ units and OS $= \sqrt{5}$ units.

The ratios of the squares of the radii of the circles are, from greatest to least are $\frac{2^2}{(\sqrt{2})^2} = \frac{4}{2} = 2$, $\frac{(\sqrt{2})^2}{1^2} = \frac{2}{1} = 2$ and $\frac{(\sqrt{5})^2}{2^2} = \frac{5}{4}$. The least ratio greater than 1 is $\frac{5}{4}$.

Problem 2: The sector cut from the circle has a 120° angle, so the circumference and the area of the sector are $\frac{120}{360} = \frac{1}{3}$ of the circumference and area of the whole circle. The circumference of $\widehat{AQB} = \frac{1}{3}(12\pi) = 4\pi$ inches, and this is the circumference of the base of the cone. Hence, the radius of the base of the cone is $\frac{4\pi}{2\pi} = 2$ inches.

The slant height of the cone is the radius of the original circle, which is 6 inches. In $\triangle$ACQ, since one leg is 2 inches and the hypotenuse is 6 inches, the other leg—which is the height of the cone—is $\sqrt{6^2 - 2^2} = 4\sqrt{2}$. The base of the cone has area $\pi r^2 = 4\pi$, and its height is $4\sqrt{2}$, so the volume of the cone is $V = Bh = \frac{1}{3}(4\pi)(4\sqrt{2}) = \frac{16\pi\sqrt{2}}{3} \approx 23.695$ cubic inches. To the nearest cubic inch, the answer is 24.

Problem 3: Since both the numerator and denominator are randomly selected from the same set $\{1, 3, 5, 7, 9\}$ and the set has five elements, there are only $5 \times 5 = 25$ possible fractions. Fractions with terminating decimal expressions have denominators with only powers of 2 or 5. The only possible denominator for a reduced fraction in this set is 5. Thus, any of the five fractions with denominator 5 will have a terminating decimal. In addition, any of the five fractions with denominator 1 is a whole number, which also yields a terminating decimal. Finally, we must also look at fractions that could be reduced to give a terminating decimal. These are $\frac{1}{1}$ and $\frac{5}{5}$, which have already been counted, and $\frac{3}{3}$, $\frac{7}{7}$, $\frac{9}{3}$ and $\frac{9}{9}$. Thus, there are $5 + 5 + 4 = 14$ fractions that terminate, so there are $25 - 14 = 11$ fractions which do not. Therefore, the probability that the decimal representation of the fraction is not a terminating decimal is $\frac{11}{25}$.

Problem 4: If 373 0n5 is divisible by 21, then it is divisible by both 3 and 7. For a number to be divisible by 3, the sum of the digits must be divisible by 3. Thus, the value of n must make $3 + 7 + 3 + 0 + n + 5 = 18 + n$ divisible by 3. Because 18 is divisible by 3, n must also be divisible by 3, so n can be 0, 3, 6 or 9.

Using a calculator, we can check to see which of 373 005, 373 035, 373 065 or 373 095 is also divisible by 7. $373\,065 \div 7 = 53\,296$, so the answer is $n = 6$.

Problem 5: Since the base has area $9\,\text{in}^2$, the base edges must each have length $3\,\text{in}$. Consequently, since the faces are equilateral, all edges must have length $3\,\text{in}$. By Heron's Formula, each triangular face has area $\dfrac{9\sqrt{3}}{4} \approx 9.3897$. The total surface area of the four lateral faces is $4 \times \frac{9\sqrt{3}}{4} = 9\sqrt{3}$. Including the area of the base gives a total surface area of $9\sqrt{3} + 9 \approx 28.59$.

Problem 7: By folding the corners of the larger square in to the center, the folded parts will exactly cover the smaller square, showing that the area of the larger square is twice the area of the smaller square. The perimeter of the original square is 40 centimeters, so each side has length $10\,\text{cm}$, and the area is $100\,\text{cm}^2$. The area of the smaller square in Stage 2 is $\frac{1}{2}(100) = 50\,\text{cm}^2$, the area of the smallest square in Stage 3 is $\frac{1}{2}(50) = 25\,\text{cm}^2$, and so forth, so that the area of the smallest square in Stage 5 is $(\frac{1}{2})^4 \cdot 100 = 6.25\,\text{cm}^2$.

Problem 8: Given that $x! = 30y!$, we want the least value of $x + y$. To find values of x and y, a product of 30 must be contained in $x!$ exactly once more than in $y!$. This can happen when $x = 6, y = 4$ or when $x = 30, y = 29$. Of these possibilities, $x + y = 6 + 4 = 10$ is the least.

Problem 9: Placing one key on the ring, there are four ways to choose a key for the second spot, three ways to choose a key for the third spot, two ways to choose a key for the fourth spot, and one way to choose a key for the last spot. (Note that ABCDE is equivalent to BCDEA, CDEAB, DEABC and EABCD because this is a circular key ring. Hence, the first key must be placed, not chosen from the five available keys.) That gives $4! = 24$ ways. However, the key ring can be flipped to show that ABCDE and EDCBA are the same arrangement. Hence, every arrangement has been counted twice. Thus, there are $\frac{24}{2} = 12$ different arrangements.

Problem 10: Simplify:

$$\sqrt{900} \times \frac{1}{(0.04)^2} \times \frac{1}{0.75} = 30 \times 625 \times \frac{4}{3} = 25\,000$$

WORKOUT 6

Problem 1: After the first reduction of 40%, the price was 60% of the original price. The second reduction was 30% of the reduced price, which is an additional reduction of $60 \times 0.3 = 18\%$. The total reduction was $40 + 18 = 58\%$.

Problem 2: In palindromic numbers, the first and last digits are the same. Since the units digit in $n+1$ and $n-1$ will always be different, the first digit always has to be different as well. Also note that 2 must be added to $n-1$ to get $n+1$. Thus, we want numbers such that adding 2 to the units digit will also change the first digit. This can only happen when n is a power of 10. From 6 to 6 000 000, there are six powers of 10, namely 10, 100, 1000, 10 000, 100 000 and 1 000 000, so the answer is 6.

Problem 3: Because F is the midpoint of $\overline{\text{AE}}$, the height of $\triangle$ABF is equal to $\frac{1}{2}$ the length of $\overline{\text{BE}}$, which is the base of $\triangle$BEF. Further, the height of $\triangle$BEF is equal to $\frac{1}{2}$ the length of $\overline{\text{AB}}$, which is the base of $\triangle$ABF. Consequently, the areas of the triangles are equal, so the difference is 0 square units.

Problem 4: We know that the number that we are squaring needs to be even since the units digit needs to be even. Try even integers counting down from 100 until one is found that works. The first square with all even digits is $92^2 = 8464$.

Problem 5: In the first 24 numbers, the multiples of 3 would be eliminated: $3, 6, 9, \ldots, 24$. The multiples of 8 (which are not multiples of 3) are also eliminated: $8, 16$. The number of terms eliminated in the first 24 numbers will be the same number of terms eliminated in every subsequent set of 24 numbers. In the first 24 numbers, 10 terms have been eliminated, leaving 14 terms. Through 48 numbers, then, 20 terms will have been eliminated, leaving 28. Thus, the 34th term will be the 6th term in H after 48. The sequence continues $49, 50, 52, 53, 55, 58$, so 58 is the 34th term.

Problem 6: There are eight spots in which to place coins, and four of these spots will be filled with pennies. Hence, there are $\binom{8}{4} = 70$ ways to arrange the coins. If a penny is at each end, there are six spots in between, and four of these in-between spots will be filled with dimes. So, there are $\binom{6}{4} = 15$ ways to arrange the coins with a penny at each end. The probability of a random arrangement having a penny at each end, then, is $\frac{15}{70} = \frac{3}{14}$.

Problem 7: The first term and seventh terms are both 8. Arbitrarily, we can call the second term x. Then, the third term is $8 + x$, the fourth term is $8 + 2x$, the fifth term is $16 + 3x$, the sixth term is $24 + 5x$, and the seventh term is $40 + 8x$. But the seventh term is 8, so $40 + 8x = 8$, or $x = -4$. The fifth term is $16 + 3x = 16 + 3(-4) = 4$.

Problem 8: Four radii make up the length 20 cm, so each radius is 5 cm. Hence, the width of the box is 5 cm, and the area of the box is $5 \times 20 = 100\,\text{cm}$. The area of the two semicircles is equal to the area of a circle of radius 5 cm which is $25\pi \approx 78.5\,\text{cm}^2$. The area of the shaded region is $100 - 78.5 = 21.5\,\text{cm}^2$.

Problem 9: Work backwards through the fraction.

$$4 + \frac{8}{1 + \frac{1}{2 + \frac{1}{4 + \frac{1}{2}}}} = 4 + \frac{8}{1 + \frac{1}{2 + \frac{2}{9}}} = 4 + \frac{8}{1 + \frac{9}{20}} = 4 + \frac{160}{29} = \frac{116 + 160}{29} = \frac{276}{29} \approx 9.52$$

WORKOUT 7

Problem 2: Most of the numerators and denominators cancel out, leaving $\frac{1}{56} = \frac{1}{2x}$, so the solution is $x = 28$.

Problem 3: Perform the unit conversion:

$$\frac{4400\,\text{ft}}{1\,\text{min}} \times \frac{60\,\text{min}}{1\,\text{hr}} \times \frac{1\,\text{mi}}{5280\,\text{ft}} = \frac{4400 \times 60}{5280} = 50\,\text{mph}$$

Problem 4: The area of the entire equilateral triangle is $3 \times 30 = 90\,\text{cm}^2$. Let x be the length of a side of the equilateral triangle. Using Heron's formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of $\triangle$ABC, we get

$$90 = \sqrt{\frac{3x}{2} \cdot \frac{x}{2} \cdot \frac{x}{2} \cdot \frac{x}{2}}$$
$$90 = \sqrt{\frac{3}{16}x^4}$$
$$90 = \frac{x^2}{4}\sqrt{3}$$
$$\frac{360}{\sqrt{3}} = x^2$$
$$120\sqrt{3} = x^2.$$

So, $x = 120\sqrt{3} \approx 14.4$, and the perimeter is approximately 43 cm.

Problem 5: The probability of getting a red on the first draw is $\frac{3}{18}$; the probability of getting a green on the second draw is $\frac{5}{17}$; the probability of getting a blue on the third draw is $\frac{6}{16}$; and the probability of getting a white on the fourth draw is $\frac{4}{15}$. Multiplying these gives $\frac{1}{204}$. However, this is the probability that one of each color was selected *in the order stated.* We could have gotten one of each color in many different ways. In fact, there are $4! = 24$ ways to get one of each color. Thus, the probability is $24 \times \frac{1}{204} = \frac{2}{17}$.

Problem 6: If each edge has length 4 in, then each face is an equilateral triangle with area $A = \frac{1}{2} \times 4 \times 2\sqrt{3} = 4\sqrt{3}\,\text{in}^2$. Since there are eight faces, the total surface area is $8 \times 4\sqrt{3} = 32\sqrt{3} \approx 55.4\,\text{in}^2$.

Problem 7: Study the squares of one- and two-digit numbers to find which digits could be repeated in the ones and tens places. Perfect squares with units digit 1 always have an even number in the tens place. No digit squared gives 2, 3, 7 or 8 in the units place. Perfect squares with units digit 5 always have a 2 as the tens digit. Perfect squares with units digit 6 always have an odd number as the tens digit. Perfect squares with units digit 9 always have an even number as the tens digit. This logic leaves only 4 as a possibility. Note that perfect squares with units digit 4 always have an even number in the tens place, so 4 could work. Can a six-digit perfect square be found that has the digit 4 in the hundreds, tens and units places?

If such a number exists, it lies between 100 000 and 999 999. The square root of the number lies between $\sqrt{100\,000} \approx 316$ and $\sqrt{999\,999} \approx 999$. The square root must have a 2 or 8 in the units place since its square ends in 4. Further, if the square root ends in 2, it must end in 12 or 62, and if it ends in 8, it must end in 38 or 88. Thus, there are four possibilities with each hundreds digits 3–9. There are three such numbers: $538^2 = 289\,444$, $462^2 = 213\,444$, and $962^2 = 925\,444$. Hence, the tens digit is 4.

Problem 8: To maximize the effect of the decimal point, we should divide some preliminary result by .1, which is equivalent to multiplying by 10. Further, the 0 is neutralized if it is subtracted. After using 0 and 1, the least value remaining is 2, so we should add 2. It may seem weird that we want to add a small value, but addition is not nearly as effective as multiplication in attaining a large result. Adding 2 leaves the digits 3–9 for multiplication. Trial and error with a calculator yields that $964 \times 8753 = 8\,437\,892$ is the largest product that can be reached with these digits. Hence, the greatest attainable value is $964 \times 8753 \div .1 + 2 - 0 = 84\,378\,922$.

Problem 9: There are two direct paths from Annandale to Carderock, and three ways from Carderock to Ballston, giving six ways. There is one direct route to Epps and three ways from Epps to Ballston, adding an additional three ways, taking the total to nine. Finally, there are six more ways through Dale City (symmetrical to the ways through Carderock), so there is a total of 15 ways.

Problem 10: Start with the distance formula:

$$\sqrt{\left(3-(-2)\right)^2 + (p-4)^2} = 5\sqrt{5}$$
$$\sqrt{25 + p^2 - 8p + 16} = 5\sqrt{5}$$
$$\sqrt{41 + p^2 - 8p} = 5\sqrt{5}$$
$$p^2 - 8p + 41 = 125$$
$$p^2 - 8p - 84 = 0$$
$$(p-14)(p+6) = 0$$

So the solutions are $p = 14$ and $p = -6$, and their sum is 8.

WORKOUT 8

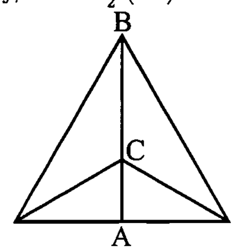
Problem 1: The ratio of the heights (a linear measure) of the cones is $\frac{2}{5}$, so the ratio of the volumes will be in the ratio $\left(\frac{2}{5}\right)^3 = \frac{8}{125}$. The volume of the larger cone is $27\,\text{cm}^3$, so the volume of the smaller cone is $27 \times \frac{8}{125} \approx 1.73\,\text{cm}^3$.

Problem 2: When an $8.5'' \times 11''$ paper is rolled vertically, a cylinder is formed that has circumference $8.5''$ and height $11''$. The circumference is given by $C = 2\pi r$. Thus, $2\pi r = 8.5$ and $r \approx 1.35$. So, $V = \pi r^2 h \approx 62.95\,\text{in}^3$.

When an $8.5'' \times 11''$ paper is rolled horizontally, a cylinder is formed with circumference $11''$ and height $8.5''$. So, $11 = C = 2\pi r$, and $r \approx 1.75$. Thus, $V = \pi r^2 h \approx 81.74\,\text{in}^3$.

The difference between the two cylinders' volumes is $81.74 - 62.95 = 18.79\,\text{in}^3$.

Problem 4: Each face of a regular tetrahedron is an equilateral triangle. The height of each face is equal in length to AB in the figure below. Let b represent the length of a base edge of the triangle below. Then the height, which is the longer leg of a 30-60-90 triangle with shorter leg $\frac{1}{2}b$, has length $\frac{\sqrt{3}}{2}b$. The area of the equilateral triangle, then, is $\frac{\sqrt{3}}{4}b^2 = 15.6$, which gives $b \approx 6.00$. Consequently, $\text{AB} = \frac{\sqrt{3}}{2}(6.0) \approx 5.20$.



Similarly, AC is the shorter leg of a 30-60-90 triangle with longer leg $\frac{1}{2}b$. Hence, $\text{AC} = \frac{b}{2\sqrt{3}} \approx 1.73$.

The volume of a tetrahedron is equal to $\frac{1}{3}Bh$, where B is the area of base (which we already know) and h is the height of the tetrahedron. The height of the tetrahedron is equal to the longer leg of a right triangle which has a shorter leg equal in length to AC and a hypotenuse equal in length to AB. Hence, $h = \sqrt{(5.20)^2 - (1.73)^2} \approx 4.90$. The volume, then, is $V = \frac{1}{3}(15.6)(4.90) \approx 25.48$, which, to the nearest tenth, is $25.5\,\text{in}^3$.

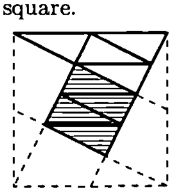
Problem 5: There are 25 pennies to 1 quarter, so simply divide 13.5 billion by 25. $13.5 \div 25 = 0.54$. Thus, the answer is 0.54 billion, which is written as 5.4×10^8 in scientific notation.

Problem 6: A diagonal of the square is $\sqrt{90^2 + 90^2} = 90\sqrt{2} \approx 127.28\,\text{ft}$. The intersection of the diagonals occurs 63.64 ft from home plate. The pitcher's mound is 60.5 ft from home plate. The distance from the mound to the intersection of the diagonals is $63.64 - 60.5 = 3.14\,\text{ft}$, and the distance from the intersection to first base is 63.64 ft. These two lengths are the legs of a right triangle, because the diagonals intersect at a right angle. Using the Pythagorean theorem, then, the distance from the mound to first base is $\sqrt{(3.14)^2 + (63.64)^2} \approx 64\,\text{ft}$.

Problem 7: There are 6 two-digit perfect squares, 16, 25, 36, 49, 64 and 81. The sums of the digits of these numbers are 7, 7, 9, 13, 10 and 9, respectively. There are $\binom{6}{2} = 15$ ways to choose two numbers from the first list. Numbers can only be chosen in two ways—if both numbers chosen have a digital sum of 7, or if both numbers chosen have a digital sum of 9. The probability, then, is $\frac{2}{15}$.

Problem 8: To guarantee receiving 10 puzzles, consider the worst case scenario. There are 10 varieties (call them A–J) which come in packages of 15 puzzles. In the worst case, there will be 2 puzzles of 5 varieties and just 1 puzzle of the other 5 varieties—arbitrarily, say that there are 2 puzzles of varieties A, B, C, D and E and 1 puzzle of F, G, H, I and J in the first package purchased. Still considering worst case, then, there will be 2 puzzles of varieties F, G, H, I and J and 1 puzzle of A, B, C, D and E in the second package. Hence, in the worst case, buying two packages guarantees that at least 3 puzzles of each variety will have been purchased. Hence, six packages will give $3 \times 3 = 9$ puzzles of each variety. Buying a seventh package will then ensure that a tenth puzzle of at least one variety has been purchased.

Problem 9: In the figure below, one-fourth of the unshaded area is divided into four congruent triangles. (These triangles were constructed by drawing lines parallel to the lines in the figure, so they are definitely congruent.) The total unshaded area, therefore, consists of 16 triangles. Similarly, the shaded area is divided into four triangles. Hence, the shaded area represents $\frac{4}{16+4} = \frac{1}{5}$ of the entire square.



Problem 10: The minimum perimeter is found in the rectangle which most closely resembles a square. Hence, the minimum perimeter is found by choosing the factors of 72 which are as close to equal as possible, which are 8 and 9. The minimum perimeter is $2(8+9) = 34$. The maximum perimeter, conversely, occurs when the factors are as unequal as possible. The maximum perimeter, then, is $2(1+72) = 146$. The difference is $146 - 34 = 112$ cm.

WORKOUT 9

Problem 1: A number is divisible by 4 if the number represented by its last two digits is divisible by 4. For instance, we know that 1252 is divisible by 4 because 52 is divisible by 4, and we know that 1233 is not divisible by 4 because 33 is not. The possibilities for the last two digits, then, are 12, 24, 32 and 52. There are four possible pairs for the last two digits. For the other three digits, there are $3 \times 2 = 6$ permutations for each number. So, there are $6 \times 4 = 24$ numbers.

Problem 2: The number $33ab6$ is between 33 006 and 33 996. The square root of 33 006 is 181.68. The square root of 33 996 is 184.38. So, the square root of 33ab6 must be 182, 183 or 184. Since the units digit of 182^2 is 4 and the units digit of 183^2 is 9, these cannot be the square root of our number. Thus, the only possibility is 184. Since $184^2 = 33\,856$, then $a = 8$ and $b = 5$, which gives $a + b = 8 + 5 = 13$.

Problem 3: Each edge of the cube is 6 cm, so the diagonal of the face is $6\sqrt{2}$ cm. Hence, the distance from A to B is the hypotenuse of a right triangle with one leg of length 6 cm (an edge of the cube) and the other leg of length $3\sqrt{2}$ cm (half the length of the diagonal on which A lies). Using the Pythagorean theorem, the distance from A to B is $\sqrt{6^2 + (6\sqrt{2})^2} \approx 7.3$ cm.

Problem 4: Given that $\sqrt{ab} = 6$ and $\frac{a+b}{2} = 10$, it must be that $ab = 36$ and $a + b = 20$. Thus, we need two integers whose sum is 20 and whose product is 36. The factor pairs of 36 are $(1,36), (2,18), (3,12), (4,9), (6,6)$. The only factors that add to 20 are 2 and 18. Their positive difference is 16.

Problem 5: There are $9 \times 9 = 81$ possible sums of two numbers 1–9. There are no numbers that can be added to 1 to yield a sum greater than 10. There is one number (9) that can be added to 2 to yield a sum greater than 10. There are two numbers that can be added to 3, three numbers that can be added to 4, and so on, up to eight numbers that can be added to 9, which yield sums greater than 10. The number of sums greater than 10 is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$. The probability is $\frac{36}{81} = \frac{4}{9}$.

Problem 6: The front row of the top level of the cube has 3 pairs of glued faces. Since we need four of these rows to make 1 level of the cube, there are 12 pairs of glued faces for each row of this level. However, we must then glue these 4 rows together to complete the level, making 12 more glued pairs. Then, each level has 24 glued pairs, and since there are 4 levels, that makes $24 \times 4 = 96$ glued pairs. Now, the levels must be glued together. There are 16 unit cubes in each level. These 16 must be paired with another 16 to glue two levels together, and this will be repeated 3 times. This makes an additional $16 \times 3 = 48$ pairs, for a total of $96 + 48 = 144$ pairs.

Problem 8: Every third setting increases by a multiple of two, so the first four represent 100, $100r$, $100r^2$ and $100r^3$ for some ratio r, where $100r^3$ is the second numbered setting, 200. Hence, $100r^3 = 200$, so $r = \sqrt[3]{2}$. Therefore, the setting after 800 is $800 \times \sqrt[3]{2} \approx 1008$.

Problem 9: A regular hexagon consists of six equilateral triangles. If each side has length $2x$, the triangle has area $x^2\sqrt{3}$. There are six triangles, so the area of a regular hexagon with side length $2x$ is $6x^2\sqrt{3}$.

When the midpoints of the hexagon are connected, a smaller regular hexagon is formed. The sides of the six triangles which comprise this hexagon have length $x\sqrt{3}$, and the area of each of these six triangles is $\frac{3x^2\sqrt{3}}{4}$. So, the area of the smaller hexagon is $\frac{9x^2\sqrt{3}}{2}$.

The ratio of the smaller hexagon to the larger hexagon is

$$\frac{\frac{9x^2\sqrt{3}}{2}}{6x^2\sqrt{3}} = \frac{3}{4}.$$

Problem 10: The five angles A–E intercept five arcs which, when combined, form a complete circle. The measure of a circle is 360°. The measure of the angle formed by two chords which intersect on the circumference of a circle is equal to $\frac{1}{2}$ the measure of the intercepted arc. The five angles combine to intercept an entire circle, so the number of degrees in the combined measure of the angles is $\frac{1}{2} \times 360 = 180°$.

Problem Index to the Warm-Ups and Workouts

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