DOCUMENT RESUME

ED 430 775 SE 061 959

AUTHOR	Berenson, Sarah, Ed.; Dawkins, Karen, Ed.; Blanton, Maria, Ed.; Coulombe, Wendy, Ed.; Kolb, John, Ed.; Norwood, Karen, Ed.; Stiff, Lee, Ed.
TITLE	Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 1 (20th, Raleigh, NC, October 31-November 3, 1998).
INSTITUTION	International Group for the Psychology of Mathematics Education. North American Chapter.
PUB DATE	1999-00-00
NOTE	409p.; For Volume 2, see SE 061 960.
AVAILABLE FROM	ERIC/CSMEE, 1929 Kenny Road, Columbus, OH 43210.
PUB TYPE	Collected Works - Proceedings (021)
EDRS PRICE	MF01/PC17 Plus Postage.
DESCRIPTORS	Algebra; Calculus; Cultural Influences; Educational Assessment; *Educational Psychology; Educational Research; Educational Technology; Elementary Secondary Education; Epistemology; Functions (Mathematics); Geometry; Higher Education; Knowledge Base for Teaching; *Mathematics Education; Number Concepts; Problem Solving; Sex Differences; Social Influences; Statistics; Teacher Education; Thinking Skills
IDENTIFIERS	Representations (Mathematics)

ABSTRACT

This conference proceedings contains three plenary session reports, 12 working group and 79 research reports, 35 short oral reports, 60 poster session reports, and two discussion group reports. The titles of all papers (excluding "short orals", "posters", and brief discussion group reports) are: (1) "On Relationships between Psychological and Sociocultural Perspectives" (Pat Thompson and Paul Cobb); (2) "On Theory and Models: The Case of Teaching-In-Context" (Alan H. Schoenfeld); (3) "Building Mathematical Structure within a Conjecture Driven Teaching Experiment on Splitting" (Jere Confrey); (4) "The Role of Advanced Mathematical Thinking in Mathematics Education Reform" (M. Kathleen Heid, Joan Ferrini-Mundy, Karen Graham, and Guershon Harel); (5) "Visions of Algebra in Diverse Instruction" (David Kirshner, Carolyn Kieran, Tom Kieren, and Analucia D. Schliemann); (6) "Representing Algebraic Relations Before Algebraic Instruction" (Analucia D. Schliemann); (7) "As It Happens: Algebra Knowing in Action: The Polynomial Engineering Project" (Tom Kieren); (8) "Theories and Experiments in Collegiate Mathematics Education Research" (Ed Dubinsky, Francisco Cordero, Joel Hillel, and Rina Zazkis); (9) "Gender and Mathematics: Integrating Research Strands" (Suzanne Damarin, Diana Erchick, Jere Confrey, Dorothy Buerk, Linda Condron, Ruth Cossey, Laurie Hart, Peter Appelbaum, and Patricia Brosnan); (10) "Geometry and Technology" (Doug McDougall, Dan Chazan, Chronis Kynigos, and Rich Lehrer); (11) "Learning to Reason Probabilistically" (Carolyn Maher, Susan Friel, Cliff Konold, and Robert Speiser); (12) "Research on Rational Number, Ratio and Proportionality" (Tom Post, Kathleen Cramer, Guershon Harel, Thomas Kieren, and Richard Lesh); (13) "Representations and Mathematics Visualization" (Fernando Hitt, James Kaput, Norma Presmeg, Luis Radford, and Manuel Santos-Trigo); (14) "Using Socio-Cultural Theories in Mathematics Education Research" (Judit

+++++ ED430775 Has Multi-page SFR---Level=1 +++++

Moschkovich, Karen Fuson, Yolanda de la Cruz, Joanna Masingila, and Marta Civil); (15) "Developing a Conceptual Framework for Mathematics Teaching" (Martin Simon, Deborah Ball, Rijkje Dekker, and Susan Jo Russell); (16) "Teacher Change in a Reform Calculus Curriculum: Concepts Related to the Derivative" (Thomas Fox); (17) "A Logo-Based Microworld as a Window on the Infinite" (Luis Moreno-Armella and Ana Isabel Sacristan R.); (18) "Sociomathematical Norms and Student Autonomy in Calculus II Honors Students" (Chris L. Rasmussen and Karen D. King); (19) "The Role of a Formal Definition in Nine Students' Concept Image of Derivative" (Michelle J. Zandieh); (20) "What Kind of Notation Do Children Use to Express Algebraic Thinking?" (Carol W. Bellisio and Carolyn A. Maher); (21) "Effects of Algebra Instruction on the Recognition of the Mathematical Structures of Word Problems" (Emily D.F. McFadden); (22) "Middle School Students' Algebraic Reasoning: New Skills and Understandings from a Reform Curriculum" (John P. Smith III, Elizabeth A. Phillips, and Beth Herbel-Eisenmann); (23) "Developing an Imagistic Basis for Understanding Binomial Multiplication" (Erna Yackel and Diana Underwood); (24) "The Effects of Staff Development Focused on Assessment Practices" (Daniel J. Brahier); (25) "Case Studies of Preservice Elementary Teachers' Development in Mathematics Assessment" (Anne Raymond); (26) "Assessing and Documenting Student Knowledge and Progress in Early Mathematics" (Bob Wright, Rita Stewart, Ann Stafford, and Richard Cain); (27) "The Dualistic Nature of School Mathematics Discourse" (Eric J. Knuth and Dominic Peressini); (28) "Examining Teacher and Classroom Supports for Student Invention" (Elizabeth S. Senger); (29) "Tracing Children's Construction of Fractional Equivalence" (Elena Steencken and Carolyn Maher); (30) "Graphing of Discrete Functions versus Continuous Functions: A Case Study" (Fernando Hitt and Orlando Planchart); (31) "Perception of Graphs and Equations of Functions and Their Relationships in a Technology-Enhanced College Algebra Course" (Armando M. Martinez-Cruz); (32) "Understanding Mathematical Experience" (Tracy Noble, Ricardo Nemirovsky, Tracey Wright, and Cornelia Tierney); (33) "Unspecified Things, Signs, and 'Natural Objects': Towards a Phenomenological Hermeneutic of Graphing" (Wolff-Michael Roth); (34) "Re-Thinking Covariation from a Quantitative Perspective: Simultaneous Continuous Variation" (Luis Saldanha and Patrick W. Thompson); (35) "Analyzing Children's Length Strategies with Two-Dimensional Tasks: What Counts for Length?" (Jeffrey Barrett and Douglas Clements); (36) "How Has Measurement Been Taught in Mexico?" (Mariana Saiz); (37) "Understanding Angles from the Perspective of a High School Cerebral Palsy Student" (Kenneth L. Shaw and Paul Durden); (38) "Analyzing Students' Learning in Social Context: A Student Learns to Measure" (Michelle Stepha); (39) "The Learning of Concepts of Stochastic Implied in the Binomial Distribution by Means of the Use of Different Representations and Contexts" (Jesus Colin Miranda); (40) "Comparing Data Sets: How Do Students Interpret Information Displayed Using Box Plots?" (Susan N. Friel); (41) "Students' Statistical Thinking" (Graham Jones, Carol Thornton, Cynthia Langrall, Edward Mooney, Bob Perry, and Ian Putt); (42) "Tracing the Origins and Extensions of Mathematical Ideas" (Regina Kiczek and Carolyn Maher); (43) "Justifications to 5-8 Year-old Students' Responses to Decision Problems" (M.en C. Araceli Limon Segovia); (44) "Supporting Students' Reasoning about Data" (Kay McClain and Paul Cobb); (45) "An Analysis of Students' Statistical Understandings" (Maggie McGatha, Paul Cobb, and Kay McClain); (46) "Middle School Students' Misuse of the Phrase '50-50 Chance' in Probability Instruction" (James E.

Proceedings of the Twentieth Annual Meeting

North American Chapter of the International Group for the

Psychology of Mathematics Education

VOLUME 1

North Carolina State University
Raleigh, North Carolina, USA

ERIC® *Clearinghouse for Science, Mathematics, and Environmental Education*

Proceedings of the
Twentieth Annual Meeting

North American Chapter
of the International Group
for the

Psychology of Mathematics Education

Volume 1

PME-NA XX

October 31-November 3, 1998
North Carolina State University
Raleigh, North Carolina U.S.A.

Editors:
Sarah Berenson
Karen Dawkins
Maria Blanton
Wendy Coulombe
John Kolb
Karen Norwood
Lee Stiff

Published by
The ERIC Clearinghouse for Science,
Mathematics, and Environmental Education
Columbus, OH

Cite as:

Berenson, S.B,, Dawkins, K.R., Blanton, M., Coulombe, W.N., Kolb, J., & Norwood, K, Stiff, L. (Eds.). (1998). *Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Document development:

David L. Haury, ERIC/CSMEE Executive Editor
Linda A. Milbourne, ERIC/CSMEE Copyeditor
Niqui Beckrum & David Haury, Layout

Accession number: SE 061 830

This document and related publications are available from ERIC/CSMEE Publications, The Ohio State University, 1929 Kenny Road, Columbus, OH 43210-1080. Information on publications and services will be provided upon request.

ERIC/CSMEE invites individuals to submit proposals for monographs and bibliographies relation to issues in science, mathematics, and environmental education. Proposals must include:

- A succinct manuscript proposal of not more than five pages.
- An outline of chapters and major sections.
- A 75-word abstract for use by reviewers for initial screening and rating of proposals.
- A rationale for development of the document, including identification of target audience and the needs served.
- A vita and a writing sample.

History and Aims of the PME Group

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and of the North American Chapter (PME-NA) are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;

2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

Reviewers
1998 PME-NA Proceedings

Sergei Abramovich
Claudia Acuna
Sunday Ajose
Leslie Aspinwall
Cynthia Barb
Jeffrey Barrett
Babette Benken
George Bright
Debbie Bryant
Marilyn Carlson
Glenda Carter
Olive Chapman
Jennifer Chauvot
Victor Cifarelli
Marta Civil
Lawrence Clark
Jose Contreras
Franciso Cordero
Kelly Costner
Ann Crawford
Carlos Armando Cuevas
Suzanne Damarin
Jose De Leon Solorzano
Eugenio Diazbarriga
Thomas Edwards
David Feikes
Maria Fernandez
Eugenio Filloy
Jayne Fleener
Thomas Fox
Susan Friel
Aurora Gallardo
Melissa Hanzsek-Brill
Michael Hardy
Lynn Hart
Victor M. Hernandez L.
Susan Hillman
Fernando Hitt
Kimberly Hufferd-Ackles
Kathy Ivey
Graham Jones
Ann Kajander
Rochelle Kaplan
Richard Kitchen
Eric Knuth
John Kolb
Carolyn Krohn
Andrea Lachance
Susan Lamon
Hector Lara-Chavez
Lawrence Lesser
Yeping Li
M.en C. A. Limon Segovia
Joanne Lobato
Ana Locicero
Cheryl Lubinski
Joan Lukas
Armando Martinez-Cruz
Marguerite Mason
Kay McClain
Douglas McDougall
Maggie McGatha
Jean McGehee
Rodney McNair
Betsy McNeal
David Meel
Denise Mewborn
Diane Miller
Simon Mochon
Joyce Moore
Luis Moreno-Armella
German Munoz-Ortega
Patrick Mwerinde
Ricardo Nemirovsky
Tracy Noble
Karen Norwood
Rodolfo Oliveros
Albert Otto
Catherine Pearn
Erkki Pehkonen
Barbara Pence
John Penick
Jorge Peralta Samano
Fred Peskoff
Roland Pourdavood
David Pugalee
Luis Puig
Anne Quinn
Anne Raymond
Anne Reynolds
Beverly Rich
Martie Rockenbach
Teresa Rojano
Denise Rowell
Fabiola Ruiz
Alan Russell
Adalira Saenz-Ludlow
Manuel Santos
Roberta Schorr
Annie Selden
John Selden
David Shaffer
Kenneth Shaw
Paul Shotsberger
Rose Sinicrope
Jack Smith
Michelle Stephan
Lee Stiff
Frances Stroup
Sheryl Stump
Cheryl Thompkins
Carol Thornton
Guenter Toerner
Ron Tzur
Saleh Umaru
Marta Valdemoros
Laura Van Zoest
Draga Vidakovic
Sharon Walen
Lewis Walston
Janet Warfield
William Waters
Larry Watson
David Webb
Eric Weibe
Gideon Weinstein
Susan Westbrook
Grayson Wheatley
Eileen Williams
Terry Wood
Erna Yackel

PME-NA Steering Committee (1997-1998)

Sarah Berenson	Chair
Jane Swafford	Past-Chair
Barbara Pence	Treasurer
David Reid	Membership Secretary
Catherine Brown	Newsletter Secretary
Fernando Hitt	Member at Large
Manuel Santos	Member at Large
Maria Fernandez	Member at Large

ACKNOWLEDGEMENTS

The editors extend their heartfelt thanks to CRMSE staff members—Linda Steelman, Philomena Karol, and Claire Fraser—for their dedication and commitment to the preparation of these *Proceedings*.

1998 Program Committee

Sarah Berenson, Program Chair

Wendy Coulombe and Karen Dawkins, Conference Coordinators

Maria Blanton
John Kolb
Denise Rowell
Cheryl Thompkins
William Waters
Eileen Williams
Glenda Carter
Karen Norwood
Lee Stiff
Draga Vidakovic
Larry Watson
Lawrence Clark
John Penick
Frances Stroup
Lewis Walston
Susan Westbrook
Sunday Ajose
Jose Contreras
Maria Fernandez
Jeannie Hollar
Rick Kitchen
Douglas McDougall
Ron Preston
Anne Reynolds
Rose Sinicrope
Darlene Whitkanack
Leslie Aspinwall
Ann Crawford
Susan Friel
Robert Hunting
Anita Kitchens
Mercedes McGowen
Sid Rachlin
Manuel Santos
Jane Swafford
George Bright
Thomas Edwards
Fernando Hitt
Kathy Ivey
Armando Martinez-Cruz
Barbara Pence
David Reid
Ken Shaw
Tad Watanabe

PME Proceedings in the ERIC Database

ERIC Document Reproduction Service: (800) 443-ERIC
ERIC Clearinghouse for Science, Mathematics,
and Environmental Education (800) 276-0462

North American Chapter

1980	PME-NA 2/PME 4	California	ED 250 186
1981	PME-NA 3	Minnesota	ED 223 449
1982	PME-NA 4	Georgia	ED 226 957
1983	PME-NA 5	Montreal	ED 289 688
1984	PME-NA 6	Wisconsin	ED 253 432
1985	PME-NA 7	Ohio	SE 056 279
1986	PME-NA 8	Michigan	ED 301 443
1987	PME-NA 9/PME 11	Montreal	SE 055 633
1988	PME-NA 10	Illinois	SE 056 278
1989	PME-NA 11	New Jersey	SE 057 133
1990	PME-NA 12/PME 14	Mexico	Not yet avail.
1991	PME-NA 13	Virginia	SE 352 274
1992	PME-NA 14/PME 16	New Hampshire	SE 055 811
1993	PME-NA 15	California	ED 372 917
1994	PME-NA 16	Louisiana	SE 055 636
1995	PME-NA 17	Ohio	SE 057 176
1996	PME-NA 18	Florida	SE 059 001
1997	PME-NA 19	Illinois	SE 060 729

International

1978	PME 2	W. Germany	ED 226 945
1979	PME 3	England	ED 226 956
1980	PME 4	California	ED 250 186
1981	PME 5	France	ED 225 809
1982	PME 6	Belgium	ED 226 943
1983	PME 7	Israel	ED 241 295
1984	PME 8	Australia	ED 306 127
1985	PME 9	Netherlands	SE 057 131
1986	PME 10	London	ED 287 715
1987	PME 11	Montreal	SE 055 633
1988	PME 12	Hungary	SE 057 054
1989	PME 13	Paris	SE 056 724
1990	PME 14	Mexico	SE 058 665
1991	PME 15	Italy	SE 055 792
1992	PME 16	New Hampshire	SE 055 811
1993	PME 17	Japan	SE 055 789
1994	PME 18	Portugal	SE 055 807
1995	PME 19	Brazil	SE 057 135
1996	PME 20	Spain	SE 059 578
1997	PME 21	Finland	SE 061 119-121
1998	ONE 22	South Africa	Not yet avail.

Preface

The program for PME-NA XX was discussed and changes proposed by PME-NA members attending the October, 1997, annual meeting at Illinois State University. The theme of this conference is the richness and power of students' mathematical ideas where students are broadly defined to include children, adolescents, and adult learners. This theme is the focus of three plenary papers. Pat Thompson and Paul Cobb debate the merits of psychological frameworks versus sociocultural frameworks in their paper. Alan Schoenfeld discusses the theoretical implications of the Berkeley model for Teaching-in-Context along with standards to judge such models. Jere Confrey explicates a splitting-based analysis of multiplicative structures with examples of students who successfully build mathematics structure.

At the 1997 annual meeting, the Steering Committee proposed that preliminary proposals could be submitted in either English or Spanish, and this change in procedure was approved by the general membership. Another change proposed was to create 10 Working Groups with an appointed organizer and panel members as a means of increasing PME-NA attendance by senior researchers. The purpose of these working groups was to establish a community of researchers with common areas of expertise. Organizers of each working group have established goals and strategies to increase the scholarly activities within each of these 10 communities. It was expected that many of the working groups will continue to collaboratively pursue common research interests over the course of this year. The following working groups and organizers were established for pilot in 1998:

- Advanced Mathematical Thinking - Kathleen Heid
- Algebra - David Kirschner & Carolyn Kieran
- Collegiate Mathematics - Ed Dubinsky
- Gender and Mathematics - Suzanne Damarin & Diana Erchick
- Geometry and Technology - Douglas McDougall
- Probability and Statistics - Carolyn Maher
- Rational Number, Ratio, and Proportionality - Tom Post
- Representations and Visualization - Fernando Hitt
- Socio-Cultural Theories - Judit Moschkovitz & Karen Fuson
- Teacher Education - Martin Simon

Beyond the papers of the 4 plenary speakers and 10 working groups, there are papers from 2 discussion groups, 79 research reports, 40 short oral reports and 50 poster sessions. There were 232 proposals submitted for review. The acceptance rate for research reports was 56%. The research reports, discussion groups, short orals, and poster presentations are organized by topic following the pattern begun with the Proceedings of the 1994 PME-NA meeting. Proposals for all categories were blind reviewed by three reviewers with expertise in the topic of submission. Cases of disagreement among reviewers were refereed by a subcommittee of the Program Committee at North Carolina State University.

Submissions for the Proceedings were made on disk; read, edited, and formatted by the editors. The format of the papers was adjusted to make them uniform and to conform to the page limit specified in the documentation for manuscript submission.

The editors wish to express thanks to all those who submitted proposals, the reviewers of proposals, the PME-NA XX Steering Committee, and the PME-NA XX Program Committee. The Program Chair would like to extend special thanks to the mathematics and science education faculty at North Carolina State University for their support and generous contributions to make this a successful professional experience for the community of mathematics education researchers.

The Editors

Sarah B. Berenson, Chair of PME-NA
Karen R. Dawkins
Maria L. Blanton
Wendy N. Coulombe
John R. Kolb
Karen Norwood
Lee V. Stiff

CONTENTS

Contents of Volume 1

16

Geometric Thinking

18

Contents of Volume 2

24

28

PLENARY PAPERS

ON RELATIONSHIPS BETWEEN PSYCHOLOGICAL AND SOCIOCULTURAL PERSPECTIVES

Pat Thompson
Vanderbilt University
Pat.Thompson@Vanderbilt.edu

Paul Cobb
Vanderbilt University
CobbP@ctrvax.Vanderbilt.edu

Discussions of sociocultural perspectives of education and educative processes continue to occupy center stage in educational research and mathematics education. In part, this is a natural reaction to the not-so-distant predominance of strongly reductionist psychological theories drawing upon a correspondence between mental representation and an external mathematical reality. Also, one of us has written extensively about the difference between claiming that a perspective "has it right" and acknowledging the possibility of adopting different stances in regard to a given observation (Cobb, 1990, 1991, in press; Cobb, Yackel, & Wood, 1992). But tensions do exist in trying to reconcile the two perspectives so that one need not suffer "split-brain" syndrome in order to use both without contradiction (Confrey, 1991, 1995; Steffe, 1995).

In this paper we make public an ongoing discussion related to the compatibility of psychological and sociocultural perspectives, and our discussions of how either might be rethought to be more compatible with the other.

Pat

The movie *Contact* opens showing us the earth as seen from an orbiting satellite. The camera backs away slowly at first, then increasingly rapidly, showing the moon in orbit around the earth, then the earth-moon pair orbiting the sun. Mars appears to our left, then Jupiter, then Saturn. The planets and sun diminish in size as we leave the solar system, which itself becomes a speck against a sparkling background as we back away further, passing Alpha Centauri. We pass through interstellar dust as we approach the Milky Way's edge, then we exit the Milky Way and continue backing away until we see thousands of galaxies, then nebulae, and so on.

To me, a significant aspect of this opening was that at no moment did I feel like I'd made a jump in perspectives. I always had the feeling of moving through a continuous transition. Not once did I wonder about what I was seeing or how it fit within the overall transition. It was only when I thought of fixed states within a larger overall transition, such as from an image of a single cell to an image of thousands of galaxies, that I was startled by a sense of apparent disconnection. But the sense of apparent disconnection dissipates when we can imagine "zooming" continuously from one state to the next, keeping a coherent image of the transition.

We might describe any one perspective in isolation of the others in structural terms related to human experience (e.g., swirls, columns, dust, clouds, etc.). At the same time, it would be a challenge to describe the mechanics of an exploding star using cell-level vocabulary. But we can aspire to develop theories which articulate well enough across observations differing in orders of magnitude that we can translate among them while keeping a sense of underlying or overarching phenomena.

I find this image, of "zooming out continuously", to work metaphorically for making a distinction between a unified perspective and the coordination of multiple perspectives. A unified perspective is one which enables us to transition among seemingly disparate phenomena — phenomena which seem to need their own theories. Thermodynamics is one example of a unified perspective. For many years, heat energy, energy of falling objects, and the work of physical labor were treated as unrelated quantities. The genius of thermodynamics was that it reconceived the idea of energy so that measures of one form could be transformed into an equivalent number of units appropriate for another form (Klein, 1974).

The distinction between unified and multiple perspectives isn't a huge distinction, and it isn't new. The unification of quantum mechanics and the general theory of relativity was one of Einstein's major, unfulfilled efforts (Fritzsch, 1994; Hawkings, 1988), and that one of the obstacles to the unification is an absence of appropriate imagery in which phenomena in both might be grounded (Miller, 1987). Newell (1973) pointed out that what one takes as an object versus what one takes as process varies with one's grain of analysis. Paul's enormously powerful work on coordinating psychological and social perspectives (Cobb, 1990, 1995) makes a parallel point about coordinating different background theories as a way of looking at classrooms from different points of view.

What might be a little new in the above is an orientation toward establishing ways of thinking about phenomena that enable shifts between perspectives to be more like continuous zooming. Making the attempt to think of unifying metaphors may be useful for framing current questions about psychological versus social perspectives in mathematics education. To achieve a unification of psychological and social perspectives would mean that we become able to "zoom" out or in with respect to a set of problems without loosing sight of where we started. In particular, we could "zoom in" from what we see as patterns of sociocultural activity to seeing that same activity as an expression of a hugely complex set of interactions among reflectively acting, cognizing, remembering, interpreting, feeling individuals. Also, we could zoom out from what we see as a collection of individuals who vary (or not) with respect to some set of characteristics we've deemed of interest, and who we imagine interacting by various means and with various resources, to seeing that same collection as having stable and persistent characteristics that appear to be independent of individual participants.

Paul

Pat, I find your metaphor of zooming for coordinating perspectives (or perhaps levels of analysis) helpful. I assume that you have something like this in mind:

1. A student,
2. located in ongoing small group interactions,
3. located in an emerging classroom microculture,
4. located in the activity system of the school,
5. located in the practices of the local community,
6. located in the broader policy environment.

To start the conversation, I want to make two observation sparked for me by the metaphor. The first is to differentiate zooming and the nesting of settings from an alternative slant on the coordination of perspectives. As an illustration, imagine that we are analyzing video-recordings of a one-on-one teaching session between a researcher and a student. We might focus on the ways in which the student reorganizes his or her mathematical reasoning while interacting with the researcher. A psychological constructivist analysis of this type is, in effect, made from inside the interaction and is concerned with the student's interpretations of the researcher's actions. Alternatively, we might analyze the same video-recording by focusing on patterns and regularities in the ongoing interaction, and on the taken-as-shared meanings that the researcher and students jointly establish rather than on the student's (or teacher's) personal interpretations. A symbolic interactionist interpretation of this type is established from the outside and makes the interaction between the researcher and student and explicit object of analysis. As a further possibility, we might view the researcher and student representatives of different cultural traditions who are attempting to communicate. For example, we might contrast the suppositions and assumptions that the student makes as a consequence of her history of participation in particular cultural practices with those that the researcher makes about the teaching session as a consequence of her induction into a particular research tradition in graduate school. In an analysis of this type, which might be characterized as sociohistorical in nature, our position is not merely outside the local interaction, but is outside entire communities of practice.

My point in giving this example is to illustrate a case of coordinating perspectives in which the scale of the phenomenon to be explained does not change (at least on the surface). To be sure, zooming is still involved — from inside the ongoing interaction, to outside the local interaction, to outside broad cultural traditions. However, it is accomplished (usually implicitly) by the analyst as she switches from one perspective to another. If we think about the coordination of theoretical perspectives in such cases, the challenge of developing a "unified theory" involves integrating a number of well-established theoretical perspectives such as psychological constructivism, symbolic interactionism, and sociohistorical theory. The result would be something akin to a cosmology that purports to provide a way of explaining almost everything independently of situation and purpose. Aside from the issue of feasibility in light of conflicting epistemological assumptions, the quest for an over-arching theoretic scheme of this type is of little interest to me as a mathematics educator. To explain why, I turn to the second observation sparked by your zooming metaphor.

As I read the analogies you draw with the development of theory in physics, I found myself reflecting on a characteristic of official, public scientific (and mathematical) discourse that is incidental to your argument. As we are both aware, this discourse assumes an agent-less voice that masks the interests and purposes for which a theory was developed, and instead portrays the theory as the result of reading of the Book of Nature. At times, when I read attempts to synthesize, say, Piagetian and Vygotskian theory, I have a sense that this same orientation is involved. In my view, this orientation, which Shotter (1995) referred to as the lure of cosmology, should be avoided by mathematics educators. The type of work we do as we seek to contribute to the continual improvement of the learning and teaching of mathematics is not a spectator sport. Instead, co-participation is at the core of work in our field. We might, for example, co-participate in mathematical reasoning with a student during a one-on-one teaching session, or we might co-participate with a teacher and his students is the learning and teaching of mathematics during a classroom teaching experiment, or we might co-participate with a group of teachers in the development of a professional teaching community, or we might co-participate in the restructuring of a school or school system as we attempt to forge a common agenda with teachers and administrators. For me, it is essential that the theoretical constructs we use to make sense of what is happening in any of these cases capture our co-participation in the process of educational improvement. To put the matter even more directly, we have to avoid what might be termed split-brain syndrome wherein we co-participate in the educational process with students, teachers, and administrators, but then describe the experience of doing so in the agent-less voice of the ultimate observer.

Against the background of these observations (some would say diatribes), let me return to your zooming metaphor and formulate the issue as I see it. I hope it is clear that the various forms of co-participation listed above can easily be brought into correspondence with nested settings that I listed at the outset. On my interpretation, the issue you raise is that of developing a coherent set of interrelated theoretical constructs that enable us to make sense of the various levels of activity in which we might co-participate as we seek to contribute to the ongoing improvement of mathematics teaching and learning. In this context, coherent means that analyses of one level of activity can, at least in principle, be recast in terms of analyses of activity at other levels. This is, I believe, consistent with the spirit of your proposal. I would also add that an important criterion for me is that analyses of any level of activity feed back to inform our own (and hopefully others') decisions and judgments as we strive for improvement. The theoretical constructs used to develop such analyses therefore have to do work. They might best be viewed as conceptual tools that are specifically designed to support the ongoing process of change and innovation. And, in this process, the theoretical constructs would be modified and adapted in response to the pragmatic concerns and interests that are encountered. In addition to grounding theory to the multiple settings of mathematics learning and teaching, this openness to pragmatic concerns serves as an antidote to the lure of cosmology.

Now it's your turn at the plate. Does the issue as I have formulated it provide an adequate basis for our continuing conversation, or do you want to tweak it a little?

Pat

Thank you, Paul, for helping me elaborate my original idea. In doing so I think you take the conversation in interesting and productive directions I hadn't considered. I'd like to begin with your last point, on co-participation, then your early example of analyzing a videotaped interview from multiple perspectives, then my original zooming metaphor. This will be with the aim that we tease out some of the details needing attention if we're to actualize the connections we have in mind.

I remember learning in college that, to write scientifically, I should write with authority, and to write with authority often translated into writing in the passive voice. It is by employing this simple grammatical trick, writing in the passive voice, that we, as researchers and observers, turn our (certainly powerful!) personal insights into seemingly general truths read from "The Book of Nature." You said:

> For me, it is essential that the theoretical constructs we use to make sense of what is happening in any of these cases capture our co-participation in the process of educational improvement. To put the matter even more directly, we have to avoid what might be termed split-brain syndrome wherein we co-participate in the educational process with students, teachers, and administrators, but then describe the experience of doing so in the agentless voice of the ultimate observer.

This reminded me of Steir's (1991, 1995) proposal that people doing research in social settings attempt to capture their own contributions to the phenomena they investigate, being open to the possibility that there might not be anything to investigate had they not participated in creating the phenomena being studied. But even more, I read your statement as a call that we always attempt to speak in ways that allow readers to know where we, as observers, have positioned ourselves relative to what we are describing. Put another way, I interpret your suggestion as one that calls on us, as researchers, to always make clear in our text for whom we imagine ourselves speaking. In regard to viewing a videotape of a researcher and child and reporting our observations and analyses, we could imagine ourselves speaking for:

- a participant in a dialog, conveying that person's meanings and motivations;
- an observer of a dialog who has access to the participants' personal meanings and to each participant's ongoing interpretations of the other;
- an observer of a dialog who has access to the participants' personal histories and to the histories of groups with whom they identify themselves.

Your suggestion is especially powerful in its implications for making research easier or harder for outsiders to read. By striving to make clear for whom our text speaks, we help readers position themselves as they build images of the phenomena we describe. This, with one small exception, is consistent with your notion that sometimes we choose from among different theoretical perspectives without "zooming" between scales of analysis. The exception is one you noted yourself — that in moving from one perspective to another "...zooming is still involved — from inside the ongoing interaction, to outside the local interaction, to outside broad cultural traditions. However, it is accomplished (usually implicitly) by the analyst as she switches from one perspective to another."

So, in proceeding it might be useful to state what I see is our common position and restate the issue I'd hoped to raise with the metaphor of "zooming" among perspectives. I see our common position being that we need a way to imagine the amalgam of settings in which we situate our activities, observations, and analyses so that we can move across levels of analysis — across what you've described as psychological constructivism, symbolic interactionism, and sociohistorical theory — without lapsing into the passive voice, and thereby avoiding "agentless descriptions given by a universal observer". The issue I'd hoped to raise is that these "ways of imagining ..." will not come free. We must discuss and debate possible "ways of imagining" explicitly (Miller, 1987). I should also agree explicitly with another point you made — that, as mathematics educators, we should not lose sight that *our* actions are tightly bound up with a set of core problems having to do with the improvement of individuals' mathematics education. This, I believe, will keep us from trying to develop, as you say, a theory of everything.

I do not have in mind a particular "way of imagining", but I can suggest a starting point for the discussion. In fact, you suggested it by way of example:

> We might contrast the suppositions and assumptions that the student makes as a consequence of her history of participation in particular cultural practices with those that the researcher makes about the teaching session as a consequence of her induction into a particular research tradition in graduate school. In an analysis of this type, which might be characterized as sociohistorical in nature, our position is not merely outside the local interaction, by is outside entire communities of practice.

I suspect that analyses of participation and of practice will be particularly rich in possibilities for elaborating "ways of thinking" which enable one to move between levels of analysis in ways that insights drawn at one level inform our analyses at another. The reason I think this is that, in its common usage, "to participate," in its intransitive form, suggests an interface between an actor and a setting. At the same time, "practice", as a noun, suggests a stable form of activity within a group which need not be a common form of activity among members, but rather is a state of dynamic equilibrium among its inter-acting members. So, to me, by focusing on how we might understand or come to understand the ideas of participation and practice we address explicitly the question of how we can imagine individuals' activities and groups' characteristics in mutually supportive, compatible ways.

In closing this piece, I'd like to make explicit to persons reading our exchange something said by Salomon (Salomon, 1993). It is that sociocultural and scientific theorists tend to think of explanations differently. Sociocultural explanations tend to be oriented toward descriptions of intact systems having certain observed characteristics, where descriptions tend not to appeal to internal mechanisms which produce the observed characteristics. Scientific explanations tend to be more mechanistic — in the sense of aiming to produce models having components that interact according to certain principles and which produce, through interaction, the observed phenomenon. This is not to be confused with strong information processing models. Maturana captured the essence of modeling when he described it as rethinking the observed phenomenon so that you imagine from whence it arose.

> As scientists, we want to provide explanations for the phenomena we observe. That is, we want to propose conceptual or concrete systems that can be deemed intentionally isomorphic to the systems that generate the observed phenomena. (Maturana, 1978, p. 29)

I, personally, find the modeling perspective to be useful in that explanations we give tend to be less ad hoc than the former. The scientist's production of models also reflects the high value scientists place on explanations which support prediction. A byproduct of adopting a modeling point of view is that it forces us to examine our basic constructs, to ask "what do we mean" by such basic terms as "participation" and "practice."

I suppose I leave my part with the question to you, Paul, of whether you see this as a productive direction.

Paul

Pat, judging from your comments, I think that we are very much on the same page. In this response, I am going to address the last you point you raise first by discussing what we might means by such basic terms as participation and practice. My motivation for doing so is to attempt to clarify as much for myself as for others what we might be talking about when we throw these terms around. Given that this is an ongoing project, I would certainly welcome further probing and pushing on your part. Having taken a stab at addressing this issue, I will then focus on the points you make about modeling and explanation.

As you know. I and several colleagues[1] have been trying to develop the notion of a classroom mathematical practice for the last few years. Our motivation for doing so stems directly from the problems and issues we have encountered while conducting classroom teaching experiments. For example, in preparing for a teaching experiment, we outline a possible sequence of instructional activities by envisioning how students' mathematical learning might proceed as the potential sequence is enacted in the classroom. In doing so, we develop testable conjectures about both 1) possible learning trajectories, and 2) the specific means that might be used to support and organize that learning (Gravemeijer, 1994). The important point for our discussion is that these conjectures cannot be about the anticipated learning of each and every student in a class for the straight-forward reason that there are significant qualitative differences in their mathematical thinking at any point in time. In my view, descriptions of planned instructional approaches written so as to imply that all students will reorganize their reasoning in particular ways at particular points in an instructional sequence involve, at best, questionable idealizations. A problem that has arisen for us is therefore that of figuring out how to characterize the envisioned learning trajectories that are central to our work as instructional designers. In particular, if it does not make

[1] These colleagues include Janet Bowers, Koeno Gravemeijer, Kay McClain, Michelle Stephan, Joy Whitenack, and Erna Yackel.

sense to view them as trajectories for the learning of individual students, then what might they be trajectories of? Our current (and potentially-revisable) solution is to view a hypothetical learning trajectory as consisting of conjectures about the collective mathematical development of the classroom community. This proposal in turn indicates the need for a theoretical construct that enables us to talk explicitly about the mathematical learning of a classroom community.

- If we think of theoretical constructs as tools that are developed for particular purposes and interests, then additional design specifications for the theoretical tool that we need include. It should enable us to think about communal mathematical development over the extended periods of time that are covered by instructional sequences.
- It should enable us to make sense of what might be happening in classrooms over these time periods in such a way that the resulting analyses feed back to inform the ongoing instructional design effort.
- It should enable us to relate the collective mathematical activity of the classroom community to both the developing mathematical reasoning of the participating students and to the broader activity system of the school (see the nesting of settings discussed earlier).

My reason for suggesting this last criterion is again pragmatic. For example, when we make pedagogical decisions and judgments during a teaching experiment, we find it essential to attend to students' qualitatively different ways of interpreting and solving tasks, and in fact view that diversity as a primary resource upon which to capitalize when attempting to advance our pedagogical agenda. Further, we are all too aware from personal experience that events in the classroom do not occur in a social vacuum. Influences that we have found it necessary to take into account over the years include the students' prior instructional histories, the institutionalized procedures for assessing both students and teachers, the established norms of participation for teachers within the school community (i.e., their obligations to other teachers, administrators and parents), and the students' developing identities as members of groups within the student body. In light of these issues, I hope it is clear that while I am interested in coordinating levels of analysis, the purpose for me is not to develop an encompassing theoretical scheme as an end in itself. Instead, it is to come to grips with the types of issues that we find ourselves addressing in the course of our work.

It is against the background of these and other considerations that we have attempted to "hammer out" the notion of a classroom mathematical practice. Described in terms of this construct, an envisioned learning trajectory consists of an anticipated sequence of classroom mathematical practices together with conjectures about the means of supporting their evolution from prior practices. To clarify what we might mean by a mathematical practice, I will focus of three interrelated aspects: 1) The taken-as-shared purpose, 2) the norms for mathematical argumentation, and 3) the taken-as-shared ways of reasoning with tools and symbols. In doing so, I am going to give a brief example from a seventh-grade teaching experiment with which you are familiar that are focused on statistical data analysis. During this experiment, the students routinely used computer minitools prototyped in Java to compare univariate data sets. In an analysis of this experiment, we argued that the first mathematical practice that emerged as the students used one of these minitools involved exploring qualitative characteristics of collections of data points. In giving this characterization, we are claiming that the taken-as-shared purpose for analyzing data sets in this classroom was to identify qualitative trends or patterns. For example, in one instructional activity, the students attempted to determine which of two brands of batteries was superior by analyzing data on the life spans of ten batteries of each brand. A pattern identified by one student that was treated as significant by the teacher and other students was that all the batteries of one brand lasted more than 80 hours whereas two batteries of the other brand lasted considerably less than 80 hours. As this illustration indicates, part of the challenge when describing a mathematical practice is to clarify what mathematical activity might be about in a particular classroom at a particular point in time.

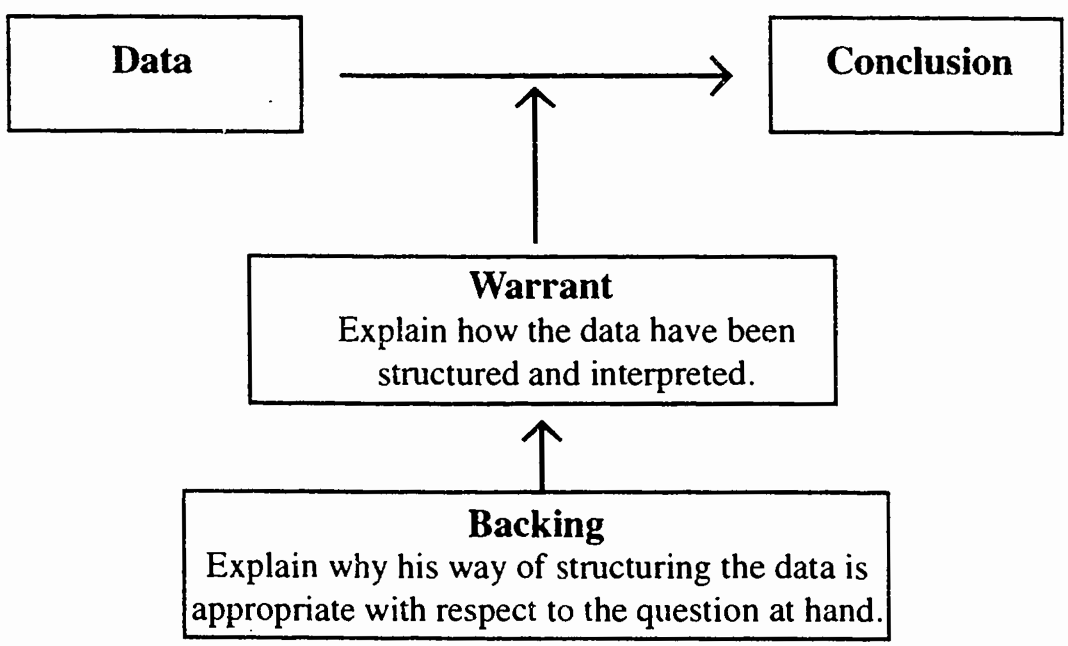


Figure 1. Toulmin's Justification Scheme

In this same analysis, we also argued that the teacher and students negotiated particular norms for mathematical argumentation. In terms of Toulmin's (1969) scheme, this can be represented as shown in Figure 1.

In the case of the sample solution, the student concluded from the data that one of the brands of batteries was superior. He also gave a warrant that explained why the data supported the conclusion when he said that he had used the computer minitool to partition the data sets at 80 hours and had noticed that two batteries of one brand lasted less than 80 hours. In addition, he gave a backing to indicate why his warrant and thus his method for comparing the two data sets should be accepted as having authority when he explained that he wanted a consistent battery that would last at least 80 hours. The norms of argumentation exemplified by this explanation both reflect the taken-as-shared purpose for analyzing data outlined above and serve to further clarify that purpose. In particular, searching for patterns was not an end in itself. Instead, the patterns identified by structuring data in a certain ways had to be justified with respect to the question at hand.

The norms for argumentation also relate to the third aspect of the mathematical practice we analyzed, the taken-as-shared ways of reasoning with the computer minitool, in that the students were developing arguments for a decision or judgment when they used the minitool to analyze data. In general terms, this last aspect of a mathematical practice is concerned both with the ways of using tools and symbols that are treated as legitimate in the classroom, and with what is reasoned about while doing so. In the case of the statistics teaching experiment, taken-as-shared ways of using the minitool to organize data included partitioning data sets, bounding the data points in particular intervals, and bounding clusters of data points. Further, the taken-as-shared ways of reasoning about data that was organized in these ways appeared to be additive rather than multiplicative (see Cobb, in press, for a more detailed discussion). It was for this reason that I in fact spoke of the students exploring qualitative characteristics of collections of data points rather than, say, of distributions. In our estimation, as we looked at public classroom discourse at the beginning of the teaching experiment, there was no indication that the teacher and students were concerned with how data sets were distributed in a statistical sense (cf: Konold, Pollatsek, Well & Gagnon, 1996). Instead, classroom discussions focused on the number of data points in particular intervals, or above or below a particular values.

Well, Pat, that is the best that I can currently do to say what I mean by a classroom mathematical practice. I should clarify that we have refined this notion as we have conducted a number of specific analyses. Part of the difficulty is therefore that of trying to explicate what we actually do in action while making sense of what might be going on in the classrooms in which we work. I would therefore anticipate that there are a number of implicit suppositions and assumptions that we are yet to dig out. In your language, how adequate is the above account in helping you build imagery for the phenomena we are attempting to describe (and how adequate is the construct itself given the purposes for which it is being developed)?

In considering the other term you mention, "participation", we have to address the issue of coordinating individual and communal perspectives on classroom events head on. In the way that I currently look at what is going on in classrooms, a student's mathematical reasoning *is* his or her way of participating in communal classroom practices. Obviously, this statement needs to be unpacked. When I speak of a student's mathematical reasoning, I am taking a psychological constructivist perspective that brings qualitative differences in students' thinking to the fore. In contrast, when I speak of participation in communal practices, I am taking a social perspective that situates the student's reasoning within an evolving classroom microculture. The above statement therefore involves a claim about how these two perspectives on a student's mathematical activity might be coordinated (and thus how the collective mathematical activity of the classroom community might be related to the developing mathematical reasoning of the participating students). We in fact take the relation between the two perspectives to be reflexive. This is an extremely strong relationship and does not merely mean that individual students' reasoning and the practices in which they participate are interdependent. Instead, it implies that one literally does not exist without the other (Mehan & Wood, 1975). What, from one perspective, is viewed as an individual act of reasoning is, from the other perspective, viewed as an act of participating in the communal practices of the classroom community.

I hope that it is clear from this brief account that the coordination at issue is not between individual students and the classroom community viewed as separate, sharply defined entities. Instead, the coordination is between two alternative ways of looking at and making sense of what is going on in classrooms. In other words, we are coordinating different ways in which we can interpret classroom events. What, from one perspective, are seen as the norms and practices of a single classroom community is, from the other perspective, seen as the reasoning of a collection of individuals who mutually adapt to each others actions. Whitson (1997) articulates this point as clearly as anyone when he proposes that we think of ourselves as viewing human processes in the classroom, with the realization that these processes can be described in either social or psychological terms. This formulation is, I believe, consistent with your discussion of the need to allow readers (or conversation partners) know where we have positioned ourselves relative to what we are describing.

In turning to consider your comments about modeling and explanation, I should clarify that my commitments as I try to understand what might be going on in classrooms are more than just to understand the particular case at hand. Instead, specific classrooms serves a paradigm cases as I try to develop ideas that might have more general relevance and yet remain rooted in the settings in which we co-participate with teachers and students. In the case of the statistics teaching experiment, for example, our more general concerns were to further develop the notion of a classroom mathematical practice and to deepen our understanding of the role that tool and symbol use can play in students' mathematical development. I would note that in an approach of this type, theorizing is not an abstract, esoteric game. Instead, it is a means of attempting to be more effective in supporting students' mathematical learning. As a consequence, the perennial problem of bridging the gap between theory and practice fails to materialize in that the resulting theoretical ideas do not stand apart from practice but are instead developed in the context in which they will be used.

As a further point, I want to question whether models of the type you describe that involve "components that interact according to certain principles and which ... produce the observed phenomenon" are necessarily the most useful for our purposes as mathematics educators. I assume that the primitives in such a model of a classroom community might be the teacher's and students' ways of interpreting each others' actions. When the model is "turned loose", broad pattern such as those that we point to when we speak of classroom mathematical practices might then emerge as epiphenomena in much that same way that patterns emerge in statistical data at the macro-level. A difficulty for me concerns what might be taken as a primitive in such a model. Earlier, I clarified that I take the relation between psychological and social perspectives and thus between individual students' reasoning and the practices in which they participate to be reflexive. Given this theoretical commitment, teachers' and students' reasoning are not seen to exist apart from their participation in communal practices, just as the practices are not seen to exist apart from their continual regeneration as teachers and students mutually adapt to each others' activities. Thus, in adopting this view, I would not treat individual students' reasoning as being more primitive than communal mathematical practices, or vice versa. Just as I would have difficulty with a theoretical position that portrayed students' reasoning as being determined by their participation in communal practices, so I would question an approach that treats communal practices as mere epiphenomena. In classroom teaching experiments, for example, our understanding of students' history of participation in classroom mathematical practices helps us explain their reasoning in exit interviews. This attention to history does not appear to be as relevant to the concerns of physicists and biologists when they think through how the primitives in the systems that they study will behave.

I think that I have said more than enough at this juncture. As you can see, I certainly found you comments both stimulating and provocative. Hopefully, this response will serve to move the conversation along.

Pat

Well, Paul, you certainly did move the conversation forward. Let me see if I can recap what I last said and your response to it. I suggested that it might be productive for us to discuss "ways to imagine" social and psychological phenomena that would support our ability to "zoom" between individual and social perspectives so that each truly becomes background for the other. I suggested that the ideas of participation and practice might be productive sites for this discussion, because the idea of participation seemed to entail a relationship between individuals and a group in which they are members, and activity of the group in which the individuals participate. I also drew a distinction between explanations common to sociocultural theories, which tend to describe social systems as unanalyzed wholes having various properties, and scientific explanations which are more analytic, breaking a system into component parts and postulating how those components might interact to produce the observed phenomena.

You agreed with the general thrust or my proposal, elaborating your ideas of practice and participation, and then you went further to explain by way of example that the idea of practice is important in your own work because it supports your goal of understanding and affecting what happens in classrooms. You closed by wondering what utility the activity of scientific modeling might have for mathematics education, stating your strong dislike for any approach that proposes communal practices as "mere epiphenomena".

To respond to all this is a challenge! I'll jump around a bit by first touching on the example of practice as employed in your current project and its affiliation with the notion of "taken as shared". Then I'll respond to your question about the utility of analytic models, and in doing so try to explicate a confusion I have which stems from your remarks about epiphenomena.

If I understand you correctly, you use "practice" in two senses — one which supports your attempt to express what you hope an instructional sequence produces. I had a horrible time formulating the previous sentence so that I wouldn't say something to which you would take immediate exception. My original inclination was to say, "... what you hope students learn," not meaning that you expect every student to learn what you state, but rather that you would be delighted if they did. In my understanding, you use "practice" in the context of instructional design almost heuristically — as a way to finesse the sticky problem of saying what you hope students learn without committing yourself to the impossible objective that every student learn it. It provides a way to imagine "the class, collectively" as if it were one person who could participate in every setting you might imagine being pertinent to the ideas and dispositions you want to address. If my interpretation is consistent with your intention, then this sense of "practice" is consistent with what I would, in other settings, call "cognitive goals of instruction" — the imagery, orientations, dispositions, mental operations, schemes, etc. that would enable a person to contribute to and partake of classroom conversations, activities, and tasks productively. I must stress again, though, that I am talking about *intention*, not expectation. I find this sense of "practice" quite powerful, for it allows me to think about not just what I want students to know, but also to think about ways they might think of the settings in which they find themselves that will be supportive of their desire to participate in ways which will contribute to other students' intellectual growth.

But I need help understanding what you mean by communal mathematical development and collective mathematical activity. On one hand, I can understand these phrases as referring to phenomena I might observe within the confines of a classroom which, when I leave them unanalyzed, strike me as having certain features. I see this as being parallel with observing a particular house not as a structure that evolved over time, emerging from the joint efforts of its builders and designers, but as an object having a certain color, shape, and size, and having certain accoutrements. A consumer could function quite adequately with the latter perspective; a designer could not. I also suspect that an experienced designer *could not* look at a house without a background image of the activity producing it. That is, I suspect that the notion of collective mathematical activity has at its center the characteristic of being an epiphenomenon. I'll return to this later. But first I'd like to point to another example.

Bransford et al. (in press) described two boys, one of whom could not read and one of whom suffered attention deficit disorder. The two cooperated in a cooking club by each relying on the other to compensate for his own deficit. In this setting, we could view the two as, communally, constituting a pretty effective cook. How does this example differ from the example of a house emerging from the communal efforts of its builders. In two ways. First, the house is the product of a group's activity, but we never thought of the house as somehow constituted by the crew. The house is analogous to a meal the boys produce. But another difference is that we view the house as having a permanence that the two boys acting together do not. The two boys acting together is more like the crew which produced the house. We imagine the crew as having, too, a permanence in the sense that we expect some members to leave and others to join without affecting the crew's overall competence. That is, we attribute a permanence to the crew — at least in terms of continuing competence and skill. But we don't attribute the same permanence to the communal competence of Bransford's boys, largely because we don't expect them to stay together in settings other than the cooking club, nor do we expect either boy's deficiency to be redressed by his activities in the cooking club.

My confusion, I think, is that I don't know how to think about communal activities as *other* than epiphenomena, at least in regard to the goal that instruction have some lasting effect. Being epiphenomenal, then, I won't know whether the communal activity is valuable unless I know something about the changes taking place within individual children so they may contribute to it. It seems possible, in principle, that a desired communal practice emerges, but few students are affected in ways that will allow them to contribute in other settings to making it emerge again.

In the same way that I don't know how to think of communal practice as other than epiphenomenal, at least to think of it in ways that matter to instructional design and students' learning, I'm afraid I don't know how to think of "taken as shared". On one hand, we could, like Voigt, (1994, 1996), mean that it is a statement about what an individual person thinks. In Voigt's usage, an idea is "taken as shared" when an individual person presumes other people think the same way as she does about some meaning or idea. It is the observed actor who we claim is doing the taking.

On the other hand, we could mean something in line with Lave (Forman, 1996; Lave, 1991), that when we imagine some meaning or practice as being "taken as shared", that we are making no claim at all about what members of a group think, believe, or mean. Rather, the claim that something is "taken as shared" is a claim that the group, as a single entity, seems to act *as if* it were one entity which thinks in some particular way. In other words, it is the *observer* who does the taking. "I take this group's behavior as if ..."

The examples from your statistics teaching experiment are helpful in one way, in that they clarify for me the kinds of things you see happening communally which inform your assessment of potential learning trajectories (i.e., they inform your evaluation of instructional design). But they are unhelpful in a very important way. On one hand, you contend that we cannot think of classroom mathematical practices as constituting something that each and every student will learn. On the on the other hand, you offer one student's activity as being illustrative of a practice you claim developed. It is in this sense that I see a misfit between theory and implementation. I would think that, to implement your idea of classroom mathematical practices in the conduct of mathematics education research, we would attempt to identify in classroom activities aspects of the class' taken-as-shared (in Lave's sense) activity that emerges because of a collage of behavior emanating from an interaction among students' taken-as-shared (in Voigt's sense) meanings. But this seems to point again to the need to think of mathematical practices as epiphenomena. Now, it may be that we must clarify our personal meanings of "epiphenomena". To me, it points to thinking of an observation as being the *result* of something else. I must be careful lest you think that I'm attributing reality to individuals in interaction and not to communal activity. Far from it. In that regard, I think Bishop Berkeley's (1963) famous dictum, "To exist is to be perceived", is very helpful. When we *see* communal activity, it exists.

In closing, I must say I couldn't agree more with your characterization of the reflexive relationship between social and psychological perspectives. However, for our purposes I think that, with respect to modeling and designing, the psychological perspective is more fundamental. This is for the simple reason that the groups within which children act do not persist. Students act within many groups, and they will join many others over their life. Therefore, we would be remiss not to address the question of how we hope to affect individual children so they are able to act productively in a variety of settings. That is, it is individual children who will persist over time, not the classes in which we view them or in which they act for relatively short periods of time. That is why while I agree completely with your characterization that, as perspectives, psychological and social perspectives are mutually constitutive — one perspective cannot exist without the other —I choose to view the psychological perspective as more fundamental. It aligns more explicitly with what I take as our fundamental goal of making a positive, lasting difference in students' lives after they leave our classrooms.

Paul

Wow, Pat, my immediate reaction is to disappear for a month and to develop a position paper as a means of clarifying my own thinking on the issues you raise. However, as we are under that gun, I will try to give quick responses to the various points you raise.

In talking about our use of the term practice in the context of instructional design, you say that:

> It provides a way to imagine "the class, collectively" as if it were one person who could participate in every setting you might imagine being pertinent to the ideas and dispositions you want to address. If my interpretation is consistent with your intention, then this sense of "practice" is consistent with what I would, in other settings, call "cognitive goals of instruction" — the imagery, orientations, dispositions, mental operations, schemes, etc. that would enable a person to contribute to and partake of classroom conversations, activities, and tasks productively. I must stress again, though, that I am talking about *intention*, not expectation.

Here, I believe that there is a mismatch in our interpretations in that, from my point of view, you have recast the notion of a classroom mathematical practice in individualistic terms. To tease out these difference, I will given an example from our ongoing work. Earlier, I mentioned a seventh-grade teaching experiment that focused on statistical data analysis. We are in fact currently in the process of planning for a follow-up eighth-grade teaching experiment that we will conduct with the same group of students in fall 1998. One of the mathematical ideas that we will focus on is that of co-variation, which includes but is not limited to correlation. An image that I have in mind as I think about possible instructional goals concerns how scatter plots might be talked about and used in public classroom discourse. In particular, we (currently) want scatter plots to be talked about and referred to as texts about the situations from which the data were generated. If this occurs, then it will be taken-as-shared that the aspects of a situation that were judged to be significant and were measured when generating the data co-vary, and that the specific nature of that co-variation is shown by the graph. This formulation of the instructional goal provides an initial orientation for myself and my colleagues as instructional designers and teachers. For example, it suggests that the cloud of dots on a scatter plot should explicitly be spoken about in classroom discussions as measures of aspects of a situation that are distributed in a (two-dimensional) space of values. We therefore have an initial, provisional sense of the types of conversations that we might want to support in the latter part of the teaching experiment.

I hope it is clear that in stating our instructional intent in this way, I am not thinking about the classroom community as if it were one person. Instead, I am thinking about what the teacher and students might be doing collectively. And in doing so, I am attempting to articulate my (potentially-revisable) image of the immediate social situation of individual students' mathematical development at the end of the experiment. In addition to formulating goals, part of the challenge when planning an experiment is to think through possible means of achieving these goals. In this regard, I noted earlier that this involves outlining both 1) a learning trajectory that might culminate with the mathematical practices that constitutes the envisioned goal, and 2) the specific means that will be used to support and organize that learning. In the case of the eighth-grade experiment, for example my colleague Koeno Gravemeijer has sketched such a trajectory and, at the time of writing, we are programming two computer-minitools that we hope will be effective means of supporting the mathematical learning of the classroom community and of the students who participate in it. I mention this to stress that, in contrast to your example of the house, we take a developmental point of view when we think of classroom mathematical practices. Consequently, in the planning process, we attempt to envision how practices might emerge as reorganizations of prior practices. This developmental emphasis is, I hope, also evident in our analyses of what actually transpires in the classroom when we conduct a teaching experiment. For me, an analysis that merely lists a number of practices without describing the process of their emergence from prior practices would be woefully inadequate given that a primary objective when conducting a teaching experiment is to investigate the means of supporting the development significant mathematical ideas.

Pat, a second observation you made allows me to be a little more specific about the process of analyzing classroom events in terms of mathematical practices. You say that the examples I gave from the seventh-grade teaching experiment in my last response:

> are unhelpful in a very important way. On one hand, you contend that we cannot think of classroom mathematical practices as constituting something that each and every student will learn. On the on the other hand, you offer one student's activity as being illustrative of a practice you claim developed. It is in this sense that I see a misfit between theory and implementation.

You are right, I did focus on one student's explanation. However, in doing so, I indicated that this explanation was treated as legitimate by the teacher and other students. Thus, for me, it was an example of what counted as an acceptable explanation in this particular classroom. My focus was not on the reasoning of the student who gave the explanation (psychological perspective), but on the status of the explanation in this classroom community (social perspective). And, I contend, its constitution as a legitimate explanation was a collective accomplishment. As a caveat, I should add that we would not in practice (that word again) claim that certain norms of argumentation had been established on the basis of one isolated case. For example, from what I said, you do not know whether the other students were bored and had no interest in the discussion, or whether they did not view it as their role to question each others' contributions. In general, when we make the inference that something is normative in a classroom (e.g., a particular form of argumentation or a way of reasoning with tools and symbols), we are claiming that members of the classroom community will object when they perceive that those norms have been breached. Thus, methodologically, when we conjecture that something is normative in a classroom, we look for instances where a student's contribution violates those norms and examine whether or not that contribution is constituted as legitimate by the classroom community. In the case of the seventh-grade teaching experiment, there were in fact occasions when students objected when they perceived that the scheme of argumentation I illustrated had been violated (Cobb, in press). This constitutes reasonably strong evidence that the standards of argumentation I described were normative.

A third point that you make brings us to the core issue, the types of theoretical tools at might facilitate our attempts to contribute to the continual improvement of the learning and teaching of mathematics. Framed in this way, the issue at hand is not to decide whether communal practices are epiphenomena in an ontological sense. Instead, it is to clarify whether it more useful for our purposes to think about them as epiphenomena or as phenomena in their own right. In this vein, you comment that you will not know whether communal activity is valuable unless you:

> know something about the changes taking place within individual children so they may contribute to it. It seems possible, in principle, that a desired communal practice emerges, but few students are affected in ways that will allow them to contribute in other settings to making it emerge again.

Later, you reiterate this point when you say that "we would be remiss not to address the question of how we hope to affect individual children so they are able to act productively in a variety settings". From this, you conclude that "the psychological perspective as more fundamental, because it aligns more explicitly with what I take as our fundamental goal of making a positive difference in students' lives". I could not agree more strongly with your statement of our overall goal as mathematics educators. It is for this very reason that we have struggled so hard to develop a way of talking about the mathematical learning of classroom communities. I contend that what we need if we are to continually improve our instructional designs are accounts of students' learning that are tied to analyses of what happened in the classrooms where that learning occurred. An analysis of the classroom mathematical practices established by a classroom community provides a way of describing what transpired in the classroom over an extended period of time. In addition, it enables us to specify the evolving social situations in which the students' mathematical development occurred. To be sure, this analysis of communal learning should be coordinated with a psychological analysis of the qualitatively different ways in which students participated in communal practices and what they learned when doing so. What we then end up with is a situated account of students' learning, one that directly relates the process of their learning to the means by which it was supported. As a consequence, we can immediately develop testable conjectures about how we might be able to improve those means of support. This in turn enables us to engage in educational reform as an ongoing process of improvement in which we continually learn from our experiences of experimenting in classrooms in collaboration with teachers.

In your argument for the primacy of the psychological perspective, I was also struck by your suggestion that although a desired communal practice could emerge, only a few of the participating students might learn in significant ways. First, I should clarify that the establishment of a classroom practice is, for me, a collective accomplishment to which students actively contribute by reorganizing their reasoning. Consequently, a practice cannot, by definition, become established if only a few students learn. Particular purposes, ways of arguing, and ways of reasoning with tools and symbols simply would not become taken-as-shared. It could, however, be the case that the ways in which the students reorganize their reasoning are not as significant as we had intended. This is a question that has to be addressed empirically. In the case of the seventh-grade teaching experiment, for example, we claim that a particular practice that involved reasoning multiplicatively about data emerged during the last part of the experiment. Our classroom observations of the students' reasoning as they participated in this practice indicate that most came to think about data in relatively sophisticated ways. To check the validity of this inference, Cliff Konold, in his role as an independent evaluator, is currently analyzing the individual exit interviews that we conducted with the students. His findings could well lead us to revise our interpretations of our classroom observations. In addition, his analysis, when combined with the analysis of the classroom mathematical practices, will enable us to consider how our instructional design might be improved.

In closing, I want to make a final comment that draws on my involvement in the teaching during the seventh-grade teaching experiment. This proved to be an significant experience for me in that I had not taught in a school classroom for 15 months. I found that I was making sense of what was happening in the classroom differently than I had done previously. In particular, I truly saw a classroom community in which I was a participant. For example, I found that I was looking at ongoing discussions as collective social events. In doing so, I was able to think about and influence the taken-as-shared ways of talking about and reasoning with graphs and other inscriptions, and thus the issues that emerged as topics of conversation. This gave me a greater sense of efficacy as I sought to achieve a pedagogical agenda in that I could attempt to influence the social situation of all the students' learning.[2] I should stress that in doing so, I tried to build on the students' thinking (although a viewing of the video-recordings indicates that there is considerable room for improvement in this regard). However, rather than looking at students contributions in purely individualistic terms, I found

[2] The conjectured learning trajectory for the instructional sequence was also important in that it, in effect, provided a big picture that served to frame the local pedagogical decisions and judgments that I and my colleagues made.

that I was viewing them as acts of participation in ongoing social events. This seemed to make the challenges of teaching more tractable. Perhaps you can keep track of and attempt to influence the reasoning of 30 different individual students simultaneously. Such a feat is beyond my limited capabilities. However, I found that I could (to some extent) monitor both what we were doing as a community and three or four qualitatively distinct ways in which the students were participating in these collective activities. As a consequence, classroom situations that seem almost overwhelming complex when viewed in purely individualistic terms became more manageable.

Conclusion

This conclusion is written by just one of us (PT). Paul and I were supposed to write the conclusion while attending a meeting in Amsterdam. Unfortunately, I sat, stranded, in Washington Dulles airport while Paul was in Amsterdam, and the deadline for final manuscript passed. As such, Paul can't be held accountable for whatever conceptual errors I reveal in these last few paragraphs.

We began this paper with a metaphor — zooming between perspectives without losing the overarching or underlying phenomena revealed at various scales of observation. We then explored how we might think of psychological and sociocultural perspectives regarding mathematical understanding/learning/activity to realize promising directions for resolving conceptual conflicts between them, and in the process revealed what appear to be different commitments to forms of explanation and justification.

We did agree that the notions of practice and participation seemed promising sites for making a connection between psychological and social perspectives, but we ran into problems on how to view social activity. Pat prefers to think of interaction and mutual influence among individuals as being fundamentally constitutive of social activity. This is in the same way that we think of chemical and molecular interactions as being fundamentally constitutive of organic matter. We may not understand the interactions in all their details, nor may we keep track of them in real time. But we never pretend that perspectives of molecular interactions and of organic matter are mutually, reflexively constitutive. Organic matter "emerges" through special types of molecular interaction.

At the same time, Paul prefers to think of perspectives of individual interaction and sociomathematical activity as being mutually, reflexively constitutive. This is not to say that individuals and groups are mutually constitutive. Rather, he prefers to adopt a perspective of social activity that is fundamentally individualistic and adopt a perspective of individual activity that is fundamentally social. One cannot conceive either without having adopted the other.

We still agree that continued efforts to explicate the ideas of participation and practice will be productive. These are ideas that seem to embody, at heart, significant aspects of both psychological and sociocultural perspectives simultaneously.

References

Berkeley, G. (1963). *A treatise concerning the principles of human knowledge*. La Salle, IL: Open Court.

Cobb, P. (1990). Multiple perspectives. In L. P. Steffe & T. Wood (Eds.), *Transforming children's mathematics education*, (pp. 200–215). Hillsdale, NJ: Erlbaum.

Cobb, P. (1991). Some thoughts about individual learning, group development, and social interaction. In F. Furinghetti (Ed.), *Proceedings of the Fifteenth International Conference on the Psychology of Mathematics Education* Vol. 1 (pp. 231–238). Assisi, Italy: PME.

Cobb, P. (1995). Cultural tools and mathematics learning: A case study. *Journal for Research in Mathematics Education, 26*(4), 362-385.

Cobb, P. (in press). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical Thinking and Learning*.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23*(1), 2–33.

Confrey, J. (1991). Steering a course between Vygotsky and Piaget. [Review of Soviet Studies in mathematics education: Volume 2. Types of Generalization in instruction.]. *Educational Researcher, 20*(8), 28–32.

Confrey, J. (1995). How compatible are radical constructivism, sociocultural approaches, and social constructivism. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education*, (pp. 185–223). Hillsdale, NJ: Erlbaum.

Forman, E. A. (1996). Learning mathematics as participation in classroom practice: Implications of sociocultural theory for educational reform. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning*, (pp. 115-130). Mahwah, NJ: Erlbaum.

Fritzsch, H. (1994). *An equation that changed the world: Newton, Einstein and the theory of relativity* (Karin Heusch, Trans.). Chicago, IL: University of Chicago Press.

Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education, 25*(5), 443–471.

Hawkings, S. W. (1988). *A brief history of time*. New York: Bantam.

Klein, A. H. (1974). *The world of measurements: The definitive book on the units and concepts by which we measure everything in our universe*. New York: Simon & Schuster.

Konold, C., Pollatsek, A., Well, A., & Gagnon, A. (1996). Students' analyzing data: Research of critical barriers, *Roundtable Conference of the International Association for Statistics Education*. Granada, Spain.

Lave, J. (1991). Situated learning in communities of practice. In L. B. Resnick, J. M. Levine, & S. D. Teasley (Eds.), *Perspectives on socially shared cognition*, (pp. 63-82). Washington, DC: American Psychological Association.

Maturana, H. (1978). Biology of language: The epistemology of reality. In G. A. Miller & E. Lenneberg (Eds.), *Psychology and Biology of Language and Thought*, (pp. 27–63). New York: Academic Press.

Mehan, H., & Wood, H. (1975). *The reality of ethnomethodology*. New York: John Wiley.

Miller, A. I. (1987). *Imagery in scientific thought*. Cambridge, MA: MIT Press.

Newell, A. (1973). A note on process-structure distinctions in developmental psychology. In S. Farnham-Diggory (Ed.), *Information processing in children*, (pp. 125–139). New York: Academic Press.

Salomon, G. (1993). No distribution without individuals' cognition: A dynamic interaction view. In G. Salomon (Ed.), *Distributed cognitions*, (pp. 111-138). New York: Cambridge University Press.

Shotter, J. (1995). In dialogue: Social constructionism and radical constructivism. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education*, (pp. 41–56). Hillsdale, NJ: Erlbaum.

Steffe, L. P. (1995). Alternative epistemologies: An educator's perspective. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education*, (pp. 489–523). Hillsdale, NJ: Erlbaum.

Steier, F. (1991). Reflexivity and methodology: An ecological constructionism. In F. Steier (Ed.), *Research and reflexivity*, (pp. 163–185). London: Sage.

Steier, F. (1995). From universing to conversing: An ecological constructionist approach to learning and multiple description. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education*, (pp. 67–84). Hillsdale, NJ: Erlbaum.

Toulmin, S. (1969). *The uses of argument*. Cambridge, UK: Cambridge University Press.

Voigt, J. (1994). Negotiation of mathematical meaning and learning mathematics. *Educational Studies in Mathematics, 26*(2-3), 275–298.

Voigt, J. (1996). Negotiation of mathematical meaning in classroom processes: Social interaction and learning mathematics. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning*, (pp. 21-50). Mahwah, NJ: Erlbaum.

Whitson, J. A. (1997). Cognition as a semiotic process: Grounding, mediation, and critical reflective transcendence. In D. Kirshner & J. A. Whitson (Eds.), *Social, semiotic and neurological perspectives*. Hillsdale, NJ: Erlbaum.

ON THEORY AND MODELS: THE CASE OF TEACHING-IN-CONTEXT

Alan H. Schoenfeld
University of California at Berkeley
alans@socrates.berkeley.edu

The Teacher Model Group at Berkeley has, for some years, been working toward the development of a theoretically driven model of the teaching process. The idea is to characterize, with the kind of precision typically used in cognitive models, how and why teachers do what they do "on line" – that is, during the act of teaching. The main questions involved in constructing this kind of model of teaching-in-context are as follows: What knowledge does the teacher have? What beliefs? What goals? How does the teacher perceive unfolding events in the classroom? What decisions does he or she make, for what reasons? How does all this fit together, in fine detail, at a level of mechanism?

We see this kind of work on modeling teaching as having both practical and theoretical implications. I will not pursue the practical side of things in this paper, although I hope some of the pragmatic implications are obvious. Simply put, the better you understand any process, the more potential you have to make it work better. Doing so may be far from easy – consider how much work it has taken to translate research on problem solving into productive problem-solving instruction – but as the case of problem solving shows, improved understanding can indeed yield improved performance. Details regarding the pragmatic implications of our work in modeling teaching, and on what it may take to translate this kind of theoretical work into practice, may be found respectively in Schoenfeld (in press-a) and van Zee and Minstrell (in press).

My focus in this paper is primarily metatheoretical. Main concerns are questions of what theories and models of cognitive/behavioral phenomena such as "teaching-in-context" might look like, and establishing standards by which to judge work of this type. Within the space allotted for this paper, I can only suggest the dimensions of the model itself and of the cases we have worked through in detail. This will (just barely) convey some of the flavor of the work. Then I shall suggest how well the model measures up to the standards that have been elaborated. Though there is scant room for detail here, extensive detail can be found in a forthcoming volume of *Issues in Education,* which includes an extended discussion of the model (Schoenfeld, in press-a), a series of commentaries on it. and a response to the commentaries (Schoenfeld, in press-b), and in two papers that offer case studies (Schoenfeld, Minstrell, & van Zee, 1996; Zimmerlin and Nelson, 1996). I begin by providing some brief examples of situations that we have modeled.

Cases in point: Typical situations that we try to model.

Example 1: Jim Minstrell. James A. Minstrell teaches physics at Mercer Island High School in Washington state. Minstrell is an award-winning teacher who has written extensively about his goals and practices (see, e.g., Minstrell, 1989, 1992; van Zee & Minstrell, 1997a, 1997b.). It is the fourth day of the school year. Minstrell is teaching a lesson of his own design, one of a sequence of introductory lessons carefully constructed to introduce students to some of the important themes underlying the course. He wants students to experience physics as a sense-making activity, and to understand that even in rather simple and ostensibly "objective" judgments, there are multiple issues of discretion – e.g., in deciding which data to collect, which data to "count" after they have been collected, and how to combine and interpret those data.

The topic under discussion appears simple: what is the width of a particular table in the classroom? Eight students in the class have taken measurements, in centimeters, and have produced the following numbers: 106.8; 107.0; 107.0; 107.5; 107.0; 107.0; 106.5; 106.0. The class has discussed various issues, such as: Should all or only some of the data be included? How might the data be combined, and which method of combining them would yield the "best number" to represent the width of the table? In the classroom discussion, students have mentioned and discussed the possibility of using the arithmetic average (defined by a student as "Add up all the numbers and then divide by whatever amount of numbers you added up") and the mode ("the number that shows up most frequently"). At that point a student says: "This is a little complicated but I mean it might work. If you see that 107 shows up 4 times, you give it a coefficient of 4, and then 107.5 only shows up one time, you give it a coefficient of one, you add all those up and then you divide by the number of coefficients you have."

Here are the key questions in terms of the model. Assume we have studied Minstrell carefully – read his papers, interviewed him, perhaps even seen him teach previous versions of this course. We have a good sense of what he thinks is important, what his agenda for the class that day is, and what he knows. He is in the middle of teaching, and something unusual has just happened. Can we say how Minstrell is likely to respond? More importantly, can we say what leads him to respond that way – what beliefs, goals and knowledge shape his decision, and how their interplay results in his choosing to act the way he does?

Examples 2, 3, and 4: Mark Nelson, Deborah Ball, and Alan Schoenfeld. Here are some parallel cases, covering a wide variety of teaching "territory." Mark Nelson is a student teacher teaching an algebra lesson on reducing exponents in expressions like (x^3y^5/xy^2). This is the first time he is teaching the lesson, so he has little by way of pedagogical content knowledge (Shulman, 1986) related to the topic, though his knowledge of the mathematics is secure. He has had students work some problems at their desks, and is about to convene the class for a whole-class discussion of the problems. We know his intentions and expectations, as well as his classroom routines. If you "feed" us the class's responses to his questions, one by one, can we predict what he will say, and how the discussion will go? Failing prediction – the toughest standard for any model – can we at least explain, post hoc but in a principled way grounded in the mechanisms of the model, why things evolved the way that they did?

Expanding the problem space, consider a lesson taught by Deborah Ball (the "Shea Numbers" tape of her third grade classroom on January 19, 1990). Ball enters the classroom with a specific item high on her agenda – to have the students reflect about they ways they learn and what they take as evidence for mathematical "truth" – as a follow-up activity to a meeting they had had the previous day with a fourth grade class. The classroom discussion keeps tending away from this kind of "meta-level" conversation to mathematical specifics: is the number zero even, odd, or special; can a number be even *and* odd; and so on. How will she act, and why? Or, consider the opening days of my problem solving course (see Arcavi, Kessel, Meira, & Smith, 1998). The course is largely interactive, with many of the ideas we work with generated by the students. Is it possible to model my teaching – to say in advance, on a principled basis, how and why I will react to the comments and suggestions made by students? Can this be done in such a way that it "explains" my actions, from the moment I enter the class on any given day to the moment the class session ends? [N.B. The presentation at the conference will allow for elaboration in detail, including a line-by-line discussion of transcripts, that is precluded here by space constraints.]

How the model works

What follows is a brief suggestion of the mechanism by which the model works – for detail on the specifics of the case presented see Schoenfeld (in press-a, in press-b) and Schoenfeld, Minstrell, & van Zee (1996). As noted, the core idea is that the decisions made by the model of the teacher are a function of the teacher's attributed beliefs, goals, and knowledge. Here is how they play out in the case of example 1 described above.

Figure 1, which represents a small part of the complete parsing of Minstrell's lesson, provides a rough characterization of what Minstrell did and why in response to the student's suggestion of a "complicated" way to arrive at a best value for the width of the table. The whole of our lesson representation starts with a box representing the lesson, marked [1] in its upper left-hand corner. In this case the analysis indicates that the lesson can be decomposed into four major "chunks" (segments of the lesson that cohere phenomenologically in some way), which are denoted [1.1], [1.2], [1.3], and [1.4] respectively. The labeling continues in that way. Here, the segment of the lesson catalyzed by the student's comment is labeled [1.2.2.3]

– already fairly deep in the nested structure of the lesson. In the upper right-hand corner of each box in Figure 1 we identify the numbers of the lines of transcript corresponding to each transcript chunk. Chunk [1.2.2.3] extends from lines 164 through line 271 of the transcript, which is 517 lines long. It is further decomposed into chunks [1.2.2.3.1]. [1.2.2.3.2], etc. Inside each box, which represents a chunk of the lesson, we briefly describe the following: triggering and terminating events (what caused the teacher to embark on this path, what caused it to be terminated); high priority beliefs related to this episode; goals that the teacher's decision was intended to achieve; relevant knowledge on which the teacher's actions are based and decisions are made; the nature of the chunk (e.g., standard pedagogical routine or script).

Here is a summary description of Minstrell's initial decisions and actions in response to the student comment. In terms of content, Minstrell believes that the class should serve as a sense-making community, in which students explore physical phenomena in reasoned ways. In terms of pedagogy, he believes that he should be responsive to student initiatives that are "in the ballpark." Here the student comment, a proposed way to compute the "best value," is reasonable and germane. Thus the model says that Minstrell will decide to pursue it – even if the short-term cost is to defer other topics he'd planned on doing next in the lesson. But, how will he pursue it? First, it is important to note that Minstrell recognizes that one possible interpretation of what the student says is the standard formula for "weighted average" of a collection of numbers. Hence there is the potential to relate the student's suggestion directly to an earlier discussion of "average." It is also important to know how Minstrell tends to introduce issues into discussion. Minstrell employs a rhetorical device he calls "reflective tosses" in which he "catches" the intellectual content of an idea and "tosses" it back to the students for clarification, elaboration, or comment. Thus the model predicts that, having decided to attend to the issue and having the relevant knowledge, Minstrell will first ask the student to clarify her statement (thus making it public, and open for classroom discussion) and will then work with the class to explore it. This is what he does – in fact, by asking the student who had first proposed a definition of "average" to comment on this new proposal, setting the stage for a comparison of the usual definition of average ("Add up all the numbers and then divide by whatever amount of numbers you added up") and the formula for weighted average that he has written on the board. When this is resolved (bringing chunk [1.2.2.3.1] to a close), a comment by the student leads (as the model predicts) to a second round of clarifications, where the class compares weighted and unweighted averages. At that point, having dealt fully with the student's comment, Minstrell returns to his agenda for the lesson. [For more detail on this and the other cases, see the papers cited above.]

Theoretical underpinnings

The Teacher Model Group's work is situated in the "cognitive science" approach to cognition – specifically in what Greeno (in press) calls "the standard framing assumptions of cognitive theory." Our intention (for now) is to construct the architecture of a model that, in some meaningful way, captures the thinking and decision-making that teachers make "on line." The specific goal of any particular model (of a particular teacher-in-context) is to delineate the beliefs, goals, and knowledge of the teacher, and, using these constructs, to characterize the decision-making of the teacher as events unfold in the classroom. We are, then, studying what goes on "in the head" of particular teachers. Our constructs are mental entities – in the model, representations of beliefs, goals, knowledge (in the form of action plans or other schemata), etc. The decision-making mechanism is akin to that of AI-like models: one can think of a goal-driven architecture using a spreading activation network. (Rough translation into everyday English: When one or more goals that a teacher has are of highest priority at the moment, and some action or sequence of actions within the teacher's repertoire is likely to do the best job of meeting those goals, then that is the action or sequence of actions the model says the teacher will take.)

Our modeling work draws upon the vast literature on teaching (see, e.g., Borko & Putnam, 1996; Calderhead, 1996; Clandinin, 1986; Clark & Yinger, 1987; Fenstermacher, 1994; Grossman & Stodolsky, 1994; Shulman, 1986, 1987; Thompson, 1992) and a more specific, cognitively-oriented corpus of research that attempts to describe the mental constructs that support teaching and how they interact (see., e.g., Berliner, 1994; Clark & Peterson, 1986; Leinhardt, 1993; Leinhardt & Greeno, 1986; Shavelson, 1986). I see the Teacher Model Group's work as a logical extension of the past few decades' work on thinking, learning, and problem solving – as one point on a continuum where the ultimate goal is to explain (individuals') thoughts and actions in complex social settings. This work is in many ways a direct extension of my work on problem solving, and a reflection of the field's increasing capacity to model complex behavior. In the early years we brought people into the laboratory to watch them working on problems, in isolation–the reason being that the tools researchers had for understanding cognition were so limited that we needed to control the environment as much as possible. As the field's understandings of things such as the knowledge base, strategy use, metacognition, and beliefs grew, it moved toward the study of cognition in more "natural" settings, e.g., in classrooms. As the capacity to model interactive decision-making grew, studies of tutoring and teaching-in-context became feasible. We are now, as the research under discussion shows, capable of modeling such complex behavior. Yet, this work is still quite constrained, and its limitations should be noted.

From an "internal" perspective (that is, living within the framing assumptions of cognitive theory), there are at least two major issues to con-

[1.2.2.3] (164-271)

Major Unplanned Excursion: Exploration of an alternative formula for arithmetic average

Initiating event (trigger):
A student suggests an idea that "is a little complicated" but "might work."

Beliefs:
- Teacher should follow students' lead in thinking where appropriate.
- Content is relevant and appropriate.

Goals:
- All overarching goals are active (see narrative).
- Explore and clarify the the properties of the student's proposed formula.
- Have content emerge from students if possible.
- Note: Second goal especially is emergent (unplanned).

Method/action plan:
- Interactive elicitation using reflective tosses.

Chunk type:
- As above, interactive elicitation (a familiar classroom routine).

Knowledge base:
- Immediate recognition and understanding of proposed student formula.
- Memory of which students were familiar with which ideas.

Terminating event:
- Satisfactory elaboration of student's method and comparison with standard and alternative methods.

This entire episode was unplanned. The time and effort spent on it reflect Minstrell's commitment to the goal of taking student ideas seriously and pursuing them where possible.

→ [1.2.2.3.1] (164-225)

Impromptu Excursion: Clarifying the nature of a "complicated" formula proposed by a student

Initiating event, beliefs, goals, method, and chunk type are all as identified in Chunk 1.2.2.3.

Specific content goal:
- Have students come to conclusion that the "complicated" formula yields the arithmetic average.

Terminating event:
- The specific content goal immediately above is achieved.

→ [1.2.2.3.1.1] (164-167)

Student Comment

→ [1.2.2.3.1.2] (168-169)

Clarifying What the Student Suggested

Specific (emergent) content goal:
- Make sure the class understands the nature of the proposed formula.

Method:
- Interactive elicitation using reflective tosses.

Terminating event:
- The content goal is achieved.

→ [1.2.2.3.1.3] (200-225)

Showing the "Complicated" Formula is the Arithmetic Average

Specific content goal:
- Have class conclude the formulas are the same.

Method:
- Interactive elicitation, calling on a specific student to provide content.

Terminating event:
- The content goal is achieved.

→ [1.2.2.3.2] (226-271)

Impromptu Excursion: Comparing "weighted" and "unweighted" formulas for the average

Initiating event:
- Student comment about possible confusion between the two formulas.

Beliefs, goals, method, and chunk type are all as identified in Chunk 1.2.2.3.

Specific content goal:
- Clarify the difference between the two formulas.

Terminating event:
- The specific content goal immediately above is achieved.

→ [1.2.2.3.2.1] (226)

Student Comment

→ [1.2.2.3.2.2] (227-241)

Framing and Clarifying the Comparison

Specific (emergent) content goal:
- Make sure the issue is clear to the students.

Method
- Interactive elicitation using reflective tosses.

Terminating event:
- Student consensus (by assertion) they are not the same – goal achieved.

→ [1.2.2.3.2.3] (242-271)

Framing and Clarifying the Comparison

Specific (emergent) content goal:
- Work through compelling example to make sure the difference is understood.

Method:
- Interactive elicitation using reflective tosses.

Terminating event:
- Content goal clearly achieved; teacher summarizes with mini-lecture.

Note:

The next level of detail, which would consist of elaborating on Minstrell's use of interactive elicitation to achieve the goals specified in the following chunks:
[1.2.2.3.1.2],
[1.2.2.3.1.3],
[1.2.2.3.2.2], and
[1.2.2.3.2.3],
is not represented here. A detailed description of the ways in which Minstrell interacts with the students using that method is given in Schoenfeld, Minstrell, and van Zee (1996).

Figure 1. A Representation of Part of Minstrill's Decision-Making
Reprinted with permission from Schoenfeld, in press-a

BEST COPY AVAILABLE

sider. The first is whether such modeling makes unwarranted assumptions about the phenomena being studied, and thus distorts them. As Leinhardt (in press) observes, physics makes progress by virtue of idealizations: "consider a spherical cow" is not a bad assumption with which to begin solving some physics problems. But the same hypothetical spherical cow, in a biology lab, might be problematic. Is there the danger of introducing such beasts into the classroom, via models such as ours? The second has to do with very stringent constraints on the model, which should be stressed – the model is a model of teaching-in-context, in the "here-and-now." We do not yet model *history* – except as in the mind of the teacher, whose knowledge includes his or her memories of previous experiences with the content, with these students, etc. We do not model *context* – except for the teacher's perceptions of the context, and of the supports and constraints within it. We do not model mechanisms of change – for example, how and why the teacher thinks differently after a lesson, or a unit, or the year is over. All of these are limitations of the current model – but the kinds of things that might be overcome, within the framing assumptions of cognitive theory, over the next few decades.

From an "external" perspective, the challenge can be raised (see., e.g., Greeno, in press) that the lens through which this kind of model views the classroom – the teacher's – is all too distorted. The classroom is a highly interactive environment in which there are multiple actors; the teacher is only one (albeit an important one) whose view may or may not "explain" much of what takes place. Moreover, the totality of the classroom may supersede the perspectives of the individual actors, rendering individual perspectives inadequate as versions of what takes place. In short, I agree. The issue here is to see how far we can push this kind of model, and how much it can explain under various circumstances – not to claim that what the teacher sees, and how much of it we can model, represent "reality." In a fashion similar to Greeno, others may argue that the "interior lens," which only accounts for the teacher's perspective of context (constraints, supports, etc.) and not for the "real thing," must perforce be inadequate. Perhaps so – but again, the teacher's view of context (including the teacher's sense of what materials might or might not be accessible, what flexibility there is with regard to curriculum, and what the "abilities" of the students might be) is surely a significant factor in shaping what happens in the classroom. The goal is to see what can be explained with this kind of model, and then to transcend it.

Metatheoretical Notes

If only for a moment, it is worth stepping outside the space of current assumptions to point out that the terms "theory" and "model" have very different meanings in different fields. Consider Table 1, for example.

People with backgrounds in mathematics and physics expect theories and models to have very specific entailments. In those domains, a theory

Table 1
Aspects of Theories and Models in Different Subject Areas

Subject:	Math, Physics	Biology	Education, Psychology
Theory of...	Equations, Gravity	Evolution	Mind
Model of...	Heat Flow in a Plate	Pedator-Prey Relations	Problem Solving

(e.g., the mathematical theory of equations or an inverse-square law of gravitational attraction) is a precise statement of "what counts," and a model embodies that theory in very specific computational terms. Moreover, in both domains, theories and models support a precise form of *prediction*. By that standard, educational/psychological theories and models are often found seriously wanting – although the rejoinder, that spherical cows don't necessarily represent real objects very well, should not be lightly dismissed. In my opinion, theories and models from the biological sciences (which may also be disdained by some mathematicians and physicists) may provide quite appropriate parallels to the kinds of theories and models that are appropriate in psychology and education. Consider theory, for example. The theory of evolution is not "provable" in the mathematical sense, but evidence can be brought to bear on its validity. And, the theory can be and is held to strict scientific standards, for example a kind of a posteriori "prediction": while evolution moves too slowly for predictions of the future to be tested, the theory does imply that as yet undiscovered fossil records will have certain properties, and will not have others. Equally important is the stance toward models. One can take biological models (whether of predator-prey relations, or of specific organs such as the heart or even of the human body) as approximations, in the sense that actuarial tables are approximations – what they predict may best be thought of as a range of outcomes, with probability values attached. (Such a distribution is, of course, the precise form of genetic predictions using Punnett squares.) In many contexts, it may be that the appropriate form for the predictions of educational and psychological models can most productively be thought of as probability distributions of outcomes.

Standards for judging models and theories

In keeping with the above comments, I propose that four major criteria are appropriate for judging theories and the models that embody them: descriptive power, explanatory power, predictive power, and scope. Descriptive power refers to how well the theory and model seem to capture the situation being characterized. Are important aspects of the situation represented, and do they interact in the theory and model ways that seem to correspond to the ways they interact in "reality?" Explanatory power takes things a step further. Do the theory and the model provide a sense of mechanism that explains how and why things fit together, above and beyond providing descriptions of their interactions? The notion of predictive power is almost self-explanatory. What is obvious is that the more accurately the theory and models derived from it predict outcomes, the more confidence one will have in the robustness of the theory. Somewhat less obvious is the nature of *appropriate* predictions – see the comment in the preceding paragraph about psychological predictions being conceptualized as probability distributions of outcomes. Finally, on scope: the issue is, what range of phenomena do the theory and model cover? A theory of equations that covers only linear and quadratic equations is not of much interest; nor is a theory of teaching that applies only to didactic lectures.

A preliminary assessment of the theory and the model

Lacking the space to examine Examples 1 through 4 in detail, I can only argue here by assertion. The detail does exist. Minstrell's lesson segment is analyzed in depth in Schoenfeld (in press-a) and in Schoenfeld, Minstrell, & van Zee (1996); Nelson's in Schoenfeld (in press-a) and in Zimmerlin & Nelson (1996); Schoenfeld's in Arcavi, Kessel, Meira, & Smith (1998) and in Schoenfeld (in press-a); and Ball's in Schoenfeld (in press-b).

Broadly speaking, the model does well on the criteria of descriptive and explanatory power. In all of the examples above, the teacher being modeled has been an informant on the research and has provided substantial information regarding the work. In some cases, such as Minstrell's, we did a preliminary analysis and then ran it by the person being modeled – providing that person the opportunity to say that the assertions we made were wrong, or that we had missed something important or emphasized the wrong things. Thus far the analyses have held up rather well. They seem to take into account what is important, both from the perspective of cognitive theory (after all, the constructs in our models are derived from the main constructs of cognitive theory) and from the perspective of our informants/ colleagues. In the case of Nelson, for example, the model predicts that he will run into difficulty when an explanation he offers the students does not clear up their (expected) confusion as he thinks it will. Moreover, the model explains why he gets into that difficulty by providing a detailed description of the specific cognitive and pedagogical resources Nelson has at his disposal, and showing how those resources are insufficient to deal with the situation he finds himself in.

At a "face value" level, the model does fairly well by way of prediction – at least in those cases (Minstrell, Nelson, Schoenfeld) where we have felt confident that the model captures the teacher's decision-making. Here the issue of scope becomes central. On the one hand, the three cases just mentioned cover a fair amount of territory: Minstrell is an experienced high school physics teacher who was teaching an innovative lesson of his own design, Nelson a beginning high school mathematics teacher working through a traditional lesson for the first time, and Schoenfeld (like Minstrell) an experienced teacher working through a college mathematics class of his own design. I feel comfortable asserting that the model covers mathematics and science, secondary and collegiate, traditional and innovative – as long as the lesson is *agenda-driven*. In all of these cases, the teachers had fairly clear ideas of where they wanted the lessons to go. Although there was wide variation in how and with what success these teachers deviated from the original agendas in response to classroom contingencies, there is no question that, by and large, the teachers' agendas were the primary driving forces in shaping what took place in the lessons modeled. I have little doubt that agenda-driven instruction, in general, can be modeled – and that when it is, the models will fare rather well with regard to prediction.

Things get more complex, however, when one considers some of the things that happen in Deborah Ball's January 19, 1990 third grade class. There the teacher is highly sensitive to developmental as well as content concerns, making for a more complex initial agenda than is apparent in the lessons that we have analyzed at more advanced levels. Perhaps more importantly, the directions of that lesson evolve substantially in response to unpredictable issues that arise during the class session. This kind of *emergent agenda* has been much more difficult for us to model; it may, ultimately, be where the model will break down. It is not yet clear that it will: recently (Schoenfeld, in-press-b) we have had some success in analyzing why (we believe) Ball makes some of the choices she does in that class, and we may ultimately become successful at modeling that lesson. If we do, we will have shown that the model has very large scope – the teaching in these lesson segments spans a pretty large teaching space. If we do not succeed, so be it. Where it is known to work – which already covers a fair amount of territory – the model does well along the dimensions of descriptive, explanatory, and predictive power. And when we discover where it doesn't, we will have an important set of phenomena to explore further.

References

Arcavi, A., Kessel, C., Meira, L., & Smith, J. P. (1998). Teaching mathematical problem solving: An analysis of an emergent classroom community. In A. H. Schoenfeld, J. J. Kaput, & E. Dubinsky (Eds.),*Research in Collegiate Mathematics Education, Volume III* (pp. 1-70). Washington, DC: Conference Board of the Mathematical Sciences.

Berliner, D. C. (1994). Expertise: The wonder of exemplary performances. In J. M. Mangier & C. C. Block (Eds.), *Creating powerful thinking in* *teachers and students: Diverse perspectives* (pp. 161-186). Fort Worth, TX: Holt, Reinhart, & Winston.

Borko, H., & Putnam, R. (1996). Learning to teach. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 673-708). New York: Macmillan.

Calderhead, J. (1996). Teachers: Beliefs and knowledge. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 709-725). New York: Macmillan.

Clandinin, D. J. (1986). *Classroom practice: Teacher images in action.* Philadelphia: Falmer Press.

Clark, C., & Peterson, P. (1986). Teachers' thought processes. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd edition, pp. 255-314). New York: Macmillan.

Clark, C., & Yinger, R. (1987). Teacher planning. In J. Calderhead (Ed.), *Exploring teachers' thinking* (pp. 84-103). London: Cassell.

Fenstermacher, G. (1994). The knower and the known: The nature of knowledge in research on teaching. In L. Darling-Hammond (Ed.), *Review of research in education*, Volume 20 (pp. 3-56). Washington, DC: AERA.

Greeno, J. G. (in press). Where is teaching? *Issues in Education.*

Grossman, P. L., & Stodolsky, S. S. (1994). Considerations of content and the circumstances of secondary school teaching. In L. Darling-Hammond (Ed.), *Review of Research in Education*, Volume 20 (pp. 179-222). Washington, DC: AERA.

Leinhardt, G. (1993). On teaching. In R. Glaser (Ed.), *Advances in instructional psychology* (Volume 4, pp. 1-54). Hillsdale, NJ: Erlbaum.

Leinhardt, G., & Greeno, J. G. (1986). The cognitive skill of teaching. *Journal of Educational Psychology, 78* (2), 75-95. Also appears in P. Goodyear (Ed.) (1991), *Teaching knowledge and intelligent tutoring.* Norwood, NJ: Ablex.

Minstrell, J. (1989). Teaching science for understanding. In: L. B. Resnick & L. E. Klopfer (Eds.), *Toward the thinking curriculum: Current cognitive research* (pp. 129-149). Washington, DC: Association for Supervision and Curriculum Development.

Minstrell, J. (1992). Facets of students' knowledge and relevant instruction. In R. Duit, F. Goldberg, & H. Niedderer (Eds.), *Research in physics learning: Theoretical issues and empirical studies* (pp. 110-128). Kiel, Germany: Institute for Science Education.

Schoenfeld, A. H. (1996). Elements of a model of teaching. Paper presented at the annual meeting of the American Educational Research Association, New York, April 8-12, 1996.

Schoenfeld, A. H. (in press-a). Toward a theory of teaching-in-context. *Issues in Education.* This paper is available at <http://www-gse.berkeley.edu/Faculty/aschoenfeld>.

Schoenfeld, A. H. (in press-b). On modeling teaching. *Issues in Education.*

Schoenfeld, A. H., Minstrell, J., & van Zee, E. (1996). *The detailed analysis of an established teacher carrying out a non-traditional lesson*. Paper presented at the annual meeting of the American Educational Research Association, New York, April 8-12, 1996.

Shavelson, R. (1986). *Interactive decisionmaking: Some thoughts on teacher cognition*. Paper presented at I congreso internacional, "Pensamientos de los profesores y toma de decisiones," Seville, Spain, June 19-21, 1986.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 17*(1), 4-14.

Shulman, L S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57, 1-22.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

van Zee, E., & Minstrell, J. (1997a). Reflective discourse: Developing shared understandings in a physics classroom. *International Journal of Science Education 19*, 209-228.

van Zee, E., & Minstrell, J. (1997b). Using questioning to guide student thinking. *Journal of the Learning Sciences, 6*(2), 227-269.

van Zee, E., & Minstrell, J. (in press). The promise of a theory of teaching-in-context. *Issues in Education.*

Zimmerlin, D., & Nelson, M. (1996). *The detailed analysis of a beginning teacher carrying out a traditional lesson*. Paper presented at the annual meeting of the American Educational Research Association, New York, April 8-12, 1996.

BUILDING MATHEMATICAL STRUCTURE WITHIN A CONJECTURE DRIVEN TEACHING EXPERIMENT ON SPLITTING

Jere Confrey
The University of Texas at Austin
jere@mail.utexas.edu

It's hard to imagine that another study involving ratio and proportions or fractions could add to our understanding of students' learning. And yet, even with the significant amount of research on this topic (Behr, Harel, Post, & Lesh, 1992; Harel & Confrey, 1994; Hart, 1984; Nesher, 1988; Noelting, 1980a; Steffe, 1994), students continue to perform poorly on these fundamental concepts. As a direct result of not mastering the concepts of fractions, ratio and proportion, decimals, and percents, many students are at risk in an "algebra for all" middle school movement of becoming part of the failure statistics. Potentially robust alternative approaches to this array of topics are still needed.

A second goal in discussing this study is to consider in some detail one crucial aspect of a "modeling approach" to classroom instruction (Confrey, 1996; Confrey & Doerr, 1996a; 1996b). This "modeling approach" to instruction includes: 1) tool-based investigations grounded in modeling activities, 2) support for the articulation of student voice within heterogeneous groups and whole group discussion, 3) careful elaboration, refinement, and differentiation of concepts as intellectual tools through intra-, and inter-activity sequencing, and 4) the use of reflection and practice to stabilize student knowledge and promote teacher's self-examination of their own perspective. In this paper, I seek to clarify the meaning of the component of "elaboration, refinement, and differentiation of concepts as intellectual tools" by discussing the critical moments of conceptual development in this research. Specifically, I seek to focus on how to "build mathematical structure." Too often, when one observes teachers implementing constructivist instructional practices using small groups, the articulation of student method, or manipulatives, one fails to see them attain "coherence" (Stigler & Perry, 1988) or mathematical acuity and power (National Council of Teachers of Mathematics, 1989). Though critics will discuss this as a lack of attention to basic skills, I will argue that too often the inadequacy lies in teachers' failure to assist students in building mathematical structure. If done correctly, students learn basic skills, not as memorized facts, but as interconnected ideas within a network of mutually supporting argumentation.

To accomplish these two goals of explicating a splitting-based analysis of "multiplicative structures" and of discussing how these students successfully "build mathematical structure," I will report on the results of a three year teaching experiment with approximately twenty children during their third, fourth, and fifth grade years. From three to six months a year, for 45 minutes to one hour a day, I taught them mathematics daily. It was a school near a major university, that drew students from the university faculty and staff, but also from a poorer section of town. The study was designed to develop and explore "the splitting conjecture" (Confrey, 1988; Confrey & Smith, 1995). This conjecture states, there are at least two independent but connected primitive constructs that lead to a robust understanding of numeration; one is counting and the other is splitting. Splitting has its roots in activities like sharing, magnifying, shrinking, copying, and reproducing and is the primitive that leads to the development of multiplication, division, and ratio. There are fundamental, early, and essential ties between ratio and two-dimensional space that make a set-based approach to splitting inadequate, and necessitate careful ties to area, slope, rate, and similarity (Confrey, 1988, pp. 255-259).

This work differs from related work in multiplicative structures in that it tackles the entire array of related ideas of multiplication, division, and ratio and proportion and relates them within a single conjecture: that multiplication, division, and ratio co-define each other and should be introduced as a trio in the early elementary grades. This approach pioneered two basic claims: 1) to develop this reasoning, students must learn to work in two dimensional mathematical space which means careful and extensive attention to geometry and graphing. Simultaneously, this conjecture rejects the reduction of elementary mathematics to the construction of the rational number line and its operations—either as a set of fractions and mixed numbers or decimals (Confrey, 1995a); 2) to make ratio and proportion (and percent) primary and to relegate fractions and decimals to its subset. I claim that though treating multiplication as repeated addition and division as repeated subtraction is necessary to link the splitting and counting worlds, the overreliance on these methods of multiplication as repeated addition and division as repeated subtraction as the dominant basis for multiplicative reasoning creates spindly networks of mathematical reasoning on ratio, rate, and later functions. Given the critical significance of these ideas to advanced mathematics, neglect of the independence of splitting from counting and lack of attention to the internal structure of the splitting world will continue to hobble our students mathematically. On examination, it appears that Japanese students do experience an early introduction to multiplication and division facts in a curriculum with a conceptual structure that considers two models for multiplication, that of scaling and one and two dimensional arrays, both of which are structurally more consistent with this conjecture and are not simply repeated addition.

The Results

Data presenting the success of these students has been presented elsewhere (Confrey & Scarano, 1995) demonstrating that by fifth grade, seventeen of the twenty students scored in the nineties on the California test with scores between 60 and 80 by the other three. By comparing the student scores on entry with their outcome measures, we found a decreasing gap between high and low achievers, showing the viability of a modeling approach with heterogeneous groups. By using comparison tasks from CSMP, we demonstrated that our students showed remarkably low incidence of additive errors, and that their performance relative to other groups improved as the problems increased in difficulty.

Critical Moments

In order to focus on "building mathematical structure" within the elaboration and differentiation of intellectual tools, I have chosen to identify a sequence of nine critical moments from the splitting curriculum. Critical moments one and two come from the pilot studies with children in the first and third grade. Three and four come from third grade, where we began work with our twenty students at the level where the schools perceive multiplication and division are finally to be fully introduced. The last five examples were drawn from a six month period in fourth grade during which we focused on ratio and proportion making fractions a subset. Critical moments have been selected based on their subsequent reappearance as essential intellectual tools for further reasoning.

1: Doubling and Halving and Basic Sharing

As shown by Pothier (1983), doubling and halving develop very early with a high degree of ease and accuracy. Furthermore, children in primary grades demonstrate high interest and accuracy in splitting and sharing. In 1993, we undertook studies with first and third graders using play dough cakes and cookies and poker chips to show that young students could effectively use a variety of strategies to share fairly across a variety of tasks and that the language of splitting occurs repeatedly as they do so. Relying on dealing into piles, they could undertake any kind of split. Occasionally, while dealing, children would spontaneously line up their objects to create arrays. A notable outcome of fair sharing was the children's ability to claim the equivalence of the set without the necessity of counting, and used counting to confirm and name their results, evidence of the independence of the two structures of counting and splitting. In the continuous case, a heavy reliance on symmetries in rectangular objects made even splits easier than odd ones. The critical underlying construct is equal sized pieces and equal sized groups.

2: Similarity

In another study of first and third graders, we experimented with young children's understanding of similarity. We demonstrated that large numbers of young children chose to sort sets of circles, squares, triangles (acute, right, and obtuse) and rectangles (three different ratios of length to width) as "the same" based on similarity. These young children achieved a high degree of accuracy on the first three cases, with a significantly lower accuracy on rectangles which are notably harder even for adults. A few first graders spontaneously made the argument that matching two angles was sufficient to convince them of the similarity of triangles. From this study, we realized that spontaneous experience with similarity in perception of depth and motion provides a rich and untapped resource in early reasoning on ratio and proportion.

3: Relating Multiplication, Division, and Ratio: Unit Ratios

In introducing third grade students to multiplication facts, we used a variety of problems including some of Marilyn Burns', like her chopsticks problems where there is a well-known underlying and constant ratio (two chopsticks per guest). For instance, how many legs if there are seven horses? A first characteristic of the students' solutions was our requirement that: 1) as they worked such problems, they would be asked to practice, write, and discuss a multiplication fact, division fact, and underlying ratio. Hence the problem mentioned would have coded with it: 4 x 7, 28/4 and 4:1. Helping students see that across the entire set of problems concerning horses and their legs, there was an invariant relationship of 1:4 , and this formed the basis of their early use of the term ratio. We refer to this as a unit ratio (where one of the two comparisons is always one). Because all examples had external referent, we did not preference the form a:1 or 1:a requiring only that the associated units were clarified. Although we did not realize it at the time, introducing students to this trio should have included attention to the construction of the language, 7 is one-fourth of 28. Moving between construct of the ratio 1:4 and its use as an operator of "1/4 of" came to our attention during the analysis phase of the work as we found students using the phase "1/nth" of spontaneously but without our notice. Confrey (1995a) analyzed it as a key structural element as an operational inverse to partitive division. Its role in the construction of multiplication and division of rational numbers will become evident in Critical Moment 8.

A second, and notably different, characteristic was that we introduced the multiplication facts in an unusual order: by 2's, 10's, 5's, 4's, 3's, 8's, 6's, 9's, 7's. This ordering reflects the multiplicative ease of the numbers relative to their prime factors, rather than consecutive order. For instance, multiplying times 4 is carried out as double, doubling and times 8 as double, double, doubling. Five can be viewed as times ten and halving or counting by fives. This is an elementary example of what I mean by helping students to move flexibly in multiplicative space, a skill which is badly neglected in traditional curricula. In contrast, in the typical third grade treatment, multiplication is introduced separately from division, as repeated addition and skip counting, and there is virtually no discussion of ratio except as the identification and perhaps equivalence of basic fractions. Typically, multiplication tables are presented in consecutive order and the patterns that are explored, using multiplication tables or skip counting, are additive rather than multiplicative. This is an example of how the additive world and the development of one-dimensional number line implicitly undergirds the elementary curriculum to the detriment of multiplicative reasoning.

4: Linking Partitive and Quotative Division

A critical moment occurred as the children discovered the equivalence of results of partitive and quotative operations. To children the problem, "if I have 128 leaves and want to make four leaf clovers" is very different than to have "128 candies and share them among four children" despite obtaining the same numeric answer. A critical moment arose as the class struggled to understand why the same answer is produced. The resolution came as the children worked with Dienes blocks and flats to form their solutions into rectangular arrays. At that moment, children began to see that arrays portray the same group as 32 sets of 4 leaves or as 32 candies for 4 children. This was a critical moment in abstraction, in that the children began to view 128 divided by 4 as equally coding a partitive or quotative action—as "its just division" they would later say. As I have argued elsewhere (Confrey, 1995b), abstraction is the ability to see likeness in things apparently dissimilar—rather than as an act of disassociation with context as so many try to argue.

5. Moving in Multiplicative Space

In fourth grade, we began our experiments in January, and wanted students to become flexible in moving in multiplicative space. To do this, we created challenges we called "daisy chains" (Confrey & Scarano, 1995). Students were challenged to write as many ways as they could think of to use multiplication and division to go from one number, say 12, to another, say 20. They might write 12/ 3=> 4 x 5 => 20; or 12 x 5=> 60 / 3 =>20. Exercises required students to make longer and shorter chains. In a whole class discussion, it came out that for any problem there were two algorithmic solutions: 1) to go from a to b, divide by a and multiply by b or 2) multiply by b and divide by a. This inquiry set up two critical structural elements used extensively in later work: the importance of the identity element, one, in multiplicative and divisional reasoning; and the possibility of moving among numbers using a combination of multiplication and division, a process which I argue later is the basis for the construction of multiplication and division of rational numbers (also see Behr, Harel, Post, & Lesh, 1994).

6: LCM and GCF and Prime Factors

A further elaboration of the joints in multiplicative and divisional space came from a clapping exercise where the children were asked to predict when one person clapping every fourth beat and one clapping every third beat would clap at the same time (Confrey, 1993). Using manipulatives, children explored this problem which led to a series of exercises on prime factors and on using Venn diagrams of common and non-common factors to identify the LCM (union of factors) and GCF (intersection of factors). The underlying structure here is to develop an efficient means of analyzing a number into its multiplicative constituents and seeing how these are useful in predicting common splitting elements or multiplicative joints.

7: Comparing and Equivalence of Ratio: Ratio Units

One essential element of the modeling approach is creating a need for an idea and a problematic which acts as a roadblock to where children want to be (Confrey, 1991). With this goal to begin our discussions of ratio, we had students conduct a poll concerning a topic of widespread interest and controversy. In one set of results, the children realized that there was a gender difference they wanted to compare, but they lacked a tool to quantitatively express this comparison. This open question led us to the question of when two ratios are "the same" in the context of mixing lemonade (represented by orange and white ping pong balls). Rather than "building up" to create equivalent ratios, students were asked to work problems with higher numbers of orange and white balls and to find ways to increase and decrease the total amounts without changing the "taste." They used doubling and halving, and out of these discussions the students created a critical construct which they named "the little recipe" (or as one child suggested, the base combination). This littlest recipe describes a critical concept in ratio and proportion reasoning, the smallest integral comparison to describe a given set. Theoretically, we refer to it as the ratio unit (a:b) as opposed to the unit ratio (a:1). The little recipe represented two fundamental ideas: 1) it gave the simplest formula for building equivalent ratios and 2) it expressed and coded the commonality across the different instances. In other work, I examine in more detail the development of the "little recipe" through the use of tables of data, examination of dot drawings, and notation including prime factoring (Confrey, 1995a).

8: Relating Slopes to Ratio and Interpolation

To help students understand the idea of ratios as invariance across a set of equivalent proportions, we taught them to make two dimensional graphs from the tables of values of equivalent ratios. Coordinates that fell along the same line had a common ratio or, put more formally, they were vectors that shared a common basis. In related historical work on Greek mathematics, we (Confrey & Smith, 1989) defined a ratio as "an invariance across a set of proportions," a view of ratio that was mirrored in the teaching experiment. This reverses the common stance that proportions are composed of equivalent ratios. Viewing pi as an expression of the invariant relation between circumference and diameter often helps people to understand this definition. The use of the two-dimensional plane gave the fourth graders a way to see ratio as a commonality among variation. It also supported a variety of strategies to compare ratios. Graphically they could build to a common x value and compare y's, build to a common y and compare x's, or look for differences in the angle of rotation of the vector line.

At this point, students had developed a discrete, rather than dense, notion of equivalent ratios. For a ratio unit of a:b, they could find the set of equivalent proportions of the form na:nb by incrementing and decrementing or doubling and halving. The next step was for them to be able to solve missing values problems for any target value. This was approached gradually through the use of a number of contextual problems and special representations. Contextually, we chose to work with the idea of constant slope and gave the children a design challenge to build a handicapped access ramp. This design challenge took the students about six weeks to work on. They began by finding ways to describe the slope of an incline they could safely traverse in a wheel chair. During this six weeks, they worked on a number of sub-problems such as how to copy a slanted line from one sheet to another without tracing and how to predict the height of a tree from its shadow length. Students made extensive use of tables in this work and from their tables, we developed and elaborated the idea of the "ratio box." A ratio box is a two by two table which students used to analyze the underlying ratio and eventually to solve missing value problems. Its strength in this context was that students viewed it as an abbreviated or collapsed set of table entries, so they saw it as representing an invariance which they already believed in. By exploring various ratio tables, students were able to find that not only does the same ratio relate x_1 to y_1 as x_2 to y_2 but x_1 to x_2 as y_1 to y_1.

This set of relations described by Noelting (1980b) and Vergnaud (1988) distinguished as "within" (or scalar) vs. "between" (or functional) ratios are brought into parallel and equivalent structure using the ratio box. An advantage of the ratio box lies in its symmetry in displaying the two kinds of ratios. In contrast, in traditional notation $x_1 / y_1 = x_2 / y_2$, this symmetry is lost. Though the ratio box facilitated the transition to mulitplicative reasoning, it was the modeling challenge that propelled the students towards a facility in missing value problems as it required them to extrapolate to a missing value with a large magnitude. Building up using ratio units was too cumbersome. The problem required students to use their calculation of the ratio of the shadow to the height of a ruler to find the height of a tree whose shadow was about 75 feet. The value of the contextual problems was that they assisted students in keeping straight the differences in the two kinds of ratios, while applying the same abstract reasoning.

Finally, and perhaps most importantly, the students developed a unique method of describing the movement from one cell to another horizontally or vertically (the development of cross multiplication as a strategy evolved later). Drawing upon their previous work with daisy chains, students described the movement from one cell to another (horizontal or vertical) using a daisy chain and used this same method to find the missing value. For instance, if a ratio box had 3 and 5 as x_1 and y_1 and 10 as x_2 they would search for y_2 by first arguing that to go from 3 to 5, you divide by 3 and multiply by 5, hence to find the missing value, you divide 10 by 3 and multiply by 5 to get 50/3 or 16 2/3. Sometimes they would check by going in the other direction, in this case, from 3 to 10 by dividing by 3 and multiplying by 10, applied this to 5 to confirm their prediction of 50/3. From this work, a deep-seated belief among students developed that if they were given three values of a ratio, a fourth one existed. There is more to the story of how this initial work with missing values developed into an understanding of multiplication and division of rational numbers as inverses, but space limitations do not permit this explanation here.

9: Differentiating Adding Fractions from Combining Ratio Units

A critical issue in the literature has been to clarify the distinction between ratios and fractions. The resolution of this within the splitting conjecture surfaced in as students undertook a jigsaw puzzle expansion problem (Douady, 1991). Each member of a small group was assigned a particular puzzle piece and required to make replacement pieces that were double the lengths of the sides. If each member acts correctly, the pieces fit together to create a similar puzzle with new dimensions. The question came up as to whether 1/4 + 1/4 = 2/8 which was 1/4. Students then needed to figure out when they were *combining ratio units*, in which case they added numerators and denominators and when they were adding fractions and required to find and keep a common denominator as they added the numerator. This discussion and one other in which the children debated whether 3 pizzas shared among 10 people produced 3/30 (each student got 3 pieces of the ten pieces per pizza or 3 of 30 total pieces) or 3/10. We determined that the critical issue was the identification of the whole, or the unit 1, thus it was 3/30 of 3 pizzas or 3/10 of 1 pizza. This led us to postulate that the difference between a fraction and a ratio is that within the context of fractional units, a share unit of one must be presumed. Thus, in splitting terms, 2/3 as a fraction means 2/3 of 1. Since the 1/n of 1 develops as an early construct in splitting, then 2/3 of 1 is not too advanced a concept. Confrey (1995a) illustrated this distinction on the two dimensional plane. A fracture of the plane along the line x=1 or y=1 produces a linear mapping of the set of ratios built as vectors onto a single one-dimensional line. This line has the structure of the number line, and the numbers along it, by virtue of sharing the line, share the unit of one. This number line has on it fractions, which can be seen as a subset of the ratios represented in the plane. Any vertical or horizontal line or line segment can be used to compare fractions provided one is clear about the assumed units.

Implications for Additional Topics

Our work in the fifth grade began with multiplication and division of rational numbers. We extended the work into an introduction to decimals and percent. This study, (Lachance, 1996; Lachance & Confrey, 1996), is important in that it demonstrates that the literature on decimals is overly reliant on its relationships to base ten to the neglect of the underlying ratio relations. In our studies, Lachance and I used an international weight comparison system based on varying ratios to create the need for a standard unit of comparison across weights. Decimals met this need. As a result, students demonstrated a stronger understanding of place value and a strong ability to describe underlying relationships as a result of the intervention. When introduced to percents, our students found the underlying reasoning easily compatible with their ratio work. On reflection, we realized that there is an isomorphism between the relationship between percentage and decimal that mirrors the relationship between ratio and fraction. This is evident in the parallel in a/b of n and c% of n. Moss and Case (1998) use percentage as a means to introduce multiplicative reasoning and it has the advantage of being a common cultural tool. We see considerable commonality in our approaches. Though we would not support an exclusive reliance on percentage to the detriment of ratio, we would consider the possibility of uniting these approaches.

Our students also demonstrated a deep understanding and intuition for rate relationships and for an easy transition to algebra. Lachance (1996) reports on their exploration of a problem involving falling dominoes in which they generated spontaneously explanations and predictions about acceleration and its appearance on graphs as curvature. Finally, in related work, not directly with these students, we have argued extensively that student understanding of exponential and logarithmic functions is greatly facilitated if these are viewed as covariation of additive and multiplicative structures (Confrey & Smith, 1995). This is an easy transition if the counting and splitting worlds are built to be independent but interrelated.

Conclusions

Since the time of the original splitting conjecture, researchers have begun to describe the relationship of additive and multiplicative reasoning as parallel rather than as prerequisite (Carpenter, Fennema, & Romberg, 1993; Kaput & West, 1994; Thompson, 1994). I believe this research on the splitting conjecture and on exponential and logarithmic functions has contributed the foundations for this change and made the most extensive and explicit arguments in its favor. Furthermore, by arguing for the importance of two-dimensionality to reasoning on multiplicative structures via splitting, we have been able to challenge the underlying dominance of elementary curriculum's implicit construction of the rational number line. Our work has established the importance of geometric reasoning and anticipation of algebraic and rate reasoning throughout the elementary curriculum. This complements the work of early algebra researchers (Kaput & West, 1994; Nemirovsky, Tierney, & Wright, 1995; Stroup, 1998; Thompson, 1994) and adds ratio to their work as a basis for an analytic passage into rate concepts. Finally, it is hoped that through the careful elaboration of the progressive path students followed and through the introduction of new representational forms (daisy chains, ratio boxes), the critical moments have documented the need for careful attention to building mathematical structure from student ideas and contextual challenges in modeling.

References

Behr, J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grows (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan Publishing Co.

Behr, M., Harel, G., Post, T., & Lesh, R. (1994). Units of quantity: A conceptual basis common to additive and multiplicative structures. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 123-176). Albany, NY: SUNY.

Carpenter, T. P., Fennema, E., & Romberg, T. A. (Eds.). (1993). *Rational numbers: An integration of research*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Confrey, J. (1988). Multiplication and splitting: Their role in understanding exponential functions. In M. Behr, C. LaCompagne, & M. Wheeler (Eds.), *Tenth Annual Meeting of International Group for the Psychology of Mathematics Education-NA*, Nov. 3-5 (pp. 250-259). Dekalb, IL: Northern Illinois University.

Confrey, J. (1991). Learning to listen: A student's understanding of powers of ten. In E. von Glasersfeld (Ed.), *radical constructivism in mathematics education* (pp. 111-138). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Confrey, J. (1993). Learning to see children's mathematics: Crucial challenges in constructivist reform. In K. Tobin (Ed.), *The practice of constructivism in science education* (pp. 299-321). Washington, DC: American Association for the Advancement of Science.

Confrey, J. (1995a). Student voice in examining "splitting" as an approach to ratio, proportion, and fractions. In L. Meira & D. Carraher (Eds.), *Proceedings of the Nineteenth International Conference for the Psychology of Mathematics Education*, *1* (pp. 3-29). Recife, Brazil: Universidade Federal de Pernambuco.

Confrey, J. (1995b). A theory of intellectual development: Part III. *For the Learning of Mathematics, 15*(2, June), 36-44.

Confrey, J. (1996). *Strengthening elementary education through a splitting approach*. Paper presented at the annual meeting of the American Educational Research Association. New York, NY.

Confrey, J., & Doerr, H. (1996a). Student Modelers. *Interactive learning environment, 4*(3).

Confrey, J., & Doerr, H. M. (1996b). Changing the curriculum to improve student understandings of function. In D. F. Treagust, R. Duit, & B. J. Fraser (Eds.), *Improving teaching and learning in science and mathematics* (pp. 162-174). New York: Teachers College Press.

Confrey, J., & Scarano, G. H. (1995). Splitting reexamined: Results from a three-year longitudinal study of children in grades three to five. In D. T. Owens, M. K. Reed, & G. M. Millsaps (Eds.), *Proceedings of the Seventeenth Annual Meeting for the Psychology of Mathematics Education-NA, 1* (pp. 421-426). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Confrey, J., & Smith, E. (1989). Alternative representations of ratio: The Greek concept of anthyphairesis and modern decimal notation. In D. E. Herget (Ed.), *Proceedings of the First History and Philosophy of Science in Science Teaching*, November (pp. 71-82). Tallahassee, FL: Science Education & Department of Philosophy, Florida State University.

Confrey, J., & Smith, E. (1995). Splitting, covariation and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Douady, R. (1991). Tool, object, setting, window: Elements for analysing and constructing didactical situations in mathematics. In A. J. Bishop & S. Mellin-Olsen (Eds.), *Mathematical knowledge: Its growth through teaching*. Dordrecht: Kluwer Academic Publishers.

Harel, G., & Confrey, J. (Eds.). (1994). *The development of multiplicative reasoning in the learning of mathematics*. Albany, NY: State University of New York Press.

Hart, K. M. (1984). *Ratio: Children's strategies and errors*. Windsor, Berkshire: Nfer-Nelson Publishing Co., Ltd.

Kaput, J., & West, M. M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 237-287). Albany, NY: SUNY.

Lachance, A. (1996). *From ratios to decimals: Introducing students to decimal notation through ratio and proportion*. Masters thesis, Cornell University.

Lachance, A., & Confrey, J. (1996). Mapping the journey of students' explorations of decimal notation via ratio and proportion. In Annual Meeting of the American Educational Research Association New York, NY.

Moss, J. & Case, R. (1998). *Developing children's understanding of the rational numbers: A new model and an experimental curriculum*. Manuscript in preparation, University of Toronto.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Nemirovsky, R., Tierney, C., & Wright, T. (1995). *Body motion and graphing*. A paper presented at the Annual meeting of the American Educational Research Association, San Francisco, CA:

Nesher, P. (1988). Multiplicative school word problems: Theoretical approaches and empirical findings. In M. Behr & J. Hiebert (Eds.), *Number concepts and operations in the middle grades* (pp. 19-41). Reston, VA: NCTM and L. Erlbaum Associates.

Noelting, G. (1980a). The development of proportional reasoning and the ratio concept Part I: Differentiation of stages. *Educational Studies in Mathematics, 11*, 217-253.

Noelting, G. (1980b). The development of proportional reasoning and the ratio concept Part II: Problem-structure at successive stages, problem-solving strategies, and the mechanism of adaptive restructuring. *Educational Studies in Mathematics, 11*, 331-363.

Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education, 14*(4), 307-317.

Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3-39). Albany, NY: State University of New York Press.

Stigler, J. W., & Perry, M. (1988). Cross cultural studies of mathematics teaching and learning recent findings and new directions. In *Effective mathematics teaching* (pp. 194-225). Hillsdale, NJ: Erlbaum Associates & NCTM.

Stroup, W. (1998). *A structural analysis of learning qualitative calculus in graphical environments: An overview and analysis of more than a decade of research*. Research Presession of the 76th Annual Meeting of the National Council of Teachers of Mathematics, Washington DC.

Thompson, P. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp.181-234). Albany, NY: SUNY.

Vergnaud, G. (1988). Multiplicative structures. In M. Behr & J. Hiebert (Eds.), *Number concepts and operations in the middle grades* (pp. 141-161). Reston, VA: L. Erlbaum Associates.

WORKING GROUPS

THE ROLE OF ADVANCED MATHEMATICAL THINKING IN MATHEMATICS EDUCATION REFORM

Working Group Organizer
M. Kathleen Heid
Pennsylvania State University
ik8@psu.edu

Working Group Panel

Joan Ferrini-Mundy
MSEB & University of New Hampshire
jferrini@nas.edu

Karen Graham
University of New Hampshire
kjgraham@hopper.unh.edu

Guershon Harel
Purdue University
harel@math.purdue.edu

Traditionally, the area of "advanced mathematical thinking" research has been the province of groups and individuals concerned primarily with the education of students at postsecondary levels. Links to questions of policy and practice in school mathematics education are not necessarily established. However, recent trends in mathematics education point us toward a case for more strategic thinking about the relationship between K-12 mathematics education and mathematics learning as it occurs in higher education, especially in the formulation and enactment of research agendas. Issues of articulation between school and postsecondary education continue to present serious practical challenges, and stronger synergy in the research enterprise would be worthwhile.

There is currently a proliferation of remedial mathematics courses at major universities along with increasing numbers of students taking advanced placement courses and coming to higher education with some background in calculus, the traditional starting point for the undergraduate program. Policies and practices that are established around the resulting array of articulation issues have direct implication for the background of students who will be involved in advanced mathematical thinking areas, and for the faculty who will teach them.

Closely related to these complex problems of articulation is the matter of the direction of reform in K-12 mathematics education. Standards-based efforts to improve mathematics education are underway–at least at the policy level–through state frameworks, new curriculum materials, and new expectations for teacher professional development. The foundations potentially provided through standards are critical to what will be possible later for students engaged in "advanced mathematical thinking" at the undergraduate level. One of the major criticisms of the NCTM *Standards* and related implementation efforts is that such efforts may not be sufficiently committed to advanced mathematical thinking, which many presume grows from experience with mathematical formalisms, understanding of mathematical structures, early experience with logical reasoning and proof, etc. The K-12 mathematics education community might be able to profit quite substantially from closer collaboration with those who consider questions about "advanced mathematical thinking." The K-12 community faces earlier versions of the same questions: What constitutes advanced mathematical thinking? What early experiences will help children move toward advanced mathematical thinking? Is advanced mathematical thinking in some sense "generic," or does it have particular characteristics within particular content areas of mathematics? "Backwards mapping" from some of the insights and issues considered by researchers at the postsecondary level would be a useful means for research to play a role in bridging the K-12/undergraduate gap. Thinking that is underway at postsecondary could help to guide research and practice at earlier levels, and vice versa.

A third area of intersection between "advanced mathematical thinking" and K-12 mathematics education has to do with the preparation of future teachers of mathematics. Although questions of teachers' understanding of mathematical content have been of interest within the "advanced mathematical thinking" arena, it would be appropriate to capitalize on heightened national attention to issues of mathematics teachers' content knowledge (several projects are underway, including one to produce new "standards" for teachers' mathematical content expertise). Research directions that seem especially promising might include careful examination of the ideas that teachers can learn useful mathematical content for "sites of practice." Although there is considerable support and momentum for this idea in arenas of policy and practice, the research and foundation from which more informed curriculum and program development might occur is thin.

Complementing the reform in K-12 mathematics education have been efforts to improve the teaching and learning of undergraduate mathematics. A national conference held at Tulane University in 1986, (Douglas, 1986) marked the formal beginnings of a movement to reform the teaching of calculus. The methods workshop of the Tulane conference listed several goals for calculus instruction which appear relevant to this working group's discussion (Davis et al., 1986) of "advanced mathematical thinking." According to these goals calculus instruction should focus on the development of conceptual understanding and flexibility in applying the subject matter, on improving students' ability to articulate mathematical ideas, and on develop students' abilities of construct logical arguments. Many curriculum development and implementation projects have been funded since the Tulane Conference and there remains continuing national interest in the area. Efforts have also begun in differential equations, linear algebra, and abstract algebra. All of these efforts could benefit from a deeper understanding about how students learn advanced mathematics.

Data from the CCH Evaluation and Documentation Project (Ferrini-Mundy, 1994) suggest that questions surrounding appropriate goals for calculus instruction still exist as do questions related to appropriate ways to implement these goals. One of the survey questions asked participants to respond how well they thought the CCH materials accomplish the following goal as stated in the instructor's manual, "Let formal definitions and proofs evolve from a long process of common sense investigations, rather than to start with abstract definitions." (Please note that although this information was collected as part the CCH documentation and evaluation effort, similar statements could most likely be made about many of the reform-based projects.) The collection of responses indicated that there was no general consensus on the issue. Respondents disagreed in a variety of ways on how mathematical thinking evolves. Another question in the survey asked the participants to state their own definition of "mathematical rigor." Here again, there was no overall consensus. Answers ranged from one-word definitions such as "proofs" to more extensive definitions such as:

- Mathematical rigor: highly symbolic methods that rely on fundamental axioms or theorems. Intuitive reasoning is not allowed.
- An argument that convinces another knowledgeable person.
- Establish with enough certainty to bet your life on.
- APPROPRIATE choice, use of tools (functions, graphs, logs, etc.) and solid understanding of the real-life meaning of results.

Participant responses to both of these questions raise important issues that need to be discussed and researched by individuals interested in the area of "advanced mathematical thinking." What constitutes "advanced mathematical thinking" at the level of calculus and beyond ? What constitutes "mathematical rigor" at the level of calculus and beyond ? How are the two quantities related ? What are the cognitive and developmental characteristics of this type of thinking? What are the most appropriate learning experiences for developing this thinking? What is the role of definitions and proof in developing such thinking processes? How can research in this area inform curriculum development in calculus and other more advanced courses in mathematics ?

There are powerful examples of the relationship of advanced mathematical thinking to school mathematics. Following are examples of ways of mathematical thinking (a) that are essential to the learning of advanced mathematical content and (b) whose development must start in an early age when elementary mathematical contents are taught. These examples will give rise to additional questions about advanced mathematical thinking.

Multiple ways of understanding. Most students' repertoires of reasoning do not include the understanding that a concept can be understood in different ways, that it should be understood in different ways, and that it is advantageous to change ways of understanding of a concept while attempting to solve a problem. In linear algebra, a course which requires multiple ways of understanding, one must understand, for example, that problems about systems of linear equations are equivalent to problems about matrices, which, in turn, are equivalent to problems about linear transformations. Students who are not equipped with these ways of thinking are doomed to encounter difficulties (See Harel, 1998.). At the precollege level, there are various opportunities to help students think this way. The study of fractions provides one such opportunity. Students should learn, for example, that the fraction 3/4 can be understood in different ways: 3 individual objects, each of quantity 1/4; the result when 3 objects are shared among 4 individuals; the portion of the quantity 4 that equals the quantity 3; and the number 3/4. Students should also learn that depending on the nature of the problem, some interpretations are more advantageous than others.

Transformational reasoning. Harel and Sowder (1998) coined the phrase "proof scheme" to refer to what convinces a person, and to what the person offers to convince others. They provided a system of three classes of proof schemes, which were derived from observations of the behavior of college students working in different mathematical domains. Key to the concept of mathematical proof is the transformational proof scheme—a scheme characterized by consideration of the generality aspects of the conjecture, application of mental operations that are goal oriented and anticipatory, and transformations of images as part of a deduction process. The education of students toward transformational reasoning must not start in college. Years of instruction which focus on the results in mathematics, rather than the reasons behind those results, can leave the impression that only the results are important in mathematics, an opinion sometimes voiced even by mathematics majors. Harel and Sowder argue that instructional activities that educate students to reason about situations in terms of the transformational proof schemes are crucial to students' mathematical development, and they must begin in an early age.

Practicing reasoning rather than mere application. Research has shown that repeated experience, or practice, is a critical factor in enhancing, organizing, and abstracting knowledge. The question is not whether students need to remember facts and master procedures but how they should come to know facts and procedures and how they should practice them. Consider the following examples (from Harel, 1998), one from elementary mathematics and one from linear algebra.

Two elementary school children, S and T, were taught division of fractions. S was taught in a typical method, where he was presented with the rule (a/b)(c/d)=(ac)/(bd), and the rule was introduced to him in a meaningful context and with an adequate justification that he understood. T, on the other hand, was presented with no rule. Each time she encountered a division of fraction problem, she explained its meaning and the rationale of her solution. S and T were assigned homework problems on division of fractions. S solved all the problems correctly, and gains, as a result, a good mastery of the division rule. It took T much longer time to do her homework. Here is what T—a real third-grader—said when she worked on (4/5)/(2/3):

How many 2/3s in 4/5? I need to find what goes into both [meaning: a unit-fraction that divides 4/5 and 2/3 with no remainders]. 1/15 goes into both. It goes 3 times into 1/5 and 5 times into 1/3, so it would go 12 times into 4/5 and 10 times into 2/3 (She writes: 4/5=12/15; 2/3=10/15; (4/5)/(2/3)=(12/15)/(10/15). How many times does 10/15 go into 10/15? How many time do 10 things go into 12 things? One time and 2/10 of a time, which is: 1 and 1/5.

T apparently had opportunities for reasoning of which S was deprived. T practiced reasoning and computation, S only computation. Further, T eventually discovered the division rule and learned an important lesson about mathematical efficiency (which is part of algorithmic thinking)—a lesson S had little chance to learn.

Similarly, ready-made theorems, formulas, and algorithms, even when motivated and completely proved, are hastily introduced in undergraduate mathematics courses. In Harel and Sowder's teaching experiments, an interesting phenomenon was observed. It illustrates the importance of practicing mathematical reasoning. Until a mathematical relationship is declared a theorem, the students continue—either voluntarily when they needed to use the relationship or upon request—to justify it. Once the relationship was stated as a theorem, there seemed to be a reduced effort, willingness, and even ability with some of the students to justify it. This phenomenon was explained in terms of students' authoritarian view of mathematics: For them the label "theorem" renders the relationship into something to obey rather than to reason about. Or, possibly, these students had not practiced enough the reasoning behind the theorem.

The ways of mathematical thinking we have identified here can be translated into essential cognitive objectives–objectives that would position elementary mathematics content for the successful subsequent learning of advanced mathematical content. But what is the complete set of such ways of thinking? Is the set a mere list, or does it have an underlying structure and is it guided by a small number of principles? Advanced mathematical thinking research can and should take the lead in answering these critical questions.

As we study the early development of advanced mathematical thinking (or the conceptual underpinnings of advanced mathematical thinking or the cognitive roots of advanced mathematical ideas), we should do so with an awareness of current theories and debates about the nature of mathematical thinking. Some of these theories, related to cognition or to concept images, suggest that our attention should be turned to ways that mathematical knowledge might be organized. Other theories (debated by mathematics education researchers like Dubinsky, Sfard, Tall and Gray, and Confrey) remind us to proceed with caution as we examine the qualitative evolution of students' understandings of mathematical entities, especially as our inquiry centers on the relationship between process understanding and object understanding. As we examine advanced mathematical thinking as it relates to mathematics education reform at the school and college levels, we might draw on the results of research programs, like that reported in a recent special issue of *Journal of Mathematical Behavior*, which have aimed at accounting for the current evolution of particular mathematical ideas (in this case, those arising in abstract algebra) in students. An understanding of the possible evolutions of mathematical understandings could inform the content and sequencing of advanced mathematical content. We might extend our understanding of students' advanced mathematical understandings by applying the results of researchers who, like Moore, Harel, and Sowder, have studied the development of an understanding of proof, of Edwards, who has examined the ways in which students' understandings of definition develop in the context of an advanced mathematics course, and of Williams and Zadieh who have studied the development of understanding of particular mathematics concepts like limit and derivative. In the current mathematics education reform arena, advanced mathematical thinking takes a center-stage role. We need to develop research agendas in advanced mathematical thinking that can serve to inform that reform.

References

Davis, R.B., Epp, S.S., Kenelly, J.W., Layton, K.P., Schoenfeld, A.P., Steen, L.A., Stein, S.K., Terry, S.S., & van der Vaart (1986). Notes on teaching calculus: Report of the methods workshop. In R.G. Douglas (Ed.). *Toward a lean and lively calculus: Report of the conference/workshop to develop curriculum and teaching materials for calculus and the college level* (MAA Notes Series No. 6, pp. xv -xxi). Washington, DC: Mathematical Association of America.

Douglas, R. (Ed.) (1986). *Toward a lean and lively calculus: Report of the conference/workshop to develop curriculum and teaching materials for calculus and the college level* (MAA Notes Series No. 6). Washington, DC: Mathematical Association of America.

Ferrini-Mundy, J. (1994). *CCH evaluation and documentation project.* Durham, NH: University of New Hampshire.

Harel, G. (1998). Two Dual Assertions: The First on Learning and the Second on Teaching (or Vice Versa). *The American Mathematical Monthly*, Volume 105, Number 6, p. 497.

Harel, G., and Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *CBMS Issues in Mathematics Education*, 6, 234-283.

VISIONS OF ALGEBRA IN DIVERSE INSTRUCTION

Algebra Working Group Organizers
David Kirshner, Louisiana State University, dkirsh@lsu.edu
Carolyn Kieran, University of Quebec at Montreal
kieran.carolyn@uqam.ca

Algebra Working Group Panelists
Tom Kieren, University of Alberta
Analúcia D. Schliemann, Tufts University

In the past decade or so, the vision of school algebra has gradually been widening to encompass activities and perspectives beyond the traditional study of literal symbols and operations on such symbols. This Working Group will sample the diversity of current approaches to algebra instruction, taking as its problematic the vision of algebra underlying each. This mission contributes to ongoing efforts to redefine and recharacterize school algebra as a more stimulating and more potent educational strand (Bednarz, Kieran, & Lee, 1996; Sutherland & Rojano, in press). However, we do not envision, nor seek, a unitary answer to the question "What is algebra?" Rather we will revel in the diversity of current practices, coming eventually to weigh the tensions and relationships between the visions of algebra they entail. This process, we feel, is the most productive route to informing the coming generation of school algebra initiatives.

The Working Group Plan

1. Preparations

The panelists each have prepared a one page summary of an approach that they have developed for teaching algebra (below). Similarly, we invite intending Algebra Working Group participants to contribute a one page summary of an approach that they have developed, used, or are interested in (see Guidelines for Submissions to Algebra Working Group, below). We will receive these contributions by September 30, and select a small number of them for presentation in the Working Group. Five approaches to algebra including those of the panelists, participants, and possibly one or both of the organizers will be used to structure the work of the Working Group.

2. First Day

After brief introductions, Working Group participants will view five Poster Presentations on the five approaches to algebra. During this time participants will be asked to sign up to study and discuss one approach of interest to them. (To help ensure an equal distribution of people to groups, there will be opportunity for cooperative participants to move to less heavily subscribed groups.) This period of time also is viewed as a social occasion to mingle and become acquainted with other Working Group members.

The remainder of the first day will be spent in the 5 subgroups established through the above process. Each subgroup will begin its work with a brief presentation by the Poster author. This presentation will focus on how the approach is supposed to be implemented, not on the vision of algebra that may have motivated that instructional approach. *It will then be the responsibility of the subgroup as a whole, and not just the author of the approach, to deliberate on the underlying vision of algebra.* The subgroup must prepare a comprehensive five minute presentation for the Working Group that describes the vision of algebra, as they see it embedded in the instructional approach. A member (or members) of the group, other than the Poster author should be selected to present the position of the subgroup to the Working Group. Overhead transparencies will be provided for each subgroup; however, we hope that the transparencies will be prepared, outside of Working Group time. A written record of the subgroup's position (possibly just a typed version of the overhead transparencies) will need to be prepared for distribution by the Working Group leaders.

3. Second Day

This day will be devoted to presentation and discussion of the subgroup reports. Each subgroup will be allocated a 20 minute time slot. We recommend the following breakdown of time use: 1) no more than 10 minutes for the Poster author to present the basic approach, 2) 5 minutes for the subgroup representative(s) to present the vision of algebra they see embedded in the approach, 3) the remaining 5 or more minutes for questions/ discussion with the audience. The text of the instructional approach and the summary of its underlying algebraic vision for each subgroup will be photocopied and distributed to all participants on the third day.

4. Third Day

On our final day we will consider the composite picture of algebra represented by the diverse instructional approaches previously considered. We will begin by creating an Algebra Collage by gluing statements, pictures, icons, etc. onto a large beach ball supplied for that purpose. Most of the rest of the session will be spent tossing around the algebra ball, as it were; considering the tensions and harmonies between alternative visions, and reflecting on the current state of thinking about algebra as reflected in our joint work together. If interest expresses itself, time will be taken at the close of the third day to consider future communication/projects of the Working Group.

The success of the Working Group depends largely on the quality of the five approaches selected as the focus for our activities. The approaches of the panelists are outlined below, in sample one page summaries. Each of the Working Group organizers (both experienced algebra researchers) is prepared to step in and fill another of the five slots, as needed. However, we hope that readers of this notice who are interested in participating in the Algebra Working Group will contribute a one page outline of an approach that they would like to have featured at the Working Group. Please see the following guidelines.

Guidelines for Submissions to Algebra Working Group

Intending participants are invited to submit a one page summary of an instructional approach to algebra that they have developed, used, or are interested in to David Kirshner by September 30, 1998. Mailing address: Department of Curriculum & Instruction, Louisiana State University, Baton Rouge LA 70803-4728; Email address: cikirs@lsuvm.sncc.lsu.edu; Fax number: (504) 334-1075; Phone: (504) 388-2332. If possible, please provide an email address when submitting your page.

Those persons whose submissions are accepted must be willing to:

(1) prepare a Poster Presentation of your approach to be viewed by the Working Group on the first day. It is essential that this poster describe only the instructional approach, and not the underlying vision of algebra.

(2) use your poster to give a presentation to your subgroup on your instructional approach. This will be followed by what is primarily an opportunity for you to hear from *others* what they see as the underlying vision of algebra embedded in your approach. As a responsible moderator you need to be willing to withhold your own comments until the ideas of others have been fully expressed and explored.

(3) use your poster (perhaps supported by overhead transparencies) to give a presentation on your instructional approach (maximum 10 minutes) to the Working Group.

Our criteria will include the conceptual similarity/difference of the submissions to one another, so selection or rejection should not be construed as a judgment as to the merits of the approach.

Pre-Reading

For intending Algebra Working Group participants who wish to do prereading, we recommend Kieran (1996), *The Changing Face of School Algebra*, available on the internet at http://www.math.uqam.ca/_kieran/, or through regular postage from David Kirshner, address above.

REPRESENTING ALGEBRAIC RELATIONS BEFORE ALGEBRA INSTRUCTION

Analúcia D. Schliemann, Tufts University

Research in mathematics education has consistently found that students have enormous difficulties with algebra. Paradoxically, more recent research shows that even seven year-olds understand the basic logical principles underlying transformations on equations (Schliemann, Carraher, Pendexter, & Brizuela, 1998) and that children in elementary school classrooms use algebraic reasoning while they interact with their peers and the teacher to solve relatively complex, open-ended problems (Schifter, 1998). Moreover, extremely successful attempts to teach algebraic representation from grade one are found in Bodanskii's (1991) work with Russian children.

These new findings suggest that it is time to seriously consider deep changes in the elementary and middle school curriculum and the possibility of having children discussing, understanding, and dealing with algebraic concepts and relations from the earlier grades. But such radical change demands analysis of children's own ways of approaching and representing algebra problems in different contexts and of the most adequate instructional models for initiating algebra instruction.

In a series of interviews and classroom activities we have explored how third and fifth graders intuitively produce notations to solve verbal problems. Our first findings show that third graders do understand that equal transformations on the two sides of an equality do not destroy the equality and that equal unknowns on the two sides of an equality could assume any value without destroying the equality. But they must overcome two main difficulties to solve algebra problems, namely, to accept to work out a solution from unknown quantities and to develop a notation for the unknowns.

To overcome these difficulties, we have used two approaches that seem promising. The first (Schliemann, Carraher, Pendexter, & Brizuela, 1998) involved discussions with an interviewer, guiding children to develop a consistent notational system for knowns, unknowns, and their relationships in events described in verbal problems. In this process, their use of circles and shapes to represent collective bunches appeared as a meaningful transitional notation between measured quantities and unknown quantities. The second (Carraher & Schliemann, in preparation) uses a computer software (The Visual Calculator™) where quantities are represented as directed line segments. The software help students visualize and discuss what happens to the lines when they are subjected to arithmetical operations. But the problems discussed in this environment cannot be solved through computational routines alone. Instead, they require reasoning about the relationships between numbers and physical quantities, and their representation in algebraic statements. We found that, as children deal with the problems they spontaneously use algebraic notation, write equations, and meaningfully discuss algebraic relations.

AS IT HAPPENS: ALGEBRA KNOWING IN ACTION THE POLYNOMIAL ENGINEERING PROJECT

Tom Kieren
University of Alberta

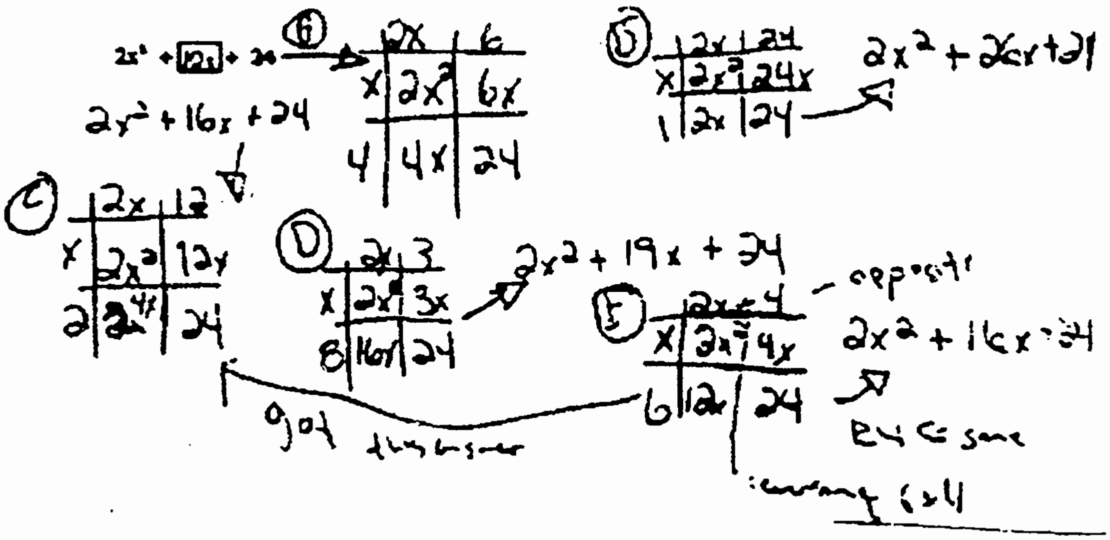
This project arose out of an attempt to understand in practical classroom terms what might be meant by the following view of mathematical knowing adapted for mathematical educational purposes from the work of Maturana and Varela(1987): "Knowing occurs in action as a students bring forth a world of algebraic significance determined each by their own structures or histories of actions, *with other students and a teacher, in a sphere of behavioral possibilities.* This view suggests that algebraic knowing is a coemergent phenomenon. Thus a teacher needs to understand all at once both how students knowing in action and how elements of the environment act as occasions for knowing.

To study such knowing we have interacted with students through some 30 hours, during which they are introduced to ideas from the algebra of polynomials. During this study students worked with two and three dimensional models of polynomials and were guided by some 60 lesson set-ups each of which was developed based on our observation of and our being occasioned by the student knowing exhibited in previous lessons. It was intended that these materials and lessons would allow the students (of very varied mathematical backgrounds) to experience and use polynomial concepts, language and computations to design and describe algebraic objects and to look for relationships in so doing. The algebraic actions in which students engaged included actions on physical materials; actions in which they developed images (both mental and on paper); actions with informal schemes which were sometimes used descriptively and sometimes used more conceptually; actions on standard formal expressions; and actions in which students deliberately inter-related physical, image, scheme and standard formal re-presentations of their thinking (Kotagiri (1992)). Deliberate attempts were made to have students illustrate their ideas for others and to explain their thinking in various modes. It is certainly beyond the scope of this note to capture either the instruction/learning aspects of this work or the research on mathematical knowing and dynamical understanding which is occurring within it. But a brief example might help:

Donny and Jennifer (2 students with very weak performance histories in school mathematics) are working on this prompt: **The following polynomial is known to form a rectangular design or tiling with pieces from the Polyset. However it is missing a term. What might the term be. Offer a possibility or a list of possibilities to go over with your partner.**

$2x^2$ + ////x +24

89



Donny to look for algebraic relationships among schemes (see arrows and notes). This approach appears to allow students with varying backgrounds to act together in bringing forth a world of algebraic significance and allows teachers and researchers to observe that knowing in action as it happens.

References

Bednarz, N., Kieran, C., & Lee, L. (Eds.). (1996). *Approaches to algebra: Perspectives for research and teaching*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Kieran, C. (1996). *The changing face of school algebra*. (Invited lecture, ICME 8 Congress, Spain, July, 1996, to be published in: *Selected lectures from the 7th International Congress on Mathematical Education*, edited by C. Alsina et al., Seville, Spain: FESPM)

Sutherland, R., & Rojano, T. (Eds.). (in press). *Algebraic processes and structure*. Dordrecht, The Netherlands: Kluwer Academic Publishing.

THEORIES AND EXPERIMENTS IN COLLEGIATE MATHEMATICS EDUCATION RESEARCH

Working Group Organizer
Ed Dubinsky
Georgia State University, USA
edd@cs.gsu.edu

Panelist: Francisco Codero
Centro de Investigación y
Estudios Avanzados del IPN,
Mexico
fcordero@mail.cinvestav.mx

Panelist: Joel Hillel
Concordia University, Canada
JHILLEL@Vax2.Concordia.CA

Panelist: Rina Zazkis
Simon Fraser University, Canada
Rina_Zazkis@sfu.ca

The structure of this paper follows the structure of the three two-hour sessions of the Working Group. The paper consists of four sections written, respectively by Ed Dubinsky, Rina Zazkis, Joel Hillel and Francisco Cordero. The first two sections will form the basis of discussion at the first Working Group session and the second two sections will be considered in the second session. The third session will be devoted to general discussions and topics which are proposed by members of the Working Group.

The format of the first two sessions will consist of one hour devoted to each of the sections in this paper. The first 20 minutes of the hour will be a presentation of the material in the corresponding section of this paper followed by 30 minutes of general discussion. There will be a 10-minute break after the first discussion, and after the second discussion we will consider our activities for the third session.

In the first section, Dubinsky raises the question of what a theory of learning might be and how it can be used in collegiate mathematics education research. To set the discussion going, he describes how his own work is one possible response to these questions.

In the second section, Zazkis describes a model, which is preliminary to developing a theory, based on "Fuzzy Logic" which is an alternative to standard logic. It is a tool for understanding students' thinking and can serve as an aid in introducing students to mathematical conventions.

In the third section, Hillel describes in some detail the design of an experiment using a dynamic geometry computer environment to study students' understanding of eigenvalues and eigenvectors. His research group found it necessary to first investigate understanding vectors, linear combinations, and linear transformations. Having created an instructional design for fostering the learning of these concepts, they found some unexpected confusion by the students between the dynamic aspect of the geometry (e.g. dragging vectors) and linear algebra operations such as taking linear combinations.

Finally, in the fourth section, Cordero considers certain categories of mathematical knowledge that form an epistemology for Calculus and Analysis. This is the beginning of a comprehensive program for organizing collegiate mathematics and investigating students' learning in this domain.

Acknowledgments. Contributions to the discussion leading to this paper were made by the Working Group Advisory Board consisting of Broni Czarnocha, Bill Martin, Alan Schoenfeld, Pat Thompson, Draga Vidakovic, and Joe Wimbish. The work described by Dubinsky was participated in by various members of RUMEC. The project discussed by Hillel is a joint work with Anna Sierpiska and Tommy Dreyfus. Jana Traglova also participated in the design and implementation of the experiment. Some of the results reported by Cordero were obtained by Ricardo Cantoral and Rosa María Farfán.

1 Use of theory in studying how collegiate mathematics can be learned

The purpose of this first discussion is to consider application to collegiate mathematics of the general notion of using a theory to investigate how learning mathematics can take place. First, we will consider what might be meant in education by the term "theory". Next is the question of how a theory might be used and I will start off the discussion with an example from my own work. After a very brief description of the theory I prefer to work with, I will indicate some examples of its use. Finally, I think we must consider the question of research in which a theoretical perspective may not be helpful, or at least be premature.

Much of what I have to say here will be focused on one particular theoretical perspective — the ones I use. I do this in the spirit of hoping to hear about alternatives in the discussion.

What can be meant by the term "theory"? According to Alan Schoenfeld in "Toward a theory of teaching-in-context", to appear in *Issues in Education*, models and theories support prediction, have explanatory power, and are applicable to broad ranges of phenomena. To this I would add that a theory can help organize one's thinking about complex, interrelated phenomena, serve as a tool for analyzing data, and provide a language for communication of ideas about learning that go beyond superficial descriptions.

This is one view of what a theory might be. The Working Group can discuss other views and possibly organize a critique and / or synthesis of various ways of understanding the meaning of theory in educational research.

A research paradigm for using a theory. Aside from what might be meant by theory is the question of how a theory is used in actually conducting research. Again there are different views about this which the Working Group can discuss. In my presentation I will describe the way in which theory relates to the research I do.

I think that mathematics education research in its fullest sense ought to be some combination of theoretical analysis, design and implementation of instruction, and the gathering and analysis of data. Things get a little more interesting when you ask about the nature of the combination. The way in which I combine them is illustrated in Figure 1.

As a constructivist, I think that the role of theoretical analysis should be to propose mental constructions a person might make in order to learn and understand a mathematical topic. Instruction should be designed so as to foster these constructions in the context of the mathematics to be learned. Implementation may require pedagogical strategies and tools other than what is traditionally used. Data of all sorts need to be gathered using qualitative methods, quantitative methods and combinations thereof. Triangulation should be an important goal. Finally, the relation between theory and the analysis of data is two-fold. The theory can direct the analysis of an often unmanageable mass of data by helping to focus on specific questions. Conversely, a theory lives or dies by the data. Did the students make the mental constructions proposed by the theory and if so, did they learn the desired mathematics? In my view, the value of a theory should be almost totally based on the answers to these questions.

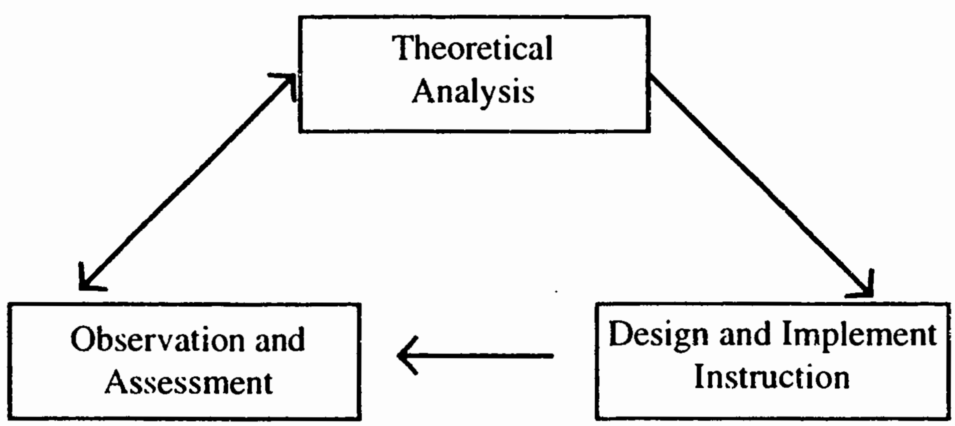


Figure 1. Relationship of Theory to Research

APOS Theory: one example. The APOS theory, put forward, developed and used by a Research in Undergraduate Mathematics Education Community (RUMEC), postulates that learning a mathematical concept consists in constructing certain actions, processes, and objects and organizing them in schemas. An action is a mental or physical transformation of mathemat- ical objects that is directed externally by a set of explicit instructions. An action can be reflected upon and interiorized into a process in which the transformation takes place internally under the control of the subject, perhaps in her or his imagination. A process can be encapsulated into an object to which actions or processes can be applied. The richest form of such a construction will allow the subject to alternate in interpretation between process and object. Finally, a schema is a collection of actions, processes, objects and other schemas which is coherent in the sense that there is some general criteria for knowing in advance if a previously unmet phenomenon fits in the schema. Schemas can also be thematized so as to become objects.

We can see how this relates to what I mean by theory. The explanatory power of APOS Theory, which also provides a language for communication, lies in its expression of understanding a mathematical concept in terms of very specific mental construcions. This can explain student difficulties in the sense that fine-grained comparisons of successful and unsuccessful student performances are related to making, or not, the specific constructions proposed by the theory. It is here that APOS theory supports prediction: a falsifiable assertion is that if a student makes certain specific mental constructions then he or she will be successful in a certain mathematical situation. The theory provides a tool for analyzing data by focusing on specific questions to ask of the data. There is no doubt that in using APOS theory, researchers organize their thinking in terms of actions, processes, objects, schemas and their interrelationships. Finally, there is really now a fair-sized (and growing) body of published research into a broad collection of topics in which APOS theory is applied. This includes mathematical induction, predicate calculus, thefunction concept, elementary set theory, slopes, limits in calculus, the chain rule, cosets, Lagrange's theorem and quotient groups, permutations and symmetries, number theory, elementary concepts in statistics, place value in arithmetic and fractions. It is used both by those involved in developing the theory as well as others who have read about it.

Using APOS Theory. One fairly simple, but very effective application of APOS theory is to the concept of coset of a subgroup in abstract algebra and Lagrange's theorem. Both of these can be very difficult for students, although to working mathematicians they appear quite simple. The difference, according to APOS theory, is the ability to go beyond an action conception of coset to understand the process of forming cosets and then to encapsulate this process into an object.

More specifically, an individual's understanding of coset is restricted to action when he or she can only think in terms of some formula such as the set of integers whose remainder on division by 4 is 2. This will work only in such simple cases as groups of integers and subgroups of all multiples of a single integer. For more complex groups such as the group of all permutations of *n* objects, it is necessary to understand a coset of a subgroup as a process of applying the group operation with a single, fixed element of the group and all elements of the subgroup. Finally, this process must be encapsulated to see a coset as an object to which actions and processes can be applied, but always based on the process of its formation.

Our research has suggested that students often do not see cosets as objects in this sense. When they do, counting the number of cosets, comparing their size and checking their common elements becomes fairly simple actions on these objects, also, if necessary, interpreted as processes. Since such actions amount to a proof of Lagrange's theorem, it is not surprising that when we design instruction (using cooperative learning, computer programming, and active learning methods) focusing on mental constructions of cosets as processes and objects, students appear to find Lagrange's theorem more accessible than in standard abstract algebra courses.

Sometimes not. Let me close this essay by making clear that I am not suggesting that everyone uses a theory in this or some other sense in all research. A theory is a tool that, in each case, one chooses to use or not. Although I am strongly convinced of the value of using theory in general, and APOS Theory in particular, this is not a dogma and I have engaged in research related to visualization as well as predicate calculus (in studies other than ones in which APOS Theory *has* been used) in which there is either a different kind of theoretical analysis, or none at all — for the present.

2 What is "true" in mathematics?- a Fuzzy Logic perspective on students' arguments.

I would like to suggest a model (not yet developed enough to be called a "theory") for interpreting some of the students' arguments in making decisions about the "truth" value of mathematical statements. The context I refer to is a course "Foundation of mathematics for elementary school teachers", which is a core course in many teacher education programs.

Consider the following statements:

(1) Prime numbers are odd.

(2) Even numbers are divisible by 4.

(3) Numbers that have the last digit of 5 are even.

In a mathematical convention each one of the statements is "false". However, a repeating "error pattern" in the responses of my students was to label the statement (2) as "partly-true, partly-false" and label the statement (1) as "almost true" or "true with one exception". Although such students' responses do not receive credit on any graded assignment, they are much more informative than the conventionally correct, "false". To consider the loss of information in a traditional approach of standard bivalent Aristotelian logic, a new domain of mathematics was introduced in the late 60s, called Fuzzy Logic.

In my presentation I will explain the motivation for developing Fuzzy Logic and Fuzzy Set theory. Briefly, Fuzzy Logic is a multivalent logic, which sees the True/False dichotomy of standard logic as a continuum, where the truth value of a statement is represented by a number in the closed interval [0,1]. A statement can be "true to a degree" and "false to a degree", which means "true and false" at the same time. Therefore, from a perspective of a Fuzzy Logic statement (2) is indeed 50.

It has been my conjecture that mathematical decision making by many students does not correspond to standard logic and can be modeled using Fuzzy Logic. I am conducting a research project that attempts to investigate decision making of pre-service elementary school teachers regarding the truth value of mathematical statements. I will share results of this research and present several questions for discussion.

After being submerged in mathematical culture for many years we take for granted that in order to claim that a given mathematical statement is true one has to provide a convincing argument — a proof, whereas in order to claim that a statement is false, a single counterexample is sufficient. Is this a prevalent way of decision making among individuals aiming at a teaching career? After the mathematical convention had been discussed in class the following story was presented to a group of 58 preservice elementary school teachers, seeking their written response:

> Jennie has been asked to decide whether a given mathematical statement was true or false. She checked several examples at random. In three cases the statement was true, in two other cases it was false. Can you help Jennie make a decision? What will be your advice?

About one half of the students claimed that the statement was false because one example was sufficient to disprove it, and, in fact, Jennie found two such examples. The responses of the other half varied; several typical arguments follow:

- I suggest that Jennie keeps checking more examples to see on what side (true or false) you have more.
- When you don't get true in all the cases or false in all the cases you have to guess. In Jennie's case she should guess "true" because then her chances of getting it right are 3 to 2.
- The statement is both true and false because you've got examples of both cases.
- My advice will be to ask Rina or one of the assistants.

In students' responses (the first three) we can recognize applications of fuzzy arguments to a situation where they are not applicable or not invited.

(By "fuzzy" I mean here multivalent rather than imprecise or vague.) While in standard logic a statement is either true *or* false, the ideal of true ***and*** false is acceptable in Fuzzy Logic. Therefore a Fuzzy Logic model has a strong p ower to explain students' responses. Using this model we identified several cases in which students' errors are not the result of misunderstanding concepts, but rather the result of a failure to apply a standard logical argument.

Excerpts from clinical interviews with students will be presented. I will discuss students' responses for mathematical decision making as well as the influence of quantifiers on determining the truth value of a statement. For example, students' responses to "even numbers are divisible by 4" were quite different from their responses to "all even numbers are divisible by 4".

I believe that acknowledging students' intuitive fuzzy perspective on mathematics can help in understanding students' thinking and can serve as a springboard in introducing students to mathematical conventions. I wonder how this model can be developed further to become a theory for explaining students' learning of predicate calculus.

3 Eigenvalues and eigenvectors: genesis of a research project in linear algebra.

In a typical first linear algebra course, eigenvectors of a linear transformation are defined, then illustrated geometrically by means of co-linearity of a vector and its image, and finally, students are taught the procedure of finding eigenvalues and eigenvectors. Yet, when asked at the end of the course to define an eigenvector, students' answers generally go like this "it is when you find the characteristic equation and solve for the roots and...". Or, when asked to decide whether a particular vector X_0 is an eigenvector of a matrix A, students still go through the whole general procedure. In other words, it seems that the majority of students have not acquired the concept of a eigenvector at the end of their linear algebra course.

Given the centrality of the notion of eigenvectors in linear algebra a research project probing into students' conceptions seemed to us timely and sufficiently focused. However such probing quickly opened up the proverbial can of worms of other difficulties which, in retrospect should not have come as a surprise. Eigenvectors are linked to the notion of both vector and linear transformation and very quickly we found ourselves asking what our students' conceptions of both of these notions were. Some of the more salient aspects that emerged were that the concept image of a vector was that of an n-tuple; that there was a lot confusion about the status of a vector when one n-tuple was represented by its coordinate vector relative to a basis; that matrices were often looked at in a static way as representing system of equations rather than transformations; and that the model of vectors as arrows in the usual coordinate 2- and 3-dimensional space was not well understood (for example, when asked to show vectors (x_1, x_2) such that $x_1 + x_2 = 1$, students drew the line $x_1 + x_2 = 1$ and drew some arrows lying on the line).

Our initial aim of looking at eigenvectors and designing a brief instruction sequence to foster their construction by students led us to a longer term design starting with the notions of vectors, basis, linearity, and transformation. The design was to be based both on our findings about students' conceptions as well as on epistemological analyses of basic mathematical concepts and of the modes of reasoning of linear algebra (see chapters VI and VII in Dorier's recent book on research in linear algebra, 1997 Dorier, J.L. (Ed.), L'enseignement de l'algébre linéaire enquestion, Bibliothèque Recherches en Didactique des Mathématiques). We also looked at instructional practices — for example a series of videotapes of four colleagues teaching eigenvectors showed how easily we tend to slide back and forth between the abstract, the arithmetic (n-tuple), and the geometric (arrows and points) modes of description of vectors without ever making these shifts explicit to the students.

In our project we decided to try to restore the geometric roots and thinking in linear algebra at the intuitive and heuristic levels without downplaying the power of analytic methods. Our decision to start with a geometric model was influenced by the possibility of creating such models within a dynamic geometry software such as Cabri. We projected an instructional design in three phases: the phase of geometric intuitions in the plane, the phase of arithmetization and generalization to higher dimensions, and the phase of applications. In the first phase, the main goal is the students' interaction with a geometric model of the two-dimensional vector space, constructed in Cabri. The notion of linear transformation in the two-dimensional space is introduced in a geometric way as a generalization of the one-dimensional notion of proportionality. Eigenvectors are modeled by the idea of the invariant line of a linear transformation, and are visualized by the co-linearity of a vector and its image. The arithmetization process in the next phase starts only after students have gained some geometric intuition for vectors, bases and transformations. It starts with the idea that a linear transformation is completely determined by its values on a basis. In the final phase, applications are emphasized in several contexts, not only geometric ones.

But even a decision to begin with a coordinate-free geometric model for vectors necessitatedmaking a choice among: 1. Vectors as directed line segments emanating from a fixed point; 2. Vectors as dots; 3. Vectors as equipollent line segments; and 4. Vectors as translations. We considered the tradeoff among these possibilities and opted for the first. We created CR2, a Cabri model of 2-dimensional vector space (including vectors and a scalar line) and designed an instructional sequence for the first phase of the study. While we have given a lot of thought to the different aspects of the design (which includes the geometric environment, scripted interventions by the instructor, and students' activities), our pilot project with a pair of students produced very inconclusive results.

A pair of (relatively weak) students does not make a reasonable scientific experiment but it was sufficient to wave some red flags about several problems, some of our making, others perhaps inherent to a dynamic geometry environment. For example, a vector v with a fixed initial point can be dragged on the screen - it is still labeled v so our students' initial idea was that the only attribute of vector worth attending to was its length - vectors were not equal only if they have different lengths (discussing equality of vectors and, later on, equality of linear transformations, was not in our design). This looks like a fixable problem. More serious was the implicit idea that a single vector v "generates" the whole space since every other vector can be obtained from v by dragging. This problem only became apparent when we started to work with bases and linear transformations. In particular, our students had trouble understanding why two (basis) vectors are required in order to determine a transformation. A Cabri vector v was looked upon as a variable vector and the implicit change to an 'arbitrary but fixed' status (which essentially meant that dragging was not a legitimate option) created serious obstacles. Certainly our faith in the potential of a dynamic geometry environment to provide solid intuitions for basic linear algebra concepts has been shown to be a bit naive.

We have now changed our design and are about to experiment with 3-4 pairs of students. Our new design is more elaborate and, at this stage, touches only the concepts of vectors, linear combinations, and linear transformations. So eigenvectors, which were the original raison-d'etre for our research, have been put on the back burner. Thus we have begun with a specific concept and with the idea of being able to design some practical instructional interventions. But research seems to have taken its own course. It has generated new questions and led us to design a new learning environment — one which will enable us to answer some questions but which is impractical as an instructional design in real classroom situations (it is too elaborate). We feel that we gained much better understanding of some of the conceptual difficulties that students have, and the possible shortcomings of dynamic geometry models. But we have gotten further away from our initial concern with eigenvectors and, at this moment, it is not clear if we will be able to reestablish that connection.

A question we might consider in the Working Group is whether the kind of evolution of the research project which I have described is more or less inevitable whenever one tries to examine a complex mathematical concept which is linked to other concepts or are there ways to avoid the process of "infinite regress"?

4 The role of some categories of mathematical knowledge in the teaching and learning of collegiate mathematics.

The purpose of investigating categories of mathematical knowledge in relationship with teaching and learning in collegiate mathematics is to find an epistemological basis for explaining all inter-actions among knowledge, teacher, student, classroom, institution, social situation and cultural frameworks. This basis could be a comprehensive program to organize collegiate mathematics.

I am going to consider an example from Calculus and Analysis. I will use some results that my colleagues and I have achieved in the past several years. With this example I will point out the main aspect of our point of view. And finally, I would like to reflect on how this kind of view could modify the sources from which the individual makes mental constructions. In this sense the focus of research will be on mathematical content in that it means to pay attention to the relation of knowledge to the physical and social characteristics of situations to which the student must attend.

I will start with the example but first, it is necessary to describe our epistemological position on the nature of Calculus and Analysis. I will explain some aspects of the basis of our epistemological position through a set of questions.

What does the abstraction of properties and relations of operations in Calculus and Analysis give or offer? Certainly mathematical knowledge, but why does that mathematical knowledge exist? In other words, what is the relation of mathematical knowledge to physical and social characteristics of situations?, What is an adequate and careful selection of situations that bring out mathematical knowledge?, What are the relations between problems to be solved and specific conceptions?

Calculus and Analysis would not exist if they did not provide an acceptable model of some reality, such as physical entities, and if it did not help to work on empirical problems. Therefore, the categories of mathematical knowledge that make up an epistemology perhaps should be in a functional framework. This means that the individual establishes relations between processes and objects through meanings. Thus, these categories do not correspond to logical operations but rather with modeling and use. Certainly these could be the basis of those categories.

Some categories that we have found in our epistemological studies, so far, are the following: prediction notion, due to Cantoral, accumulation notion and tendency behavior of functions due to Cordero, and permanent state notion due to Farfán. And each one is in relation with the mathematical structure of Calculus and Analysis: approach, derivation, integration and convergence.

The functional aspect focuses the kind of categories that contrast with categories found in the mathematical structure. The contrast consists of procedures obtained, on one hand, for representations, and on the other, for formal operations. In this sense, the important aspect of categories, in mathematical didactics, do not consist in establishing a mathematical definition but in establishing or identifying all relations of mathematical content (tool and meaning) through representations and procedures in this framework.

The nature of these relations relates more to the manner of construction of processes and objects than to processes and objects themselves. This corresponds to cognition, that is, variability of the frameworks and multiplicity of representations affect the method of construction. To know how they are affected we must analyze the progress and restriction of mental constructions.

The inter-relation of mathematical content, representations and procedures are elements that we want to see in a "developmental understanding structure". For that, we must design situations that take into account inter-relations of context organized by those categories.

The methodology that we use for the design of situations is composed of three dimensions: epistemological, cognitive and didactic. The epistemological dimension establishes the reference framework of mathematical content. This is necessary for the cognitive dimension where the plans of representations and procedures of the student appear. Also both the framework and the representations and procedures are necessary for the didactic dimension to establish arguments or explanations of what is organized there. In this sense the categories transform to a program that organizes contents, concepts and ideas.

Thus, the epistemological explanations with their social interactions (society, school, institution and culture) are based on the following elements: meanings, symbolical systems, procedures, processes and objects, and arguments. These elements together we call construction of representations.

Moreover, the construction of representations covers, so far, three groups: variability of the variables, graph of the function and formal expression. Each group is an epistemological framework of mathematical knowledge and also the students construct representations and apply procedures in relation with the operations that they are able of capturing and transforming, and in relation with the conditions that are being built in progress. For example, following horizontally each group the procedures that have been obtained of the students are: comparison of two states, variation of variable and coefficients of a transformations and formal operations. The process and objects on which they work are: quantity, form of the graph of function and function. And arguments that they have generated are: taking the differential element, tendency behavior of function and analytic function.

This kind of program not only orients what should be Calculus and Analysis for teaching and learning, but also composes a program of investigation of the collegiate mathematics in the field of mathematics education. The orientation consists in taking tools such as a base of these categories. This would be to pay attention to the progress of a student's executions or in the development of new or better methods of use. Here structure is not a priori a set of arrangements, but rather is the result of the kind of activities and actions that a tool permits when used in a certain way. This carries out the act of seeing similarities in structures and these are different contexts that could be the source or the basis of abstraction. Concretely, we are referring to all the relations and interactions that could be established among the three groups referred to above.

GENDER AND MATHEMATICS: INTEGRATING RESEARCH STRANDS

Working Group Organizers

Suzanne K. Damarin
Ohio State University
damarin.1@osu.edu

Diana B. Erchick
Ohio State University
erchick.1@osu.edu

Dorothy Buerk
Ithaca College
buerk@ithaca.edu

Linda Condron
Ohio State University
condron.1@osu.edu

Jere Confrey
University of Texas
jere@mail.utexas.edu

Ruth Cossey
Mills College
rcossey@mills.edu

Laurie Hart
University of Georgia
lhart@uga.cc.uga.edu

Peter Appelbaum
William Patterson College
pappel@ frontier.wilpaterson. edu

Patricia Brosnan (Convenor)
Ohio State University
brosnan.1@osu.edu

Our working group/discussion session constitutes a coming together of researchers in mathematics education in order to weave together the findings of various strands in research and understanding of issues in gender and mathematics. Over the past several years there has been a significant turn in the study and understanding of women's (and girls') relationships to the study and use of mathematics. Fifteen to twenty years ago, two kinds of studies dominated research on women and mathematics: (1) studies of sex differences in mathematics ability, study, achievement, and use, and (2) causal modeling studies which sought to identify social and psychological variables which interacted to predict female success in mathematics.

While these lines of research provided interesting findings and led to modest changes in some practices of mathematics education, they were insufficient to address the issues of gender and mathematics in several regards. The statistical findings on sex differences in predictive (independent) variables were of a much smaller magnitude than the observable gender differences in mathematics participation by women, and the purported explanations for the lesser participation and success of women in mathematics lacked credibility in the views of many women (both mathematically successful and not).

Beginning with Dorothy Buerk's 1983 qualitative study of women in college mathematics classes, many researchers interested in gender and mathematics initiated new modes of research on the phenomena. Over the past fifteen years, many studies and theory building activities based on diverse research methods and theoretical bases have been conducted in the area of gender and mathematics. These have been conducted in parallel with two other developments: (1) the rethinking and re-examination of gender issues by feminist psychologists (e.g., Bem, 1993; Kimball, 1995), and (2) increased attention to the philosophy of mathematics and the sociology of knowledge within the mathematics education community (e.g., Ernest, 1994; Restivo et al., 1993). These broader considerations of gender and of knowledge building lend support to new lines of research and provide impetus for work which can weave them into a more integrated understanding of the current status of research on gender and mathematics.

In this context the purposes of this discussion at PME-NA XX were: (1) to bring together in one discussion reports on the status of research of gender and mathematics conducted within several research frameworks; (2) to identify commonalities and conflicts in the research findings; (3) to identify critical questions that must be addressed; (4) to work toward the development of an edited volume conceived as a handbook of current research, findings, and issues in the area of gender and mathematics; and (5) to provide for the mathematics education community and other interested researchers and practitioners a comprehensive view of the current state of knowledge about gender and mathematics by weaving together the various strands of knowledge and examining the resulting fabric.

The researchers participating in this session provided the following contributions to form the foundation for our integrated perspective. We examine what we learn through (1) research based on *Women's Ways of Knowing* (Erchick); (2) research based on critical theory and media studies (Appelbaum); (3) applications of feminist theories (Damarin); (4) a current re-reading of the "classics" of gender and math (Hart); (5) studying the intersections of gender and race (Cossey); (6) studies of women in math-using fields (Condron); (7) studies of equity and K-12 curriculum (Confrey); and (8) talking with women in college mathematics classes (Buerk).

References

Bem, S.L. (1993). *The lenses of gender.* New Haven, CT: Yale University Press.

Ernest, P., (Ed.). (1994). *Mathematics, education, and philosophy.* London: Falmer Press.

Kimball, M. M. (1995). *Feminist visions of gender similarities and differences.* NY: Haworth.

Restivo, S., Van Bendegem, J.P., & Fischer, R.,(Eds.). (1993). *Math worlds.* Albany: SUNY.

GEOMETRY AND TECHNOLOGY

Working Group Organizer
Doug McDougall
University of Toronto
dmcdougall@oise.utoronto.ca

Working Group Panel

Don Chazan
Michigan State University
dchazan@pilot.msu.edu

Chronis Kynigos
University of Athens, Greece
kynigos@cti.gr

Rich Leher
University of Wisconsin-Madison
lehrer@macc.wisc.edu

Introduction

The working group on Geometry and Technology will discuss the integration of geometry and technology from the student and teacher perspective. This paper will provide a framework for some of our discussions and to identify researchers who have contributed to the conversation. We will first discuss the student perspective, followed by the teacher perspective.

The use of computers in a geometric environment has been investigated by a number of researchers (e.g. Kilpatrick & Davis, 1993; Laborde, 1993; Noss, Hoyles, Healy & Hoelzl, 1994; Schwartz, 1994; Kajander, 1990, 1989). Computers, especially with their graphic capabilities, may facilitate the construction of geometric concepts (Clements & Battista, 1992, 1994). Since computers have been introduced into the teaching and learning of mathematics, several software packages have been developed aimed at improving learning of mathematics in general and geometry in particular (Laborde, 1993).

One of the first software programs developed to investigate geometric relationships was LOGO, beginning with Papert in 1980. Many researchers have used LOGO with varying degrees of success in mathematics (Johnson-Gentile, Clements & Battista, 1994; Clements & Battista, 1994; Cohen, 1987; Hoyles, 1987; Hillel, 1986; Goldstein, 1985). Recently, Johnson-Gentile et al. (1994) used LOGO motion with two groups: one group used paper and pencil for the transformations while the other group used LOGO on a computer. They observed that the computer group developed better geometric thinking skills. Based on these and other results, it is important that we investigate the use of other software programs in the learning of geometric relationships.

A growing number of teachers have used the dynamic geometric software programs as the basis for geometric construction in place of a compass and straightedge. Dynamic geometric software allows the user to explore geometric properties and relationships and to manipulate images on the screen to investigate relationships and patterns in geometric constructions. This makes it possible for the user to make and explore conjectures about the generality of their observations (Chazan & Houde, 1989). Because students will construct adequate drawings with the computer, they can concentrate on the exploration and investigations of geometric relationships. This new focus on exploring conjectures can have the effect of bringing curiosity, inquiry, and research into the mathematics classroom (Schwartz, 1994).

The use of dynamic geometric software began in 1985 when Judah Schwartz and Michal Yerushalmy developed the Geometric Supposers (Schwartz & Yerushalmy, 1988). In their previous research on geometric constructions and proof, Yerushalmy et al. (1987) found that students had difficulty with problem posing and with inductive thinking. They felt that students would be better inductive thinkers if they could manipulate the geometric figures on a computer screen. The Geometric Supposers software program was therefore developed to give students the opportunity to test conjectures quickly without the difficulties that arise when one uses a compass and straightedge. The Geometric Supposers program is a member of the first generation of geometric dynamic software.

Since 1987, two other programs, Cabri-geometrie and Geometer's Sketchpad, have been developed in addition to the Geometric super Supposer program. Laborde et al. (1988) developed Cabri-geometrie as a second generation program (presented at the ICME IV meeting in Budapest). The Geometer's Sketchpad program was developed by Nicholas Jackiw, in early 1990, also as a second generation of educational geometry software. These dynamic geometric software programs enable the student to perform constructions and to observe the changes while they manipulate geometric shapes on the computer screen. The transformation and scripting ability of these programs has broadened the scope of what can be done with geometric software.

There is a considerable amount of curriculum materials available to the dynamic geometric software user. A high school geometry textbook by Serra (1989) encourages the students to create their own geometric constructions and formulate the mathematics for the relationships that they discover. Students can work individually or in groups to do investigations and discover geometric properties. Students are encouraged to look for patterns and use inductive reasoning to make conjectures (Jackiw, 1992). Other manuals and textbooks (Serra, 1989; Bennett, 1996) have additional activities for students and teachers.

Researchers are asking important questions about what teachers can do to better integrate geometry into mathematics classrooms. In Linn and Pea (1994), Linn asked a key question about teachers in a geometric computer environment: "What kind of support is really needed to create the kind of experimenting society where teachers really think that they can try out a curriculum, listen to what students have to say, make some adjustments and try it again?" (p. 12). The call to investigate geometric computer tools has come from many researchers and reformers (Clements & Battista, 1994; Linn & Pea, 1994; ICMI Discussion Document, 1994; Davis, 1992; Chazan & Houde, 1989). The computer can provide an exploratory environment in which teachers and students can explore mathematics. It can also be used to investigate what needs teachers have to enable them to change their mode of delivery. Clements and Battista (1994) summarized a number of studies that suggest that geometric computer environments can help develop students' thinking in geometry. According to these studies, students can make conjectures, evaluate visual manifestations of those conjectures, and reformulate their thought (p. 188). Kilpatrick and Davis (1993), in their review of computers in mathematics education, found that if students use computers to test conjectures, then the demands on the teacher are increased and more effort is necessary by the teachers and students. Schoenfeld suggested that "we need to change the atmosphere in the classroom, to establish a different kind of classroom dynamic that would ultimately affect the student's habits of mind" (cited in Linn & Pea, 1994, p. 11). Teachers play an important role in mathematics education. Their enthusiasm and interest can influence student interest and excitement for geometry (Mason, 1991). If teachers become uninterested or unimpressed with geometric relationships and facts, they tend to maintain a teacher-directed pedagogy. That is, they determine what questions are important to ask and what geometric facts are important to 'discover'. Use of dynamic geometric software programs can help teachers to develop or redevelop an enthusiasm for investigating geometric relationships. Researchers have suggested that computers will change the way teachers teach. Pea (1987) stated that it is difficult to predict how the role of the teacher may change due to the increased use of technology. Yerushalmy (1987) called for a new type of teacher and a new type of teaching. She states that this new type of teaching must support and integrate exploration, inquiry, and ideas into the mathematics classroom. Furthermore, Schoenfeld, in questioning what researchers can do to investigate teachers in exploratory classrooms, asks "What makes the magic happen when it happens?" (cited in Linn & Pea, 1994, p. 11). Therefore, the challenge is to assist teachers in using the computer so that they can allow students to work in an open way and still be comfortable in managing the students, their classroom activities, and their time.

LEARNING TO REASON PROBABILISTICALLY

Probability & Statistics Working Group Organizer
Carolyn A. Maher, Rutgers University

Working Group Panel

Robert Speiser	Susan Friel	Cliff Konold
Brigham Young University	University of North Carolina-Chapel Hill	TERC

What are the issues?

In the recent past, discussion has tended to focus on documenting student "errors" and "misconceptions" in probabilistic reasoning. These are certainly widespread. Given this awareness, we need to know more about how students do learn to reason probabilistically. Here, therefore, we focus our attention on precisely how, on the one hand, learners build mathematical models and how, on the other hand, these models interrelate both with each other and with data. For this reason, our focus is on how students build and work with information.

Theoretical Perspective

Children, in appropriate environments, conduct strong inquiries. Probabilistic reasoning is complex, as models can extend distortions as easily as they can contribute to the growth of understanding. Often they do both. Indeed, the variety of representations which children find useful, and the complex relationships between the models children build and the data which they seek to explicate provide rich opportunities for discussions focused centrally on sense and meaning. Because of this complexity, the development of probabilistic thinking requires careful building over time, in which earlier inquiries are revisited, reconsidered, extended and reformulated. The tools available, the ways these tools are used, the ways in which ideas and information move among the learners, the teacher's questions, ideas and interventions also contribute (or fail to contribute) in important ways. Both research and teaching need to take the long-term building into account, as well as the complexity, at any given moment, of the work and discourse, including adult intervention, in each given setting.

Background

A cross-cultural investigation concerning the emergence of statistical reasoning in children and adult learners was carried out by colleagues from three countries (Brazil, Israel, and the United States) working together (Amit, 1998, Kaufman-Fainguelernt, E. & Bolite-Frant, J., 1998; Maher, 1998; Speiser & Walter, 1998; Vidakovic, Berenson, & Brandsma, 1998). The studies examined, in detail, aspects of the development of statistical ideas about dice games, the student's visual and symbolic presentations of those ideas, and the process by which the students were engaged in discourse in convincing each other of the validity of their thinking. The studies center around students' investigations of these same tasks, across several grade levels—elementary to college level— and from each of the respective communities. A central idea is the assignment of correct probabilities to sample points—whatever the model— and how these assignments interrelate. Another outcome is the recognition that models and data interact, that statistical prediction, therefore, is subtle and hence takes time to develop. Students' work offers wide variety of representations but—it seems—with very interesting constraints.

An Example

All students were challenged with the same two problem tasks.[1] Students built sample spaces, in effect, while playing dice games. The data describe learners' ideas about chance, sampling, sample space, probability, and fairness. Also, they provide information about student theories about the fairness of the games and what students regarded as evidence for their theories. The game with 2 dice triggers two models, based on different sample spaces [see Appendix] which we denote by A (36 elements) and B (21 elements). Learners were videotaped playing the following dice games:

Game 1: A Game for Two Players: Roll one die. If the die lands on 1, 2, 3 or 4, Player A gets 1 point (and Player B gets 0). If the die lands on 5 or 6, Player B gets 1 point (and Player A gets 0). Continue rolling the die. The first player to get 10 points is the winner. Is this a fair game? Why or why not?

Game 2: Another Game for Two Players: Roll two dice. If the sum of the two is 2, 3, 4, 10, 11, or 12, Player A gets 1 point (and Player B gets 0). If the sum is 5, 6, 7, 8, or 9, Player B gets 1 point (and Player A gets 0). Continue rolling the dice. The first player to get 10 points is the winner. Is this a fair game? Why or why not?

Data

The following transcripts from video excerpts of fifth-grade children (Maher, 1998) are presented as metaphors for conversation among working group participants.

---

[1] The tasks were developed as part of longitudinal study of children's thinking for the strand of probabilistic and statistical reasoning (Maher, 1995). The research was funded, in part, from a grant by the National Science Foundation (#MDR-9053597). Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

First play the game and see what happens. All the learners, almost immediately, asserted that the first game was unfair. The groups proceeded to modify the game in various ways so that each player would have the same number of opportunities for a score.

Initial responses to the second game ranged widely. Several students claimed that Player A had the advantage, since there were 6 opportunities for a point (as compared to 5 for Player B). Others suggested that this game was "probably fair", on the grounds that Player B had certain numbers which they felt were easier to roll, making up for the fact that Player A had an extra number. Individual students asserted that even sums were harder to roll than odd and that higher sums were easier to get than lower ones. One student, Jeff, remarked to his classmate, Romina, that "snake eyes" (2) and "boxcars" (12) were the most difficult sums to roll and that 7 was the easiest. The students in each group played the game several times, recording scores. According to these data, Player B won far more often than Player A did, so many students began to revise their initial hypotheses. At the end of the session, the students were asked to think about the game, to play it as many times as they liked, and to return to the second session with any results, hypotheses, and/or explanations that they could then develop.

Moral: model dice, not sums. The following episode, as this group collaborated to develop a "fair" game, gives evidence about how Jeff might represent his outcomes, and suggests potential implications for constructing a "fair" game:

Romina: I did this - I don't know. I put 2, 3 and 4 - and 2 and 3 only have one possibility, and so does 11 and 12.... And I put that Player B has more.

Jeff: Well, if we get - They (Player B) have 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,13 chances of hitting their number. So they have more chances - and giving more points wouldn't work though -

Jeff produced a chart with each row recording what he considered to be the possible outcomes resulting in a particular sum from 2 to 12, indicating beside each row the number of outcomes which resulted in each given sum. His sample space is clearly B, with 21 elements, in which Player B receives 13 sample points, as opposed to Player A, who receives 8. Because 21 is odd, there is no way to partition the events (that is, the possible sums) equally unless some events, somehow, can be omitted from consideration.

Stephanie (joined by Ankur and Milin) works with a different model, based on the sample space A, which has 36 elements. She, too, has made charts to represent her possibilities, including a colored histogram showing the triangular frequency distribution for the various sums. At this point in the discussion, both Jeff and Stephanie appear to view the points of their sample spaces as equally probable. On the one hand, both models of the dice lead to the conclusion that Game 2, as originally posed, favors Player B. On the other hand, the two competing models—by *design*—seem to offer somewhat different possibilities for making the game fair. But on what considerations, as these children see it, might such possibilities depend?

Teacher: Do you think you could give some insight into why B has an advantage and even more so what kind of an advantage B has? Can you give some insight into the advantage of B and even tell me a little more about the kind of an advantage?

The responses from two of the three groups suggested reasoning based on sample space B. This reasoning led to Amy's prediction that there were 3 ways to get a six.

Amy: Well I think that B has the advantage because he has like the numbers that a lot of people get like if they're playing a dice game...they usually get those kind of numbers instead of like a 12 or an 11 they usually get 7's or 6's or 8's, 9's.

Teacher: Why's that?

Amy: Because they have...a like they have different pairs that can add up to the numbers...like 6...3 and 3 or 4 and 2...

Teacher: OK. So you're telling me there are two ways you can get 6...3 and 3 and 4 and 2...so is that what you're telling me?

Amy: Yeah.

Teacher: Well 3 you can get...how many ways?

Amy: One.

Teacher: OK so Amy is telling me there is one way you can get three and what's that way?

Amy: 2 and 1.

Teacher: Amy says you can get 3 by 2 and 1 ...one way...and she can get 6 by?

Jeff: 2 and 4...3 and 3...5 and 1.

Teacher: Three ways. Do you all agree with that?

Ankur: No.

Teacher: Ankur...Ankur doesn't agree with that.

Ankur: I say...I say for three there's 2 and 1 and 1 and 2...because 2 is on one die and 2 is on the other die and 1 is on the one die and 1 is also on the other die.

This disagreement, which seems to depend on two quite different representations, needed to be explored.

Teacher: Okay, so we have some disagreement here. Can somebody tell me what the disagreement is? Who can summarize what the disagreement is? Michelle?

Michelle : He's saying that you have 1 on one die and 2 on the other...but you can also have 2 on one die and 1 on the other...but it is the same thing. We're working with what it equals up to not the numbers that are on the die... we're working with what it equals not what...

Jeff: Unfortunately he makes somewhat sense because actually you do have two chances of hitting it.

Stephanie: What?

Ankur has advocated Stephanie's idea, Jeff has listened, Stephanie has alertly taken note. Now Jeff responds:

Jeff: See look because...if you roll...if this die might show a 1 and this might show a 2...but next time you roll it might be the other way around. [Jeff is demonstrates with dice.]

Stephanie: What, Jeff? [Stephanie is at the overhead throughout this discussion, presenting her table and chart.]

Jeff: And that makes it two chances to hit that even though it's the same number. It's two separate things on two different dies.

Stephanie: Therefore there's more of a chance ...therefore there two different ways...therefore there are two ways to get 3.

Jeff: And that throws a monkey wrench...and that just screws up everything we just sort of worked on for about the past hour.

It's a game of luck. For the second game, discussion began after the students have had a chance to play the game. When the question about fairness of the game was posed initially, there was little immediate response. Once the students had a chance to play the game, however, ideas of "fairness", "luck", and "chance" became apparent. Stephanie, Michelle, and Jeff reluctantly admit that player A has an advantage. Their reluctance stems, as they describe it, from belief in "luck". These children seem sharply aware, through reflection on their own experience, that carefully reasoned theoretical expectations about what *should* happen often differ from what *does* happen. Their conflict, which reflects historic controversies, seems profound. At the same time, despite their hesitation, these children do connect their theories to their data, continuing to wonder and reflect.

Discussion

As we think about cognition, we, too, attempt to learn from special cases, taking care to note the special nature of each case as well as its relationship, often strongly reciprocal, to models which we seek to build. Probability, indeed, is subtle and complex, and so is children's thinking about it. Quite simple tasks produce rich data and important questions for investigation. The transcripts above suggest that children's social interaction deserves as much study as their mathematics in this context. The videotape and accompanying transcripts—and their interpretation— are offered as major counterexamples to widely held points of view. We invite discussion and collaboration.

References

Amit, A. (1998). Learning probability concepts through games. In L. Pereira-Mendoza, L. Seu Kea, T. Wee Kee, & W. Wong, (Eds.), *Proceedings of the Fifth International Conference on Teaching Statistics*, 1, (pp. 45-47). Singapore.

Kaufma-Fainguelernt, E. & Bolite-Frant, J. (1998). The emergence of statistical reasoning in Brazilian School Children. In L. Pereira-Mendoza, L. Seu Kea, T. Wee Kee, & W. Wong, (Eds.), *Proceedings of the Fifth International Conference on Teaching Statistics*, 1, (pp. 49-52). Singapore.

Maher, C. A. (1998). Is this game fair? The emergence of statistical reasoning in young children. In L. Pereira-Mendoza, L. Seu Kea, T. Wee Kee, & W. Wong, (Eds.), *Proceedings of the Fifth International Conference on Teaching Statistics*, 1, (pp. 53-59). Singapore.

Maher, C. A. (1995). Children's development of ideas in probability and statistics: studies from classroom research. *Bulletin of the International Statistical Institute*, 2, Beijing.

Speiser, B. & Walter, C. (1998). Two dice, two sample spaces. In L. Pereira-Mendoza, L. Seu Kea, T. Wee Kee, & W. Wong, (Eds.), *Proceedings of the Fifth International Conference on Teaching Statistics*, 1, (pp. 61-66). Singapore.

Vidakovic, D., Berenson, S., & Brandsma, J. (1998). Children's intuition of probabilistic concepts emerging from fair play. In L. Pereira-Mendoza, L. Seu Kea, T. Wee Kee, & W. Wong, (Eds.), *Proceedings of the Fifth International Conference on Teaching Statistics*, 1, (pp. 67-73). Singapore.

APPENDIX: Sample spaces for Game 2.

There are two. The first, A, is defined to be the set of all possible ordered pairs (x, y), where x and y are integers from one to six. The second sample space is B. We define B to be the quotient A/s, where s denotes the symmetry s: A–>A defined by s(x, y) = (y, x). Each sample point in B can be uniquely represented by an ordered pair (x, y) in A, with the restriction, now, that we have x ≥y.

Events in A, their elements, their cardinalities:

2:	(1,1)		1*
3:	(1,2), (2,1)		2*
4:	(1,3), (2,2), (3,1)		3*
5:	(1,4), (2,3), (3,2), (4,1)	4	
6:	(1,5), (2,4), (3,3), (4,2), (5,1)	5	
7:	(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)		6
8:	(2,6), (3,5), (4,4), (5,3), (6,2)	5	
9:	(3,6), (4,5), (5,4), (6,3)	4	
10:	(4,6), (5,5), (6,4)		3*
11:	(5,6), (6,5)		2*
12:	(6,6)		1*

Events in B, their elements, their cardinalities:

2:	(1,1)	1*
3:	(1,2)	1*
4:	(1,3), (2,2)	2*
5:	(1,4), (2,3)	2
6:	(1,5), (2,4), (3,3)	3
7:	(1,6), (2,5), (3,4)	3
8:	(2,6), (3,5), (4,4)	3
9:	(3,6), (4,5)	2
10:	(4,6), (5,5)	2*
11:	(5,6)	1*
12:	(6,6)	1*

Stars, above, mark Player A's events.

RESEARCH ON RATIONAL NUMBER, RATIO AND PROPORTIONALITY

Working Group Organizer

Thomas Post, University of Minnesota, postx001@maroon.tc.umn.edu

Working Group Panel

Kathleen Cramer
University of Wisconsin-River Falls

Guershon Harel
Purdue University

Thomas Kieren
University of Alberta

Richard Lesh
Purdue University

The purpose of this series of three two hour sessions will be to help frame current and future research on rational number, ratio and proportionality within the perspective of earlier important studies and the projects which have conducted them. This will be accomplished with a series of presentations providing overviews of several of these earlier studies along with implications for further study and investigation. Ample time for discussion (about half) will be provided. Persons interested will receive a bibliography with suggested readings prior to the conference.

Major areas to be considered include: multiple interpretations of rational number (subconstructs), children's partitioning schemes, observed blockages to efficient learning, translations within and between modes of representation, concept of unit considerations, ratio-a measure space interpretation perceptual distracters, evaluation—criteria for understanding, and connections between rational number, ratio and proportionality. Teachers knowledge in these areas will be addressed. A survey of rational number, ratio and proportionality related research reported in past PME proceedings will be discussed. Guidelines for the construction of theory based instructional student/teacher materials will be considered. Time will also be devoted to understanding innovative research designs for research on rational number constructs. Importantly, implications for our future work in these domains will also be explored relying on participant discussion and contributions from the floor.

We have lived through a series of "content free" programs which have captured the attention of teachers and school administrators nationwide. As a result, teachers have not spent the needed time to learn about actual mathematical content and students struggles to learn it. An elementary school administrator recently (and proudly) had boasted that every elementary teacher in her district (over 1000) had 30 hours of Madeline Hunter. There is nothing wrong with Hunter's ideas, and the intent here is not to demean those ideas but unless they are applied to meaningful mathematical (or other) content there is no a-priori reason to expect students achievement levels to prosper. That time would have been better spent applying Hunter's ideas to children's learning of rational number concepts or some other topical domain. Similar things have happened with cooperative learning and radical constructivism. All sound ideas, but useful only to the extent they are applied to high quality (useful and theoretically sound) mathematical and pedagogical content. It is this latter category of concern which has proven to be elusive to many teachers and administrators in school situations who are attempting to improve the school mathematical experience for children. Site based management where individual school based teachers and administrators make all curricular decisions does not help the situation either.

This session will attempt to reassert the importance of accumulating knowledge about teachers and students learning of rational number, ratio, and proportionality. We will do this by examining the work of several projects, groups and individuals who have made important contributions to the field. We will then use these "classical" studies as a springboard to identify additional investigations to further our knowledge and understandings in these areas.

The Rational Number Project (RNP) is a program of cooperative research which has been funded continuously by the National Science Foundation (NSF) since 1979. It is thought to be the longest lasting cooperative research project in the history of mathematics education. To date, there have been six separate multi-year NSF grants involving the Universities of Minnesota, Wisconsin at River Falls, Northern Illinois, Louisiana State, Northwestern, Massachusetts at Dartmouth and Purdue. The project bibliography contains eighty or so entries including books, research reports, book chapters and technical reports. There also have been a like number of presentations at regional, national and international meetings. The RNP bibliography will be shared with participants at one of our sessions. The RNP is generally considered to have made important contributions to our understanding of children's rational number thinking.

The first of the RNP grants was obtained in 1979 to examine the impact of manipulative materials on children's understanding of rational number concepts. Later grants have extended our study of fractions to the study of proportionality in the middle grades. We are now working with middle grade teachers in a teacher enhancement program to facilitate the implementation of new NSF middle school curricula, including the RNP fraction lessons.

This latest grant (1994-98) is primarily concerned with the development and testing of a model for re-educating middle grades teachers. The model used attempts to integrate teachers mathematical, pedagogical and psychological content knowledge. (a la Shulman)

Behr, Cramer, Harel, Lesh and Post have been the mainstays of the RNP over the years although others have worked with us for shorter periods of time. (Bright, Khoury, Silver and Wachsmuth and many, many graduate students through the years). In addition, RNP project personnel have worked with many of our colleagues in mathematics and in psychology on issues of mutual interest.

1) Tom Kieren has been interested in the domain of rational number throughout much of his career at the University of Alberta. His work on partitioning with Pothier have contributed important understandings into the partitioning related behavior of young children. Tom is fond of saying that mathematics is "about something." "About something" is an important idea which has been lost on many who have over the years driven students to premature abstraction and to a preoccupation with computational algorithms. Tom has provided an important conceptual framework for the RNP with his paper "On the mathematical cognitive and instructional foundations of rational numbers" (1976) in which he suggests important components of rational number understanding and his belief that a mathematically literate person in the rational number domain has an integrated view as to how the various subconstructs part-whole, ratio, decimal, indicated division, measure and operator interact and are related to one another. He argues for a research program which acknowledges the interrelated nature of these subconstructs. Tom will comment on his current views of these subconstructs and related rational number issues.

2) Vergnaud (1983, 1988) coined the term "Multiplicative Conceptual Field" (MCF) to refer to a web of multiplicatively related concepts, such as multiplication, division, fractions, ratio and proportions, linearity, and multilinearity. Similar to Kieren's' subconstructs of rational number, a key idea of the MCF is the observation that its content is not a mere collection of isolated concepts but rather an interconnected and interdependent complex structure. Its complexity is both mathematical and developmental.

Until the mid 80s, research in mathematics education looked, to a large extent, at the development of individual multiplicative concepts, without explicit attempts to deal with the interrelations and interdependencies within, between, and among these concepts. For example, research on the rational number concept did not take into account children's conceptions of multiplication and division, and vice versa. Similarly, research on the learning of the decimal system was quiet separate from research on fractions and proportionality. During the middle-to-late 80s, the RNP advanced the research on the MCF by exploring the interconnectivity of the cognitive development of multiplicative concepts.

As an example, we mention one important result of this research:

In a study with inservice elementary school teachers, we found that teachers use four solution strategies to multiplicative problems that do not conform to Fishbein's intuitive models:

(a) The Multiplicative strategy, involving the concept of proportionality.

(b) The Pre-multiplicative strategy, reflecting an early stage toward proportionality.

(c) The Operation-search strategy, based on a trial-and-error approach.

(d) The Keyword strategy.

We found that teachers who solved the problems correctly and relationally reasoned in terms of ratio and proportion concepts, whereas teachers who arrived at correct solutions without these concepts did so by a trial-and error like methods. This suggests that multiplicative problems that do not conform to Fishbein's models require for their solution a scheme that includes the concepts of ratio and proportion. This finding suggests that the formation of ratio and proportion concepts can be powerful tools in dealing with multiplicative problems. The question of whether these concepts are necessary tools for multiplicative problems that do not conform to Fishbein's intuitive models is still open and needs further research.

Within the enormous structure of the MCF, we have identified smaller structures that, although interconnected, are, to some extent, autonomous. An example of such a substructure consists of a three-stage development of the concept of multiplication: from an early stage of whole-number multiplication, to an operation non-conservation stage, and to an operation conservation stage. We have called this structure, a Multiplicative Conceptual Subfield (MCS) because it represents a closed unit within the greater structure of the MCF. Our semantic analyses of the different MCS's-drawing heavily on the work of many other researchers, particularly, the work by Kieren, Steffe, Thompson, and Kaput—revealed critical deficiencies in the mathematics curricula of elementary and middle schools. The areas where the teaching of [MCS's] is deficient include composition, decomposition, and conversion of units, operation on numbers from the perspective of mathematics of quantity, and mathematical variability.

Guershon Harel will discuss the implications of these issues for school rational number curricula and for further research in the area.

3) Innovative Research Designs Needed for Research on Rational Numbers Constructs

New research designs have been developed are based on new assumptions about the nature of students' knowledge, problem-solving, learning, and teaching, they frequently involve lines of reasoning that are fundamentally different from those that applied to industrial-era factory metaphors for teaching and learning. Therefore, new standards of quality often are needed to assess the significance, credibility, and range of usefulness of the results that are produced by such studies. But, in general, the development of widely recognized standards for research has not kept pace with the development of new problems, new theoretical perspectives, and new approached to the collection, analysis, and interpretation of data.

High-quality studies may be rejected because they involve unfamiliar research designs, and because inadequate space is available for explanation, or because inappropriate or obsolete standards of assessment are used (similar to a Type I error).

Low quality studies may be accepted in which innovative research designs are done poorly, or in which traditional research designs even though they are based on obsolete assumptions about the nature of teaching, learning, and problem-solving (or about the nature of program development, dissemination, and implementation) (similar to a Type II error).

In research on rational numbers and proportional reasoning, as in other areas of mathematics education research, some of the most important products of research has involved the development of new tools and research methodologies. Yet, these products seldom are reported in research journals. In a series of projects known collectively as the Rational Number Project, some of the most useful research designs that we've developed involve the integrated use of qualitative and quantitative methods. Also, because we often are interested in going go beyond investigating typical development in natural environments to also focus on induced development within carefully controlled and mathematically enriched environments, we have had to develop new approaches to research that involve:

1) Action research: in which teachers participate as co-researchers.

2) Multi-tiered teaching experiments: investigating the interacting development of students and teachers often over time periods involving several months or years.

3) Carefully structured clinical interviews: in which it is important to minimize uninteresting interventions by the researcher.

4) Iterative videotape analyses: in which it is important to take into account interpretations involving a variety of theoretical and practical perspectives.

5) Ethnographic observations: in which it is important to avoid needlessly distorting the perspectives of the people being observed.

6) New approaches to assessment: that focus on deeper and higher-order understandings, and which go beyond simplistic assumptions that underlie most standardized testing programs.

Dick Lesh will lead a session which will deal with each of the preceding approaches to research. Examples will be taken from relevant research on rational numbers and proportional reasoning, and resources will include selections from a new book, edited by Kelly & Lesh (in press), on Research Design in Mathematics and Science Education.

References

To be distributed at or prior to the PME-NA Meeting.

REPRESENTATIONS AND MATHEMATICS VISUALIZATION

Working Group Organizer

Fernando Hitt

Cinvestav-IPN, Mexico, fhitta@data.net.mx

Working Group Panel

James Kaput
University of Massachusetts, USA

Luis Radford
Laurentian University, Canada

Norma Presmeg
Florida State University, USA

Manuel Santos-Trigo
Cinvestav-IPN, Mexico.

Introduction

The themes Representations and Mathematics Visualization have appeared in the recent literature as fundamental aspects to understand students' construction of mathematical concepts and problem solving (diSessa, 1994; Dubinsky, 1994; Duval, 1995; Eisenberg & Dreyfus, 1990; Dreyfus, 1991; Glasersfeld, 1991; Janvier, 1987; Kaput, 1994; Presmeg, 1986; Steinbring, 1991; Vergnaud, 1987; Vinner, 1989; Zimmermann & Cunningham, 1990). It is interesting to observe that there are different lines of research in which both theoretical and empirical approaches have been developed.

An important objective of our working group on Representations and Mathematics Visualization is to promote an open discussion of the relevant theoretical orientations endorsed by different authors, and their influence in empirical research that intend to improve our understanding of the learning of mathematics. Particularly, there is interest to discuss how these research results can be interpreted and finally applied into classroom settings.

The importance to discuss and contrast present theoretical orientations is based on a well known fact that often authors follow different theoretical approaches to tackle the learning phenomena. Here, it may occur that the different perspectives are complementary but other times these perspectives are irreconcilable. It could also be that one theoretical orientation permit to have a better explanation of a phenomenon that other orientation. Hence, it is important to ask: (i) To what extent one should adopt or follow a theoretical orientation? (ii) Which orientation will help us to better explain a learning phenomenon? The well organized discussion of these types of questions during the working group sessions might be useful for us to study in depth what happens in the students' process of learning.

One aspect that the group should include is the discussion of research related to problem solving in terms of searching what kind of mathematical representations students use when solving a problem and the role of mathematics visualization in this problem solving process.

We consider important in general that the working group should focus on the discussion on four relevant aspects:

1) The relevance of doing empirical research linked to representations and mathematics visualization.
2) The importance of the theory in pursuing research on representations and mathematics visualization.
3) The application of research's results that links the students' learning of mathematics and the use of multiple representations within a theoretical frame.
4) The influence of technology-based multiple linked representation in the students' construction of mathematical concepts.

1) The relevance of doing empirical research linked to representations and mathematics visualization.

Mathematics instructors, at all levels, traditionally have focused their instruction on the use of algebraic representations with the intention to avoid confusion between mathematical objects and their representations, they normally do not take into account geometric and intuitive representations. This is because they think that the algebraic system of representation is formal and the others they are not. Perhaps, some students difficulties in the construction of concepts are linked to the restriction of representations when teaching. Nevertheless, it is known by empirical research that the students' construction of a mathematical object is based on the use of several semiotic representations. The students' handling of different mathematics representations will permit ways of constructing mental images (a concept image in Vinner and Tall' sense) of a mathematical concept. The richness (or lack of) of their concept image will depend on the students' handling of the representations used. However, the tendency to remain in an arithmetic or algebraic system of representation is well documented in the bibliography. This tendency will produce errors in problem solving situations. In this context, Santos (1996, p. 275) shows us an example in which *a tennis ball problem* was given to some students (How many tennis balls do you need to fill your classroom?). Santos says: *For the tennis balls problem, the students* [35, grade 10] *experienced difficulty in estimating the dimensions of the classroom and the tennis balls. All the students asked the interviewer to provide the dimensions and when they were asked to estimate, some of the students asked for a meter stick. The most common approach was to divide the volume of the classroom by the volume of a tennis ball....* Why are these children not worried about the answer? Why did they remain in the arithmetic system of representation?

Also related to this part, Goldenberg (1995, p. 155) quotes: *We cannot expect to understand that understanding if we look only at the student's* *facility with one representation, or even the quality of a student's handling of each representation in isolation. But, as we observe students juggling the interaction among representations, we get a glimpse of the rich internal models they construct in their attempt to understand the bigger picture.*

Are the students' errors a product of a deficient handling of a representation when transforming it in the same semiotic system of representation or when converting it to another system? That is to say, the difficulties faced by the students could be explained as a lack of coordination between representations? A plausible answer seems to be that not all the difficulties could be explained in terms of a lack of articulation between representations. There exist epistemological obstacles that are explained by other ways, for example through the evolution of mathematical ideas (see Hitt related to functions, 1994); however, the identification of students' errors when handling different representations could give us a glimpse of the concept image they have.

2) The importance of the theory in pursuing research on representations and mathematics visualization.

There are several questions that need to be addressed here. For example, how can we explain the lack of success of first year university students in Selden et al. studies (1989, 1994) when solving calculus nonroutine problems? In this context, Zimmermann (1991, p. 136), state that: *Conceptually, the role of visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject. This is especially true if the course is intended to stress conceptual understanding, which is widely recognized to be lacking in many calculus courses as now taught. Symbol manipulation has been overemphasized and ... in the process the spirit of calculus has been lost* (Aspinwall et al., 1997, p. 301).

Why do some authors consider so important the study of the semiotic system of representations on the construction of mathematical concepts? For example, Duval (1993, p. 38) quotes: *...we are then in front of what we could call a cognitive paradox of the mathematical thinking: On one hand, the* ***apprehension*** *of the mathematical objects can only be possible as a conceptual apprehension, and on the other hand, only mediated by semiotic representations it is possible an activity on the mathematical objects.*

Other authors emphasize the relations of symbols and ideas, Radford & Grenier (1996) assert: *...the relation between symbols and ideas can not only be considered as an interaction which consist to put in contact an object (or an idea) changeless and extern to the individual with the representations of that object, ... Far away from this, the interaction between the symbols and the ideas must be, we think, seen as a system of relation constructed by the individual himself in his intellectual path, at the same time socially and individually.*

By following the above ideas, we can pose other related questions:

a) What is the role of the mathematical intuition in this context?

b) To what extent mathematics visualization has to deal with certain abilities related to the conversion of representations from one semiotic system to another?

c) Is the comprehension of a mathematical concept related to the use of different semiotic representations of the concept in question?

d) Should conceptual knowledge be taken as an invariant of the multiple semiotic representations?

e) The cognitive knowledge being mediated semiotically, bring us a culturalized vision?

f) What is the nature of the interaction between external and internal representations?

g) How do individuals construct internal representations?

h) How do we infer internal representations?

These questions could be used as a framework to approach:

- Theoretical aspects of the learning of mathematics which take into account the role of the semiotic representations on the construction of mathematical concepts.
- Theoretical aspects related to semiotic representations dealing with a social epistemology of mathematical knowledge with applications to didactical situations in the classroom.
- An analysis of the mathematical ideas related to a concept through the history of mathematics to detect epistemological obstacles.

3) The application of research's results that links the students' learning of mathematics and the use of multiple representations within a theoretical frame.

By taking into account different systems of representations, we can identify specific variables related to cognitive contents and, by this way, organize didactical proposals to promote the students articulation of different representations.

In this context, Eisenberg & Dreyfus (1990, p. 25) state that: *although the benefits of visualizing mathematical concepts are often advocated, many students are reluctant to accept them; they prefer algorithmic over visual thinking... Indeed, with respect to problem nine* [Given: f a differentiable function such that $f(-x) = -f(x)$. Then, for any given a: A) $f'(-a) = -f'(-a)$ B) $f'(-a) = f'(a)$ C) $f'(-a) = -f'(a)$ D) none of the above] *one typical calculus teacher (who also happens to have authored a calculus textbook) wrote:* $f'(-a) = (f(-a))' = (-f(a))' = -f'(a)$. It might be that, this teacher is totally convinced of his/her algebraic process and considers irrelevant a mental construction of one example of a function and the derivatives [for example, the graph of $f(x) = x^3$] that he/she may realize that the proposed solution is not correct. Similarly, as the case related to the tennis ball problem, mentioned before, Why did it happen that individuals get attached to the algebraic system of representation?

Dreyfus (1991, p. 42) quoted: *Mathematics educators seem to have recognized the potential power and the promise of visual reasoning; but in spite of this, implementation is lagging: Students tend to avoid visual reasoning.* It seems that teachers continue emphasizing their instruction on nonvisuals methods.

4) The influence of technology-based multiple linked representation in the students' construction of a mathematical concepts.

Related to this point, Kaput (1994, p. 387) states that: *More subtle examples of notation modification include such strategies as enabling students to act on traditional mathematical notations in more natural ways, as when in a computer environment, for example, one uses a pointing device and graphical interface to act directly on coordinate graphs by sliding, bending, reflecting, and so forth (as with Function Probe,...). This is a subtle exploitation of the rich knowledge based in kinesthetic experience to act on mathematical notations, and hence to effect mental operations on mathematical objects, that is, functions. Another example is the direct manipulation of algebraic objects used in Theorist... Yet another example, reflecting the dynamic, interactive properties of the computer medium, is offered by CABRI-Geometry....*

How can we develop new external systems of representations that foster more affective learning and problem solving? More questions and lines of research may appear during the development of the discussion.

References

Aspinwall, L., Shaw, K., & Presmeg, N. (1997). Uncontrollable mental imagery: graphical connections between a function and its derivative. *Educational Studies in Mathematics*, 33, 301-317.

diSessa, A. (1994). Comments on Ed Dubinsky's chapter. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving*. Hillsdale, NJ: Lawrence Erlbaum.

Dreyfus, T. (1991). On the Status of Visual Reasoning in Mathematics and Mathematics Education. *Proceedings of the XV Meeting of the International Group fot the Psychology of Mathematics Education, Assisi, Italy*, Vol. I, pp. 33-48.

Dubinsky, E. (1994). A theory and practice of learning college mathematics. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving*. Hillsdale, NJ: Lawrence Erlbaum.

Duval, R. (1993). Registres de représentation sémiotique et functionnement cognitif de la pensée. *Annales de Didactique et de Sciences Cognitives*, 5, 37-65.

Eisenberg, T., & Dreyfus, T. (1990). On the reluctance to visualize in mathematics. In W. Zimmermann, & S. Cunningham (Eds.), *Visualization in teaching and mathematics*, MAA Series, USA.

Glasersfeld, E. (Ed.). (1991). *Radical constructivism in mathematics education*. Netherlands: Kluwer.

Goldenberg, E. (1995). Multiple representations: A vehicle for understanding. In *Software goes to school. Teaching for understanding with new technologies*.

Hitt, F. (1994). Teachers' difficulties with the construction of continuous and discontinuous functions. *Focus on Learning Problems in Mathematics*, 16(4)10-20.

Hitt, F. (In press). Difficulties in the articulation of different representations linked to the concept of function. In G. Goldin, & C. Janvier, (Eds.), Special Issue. *Journal of Mathematical Behavior*,17(1).

Janvier, C. (1987). Representation and understanding: The notion of function as an example. In C. Janvier (Ed.), *Problems of representations in the teaching and learning of mathematics* , pp. 19-26. Hillsdale, NJ: Lawrence Erlbaum.

Kaput, J. (1994). The representational roles of technology in connecting mathematics with authentic experience. In Biehler et al. (Eds.), *Didactics of mathematics as a scientific discipline*. Netherlands: Kluwer.

Presmeg, N. (1986). Visualisation and mathematical giftedness. *Educational Studies in Mathematics*, 17, 297-311.

Radford , L., & Grenier, M. (1996). Entre les choses, les symboles et les idées... une séquence d'enseignement d'introduction à l'algèbre. *Revue des Sciences de l'Education*, 23(2), 253-276.

Santos, M. (1996). An exploration of strategies used by students to solve problems with multiple ways of solution. *Journal of Mathematical Behavior*, 15, 263-284.

Selden, J., Mason, A., & Selden, A. (1989). Can average calculus students solve nonroutine problems? *Journal of Mathematical Behavior, 8* (1989), pp. 45-50.

Selden, J., Mason, A., & Selden, A. (1994). Even good calculus students can't solve nonroutine problems. In J. Kaput, & E. Dubinsky (Eds.), *Research issues in undergraduate mathematics learning*, MAA Notes, 33, pp. 19-26.

Steinbring, H. (1991). Mathematics in teaching processes: The disparity between teacher and student knowledge. *Research en Didactique des Mathématiques*,11(1), 65-108.

Vergnaud, G. (1987). Conclusion. In C. Janvier (Ed). *Problems of Representations in the Teaching and Learning of Mathematics*, pp. 227-232. Hillsdalc, NJ: Lawrence Erlbaum.

Vinner, S. (1989). The avoidance of visual considerations in calculus students. *Focus on Learning Problems in Mathematics, 11*, 149-156.

USING SOCIO-CULTURAL THEORIES IN MATHEMATICS EDUCATION RESEARCH

Working Group Organizers

Judit Moschkovich
TERC
judit_moschkovich@terc.edu

Karen Fuson
Northwestern University
fuson@nwu.edu

Working Group Panel

Yolanda de la Cruz
Arizona State University

Joanna Masingila
Syracuse University

Marta Civil
University of Arizona

History of the Working Group

This Working Group began as a discussion group at PME-NA 1997. During that discussion group several researchers presented summaries of their research framed by socio-cultural theories. Out of these discussions grew an interest in pursing these issues further and organizing future collaborations around this theme. The presentations in 1997 were: Judit Moschkovich (organizer), Internalization and appropriation as metaphors for learning; Marta Civil, Project BRIDGE: Linking home and school mathematics; Yolanda de la Cruz, Applying the Zone of Proximal Development to mathematics learning; Karen Fuson, Building relations between spontaneous and scientific concept learning in the classroom; Lena Licon-Khisty, Understanding equity issues from a socio-cultural perspective; Joanna Masingila, Learning and doing mathematics in and out of school.

Focus and Aims of the Working Group

Although socio-cultural theoretical perspectives (including Vygotsky, Leontiev, Luria, and Bakhtin) have recently received attention in the literature on mathematics education (see for example, Educational Studies in Mathematics, September 1995) many important questions remain regarding how to apply these perspectives to research design, data analysis, curriculum development, and teacher professional development. Before applying socio-cultural theories to research questions in mathematics education it is important to a) clarify which specific versions, aspects, or concepts of socio-cultural theories are being invoked, b) unpack and question key analytical concepts, and c) explore which aspects and concepts can be useful for framing research on learning and teaching mathematics.

The focus of this working group will be the connection between socio-cultural theories and research in mathematics education. This working group will serve as a forum where participants can discuss the issues involved in using socio-cultural theory to frame mathematics education research. The activities and discussion will address several areas where socio-cultural theory might be applied: design of studies on learning or teaching, analysis of data, design of teacher professional development, and curriculum design.

The central goals of the Working Group are to:

1) Develop a more shared sense of the contribution that socio-cultural theories can make to research in mathematics education by discussing research conducted using these theoretical perspectives and by analyzing sample data using concepts from these perspectives.
2) Develop a plan for a set of related projects using socio-cultural theory to explore questions about learning or teaching mathematics.

Session 1-Judit Moschkovich

1) Introduction and overview of the Working Group.
2) Two brief (5-10 minutes each) presentations by panel members providing overviews or examples of how researchers have used socio-cultural theory in their research. The purpose for these short presentations is to provide examples of how socio-cultural theories have been applied and show several different perspectives in a structured way.
3) Participants will analyze and discuss a segment of videotape data from a variety of socio-cultural perspectives, sharing their own experiences in data analysis as part of the discussion.

Session 2-Karen Fuson

1) Two brief (5-10 minutes each) presentations by panel members (providing overviews or examples of how researchers have used socio-cultural theory in their research.
2) Discussion in small groups (or if the group prefers, as a whole group) focusing on the following questions: a) What aspects of socio-cultural theories have participants used in mathematics education research? b) In what areas that have not been linked to these perspectives might socio-cultural theories be useful? c) What are some of the basic characteristics of a study conducted from a socio-cultural perspective? d) How might this theoretical perspective inform participants' research projects?

Session 3-Judit Moschkovich and Karen Fuson

Participants will discuss and plan future activities for the Working Group. The anticipated follow-up activities from this discussion group include planning for a continuation of the Working Group at PME-NA 1999 and ultimately organizing a collaborative writing project on this topic. Some possible products from this Working Group are: 1) A collective list of basic characteristics of studies conducted from a socio-cultural perspective; 2) Plans for related research on mathematics teaching and learning applying socio-cultural theories; and 3) Suggestions for how socio-cultural theories might inform teacher professional development.

Below are brief descriptions of panel members' research projects and a list of relevant references.

The Zone of Proximal Development and Collective Learning
Yolanda De La Cruz, Arizona State University West

The Family Mathematics Learning Program is strongly informed by Vygotsky's position on the social nature of learning and development. Vygotsky proposed that learning was socially, culturally, and historically determined. From this perspective, language is seen as a vital instrument used to regulate behavior and organize thinking, thus creating new perceptions, memories and thought processes. The perspective taken by the Family Mathematics Learning Program is that parents and children learned collaboratively. Rather than taking the ZPD to be a factor in how individuals learn, this project has investigated how collectives of people learn and develop through the self-conscious utilization of the ZPD.

Almost all learning observed during this project was in a sense collaborative, typically through interaction between a more expert agent and a novice. Children challenged their parents by asking penetrating questions which demanded a simple explanation of an abstract and complex concept. Parents learned mathematical concepts while also learning strategies and techniques to help their children in the home. The support systems in the Family Mathematics Learning Program provided scaffolding for parents while they learned to teach mathematics to their children.

Connecting In-school and Out-of-school Mathematics Practice
Joanna O. Masingila, Syracuse University

My view of knowing, doing, and learning is based on sociological and anthropological research which suggests that (a) people make sense of things mathematically through situations and contexts, drawing on their prior experiences and social interactions, and (b) how one knows mathematics is connected with how one learned it (Bishop, 1988; Masingila, 1994; Millroy, 1992; Saxe, 1988). My research work has been strongly influenced by Saxe's (1991) Emergent Goals framework. Saxe has delineated a "research framework for gaining insight into the interplay between socio-cultural and cognitive developmental processes through the analysis of practice participation" (p. 13). Saxe's (1991) framework consists of three analytic components: (a) goals that emerge during activities, (b) cognitive forms and functions constructed to accomplish those goals, and (c) interplay among the various cognitive forms. The theoretical underpinnings of the framework are based on both Piaget and Vygotsky, but the framework moves beyond them in considering this interplay. Although Saxe's framework is a method for studying the interplay between socio-cultural and cognitive developmental processes, I find it helpful in thinking about working towards connecting in-school and out-of-school mathematics learning and practice.

My current research work is part of a four-year National Science Foundation-funded project entitled, *Connecting In-school and Out-of-school Mathematics Practice*. During this project we are (a) investigating how middle school students use mathematics concepts and processes in a variety of out-of-school situations, and (b) working with a middle school teacher to develop and implement curriculum materials for building on students' out-of-school mathematics practice. All the data are being analyzed through inductive data analysis procedures using Saxe's Emergent Goals framework.

Socio-cultural Contexts for Learning Mathematics
Marta Civil, University of Arizona

Our research project aims at developing mathematics teaching innovations in which students and teachers engage in mathematically rich situations through the creation of learning situations that build on students' and their families' knowledge and experiences in their everyday life. A key premise for us is the development of learning environments that allow students to participate in activities that are meaningful to them and which at the same time allow students to advance in their learning of academic mathematics. In this sense, we agree with van Oers (1996) characterization of mathematical apprenticeship in the classroom by aiming for activities that are "recognized as 'real' by the mathematical community of our days" and by immersing the mathematical activity in "a socio-cultural activity that makes sense for the pupils" (p. 106).

One of our goals is to characterize the mathematics embedded in the practices of these students' parents (all of whom are working-class and Hispanic, primarily Spanish speaking). In analyzing these interviews in our regular study group sessions, we hope to address issues of values and beliefs as we discuss what counts as mathematics (Abreu, 1995). At the classroom level, three teachers are currently developing curriculum themes based on some of the findings from these occupational interviews. Another goal of the project is to engage the students (and maybe their parents) in mathematical activities that use these everyday practices as starting points (this is similar to the work done by one of the BRIDGE researchers, Fonseca, 1997).

In my presentation at this working group, I will provide examples from three very different schools in which our research project takes place. One is a school where a garden theme was developed; another follows a reform-based curriculum for mathematics. Our project attempts to embed academic mathematics in the socio-cultural context of these classrooms. My goal in bringing these examples is to engage the Working Group participants in our ongoing discussion of questions such as "what is learning?" and "how does it take place?" and share our perspective on issues that arise often in our study group sessions, such as context (Jacob, 1997), individual cognition (Sfard, 1998), the role of interaction, and learning as transformation of participation (Rogoff, 1994).

Socio-Cultural Foundations for the Children's Math Worlds Project
Karen C. Fuson, Northwestern University

Children's Math Worlds is a 6-year action-research project directed toward designing a conceptually complex and challenging K-3 math curriculum that builds on the individual experiences, interests, and practical math knowledge that diverse children bring to our classrooms. Our collaborative-research project has been and is being carried out in urban schools of under-represented minorities, mostly Latino English-speaking and Latino Spanish-speaking children, as well as in English-speaking upper-middle class schools. Our ambitious math curriculum is based on combining the following: (1) pervasive use of enacted, linguistic, drawn representational, and object conceptual supports to facilitate individual understanding and communication, (2) teaching/learning activities that help children through developmental sequences of conceptual structures so that they continually build more-advanced conceptual structures, and (3) classroom mathematizing methods using children's personal stories to give teachers and other children insight into their lives and culture and to provide familiar real-world bases for mathematical thinking. Through active, social, and affective teaching-learning processes that engage the participants as persons, teachers and children re-construct and co-construct mathematical knowledge through individual thinking and reflection and interpersonal interaction.

The theoretical work of our Children's Math Worlds project focuses both on particular domains of mathematics and on more general teaching and learning models. We carry out domain analyses of real-world situations that need to be discussed or introduced in the classroom, develop full quantity conceptual support nets to facilitate children's understanding of mathematical linguistic and notational tools, and ascertain progressions in children's thinking that can be the basis for teaching efforts. Two of the more general models are Vygotskiian. One describes how a referential classroom bridges from children's spontaneous concepts to cultural mathematical scientific concepts (Fuson, Lo Cicero, et al., 1997). Another specifies an Equity Pedagogy that outlines attributes of a referential classroom which can help teachers support students in building on their initial personal meanings and experiences to create advanced and ambitious mathematical concepts, notations, and methods (Fuson, De La Cruz, et al., 1997)

References

Abreu, G. de (1995). Understanding how children experience the relationship between home and school mathematics. *Mind, Culture, and Activity*, 2, 119-142.

Bishop, A. J. (1988). Mathematics education in its cultural context. *Educational Studies in Mathematics, 19*, 179-191.

De La Cruz, Y. (1996). Vygotskian perspectives illuminating a mathematics tutoring program for inner-city children. In, Jakubowski, E., Watkins, D., and Biske, H. (Eds.). *Proceedings of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Columbus, OH: ERIC Clearinghouse for the Science, Mathematics and Environment Education.

Educational Studies in Mathematics (September, 1995). Socio-cultural approaches to mathematics teaching and earning. Lerman, S. (Ed). Volume 31, Nos. 1-2.

Fonseca, J. D. (1997). Build your dream home: An ethnomathematical approach to motivate the learning of mathematics. In J. A. Dossey, J. O. Swafford, M. Parmantie, & A. E. Dossey (Eds.), *Proceedings of the Nineteenth Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol 1, p. 315). Columbus, OH: ERIC.

Fuson, K. C., De La Cruz, Y., Smith, S. B., Lo Cicero, A., Hudson, K., & Ron, P. (1997). *Toward a Mathematics Equity Pedagogy: Creating ladders to understanding and skill for children and teachers*. Manuscript under revision.

Fuson, K. C., Lo Cicero, A. M., Ron, P., De La Cruz, Y., & Zecker, L.(1998). *El Mercado: A fruitful narrative for the development of mathematical thinking in a Latino first- and second-grade classroom.* Manuscript under revision.

Jacob, E. (1997). Context and cognition: Implications for educational innovators and anthropologists. *Anthropology & Education Quarterly*, 28(1), 3-21.

Masingila, J. O. (1994). Mathematics practice in carpet laying. *Anthropology and Education Quarterly, 25* (4), 430-462.

Millroy, W. L. (1992). An ethnographic study of the mathematical ideas of a group of carpenters. *Journal for Research in Mathematics Education, Monograph No. 5.*

Moschkovich, J. N. (1996). Moving up and getting steeper: Negotiating shared descriptions of linear graphs. *Journal for the Learning Sciences 5(3):* pp. 239-277.

Rogoff, B. (1994). Developing understanding of the idea of communities of learners. *Mind, Culture, and Activity*, 1, 209-229.

Saxe, G. B. (1988). Candy selling and math learning. *Educational Researcher, 17* (6), 14-21.

Saxe, G. B. (1991). *Culture and cognitive development: Studies in mathematical understanding*. Hillsdale, NJ: Lawrence Erlbaum Associates.
Secada, W. & De La Cruz, Y. (1996). Teaching mathematics for understanding to bilingual students. In, LeBlanc Flores, J. (Ed.). *Children of la frontera*. Charleston, WV: ERIC Clearinghouse on Rural Education and Small Schools.
Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher,* 27(2), 4-13.
van Oers, B. (1996). Learning mathematics as a meaningful activity. In L. Steffe and P. Nesher (Eds.), *Theories of mathematical learning* (pp. 91-113). Mahwah, NJ: Lawrence Erlbaum.
Vygotsky, L. S. (1986). *Thought and language*. (A. Kozulin, Ed. and Trans.). Cambridge, MA: MIT Press. (Original work published 1934).

DEVELOPING A CONCEPTUAL FRAMEWORK FOR MATHEMATICS TEACHING

Statement of the Working Group on Mathemati cs Teacher Education

Working Group Organizer

Martin A. Simon, Penn State University, msimor.'@psu.edu

Working Group Panel

Deborah Ball
Michigan State University
dball@pilot.msu.edu

Rijkje Dekker
Freudenthal Institute
Netherlands

Susan Jo Russell,
TERC

In many countries of the world, there is a major attempt to reform and improve mathematics education in ways that result in powerful mathematics for all students. Emphases are on mathematical thinking and problem solving, conceptual understanding, and effective communication of mathematical ideas. Key to this reform is the mathematics classroom teacher. In order for teachers to teach in ways which are radically different from the ways that they were taught, they must undergo a transformation in the ways that they conceptualize mathematics and mathematics learning and teaching.

In the United States, large scale efforts are being funded to reform teaching practice through a variety of systemic initiatives and collaborative projects. One can ask whether the empirical and theoretical base is sufficient to guide these ambitious efforts. The empirical and theoretical base can be divided into two broad categories: detailed articulation of the envisioned teaching, the goal of such teacher education efforts, and articulation of a framework for fostering the development of such teaching. Clearly the former is prerequisite to the latter.

The working group, Research on Mathematics Teacher Education, will focus on the first of these two categories. Specifically, it will work towards the development of a conceptual framework with respect to mathematics teaching that can provide direction for mathematics teacher education and research on mathematics teacher development. Towards this end, I will be joined by an outstanding panel of Deborah Loewenberg Ball, University of Michigan; Rijkje Dekker, University of Amsterdam; and Susan Jo Russell, TERC as well as 26 mathematics education researchers attending this working group.

The Need for a Conceptual Framework for Mathematics Teaching

The large-scale reform effort in mathematics education has its roots in several phenomena: a general dissatisfaction with the results of teaching mathematics as predominantly telling and showing, the adaptation by mathematics educators of important theories of knowledge development (e.g., constructivism, socio-cultural theories), and recent research on the mathematical conceptions of learners. The result of these three factors is a desire to change mathematics teaching in ways that are more consistent with and supportive of the ways that learners think and learn.

Although these factors are a reasonable and significant impetus for change, they do not in themselves provide a vision or conceptual framework for a new mathematics pedagogy. How do we make use of theories of knowledge and greater insight into learners to promote powerful mathematics learning? In short, what new model(s) of teaching can take the place of the traditional tell-and-show model?

I argue that not only has the work on this question been insufficient, but that relatively few mathematics educators see this as an important unanswered question. There seem to be two reasons for this. First, there is a common conception that theories of learning related to constructivism define an alternative to traditional teaching through the injunctions "students should be active in their learning" and "students should figure things out for themselves, not be told by the teacher." Second, useful teaching strategies are often seen as the new pedagogy itself. Use of collaborative groups, manipulatives, calculators, and large group problem solving discussions can be powerful tools for teaching, but they do not in themselves define an approach to promoting powerful mathematics learning.

Mathematics teacher education if it is to be effective, and particularly if it is going to be transformative (promote a reform in mathematics teaching), must be guided by a clearly articulated conceptual framework that specifies the nature of mathematics teaching that is its goal.

What Must Constitute the Framework?

Advocating a conceptual framework for mathematics teaching does not imply a need for a monolithic view of mathematics teaching. There seems to be some consensus on broad goals for mathematics instruction (e.g., to help all students develop as powerful mathematical thinkers, to value mathematics, and to be able to use mathematics in a variety of settings). However, beyond these broad goals, determining where consensus exists and where ideas diverge is part of the process of specifying a new pedagogy. Divergence in goals, theories of learning, and emerging theories of teaching may stimulate the development of multiple models.

Any new model of mathematics teaching must:

1. elaborate the goals for instruction on which it is based (including an articulated view of mathematics),

2. articulate an integrated conceptualization of teaching and learning,
3. define a technology for teaching students powerful mathematics.

These three exigencies of a new conceptualization of mathematics teaching are interrelated. Let me explicate the second and third points, which may not be distinct points at all.

An integrated conceptualization of teaching and learning must be based on the goals of instruction. (What constitutes a useful conceptualization of teaching and learning for memorizing formulae is likely not to be adequate for promoting conceptual understanding.) The integrated conceptualization of teaching and learning describes a reflexive relationship between these two aspects. Thus, the conceptualization of teaching builds on and is consistent with the perspective on students' learning of mathematics and the conceptualization of learning reflects a perspective on learning in the context of a particular conceptualization of teaching.

By "defining a technology for teaching," I am not referring to electronic technology, but rather a conceptualization of mathematics teaching that specifies the role of pedagogy in promoting powerful mathematics in learners. The rejection of traditional teaching, telling and showing as the predominant approach, has left a void in the technology of teaching. The technology of traditional teaching was well-defined: tell and show the students what they should know and be able to do. What was to be learned was put out by the teacher or text materials and taken in by the students. How is learning promoted if this traditional technology is rejected? To assert that students will learn if they are given non-routine problems, hands-on materials, and encouraged to communicate in small and large groups reflects the absence of a technology for intentionally promoting mathematical ideas and ways of participating in mathematics.

What is the Nature of this Task and the Role of the Working Group?

The development of elaborated goals for mathematics teaching and an integrated conceptualization of mathematics teaching and learning is likely to be accomplished by an ongoing cycle of theoretical and empirical work, what Gravemeijer refers to as "developmental research". The theoretical work informs and guides the next iteration of enacting effective pedagogy which in turn is analyzed to advance the theoretical. Such developmental research differs from one that seeks out extant examples of reform teaching in order to analyze and conceptualize the teaching portrayed by these examples.

Each of the panel members is engaged in some form of developmental research that has the potential to contribute to elaborating a conceptual framework with respect to mathematics teaching. Over the course of the 3 working group sessions, each of the 4 panel members will make a short presentation of ideas deriving form his/her work. The majority of the time will be spent discussing, questioning, challenging, synthesizing ideas of the panel, participants, and the larger mathematics education community. Participants are encouraged to come to the working group sessions ready to contribute to the conversation ideas from their work or the work of others.

About the Panel Members

Deborah Loewenberg Ball has been conducting research centering on her own teaching of third grade. She uses this research site to develop theoretical constructs that elaborate important aspects of mathematics teaching. Her recent focus has been seeking to unearth, name, and analyze the mathematics entailed in elementary school teaching and learning. Her work examines also the dilemmas embedded in teaching mathematics in ways that honors both the mathematics and students' thinking.

Rijkje Dekker conducts research on teaching in the Netherlands. Dutch researchers have developed Realistic Mathematics Education over the last two decades, an integrated approach to mathematics teaching and learning that builds on fundamental notions of Freudenthal and Piaget. Her recent research involves analysis of the impact of different teacher roles on the learning and participation of students.

Susan Jo Russell has used her role as a curriculum designer to study important issues of mathematics pedagogy. Her work in producing *Investigations in Number, Data, and Space* has involved her in reconceiving classroom mathematics teaching and in thinking about the role of curriculum in supporting and fostering a new pedagogy.

Marty Simon has been studying teaching in three contexts as part of research projects on mathematics teacher development: the classroom practice of participating teachers, his teaching of mathematics for teachers, and the thinking/learning of teachers in his mathematics pedagogy courses. Each of these contexts provides opportunities for making distinctions about mathematics teaching.

Recommended Pre-Conference Readings

Ball, D. L. (1997, April). *Crossing boundaries: Probing the interplay of mathematics and pedagogy in elementary teaching*. Paper prepared for the Research Presession, annual meeting of the National Council of Teachers of Mathematics, Washington, DC.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93,* 373-397.

Dekker, R., & Elshout-Mohr, M. (1998). A process model for interaction and mathematical level raising. *Educational Studies in Mathematics, 36*, 303-314.

Freudenthal, H. (1978). *Weeding and sowing*. Dordrecht: Reidel.

Russell, S. J. (1997). The role of curriculum in teacher development. In S. N. Friel & G. W. Bright, (Eds.), *Reflecting on our work: NSF teacher enhancement in K-6 mathematics* (pp. 247-254). Lanham, MD: University Press of America, Inc.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114-145.

Simon, M. (1997). Developing new models of mathematics teaching: An imperative for research on mathematics teacher development. In E. Fennema & B. Nelson. (Eds.), *Mathematics teachers in transition* (pp. 55-86). Hillsdale, NJ: Lawrence Erlbaum Associates.

Streefland, L. (1990). Realistic Mathematics Education (RME). What does it mean? In K. Gravemeijer, M. van den Heuvel, & L. Streefland (Eds.), *Contexts free productions tests and geometry in Realistic Mathematics Education* (pp. 1-9). Utrecht, the Netherlands: Research Group for Mathematics Education and Educational Computer Centre State University.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education: A paradigm of developmental research.* Dordrecht: Kluwer.

Discussion Groups

MATHEMATICS PEDAGOGY IN SOCIAL AND CULTURAL CONTEXTS

Yolanda De La Cruz
Arizona State University West
ydelacruz@asu.edu

This paper reports an overview of our Children's Math World project work with families. We found it to be critical for teachers to involve families in their children's mathematical education. This paper presents a modification, based on the experiences of our project, of the Epstein (1989, 1995) model of building effective and supportive home-school relationships. We have concentrated on efforts that will help families support their child's daily learning in the mathematics classroom, their practicing of needed skills at home in family games, and their finding and talking about mathematics in the real world outside of school. We found that when teachers make strong efforts to establish home/school connections, human resources can be found within the homes of almost all children to support their mathematics learning. Teachers found it to be important to appreciate and affirm all of the many efforts parents were making in raising their children. The building of home/school links also needs to be viewed by all participants as information-sharing, as building mutual adaptations between the school and home settings, and as involving joint working toward the common goal—mathematical learning by the family's child.

To achieve the high level of mathematics understanding and skill that was our goal, children needed to complete daily homework. Our urban children and families were very supportive of such homework. It made the children feel grown-up, and most of them enjoyed doing most of the homework. Families were involved by identifying a Math Helper in each home to be responsible for monitoring the child's homework completion and to help if necessary. Identifying a particular person for this responsibility was helpful in homes where many pressures create difficult on-going and changing life demands. When families understood that it was important to do so, almost all would organize themselves to identify such a helper. This might be a parent, an older sibling, an uncle or grandparent who lived with the family or lived nearby, or a neighbor. Because of language differences, no phone in the home, and distance from the school, teachers had to be resourceful and persistent in order to communicate the need for a math helper to some families.

Most homework involved repetitive mathematical experiencing so that the demands on family helpers and on children were minimized. Some elements of CMW are new (e.g., the use of MDMs) and were explained in letters home, parent nights, and by the children themselves. Wherever possible, the homework was designed not to be too different from homework the home helper might have had. In some reform-math curricula, parents are asked to do many different kinds of activities with their child. The reading level is very high, too high for many of our families. Teachers did have to work hard with some children to establish patterns of effective return of homework. But most teachers who worked at it were able to achieve high rates of such return, rates that were surprising to them and considerably higher than in the past.

Because many Latino families are family-centered, we designed games that family members could play to help their child practice important math competencies. Some families and some children had no previous experience in playing board games or games with cards, so teachers organized family nights in which they taught families to play the games. These were generally well-attended and enjoyed by all. One school librarian made fancier versions of the games and made them available to family members in the school library so that they could learn the games there. The feedback from families involved in using the games was quite positive.

PERFORMANCE ASSESSMENT OF K-12 PRESERVICE TEACHERS' MATHEMATICAL CONTENT KNOWLEDGE

Thomas B. Fox
Ball State University
tfox@wp.bsu.edu

Kay Meeks Roebuck
Ball State University
Kroebuck@wp.bsu.edu

Sheryl L. Stump
Ball State University
sstump@wp.bsu.edu

Both the Mathematical Association of America (MAA, 1991) and the National Council of Teachers of Mathematics (NCTM, 1991) have published sets of specific recommendations for the mathematical content preparation of preservice elementary, middle school, and high school teachers. However there is little published discussion of how to *assess* teachers' mathematical content knowledge. There is a growing need for such discussion. For example, the emphasis of the National Council for Accreditation of Teacher Education (NCATE, 1998) is moving toward evaluation of the performance of preservice teachers rather than evaluation of merely the curriculum of teacher education programs. Through the New Professional Teacher project, NCATE is working with its subject matter professional associations to develop performance expectations for teacher preparation. Many states are revising their state teaching licensure and certification requirements to reflect such an orientation. In response, an increasing number of teacher education programs will most likely require their students to develop assessment portfolios. These portfolios must include items that demonstrate the preservice teachers' mathematical content knowledge. Opportunities to develop these materials must be provided not only in mathematics methods courses but also in mathematics content courses for preservice K-12 teachers. This major paradigm shift in evaluating an individual's readiness to begin teaching raises new challenges for teacher educators as well as those less directly involved in teacher education, including mathematicians who teach content courses for preservice teachers.

The topic for this discussion group is performance assessment of preservice K-12 teachers' mathematical content knowledge. The discussion will focus on three main issues: 1) What research is in the existing literature on performance assessment of preservice K-12 teachers? 2) What suggestions for practice do the participants bring from their own experience? 3) What are some questions for research on performance assessment of preservice K-12 teachers' mathematical content knowledge?

The discussion will begin with an overview of the existing literature on performance assessment of preservice K-12 teachers. Next participants will form small discussion groups to focus on suggestions for practice and questions for research. Discussion questions may include the following: What types of performance mathematics activities have been successfully used? How can mathematics assessment portfolio items be developed? How are they evaluated? What constitutes a successful performance-based activity? What supports will be necessary for those less familiar with teacher education and its goals, including many mathematicians who will now be required to design and administer such assessment tasks? How might these assessments tasks enhance current mathematics instruction? Do we have to change what we have been doing in our mathematics content courses to better ensure that students will be able to meet new standards? How do we assess performance assessment? As questions arise, do they require answers from research? The small groups will report summaries of their discussions to the larger group.

The group will then make plans for follow-up activities. To facilitate these follow-up activities, participants will be asked to provide information about their research interests and data sources and to collaborate on research projects involving performance assessment of preservice teachers' mathematical content knowledge.

Reference

Mathematical Association of America. (1991). *A call for change: Recommendations for the mathematical preparation of teachers.* Washington, DC: MAA.

National Council for Accreditation of Teacher Education. (1998). *Developmental projects: New professional teacher project.* [On-line]. Available: htpp//www.ncate.org/projects/npt/

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: NCTM.

Advanced Mathematical Thinking
Research Reports

TEACHER CHANGE IN A REFORM CALCULUS CURRICULUM: CONCEPTS RELATED TO THE DERIVATIVE

Thomas B. Fox
Ball State University
tfox@bsu.edu

An experienced high school calculus teacher was studied to see what extent, if any, her practice changed during her implementation of a reform calculus curriculum relative to her previous instruction using a more traditional curriculum. Results showed that her focus on concepts relative to the derivative increased in important ways during her implementation of the reform curriculum.

Earlier this decade, implementation of reform calculus curricula began on a large-scale basis at many universities. Today many secondary schools are now using these materials. The changes called for in this movement parallel those advocated for secondary mathematics in general (National Council of Teachers of Mathematics [NCTM]; 1989, 1991). Teachers are the ones that ultimately implement or reject advocated changes. Reforms such as these are not easily accomplished, and high school mathematics teachers can be among the most resistant to change (Wasley, Donmoyer, & Maxwell; 1995). If reform is to take place, more knowledge about the process and psychology of teacher change is needed in order to better facilitate it: "In light of the significant challenges teachers face to modify existing routines and procedures, it is crucial that we investigate how teachers deal with . . . calls for reform" (Lloyd, 1996, p. 200).

In this era of mathematics education reform, curricular materials can serve as change agents: "Because many teachers rely on textbooks as a core for their teaching, a textbook is a reasonable candidate for communicating and providing guidance for change" (Ball, 1990, p. 257). In the history of education, texts have been used with varying degrees of success to bring about change. Ball (1990) believes that texts "clearly *can* provide guidance to teachers . . . in selecting better mathematical tasks, and in creating different kinds of activities" (p. 257), yet they will not produce all the changes espoused by reform movements. However, they reach more teachers than reform documents and are more likely to be read by them (Ball, 1990). The text used by the teacher in this study, *Calculus* (Hughes-Hallett et al., 1994) [to be referred to as HC], aspires to this goal. It is one of many reform calculus texts commercially available and is often considered a moderate attempt at reform.

The purpose of this research is to document any changes in practice of one high school calculus teacher during her first-year implementation of HC relative to her practice using a more 'traditional' curriculum (Larson &

Hostetler, 1987) [to be referred as LH]. The specific research question addressed is: Regarding the conceptual development of the derivative, what is the nature of any changes in the teacher's practice during her first-year implementation of HC relative to her instruction in LH?

Research Design and Methodology

A case study of teacher change was constructed in this study and is part of a larger study that examined changes in practice and beliefs of a high school mathematics teacher as she implemented HC. The teacher-participant in this study has taught mathematics for 13 years during 7 of which she has taught one section of a non-Advanced Placement [AP] calculus course. She was the only calculus teacher in her small, rural, Midwestern high school. The teacher was the main influence on the school's choice of the reform curriculum materials and expressed a desire to change her instruction. Her class in the 1996-1997 school year consisted of 10 students. The teacher described these students as "atypical" of calculus students she had taught in the past, believing that they complained more and seemed less studious than her former calculus students.

The data were collected as follows. In the summer of 1996, baseline interviews with the teacher were conducted. These interviews focused on teacher beliefs, instructional practices, and on reconstructing her lessons from LH. The following school year, data collection included observations of the teacher's instruction in HC. All lessons pertinent to the conceptual development of the derivative were observed and videotaped. Teacher interviews were also conducted after all observations, before and after each chapter in the text, and at the end of both semesters. These interviews were tape recorded and transcribed. Artifact collection included teacher lesson planning notes, student class notebooks, and handouts from both curricula.

Qualitative methods were used to analyze the data. Data were analyzed using grounded theory (Strauss & Corbin, 1990). The particular manner in which the data were used is as follows. Field notes, interviews, and written documents were coded. Coding the data helped the researcher find commonalties. Initially, these data were Open Coded for rough categorization. During the coding process, the focus was on mathematics content, teacher actions and beliefs, assessment, technology use, and representations used in instruction. Axial Coding techniques were then used to relate categories and their subcategories discovered during Open Coding. Relationships between the different categories were then examined to determine the presence of more abstract concepts that might link less abstract categories (Strauss & Corbin, 1990). Direct comparison of instruction in both curricula was also made. Multiple data sources were used to validate trends in the data. The theory that emerged was used to construct the case study.

Results and Conclusions

In general when compared to her instruction in LH, the teacher maintained a relatively similar mode of instruction in HC. Her practice remained largely teacher-centered and focused on demonstrating prototype examples. However, significant changes in her practice included an increased focus on conceptual knowledge that included, but was not limited to, the derivative. In the following discussion, all names used are pseudonyms. An 'O' indicates data from an observation, and an 'I' indicates data from a teacher interview.

Comparison of the Teacher's Instruction in Both Curricula

The major difference between the teacher's presentation is the lack of conceptual development in LH. Her presentation of the derivative in LH began with the limit-based definition of the derivative and rules for calculating the derivative of functions:

The chapter before we did limits, and they relate the derivative to a limit, and so some of it is just algebra. . . . They'll have us look at a four step process to finding the derivative [using the limit-based definition], and I do that just a little bit . . .and then I go ahead. . . and show them how to do simple derivatives even the first day. Like the derivative of $6-2x$ is -2. Then I relate that to 'well that's a line. The slope is. . . -2'. (I)

The teacher did not discuss the idea of a derivative as a rate of change until the third section in the LH chapter, after she had presented the common procedures for differentiation. Furthermore, opportunities for conceptual development existed in the first section of the LH text. However, the teacher, who focused instead on the more procedural problems, omitted these exercises. Data from student notebooks in LH confirmed this.

By contrast, the teacher used a more conceptual, real world example to introduce the derivative her implementation of HC. This application involved examining the average and instantaneous rates of change of the distance traveled by a thrown object with respect to time. Following this discussion that lasted several class periods, her instruction continued to focus on concept related to the derivative. For several weeks, her instruction focused on looking at numeric, algebraic, and geometric notions of the derivative. No procedures for finding derivatives were taught until after this development. The reform calculus text was very influential in these changes as it provided the overall structure for her instruction.

Further strengthening a conceptual notion of the derivative, the teacher emphasized in her implementation of HC problems that focused on the derivative's meaning in applied settings. Here the focus was not on calculating derivatives, but on interpreting them as a rate of change:

The teacher presented a problem from the text that asked for the sign of the derivative of a function, f at a time t, that represented the temperature of a yam as it was heated in an oven. She asked a student, Sammy, what the sign of $f'(t)$ would be. When she received no response from her, she told her that the yam was cold when it was put in the oven. Sammy responded "up". The teacher then asked her what the corresponding derivative's sign would be and Sammy said "positive". After this response, she asked her student why this was true, and Sammy told her why. . . .The problem then asked for the practical meaning of $f'(20) = 2$. The teacher asked Jimmy to answer this, and he responded "at 20 minutes it goes up 2 degrees". (O)

The teacher believed that text problems such as this influenced her focus on concepts:

The new book, they want you to really think about the problems. If I think about the meaning. . . of a derivative. They actually had us write out, 'Okay, what's that mean in English?'. Well, in the old book, it was 'Okay, you have a line $y = 2x$. Give me the derivative'. Not anything about 'well what does this mean?'. . . . So the new book is so much more conceptually [orientated] than the old book. (I)

Furthermore, in several instances the teacher spent a significant amount of class time discussing problems from HC of a highly conceptual nature. Her emphasis on a particular problem was a conscious choice on her part and not a decision highly influenced by the text. This is a stronger indicator of more important teacher changes.

In her implementation of HC, the teacher's conceptual development of the derivative's geometric and numeric notions was far greater. While little time was spent developing a geometric notion of the derivative in LH, she devoted several days in her implementation of HC to sketching the derivative, $f'(x)$, from the graph of $f(x)$. Further strengthening this interpretation, graphing calculator technology was utilized to study local linearity of a function, a topic not studied in the LH. In both curricula, the teacher developed a numeric approach to the derivative. However, in HC, she emphasized it to a greater extent and further developed it using the limit-based definition of a derivative and local linearity.

The teacher's development of the derivative's algebraic notions was similar in both curricula. The limit-based definition was used to develop this. However, in HC she used this definition to find the derivatives of less complicated functions. Approximately the same amount of class time was devoted to the limit-based definition of the derivative in both years, yet she placed less emphasis on it as a "four step process" in her instruction in HC.

This study has two major limitations. The first involves generalizability. As this is a highly contextual case study, sweeping implications are not warranted. However, implications are possible for cases that closely resemble this one. A second limitation is the lack of formal observation data of the teacher's instruction in LH. An attempt was made to acquire knowledge of the teacher's instruction using student notebooks and interviews designed to recreate instruction as it took place in her implementation of the LH. While these methods generated important data, they cannot re-place formal observations. Any conclusions based on these data must be constructed with this in mind.

This study suggests several future directions for research. In her implementation of HC, the teacher focused on concepts to a greater degree than she had in the past. What connections exist between an increased teacher focus on concepts and students' acquisition of such knowledge? Her instruction focused more on multiple representations of the derivative in HC than in her instruction in LH. In what specific ways does this emphasis influence students' acquisition of conceptual knowledge? As the student procurement of conceptual knowledge is a major tenet of the reform calculus movement, it is important to further investigate the degree to which students acquire such knowledge.

References

Ball, D. L. (1990). Reflections and deflections of policy: The case of Carol Turner. *Educational Evaluation and Policy Analysis, 12* 247-259.

Hughes-Hallett, D., Gleason, A. M., Flath, D. E., Gordon, S. P., Lomen, D. O., Lovelock, D., McCallum, W. G., Osgood, B. G., Pasquale, A, Tecosky-Feldman, J., Thrash, J. B., Tucker, T. W. (1994). *Calculus.* New York: John Wiley & Sons.

Larson, R. E., & Hostetler, R. P. (1987). *Brief calculus with applications: Second edition.* Lexington, MA: D. C. Heath and Company.

Lloyd, G. M. (1996). Change in teaching about functions: Content conceptions and curriculum reform. In E. Jakubowski, D. Watkins, & H. Biske (Eds.), *Proceedings of the Eighteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Vol. 1* (pp. 200-205). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques.* Newbury Park, England: Sage Publications.

Wasley, P. A., Domoyer, R., Maxwell, L. (1995). Navigating change in high school science and mathematics: Lessons teachers taught us. *Theory into Practice, 34*, 51-59.

Note. This work is part of a doctoral dissertation completed by the author at Illinois State University under the direction of Beverly S. Rich.

A LOGO-BASED MICROWORLD AS A WINDOW ON THE INFINITE

Luis Moreno Armella
Departamento de Matemática Educativa, Cinvestav
lmorenoa@data.net.mx

Ana Isabel Sacristán R.
Departamento de Matemática Educativa, Cinvestav
asacrist@mail.cinvestav.mx

Introduction

We want to use this opportunity to introduce some issues that are important to articulate a reflection on the ways in which computational environments mediate the construction of mathematical concepts. We adopt the perspective that learning involves the construction of representations. That it is through the construction of models, which serve to represent an observed phenomena, idea or concept, that we make sense of the world, including mathematical objects. Models become tools for understanding. Indeed, they are mediational tools for the construction of knowledge.

Based on this premise, we built a microworld which could simultaneously provide its users with insights into a range of infinity-related ideas and offer the researchers a window into the users' thinking about the infinite. An important research issue was to look at the ways in which the different forms of representations were coordinated and integrated through their interaction with the procedural code.

Papert (1993; p. 142) points to the importance of the use and construction of external representations in the process of knowledge construction:

"One of my central mathematical tenets is that the construction that takes place "in the head" often happens especially felicitously when it is supported by construction of a more public sort "in the world" [...]. Part of what I mean by "in the world" is that the product can be shown, discussed, examined, probed, and admired. It is out there".

There is extensive research into the idea that writing a computer program provides a means for sketching half-understood ideas. In fact, much of the original research in this area involved programming with Logo and such work still continues (See Noss & Hoyles 1996). Computers bring a tool that incorporates the visual dimension into mathematics in ways that were not previously possible. For instance, they allow a process to be seen as it evolves or is generated in time (Moreno & Sacristán, 1995). Thus the dynamics and the behavior of the process as opposed to its result can be observed. The writing of a procedure is, in a sense, similar to the construction of an isomorphism between any two models of a mathematical concept. The interaction between these models facilitates the construction of situated abstractions (Noss & Hoyles, 1996).

The Microworld: Activities and Some Relevant Results

The design of the study involved several phases: an exploratory study was used for defining the activities and makeup of the microworld (to be used in the main study) as well as the research methodology. The main study involved case-studies of 4 pairs of Mexican students of varying ages and backgrounds[1], paired up by age-group. The analysis of student's experiences was carried out by working with one pair of students at a time, using a clinical interview style, with approximately 15 hours of work for each pair. A selection of the activities for the main study was made on the richness of possibilities of each activity, taking into account those which produced interesting results as well as the simplicity with which they could be approached.

The activities finally chosen for the microworld in the main study were the following:

i) Explorations of infinite sequences, and

ii) Exploration of fractal figures ("limit objects").

The former included the sequences $\{1/2^n\}$, $\{1/n\}$ and $\{1/n^{1.1}\}$ and the sequences of their corresponding partial sums, studied through visual models–mainly spirals (e.g. Figure 1), bar graphs and staircases (See Sacristán, 1997, for further details).

The exploration of fractal figures centered mostly on the study of the recursive structures of the Koch curve and snowflake (Figure 2) and the Sierpinski triangle (Figure 3), and involved the encounter of some apparent paradoxes such as that of a finite area bounded by an infinite perimeter. Through these sequences and fractals activities we intended to confront students with the idea of "what happens in the infinite" by allowing them to visualize an infinite process through the dynamics–as the procedures were running α computer-based approximations.

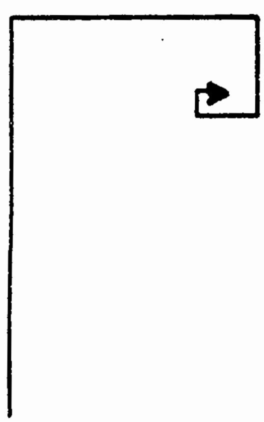


Figure 1. *Spiral model for the sequence $\{1/2^n\}$*

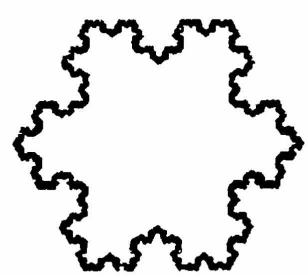


Figure 2. *The Koch snowflake*

[1] Since the research objectives did not make any particular demands on the sample, we used students of different age groups (a pair of 14 year old students, a pair of high-school students, a pair of college students, and a pair of math teachers in their mid-thirties) as it was also interesting to compare the conceptions and ways of working of younger students with those of older students (and teachers).

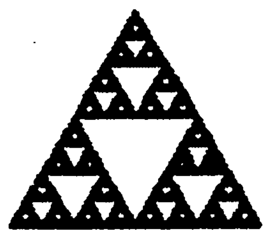


Figure 3. *The Sierpinski Triangle*

As explained in the introduction above, an essential aspect of the functioning of the microworld was the programming activity on the part of the students. All of the students therefore wrote their own procedures using the Logo programming language, such as for instance those which produced the visual models, although the activities were suggested by the researchers (who also served as guide).

In the initial activity, the subjects were given the following procedure (from which students derived the procedures for the activities that followed):

```
TO DRAWING :L
PU
FD :L
RT 90
WAIT 10
DRAWING :L/2
END
```

This procedure makes Logo's turtle walk (without leaving a trace _ the Pen is up) through a spiral with arms each having half the length of the previous one (see Figure 1 above). It should be noted that as there is no stop condition, the procedure continues indefinitely[2]. It was designed to induce students to reflect on the behavior of the turtle and the procedure itself.

This initial activity produced interesting results: in particular, the students had to visualize the actual pattern without relying on the computer drawing; it induced students to try to make sense of the relationships between the code and the graphical output: For instance, most students did not expect to see the turtle endlessly spinning without leaving a trace. In order to explain to themselves this unexpected behavior and make sense of why the turtle was endlessly spinning, the students had to re-examine the procedural code. Victor was one student who immediately remarked that the procedure would never stop because the recursive structure of the code represented an infinite process. He explained it was because the procedure called itself without anything telling it to stop, so it never would; the process of turning and walking half the previous distance would continue repeating itself and would never stop. By analyzing the code Victor was able to connect to it the behavior of the visual output (in this case the *move-*

[2] Later, when the students became aware of the recursive structure, all of them eventually added a stop condition, which also served as an important investigation tool for making sense of the relationship between code and figure, as is described in Moreno & Sacristán (1995) and Sacristán (1997).

ments of the turtle) since he correctly predicted the outcome and was able to justify that visual behavior through the code. He *linked the recursive structure of the code with the infinitude of the process.*

A modified procedure (with the Pen down) produced an inward spiral with the turtle then turning endlessly in its center. Victor and Alejandra pointed out that although the turtle seemed to be just turning in the same spot, in reality there was "a variation". There were two factors here: a) The turtle kept turning and b) the turtle turned at same spot. The first factor could have served as an indicator that the process continued, but it was the fact that the students seemed to be able to disregard the *visual appearance* of the turtle — spinning in apparently the same spot — that suggests that they understood that the underlying (mathematical) process continued, and that they were able to link the output with the code and the process.

It was thus that students were able, via a process of experimenting backwards and forwards from code to figure, to make sense of the behavior of the turtle which seemed to be spinning on the same spot realizing that the amount that the turtle moved each time was halved. The key point here is that the analysis of the code allowed them to:

1) Recognize in the recursive structure a potentially infinite process.
2) To quantify the movement, to explain that although the turtle seemed to be turning without moving forward, in reality there *was* a variation.

Thus by coordinating the visual and symbolic—in the order visual to symbolic to visual—and later complementing it through numerical explorations, their understanding of the process became integrated and potentially misleading visual appearances could be ignored.

The fractal explorations also produced interesting results. For instance, during the Koch curve and snowflake explorations, some students, when confronted with the fact that the perimeter of the snowflake tended to infinity but was contained in a finite area, found the conjunction of these two elements at least counter-intuitive. In the case below the dilemma was solved when one of the students (Manuel) pointed to the significance of the *shape* of the figure as the determinant factor:

Jesus: It is incredible that it has an infinite perimeter and that it comes to a point where the area is limited.

Manuel: Well, not so incredible since...

Jesus: Well, it is unusual. What other figure do you know that has an infinite perimeter with a limited area?

Manuel: Well, what happens is that the perimeter is growing and growing but it is somehow folding inside the (circumscribing) circle, and that is why the area is almost constant... looking at it this way I don't find it so incredible...

The experiences with the Sierpinski Triangle explorations were similar. We asked the students to imagine what would remain after removing the central triangles. Some students predicted from the beginning that the remaining area would be zero. The younger students (Consuelo and Veronica) initially suggested that if they rearranged the remaining areas they might obtain a triangle the size of the initial central triangle. However, when Veronica reflected on the fact that after level 7 of the process, all the subsequent figures looked the same but had less area, thinking aloud she exclaimed that it would be the entire triangle which would be removed.

The specific examples serve to illustrate and analyze some of the ways in which students used and coordinated the elements of the exploratory medium to construct meanings for the infinite. We divided the findings into three categories:

i) The construction of meaning through programming;
ii) The use of the medium as a "mathematical laboratory" ; and
iii) The relationship between the activities and tools on the environment and students' conception of the infinite.

An interesting finding is that the younger students focused more on open-ended explorations, while the more mathematically experienced students (the pair of teachers) tried to make connections with the official mathematical knowledge. Another interesting remark is that when the students were unable to see the deeper levels in the visual representations, some of them blamed this on the resolution but were able to compensate for the deficiencies by using information provided by the symbolic structures of the procedures. For instance, one of the older students (Martin) explained the behavior of the spiral as follows:

Martin: What happens is that there is a part that our eyes can no longer perceive. Inside (the spiral) it continues the same way, because it is the same process that continues...If we used a magnifying glass and looked at that little square there, we would see like all this part (the full spiral).

We want to emphasize that it was the interplay (and linking) between the programming code, the endlessness of the dynamical process, and the turtle's movements which led students to make sense of what they observed. This includes confrontation with prediction, recursion and change of codes.

References

Moreno, L., & Sacristán, A. (1995). On visual and symbolic representations. In R. Sutherland and J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education.* New York: Springer-Verlag, Nato ASI series, vol. 138.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meaning. Learning cultures and computers.* Dordrecht, The Netherlands: Kluwer.

Papert, S., (1993). *The children's machine*, Basic Books, New York.

Sacristán, A.I. (1997). *Windows on the infinite: Constructing meanings in a logo-based microworld.* (Doctoral dissertation, Institute of Education, University of London).

SOCIOMATHEMATICAL NORMS AND STUDENT AUTONOMY IN CALCULUS II HONORS STUDENTS

Chris L. Rasmussen
Purdue University Calumet
raz@nwi.calumet.purdue.edu

Karen D. King
San Diego State University
kdking@mail.sdsu.edu

This study compared student performance and the nature of students' intellectual autonomy in two sections of a second semester calculus course. An inquiry-based approach to instruction and the TI-92 graphing and symbolic calculator was used in an experimental section while a more traditional mode of instruction and standard graphing calculators were used in the control section. At a macro level, there was no significant difference in students' performance on the common final exam. However, individual student interviews revealed a striking difference in the nature of students' intellectual autonomy. In the experimental section, the two students interviewed reasoned in multiple ways regarding the viability of their solution. In contrast, the two students from the control section appeared to be limited to checking their calculations and looked to the interviewer for confirmation that their solution was correct. In part, we attribute these important differences to classroom norms regarding what constitutes an acceptable mathematical justification.

In the past decade there has been considerable discussion of and changes to the teaching and learning of college calculus. Changes in modes of instruction and assessment, the use of technology, an increased emphasis on conceptual understanding, the development of concepts graphically, numerically, and symbolically, and a greater emphasis on applications are a few of the hallmarks of the calculus reform movement (Roberts, 1996). Although these changes have received national recognition and widespread implementation, there is a growing debate regarding the effect of these changes on students' knowledge, beliefs, and values. The study reported here examines the impact of one approach to reform in a second semester calculus course at a large mid-Atlantic university.

Theoretical Framework

The theoretical orientation employed in this study is based on the emergent perspective as described by Cobb and Bauersfeld (1995). This framework strives to coordinate the individual cognitive perspective of constructivism (von Glasersfeld, 1995) and the sociocultural perspective based on symbolic interactionism (Blumer, 1969). In this view, the development of students' reasoning and sense-making processes cannot be separated from their participation in the specific microculture of the classroom.

A central construct in this perspective is that of sociomathematical norms. These norms include interactively constituted understandings such as what counts as mathematically efficient, mathematically elegant, and what constitutes an acceptable mathematical justification and explanation. Such norms are assumed to be reflexively related to the development of students' mathematical conceptions, mathematical beliefs and values. In particular, sociomathematical norms are a useful and clarifying construct in accounts regarding how students develop intellectual autonomy in mathematics (Yackel & Cobb, 1996).

Method

The students were enrolled in two sections of second semester honors calculus, MATH 2H, at a large state university in the Mid-Atlantic region. The two sections of the course met at the same time, three times per week, twice for fifty minutes and once for one hour and fifty minutes. A professor of mathematics at the university and co-author of the standard text used for all calculus courses for mathematics and science majors taught the control section with a traditional lecture approach. He allowed graphing calculators (e.g., the TI-82), but did not integrate them into the course. Rasmussen taught the experimental section, incorporating use of the TI-92 computer algebra system and alternative instructional strategies compatible with the calculus reform movement. The instructors used the same textbook, however the experimental section altered the order of topics, beginning with applications of the integral and numerical techniques followed by standard analytic techniques. This alteration was possible due to the symbolic computation of the TI-92. While the homework sets were drawn from the textbook, the experimental section incorporated an additional assignment by requiring that students provide justifications for three to five specified traditional homework problems.

Two students from each class agreed to participate in an individual interview at the end of the semester. The instructors of the two sections of the course chose the students based upon the level of articulation shown during their interactions. The two students, Donald and Aaron, from the control class are both Caucasian males with A averages in the course. From the experimental class, one student, Susan, is a Caucasian female with an A average in the class and the other, Debra, was an Iranian-American female with a B average in the course.

King observed the experimental section approximately once per week on varying days, during which she took extensive field notes focusing on the instructor's interaction with the students. The final examination was a uniform test given to all sections of the course, honors and non-honors. Students in both sections took the exam under uniform conditions at the same assigned time. A person who taught a section of MATH 2 or MATH 2H graded one problem per page of the test to eliminate instructor bias.

King conducted interviews in a Mathematics building office during the last week of the semester. Videotaped interviews were about an hour in duration. The students responded to general questions about their backgrounds and the course, then "thought aloud" as they worked several calculus problems on an overhead projector. The problem discussed in this paper is

Results and Discussion

At a macro level, there was no statistical difference in students' performance on the common final examination ($p = 0.902$). This is significant since a common concern among some educators is that the use of technology, in particular computer algebra systems, will result in loss of procedural ability. As was the case in other studies (e.g., Heid, 1984), students in the reform-oriented section performed as well as students in the control class on the final examination where only graphing calculators were allowed.

Even more significant, however, were the results from the more in-depth analysis of the student interviews. This analysis revealed a striking difference between the nature of the reform-oriented students' mathematical justifications and their level of conviction in their answers. In particular, the two students interviewed from the traditional section looked to the interviewer for confirmation that their solution was right. The only strategy they had for justifying their answer was to review their previous computations. This strategy proved to be insufficient for one of the students to find an error in his work. In contrast, the two students from the reform-oriented section had multiple ways to justify the appropriateness of their response, including numerical approximations, the TI-92, and graphical approximations. These multiple strategies provided an opportunity for these students to find errors in their computations and appeared to give them a distinctly different personal level of conviction in their responses.

These results suggest that the students in the reform-oriented class had a more conceptual vs. calculational view (Thompson, Philipp, Thompson & Boyd, 1994) of mathematical justification and were certain of their answers without confirmation from the interviewer. For example, while Aaron, one of the students in the control class, was working on the problem, he asked if he was doing the problem correctly and when he finished the problem he asked the interviewer, "Is this right?" When asked to verify how he would know that his answer was correct he went through all of the techniques that he had learned and decided he had chosen the correct technique. This process-of-elimination strategy to justify his results resembles the problem solving strategies shown by students on story problems (Sowder, 1989). Instead of using a mathematical basis for his explanation and justification, Aaron based his justifications on authority and on procedures provided by his instructor and textbook (Voigt, 1996; Yackel & Cobb, 1996).

Aaron then went through a process of "checking his work," focusing on the procedures and calculations. He thought that his calculations were correct, but was still unsure that his final answer was right. His justifications focused on procedures, leading to our categorization of his view of justification as calculational. The other student in the control class also exhibited this same focus on procedures and calculations. In contrast to intellectually autonomous students, Yackel and Cobb (1996) describe intellectually heteronomous as students "who rely on the pronouncements of an authority to know how to act appropriately" (p. 473). The students from the control section relied on the interviewer, an authority, to decide whether their solutions were correct.

In contrast, one of the students in the reform-oriented section, Susan, noted she could use the TI-92 to do the calculation and check her work. However, she said that the class did not focus on this method and doing the calculation with the TI-92 was "boring." Instead, she focused on the concept of the definite integral as area under the curve. Susan was able to reason about the definite integral as an experientially real mathematical object, the area under the curve (Yackel & Cobb, 1996). After estimating the area under the curve, she realized that her previous result was in error and proceeded to find her computational error. Once she corrected her error, she was certain that her answer was correct and did not rely upon the interviewer to confirm or deny the result. This level of student mathematical autonomy was also observed in Debra. These students "are aware of, and draw on, their own intellectual capabilities when making mathematical decisions and judgments" (Yackel & Cobb, 1996).

In part, we attribute these differences to student-teacher negotiations regarding what constitutes an acceptable mathematical justification and what the members of the classroom community should view as mathematically different solution strategies. In the experimental class, this negotiation process occurred at the level of classroom discussion, where the teacher encouraged students to make sense of the mathematics and to understand the mathematics of other students. The teacher reiterated these negotiations through assessment of homework requiring justifications for certain problems.

Conclusion

The construct of sociomathematical norms provides a useful way to view reform-oriented instruction and the impact of this instruction on student knowledge, beliefs, and disposition. Moreover, it is conjectured that awareness of and explicit attention to the interactive constitution of sociomathematical norms by college mathematics instructors provides a means for educators to achieve some of the goals of the calculus reform movement. Finally, the research reported here illustrates how constructs such as sociomathematical norms, which were developed through a research and development project at the elementary school level, cut across content and grade levels. This generalization enables and establishes a line of communication needed for the successful systemic reform of mathematics education at all levels.

References

Blumer, H. (1969). *Symbolic interactionism*. Engelwood Cliffs, NJ: Prentice-Hall.

Cobb, P. & Bauersfeld, H. (1995). *The emergence of mathematical meaning*. Hillsdale, NJ: Erlbaum.

Heid, K. (1984). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education, 19*(1), 3-25.

Roberts, A. W. (Ed.). (1996). *Calculus: The dynamics of change*. Mathematical Association of America Notes # 39.

Sowder, L. (1989). Research into practice: Story problems and student strategies. *Arithmetic Teacher, 36*(9), 25-26.

Thompson, A.G., Philipp, R.A., Thompson, P.W., & Boyd, B.A. (1994). Calculational and conceptual orientations in teaching mathematics. In D.B. Aichele (Ed.), *Professional development for teachers of mathematics*, The 1994 Yearbook of the National Council of Teachers of Mathematics (pp. 79-92). Reston, VA: National Council of Teachers of Mathematics.

von Glasersfeld. E. (1995). A constructivist approach to teaching. In L. P. Steffe, & J. Gale (Eds.), *Constructivism in education* (pp. 3-15). Hillsdale, NJ: Erlbaum.

Voigt, J. (1996). Negotiation of mathematical meaning in classroom processes: Social interaction and learning mathematics. In L. P. Steffe, P. Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 21-50). Mahwah, NJ: Erlbaum.

Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458-477.

THE ROLE OF A FORMAL DEFINITION IN NINE STUDENTS' CONCEPT IMAGE OF DERIVATIVE

Michelle J. Zandieh
Arizona State University
zandieh@math.la.asu.edu

The purpose of this paper is to explore the relationship between a student's understanding of the concept of derivative and the student's knowledge of a formal symbolic definition for derivative. This research is part of a larger study that examines the evolution of nine students' understanding of derivative over the course of a nine-month school year.

Theoretical Framework

For this research a student's understanding of the concept of derivative is initially defined by Tall and Vinner's (1981) notion of *concept image*, "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). To give this notion more detail, consider Hiebert and Carpenter's (1992) notion that understanding consists of the number and strength of connections or links that a person has between various nodes of information. A key to giving a more detailed description of the understanding of a particular concept such as derivative is to determine the nature of the various nodes and links. The work of Sfard (1991, 1992) on the process-object duality for mathematical concepts provides some of this detail.

The process-object duality is the notion that each mathematical concept may be considered as both a dynamic process and as a static object. Sfard's theory holds that historically and psychologically the process conception is developed first and then consolidated into an object which can be acted on by another process.

In Sfard's theory processes are operations on previously established objects. Each process is reified into an object to be acted on by other processes. This forms a chain of process-object transitions.

For this research I note that understanding the concept of derivative consists of three process-object transitions. The ratio process takes two objects (two differences or increments such as two lengths or a distance and a time) and acts by division. The object created by the ratio process, a slope or velocity or other ratio, is used by the next process, that of taking a limit. The limit may be envisioned as a process of "passing through" infinitely many of the ratios, getting "closer and closer" to a particular value. This limiting value in the case of the derivative is the value of the slope at a point on a curve or the instantaneous velocity. This object, the limit, is used to define each value of the derivative function. The derivative function acts as a process of passing through (possibly) infinitely many input values and for each determining an output value given by the limit of the difference quotient at that point. The derivative function may also be viewed as an object, just as any function may.

I will refer to each of these process-object entities — ratio, limit, and function — as a layer of the derivative concept. Each layer may be observed in multiple contexts: graphical (slope), verbal description (rate of change), kinematic (e.g. velocity or acceleration), and symbolic (the symbolic difference quotient definition of derivative). A student's understanding of the concept of derivative may be seen as a matrix denoting which layers a student is aware of, in which representations, and the connections a student sees or does not see between the layers or representations.

One further detail of this structure should be considered here. Suppose a student has not developed a structural conception of one of the layers. How can that student consider the next process in the derivative structure without an object to operate on? One simple solution is to use what Sfard (1992) calls a *pseudostructural* conception. A pseudostructural conception may be thought of as an object with no internal structure. In fact, even for a person who can conceptualize each layer as both a process and an object, it is often simpler to describe a process by having it operate on a pseudostructural "object."

One example of a pseudostructural conception would be for someone to think of speed simply as how fast something is traveling without considering speed as a ratio or quotient of distance over time. Both Confrey and Smith (1994) and Thompson (1995) report examples of young children thinking of speed as an object without considering any associated ratio. Whether or not a person is aware of the ratio involved in the concept of speed or the limiting process involved in finding the speed at one instant in time, one may consider the derivative function as a process that gives us the speed at each point, like a car's speedometer. For this description a student can concentrate on the function process and it's output without, for the time being, working with the complications of the underlying limit or ratio processes.

Methodology

The methodology for the larger study is a multiple case study. Each of the case studies covers one of the nine students in the upper level (BC) advanced placement calculus class at a suburban high school. The students will be referred to as Alex, Brad, Carl, Derick, Ernest, Frances, Grace, Helen and Ingrid. All the student except Alex had been enrolled in the same math and science classes during the 9th through 11th grade years. The students also excelled academically in areas other than math and science. Six of the nine students, Carl, Derick, Grace, Frances, Helen, and Ingrid, were National Merit Finalists.

Each if the nine students were interviewed five times during the academic year. The first, second and fifth interviews provide the most fundamental information for the larger study. Each of these interviews asks the students a diverse enough and complete enough set of questions about the concept of derivative so that a student's responses may be taken as an approximate snapshot of his or her concept image of derivative at that point in the course. The third and fourth interviews serve more limited purposes. The third interview examines what aspects of the concept of derivative come into the discussion of the relationship of derivatives to integrals. The fourth interview focuses on open-ended questions and problems that are related specifically to rate of change.

This paper considers results from all the interviews to examine the role of a formal definition in each of the students' understanding of the concept of derivative.

Results

Even though most of the students in this class eventually learned to state the formal definition, only Helen regularly mentioned the formal definition as one of her initial response to "What is a derivative?" She also gave the formal definition as how she would explain the derivative to a student in a precalculus class. The other students mentioned the formal definition much more rarely, and then only after they had stated other answers to "What is a derivative?"

In terms of ability to relate the formal definition to other representations or contexts, by the end of the academic year, the students broke into three broad categories. The first group each knew the formal definition and could related it accurately to at least one other representation for the concept of derivative (Alex, Derick, Frances, Helen). The second group had the formal definition memorized but could not accurately relate it to other aspects of their understanding of the concept of derivative (Ingrid, Brad, Grace). The third group refused to memorize the formal definition and could only guess at inaccurate fragments of it when asked (Carl, Derrick).

The first group had the ability to generate the formal definition itself or related ratios such as the average needed for the Mean Value Theorem by remembering a graph or the concept of velocity and using that to reconstruct the knowledge. In the case of the Mean Value Theorem, each of these students recalled the graph involved first and were able to construct the symbolic from the graph and the idea that the slopes are parallel. The middle group had memorized many appropriate phrases, but did not always understand the meaning behind them. In particular Ingrid did not seem to be aware of the notion of ratio involved in the derivative concept. Throughout the three interviews the students were given multiple opportunities to demonstrate an awareness that the derivative involves a ratio of two quantities. Each of the other students stated a ratio in a context other than restating the formal definition. Ingrid never did. Ingrid could state that the derivative represents slope, rate and velocity, but she never mentioned the ratios involved in these quantities. A particularly obvious missed opportunity occurred in the fourth interview when each student was asked to approximate f'(2) given a table of values for the function with the inputs incrementing by .1. Each of the other students in the class were able to estimate the derivative value by calculating an appropriate ratio. Ingrid was not. She said, "I feel like I need an equation to find it." If she had an equation for the function, she would "take the derivative and plug in 2." She was unable to state how she would find an estimation without an equation. When prompted with the suggestion of sketching a graph of the points, she did so but did not think of calculating the slope at a point.

The final two students stated in their opinion of the formal definition in interviews. Carl said, "If you have to go with formal definitions, I don't know those things. I know my own definition in my head of what they are, what they do, and I can do problems like that, but when a teacher's asking for a formal definition, I go crazy." Ernest said more simply during the second interview, "I never was good at textbook definitions and stuff. ... I can't give textbook definitions." Neither of these two students found the symbolic formalism of the definition relevant to their understanding of the concept of derivative.

Discussion

As described above, the ability of the students to relate the formal definition to other aspects of their understanding of the concept of derivative varied from an ability to make many connections, to a weakness in making meaningful connections, to a disinterest in making any such connections. This variety of responses came from students who had studied calculus and other mathematics and science subjects together for the past four years. Hence, one suspects that individual factors such as beliefs and abilities contributed substantially to these differences.

What was the role of instruction? The teacher for the senior year calculus course, Mr. Forrest, emphasized the relationship between the limit of the difference quotient definition and a graphical representation with secant lines approaching a tangent line. During a two week period toward the end of September, Mr. Forrest presented the relationship between the formal definition and its graphical interpretation on the board on three different class days. On a fourth day, the limit of the difference quotient was discussed without the secant line picture. During these two weeks the students were assigned homework problems involving computing and estimating derivative values using the full formal definition and just the ratio. At the end of this period the students took a test on the material and the questions on the test were discussed in class the following day.

This instruction had some positive effects, but the results were not as complete as one might hope. One day after the discussion of the test questions the students were asked to write what they understood about derivatives at this point that they did not understand from their study of derivatives at the end of their junior year. Five of the students mentioned the formal definition in some sense. During the next interview several weeks later, six students were able to state the formal definition correctly. However, only three students in the second interview were able to correctly relate the formal definition of derivative to the secant line picture. Two others could explain that the ratio is the slope without connecting the limiting process. However, three students (Brad, Carl and Ernest) still could not state the formal definition correctly and a fourth student, Ingrid, who could state it correctly and even included it in her answer to "What is a derivative?" could not relate the two.

This study did not attempt to devise or test any other methods to influence student understanding of the formal definition of derivative. However, two observations may be made concerning the students who best connected the formal definition to other aspects of their understanding of derivative. First, these students had an understanding of the ratio and the limiting processes in a context other than the symbolic, a context such as slope or velocity. Second, the symbolic notation system had meaning for the students outside of the symbols themselves, outside of the knowledge that f(x) means to plug the x value into the expression for the f function. In other words, the students were able to use the symbols as a language that expressed their knowledge in another context.

These observations point to several potential obstacles in student understanding of the formal definition. First, the students must understanding the processes underlying the concept of derivative. In this group of nine students, only Ingrid had not made the connection that a derivative value could be approximated using a ratio. However, this phenomenon is likely to be more wide-spread in classes with weaker students. Second, students must place some value on having a symbolic representation for this ratio. Both Carl and Ernest found such an expression unnecessary. Third, students must think of symbolic expressions as having meaning in terms of their experiences in other contexts.

A question that follows from the attitudes of Carl and Ernest is whether, or to what extent, an understanding of the formal definition is necessary to the study of calculus. To be even more extreme, one may consider to what extent understandings of the ratio or limiting processes are necessary to the study of calculus. Although knowledge of the these processes and the formal definition are certainly needed for a robust understanding of the concept of derivative, many derivative problems may be solved without them. The pseudostructural knowledge of derivative as the steepness of a function at a point or the speed at an instant in time combined with efficient short cuts for taking the derivative such as the chain rule or product rule allow students to solve many problems without the complications of the formal derivative or its underlying ratio in other contexts.

References

Confrey, J. & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26,* 135-164.

Hiebert, J.& Carpenter, T. (1992). Learning and teaching with understanding. In Grouws, D. (Ed.), *Handbook of research on mathematics teaching and learning.* (pp. 65-97). New York: Macmillan

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22,* 1-36.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification — The case of function. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (MAA Notes, Vol. 25). Washington, DC: Mathematical Association of America.

Tall, D. O. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12,* 151-169.

Thompson, P. W. (1995). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics.* (pp. 179-234). Albany, NY: SUNY Press.

ADVANCED MATHEMATICAL THINKING SHORT ORALS

165

REPRESENTATION AND VISUAL THINKING RELATED TO DERIVATIVE CONCEPT AT HIGH SCHOOL LEVEL

Patricia E. Balderas-Cañas
Universidad Autónoma de México
empatbal@servidor.unam.mx

This paper reports some findings documented in Balderas (1998). The main objectives of this investigation were to study high school students' processes of internal representation while teaching and learning mathematics (derivative), according to representations generated in graphing calculators and to design a didactic proposal for differential calculus (high school level) based on the findings of this research. Research questions of the study were: (1) how is the mental representation process of related concepts to the derivative concept when graphing calculators are readily accessible?, and (2) which relations does the student establish between different representations of concepts related to the derivative concept?

The framework of the study was influenced by Goldin and Kaput (1992) point of view about cognitive integration between two or more representations of mathematical concepts. A naturalistic paradigm was considered appropriate for the holistic construction of the classroom reality. The study took place over a period of three years and involved seven students.

A working hypothesis was that fostering a broad repertoire of schemas in students promotes the integration between concept images of different representations. The study included graphic, numerical, algebraic or symbolic, table and text representations. The study variables were the connections among five representations which are shown by the students when they solve problems. Two categories were used bidimensional and tridimensional connections. The data analysis used the Propositional Analysis Model (Campos & Gaspar, 1995) to identify concepts and relations in the answers' students.

Some results showed how difficult is to separate the content from the representation in the answers, and how the behavior of oscillation among representations in the study are determined by the relations among concepts more than the concepts themselves. Few connections of the symbolic representations with the other representations were found. Hence teaching must emphasize the correct use of algebraic language. Results suggests that in order to promote connections between symbolic representations and other representations, it is necessary to provide the students with materials that foster the use of symbolic representations to express ideas. The data suggested some teaching goals, for example, (1) to achieve that students progress smoothly and in the right direction towards the integration of different representations and (2) to make necessary efforts to lead teaching in that direction considering that calculus students need considerable time working with several representations of the concepts involved before working with symbolic algorithms in order to establish relationships between the concepts. This will help them to develop more powerful representation systems to solve variation problems.

References

Balderas, P. (1998). *La representación y el razonamiento visual en la enseñanza de la matemática.* Doctoral Dissertation under the direction of Dr. Armando Martínez Cruz, Universidad Nacional Autónoma de México, México.

Campos, M. & Gaspar, S. (1995). *The propositional analysis model: A concept-link approach to text-based knowledge organization analysis.* UNAM, Mexico, Research Reports (5), 46, June.

Goldin, G.A. & Kaput, J. J. (1992). *A joint perspective on the idea of representation in learning and doing mathematics.* Draft.

CALCULUS STUDENTS' INTUITION OF DEFINITE INTEGRAL AND HISTORICAL DEVELOPMENT OF THE CONCEPT

Bronislav Czarnocha
City University of New York
bczho@mail.hostos.cuny.edu

Ed Dubinsky
Georgia State University
edd@zeus.cs.gsu.edu

Sergio Loch
Grand View College
sloch@gvc.edu

Vrunda Prabhu
William Woods University
vprabhu@wc.stephens.edu

Draga Vidakovic
Georgia State University
draga@cs.gsu.edu

The goal of our study is to learn about student intuition and understanding of a definite integral and to "compare" it to the historical development of this concept.

We find striking connections between the intuitions of Calculus students at a large mid-western university and those of well known mathematicians in history. There appears to be a deviation of some students' intuitions from those that current teaching practice is trying to develop. Although current teaching practice is mainly focused on the chopping up intuition, students' understanding, as seen from interviews in our study, includes both a chopping up and a sweeping out intuition. This finding coincides with the historical development of the concept. Historically, Archimedes employed the sweeping out and chopping up intuitions simultaneously as early as the 3rd century BC. During the 17[th] century the same ideas were rediscovered by European intellectuals such us Gregory St. Vincent, Cavaliery, Ferma, Cauchy, Roberval, Wallis, Pascal, etc.

We will also discuss the question of pedagogical practice in teaching the concept of the definite integral as directly related to the results of this particular study.

COGNITIVE DIFFICULTIES: THE CASE OF RELATED RATES

David E. Meel
Bowling Green State University
meel@math.bgsu.edu

This research study qualitatively examined both written and verbal protocols focused on the concepts of function, function composition, differentiation, and chain rule as well as solving arithmetic, algebra, and related-rate based word problems. In particular, this study investigated the difficulties faced by thirteen Calculus I students who completed the course under the same instructor with a grade of at least a C (3 - A's, 7 - B's, and 3 - C's). Analysis of the participants' responses revealed many of the same cognitive obstacles identified by prior research into the content that underlies the topic of related rates, especially that of differentiation and functions. In addition, the contextualization of the related rates problems into word problems is an additional area of contention. This conclusion was consistent with the results identified by Mestre (1988) and Kieran (1992) that word problems require students to translate verbal descriptions into mathematical models and the process of translation can act as a barrier to students' comprehension. Besides the translation and modeling aspects of the topic, another potential obstruction occurs from the required utilization of functional relationships. This study has found that in solving related rate problems the visualization of complex dynamical situations is an additional obstacle to problem solution. In obtaining a solution, students must manipulate the static model through differentiation and the chain rule to characterize the movement and express the related rates.

References

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390- 419), New York: MacMillan.

Mestre, J. (1988). The role of language comprehension in mathematics and problem solving. In R.R. Cocking & J.P. Mestre (Eds.), *Linguistic and cultural influences on learning mathematics* (pp. 201-220). Hillsdale: Erlbaum.

NEW TECHNOLOGIES IN THE TEACHING OF THE DERIVATIVE TO HIGH SCHOOL STUDENTS

Jorge Peralta Sámano
Universidad Autónoma del Estado de Morelos
jorgep@hat.ubm.uaem.mx

In the present project we used the Didactic Engineering as investigation methodology in base to the papers of Artigue (1995), Duady (1995), and Brusseau (1986) on didactic of mathematics, and those of Zimmermann and Cunningham on Mathematics visualization (1991). It is shown a pilot study accomplished with high school students of the 'Escuela de Técnicos Laboratoristas' of the Universidad Autónoma del Estado de Morelos, where is made an emphasis on the use of new technologies in the teaching of the mathematics, enforcing and/or learning of the derivative concept, from a visual point of view.

Exist some attempts to solve the problem to introduce students to differential calculus and in specific to the derivative concept, this is observed in the textbooks that they have been published recently, which defer concerning their didactic proposals, since they use the limit concept as base, until those which not use it. But furthermore, it has the problem of carrying these proposals in an effective and successful way to the classroom.

The use of new technologies such as the graphic calculators, computers, and the development of software applied to the mathematics, it has given cause for several studies designed in order to carry these technologies to the classroom with the intention of giving a different approximation to calculus teaching (Tall, 1989).

Of it previously exposed was outlined the following investigation hypothesis: The use of new technologies in the classroom permits pupil to visualize and to conceptualize the derivative.

References

Artigue M. et. al. (1995). *Ingeniería didáctica en educación matemática.* Grupo Editorial Iberoamerica, Colombia.

Zimmerman W. & Cunngham S. (1991). *Editors' introduction: What is mathematical visualization?.* In *Visualization in teaching and learning mathematics* (Simmermann & Cunningham, editors). Mathematical Association of America, pp.1-8, U.S.A.

Tall D. (1998). Concept images, generic organizers, computers and curriculum change. *For the Learning of Mathematics*, 9(3), 37-32.

ADVANCED MATHEMATICAL THINKING
POSTERS

THE IMPACT OF COOPERATIVE LEARNING ON STUDENT ATTITUDES AND SUCCESS IN COLLEGE ALGEBRA

Jane Ann Brandsma
North Carolina State University
jabrands@unity.ncsu.edu

In response to recent reform initiatives in mathematics education, the author implemented a cooperative learning approach to teaching College Algebra with her community college students. The instructor assigned students to heterogeneous groups which worked together each day and remained intact throughout the quarter. Concept maps, card sorts, pair/share and jigsaw-type lessons were among the activities and cooperative learning strategies used. Assessment techniques included classwork and homework, journal writing, self- and peer-assessment of participation, individual and group tests, and a comprehensive final exam.

At both the beginning and end of the quarter, students completed an attitude survey (DeMarois, McGowen, & Whitkanack, 1996). At the end of the quarter, students were significantly more likely to *disagree* that mathematics is mostly facts and procedures to be memorized, that answers are either right or wrong, that test problems should be just like homework problems, and that good mathematics teachers show you the exact way to answer test questions. They were significantly more likely to *agree* with the statement, "When questions are left unanswered in class, I get help from other students". Compared with a traditional section of the course taught by the author at the same time the previous year, the cooperative learning class had a decrease in attrition rate and an increase in the percentage of students completing the course with a C or better.

Reference

DeMarois, P., McGowen, M. & Whitkanack, D. (1996). *Instructor resource manual for mathematical investigations: An introduction to algebraic thinking.* (Preliminary ed.). New York: HarperCollins.

EFFECTS OF INSTRUCTION ON STUDENTS' CONSTRUCTION OF PROOFS: PROSPECTIVE ELEMENTARY AND SECONDARY TEACHERS AND THE CASE OF THE ANGLE SUM IN A TRIANGLE THEOREM

José N. Contreras
The University of Southern Mississippi
jcontrer@ocean.st.usm.edu

Establishing the validity of mathematical assertions is a central activity of mathematics. In order for a proposition to be accepted as a mathematical truth, it has to pass the test of a proof. But proof is also important because of its pedagogical nature: some proofs help to explain why a particular theorem is true. This is the case of the proof of the angle sum in a triangle theorem (ASTT) that can be stated as follows: the sum of the measures of the interior angles of a triangle is 180°. Even though the notion of proof is fundamental in any area of mathematics, it tends to receive more emphasis in geometry than in algebra, trigonometry, or calculus. However, little is known about effects of instruction on students' ability to construct proofs. The objective of this paper is twofold. First I, will examine elementary and secondary majors' ways of proving the ASTT prior to instruction on the proof of ASTT. Second, I will examine whether teaching the proof of the ASTT to prospective elementary and secondary teachers help them to prove the theorem on a post-test. Both groups of teachers were enrolled or will be enrolled in their corresponding college geometry courses. As a pre-test, 13 secondary majors were asked to prove the ASTT. Then they received instruction on the proof of the ASTT. The teaching of the proof of the ASTT occurred in a natural context at the time it is proved in the textbook. As a post-test, the students were asked to prove the ASTT in a formal test. The analysis of the pretest indicated that only one student had notions about the proof of the ASTT. However, in the post-test, 3 students provided a correct proof, 5 provided a partially correct proof, and 5 constructed an incorrect proof. These findings suggest that instruction had some effects on students' knowledge about the proof of ASTT. In addition to correctness, students' ways of proving were analyzed in terms of types of justifications provided. Examples will be displayed during the poster presentation session. Similar procedures will be followed with the elementary majors in late August and early September. Then, analysis and discussion of results about elementary majors will be available in October of 1998.

A CONCEPTUAL GROUNDING FOR MATH ANXIETY

Kelly M. Costner
The Ohio State University
costner.1@osu.edu

Math anxiety is one of a number of affective constructs that lack conceptual grounding. This project presents a framework for conceptualizing math anxiety by means of a pilot project exploring the construct in individuals studying higher levels of mathematics (e.g., mathematics majors and pre- and inservice mathematics teachers).

Data sources for the pilot study included two interviews, a classroom observation, a document analysis, and a grounded survey. Analysis using NUD*IST software suggested three common themes that can be seen as factors of math anxiety in individuals in higher mathematics: mathematics test anxiety (the most prominent component), comparison/competition with others, and mathematics teaching anxiety. These results suggest an expanded definition of math anxiety that includes anxiety in math-capable individuals, and has implications for the exploration of anxiety management.

These data support a conceptual framework for math anxiety based on Fennema's (1989) generic model for research on affect in mathematics learning. Fennema's model accounts for both affective and cognitive outcomes from mathematics learning activities and processes. The model's application to the study of math anxiety fills a long-standing void in conceptualizing this research by adding the cognitive component. The modified version of Fennema's model accounts for the three factors suggested by the data in the pilot study, and shows how the factors work in concert to produce math anxiety.

References

Fennema, E. (1989). The study of affect and mathematics: A proposed generic model for research. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving*, (pp. 205-219). New York: Springer -Verlag.

TEACHER CHANGE IN A REFORM CALCULUS CURRICULUM: LIMITS AND CONTINUITY

Thomas B. Fox
Ball State University
tfox@bsu.edu

A case study of teacher change in practice and beliefs during the first-year implementation of a reform calculus curriculum was constructed. One teacher's practices during her implementation of the reform curriculum were compared to her previous instruction in a more traditional curriculum. Her practice in the key calculus content areas of limits and continuity were studied. Classroom observations, interviews, and written document data were collected to construct the case study. The primary data analysis technique used was grounded theory.

Comparing the teacher's treatment of limits in both curricula, significant differences were noted. In the traditional curriculum, the teacher focused on teaching procedures for finding the limits of a wide variety of functions. In her implementation of the reform calculus curriculum, her instruction was much more focused. She taught students enough about limits so that they could find the derivative of a function using the limit-based definition of the derivative. Geometric interpretations of the limit were also studied by examining the limit of functions both at a specific point in its domain and as x went to infinity. Technology was used extensively by her in the reform curriculum.

The teacher's instruction related to continuity revealed similarities and differences in both curricula. She always emphasized an informal definition of continuity. In the reform text, she omitted the formal definition of continuity at a point because of time constraints, and continuity was studied in the context of differentiability. In the previous year, it was not examined in any meaningful context.

VISUALIZATION OF DIFFERENTIAL EQUATIONS USING GRAPHIC BEHAVIORS

Miguel Solis Esquinca
msolis@mail.cinvestav.mx
Cinvestav IPN, Conacyt, Mexico

Francisco Cordero Osorio
fcordero@mail.cinvestav.mx
Cinvestav IPN, Conacyt, Mexico

We reported an experience carried out with university students and offered preliminary results of the project presented in PME-NA XIX. The theoretical perspective of mental constructions which are invariant in the construction of the mathematical knowledge was used and included actions, processes, objects and schemas. We found an argument in the graphs of functions that we have called behavior of tendencies. This has epistemological status and could be considered as a category of the knowledge of Calculus. It deals with the construction of a functional frame in the sense of establishing relationships between processes and objects through meaning (Cordero, In press). We designed and applied mathematical situations in order to interview 10 university students. The mathematical content dealt with first order linear differential equations with constant coefficients. There were notions involved in the situation such as variation, graphic behaviors, recognition of patterns and relationships between functions. As for the involved concepts we have found graph of a function, transformation of functions, slope of a curve, derivative, limit of a function, asymptote of a function and integral. Students worked with the equation:

$y'(x) + y(x) = F(x)$, when $F(x) = 0$, $F(x) = k$, and $F(x) = x$.

We identified strategies of local type, global type, and a synthesis of both types. The students that could synthesize the two kinds of strategies were able to recognize patterns as well. These students recognized the term F(x) of the equation as a fundamental part of the solution; they recognized the graph of the solution observing the graphic behaviors of the solution and related it with the graphic behavior of the term F(x). In the functional frame, we observed students generating ideas in order to model and simulate situation that involve differential equations.

References

Cordero, F. (In press). *El entendimiento de algunas categorias del conocimiento del c'lculo y an·lisis: el caso del comportamiento tendencial de las funciones*. Revista Latinoamericana de InvestigaciÛn en Matematica Educativa.

Algebraic Thinking Research Reports

177

WHAT KIND OF NOTATION DO CHILDREN USE TO EXPRESS ALGEBRAIC THINKING?[1]

Carol W. Bellisio
Rutgers University
Jbellisio@aol.com

Carolyn A. Maher
Rutgers Universit
carolynmaher@worldnet.att.net

A seventh grade class of students has been given numerous mathematical challenges throughout their school careers and have been videotaped and studied as they worked on problems, discussed their ideas and searched for the words and notations to justify their solutions. This paper examines their thinking and notation as they work to identify a rule that demonstrates the relationship between a set of ordered pairs of numbers.

The purpose of this paper is to examine early algebraic thinking in seventh grade students as they work in small groups to define a rule that demonstrates the relationship between a set of ordered pairs of numbers.

Theoretical Framework

For many students, algebra is a stumbling block to the study of higher mathematics. The difficulty emerges in the transition from the study of concrete arithmetic to the study of more abstract algebra. Davis (1964) has encouraged activities with younger children that might promote algebraic thinking early on as they are engaged in arithmetic investigations. He and others have maintained that students need time to build up an understanding of these concepts before their formal study of algebra in eighth or ninth grade (Davis, 1985; Speiser & Walter, 1997). This research is based on the view that elementary age students are capable of exploring algebraic ideas. The ongoing research suggests that there are important benefits when integrating activities that could introduce children to algebraic thinking before beginning the formal study of algebra.

Consistent with this position is the premise that the student actively engage in inquiries that promote transitions from arithmetic to algebraic thinking. Promoting investigative settings requires classroom organizations in which the students are encouraged to share their ideas, to challenge each other about them, and to explain and justify their solutions (Davis & Maher, 1990; Maher, 1998). The conditions necessary to make this happen must be set up by the classroom teacher.

---

[1] This research is supported in part by a grant from the National Science Foundation #MDR-9053597 to Rutgers, the State University of New Jersey. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Davis (1984) suggested that in thinking about a mathematical problem, a person cycles through a number of steps. A representation is first built for the new data. The person then tries to connect this representation with previous, relevant knowledge. A mapping is made between the data representation and the knowledge representation. As the student is trying to make the mapping, he checks the validity and adds more knowledge. This new mapping is then used to solve or try to solve the problem. This sequence, or part of it, is cycled through many times before the learner solves the problem (Davis, Maher & Martino, 1992).

Methods and Procedures

The students described in this paper were in a seventh grade class in a school in which researchers worked with teachers to integrate algebraic activities into the regular curriculum. This report will focus on the students' thinking on the third day of a three- day inquiry in which students were challenged in "Guess My Rule" activities. The students had, in previous years, worked on similar problems and they were familiar with the use of "box-triangle" notation to represent variables (see Davis, 1964). The students were given tables of values for box and triangle and were asked to write the rule that demonstrated the relationship between the variables. The students were then asked to write the inverse rule, the statement that would show how to get the box value, if the triangle value were known. The regular classroom teacher led the session. She began by reviewing the problem that had been given to the students at the end of the previous class. There were 13 students who were organized into three groups. Group 1 consisted of Bobby, Michele I, Amy-Lynn and Magda; Group 2, of Stephanie, Romina, Brian, Ankur, Michelle, and Jeff; and Group 3, of Michael, Sarah, and Angela.

Data Source

The classroom session was videotaped using a single video camera. Videotapes, student papers and researcher notes provided the data for this study. The videotape data was transcribed and verified for accuracy by graduate student researchers. The videotapes, transcriptions, student papers and notes were then discussed and evaluated by several researchers. Videotape data allows the careful study of the students' language as they work on the given problem. This is important because, in many instances, the students are able to explain their rule but have difficulty in writing it. Further, researchers are able to see the order in which students write their notation.

Results

The following table of values was given to the students at the beginning of class:

□	Δ
1	1
9	-8
1	0
2	-1
2	B

Each group produced a rule to describe the relationship:

Group 1. Bobby immediately saw a relationship between the numbers. Others in his group followed his idea and were able to write it using words, but were unable to write a rule using the symbols box and triangle. He explained his idea using the two pairs of fractions: (1/2, 1/2) and (1/3, 2/3).

Bobby: Two minus one is one; keep the two. Three minus two is one; keep the three.

Mich: Then what's the rule?

Bobby: The denominator minus the numerator.

Michele wrote: Rule is Demonidter [sic] - the numerator = Numeraterater [sic] of triangle and denominator stays the same.

The teacher suggested that Michele use a box value of two-sevenths to explain her rule.

Mich: Seven minus two equals five and then you just put the seven so it's just like this.

Michele wrote: 2/7 7 - 2 = 5/7

About ten minutes later Bobby overheard other students talking, and he wrote in notation form what he had heard.

Bobby: They have a good one, listen. Box minus one and switch from positive to negative.

Bobby wrote: □ - 1 (+/-)= Δ

The other three members in his group did not accept this rule. A teacher comment indicated to the students that they had the right idea and the students continued to pursue their original thinking.

Mich: Well, that's not like what we're doing and she [teacher] said we're on the right track.

Group 2. Students in Group 2 saw a different relationship between the numbers but had difficulty at first with the notation to describe their idea.

Jeff: If it's positive you subtract one and that number turns into a negative. Subtract one and add a negative.

Romina: So a negative five would be a positive six. A positive five would be a negative six.

No one questioned the correctness of Romina's second example.

Jeff: Then switch from what it is to the other thing.

Steph: Positive to negative, negative to positive.

Romina wrote: □ - 1 and switch from positive to negative or negative to positive = Δ .

When a graduate student asked the group to try and write an equation, Ankur wrote:

$(\square - 1) \times - 1 = \Delta$

The students checked out this equation with several values from the chart.

Class Sharing. After the students had worked in small groups for about twenty-five minutes, they shared their solutions with each other. They checked each other's rules to see whether the rules worked. Students in Group 3 demonstrated their rule first. Angela wrote:

$(\square - 1) - 1 = \Delta$

Although Angela wrote -1 Sarah indicated verbally that the box minus one should be multiplied by the negative one. The classroom teacher then intervened by placing parentheses around the negative one. Group 1 gave several examples to show their rule but explained that they could not write it. Michele attempted to write it using three variables but was not successful. Group 2 stated that their solution was the same as that of Group 3.

The teacher then presented her rule and the students checked that it worked. The rule was:

$1 - \square = \Delta$

The students were challenged to justify and explain why these rules, which appeared different, could both work. Initially, the students were unaware that the rules were equivalent. However, as they worked to explain and support their ideas, they developed a justification for the equivalence of the two rules.

Rschr: How do you get from your rule to her [teacher's] rule?

Sarah: If you switched the box and the one, like in the first piece but kept the minus sign there?

Tchr: Let's go back to a property, all right? How about this? [She wrote "distribute."]

Amy: Negative one times box is negative one box, negative box, and then negative one times negative one is one, so it's plus one equals the triangle.

The classroom teacher then asked them what property allowed them to show that minus box plus one was the same as one minus box, and a student named the commutative property. The students agreed that their rule was the same as the teacher's rule.

Conclusions

During this one class session there was a movement from students having an idea that they first verbalized, then expressed in words, and finally wrote using the symbols box and triangle. The power of the box-triangle notation, which was extensively used by the Madison Project in the 1960's

(Davis, 1964), was demonstrated later in the class when one of the students filled in the boxes and triangles in his equation with numbers to demonstrate the veracity of his rule. During the class, the students were interested in the solutions of the others and wanted to check the accuracy of their solutions. Teacher intervention challenged them to go further with their thinking and that led them to look for the equivalence of the rules they wrote. We are finding that children are able to deal successfully with algebraic ideas before the formal study of algebra. These data contribute to the growing research that suggests that children's generalization is originally expressed in ordinary language, and that with carefully designed activities and guidance, students can build algebraic ideas. Providing tools to help students in their movement to algebraic reasoning is essential.

References

Davis, R. B. (1964). *Discovery in mathematics: A text for teachers*. Menlo Park: Addison-Wesley.

Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education*. London: Routledge.

Davis, R. B. (1985). ICME-5 Report: Algebraic Thinking in the Early Grades. *The Journal of Mathematical Behavior, 4*, 195-208.

Davis, R. B. & Maher, C. A. (1990). The nature of mathematics: What do we do when we "do mathematics?" In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), Constructivist views on the teaching and learning of mathematics. *Journal for Research in Mathematics Education Monograph* (pp. 65-78). Reston, VA: National Council of Teachers of Mathematics.

Davis R. B., Maher, C. A. & Martino, A. M. (1992). Using videotapes to study the construction of mathematical knowledge by individual children working in groups. *Journal of Science Education and Technology,* 177-189.

Maher, C. A. (1998). Constructivism and constructivist teaching-can they co-exist? In O. Bjorkqvist (Ed.), *Mathematics teaching from a constructivist point of view. Reports from the faculty of education, no.3.* Finland: Abo Akademi University.

Speiser, R. & Walter, C. (1997). Performing algebra: Emergent discourse in a fifth-grade classroom. *The Journal of Mathematical Behavior, 16* (1), 39-49.

EFFECTS OF ALGEBRA INSTRUCTION ON THE RECOGNITION OF THE MATHEMATICAL STRUCTURES OF WORD PROBLEMS

Emily D.F. McFadden
e.mcfadden@pop.comap.com

This study investigated the relationship between the formal study of algebra and the recognition of the mathematical structures of word problems. Forty-one pre-algebra students and 75 algebra students participated in a Problem Solving Task and a Card Sorting Task. The results indicated that pre-algebra subjects recognized the mathematical structures of problems significantly more often than did algebra subjects. Algebra subjects appeared to have "pseudostructural conceptions" as described by Sfard and Linchevski (1994). There also appeared to be two stages of structure recognition. Subjects at the procedural stage described the structures of problems using their solution paths. Subjects at the structural stage described the structures of problems with equations or generalized statements.

Understanding how individuals organize problem information and the procedures they apply to that information is of interest to both cognitive psychologists and mathematics educators. Mathematics researchers and cognitive psychologists describe a two-stage model of problem solving that consists of a *representation stage* and a *solution stage*. These two stages have also been called the "problem comprehension process" and the "equation solving process" (Chaiklin, 1989; Hayes, 1981; Kintsch & Greeno, 1985; Mayer, 1982; Reed, 1987; Schoenfeld, 1985).

In the representation stage, the problem solver reads the problem, forms an initial representation that includes the information to be discovered in the problem and the information relevant to reach that goal, organizes the relationships in this information into mental representations, and represents the mental relationships as expressions, equations or inequalities. These mental representations that are formed are referred to as schemata (Hinsley, Hayes, & Simon, 1977; Marshall, 1995; Mayer, 1980). The schemata describe the parts of the problem that the solver considers significant. These schematic relations constitute a problem solver's understanding of a problem.

In the solution stage of problem solving, the problem solver applies operations on the schemata developed in the first stage. This involves substituting given values into formulas, or transforming the information, expressions, equations or inequalities in the first stage, in order to produce a solution. The problem solver continues to apply operations and procedures until a goal is achieved that is acceptable to the problem solver.

The main idea behind this model is that problem solvers interpret the information in word problems in terms of schematic relationships that have associated procedures for operating on the relationships. Successful problem solving depends on being able to form appropriate schemata and being able to apply those schemata across a range of related problems (Krutetskii, 1976; Mayer, 1982; Schoenfeld & Herrmann, 1982; Silver 1977, 1981). These schemata or representations formed by successful problem solvers in the first stage of problem solving consist of the mathematical or formal structure of the problem.

Recognizing that problems have similar mathematical structures is extremely important in problem solving. When problem solvers are faced with a problem that they do not know how to solve, but they can detect the mathematical structure of the problem and recognize that the new problem shares the same mathematical structure as another problem previously solved successfully, they may be able to apply the solution path for the known problem to the new problem.

The purpose of this study was to determine if the formal study of algebra facilitates recognition of the underlying mathematical structures of word problems. This study compared pre-algebra and algebra students' recognition of the mathematical relatedness among word problems in order to answer the following questions:

1. Are there differences in the problem attributes, context and structure, used by pre-algebra and algebra subjects to classify problems as mathematically similar? If so, what are those differences?
2. Are there differences between criteria used by pre-algebra and algebra subjects to classify problems as mathematically similar? If so, what are those differences?
3. Is there a relationship between classifying word problems as similar by problem attributes and subjects' abilities to solve those problems successfully?
4. Do subjects who use algebraic strategies to solve problems, classify problems as mathematically similar by structure and solve the problems successfully?

Method

The sample for this study consisted of 41 sixth grade students enrolled in a pre-algebra course, and 41 seventh grade and 34 eighth grade students enrolled in an Algebra I course. A Problem Solving Task (PST) and a Card Sorting Task (CST) were used to gather data related to the four research questions. The PST contained nine mathematical word problems that were chosen so that each problem had the same mathematical structure as two other problems and the same context as two other problems. The set of problems formed a 3x3 matrix so that no two problems shared both attributes of mathematical structure and context (see Table 1). On the PST, subjects were instructed to solve each problem and to write in words the steps they followed to solve the problem.

Table 1. Problem Solving Task Items by Structure and Context

	$\square + \Delta = S$ $\square - \Delta = T$	$\Delta = ?$ $\square + \Delta = S$ $M\square + N\Delta = T$	$\square = ?$ $\square + \Delta = R$ $\Delta + \lozenge = S$ $\lozenge + \square = T$
$\square = ?$	Distance 1. Carl and Al left school by bike at the same time. At the end of 45 minutes, Carl had traveled 2 miles farther than Al. Together they traveled a total of 12 miles. How many miles did	Al travel? 8. Sam and Joe were racing around a track. At the end of 55 minutes, they ran a combined total of 14 miles. If Joe had run twice as far and Sam had run three times as far, the sum of their distances would be 36 miles. How many miles did Sam run?	6. Jason, Robert, and Peter were each rowing a boat across a river. After 45 minutes, Jason and Robert had rowed a combined total of 10 miles, Robert and Peter had rowed a combined total of 13 miles, and Jason and Peter had rowed a combined total of 11 miles. How many miles did Jason row?
Age	4. The sum of Annie's and Lucy's ages is 60. Annie is 10 years older than Lucy. How	old is Lucy? 2. The sum of Mike's and Nancy's ages is 30. If Mike were five times older than he is now and Nancy were three times older than she is now, the sum of their ages would be 100. How old is	Mike? 9. The sum of Adam's and Betty's ages is 55. The sum of Betty's and Carla's ages is 75. The sum of Adam's and Carla's ages is 70. How old is Adam?
Money	7. A tube of tooth-paste and a bottle of shampoo together cost \$8.00. The shampoo costs \$1.00 more than the toothpaste. How much does the toothpaste cost?	5. Tim is trying to remember the cost of a cassette tape at Music World. He remembers that he paid \$21 for one cassette and one CD. He also remembers that he paid \$72 for four cassettes and three CDs. How much does one cassette cost	at Music World? 3. A mother and her son and daughter went shopping. Together the mother and son spent a total of \$65. The son and daughter together spent \$28. The mother and daughter together spent \$53. How much did the mother spend?

The CST consisted of the same nine problems from the PST. At the completion of the PST, subjects were given the CST. Subjects were directed to sort the nine problems into groups by placing problems they perceived as "like each other mathematically" in the same group and to describe the ways in which the problems in each group were alike.

Results and Conclusions

Findings of this study revealed that 1) pre-algebra subjects grouped problems by structure significantly more often than did algebra subjects ($p<.05$); 2) algebra subjects grouped problems by context significantly more often than did pre-algebra subjects ($p<.05$); 3) all but one algebra subject and only two pre-algebra subjects used algebraic strategies to solve the problems; 4) there was no significant relationship between use of algebraic strategies and recognition of the mathematical structures of problems; and 5) there was a significant positive relationship between the use of algebraic strategies and the ability to solve problems successfully ($p<.0001$). Taken together, these findings suggest that use of non-algebraic strategies contributed to the recognition of the mathematical structures of problems, whereas use of algebraic strategies, while still facilitating successful solutions, appeared to impede subjects' abilities to recognize structure.

These findings support the findings of Tabachneck, Koedinger, and Nathan (1994), and Hall, Kibler, Wenger, and Truxaw (1989) that the use of algebraic strategies result in more errors of conceptualizing relationships in problems than in manipulating symbols. These researchers suggest that an algebraic strategy can be applied almost "mechanically" to the solution of many problems. Once the relations in the problem are represented symbolically, computations and manipulations can be performed without connection or understanding of the situation of the problem. Thus, use of an algebraic strategy can lead to a correct solution but does not require recognition of structure. With informal strategies, such as guess-and-check, problem solvers do not distance themselves from the relationships of the problem, and, for this reason, conceptual errors are often less likely to occur.

Since a strong relationship existed between the use of algebraic strategies and the ability to solve problems successfully, and yet there was no significant relationship between use of algebraic strategies and recognition of the mathematical structures of problems, algebra subjects in this study could be considered to have, what Sfard and Linchevski (1994) call, "pseudostructural conceptions" of the language of algebra. That is, if algebra students do not understand the structural aspects of algebra, then they learn to manipulate symbols that have no meaning for them. Thus, algebra students may successfully solve numerous algebra problems using algebraic symbols, and appear to understand the language of algebra, but may not have made the transition to the structural perspective of algebra.

Other findings of the study revealed differences in the criteria used by pre-algebra and algebra subjects to classify problems as mathematically similar. Reasons given by subjects to classify problems as similar on the CST were placed into one of four categories: 1) mathematical structure, 2) problem context, 3) solution strategy, and 4) solution path (the algorithm or steps used to solve the problem). Comparisons of the number of reasons belonging to each of the four categories of criteria revealed that: 1) algebra subjects used criteria belonging to the category of problem context significantly more often than did pre-algebra subjects ($p<.001$); and 2) pre-algebra subjects used criteria related to their solution strategies and solution paths significantly more often than did algebra subjects ($p<.001$).

Kieran (1990, 1992), Sfard and Linchevski (1994), and Sfard (1991) claim that the transition from arithmetic to algebra involves a procedural-structural evolution. Procedural refers to arithmetic operations carried out on numbers to yield numbers. Structural refers to forming algebraic equations or expressions, and then performing operations upon those equations or expressions, but not upon numbers. The fact that pre-algebra subjects used criteria related to their procedures in order to group problems more often than did algebra subjects supports their theory that prior to experiencing algebra, students focus on the procedures and operations necessary for finding correct solutions and are therefore at a procedural stage of problem solving.

In addition to the differences in criteria used by subjects to group problems as similar on the CST, there were also differences in subjects' descriptions of the mathematical structures of problems. The mathematical structures of problems can be described in one of three ways: with equations, with generalized statements, or with solution paths (Goldin, 1984). Subjects in the present study used all three ways of describing the structures of problems. These findings contribute additional information to the procedural-structural theories of Kieran (1990, 1992), Sfard and Linchevski (1994), and Sfard (1991), and suggest the existence of two stages of structure recognition. That is, subjects who described the structures of problems using their solution paths were at a procedural stage of structure recognition. Likewise, subjects who described the structures of problems with equations or generalized statements were at a structural stage of structure recognition. Since all algebra subjects, who grouped problems together by mathematical structure, described the structures of problems with equations and generalized statements, they were considered to be at the structural stage of structure recognition. Since pre-algebra subjects, who grouped problems together by mathematical structure, described the structures of problems using all three ways, some of these pre-algebra subjects were considered to be at the procedural stage of structure recognition and others at the structural stage of structure recognition.

The fact that pre-algebra subjects used generalized statements as well as solution paths to describe the mathematical structures of problems concurs with findings of Chartoff (1976). He found that a large number of pre-algebra subjects rated two problems as extremely similar based on the fact that the same algorithm was used to solve the problems. Likewise, he found that if pre-algebra subjects were presented with two problems in which one was a generalized form of the other, then the subjects also rated the two problems as extremely similar. The fact that some pre-algebra subjects could be found at the structural stage of structure recognition in both the present study and Chartoff's study contributes evidence that there may be students who progress to the structural stage of problem solving without experiencing instruction in formal algebra. Further research that requires subjects to participate in tasks like the CST is needed in order to get a more detailed profile of students' abilities to recognize and describe the mathematical structures of problems.

References

Chaiklin, S. (1989). Cognitive studies of algebra problem solving and learning. In S. Wagner and C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 93-114). Reston, VA: National Council of Teachers of Mathematics.

Chartoff, B. T. (1976). An exploratory investigation utilizing a multidimensional scaling procedure to discover classification criteria for algebra word problems used by students in grades 7-13. (Doctoral dissertation, Northwestern University, 1976). *Dissertation Abstracts International, 37*, 7006A.

Goldin G. (1984). Structure variables in problem solving. In G. Goldin and C. E. McClintock (Eds.), *Task variables in mathematical problem solving*, (pp. 103-169). Philadelphia: The Franklin Institute Press.

Hall, R., Kibler, D., Wenger, E., & Truxaw, C. (1989). Exploring the episodic structure of algebra story problem solving. *Cognition and Instruction, 6*, 223-283.

Hayes, J. R. (1981). *The complete problem solving*. Philadelphia: The Franklin Institute Press.

Hinsley, D., Hayes, R., & Simon, H. (1977). From words to equations: Meaning and representation in algebra word problems. In P. Carpenter & M. Just (Eds.), *Cognitive processes in comprehension* (pp. 89-106). Hillsdale, NJ: Lawrence Erlbaum.

Kieran, C. (1990). Cognitive processes involved in learning school algebra. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education*, (pp. 96-112). Cambridge, MA: Cambridge University.

Kieran, C. (1992). The learning and teaching of school algebra. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan.

Kintsch, W. & Greeno, J. G. (1985). Understanding and solving word arithmetic problems. *Psychological Review*, *92*, 109-129.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children*. (J. Teller, Trans., J. Kilpatrick & I. Wirszup, Eds.). Chicago: University of Chicago.

Marshall, S. (1995). *Schemas in problem solving*. New York: Cambridge, MA: Cambridge University.

Mayer, R. E. (1980). *Schemas for algebra story problems* (Report No. 80-3). Santa Barbara, CA: University of California, Department of Psychology, Series in learning and cognition.

Mayer, R. E. (1982). Memory for algebra problems. *Journal of Educational Psychology, 74*, 199-216.

Reed, S. K. (1987). A structure mapping model for word problems. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 13*, 124-139.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. New York: Academic Press.

Schoenfeld, A. H. & Herrmann, D. J. (1982). Problem perception and knowledge structure in expert and novice mathematical problem solvers. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 8*, 484-494.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.

Sfard, A. & Linchevski, L. (1994). The gains and pitfalls of reification - the case of algebra. *Educational Studies in Mathematics, 26*, 191-228.

Silver, E. A. (1977). Student perceptions of relatedness among mathematical word problems. (Doctoral dissertation, Teachers College, Columbia University, 1977). *Dissertation Abstracts International, 39*, 02A.

Silver, E. A. (1981). Recall of mathematical problem information: Solving related problems. *Journal for Research in Mathematics Education, 12*, 54-64.

Tabachneck, H. J. M., Koedinger, K. R., & Nathan, M. J. (1994). Toward a theoretical account of strategy use and sense-making in mathematics problem solving. *In Proceedings of the Sixteenth Annual Conference of the Cognitive Science Society*. Hillsdale, NJ: Lawrence Erlbaum.

MIDDLE SCHOOL STUDENTS' ALGEBRAIC REASONING: NEW SKILLS AND UNDERSTANDINGS FROM A REFORM CURRICULUM

John P. Smith III
Michigan State University
jsmith@pilot.msu.edu

Elizabeth A. Phillips
Michigan State University
ephillip@math.msu.edu

Beth Herbel-Eisenmann
Michigan State Univesity
herbelei@pilot.msu.edu

New, problem-centered curricula provide quite different introductions to algebra than traditional courses based on symbolic manipulation. However, we know very little about what students learn from these new curricula. This paper reports a pilot study of the algebraic skills and understandings that one class of 8th grade students developed from work with one problem-centered, middle school curriculum, The Connected Mathematics Project. Results indicate that most students (1) had a solid grasp of linear relationships, (2) could distinguish linear from non-linear relationships, (3) used graphing calculators extensively to compute, analyze, and represent relationships, and (4) could relate symbolic expressions back to problem contexts.

In the United States no mathematical topic or course is currently more fluid and controversial than algebra. Whether the focus is curriculum, teaching, assessment, student learning, or policy, common ground is scarce and questions far outnumber answers. Algebra is widely seen as essential for both college-bound and work-bound students; some have called it the new civil right (Moses, 1993). But access to algebra is problematic (Silver, 1997). Traditional Algebra I is widely considered conceptually inconsistent, unnecessarily restrictive of students' mathematical growth and development, and narrowly focused on bare symbol manipulation. Reformers have instead emphasized access to algebraic ideas (Silver, 1997), K-12 development (NCTM, 1997), and broader conceptualizations of content (Kaput, 1995; NCTM, 1997). But these reforms have been recently challenged by critics, often for the absence of skill development. What is most needed (yet generally lacking) in this fluid and contentious context are assessments of the algebra knowledge and skills that students take away from different reform curricula.

The study described here was an initial step to assess the understandings of algebra that middle school students learn in one problem-centered, Standards-based middle school curriculum, The Connected Mathematics Project (CMP) (Lappan, Fey, Fitzgerald, Friel, Phillips, 1995). These materials include a substantial strand of algebraic content in the 7th and 8th grades. Students learn about different families of functional relationships (primarily, linear, exponential, and quadratic) in solving contextualized problems using a variety of tools (esp. graphing calculators) and representations. We developed four interview problems that (1) were grounded in quantitative or social interactional contexts, (2) involved both linear and non-linear families of functions, and (3) tapped key mathematical concepts, e.g., equivalence. We used these problems in interviews with pairs of students from one 8th grade CMP classroom. The middle school was a pilot site for the development of CMP, the students had three years of experience with the curriculum, and the classroom teacher was experienced and knowledgeable of the 8th grade content.

The Assessment Problems

We present our four interview problems in slightly edited form. Each presented an algebraic expression(s) or equation(s) for students to interpret and reason from. Two involved linear relationships and expressions. The Equivalent Expressions problem asked students to judge the equivalence of three linear expressions, two of which represented common manipulation errors.

> Your group is trying to find expressions equivalent to 2(5 + 3x). Don thinks 2(8x) is equivalent. Cathy thinks 10 + 3x is equivalent. They look to you for help.
>
> Are their expressions equivalent to 2(5 + 3x)? How will you decide?
>
> How would you explain your reasoning to them in a convincing way?
>
> Can you think of a context (problem) that you could model with 2(5 + 3x) or an expression equivalent to 2(5 + 3x)?
>
> What does the variable x represent? What does the expression represent?

The Trip Costs problem presented a linear equation (representing cost as a function of number of participants) and posed a series of interpretation and computation tasks.

> The total cost of taking some students on a trip is given by C = 150 + 10N, where C = total cost in dollars and N = number of students.
>
> If the total cost was $520, many students went on the trip?
>
> Describe at least two ways to figure that out.
>
> What do the numbers in the equation (150 and 10) mean in this context?
>
> Suppose another company charged the students according to equation, C = 25N. Which plan should you use if you want to keep costs down?
>
> Could the cost equation, C = -150 + 10N, make sense in this situation?

The other two problems involved non-linear relationships. The Populations problem asked students to compare the growth of animal populations modeled by linear, exponential, and quadratic equations.

> The growth patterns of three species of animals are given by P1 = 10,000 + 5x (Species 1), P2 = 10(2x) (Species 2), and P3 = 700 + 10x2 (Spe-

cies 3). P represents the number of animals of each species after x years.

Describe the growth pattern of each species. How do they differ? Pick two species.

Could the populations of these species be equal after some number of years?

Explain how you would decide.

The School Pool problem presented a diagram of a pool with an expression for its area and asked students to match terms in the expression to sections of the pool.

Your school is building a pool, part indoors and part outdoors. The plan for the indoor part of the pool is shown below. The end is shaped as a half-circle, and the rest of the indoor part is a rectangle. The exact dimensions have not been set but the area of the whole pool is given by the expression: $\pi x^2/2 + 6x^2 + x^2 = \pi x^2/4$.

Which part of the expression represents the area of the indoor part of the pool?

Which part of the expression represents the area of the outdoor part?

Make a sketch of the outdoor part, including important dimensions.

Is there one or more than one possible shape for the outdoor part? How would you decide?

The Students

We interviewed 16 of the 24 students in one 8th grade CMP class on three consecutive days in the middle of May. This group represented all students (but one) who consented to participate in the interviews. This middle school (School 1) did not track by ability, so the class included a range of student interest and ability in mathematics. After the interviews were complete, we asked the teacher to rate all the students in this class on three dimensions: (1) pencil & paper computational ability, (2) course grade, and (3) 9th grade mathematics placement. As Table 1 indicates, the sample included a range of success in mathematics, with greater participation from the more successful students.

We also administered the problems to 8th graders in two other middle schools in an in-class written format (with no interview): Four classes at School 2 and two classes at School 3. All three schools had full implementation of the CMP curriculum for 3 or more years, and most students had three full years of experience with those materials.

The Interview

We paired the students with a partner they were friendly with and interviewed the pair as a team. One member of one pair was absent on both interview days, so her partner worked the problems alone. Of the remaining 7 pairs, 4 matched two successful students (grades, 9th grade recom-

Table 1: The Teachers' Ratings of the Students

Dimension/Rating	Interview	Whole Class
Computational Ability		
Good	5	6
Adequate	8	11
Poor	3	7
Course Grade		
A/A-	4	5
B+/B/B-	6	7
C+/C	3	7
D+/D/E	2	5
9th Grade Placement		
Geometry	9	10
Algebra	6	14

mendation); 2 pairs matched a successful and a struggling student (criteria); and in 1 pair both students were struggling. All interviews were audiotaped for analysis. Students had access to two graphing calculators, blank paper, graph paper, and rulers.

Results

We were surprised by the students' competence with these problems. Six of seven interviewed pairs solved all parts of all four problems; the 7th pair solved Trip Costs and Populations, and most of Equivalent Expressions. Only the student who worked alone struggled. Trip Costs was routine for most pairs; Populations and School Pool were more challenging. The first part of Equivalent Expressions was relatively routine, but designing a context for the expression was more challenging. The pencil and paper results from Schools 2 and 3 were consistent with these general patterns. Success rates ranged from 50 to 67%, with the same relative difficulty.

Both the interview and pencil & paper results indicated that most students had developed solid understandings of linear relationships. They recognized and described slope/constant rate of change and y-intercept in tabular, graphical, and symbolic representations and moved easily between them. For example, with the equation $C = 150 + 10N$ (Trip Costs), students interpreted 150 as a fixed or flat cost and 10 as the cost per student and related these descriptions to the contents of the table and the shape of the graph. They knew (and used) multiple methods to determine if two different linear expressions were equivalent (Equivalent Expressions), including comparing their tables or graphs, evaluating the expressions at particular values, and appealing to symbolic rules, e.g., the Distributive Property. They discriminated linear from curvilinear relationships (Populations) based on their symbolic form or rate of increase in tables. They were also able to generate sensible situations when given a linear expression (Equivalent Expressions), though the factored form, $2(5 + 3x)$, was more difficult than the familiar $mx + b$ form (e.g., $10 + 6x$).

The majority of interviewed students showed a marked preference for analyzing symbolic expressions using the table function of their graphing calculators. They entered the expression and examined the pattern of increase in the Y values. Some mentioned and accessed the graph window but far less often. They used the table function to judge equivalence (Equivalent Expressions, were the Y tables identical or not?); describe rates of change (Trip Costs and Populations, what did the Ys go up by?); compare two linear relationships (Trip Costs, 150 + 10N vs. 25N); and solve systems of equations (Populations, when does P3 catch up to P1?). Significantly, most students asserted on Equivalent Expressions that the method of comparing tables would be the most effective way to convince their peers that $2(8x)$ and $10 + 3x$ were not equivalent to $2(5 + 3x)$.

The students' ways of talking about slope and y-intercept provided insight into their understanding of linear relationships. Spontaneous use of the term, slope, was rare. Instead, students talked about constant rates of change as what it [the function] goes up by. They knew that the difference between successive Y values was the coefficient of X in the general form of the linear equation, $y = mx + b$. Some students could read and compare "What it goes up by" directly from linear expressions or equations; others were more comfortable entering the expressions into their graphing calculators and examining the change in the tables. Graphical conceptions of slope, e.g., rise over run, did not generally emerge from work on and discussions of these problems. Spontaneous references to the y-intercept were more common, both in the table (the Y-value that corresponds to $X = 0$) and the equation (the b term). Even more frequently students referred to the y-intercept as the starting point, a term which had situation, graphical and tabular meaning. starting point appears a virtual synonym for y-intercept; what it goes up by, however, may or may be equivalent to the ratio definition of slope.

Students' success on the School Pool problem, which involved only quadratic terms, indicated that they were not derailed by a complex expression in a non-standard problem. They recalled the formula for the area of a circle (a 7th grade topic in CMP) and determined that the terms, $6x^2$ and $\pi x^2/2$ matched the indoor sections. Then they searched for ways to fit the shapes corresponding to x^2 and $\pi x^2/4$ onto the diagram of the indoor part. This task was made more difficult by some students' expectation that the pool should be symmetric along its length. It was an unfortunate feature of this problem that students' sense of possible solutions was influenced by design aesthetics - a problem we have found difficult to fix.

Next Steps

Buoyed by these positive and intriguing results, we expanded the scope of our assessment research. We revised and expanded our collection of problems to 16, ten of which involve linear relationships. We borrowed and adapted problems from a variety of sources, including items from TIMSS (the 7th and 8th grade assessment) and NAPE. We also expanded the number of participating schools and classrooms to include multiple CMP sites and matched contrast schools (8th grade Algebra I and Pre-Algebra). We are currently analyzing individual written responses to the problems collected at these sites in May 1998 and expect to have preliminary results to report at the conference.

References

Kaput, J. (1995, October). *A research base supporting long term algebra reform.* Invited address to the Annual Meeting of PME/NA. Columbus, OH.

Lappan, G., Fey, J., Friel, S., Fitzgerald, W., & Phillips, E. (1995). *The connected mathematics project.* Palo Alto, CA: Dale Seymour Publications.

Moses, B. (1993). *Algebra: The new civil right.* Paper presented at the SUMMAC II Conference, Cambridge, Massachusetts.

National Council of Teachers of Mathematics Algebra Working Group. (1997). *Algebra in the k-12 curriculum: Dilemmas and possibilities.* Final report to the Board of Directors. East Lansing, MI: Michigan State University.

Silver, E. A. (1997). Algebra for all: Increasing students' access to algebraic ideas, not just algebra courses. *Mathematics Teaching in the Middle School*, 2, 204-207.

DEVELOPING AN IMAGISTIC BASIS FOR UNDERSTANDING BINOMIAL MULTIPLICATION

Erna Yackel
Purdue University Calumet
yackeleb@calumet.purdue.edu,

Diana Underwood
Purdue University Calumet
diana@calumet.purdue.edu

The analysis presented in this paper focuses on one individual's reconceptualization of binomial multiplication from purely procedural to a personally-meaningful activity grounded in real-world imagery. As such, it illustrates how an approach that takes students' informal activity as a starting point can be used to support their development of conventional algebraic concepts, methods and symbolism.

Purpose

One commonly expressed concern about the current reform effort in mathematics education is that the emphasis on having students develop personally-meaningful mathematical concepts might result in their not developing methods and procedures which are generally taken as important components of conventional mathematical knowledge. The work of Cobb and colleagues (Cobb et al., 1991) shows that this concern is unfounded in the case of primary school arithmetic. Nevertheless, our experience with middle school and high school teachers indicates that this concern persists, especially in the case of algebra. In this paper we present one aspect of an ongoing research project in which we are investigating how to support students' development of meaning for basic concepts in algebra, including algebraic expressions and operations. The aspect we report here is a set of three interviews with a single individual that formed the impetus for a subsequent classroom investigation. For a discussion of the classroom research aspect of the study see Underwood and Yackel (1998).

---

[1]Some may question whether or not binomial multiplication is still a relevant component of school algebra, given current technology and emphases on conceptual understanding rather than on skill development. The position we take is that being able to operate with algebraic expressions as mathematical objects is foundational to algebraic reasoning. Consequently, we do not view binomial multiplication as a skill to be mastered but as an essential aspect of being able to navigate in a mathematical environment that encompasses variables, unknowns, and algebraic expressions as well as numerical expression (Arcavi, 1994).

The purpose of this paper is to present research that investigated an approach for developing an understanding of binomial multiplication[1] and related concepts that is grounded in the imagery of acting in a realistic situation.[2] In this regard, the approach follows the instructional design theory of Realistic Mathematics Education (RME) (Gravemeijer, 1994) and is consistent with Streefland's (1995, 1996) work on integers and comparing quantities that forms the theoretical foundation for several algebra units in the Mathematics in Context curriculum (Encyclopedia Britannica, 1997). Our approach can be contrasted with a common approach to binomial multiplication in introductory algebra textbooks that is based on an area model (DeMarios, McGowen, & Whitknack, 1998). In the latter approach, students are presented with models that "show" how to represent and interpret a product of two binomials in terms of the areas of the regions on the visual model. In the alternative approach, rather than presenting a model for a student to interpret and explore (Doerr, 1995), the student is engaged in informal activity, which through a process of progressive mathematization, forms the basis for her more formal mathematical activity later on.

Theoretical Framework

The theoretical framework used to guide the study is a version of social constructivism called the emergent perspective (Cobb & Bauersfeld, 1995). According to this perspective, interactionism and psychological constructivism are coordinated to account for learning and teaching. Interactionism is the social perspective that is taken of communal activity while psychological constructivism is the perspective that is taken of an individual's activity as he or she participates in and contributes to the development of communal activity. The emergent perspective is particularly useful to analyze learning that is designed to capitalize on an individual's participation in a meaningful realistic scenario.

The emergent perspective is consistent with the RME view of mathematical reinvention and vertical mathematizing. According to RME, students first develop models of their informal activity that later become models for more formal mathematical activity. In the process, means of notating and symbolizing the informal activity are developed. These records later take on a life of their own and become abstract quantities that can be

[2] Here we use the term *realistic* as it is used by mathematics educators at the Freudenthal Institute in describing Realistic Mathematics Education. In this usage, *realistic* does not necessarily mean a real-world situation but one that students can imagine participating in. Some aspects of the situation may be fictitious and quite contrived. The key element is that thinking about how they would act in the situation suggests how students might proceed in the instructional (mathematics) tasks posed. This use of the term *realistic* will be clarified through the example presented in this paper. The term is not to be confused with the common American use of real-world or authentic to refer to situations that might be part of one's actual experience.

reasoned with and about. In this way the symbolism and notation, that appear to be abstract, are not devoid of meaning but are grounded in real-world imagery of the informal activity from which they emerged.

Methods of Inquiry and Data Sources

The method of inquiry for the research reported here was a one-on-one teaching experiment (Cobb & Steffe, 1983) that consisted of three interview sessions of approximately two hours each with a novice mathematics teaching assistant at a midwestern university regional campus. All three sessions were conducted by one of the researchers. The teaching assistant, Monique, was teaching an intermediate level algebra course for under-prepared university students. Our initial motivation for meeting with Monique was to discuss possible instructional methods she might use to help her students develop an understanding of binomial multiplication. However, the sessions took on the character of a one-on-one teaching experiment in that the focus increasingly was on her own understanding rather than on teaching approaches she might use. Data from the three sessions include written material produced during each session, detailed notes prepared after each session, and a video-recording of the third session.

The tasks that formed the basis for the first session were motivated by an attempt to understand how Monique thought about the traditional area model for binomial multiplication. In the second session the intention was to focus on questions posed in a scenario of building and remodeling a rectangular deck, a scenario that Monique offered at the end of the first session in response to the researcher's request for a "story problem" that she could pose to her class that might be related to binomial multiplication. However, the major focus of the second session quickly shifted to the concept of area as covering. In the third session tasks were posed in the deck scenario. In the scenario, as the researcher posed it, a builder uses large squares of lumber of some specific, but unknown, dimension, as the basis for building a rectangular deck. He also has access to strips of lumber that have the same (specific but unknown) length as the squares and have width that is specified by a known number. For example, he might have a square of dimensions X units by X units and a strip of lumber X units long and 6 units wide. In addition, he can purchase additional rectangular shapes as needed to complete his project if he knows their length and width. The purpose of the scenario was to create a situation in which Monique might engage in informal activity, namely planning to build or remodel a deck, which later might form the basis for more formal mathematical reasoning. That is, her mathematical activity would have an imagistic basis.

Results

In the following we briefly discuss the analysis of the three interviews. The analysis focuses on Monique's reconceptualization of binomial multiplication from purely procedural to a personally-meaningful activity grounded in real-world imagery. The analysis shows that there were several critical steps in this process. The first step was the development of a dynamic rather than a static view of expressions such as $X+3$. The second step was the development of a concept of area as covering. The last several steps involved the shift from thinking about cutting and mounting lumber to anticipating the result of such activity, and, eventually, to talking about the dimensions and the areas of regions in her diagrams as though they were abstract mathematical quantities.

Developing a Dynamic View of Expressions Such as X+3

The most significant insight we gained from the first interview was the importance of a dynamic view of simple linear expressions in supporting a basis in imagery for binomial multiplication. At the beginning of the first interview Monique did not have the flexibility to think of an expression such as $X+3$ as a quantity X increased by 3 in the context of a binomial area model.[3] For example, when we asked her to alter a drawing of a square X units on a side to indicate a region that was $X+3$ units by $X+6$ units, she drew lines *within* the square and re-labeled the sides as $X+3$ and $X+6$. This was counter to our expectation that she would increase each dimension and draw a new rectangle by enlarging the square. This static approach taken by Monique results in a treatment of the area model as a task that involves assigning areas to existing regions in the diagram. By contrast, a dynamic view supports the activity (actual or imagined) of enlarging or decreasing a square region. The task of recording the results of that activity leads naturally to equating expressions such as $(X+3)*(X+6)$, and $X^2+3X+6X+18$, since each of these is a different way of indicating the area of the enlarged region. The importance of this approach is that equality results from the situation itself, not from formal mathematical operations.

Through a series of questions and tasks posed by the interviewer, Monique developed the flexibility to think about expressions dynamically by the end of the first interview. It was at this time that she suggested the scenario of building and remodeling a deck as one to use with her students to support a dynamic conceptualization of binomial multiplication. Monique's posing of the deck scenario was fortuitous because, not only did it provide us with a way to frame our subsequent interview questions, it also has the features that typify RME tasks. As the interviews progressed we were able to develop the scenario and simultaneously investigate Monique's thinking about area, multiplication and algebraic expressions.

---

[4] We learned through the interviews that this lack of flexibility was not due to limited mathematical ability on Monique's part but rather to limited mathematical experiences. The interviews were, therefore, of critical importance in helping us figure out what types of mathematical experiences are useful to promote this type of flexibility.

Developing a Concept of Area as Covering

A second critical step for Monique occurred in the second interview when she developed a concept of area as covering. The significance of this development is that it involved coming to conceptualize algebraic expressions as composite units (Steffe, 1992). As a result she gained the flexibility to think of an algebraic expression in a variety of different ways. For example, she could think of $2X$, as comprised of $X + X$ units of one or as two units of X. $2X$ was now a quantity for her that she could manipulate and operate with flexibly. A detailed discussion of the importance of a concept of an algebraic expression as a composite unit can be found in Underwood and Yackel (1998). For purposes of this paper, we note that the imagery of pretending to cover an X by X square with individual square tiles one unit by one unit was the critical activity that led to Monique's conceptualization of an algebraic expression as a composite unit.

Shifting From Informal Activity to Formal Mathematical Reasoning

In the third interview the intent was to explore how the deck scenario might be extended to foster an imagistic basis for concepts related to binomial multiplication, such as completing the square. To that end, we posed tasks that involved enlarging a square deck of some specific, but unknown size (X units by X units) by partitioning a specified strip of material lengthwise into two narrower strips, placing the narrower strips on adjacent sides of the square deck and adding a rectangular "corner" piece to make the resulting deck rectangular. Part of the task was to figure out the size of the additional corner piece that would be needed to complete the deck. Initially, Monique proceeded by partitioning the additional strip of material into a number of smaller strips and then drawing them in, one by one, on either of two adjacent sides of the square. At this stage she was clearly thinking about cutting and placing strips of lumber and not about the resulting dimensions. For example, one solution she offered was to split an additional 6 by X strip into three 2 by X strips and place (draw) two of them on one side of the square and the third on an adjacent side of the square. She continued with several additional solutions, each time actually carrying out the activity of splitting the additional strip and placing the portions on two adjacent sides of the square.

A shift in her activity came when she first drew strips (of yet unknown width) on adjacent sides of the square and only then figured out what widths she might use to create different rectangular decks. Our interpretation of her activity is that it represented a shift from imagining the activity of splitting the strip and laying down the pieces she created to thinking about how she wanted to split the strip. She was now anticipating the result of the activity. She no longer needed to carry it out. By the end of the interview Monique had made another shift. She was talking about the dimensions and the areas of the regions in her diagrams as though they were abstract mathematical quantities. That is, she had made the shift to more formal mathematical activity.

The shifts Monique made in the third interview are evidence that the deck scenario can be extended to provide an imagistic basis for concepts related to binomial multiplication, such as, completing the square. Further, it was apparent that through engaging in these activities Monique's conceptual understanding of multiplication of algebraic expressions was greatly enhanced. Her level of skill had not increased. She was already skillful in dealing with algebraic expressions of this type. She now had a conceptual basis for her understanding, a basis that emerged from the imagery of the scenario(s) in which the tasks were posed.

Significance

The significance of this research extends beyond the analysis of the interviews themselves. The study illustrates how an approach that takes students' informal activity as its starting point can be used to support their development of conventional algebraic concepts, such as, binomial multiplication. In the process, we address a concern that some express about inquiry approaches to instruction, namely that an emphasis on developing personally-meaningful mathematical concepts may result in students not learning conventional mathematical methods and symbolism.

References

Arcavi, A. (1994). *Symbol sense: Informal sense-making in formal mathematics.* Manuscript, Weizmann Institute of Science, Israel.

Cobb, P., & Bauersfeld, H. (1995). The coordination of psychological and sociological perspectives in mathematics education. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 1-16). Hillsdale, NJ: Erlbaum.

Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education, 14,* 83-94.

Cobb, P., Wood, T., Yackel, E., Nicholls, J., Wheatley, G., Trigatti, B., & Perlwitz, M. (1991). Assessment of a problem-centered second-grade mathematics project. *Journal for Research in Mathematics Education,* 22, 3-29.

DeMarios, P., McGowen, M., & Whitknack, D. (1998). *Applying algebraic thinking to data.* Reading, MS: Addison-Wesley.

Doerr, H. M. (1995, April). *An integrated approach to mathematical modeling: A classroom study.* Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Encyclopaedia Britannica (1997) *Mathematics in context.* Chicago, IL: Encyclopaedia Britannica Educational Corporation.

Gravemeijer, K.(1994). Educational development and educational research

in mathematics education. *Journal for Research in Mathematics Education, 25*, 443-471.
Streefland, L. (1995, April). *Developing instructional activities in which algebra may arise naturallyœequations.* Paper presented at the annual meeting of the American Educational Research Association, San Francisco, CA.
Streefland, L. (1996). Negative numbers: Reflections of a learning researcher. *The Journal of Mathematical Behavior, 15*, 57-77.
Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*, 259-309.
Underwood, D., & Yackel, E. (1998, April). *Developing a concept of an algebraic expression as a composite unit.* Paper presented at the annual meeting of the American Educational Research Association, San Diego, CA.

Algebraic Thinking
Short Orals

ON THE "MEANING" OF MATHEMATICAL EXPRESSIONS

Bruce L. Sherin
Northwestern University
bsherin@nwu.edu

The purpose of this paper is to address the question: What does it mean to "interpret" a mathematical expression? Intuitively, we might say that to interpret an equation is to state its "meaning." If so, the construction of a theory of meaning for equations would be a significant first step toward understanding the nature of interpretation. Rather than beginning with an *a priori* analysis of equation meaning, I began by looking at students' interpretive utterances as they occur in the course of problem solving; that is, I began by looking at students saying, in their own words, what equations mean. Then I progressed to generalizations concerning equation interpretation and the meaning of equations, as they were evidenced in these utterances.

The conclusions are based on an analysis of a 27-hour video corpus involving undergraduates enrolled in a third-semester introductory physics course. Five pairs of subjects participated, and each pair of students worked together to solve a pre-specified set of physics problems.

The analysis uncovered significant variety in the interpretive utterances made by students, though there were also clear regularities. Three broad classes of interpretations were identified. In *Narrative* interpretations, the interpretation describes an imaginary process in which some type of change occurs. For example, the interpretation might describe what would happen to the value of one quantity if the value of another quantity was increased. In *Static* interpretations, an equation is taken as describing a static physical situation. Finally, in *Special Case* interpretations, conclusions are drawn for cases in which the values of the quantities that appear are somehow restricted. This includes interpretations in which a limiting or extreme case is considered.

This investigation of interpretive utterances permits some generalizations about the nature of equation "meaning." The most important of these generalizations is that equation meaning, as realized in interpretive utterances, is not a simple function of the symbols that appear in the equation. Instead, interpretation appears to involve the building of a larger meaningful framework (e.g., involving an imaginary process) and embedding the equation in that framework.

PRECALCULUS STUDENTS' INTUITIONS ABOUT THE CONCEPT OF SLOPE

Sheryl L. Stump
Ball State University
sstump@wp.bsu.edu

This study examined precalculus students' intuitions about slope. Two categories of mathematical intuitions, as described by Fischbein (1987), were considered in this investigation: primary intuitions based on normal everyday experience and secondary intuitions developed by educational intervention. The research framework considered slope not as an isolated concept, but as a conceptual field (Vergnaud, 1983). From a conceptual point of view, slope is related to ratio, rate, angle, measure, and linear function. Situations involving slope include graphs, equations, formulas, physical structures, and functional situations.

Students in two high school precalculus classes used bicycles to investigate the relationship between pedal revolutions and wheel revolutions for various gears. Later, 22 students from these classes participated in individual interviews. The interview protocol contained a variety of situations involving the concept of slope, including the bicycle situation.

Four students explained how to determine the relative steepness of ski ramps. Eleven students correctly matched three numbers (slopes) with three ski ramp models, and six students correctly described the meaning of the numbers. Two students correctly explained the meaning of "7% grade." Eight students correctly interpreted the meaning of the slope of the graph of pedal rotations versus wheel rotations. Three students determined the slope of a graph of number of tickets sold versus profit, and ten students correctly interpreted the meaning of the slope.

Table 1. Test Results of Homework Project that Paired Volunteers and Algebra 1B Students.

	8th Grade Competency Test	Baseline Completion Rate	Baseline Test Score	Homework Completion Rates	Test Scores	Median State Test	Mean State Test
Callers (n=10)	166.1	61.3%	53.9	66.8%	70.1	77.5	72.9
Non Callers(n=7)	169.3	64.9%	62.8	67.1%	65.8	75	75.6

Fifteen students used the word "angle" to describe slope. Eight students mentioned "rise over run" or a similar description of slope. Three students quoted a formula for slope.

References

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach.* Dordrecht, The Netherlands: D. Reidel.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127-174). New York: Academic Press.

Algebraic Thinking
Posters

207

EFFECTS OF HOMEWORK PARTNERS ON ALGEBRA ACHIEVEMENT

Ann R. Crawford
University of NC at Wilmington
Crawforda@uncwil.edu

Sandra Johnston
West Brunswick High School
Sandi1206@aol.com

Edna H. Mory
University of NC at Wilmington
Morye@uncwil.edu

Past research has found completion of homework to be an alterable variable that can enhance learning. Likewise, high achievement of Japanese youth in mathematics has been attributed to intense efforts in extramural tutoring and study per week. Kaplan (1997) discussed successful use of homework partners in an 8th grade algebra classroom in a private school to encourage homework completion. Students discussed homework over the phone with their partner. From teacher observation, Kaplan noted that the students demonstrated improved completion rates, higher quality of homework, and improved confidence in problem solving.

This project investigated the effects of homework partners to encourage homework completion with low achieving students in an algebra IB class in a rural high school. Student volunteers were paired with a homework partner to discuss difficulties with homework or past class work over the telephone. Data were collected to compare completion rates and effects on achievement for callers and non callers. Baseline data for homework completion and test scores were recorded before the intervention. For ten weeks, homework partners were implemented with the experimental group telephoning their partner two to three times during the week to discuss difficulties. In addition, the Math Attitude Scale was given to the students and participants were interviewed. Preliminary project results show that homework partners had a positive effect on homework completion rates and class test scores for the callers.

References

Kaplan, C (1997). Homework partners. *Mathematics Teaching in the Middle Grades*, 2, 168-169.

TWO COLLEGE STUDENTS' UNDERSTANDINGS OF THE VARIABLES IN LINEAR INEQUALITY: INSIGHTS AND IMPLICATIONS FOR INSTRUCTION

Carole P. Sokolowski
Merrimack College
csokolowski@merrimack.edu

Inordinately high undergraduate enrollment in secondary-level algebra courses is evidence of college students' problems with understanding and using algebraic variables. Inequalities form a particularly rich context in which to study the concept of the variable as it is understood by undergraduate students, who have had years of experience with school algebra. Variables in inequalities are manipulated as they are when solving equations; variables in inequalities usually represent sets of numbers, however, which make them similar to variables in functions.

This study investigated six college students' conceptions of variable in linear inequality. Prose, symbolic, and graphical versions of linear inequality problems were presented in unstructured problem solving interviews, providing subjects with multiple contexts in which to reveal their understanding. Küchemann's (1981) six uses of the variable were used to categorize subjects' use of variables in inequalities.

Two of the six case studies will be described in this poster session. Both subjects exhibited difficulty with algebra, but in very different ways. Subject One demonstrated a conception of the variable as an evaluated unknown, a reluctance to work with symbolic variables, and a strong sense of inequality relationships. By contrast, Subject Two revealed that he viewed the variable primarily as an object to be manipulated, was adept at using the rules of algebra, but appeared unable to make sense of the inequality relationships in the problems. Insights gained from the analyses of these subjects' uses and interpretations of variables in inequalities and implications for instruction with respect to the teaching and learning of inequalities will be presented.

Reference

Küchemann, D. (1981). Algebra. In K. M. Hart (Ed.), *Children's understanding of mathematics: 11-16*, (pp. 102-119). London: John Murray.

ASSESSMENT RESEARCH REPORTS

THE EFFECTS OF STAFF DEVELOPMENT FOCUSED ON ASSESSMENT PRACTICES

Daniel J. Brahier
Bowling Green State University
brahier@bgnet.bgsu.edu

Two inservice projects, funded by the Eisenhower Program for Mathematics and Science, were conducted in 1996 and 1997 with approximately 100 teachers in the Midwest. After each group had a full year of staff development, teachers showed significant gains in their knowledge and use of several assessment strategies. Long-term follow-up research data showed that teachers continued to implement new assessment techniques in their classes and were much more likely to do so if they had a colleague in the building or district who was also involved in the project and committed to change. The results suggest that a team approach to staff development may be most effective.

The literature contains a great deal of insight regarding teacher change, including interesting parallels that can be drawn between the reform efforts of the National Council of Teachers of Mathematics in recent years and those of others across disciplines in the past. One might have a tendency, for example, to describe the NCTM Standards as a rebirth of the New Math of the 1960's. The assumption, then, is that the Standards will "come and go," just as New Math came and went. However, Cuban (1993) stated that reform efforts of the 1950's and 1960's were poorly-planned, noting that "materials were published and placed in the hands of teachers who . . . had little time to understand what was demanded by the novel materials or . . . to practice their use" (p. 4). Indeed, time appears to be a factor that influences the potential of significantly changing educational practices. Fullan (1982) said that innovators must assume that it takes two to three years for any significant changes to occur in education and that implementation should be gradual, placing early efforts on small scales.

Any changes in assessment, classroom strategies, or curriculum involve the attempt to change the belief system of teachers in the classroom (Ames, 1992; Cohen & Ball, 1990; Fullan, 1982). These changes not only require time, but they also assume that local educational leaders will develop a plan that represents a process of change (Fullan, 1982) and persuasion of the teachers that reform will actually benefit them and their students (Cuban, 1993). Goodlad (1984) has recommended a long-term, collaborative approach between classroom teachers and institutions of higher learning. Interestingly, National Science Foundation grants, as well as Eisenhower grants, have stipulations that the teacher training must be long-term and involve follow-up sessions. The "one shot" inservice for a district has simply not proven effective, nor have university summer workshops for teachers or district staff development days, because they generally lack congruence with the total effort of the district on making any long-term changes (Fullan, 1982). Finally, we must realize that, as Fullan (1982) noted, not all teachers are even interested in changing at all. Even when they recognize the benefits of educational innovations in student learning, they may still resist for a variety of reasons, not the least of which is that teachers tend to teach as they were taught (Goodlad, 1984). For example, many teachers who were presented with a new framework for teaching mathematics in California and were extensively inserviced still did not change their teaching styles (Cohen & Ball, 1990).

The NCTM Standards documents have emphasized the relationship between curriculum, assessment, and teaching. Essentially, it is difficult — if not, impossible — to change one of these three elements without changing the others. Therefore, a teacher who has tried to change teaching methodologies to be congruent with the Standards may also be in a position to recognize the shortcomings of traditional forms of assessment. Two Eisenhower-funded projects in the Midwest, ASPECT in 1996 and ASPEN in 1997, were designed to assist teachers in making connections between curriculum, assessment, and teaching, and to place assessment at the focal point of change. Raymond's model (1997) suggested that teacher education programs can influence teacher beliefs which, in turn, significantly affect teaching practices. These projects were directed at changing teacher beliefs about mathematics while using the lens of assessment.

Method

Participants in ASPECT and ASPEN were pre-tested in March of 1996 and 1997, respectively, prior to their involvement in the projects. They were asked to express their opinions about assessment, their understanding of a variety of forms of assessment, and the degree to which they were using those strategies. During the Spring sessions, participants wrote journal entries about the program and the progress they had made in rethinking their classroom practices, as suggested by Burk and Littleton (1995). During the Summer intensive sessions, participants were interviewed in small groups and asked a variety of questions regarding the effectiveness of the program and the degree to which they had changed their practices as a result. In the Fall sessions, the participants were, again, surveyed and asked many of the same questions they had been asked in the Spring. Furthermore, the project evaluator made site visits to classrooms of ASPEN participants in January of 1998 to compare the changes teachers reported on surveys to what is actually happening in their classrooms. Similarly, two ASPECT participants were visited and interviewed as a two-year follow-up on the program's effectiveness.

Since most of the survey questions were answered through the use of a Likert Scale, the quantifiable data were used to run t-tests to determine whether or not knowledge about and use of alternative assessment strategies had statistically shown a change. Furthermore, open-ended responses to survey questions, observations, and transcribed interview data were used to qualitatively examine the effectiveness of the program and the changes in beliefs and practices of ASPECT and ASPEN participants. Triangulation of quantitative survey data, qualitative survey data, journal entries, and interview data was used to provide a consistent "big picture" of the change process, as recommended by Patton (1990).

Results

The study involved exploration of five research questions, as listed below. The major findings for each question are described:

(a) How did the participants' knowledge about and use of authentic forms of assessment change as a result of their participation in the project?

Many of the participants stated that they were aware of the "new" assessment strategies available to teachers prior to the inservice experiences but were not comfortable with implementing the changes until having more extensive experiences with these methods. Table 1 shows the changes in knowledge level and use of alternative assessment strategies for each of the projects.

Table 1
Survey Data On Changes in Knowledge Level and Use of Various Assessment Strategies

	ASPECT (1996)		ASPEN (1997)	
Assessment Strategy	Knowledge (t-value)	Use (t-value)	Knowledge (t-value)	Use (t-value)
Portfolios	5.77 *	4.26 *	6.36 *	3.76 *
Journaling	4.63 *	4.96 *	7.74 *	7.13 *
Investigations	2.13 *	0.68	7.08 *	6.11 *
Open-Ended Questions	2.93 *	2.93 *	4.87 *	3.16 *
Interviews	5.96 *	3.14 *	7.24 *	8.95 *
Formal Observations	2.41 *	-1.46	6.90 *	3.65 *
Rubric Scoring	6.03 *	5.69 *	6.23 *	8.27 *
Writing	6.58 *	3.86 *	6.14 *	4.43 *
Performance Tasks	0.000	-0.90	4.18 *	0.86
Standardized Tests	-0.24	0.78	-3.08 *	-0.45
Multiple Choice Tests	-1.00	0.85	-2.54 *	-3.84 *
Competency Tests	0.50	0.78	-0.17	-0.27 *
Appropriate Internet Use	N/A	N/A	5.14 *	4.13 *
Appropriate Calculator Use	N/A	N/A	5.14 *	2.72 *

* significant 2-tailed t-value

The data indicate that both of the projects have been highly-successful in not only raising teacher understanding of various assessment practices but also in convincing participants to implement these strategies. The statistical values indicate that the ASPEN project was, overall, more successful. A major reason for this finding is that, overall, ASPEN participants entered into the project with a much lower level of understanding of assessment than did the ASPECT teachers; thus, their changes appear more significant.

(b) Besides instructing teachers on "how to" use alternative forms of assessment, what other effects did the projects have on the participants?

It is often thought that teachers change their assessment practices in an attempt to keep up with changes already made in curriculum and teaching. But the data from this study appears to show that some teachers, through direct experiences, "bought into" authentic forms of assessment first, and then they began to change instructional strategies to align with their assessments. They realized that "teaching to the test" was a good idea, provided that the test was a goal worth working toward. Therefore, the changes in assessment practices, for some, was the tool that brought about change in their classroom practices. It suggests another way to reform the thinking of classroom teachers—by beginning with a look at assessment and working backwards to instruction.

(c) What were the key factors that convinced teachers that they needed to rethink and/or change their assessment and teaching practices?

Teachers appeared to feel the need to change after experiencing worthwhile teaching and assessment episodes, first-hand, through the Spring and Summer sessions. The inservice strategy of giving participants experiences first and working back to the theory behind them appeared to be powerful, and much more so than a "traditional course" in which theory precedes discussions of practical classroom examples. In surveys and interviews, participants repeatedly spoke of the power of field testing assessment. Follow-up site visits with participants showed that continued field-testing and sharing with colleagues have accounted for several long-term changes in teaching practices. For example, two years after completion of ASPECT, "Rose" shared how her assessment strategies have evolved over time, noting that portfolio assessment was highly-successful in her first year but much less so in her next. She went on to explain that had her second year been her first experience with portfolios, she would have dropped them from her assessment plan, but earlier successes convinced her to hold on to the strategy.

(d) What measurable long-term effects did the project have on ASPECT participants, now two years from beginning their participation?

The most significant long-term effect observed thus far from ASPECT teachers has been their willingness and ability to bring about change in other teachers. Several participants in the project have made presentations at staff meetings, as well as state-level and regional conferences. Others have been involved in making presentations and serving on the instructional team for subsequent projects such as ASPEN. A third grade teacher from ASPECT worked on the implementation of innovative assessment strategies with the Kindergarten teacher in her building and "brought the teacher along" in the process. In fact, the Kindergarten teacher — never a direct participant in the assessment project — became a presenter in the ASPEN program in 1997. Site visits and interviews with ASPECT teachers also indicate a wealth of diverse assessment practices that are being implemented. Teachers and their students have constructed rubrics for class presentations, journal entries, homework assignments, and open-ended questions and have repeatedly modified them over time.

(e) What consistencies and inconsistencies can be noted in the measured effects of ASPECT, as compared to ASPEN, which can inform the development of Phase III in 1998?

Data collected from both projects indicate that the most significant changes in teachers appear to be in those schools in which more than one teacher participated in the project. A teacher who works with another person from the building or district forms a support network that translates into more marked changes in assessment strategies. In interviews, two teachers from ASPECT and six teachers from ASPEN revealed a feeling of support and the need for a sounding board in the building when others from the project could be consulted. As a result of this finding, the Phase III program in 1998 had an applicant requirement of signing up as a "team" of two of more teachers from the same building or district. Teachers in the latest project will continually collaborate with teaching peers to implement and self-assess new teaching and assessment techniques.

Discussion and Conclusions

Research on teacher change by Cooney, Badger, and Wilson (1993) showed that teachers will only use rich assessment tasks if the tasks are consistent with the teachers' beliefs about the nature of mathematics and if they acknowledge the task's usefulness in measuring understanding. Consequently, it is important that teacher inservice programs focusing on assessment involve participants in *doing* significant mathematical tasks and discussing the nature of mathematics while considering assessment alternatives. Both ASPECT and ASPEN were focused on changing participant beliefs and demonstrating the power of using alternative assessment strategies and, as a result, appear to have been very successful. Also, as Meisenheimer (1996) pointed-out, a key to long-term success for teacher inservice programs is providing opportunities for teachers to network and to share successes and challenges with colleagues. The assessment projects described in this paper emphasized this networking, and long-term effects have resulted.

Research questions which have yet to be answered about these projects center around the achievement of students in the classroom. Ideally, if assessment practices change with teacher beliefs, then students enrolled in classes with inquiry-based teaching methods should outperform peers in more traditional classrooms. However, research data has yet to be collected on the achievement of students in classes taught by ASPEN and ASPECT teachers.

References

Ames, C. (1992). Classrooms: Goals, structures, and student motivation. *Journal of Educational Psychology, 84*(3), 261-271.

Burk, J. & Littleton, P. (1995). Reflective journals: Enhancing mathematics staff development. *Mathematics Teaching in the Middle School, 1*(7), 576-583.

Cohen, D. K. & Ball, D. L. (1990). Policy and practice: An overview. *Educational Evaluation and Policy Analysis, 12*(3), 233-239.

Cooney, T. J., Badger, E., & Wilson, M. R. (1993). Assessment, understanding mathematics, and distinguishing visions from mirages. In N.L. Webb (Ed.), *Assessment in the mathematics classroom* (pp. 239-247). Reston, VA: National Council of Teachers of Mathematics.

Cuban, L. (1993). *How teachers taught: Constancy and change in American classrooms 1880-1990* (2nd ed.). New York: Teachers College Press.

Fullan, M. (1982). *The meaning of educational change*. New York: Teachers College Press.

Goodlad, J. L. (1984). *A place called school: Prospects for the future*. New York: McGraw-Hill.

Meisenheimer, B.K. (1996). Critical attributes of teachers who have become practitioners of authentic assessment (Doctoral dissertation, Loyola University of Chicago, 1996). *Dissertation Abstracts International, 56-12*, A4652.

Patton, M. Q. (1990). *Qualitative evaluation and research methods* (2nd ed.). Newbury Park, CA: Sage.

Raymond, A. R. (1997). Inconsistency between a beginning elementary school teacher's mathematics beliefs and teaching practice. *Journal for Research in Mathematics Education, 28*(5), 550-576.

CASE STUDIES OF PRESERVICE ELEMENTARY TEACHERS' DEVELOPMENT IN MATHEMATICS ASSESSMENT

Anne M. Raymond
Keene State College
araymond@keene.edu

This paper reports findings from a study of the development of preservice elementary teachers beliefs about and knowledge of mathematics assessment and their interpretations of assessment standards. The primary objective was to document preservice teachers' conversations about mathematics assessment as they engaged in alternative assessment practices and a variety of activities related to the purposes and standards of mathematics assessment. Focusing on one pair of participants within a larger study afforded the opportunity to capture *moments* of development as two preservice teachers came to new realizations and clarifications about mathematics assessment.

A changing definition of mathematics assessment continues to unfold (Kulm, 1990; Webb, 1993) and is stimulating a closer look at mathematics classroom practices (Moon & Schulman, 1995; Stenmark, 1991). Educators have agreed that good mathematics assessment practices ought to enhance learning, as well as reflect important mathematics, and provide coherent feedback. Consequently, assessment has moved beyond standard tests and homework to include student writing, classroom observations, portfolio assessment, formal and informal student self-assessment, and performance assessment tasks (Moon & Schulman, 1995). Because these types of alternative assessment practices are often difficult to evaluate in the form of a grade, there is a growing emphasis on the development of analytic and holistic scoring rubrics designed to appraise student performance via a myriad of assessment tasks (Stenmark, 1991). Teachers must be exposed to alternative assessment practices and to issues related to the implementation and communication of results of alternative assessment (Cain, et al, 1994). Accordingly, it is imperative to introduce preservice teachers to alternative assessment practices in mathematics and challenge them to reflect on the purposes of mathematics assessment and how to design assessment practices that reflect instructional goals.

Methods of Inquiry

Within a larger investigation (Raymond, 1996), eight preservice teachers volunteered to participate in a more in-depth documentation of their development. All participants were seniors enrolled in a one-semester mathematics methods course at a midwestern university in the semester just prior to student teaching. The eight volunteer preservice teachers were paired and videotaped while engrossed in the four paired activities, followed by interviews.

The four assessment activities they encountered included (a) distinguishing between "closed" and "open" mathematical tasks, (b) identifying children's computations errors, (c) scoring children's mathematical problem solving via analytic and holistic scoring rubrics, and (d) conducting peer portfolio review conferences. The primary purpose of all of these tasks was to stimulate discussion and reflection on issues related to mathematics teaching, learning, and assessment. Upon completion of a task, the researcher joined each pair, eliciting a videotaped conversation about the activity they had just addressed.

Another revealing source of data was an assignment in which the preservice teachers individually designed a mathematics assessment instrument and discussed the extent to which this assessment tool met the standards set forth by the mathematics education community. Discussions during the final interviews with preservice teachers centered on a reflective "look back" at the various experiences throughout the study, identifying the task that most challenged or affirmed their beliefs about mathematics assessment.

Data and Results

Herein, I provide a subset of data from two paired volunteer participants: Marissa and Sharon. Data presentation is framed around: (a) a brief description of each participant and her beliefs about mathematics assessment, (b) a description of the assessment tool developed by each participant and her interpretations of the NCTM assessment standards as they relate to her assessment tool, and (c) excerpts from the pair's conversations as they engaged in two of the four assessment tasks.

About Sharon. Sharon, was a senior completing a college degree in four years at age 22, and also was a single mother of two young children. She liked mathematics and had very clear-cut opinions, expressing, *I believe the purpose of mathematics assessment is to make sure the students are understanding what you are teaching. Mathematics is a staple of life, you have to know how to do it.* She had a very simplistic view of the relationship between mathematics assessment and instruction, claiming, *to learn it* [math] *or teach it you must assess yourself and your class to see what works and what does not. If you never assessed your class you might never know that for the past three weeks no one has understood a single thing you have said.* This last statement hints at a somewhat *teacher as teller* view of teaching and a view of assessment as *something that comes after the teaching.*

On the initial beliefs survey, Sharon indicated that she was unsure whether students could assess their own mathematical learning. However, oddly enough, when asked to design a mathematics assessment plan or instrument that she believed would be useful in the mathematics classroom, Sharon created a "Student Self Check List" which included these writing prompts: *How did you approach the problem? Was it hard or easy? Why? What did you try that didn't work? What pictures or models worked? What is this similar to that you have done in the past? What did you like or dislike about this problem? What did you learn? What are you overall feelings about the problem?* In her critique of this instrument regarding the extent to which she believed it met the characteristics of good mathematics assessment as outlined in the working draft of the *Assessment Standards* Sharon reported,

This type of assessment reflects what the student feels to be IMPORTANT MATHEMATICS. It can ENHANCE LEARNING by [their] *reflecting on it. It gives the student a chance to be EQUAL by explaining themselves. If the students tell the truth it becomes a VALID source of information.*

Sharon's interpretation reveals a suspicion regarding the validity of student self assessment. Class discussions that ensued after this assignment ultimately led preservice teachers to conclude that assessment techniques cannot necessarily meet all of the standards all of the time. However, they expressed that a teacher should consider the six standards when attempting to design and implement alternative assessment practices. They further concluded that one should question whether or not a particular assessment practice is appropriate for a given situation, asking, for example, *What can you learn about what students know from student self assessment?*

About Marissa. Marissa, age 23, was a married student who liked mathematics and was working toward a mathematics minor. Marissa believed that the primary purpose of assessment was to find out what students know as well as what they do not know. She also believed that assessment evaluated the effectiveness of teachers. She expressed that mathematics assessment and the teaching and learning of mathematics should be parallel in every way. She stated, *Whenever there is teaching, there should be assessment of how well the information was presented, which students understood the teaching, and* [whether] *the information being conveyed was the information learned*. She further commented, *Without assessment, there is no proof of teaching or learning.*

Underlying her statements about the importance of assessment were some clear messages about her views of teaching and learning, particularly her view that learning took place when information was conveyed and presented. On her initial beliefs survey, Marissa had strongly disagreed with the notion that the student's primary role in mathematics class was *to listen to and learn procedures explained by the teacher*. Thus, what she indicated initially on her survey, which was consistent with what the majority of her peers claimed, does not seem consistent with her related statements about relationships between teaching, learning, and assessment.

In her assessment assignment, Marissa, like Sharon, chose to focus on student self-assessment. She believed that self-assessment, done well, would meet all of the NCTM assessment standards. Marissa explained about the Standards in this way:

> **Consistency**. *What better way to assess students' understanding than to ask them. Students are also not shy in expressing a teachers' shortcomings when given the opportunity. By giving self-assessments, students are able to freely express what they have learned and what I* [the teacher] *have failed to teach them.*
>
> **Openness**. *By letting every students assess the learning process, and preferably discuss the results in a whole class setting, the students are able to rationalize their assessment of themselves so I* [the teacher] *am not the master of all evaluation.*
>
> **Equity.** *By letting each student fill out* [his] *own assessment,* [each will] *have an equal opportunity to express* [his] *beliefs.*
>
> **Important mathematics.** *With self-assessments students have the opportunity to express what they believe the important mathematics is. If their beliefs differ from the teacher's, then this knowledge allows a chance to review concept goals.*
>
> **Enhance learning.** *Students can express what learning needs to be enhanced.*

Above, when discussing self-assessment, Marissa again revealed beliefs that were aligned more with *the teacher-as-teller* and less with the *teacher-as-guide* viewpoint. Yet, according to her initial beliefs survey, Marissa indicated that she held the opposite belief. Data showed that like Marissa, five of the other eight volunteer participants, when put in the position of discussing a very focused topic such as how to assess a students' knowledge about a particular mathematics topic, revealed more traditional beliefs than they may have wanted to admit to, or may have been aware of, when taking the beliefs survey. This finding runs parallel to those in studies that focused on examining consistency between teachers' beliefs and instruction (e.g., Raymond, 1997).

Mathematics Assessment Tasks

When asked what assessment task made the most lasting impression on them, Sharon claimed that having to score elementary students' problem-solving work using two different scoring rubrics ... *opened my eyes to the complexities of mathematics assessment.* Marissa, on the other hand, thought the activity of interpreting students' computation errors ... *was neat because we were kind of reversing roles. Instead of students trying to figure out what the teacher was thinking, I was trying to see what students were thinking.*

When assessing student problem solving using scoring rubrics, the pair encountered many situations in which they did not know how to score the student's work. Too often, children did not show enough work in order for the pair to ascertain whether or not they really understood the problem or had a plan to solve the problem. Sharon was a particularly adamant about students showing sufficient work. At one point, when looking at a third-grader's solution to a process problem, Marissa expressed that even though the student did not have the right answer, since he attempted some plan, *He should get full points for* [demonstrating] *understanding*. Sharon retorted, *How can he have complete understanding if he didn't do it right*? The body language of frustration demonstrated on the videotape revealed that the pair was uncomfortable in scoring students' work using analytic or holistic scoring rubrics.

After the task, when asked which rubric they preferred and why, both stated in unison, *the analytic scale*. Sharon explained that it was *more cut and dry ... either you got it or you didn't get it*. When probed about the holistic scale, Marissa said, *It was unclear, I mean, sometimes I didn't know what score to give them. Plus, if it was shorter it would be easier*. When I suggested that they could devise a scoring rubric tailored-made to a specific problem, Sharon adamantly objected saying, *Who would have time to do that*? When I asked them by which scale they would rather have their own problem solving assessed, they chose the holistic scale because it allowed for a *more generous* distribution of points. Thus, when put in the role of the student, they felt one way. When trying to think from the teacher's perspective, they had different goals. They actually smiled when they realized from their responses how students and teachers approach the same situation from different vantage points.

During a task of identifying computation errors, Sharon and Marissa were able to find the errors, but often had difficulty putting the error into appropriate terms. For example, when describing an error in a student's division work, they could not recall which number should be referred to as the dividend and which as the quotient (both agreed on the divisor). What stood out in their conversations was their realization of the significant role regrouping plays in computations. They were struck by the number of procedural errors that stemmed from a conceptual misunderstanding of regrouping. They noted this in several cases where students added, subtracted, or multiplied whole numbers. At the end, Marissa concluded, *They really can't understand computations right if they don't know what carrying and borrowing is to begin with*. After having engaged in this activity, Marissa truly realized the challenges of teaching a child something as *simple* as adding or subtracting.

Concluding Remarks

This limited report of a large body of research begins to depict preservice teachers' thinking as they develop from students to teachers who must confront mathematics teaching, learning, and assessment issues on a daily basis. The study suggests value in providing preservice teachers a forum for discussing and engaging in alternative assessment practices. Left to interpretation, many of the goals of mathematics assessment reform may be lost to preservice teachers. Examination of beliefs and knowledge of mathematics assessment practices stimulated preservice teachers to reflect on the undeniable links between mathematics assessment and instruction (Cooney et al., 1993).

References

Cain, R. W., Kenney, P. A., & Schloemer, C. G. (1994). Teachers as assessors: A professional development challenge. In D. B. Achille (Ed.), *Professional development for teachers of mathematics: 1994 Yearbook* (pp. 93-101). Reston, VA: National Council of Teachers of Mathematics.

Cooney, T. j., Badger, E., & Wilson, M. R. (1993). Assessment, understanding mathematics and distinguishing visions from mirages. In N. L. Webb (Ed.), *Assessment in the mathematics classroom: 1993 yearbook* (pp. 239-247). Reston, VA: National Council of Teachers of Mathematics.

Kulm, G. (Ed.) (1990). *Assessing higher order thinking in mathematics.* Washington, DC: American Association for the Advancement of Science.

Moon, J. & Schulman, L. (1995). *Finding the connections: Linking assessment, instruction, and curriculum in elementary mathematics.* Portsmouth, NH: Heinemann.

Raymond, A. M. (1996). The development of preservice elementary teachers' beliefs about and knowledge of alternative mathematics assessment. In E. Jakubowski, D. Watkins, & H. Biske (Eds.), *Proceedings of the Eighteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 376-382). Panama City, FL: The Florida State University.

Raymond, A. M. (1997). Inconsistency between a beginning elementary school teacher's mathematics beliefs and teaching practice. *Journal for Research in Mathematics Education, 28*(5), 550-576.

Stenmark, J. K. (Ed.). (1991). *Mathematics assessment: Myths, models, good questions, and practical suggestions.* Reston, VA: National Council of Teachers of Mathematics.

Webb, N. L. (Ed) (1993). *Assessment in the mathematics classroom: 1993 yearbook.* Reston, VA: National Council of Teachers of Mathematics.

ASSESSING AND DOCUMENTING STUDENT KNOWLEDGE AND PROGRESS IN EARLY MATHEMATICS

Bob Wright
Southern Cross Univ., Australia
bwright@scu.edu.au

Rita Stewart
Southern Cross Univ., Australia

Ann Stafford
Oconee County, South Carolina

Richard Cain
Oconee County, South Carolina

A research-based method of assessing and documenting students' knowledge in early mathematics is outlined and used to report advancements of 91 first-graders in a Mathematics Recovery implementation involving 15 teachers and 13 elementary schools in North and South Carolina. Participants were selected because of their low attainment on an initial assessment, and were given daily, individualized teaching sessions of 30 minutes' duration for 10 to 12 weeks, and a final assessment. Project teachers participated in an on-going professional development program. Analysis of videotaped assessments included determination of participants' advancements in terms of three aspects: (a) early arithmetical strategies; (b) numeral identification; and (c) facility with the forward number word sequence. Participants' advancements are compared with objectives from a relevant state mathematics framework. Sixty-three participants (69%) exceeded curriculum expectations, five (6%) met expectations and 23 (25%) did not. Results are comparable with earlier implementations providing a further strong endorsement of the program.

This paper: (a) describes a novel, research-based method of assessing and documenting student knowledge and progress in early years mathematics (i.e. K-2); (b) uses the method in (a) to report results from a large-scale research and development project; and (c) compares the levels of knowledge attained by participating first-graders with levels specified by a relevant state curriculum. The method referred to in (a) above has its origins in the constructivist teaching experiment methodology developed by Steffe and collaborators (e.g. Steffe, Cobb, & von Glasersfeld, 1988). The method allows for: (a) documenting in detail, longitudinal development of students' arithmetical knowledge; (b) qualitative comparisons among students' arithmetical knowledge; and (c) comparisons both for one student and among students of developments of different aspects of arithmetical knowledge. Until 1992, this method had been used in several research projects (e.g. Wright, 1994a) but had not been used by teachers. In 1992, the method was adapted from its research orientation for use by teachers, and in the last six years, has been used by hundreds of teachers in Australia and the United States, and also by teachers in the UK. The adaptation of the method occurred as part of a four-year applied research and development project (e.g. Wright, 1994b; Wright, Stanger, Cowper, & Dyson, 1996), known as Mathematics Recovery, which focused on the development of a specialist teaching program with the goal of identification and advancement of first-graders who are judged to be low-attainers in mathematics. As well, the method forms the basis of a systemic, classroom-based project in Australia, involving several hundred schools which has been judged to be successful in terms of teachers' development (Bobis, 1996; Bobis & Gould, 1998;) and students' learning (Stewart, Wright, & Gould, 1998). A basic goal of both the classroom-based project and Mathematics Recovery is for teachers to better understand children's mathematical strategies and their development from less sophisticated to more sophisticated strategies. Both projects utilize a framework for early arithmetical learning (Wright, 1998) which incorporates the assessment method which is the focus of this paper.

The 1995-96 implementation of Mathematics Recovery. In 1995-96, the Mathematics Recovery project operated with 15 teachers and 91 participants in 13 elementary schools, in North and South Carolina. In each school, the project operated for approximately 18 weeks during each half of the school year. In the initial four weeks of each 18-week period and in the final two or three weeks, project teachers completed individualized interview-based assessments with participants and counterparts. Mathematics Recovery teaching cycles commenced after the initial assessment period. Students were instructed individually for 30 minutes daily, for up to four days per week, over a period of eight to twelve weeks. Project teachers undertook the first phase of the Mathematics Recovery professional development program prior to commencing teaching cycles. After the initial weeks these teachers attended on-going professional development meetings held every second week.

Initial and final assessments. Each of the 91 participants was administered an initial and final assessment which focused on several aspects of students' arithmetical knowledge including: (a) early arithmetical strategies; (b) facility with forward number word sequences (FNWSs); and (c) numeral identification. The assessment included simple additive and subtractive tasks involving screened and unscreened collections, saying FNWSs, stating the number word before or after a given number word, and identifying 1-, 2-, and 3-digit numerals. Each assessment was videotaped for subsequent analysis. The analysis results in a detailed profile of the student's current arithmetical knowledge and includes determination of a level for each of the three aspects of early arithmetical knowledge.

Curriculum expectations in terms of these three aspects. Curriculum expectations for first graders in this study were based on objectives specified in a relevant state mathematics framework (Draft mathematics curriculum revision, 1998). Four first-grade objectives have particular relevance for this study: (a) find sums and differences using counting strategies such as counting on and counting back; (b) rote count by ones to 100; (c) identify one more; and (d) read numerals to 100. For the purposes of this study, a student who meets objective (a) is regarded as being at Stage 3 on the model of Early Arithmetical Strategies. Similarly, meeting objectives (b) and (c) is regarded as being at Level 4 on the model of FNWSs, and meeting objective (d) is regarded as being at Level 3 on the model of Numeral Identification. A student who is at Stage 3, Level 4 (on FNWSs) and Level 3 (on Numeral Identification) is regarded as having met curriculum expectations.

Results and Discussion

Table 1 shows project results in terms of the model of early arithmetical strategies. Each row total shows the number of participants who, at the time of their initial assessment, were at a given stage, e.g. 63 of the 91 were initially at Stage 1. In each row, the cells show the numbers of participants for a given initial stage, who were at a given stage on their final assessment.

Advancements in arithmetical strategies. Fifty of the 63 participants who were initially assessed at Stage 1 reached Stages 3 or 4 by the end of the program. They had advanced from being counters of perceptual unit items to counters of abstract unit items (Steffe, von Glasersfeld, Richards, & Cobb, 1983). Seventy-five (82%) of the 91 participants advanced to at least Stage 3. As well, at the initial assessment, 90 of the 91 participants had not reached Stage 3 and thus were not performing at expected curriculum levels, and moreover, the majority of the participants (69%) were assessed initially at Stage l.

Table 1
Numbers and Percentages of Students Initially at Stage 0,1, 2, or Who Were at Stage 1, 2, 3, 4 or 5 in Their Final Assessment.

Initial Stage	Number of Students who reached a given stage 1	2	3	4	5	Total
0	1 (25%)	0	3 (75%)	0	0	4 (100%)
1	3 (5%)	10 (16%)	34 (54%)	16 (25%)	0	63 (100%)
2	0	2 (9%)	10 (43%)	10 (43%)	1 (5%)	23 (100%)
3	0	0	1 (100%)	0	0	1 (100%)
Total	4	12	48	26	1	91

Initial FNWS and Numeral Identification levels of participants initially assessed at Stage 1. Table 2 shows for 63 participants who attained Stage 1 at their initial assessment, the numbers attaining initial FNWS and Numeral Identification levels. Forty-three of the 63 participants attained neither FNWS Level 3 nor Numeral Identification Level 1 and thus were below levels of attainment for the end of the Kindergarten year (Stewart, Wright, & Gould, 1998), i.e. facility with the number word sequence and knowledge of numerals up to ten.

Advancements of participants finally assessed at Stage 3. Table 3 shows for 48 participants who attained Stage 3 at their final assessment, the numbers attaining final FNWS and Numeral Identification levels. Thirty-six of these participants exceeded expected curriculum levels for first grade in terms of FNWS and Numeral Identification levels and five attained the expected levels.

Advancements of participants finally assessed at Stage 4. Table 4 shows for 26 participants who attained Stage 4 at their final assessment, the numbers attaining final FNWS and Numeral Identification levels. All 26 attained FNWS Level 5, 24 attained Numeral Identification Level 4 and two Level 3. An additional participant attained Stage 5, FNWS Level 5 and Numeral Identification Level 4. All of these 27 participants far exceeded expected first grade curriculum levels on these three aspects.

Conclusion

The 91 participants in this Mathematics Recovery implementation clearly were low-attainers, since the majority of them were initially assessed at Stage 1 (the Stage of perceptual counting) and 43 of the 91 (47%)

Table 2
Numbers of the 63 Participants Initially Assessed at Stage 1 Who Were at Given FNWS and Numeral Identification Levels in Their Final Assessment.

	Numeral Identification Level					
FNWS Level	**0**	**1**	**2**	**3**	**4**	**Total**
1	4	10	1	1	0	16
2	5	9	2	0	0	16
3	2	16	3	4	0	25
4	0	0	1	4	1	6
5	0	0	0	0	0	0
Total	11	35	7	9	1	63

initially had levels of arithmetical knowledge more typical of Kindergarten than first grade students. As can be gleaned from Tables 6 and 7, in the final assessment, 63 participants (69%) exceeded curriculum expectations for first-grade, five (6%) attained curriculum expectations and 23 (25%) had not attained curriculum expectations on at least one aspect. Similar results were reported by Wright et. al. (1996) who found that, by and large, Mathematics Recovery participants made significantly greater advancements than did counterparts of similar initial levels, and that the vast majority of participants made advancements across all three aspects of arithmetical knowledge. The results of the 1995-96 implementation of Mathematics Recovery are comparable with the 1992, 1993 and 1994 results from Australia (Wright et al., 1994; 1995; 1996), and provide further evidence to support the viability of the program.

Table 3.
Numbers of the 48 Participants Initially Assessed at Stage 3 Who Were at Given FNWS and Numeral Identification Levels in Their Final Assessment.

	Numeral Identification Level					
FNWS Level	**0**	**1**	**2**	**3**	**4**	**Total**
3	0	1	0	4	0	5
4	0	0	0	5	4	9
5	0	1	1	7	25	34
Total	0	2	1	16	29	48

Table 4
Numbers of the 26 Participants finally Assessed at Stage 4 Who Were at Given FNWS and Numeral Identification Levels in Their Final Assessment.

	Numeral Identification Level					
FNWS Level	**0**	**1**	**2**	**3**	**4**	**Total**
4	0	0	0	0	0	0
5	0	0	0	2	24	26
Total	0	0	0	2	24	26

References

Bobis, J. (1996). *Report on the evaluation of the Count Me In Too Project.* Sydney: University of Western Sydney.

Bobis, J., & Gould, P. (1998). *The impact of an early number project on the professional development of teachers.* Paper presented at the Twenty First Annual Conference of the Mathematics Education Research Group of Australasia, Gold Coast, July 5-8.

Draft mathematics curriculum revision. (1998). Raleigh, NC: Department of Public Instruction.

Steffe, L. P., Cobb, P., & E. von Glasersfeld. (1988). *Construction of arithmetic meanings and strategies.* New York: Springer-Verlag.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application.* New York: Praeger.

Stewart, R., Wright, R. J., & Gould, P. (1998). *Kindergarten students' progress in the Count Me In Too Project.* Paper presented at the Twenty First Annual Conference of the Mathematics Education Research Group of Australasia, Gold Coast, July 5-8.

Wright, R. J. (1994a). A study of numerical development in the first year of school. *Educational Studies in Mathematics, 26, 25-44.*

Wright, R. J. (1994b). Working with teachers to advance the arithmetical knowledge of low-attaining 6- and 7-year-olds: First year results. In J. Ponte, & J. Matos (Eds.), *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education, Vol. 4, pp. 377-384.* Lisbon, Portugal: University of Lisbon.

Wright, R. J. (1998). *An overview of a research-based framework for assessing and teaching early number.* Paper presented at the Twenty First Annual Conference of the Mathematics Education Research Group of Australasia, Gold Coast, July 5-8.

Wright, R. J., Cowper, M., Stafford, A., Stanger, G., & Stewart, R. (1994). The Maths Recovery Project: A progress report. In G. Bell, R. Wright, N. Leeson, & J. Geake (Eds.), *Proceedings of the Seventeenth Annual Conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 709-716.* Lismore, NSW, Australia: Southern Cross University.

Wright, R. J., Cowper, M., Stanger, G., & Stewart, R., (1995). *Third year report of a project focusing on advancing the arithmetical knowledge of low-attaining first graders.* Paper presented at the Eighteenth Annual Conference of the Mathematics Education Research Group of Australasia, Darwin, 7-10 July.

Wright, R. J., Stanger, G., Cowper, M., & Dyson, R. (1996). First-graders' progress in an experimental mathematics recovery program. In J. Mulligan, & M. Mitchelmore, *Research in early number learning: An Australian perspective (pp. 55-72).* Adelaide: AAMT.

Assessment
Short Orals

229

STUDENT ATTITUDES TOWARD MATHEMATICS: A COMPARISON OF STUDENTS IN REFORM AND CONVENTIONAL CLASSES[1]

Lesley Wagner
University of Wisconsin-Madison
lrwagner@students.wisc.edu

Mary Shafer
University of Wisconsin-Madison
mcshafer@macc.wisc.edu

Jon Davis
University of Wisconsin-Madison
jondavis@students.wisc.edu

As part of a longitudinal/cross-sectional study of the impact of a reform-based middle school mathematics curriculum on mathematical knowledge, understanding, attitudes, and performance, an attitude inventory was developed and administered to approximately 1500 students in grades 5ñ7 who studied either a reform-based or conventional curriculum. This paper explores the results of the initial administration of the attitude inventory.

Statements about mathematics and students beliefs about themselves as learners of mathematics were organized into five subscales: effort to learn mathematics, confidence in one's abilities, interest in mathematics, usefulness of mathematics, and communication of mathematical ideas. The initial administration of the attitude inventory yielded no significant differences in the means of the two groups with respect to the usefulness of mathematics, effort to learn mathematics and communication of mathematical ideas. However, significant differences (based on independent samples t-test) favoring the group studying conventional curricula were noted in the interest and confidence subscales.

The attitude inventory was administered again to both groups after students had been using the reform curriculum for one year with various levels of implementation. These data have yet to be analyzed. We anticipate that over time, stronger agreement with the items on the scale will be evident in the group using the reform curriculum.

---

[1]This research is supported in part by the National Science Foundation #REC-9553889. The views expressed here are those of the authors and do not necessarily reflect the views of the funding agency.

PRELIMINARY RESULTS OF AN ASSESSMENT USING NAEP AND TIMSS MATHEMATICS ITEMS FOR PROGRAM EVALUATION[1]

David C. Webb
University of Wisconsin-Madison
dcwebb@students.wisc.edu

Thomas A. Romberg
University of Wisconsin-Madison
romberg@macc.wisc.edu

As part of a longitudinal, cross-sectional study of the impact of *Mathematics in Context*, a reform-based middle grades curriculum, four different instruments were used to assess various aspects of student understanding of mathematics. The external assessment system discussed in this paper was designed to measure student performance on selected eighth-grade multiple-choice and constructed-response items from the 1992 NAEP, the 1996 NAEP and the TIMSS. The results discussed here are from the pilot administration of this assessment.

The external assessment system is composed of four separate assessments for grades 5 – 8. Each assessment contains twenty-eight items evenly divided among four strands: number, geometry and measurement, algebra and patterns, and statistics and probability. In order to examine growth over time sixteen items of moderate difficulty were repeated on each test. Ten classes (265 students) in nine different schools participated in a pilot study of these tests. Three of the nine schools serve students from large urban school districts.

Results from the pilot administration of this assessment suggest that many fifth and sixth grade students are able to successfully respond to eighth-grade public release items of moderate difficulty (percent of students responding correctly [p-value] > 50). Despite the call for districts and schools to use NAEP and TIMSS items to evaluate their instructional programs, our preliminary results indicate that these items may be limited in their use as benchmarks for eighth grade mathematics performance.

[1]This research is supported in part by the National Science Foundation #REC-9553889. The views expressed here are those of the authors and do not necessarily reflect the views of the funding agency.

ASSESSMENT POSTERS

A JOURNEY THROUGH THE HISTORY OF THE IEA AND INTERNATIONAL MATHEMATICS STUDIES

James P. Dildine
University of Illinois Urbana-Champaign
jdildine@uiuc.edu

The Third International Mathematics and Science Study (TIMSS) supported by the IEA continues the tradition of cross-national educational comparisons started in the late 1950s.

This poster presentation is a journey through the history of the International Association of the Evaluation of Educational Achievement (IEA) and the international studies that were carried out. This history is presented with a focus on the surveys of the IEA that were intended to provide a cross-national examination of mathematics curricula. International studies of education have influenced and continue to influence policy changes in US education. Reactions to the TIMSS results are indicative of reactions to previous IEA mathematics studies.

This presentation offers important insights into the history that preempted the TIMSS and the questions that arose as a result of international mathematics education comparisons. The issues that surround the TIMSS are not unique; they have been discussed with each incantation of the IEA studies. Questions that arise from the TIMSS results are better viewed through a lens of previous experiences and discussions. This presentation explores those previous experiences and the events that led to the TIMSS.

References

Husen, T. (1972) International evaluation of educational achievement. *The Journal of Educational Research, 66.* (Inside Front Cover)

Schmidt, W.H., & McKnight, C.C. (1995) Surveying educational opportunity in mathematics and science: An international perspective. *Educational Evaluation and Policy Analysis, 17*, 337-353.

Willoughby, S. (1968) Who won the international contest? *The Arithmetic Teacher. 15*,623-629.

INTERDISCIPLINARY PROJECTS TO ASSESS UNDERSTANDING AND GROWTH

Kathleen G. Snook
United States Military Academy
ak7056@exmail.usma.edu

Researchers have been investigating Advanced Mathematical Thinking in terms of both mathematical thinking that is advanced in nature and thinking about advanced mathematics. In Snook (1997), I investigated calculus students' understanding of the concept of the derivative and called for further investigation into methods of authentic assessment of advanced mathematics students' understanding. I have put this research into practice and incorporated various methods of authentic assessment in the calculus course at the United States Military Academy (USMA).

Within the Calculus course at USMA instructors use a wide variety of assessment tools including problem solving, quizzes, mini computer exercises, essays, portfolios, exams and interdisciplinary projects. Each of these tools assists instructors in assessing students' procedural and conceptual understanding of topics in the course. Although initially intended to provide students with an applied problem solving experience, the interdisciplinary projects now serve as a valuable vehicle to authentically assess student understanding and growth on several levels.

Students submit written reports documenting their project solutions. Instructors require students to integrate analytic, graphic, numeric and verbal support in their solution presentations and analyses. The project assessment rubric incorporates both written and numeric evaluations of the project submission, as well as written feedback in four areas of student growth: mathematical modeling, communicating mathematics, mathematical reasoning, and scientific computing.

Reference

Snook, K. G. (1997). *An investigation of first year calculus students' understanding of the derivative.* Doctoral Dissertation, Boston University.

Discourse Research Reports

THE DUALISTIC NATURE OF SCHOOL MATHEMATICS DISCOURSE

Dominic D. Peressini
University of Colorado at Boulder
dominic.peressini@colorado.edu

Eric J. Knuth
University of Colorado at Boulder
eric.knuth@colorado.edu

The purpose of this paper is to describe a theoretical framework for analyzing mathematics education discourse. The framework—grounded in the ideas of Soviet semioticians Mikhail Bakhtin and Yuri Lotman—has guided our exploration of the nature and role of discourse in the professional development of mathematics teachers and in these teachers' high school mathematics classrooms. We also discuss how this framework provides a lens for examining the relationship between different functionings of school mathematics discourse and students' understandings of mathematics. In addition, implications for future research are presented.

The role of discourse, although always a central factor in education and learning (for example, consider the role of discourse in Socrates' teachings), is receiving increased attention as educators strive to better understand the variety of factors that lead to increased learning in classrooms. Indeed, scholars argue for—and reform initiatives underscore—the importance of teachers and students engaging in reflective discourse of various kinds (Cobb, Boufi, McClain, & Whitenack, 1997; Hiebert, 1992; Lampert, 1990; National Council of Teachers of Mathematics, 1991) and research examining different issues central to these arguments and initiatives has provided a variety of insights related to classroom discourse (cf. Bauersfeld, 1995; Pimm, 1987; Yackel & Cobb, 1996). Recognizing the significance of discourse in mathematics learning, and embracing Davis' (1997) argument "that an attentiveness to how mathematics teachers listen [and talk] may be a worthwhile route to pursue as we seek to understand and, consequently, to help teachers better understand their practice" (p. 356), we began to explore how we could frame our own research on the role of discourse in mathematics classrooms (see Peressini & Knuth, 1998). Consequently, we have turned to the work of Mikhail Bakhtin to help us make sense of the different functionings of discourse that we have identified in secondary school mathematics and to assist us in unpacking the relationship between the functioning of discourse and students' understandings of mathematics. For the remainder of this article we present a theoretical framework for examining discourse in mathematics education.

The Dualistic Functioning of Discourse: A Theoretical Framework

Any true understanding is dialogic in nature—Voloshinov, 1973, p. 102.

Situating ourselves in the perspective that students' development of understanding takes place through their participation in the social interactions of the classroom—a context in which discourse is a critical component—we draw upon Bakhtin's notion of dialogicality (or multivoicedness) to examine discourse in school mathematics classrooms. According to Bakhtin, understanding results only through the interanimation of voices, that is, when the voice of a listener comes into contact with and confronts the voice of the speaker (Werstch, 1991).[1] Such contact requires the listener to take an active and responsive attitude toward the speaker's utterance rather than simply duplicating it (Bakhtin, 1986). "For each word of the utterance that we are in process of understanding, we, as it were, lay down a set of our own answering words. The greater their number and weight, the deeper and more substantial our understanding will be" (Voloshinov, 1973, p. 102). However, the degree of interanimation of voices—and thus the understanding developed—may differ depending upon the nature of discourse in which the interlocutors engage.

According to Bakhtin (1986), discourse can be characterized—in terms of speech genres—by the nature of its utterances: "We speak only in definite speech genres, that is, all our utterances have definite and relatively stable typical forms of construction of the whole" (p. 78). Elaborating further, Bakhtin (1986) states:

> A speech genre is not a form of language, but a typical form of utterance; as such the genre also includes a certain typical kind of expression that inheres in it. . . . Genres correspond to typical situations of speech communication, typical themes, and, consequently, also to particular contacts between the *meanings* of words [and between voices] (p. 87).

In examining the nature of discourse, Bakhtin differentiated between different types of speech genres in terms of the degree to which one voice can come into contact with and interanimate another (Wertsch, 1991). Lotman (1988), recognizing the significance of Bakhtin's criterion, argues that all texts—a text being any semiotic corpus (e.g., a verbal utterance, a written script, a picture)—are distinguished by two very different functions: to convey meaning and to generate meaning.[2] "The first function

---

[1]Voice is "the speaking personality, the speaking consciousness" (Bakhtin, 1981, p. 434) and may include both internal as well as external voices. In addition, the multivoicedness of an utterance can refer to more than just the voices of the speaker and listener: "Our speech, that is, all our utterances (including creative works), is filled with others' words, varying degrees of otherness or varying degrees of 'our-own-ness'" (Bakhtin, 1986, p. 89). For our purposes in this paper, we restrict our examination of voices within classroom discourse to the speaker's and listener's voices.

[2] Although Lotman did not provide specific terminology with which to distinguish these types of discourse, Wertsch (1991) did, using the terms "univocal" and "dialogic," respectively, to represent these two functions; we use the terms in a similar fashion.

[i.e., univocal] is fulfilled best when the codes of the speaker and the listener most completely coincide and, consequently, when the text has the maximum degree of univocality" (Lotman, 1988, p. 34). Accordingly, speech genres oriented toward serving a univocal function limit the degree of interanimation of voices.

> A focus on the univocal function (i.e., on information transmission) is associated with minimal contact between voices and with a quite restricted way in which this contact can occur. One voice functions to send information and another functions to receive it (although in accordance with Bakhtin's account of understanding, this receiving voice can never be entirely passive). In speech genres organized around the univocal function of text, then, there is little room for the receiving voice to question, challenge or otherwise influence the sending voice (Wertsch & Smolka, 1995, p. 80).

The discourse in school mathematics classrooms has been traditionally dominated by the univocal function of text and is representative of a particular type of speech genre—an authoritative speech genre. More specifically, as it is described in the research literature, discourse takes on an authoritative role when teachers use their speech to transmit mathematics information to their students. Further, if students want to participate in the social interactions, they must follow the teacher's direction, and to participate successfully, they must solve the problems as expected by the teacher rather than articulating their own understanding (Voigt, 1995). Bakhtin (1981) describes the nature of discourse within an authoritative speech genre: "The authoritative word demands that we acknowledge it, that we make it our own; it binds us, quite independent of any power it might have to persuade us internally; we encounter it with its authority fused to it" (p. 342). Consequently, within the speech genre of traditional school mathematics, it is unlikely that students' voices will question, challenge, or otherwise influence (i.e., interanimate) the teacher's voice.

In contrast, speech genres that are grounded in the dialogic functioning of discourse (i.e., Lotman's second function)—the type of discourse that is called for in current mathematics education reform recommendations—can be viewed as generators of meaning rather than as conveyors of a static message. Lotman (1988) delineates the operationalization of discourse that is more dialogic in nature:

> The second function of a text is to generate new meanings. In this respect a text ceases to be a passive link in conveying some constant information between input (sender) and output (receiver). Whereas in the first case a difference between the message at the input and that at the output of an information circuit can occur only as a result of a defect in the communications channel, and is to be attributed to the technical imperfections of this system, in the second case such a difference is the very essence of a text's function as a "thinking device."

> What from the first standpoint is a defect, from the second is a norm, and vice versa (p. 36).

In adopting a social constructivist perspective, advocates for the current mathematics reform movement, in effect call for discourse in mathematics classrooms that is more dialogic in nature as they encourage teachers to become more like facilitators and students to take an active role in the learning of mathematics. Indeed, the discourse embodied in visions—and in the enactment—of reform-based mathematics education are of a more democratic speech genre where teachers and students engage in discourse that reflects a give-and-take process where everyone in the classroom is seen as having a responsibility to contribute to discussions as they explore mathematical topics.

> Speech genres grounded primarily in the dialogic function of text assume that each voice will take the utterances of other as thinking devices. Instead of viewing others' utterances as static, untransformable packages of information to be received and perhaps "stored," they are viewed as providing one move in a form of negotiation and meaning generation. In general, the possibilities for voices to come into contact are much greater and much richer in the case for the dialogic function of text than for the univocal function. Instead of being viewed as containers of information to be transmitted, received and stored, utterances are viewed as open to challenge, interanimation and transformation (Wertsch & Smolka, 1995, p.80).

At the heart of reform-based mathematics instruction is the hope that the accompanying pedagogical approaches and strategies will lead to students acquiring a deep conceptual understanding of the mathematics being studied. In a similar fashion, the essence of dialogic discourse is to arrive at a true understanding of the topic being discussed. Again, this understanding is achieved as speaker and listener strive to negotiate the speech of one another so that it fits with and extends each individual's already existing knowledge.

> Thus each of the distinguishable significative elements of an utterance and the entire utterance as a whole entity are translated in our minds into another, active and responsive, context. Any true understanding is dialogic in nature. Understanding is to utterance as one line of a dialogue is to the next. Understanding strives to match the speaker's word with a counter word. Only in understanding a word in a foreign tongue is the attempt made to match it with the "same" word in one's own language (Voloshinov, 1973, p. 102).

As we have begun to use this theory of the functional dualism of discourse to frame our research on the nature of discourse and communication in school mathematics, we have come to realize that the distinction between univocal and dialogic discourse is at times difficult to discern. Indeed, in any social interaction (mediated through discourse) each individual must decipher text and generate his or her own meaning of that text. Hence, all discourse is to some degree both dialogic and univocal; in a sense, it is helpful to think of discourse being more or less dialogic or univocal in nature. As we have continued to apply the dimensions of our theoretical framework, we have found that most text, however, is characterized primarily by one of these functions. Wertsch & Smolka (1995) concur with this finding as they suggest that "speech genres can be distinguished into general categories on the basis of the extent to which the univocal function or the dialogic function is foregrounded" (p. 80). We have often looked to the speaker's intent—in employing a particular type of discourse to transfer meaning or generate new meaning—to determine which functioning was more prevalent. We also examined the listener's intent in making sense of classroom discourse in a similar fashion.

Conclusion and Future Directions

Based on our experience thus far in examining the dualistic nature of discourse in mathematics education, we recognize the need for students and teachers to engage in more dialogic discourse as they explore mathematics in their classrooms. This recognition comes from our grounding in the social aspect of learning and the expectation that students will acquire a deeper understanding of mathematics when they use their own utterances, as well as those of their peers and teacher, as thinking devices that are closely examined and adapted to their unique understandings of mathematics. We are, however, beginning to position ourselves in a fashion similar to Thompson (1995)—in which he challenges the widespread assumption that "exposition [a form of univocal discourse] is an unacceptable teaching method" (p. 123)—as we acknowledge the importance of both univocal and dialogic discourse and search for an appropriate balance between the two. In fact, this balance to some extent is unavoidable as almost all text contains aspects of both univocal and dialogic functioning (as we discussed above). Indeed, as we continue to focus on the functional dualism of text, we have concluded that all dialogic text must contain some univocal functioning in order for clear communication to take place. Most of the secondary mathematics classroom instruction that we have observed, however, was more univocal in nature and we see the need to assist teachers in fostering discourse that is more dialogic in nature.

With respect to the functional dualism framework that we have been using to guide our research, we are pleased with the structure and direction it has offered, the insights it has revealed, and the possibilities it offers for future research. Thus far, this framework has guided our exploration of the nature and role of discourse in the professional development of mathematics teachers and in these teachers' high-school mathematics classrooms. As we carefully listened to the voices of mathematics teachers and their students, we have recognized the functional dualism of text and observed first-hand how that text shaped the social interactions among the teacher(s) and students. Future research should continue to explore the social nature of this discourse and how individuals' goals, intents, and motivations come into contact and overlap in the realm of mathematics classrooms. A focus on voice, as we have theorized it using functional dualism, allows us to do precisely this since Bakhtin's conception of voice "is concerned with the broader issues of a speaking subject's perspective, conceptual horizon, intention, and world view" (Werstch, 1991, p. 51). And most importantly, as we continue to explore these broader social issues related to discourse in the context of mathematics education, we must always maintain a focus on how classroom discourse influences students' and teachers' understandings of mathematics.

References

Bakhtin, M. M. (1981). *The dialogic imagination: Four essays by M.M. Bakhtin*, ed. by Michael Holquist, trans. by Caryl Emerson and Michael Holquist. Austin: University of Texas Press.

Bakhtin, M. M. (1986). *Speech genres and other late essays*, ed. by C. Emerson and M. Holquist, trans. by V. W. McGee. Austin: University of Texas Press.

Bauersfeld, H. (1995). The structuring of the structures. In. L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 137-158). Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education, 28* (3), 258-277.

Davis, B. (1997). Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education, 28* (3), 355-376.

Hiebert, J. (1992). Reflection and communication: Cognitive considerations in school mathematics reform. *International Journal of Educational Research, 17* (edited by Walter G. Secada), 439-456. Oxford: Pergamon Press.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer. *American Educational Research Journal, 27*, 29 - 63.

Lotman, Y. M. (1988). Text within a text. *Soviet Psychology, 24* (3), 32-51.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

Peressini, D. & Knuth, E. (1998). Why are you talking when you could be listening? The role of discourse in the professional development of mathematics teachers. *Teaching and Teacher Education, 14* (1), 107-125. [Special Issue on Professional Development].

Pimm, D. (1987). *Speaking mathematically*. London: Routledge & Kegan Paul.

Thompson, P. W. (1995). Constructivism, cybernetics, and information processing: Implications for technologies of research on learning. In. L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 123-133). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Voigt, J. (1995). Thematic patterns of interaction and sociomathematical norms. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning* (pp. 163-201). Hillsdale, NJ: Lawrence Erlbaum Associates.

Voloshinov, V. N. (1973). *Marxism and the philosophy of language*, trans. by L. Matejka and I. R. Titunik. New York: Seminar Press.

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes* (M. Cole, V. George-Steiner, S. Scribner, & E. Souberman, Eds. and Trans.). Cambridge, MA: Harvard University Press.

Wertsch, J. V. & Smolka, A. L. B. (1995). Discourse and learning in the classroom: A sociocultural approach. In H. Daniels (Ed.), *Charting the agenda: Educational activity after Vygotsky* (pp. 69-92). New York: Routledge.

Wertsch, J. V. (1991). *Voices of the mind*. Cambridge, MA: Harvard University Press.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27* (4), 458-477.

EXAMINING TEACHER AND CLASSROOM SUPPORTS FOR STUDENT INVENTION

Elizabeth S. Senger
Auburn University, Auburn, AL
sengees@mail.auburn.edu

The case study of a rural third-grade teacher's classroom during mathematics instruction revealed students actively inventing their own procedures to solve problems, and the teacher using questioning as an instructional technique. Students applied original thought and deep reasoning to the problem situations presented by the teacher. Without access to the traditional multiplication algorithm, they used a variety of problem-solving strategies. A double focus of the analysis became necessary: (a) the kinds of strategies the children used, and (b) the number and types of questions the teacher asked, as well as the quality of her dialogues with the students. The findings included several environmental factors that encouraged student invention: persistent teacher questioning, fair-dealing, power-sharing, withholding traditional algorithms, and the use of contextually familiar problems. The teacher's comfort level in regard to the meaning of the mathematics content was found to contribute to the style and format of the lesson.

When teachers' beliefs about teaching mathematics begin to move from traditional to constructivist frames, they tend to experiment with new practices in their own classrooms (Thompson, 1992; Fennema, Franke, Carpenter, Levi, Jacobs, & Empson, 1996). Research on children inventing their own novel solutions to problems suggests students gain a deeper mathematical understanding and number sense (Fuson, Wearne, Hiebert, Human, Olivier, Carpenter, & Fennema, 1997; Clark and Kamii, 1996; Carpenter, Franke, Jacobs, Fennema, & Empson, 1998). Students who were allowed to construct their own mental representations (von Glasersfeld, 1988), developed mathematical understandings and connections not found in students who were in teaching-by-telling classrooms (Hiebert & Carpenter, 1992). Where children reflect on their own internal logical ordering, the classroom can be viewed as a social context in which mathematical knowledge is negotiated and constructed. In the absence of a known solution procedure, children have to apply a good deal of "original thought or deep reasoning" (Silver & Kilpatrick, 1989, p. 179). Teachers who have such expectations of students organize learning situations in order to capitalize on the different modes of thinking children bring to the mathematics classroom. This research examines the conditions which supported students' inventiveness, creative strategies, and informal mathematical knowledge in an elementary school classroom. The purpose of this paper was to examine the setting conducive to the students inventing their own strategies to solve problems from two perspectives: (a) the quantity and types of questions the teacher asked and (b) the manner in which the mathematics instruction was conducted.

Method

This paper uses case study methods to report on Ms. Carpenter, a 26-year-old third-grade teacher, and her efforts to build mathematical structures in learners in her classroom using discourse and problem solving. Thirty hours of observation and interview tapes from ten classroom mathematics lessons taught by Ms. Carpenter were collected, transcribed, and analyzed using systematic ethnographic methods (Gumperz, 1981; Lofland, 1984). Analyses produced using grounded methods were triangulated with the two teachers with dissentions discussed to the point of agreement.

Discussion

General Description of the Lessons

The ten lessons observed contained different mathematical topics with the content varying naturally as planned by the teacher and without interference from the researcher. All the lessons had a problem-solving aspect to them (with the exception of lesson 2) and began with a contextual story problem involving the students themselves and some goodie that they were to increase, exchange, multiply, and/or divide. After the whole-class introduction, students worked either in small groups, pairs, or individually on the problem as Ms. Carpenter moved around asking questions and challenging initial results. After several minutes, she would draw the whole class' attention to the overhead screen and accept solutions from around the room demanding that all students understand each other's answers, and asking the class if they agreed. Ms. Carpenter continually asked for "different" ways of solving the problems while involving the class members in each response with questions like, "Do you see what she did?" Even when student-invented procedures were long and tedious, such as 20 added 35 times, she would write down every number and go through each step with the child demonstrating it aloud to the others. In other lessons such as changing decimal numbers to fractions, she would give a contextual situation to assist the children's understanding, such as a certain number of cars in a parking lot of a hundred spaces. Ms. Carpenter displayed an intensity in getting her students to think through to reasonable conclusions, asking probing questions in succession until they did. Yet the atmosphere was comfortable with students choosing their own partners, where they wanted to work, their own materials, and how they wanted to solve the problems. There was a mutual respect with rarely a discipline problem.

Typical Student Inventions

Ms. Carpenter gave her students the following problem to solve: "There are 18 second-grade classes and 17 third-grade classes in our school. Some classes have 20 students, some classes have 21 students, and some classes have 22 students. No class has more than 22 children. If Ms. Lindsey wants to buy every student an ice cream cone, how many does Ms. Lindsey need?"

The class worked on the problem the entire mathematics period in a whole-class, small-group, whole-class format. Afterwards, she asked her students if the problem she gave them was hard. Most thought it was and gave the following reasons: (a) "there was too much to add," (b) "we don't know how to multiply two digits times two digits yet," and (c) "we don't know exactly how many students there were in each class."

Although most saw the problem as difficult, most didn't seem to have difficulty dealing with the intensive quantity definition of multiplication where the situation deals with a quantity within a quantity. Mildred became an exception to this when she initially added 17+18+20+21+22, but her group convinced her that 96 would not be enough ice cream cones for the whole school. Although Mildred may not have understood the intensive quantity definition of multiplication embedded in the problem (number of classes times the number of students in each class), the highly contextual nature of the problem as well as the social nature of the task led her to think informally about the problem differently. Tina found 22, 35 times to be too difficult, so relying on her concept of commutativity, "just turned it around" to 35, 22 times and added. Ms. Carpenter, however was not satisfied with this and asked the class how the problem could be worked on a practical level that way. The students' original thinking included:

1. *Incorrect Additive-Thinking Strategy* –student added the numerals depicted in the problem without regard to intensity or function (Mildred).
2. *Counting-By Strategy* –(a) student counted by ones through a diagram (picture, concrete aid) the (other factor) number of times, (b) student counted by a number other than one through a diagram the other factor number of times (Bethany's Group).
3. *Repeated-Addition Strategy* –student made an addition problem of one factor added the other factor number of times (class majority).
4. *Running-Total Strategy* –student used repeated addition strategy in a binary method–adding two numbers, getting a sum, adding a third, getting a sum, etc. till the number of one factor was added the other factor number of times (Dean).
5. *Commutative Strategy* –student reversed the factors in order to make a simpler/shorter repeated addition problem (Tina).
6. *Distributive Strategy* –(a) student split one factor into 10's and 1's, multiplied each by the other factor, and added the two products (Ben), (b) student split one factor according to the context of the problem (Carol), (c) student split the factor most conducive to simplifying the problem.
7. *Incorrect Algorithmic-Thinking Strategy* –the student manipulated symbols to generate the traditional algorithm without knowledge of it, and without regard to the underlying structure of the problem

(Janet). From the richness of the students' thinking, I analyzed the data further to discover factors conducive to student invention.

Results

Questioning as a Teaching Method

Although teachers generally use questioning throughout their lessons, Ms. Carpenter's use of this manner of discourse extended beyond the traditional in both the quantity and quality of the questions asked. The number of questions from the ten transcribed video tapes were counted, labeled by type, sorted and tabulated (See Table 1). The types of questions Ms. Carpenter asked emerged from the data into categories by function. They included: (a) *reasoning* questions where the objective was thinking and coming to logical conclusions ("The problem was 80, eight times, can anybody tell me where she got 50 and 30?"), (b) *clarification* questions ("What do we know?"), (c) *computation* questions where the teacher begins an equation and expects the class to fill in the answer ("8x8 is—?"), (d) *management* questions ("Do we have enough beans for all the groups?"), (e) *redirect* questions or ways to lead the thinking in a different direction ("If this rod is a dollar, how can this cube be a dollar?"), (f) *sharing* questions ("Amy, how did you do it?")

Conclusion

A constructivist and phenomenological classroom atmosphere appeared salient and integral to the student invention episodes. Such an environment included several characteristics.

1. Ms. Carpenter persistently questioned and probed student responses to find their levels of understanding, not just of procedures, but of the meaning of the operations and the children's number sense.
2. Her fair-dealing manner in accepting all correct solutions equally and recognizing all solutions as products of student thought earned her the respect of her students.
3. Her shared power and authority with the students in regard to the content and flow of the lesson gave students the academic freedom to pursue their own questions as well as hers.
4. She was extremely honest in her reactions to the student's responses offering counter examples to incorrect responses and accepting all in a matter-of-fact manner without hyperbole.
5. She considered herself a learner too.
6. Ms. Carpenter withheld the traditional symbolic algorithm until the children could make sense of the ideas using various other forms of representation.
7. She provided stories to give a familiar context to the mathematics.

Table 1. Types of Questions Asked

	Ob.1	Ob.2	Ob.3	Ob.4	Ob.5	Ob.6	Ob.7	Ob.8	Ob.9	Ob. 10	Total
I. Reasoning	31%	11%	40%	30%	27%	20%	31%	29%	38%	20%	28%
II. Clarifying	17%	43%	16%	33%	32%	47%	35%	30%	26%	42%	32%
III. Computation	10%	28%	01%	00%	03%	07%	02%	05%	04%	09%	06%
IV. Management	03%	07%	05%	04%	03%	01%	01%	07%	01%	08%	04%
V. Redirect	16%	11%	17%	26%	17%	17%	14%	11%	08%	07%	14%
VI. Sharing	23%	00%	21%	07%	18%	08%	17%	18%	23%	14%	16%
Total	100	100	100	100	100	100	100	100	100	100	100

8. Ms. Carpenter emphasized children's reasoning and thinking by building on the children's own number sense and probing for differences in students' thought processes.

How did Ms Carpenter get to be the teacher described here? The data reveal not only direct insights, but provide grounds for building theory (Strauss & Corbin, 1990). Based on the data and analysis, I discovered a triangular or three way "pull" on Ms. Carpenter's mathematics teaching.

1. She had a *passion* for understanding the unique ways the students "invent" the content by coming up with their own ways of dealing with numbers.
2. She exhibited a *risk-taking attitude within the discomfort she experienced* with her own understanding level of the content.
3. She continually built her own *self-confidence,* first from outside support and progressively from her past successes with interactive classroom discussions at a deep understanding level of mathematics.

These findings raise further questions for teachers, researchers, and everyone interested in the progress of mathematics education. Such case studies push our own thinking beyond assumptions and "easy solutions" to the complexity and realities of active learning communities.

References

Carpenter, T. P., Franke, M. L., Jacobs, V., Fennema, E., & Empson, S. (1998). A longitudinal study of invention and understanding in children's multi-digit addition and subtraction. *Journal for Research in Mathematics Education, 29*(1), 3-20.

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education, 27*(1), 41-51.

Fennema, E., Franke, M. L., Carpenter, T. P., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*(4), 403-434.

Fuson, K., Wearne, D., Hiebert, J., Human, P., Murray, H., Olivier, A., Carpenter, T.P., & Fennema, E. (1997). Children's conceptual structures for multi-digit numbers and methods of multi digit addition and subtraction. *Journal for Research in Mathematics Education, 28,* 130-162.

Gumperz, J. (1981). Conversational inference and classroom learning. In J. Green & C. Wallat (Eds.), *Ethnography and language in educational settings* (pp. 3-23). Norwood, NJ: Ablex.

Hiebert, J., & Carpenter, T. P. (1992). Learning mathematics with understanding. In Grouws, D. (Ed.), *Handbook of research on mathematics teaching and learning*. New York: Macmillan, 65-97.

Lofland, J., & Lofland, L. (1984). *Analyzing social settings.* Belmont, CA:Wadsworth Publishing.

Silver, E. A., & Kilpatrick, J. (1989). Testing mathematical problem solving. In R. Charles & E. Silver (Eds.), *Teaching and assessing mathematical problem solving* (pp. 178-186). Hillsdale, NJ: Lawrence Erlbaum.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques.* Newbury Park, CA: Sage.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In Grouws, D. (Ed.). (1992), *Handbook of research on mathematics teaching and learning*. New York: Macmillan, 127-146.

von Glasersfeld, E. (1988). *The construction of knowledge*. Salinas, CA: Intersystems Publications.

TRACING CHILDREN'S CONSTRUCTION OF FRACTIONAL EQUIVALENCE

Elena P. Steencken
Rutgers University
steencke@rci.rutgers.edu

Carolyn A. Maher
Rutgers University
cmaher@rci.rutgers.edu

Algorithms associated with operations involving fractions usually become a large part of the mathematics curriculum in grades 5 and 6. Our interest is to explore the extent students can discover fraction ideas before they are formally introduced in schools. This report is a portion of a study centering on a one-year teaching experiment with 24 fourth grade children and focusing on one student, Meredith, and her "discovery" of equivalent fractions. She had built Cuisenaire rod models to compare fractions, discovered the equivalent idea, and reported her findings. This paper will also report on the ensuing discussion by the class on what Meredith had found.

Objective

The purpose of this paper is twofold: (1) to identify the origin of the idea of equivalent fractions among children engaged in investigations exploring fractions, and (2) to describe the conversations of other classmates as they considered the idea of equivalent fractions.

Theoretical Framework

Difficulties children have in attaining a clear meaning and understanding of ideas about fractions have been documented extensively (Streefland 1991; Behr, Lesh, Post, Silver, 1983; Alston, Davis, Maher, Martino, 1994). Children who were not taught rules, but rather invented their own 'clever calculations', demonstrated, according to Streefland (1991), greater understanding of mathematical concepts. Watson, Campbell and Collis (1993) investigated common fraction problems in students from kindergarten to grade 10. They suggested that students, taught through a symbolic approach, with ikonic experiences (use of images, reality, experience) are limited in their understanding. Maher (1998) indicated that learning is derived from, and builds upon, prior experiences and that students' mental images generated by these experiences can be utilized later in building powerful representations of mathematical ideas.

Certain conditions seem necessary to promote students' deep understanding of mathematical ideas. Davis (1997) included among these conditions the establishment of alternative learning environments in which children become active participants in experiential learning. Maher (1998) included conditions that develop a culture in which students are expected to support and represent their ideas, and discuss the ideas of others - conditions that nurture the exchange of ideas. It is therefore useful to try to identify children's representations and trace the movement of mathematical ideas within a classroom by carefully attending to student discourse.

Cobb, Boufi, McClain, and Whitenack (1997) give theoretical importance to *mathematical* or *reflective discourse*, suggesting a relationship between classroom discussions and children's mathematical development. When children listen carefully to their classmates, they have the opportunity to question, amend, validate or reject the ideas of others. Similarly, our listening carefully to children's conversations can offer insights into the mathematical representations that children build. In this report, we follow the mathematical ideas introduced by a particular child, Meredith, into the classroom community.

Background

A Rutgers University teacher-development partnership with a suburban New Jersey K-4 school district provided ongoing collaboration between researchers and teachers[1] (Maher, Martino, Davis, 1994). This alliance included mathematical explorations by the teachers, examination of student thinking, and pursuit of alternative pedagogical approaches to support children's serious engagement in doing mathematics. Teachers were involved in bimonthly workshops and summer institutes; they invited teacher/researchers into their classrooms to work with students. One outgrowth of this partnership was a one-year teaching experiment in a fourth grade classroom.

Setting

Twenty-four heterogeneously grouped fourth graders participated in mathematical investigations involving fraction ideas. Each of the 50 sessions was scheduled for extended time - ranging from 60 to 80 minutes. A teacher/researcher [T/R] led the sessions. Another researcher [R] and the classroom teacher [CT] acted as observers and passive participants in the sessions: they were instructed to refrain from imposing their ideas on students. However, they were encouraged to provide feedback from their observations and discussions with the children. Classroom organization was the responsibility of the teacher, who usually seated the students in pairs. The teacher/researcher began each session by posing a problem task, or continuing to explore a previously posed task - asking students to explain and describe their progress or difficulties. After a few sessions, children's comfort levels were apparent in their eagerness to discuss their ideas.

[1]This work was supported by grant MDR 9053597 from the National Science Foundation and by grant 93-992022-8001 from the NJ Department of Higher Education. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation or the NJ Department of Higher Education.

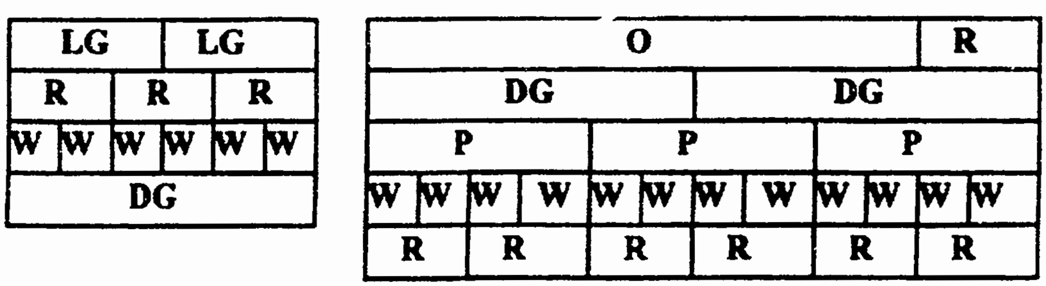
Meredith, whose discovery is reported in this paper, wrote in her journal (9/21/93), "Rutgers really worked us hard because every time someone came up with an answer Rutgers would say: 'convince us'. So we did."

Analysis

Data for this report came from the seventh classroom session, October 4, 1993, which was videotaped using 3 cameras. The research team studied videotape data as the basis for planning subsequent sessions. The T/R role was designed to elicit student response. Format for analysis is described in greater detail in Maher and Martino (1996) and in Maher, Pantozzi, Martino, Steencken & Deming (1996a). Our intent was to identify those events that indicated insight into the development of ideas, interesting wrong leaps, or particular obstacles seeming to interfere with progress in the development of an idea. This report will focus on a particular critical event, Meredith's "discovery" of equivalent fractions. Once the event was identified, three strands of inquiry were explored: (1) The nature of Meredith's representation and its development; (2) how Meredith's idea was received by the community; and (3) how her idea traveled after it was shared.

Results: Meredith's Discovery

Meredith and her partner responded to the task of solving a fraction comparison problem, "Which is bigger, one half or two thirds and by how much? by building identical Cuisenaire rod models (See Model 1 in Figure 1). Challenged by T/R, they explored whether other rod models of different lengths would support their claim that two thirds is bigger than one half



Model 1 **Model 2**

Key: The models above refer to Cuisenaire rods.
Each symbol refers to color and length.
W = white (1 cm), R = red (2 cm), LG = light green (3 cm)
P = purple (4 cm), DG = dark green (6 cm), O = orange (10 cm)

Figure 1. *Cuisenaire™ Rod Models*

by one sixth. Meredith built Model 2 (See Figure 1). In reporting her findings to R, Meredith was asked to explain why both solutions worked. In doing so, she developed an argument to justify the equivalent relationship.

Referring to the models she built, Meredith explained that two thirds is bigger than one half by two twelfths (Model 2) and that two thirds is bigger than one half by one sixth (Model 1). Questioning Meredith about her results, R probed explicitly for other number names.

R: That's interesting. Could we call the difference between the two thirds and the one half in this model [Model 2] another number name besides two twelfths?

Meredith: Um, yeah, well, maybe...

R: You said two of those little white ones were two twelfths, right?

Meredith responded by placing 6 red rods below the train of 12 white rods and then a red rod above two of the train of 12 white rods.

Meredith: Yeah, and maybe since two of these little white ones equals up to one of these [She puts 1 red rod on top of 2 white rods in the train, showing that 1 red rod is the same length as a train of 2 white rods.] Or it's one fifth, oh, I mean one sixth.

R: Okay. That's interesting, that's kind of interesting. So if you then used the reds to describe the difference, you can call this one sixth, the difference.

Meredith: Uh humm.

R: And over here [Model 1] one of the whites you say is one sixth?

Meredith: Yeah.

Meredith's Explanation Is Challenged

During a class discussion, Meredith joined 3 girls at the overhead projector. The girls' had built two models similar to those in Figure 1. The girls' models differed in that their rods were not in the same row order and model 2 did not include the 12 white rods. Meredith began by placing 12 white rods beneath the red rods of the girls' model. Michael objected.

Michael: No, they can't do that [Rising from his seat] because the, the two thirds are bigger than a half by a red. So they can't use those whites to show it.

The teacher asked Michael if he had found two thirds to be bigger than one half. He responded affirmatively and was then asked how much bigger.

Michael: [Impatiently] By one sixth.

Meredith: Or, or two twelfths.

Michael: [Shaking his head in disagreement] No.

Other children echoed Michael and responded "no". The teacher then asked

Meredith for her answer from Model 2. Meredith reaffirmed her original response of two twelfths and again some classmates disagreed. Michael voiced his objection, with Eric's support.

Michael: Yeah, but then she would have to call the two whites together one sixth.

Eric: Yeah, exactly.

Michael: She's calling the whites - one white, one sixth.

Eric: Yeah, she said.

T: She's calling one white one sixth?

Meredith: No, I'm not, I'm calling it one twelfth.

T: She's calling one white one twelfth.

Eric: Yeah, but see just the two whites together. That's right, it would be two twelfths. But you have to combine them. You can't call them, you can call them separately, but you can also call them combined and if you combine them it would be a, a, one sixth.

Michael shook his head in dissent. A student suggested that there are two answers.

Michael: [Simultaneously with Eric] No, they're the *same* [emphasis in tone] answer.

Eric: No, they're the exact same thing, except she, she took the red and divided it into half, she divided it into halves, into half and called, and called each half one twelfth. They're the exact same answer except they're just in two parts.

Eric's Response to Meredith's Idea

On the overhead projector, Meredith built a model with one red rod as the base and placed two white rods directly above it.

Eric: And she's calling a white rod one twelfth and the other white rod one twelfth and the red is really one sixth. Well, when she calls them two twelfths, the two twelfths are actually just two white rods put together to equal a red, so it should be really, it's really one sixth. Because two whites, two whites...and it's one sixth, it's one sixth.

T: All those things, are they true?

Eric: Yeah. But I don't really think you could call, call them two twelfths because two twelfths equal exactly to the same size as one sixth. Well, if you *want to* [emphasis in tone] you could call them, I guess. But I think it would be easier just to call them one sixth cause you wouldn't want to exactly call them one twelfth and another twelfth. I'd just call them one sixth. Therefore I think you just really call them one sixth.

Conclusions/Implications

Meredith's discovery of equivalent fractions contributes to a growing body of knowledge that shows children can do important mathematics in very natural ways. For this to happen, they need the freedom to explore and to talk about their ideas (Kamii, 1985; Maher and Martino, 1996). Meredith's discovery was openly discussed and debated in the classroom, and Meredith had the opportunity to respond to the challenges posed and support her ideas. Creating classroom conditions that promote opportunities for students to build meaningful mathematics is a serious challenge.

References

Alston, A. S., Davis, R. B., Maher, C. A., & Martino, A. M. (1994). Children's use of alternative structures. *Proceedings of Psychology of Mathematics Education XVIII*, (2), 248-255. Lisbon, Portugal.

Behr, M. J., Lesh, R., Post, T. R., Silver, E. A. (1983). In R. Lesh and M. Landau (Eds.), *Acquisition of mathematical concepts and processes.* New York: Academic Press Inc.

Cobb, P., Boufi, A., McClain, K., Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education*, 28(3), 258-277.

Davis, R. B. (1997). Alternative learning environments. *Journal of Mathematical Behavior*, 16 (2), 87-93.

Kamii, C. K. (1985). *Young children reinvent arithmetic* (pp.26-36). New York: Teachers College Press.

Maher, C. A. (1998). Constructivism and constructivist teaching - can they co-exist? In O. Bjorkqvist, Ed., *Mathematics teaching from a constructivist point of view. Reports from the faculty of education, no.3* (pp.29-42). Finland: Abo Akademi University.

Maher, C. A., Martino, A. M. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education*, 27 (2), 194-214.

Maher, C. A., Pantozzi, R., Martino, A. M., Steencken, E. P., Deming, L. S. (1996a). Analyzing students' personal histories: Foundation of mathematical ideas. *Paper presented to the American Education Research Association*, New York.

Maher, C. A., Martino, A. M., Davis, R. B. (1994). Children's different ways of thinking about fractions. *Proceedings of Psychology of Mathematics Education XVIII*, (3), 208-215. Lisbon, Portugal.

Streefland, L. (1991). Fractions, and integrated approach. In L. Streefland, (Ed.), *Realistic mathematics education in primary school.* (pp.93-118). The Netherlands: Freudenthal Institute.

Watson, J. M., Campbell, K. J., Collis, K. F. (1993). Multimodal functioning in understanding fractions. *Journal of Mathematical Behavior*, 12(1), 45-62.

Discourse Short Orals

ANALYSIS OF A COMPUTER-MEDIATED DIALOGIC DISCOURSE ON AN OPEN-ENDED TASK

Sergei Abramovich
State University of New York at Potsdam
abramovs@potsdam.edu

This paper is concerned with the analysis of a computer-mediated dialogic discourse (i.e., a discourse based on a direct interaction of two or more individuals in a computer environment) on an open-ended task in a classroom of elementary mathematics teachers. This type of classroom interaction was unique in the sense that from the outset, a solution (answer) to the task was not known to an instructor. Rather, a solution was gradually constructed through a group collective-comprehension activity. How does the scaffolding of such activity by an instructor take place?

The paper suggests that using the midwife metaphor (Pólva, 1981) helps to extend the notion of scaffolding to an open-ended task in which expertise in providing qualified assistance to a learner does not presuppose an a priori knowledge of an answer (solution). In such intellectual milieu, the vertical relationship between an expert and novice makes a half-turn about a task as the former assumes a new role becoming a partner in advancement for the latter. This shift in emphasis of construing a teacher's role in problem solving discourse appears to be a crucial component of any expertise required to provide such advanced scaffolding process. As the paper demonstrates, it includes the teacher's ability to control frustration from unfit guessing and support promising avenues of cognitive efforts as students are encouraged to analyze, comment, and defend each other's utterances that structure a dialogic discourse on a task with an unknown answer.

In particular such discourse occurred during the author's work with elementary teachers in a computerized setting in which interactive electronic tables were used as mediational means in search of a solution to the following problem: *How many subgrids of an n-cells grid constitute its whole number percentage part, provided that shape and location of a subgrid is not important?* This discourse was transcribed and then the utterances of the participants were analyzed. The analysis shows that whereas inductive and demonstrative phases of a solution appear to be within the teachers' zone of proximal development, the scaffolding of these phases varies significantly in terms of the level of the assistance required.

Reference

Pólya, G. (1981). *Mathematical discovery: On understanding, learning, and teaching problem solving* (Combined edition). New York: Wiley.

THE NATURE OF MATHEMATICAL DISCOURSE IN A PROSPECTIVE TEACHER'S CLASSROOM

Maria L. Blanton
University of Massachusetts
Dartmouth
mblanton@umassd.edu

Sarah B. Berenson
North Carolina State University
berenson@unity.ncsu.edu

This investigation explores the nature of mathematical discourse in a prospective middle school mathematics teacher's classroom as a window into her construction of pedagogical content knowledge. Vygotsky's sociocultural perspective provides a theoretical framework for the place of discourse in an individual's development. He maintained that one's higher mental functioning originates socially, between people, and is mediated through psychological tools such as language. It was our premise that, as the prospective teacher engages in the activity of teaching, he or she is also subject to this social formation of mind.

Data were collected during Mary Ann's (pseudonym) student teaching semester through weekly classroom observations and teaching episode interviews. Written artifacts (e.g., journal reflections, lesson plans) were collected as well. Classroom discourse data were analyzed for patterns (see e.g., Wood, 1995) as well as a univocal or dialogic function of speakers' utterances (see, e.g., Lotman, 1988). Results indicate that early classroom interactions mediated Mary Ann's languaging toward a more univocal paradigm of giving information as she funneled students' thinking. Her subsequent efforts to promote dialogic interactions with students generated a tension that positioned students as mediators of her practice. This underscores the need to extensively guide prospective teachers during their field experiences. Moreover, the primitive nature of early classroom interactions suggests that prospective teachers may need to explore ways to cultivate meaningful discourse in undergraduate settings prior to student teaching.

References

Lotman, Y. (1988). Text within a text. *Soviet Psychology 26*(3), 32-51.

Wood, T. (1995). An emerging practice of teaching. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 203-277). Hillsdale, NJ: Lawrence Erlbaum.

THE EFFECT OF A REFORM-BASED CURRICULUM ON CLASSROOM NORMS[1]

Jon D. Davis and Mary Shafer
University of Wisconsin-Madison
jondavis@students.wisc.edu, mcshafer@vms2.macc.wisc.edu

The purpose of this paper is to compare and contrast the classroom norms of teachers who are implementing a reform-based middle school curriculum, *Mathematics in Context* (MiC), with teachers who are using conventional curricula. The main source of data for this paper comes from an observation scale used in a longitudinal/cross-sectional study of the impact of MiC currently in progress.

The observation scale consists of twelve items in two categories: classroom events (e.g., conceptual understanding of mathematical ideas) and pupil pursuits (e.g., substantive conversation between students in class). Each item is numerically ranked on a scale of 1 – 4 or 1 – 3 and is supported by evidence consisting of dialogue or other information from the observed classroom. For example, on the item measuring students' explanation of solution strategies, a rating of "1" denotes a classroom where students rarely discuss their solutions to problems whereas a rating of "3" describes a classroom in which students are frequently justifying their approach to a problem and explaining their thinking.

Significant differences (on an independent samples t-test) were found between MiC teachers and Non-MiC teachers on 8 out of 12 items. The results, however, indicate room for improvement for both groups of teachers. Significant differences were also found between the average number of years of teaching experience of MiC teachers (11.21 years) and Non-MiC teachers (6.4 years). Hence, the curriculum as well as teacher experience appear to affect classroom norms.

[1] This research is supported in part by the National Science Foundation #REC-9553889. The views expressed here are those of the authors and do not necessarily reflect the views of the funding agency.

JUSTIFICATIONS, ARGUMENTATIONS, AND SENSE MAKING IN PRESERVICE ELEMENTARY TEACHERS IN A CONSTRUCTIVIST CLASSROOM

Anne Larson Quinn
Edinboro University of PA
quinna@edinboro.edu

Quinn (1997) presents a detailed case study of how preservice elementary teachers made sense of and justified their mathematical ideas before, during, and after a four-month-long constructivist course. Data included pre- and post-interviews with 13 preservice teachers in (a) three content areas (probability, number theory, and geometry), (b) small and large group discussions during the course, and (c) individual journal reactions. Over 800 arguments were analyzed.

Quantitative research showed that the preservice teachers' average level of reasoning increased from the pre- to the post-interview: from 1.96 to 3.90 in probability, from 2.77 to 3.56 in number theory, and from 2.50 to 3.77 in geometry. The teachers went from being convinced by a few examples to being able to make a variety of sophisticated arguments, including theoretical discussions of probabilities, informal versions of mathematical induction and proof by cases, and deductive geometric arguments. Qualitative research demonstrated the norms and practices that were established during this course. Teachers came to value and construct proofs, stopped relying on external authority, intertwined their justifications with their explanations, internalized the roles of an idea generator and an idea evaluator, expected others to also conform to the norms that were being established, and developed new beliefs about the nature of mathematics.

References

Quinn, A. (1997). *Justifications, argumentations, and sense making of preservice elementary teachers in a constructivist mathematics classroom.* Unpublished doctoral dissertation. Kent State University, Kent, OH.

Discourse Posters

EXAMINING ZONES OF DISCOURSE IN THE PROSPECTIVE MATHEMATICS TEACHERS' CLASSROOM

Maria L. Blanton
University of Massachusetts
Dartmouth
mblanton@umassd.edu

Susan L. Westbrook
North Carolina State University
westbrk@poe.coe.ncsu.edu

In mathematics teacher education, interest in discourse has grown beyond the practices of experienced classroom teachers to include that of prospective teachers (Blanton, Berenson, & Norwood, 1998). In this study, we have invoked Valsiner's zone theory (1987) to examine the learning environment that two prospective mathematics teachers established through classroom discourse. Valsiner expanded Vygotsky's notion of the zone of proximal development to include two additional zones of interaction—the zone of free movement (ZFM) and the zone of promoted action (ZPA). As structures through which the adult constrains or promotes the child's thinking and acting, the ZFM and ZPA interactively generate the environment in which the child develops. Westbrook, Carter, and Smith (1997) have suggested the use of this multiple zone theory to describe the complex interactions that occur in the student teacher's development.

We have considered what mathematical thinking the student teacher promotes through discourse and how discourse patterns constrain the development of students' mathematical thinking. Qualitative analysis of classroom discourse data suggested that a student teacher's actions of funneling students' thinking through leading questions was reflected in a restricted ZFM and a ZPA organized around the teacher's thinking. Such patterns of discourse appeared to promote verbalizations that provided the illusion of sense-making, yet established cognitive boundaries in the classroom.

References

Blanton, M., Berenson, S., & Norwood, K. (1998). *Mathematical discourse in a prospective teacher's classroom: The case of a developing practice.* Manuscript submitted for publication.

Valsiner, J. (1987). *Culture and the development of children's actions: A cultural-historical theory of developmental psychology.* New York: John Wiley & Sons.

Westbrook, S., Carter, G., & Smith, R. (1997). *When cultures collide: Teaching in someone else's classroom.* Manuscript submitted for publication.

THE ROLE OF DISCOURSE: AN ANALYSIS OF THE PROBLEM SOLVING BEHAVIORS AND ACHIEVEMENT OF STUDENTS IN COOPERATIVE LEARNING GROUPS

Patrick Mwerinde
University of Delaware
mwerinde@math.udel.edu

Christine L. Ebert
University of Delaware
cebert@math.udel.edu

Cooperative-learning strategies have been credited with the promotion of critical thinking, higher-level thinking, and improved problem-solving ability of students. Current research that examines behaviors that occur during group problem-solving sessions seems to indicate that groups engage in behaviors that are similar to these exhibited by expert mathematicians when they solve problems (Artz and Newman, 1990; Schoenfeld, 1987): that is, they engage in monitoring their own thoughts, the thoughts of their peers, and the status of the problem-solving process. Researchers who have studied cooperative learning at the college level generally have found that students learn just as well as in more traditional classes and often develop improved attitudes toward each other and mathematics.

In this study, we not only examined that problem-solving behaviors, strategies, *and* achievement of college students assigned to cooperative learning groups, but particularly focused on the nature of the discourse associated with the particular problem-solving behaviors of persistence and a willingness to explore alternative strategies. The subjects chosen for this study consisted of 108 students enrolled in four instructional units of College Algebra and Statistics at a major state university. Two control groups and two experimental groups were randomly selected. Throughout the semester, problem-solving behaviors, strategies, and achievement were assessed through four tasks which focused on the connections between the mathematical actions and processes and the mathematical concepts. Each of the four tasks were videotaped and audiotaped.

The results indicate that those students in the cooperative learning groups engaged in a type of mathematical discourse that would allow them to form connections between graphical and algebraic representations. In particular they were significantly more willing than their control-group counterparts to continue the discussion until they could 1) determine the graphical feature that indicated when the two objects would be worth $0 and 2) why determine setting the equations equal to zero was related to determining this graphical feature. Specifically the transcripts reveal persistence and a willingness to listen to all proposed solutions. These findings provide convergent evidence concerning the nature of discourse related to important problem-solving behaviors of persistence and a willingness to explore alternative solutions as well as suggesting the types of group activities which may facilitate higher-level thinking and improved problem-solving ability.

References

Artz, A.F., and Newman, C.M. (1990). *How to use cooperative learning in the mathematics classroom*. Reston, VA: National Council of Teachers of Mathematics.

Schoenfeld, A.H. (1987). What's all the fuss about metacognition? In A. Schoenfeld (Ed.), *Cognitive science and mathematics education.* Hillsdale, NJ: Lawrence Erlbaum.

EXPLANATION AND DISCOURSE IN 9TH GRADE MATHEMATICS CLASSES

Elaine Simmt
University of Alberta
Elaine.Simmt@UAlberta.ca

Florence Glanfield
University of Saskatchewan
glanfiel@gpu.srv.ualberta.ca

The Teaching Practices Project involved eight case researchers observing and interviewing 15 different grade nine mathematics teachers who taught in schools that have a history of strong student performance on provincial achievement tests. The researchers found that rather than simply focusing on skill development or on problem solving or on concept development, many of the teachers in this study were adept at balancing these factors in their instruction. In particular, one of the striking features of the instruction was that explanations were used as a tool for facilitating mathematical discourse; but at the same time, it was observed that classroom discourse facilitated the development of mathematical explanations. While an explanation may be the "heart of any teaching episode" (Leinhardt, 1988) many researchers have demonstrated that discourse is at the heart of learning. In some of the classes observed teachers did most of the explaining, while in other classes students did most of the explaining but in both situations students were observed to be actively participating in the discourse. It was evident, in the classes we observed, that teachers were very good at providing tasks which encouraged both discourse and explanation. In this poster we offer illustrations, taken from classroom observations, of the recursive way in which explanations facilitate mathematical discourse and mathematical discourse facilitates mathematical explanations.

References

Leinhardt, G. (1988). Expertise in instructional lessons: an example from fractions. In D. Grouws, T. Cooney and D. Jones (Eds.), *Effective mathematics teaching*. Reston, VA: Lawrence Erlbaum Associates, NCTM.

Simmt, E. (1998). *The teaching practices project: Toward an understanding of teaching practices in mathematics classrooms in Alberta schools that have a history of exceeding provincial expectations on grade 9 achievement tests*. (Final project report). Edmonton: University of Alberta.

CHANGES IN CLASSROOM DISCOURSE: SNAPSHOTS OF TWO TEACHERS

Dorothy Y. White
University of Georgia
dwhite@coe.uga.edu

The role of communication in developing childrens' mathematical thinking has gained considerable attention in recent years (National Council of Teachers of Mathematics, 1996). Mathematics educators maintain that classroom discourse centered around mathematical reasoning and sense-making allows teachers to stimulate students' thinking and reflect on students' understanding. By actively listening to students' ideas and suggestions, teachers show that they value students' contributions (Davis, 1997).

The purpose of this study was to examine how two third-grade teachers transformed their classroom practices as a result of their involvement in a teacher enhancement project. In particular, the study examined how the teachers used questioning and listening techniques as a means of exploring their students' mathematical knowledge and engaging them in mathematics lessons.

Qualitative data were collected from audio-taped classroom observations before and during the teacher enhancement, teachers' journals, and semi-structured interviews. The results from the study showed that both teachers changed the focus of their discourse from the correct answer to how students arrived at their answers. However, the degree of change varied among the two teachers. The study also showed that asking challenging questions and listening to students' answers and solution strategies are not enough to bring about significant change in classroom discourse and opportunities for students to learn mathematics. Teachers must also interpret students' responses as indicators of their levels of understanding and adjust instruction accordingly.

References

Davis, B. (1997). Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education, 28*, 355-376.

National Council of Teachers of Mathematics. (1996). *Communication in mathematics: K-12 and beyond.* Reston, VA: National Council of Teachers of Mathematics.

EPISTEMOLOGY SHORT ORALS

EPISTEMOLOGICAL AND COGNITIVE ASPECTS OF THE LINK BETWEEN THE CONCEPTUAL AND THE ALGORITHMIC IN THE TEACHING OF INTEGRAL CALCULUS

Germán Muñoz-Ortega and Francisco Cordero-Osorio
Cinvestav-IPN. Conacyt. México
gmunoz@mail.cinvestav.mx

The research problem we are going to deal with consists in analyzing the separation between the conceptual and the algorithmic aspects in the teaching of integral calculus. That is, the students are taught procedures to calculate integrals using integration methods only through drill exercises, and in a way separate from the conceptual part. It is only when they see the "applications", they study some notions related to integration. What are the causes of this? Is this rupture provoked by the teaching in which the student is involved? Does this rupture exist in the construction of the knowledge? We have an evidence for the relationship between the conceptual and the algorithmic in the construction of knowledge: There is a very close relationship between the notion of Prediction and the instrument of prediction, Taylor's series (Cantoral, 1990). So the question is, how to generate the link between the Conceptual and the Algorithmic in the teaching of integral calculus. For that, we analyze the following: a)The way with which we teach in order to generate the link. For that, we need to identify the cognitive mechanisms that act in the relationship between the conceptual and the algorithmic. b)The nature of the knowledge of teaching in order to generate the link. For instance, the notions of Prediction, Accumulation and making constant of the variable will play a main role.

One of the results of our research is: In order to create the link between the conceptual and the algorithmic in the teaching of integral calculus, we need, as a condition, to start a discussion about integration with a problem that allows thinking about integration. What are this type of problems? After reviewing some historical, epistemological and cognitive studies (Cantoral, 1990; Cordero, 1994; Piaget y García, 1994) we determined, in a certain manner, the types of problems that allow thinking about integration; summarizing: They are the specific problems that are derived from the variation or change phenomena. These specific problems don't refer to the causes of the variation phenomenon (why they vary), but how much they vary once we know how the phenomenon (dF(t)/dt) varies. That is, we ask questions about the law that quantifies (the unknown quantity F(t) that has a functional relationship with the variables in question) the phenomenon of variation. The configuration of this law depends on whether the initial conditions of the specific problem is given or not. Where is our research directed toward? To identify the genesis of the notions and rules that were mentioned before, when the student confronts with a problem situation. The method of clinical interviews will be used. All this, with the objective of identifying the cognitive mechanisms that act in the relationship between the conceptual and the algorithmic in integral calculus.

References

Cantoral, R. (1990). *Desequilibrio y equilibración. Categorías relativas a la apropiación de una base de significaciones propias del pensamiento físico para conceptos y procesos matemáticos de la teoría elemental de las funciones analíticas.* Tesis doctoral, Cinvestav-IPN, Sección de Matemática Educativa.

Cordero, F. (1994). *Cognición de la integral y la construcción de sus significados: Un estudio del discurso matemático escolar.* Tesis Doctoral, Cinvestav-IPN, Departamento de Matemática Educativa.

Muñoz, G. (1996). *Elementos de enlace entre lo conceptual y lo algorítmico, en el cálculo integral.* Tesis de Maestría en Ciencias, Cinvestav-IPN, Departamento de Matemática Educativa.

Piaget, J. & García R. (1994). *Psicogénesis e historia de la ciencia.* Siglo XXI, México, 6a. ed.

Epistemology Posters

DOUBLE TRIAD LEVELS OF PIAGET AND GARCIA IN SCHEMA DEVELOPMENT

Bernadette Baker
Drake University
bernadette.baker@drake.edu

Laurel Cooley
York College-CUNY
cooley@cims.nyu.edu

Maria Trigueros
Instituto Technologico de Mexico
trigue@gauss.rhon.itam.mx

The Triad levels of Piaget and Garcia (Intra, Inter, and Trans) were used to analyze the levels of development of students' calculus graphing schema based on their work on a non-routine calculus problem. Because the single triad levels used in other RUMEC research did not prove satisfactory to explain the development, the double triad levels related to properties and intervals was hypothesized and proved useful in explaining the students' calculus graphing schema development.

The calculus graphing schema is defined by a combination of the students' level of development in understanding the concepts of derivative, limits and continuity, as well as their precalculus ideas. The development of the calculus graphing schema can be described in terms of the level of understanding of all these concepts. A genetic decomposition of the calculus graphing schema should contain the genetic decomposition of the concept of function, limit, continuity and derivative as well as the description of the relationship between them. After analysis of the data, a model was constructed for the development of the schema in terms of Piaget and Garcia's triad. This "triad of schema development" was used to describe the students' levels of the calculus graphing schema.

The data showed that there were two important components of the problem that demanded explanation: students were not only struggling with the conditions on the function, but also with the coordination of these conditions across the intervals of the domain. It became clear that the model needed should involve the development of two different schema; one for the intervals and another for the properties.

On the one hand, students were observed at differing levels in the ability to coordinate the properties of the graph as given by the conditions. On the other hand, students were also observed at differing levels in the ability to coordinate the graph properties of the function across contiguous intervals of its domain. The development of the calculus graphing schema can be described by the interaction of these two schemas, so a two dimensional triad was used to represent the data. The two schemas that manifested themselves in the data were named the "Condition-Property Schema" and the "Domain-Interval Schema". Descriptions of the three levels of each schema as well as the double levels of the calculus graphing schema are included. Evidence of student development at eight of the nine possible levels of the double triad were found among the 41 students in this study.

Functions and Graphs Research Reports

273

GRAPHING OF DISCRETE FUNCTIONS VERSUS CONTINUOUS FUNCTIONS: A CASE STUDY

Fernando Hitt
Departamento de Matemática Educativa del Cinvestav-IPN, México
Fhitta@data.net.mx

Orlando Planchart
Departamento de Ciencias y Tecnología
Inter American University of Puerto Rico-Ponce Campus
oplancha@acpon1.ponce.inter.edu

In a general project carried out during several years, we have detected, mainly using questionnaires, the types of difficulties high school teachers of mathematics have on the construction of functions. We found in these studies that those teachers have a strong tendency to think in continuous functions expressed by one formula. In this paper we are interested in exploring the students' conception of the concept of function when they are beginning their studies at the university level. Precisely, we were interested in exploring, via an interview, a student's mathematical idea about a problem, found also in teachers who 1) related the continuous function expressed with one formula and 2) had difficulties in constructing a model related to the student's conception of function. In the interview the student could mobilize a given task through an external representation. The paper concludes with the discussion of a teaching approach.

Introduction

In studies carried out with teachers of high school mathematics, it was found (Hitt, 1994, 1998) that when the teachers worked problems that involved the construction of functions, they showed a strong tendency through the appearance of consistent errors to consider the concept of function as being associated to the idea of continuous function expressed with one formula. This is consistent with an historical analysis in which the history of intuitive mathematics ideas of the XVIII and XIX centuries gave origin to the definitions and treatments or procedures similar to those used by teachers of mathematics today (see Hitt, 1994, pp. 10-14).

In this study we are interested in analyzing the same problem, but now, our subject will be a student selected from a regular precalculus course class. What ideas are generated by the students about the concept of function during the normal process of teaching? Particularly, our intention is to document the conception one student had of the function concept and to construct a model of his ideas to understand his mathematical knowledge related to this concept. The student is an average student who has completed his first university year.

Methodology (Clinical Interview)

We selected an average student (grade B in algebra and B in precalculus, 19 years old) who was in the first year at university. In the precalculus course the teacher used the book, *Precalculus with Graphing and Problem Solving* (Smith, 1993). In general the presentation of different representations of the functions in the functions chapters (pp. 67-96) is summarized in Table 1.

Table 1.
Presentation of different representations of function in precalculus text

	Type of Representation					
Examples of Functions	**Verbal Rule**	**Table**	**Algebraic Expression**	**Graph**	**Mapping**	**Set of Ordered Pairs**
Discrete	one	two	one	one	several	one
Continuous (one algebraic expression)	none	none	several	several	one	one
Continuous (more than one algebraic expression)	none	none	none	several	one	one
Discontinuous (rational functions)	none	none	several	several	none	none

The book stresses the recognition of shapes (graphs) of continuous functions. There are no specific tasks related to the conversion from the graphic register of representation to the algebraic one. It is clear that the book shows the discrete functions only at the beginning of the chapter to make the domain and range of the function explicit using the idea of mapping and then suddenly the mapping is abandoned.

At the end of the precalculus course, the students were asked to work the problems of the book (45 to 54, pp. 75-76). The students, in particular JR (we will name him this), had been shown when solving the following task with paper and pencil, a continuous curve when converting a table (r(x) = Price of 1 lb. of round steak) to a graphic. The curve was drawn passing through the origin (0,0). Does it have any interpretation in our context?

The student was then asked to re-solve the same exercise, but this time via an interview. The intention of the interviewer was to provoke a conflict within the student with the idea that the student could feel he was in a contradictory situation and by this approach the interviewer would know about the student's cognitive obstacle in more depth. That is to say, the interviewer's intention through the interview was trying to make the student conscious that he was in a contradictory situation without saying it directly, rather the contrary. On the one hand, we were expecting the student in a contradictory situation to feel an uneasiness provoking a crucial reaction through a functional accommodation of his scheme (in Steffe sense, 1991, p.183) and the possible construction of new knowledge. In the context of a semiotic system of representations, Hitt (1998) states that: "Understanding the concept implies coherent articulation of the different representations [of the concept] which come into play during problem solving." On the other hand, we also were interested in the construction of a model related to the student's idea of functions.

The student was asked to explain all the work he was doing. The problem that provided the data for the study was:

> *Use the accompanying table (see Table 2), which reflects the purchasing power of the dollar from October 1944 to October 1984. Let x represent the year, let the domain be the set {1944, 1954, 1964, 1974, 1984}, and let* $r(x)$ = Price of 1 lb. of round steak; $s(x)$ = Price of a 5-lb bag of sugar; $b(x)$ = Price of a loaf of bread; $c(x)$ = Price of 1 lb. of coffee; $e(x)$ = Price of a dozen eggs; $m(x)$ = Price of 1/2 gal. of milk; $g(x)$ = Price of 1 gal. of gasoline. Find: $r(1954)$; $m(1954)$; $g(1944)$; ... ;$(g(1944+40) - g(1944))/40$. Graph the function r.

Table 2.
Reflection of the purchasing power of the dollar from October 1944 to October, 1984.

	Year				
	1944	1954	1964	1974	1984
Round steak (1 lb)	$ .45	$ .92	$ 1.07	$ 1.78	$ 2.15
Sugar (5-lb bag)	.34	.52	.59	2.08	1.49
Bread (loaf)	.09	.17	.21	.36	1.29
Coffee (1 lb.)	.30	1.10	.82	1.31	2.69
Eggs (1 dozen)	.64	.60	.57	.84	1.15
Milk (1/2 gal)	.29	.45	.48	.78	1.08
Gasoline (1 gal)	.21	.29	.30	.53	1.10

Interview and Comments

Through the interview JR (the student) showed abilities and difficulties to recognize the functions. In the algebraic representation he had not difficulties. He recognized the functions, he said, "because of the format f(x)." He calculated different values of the function using the table and mapping the variables "year" and "price." An important moment in the interview was when JR was asked to graph the function r. JR identified the domain and the range, graphed the points, and suddenly joined the points with a continuous curve.

Interviewer: Why do you join the lines [points]? What reason can you give to join the lines [points]?

JR: Err ... to... as I was told to graph.

Interviewer: Graph, what?

JR: The function r(x).

Interviewer: The function.

JR: Err, if we do not join the points what we get... a little pile of points that is not going to give a graph.

The student recognized the tabular representation as a function but not the discrete graphic representation. Some researchers and teachers consider the use of tables necessary in a learning process. However, regularly, the teaching process induces the use of a table, graphing point by point and, finally, the students are told to join the points (obviously with a continuous curve). But, in the teaching process, do the examples used with tables have implicitly a sense related to a discrete function and not to a continuous one? Is the student's cognitive obstacle a product of the way teaching is carried out? Do the students consider a discrete graph of a function as a representation of a function? In this context, Markovits, Eylon, and Bruckheimer (1986, p. 22) state that: "Only one student [of 400] drew the graph of the following function correctly f :{natural numbers} Æ {natural numbers}; f(x) = 3."

Table 3.
Table referenced in interview with JR.

x	y
3	6
6	9
7	15

Interviewer: For example you have this table (see Table 3). Can we consider that as a function?

JR: It could be yes... Yes, because you gave an x value and obtained y.

Interviewer: Look, if we have, for example, something like this (see Figure 1).

JR: Do I include this segment? [he points at the horizontal segment]

Interviewer: The two segments.

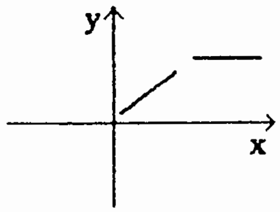


Figure 1. *Figure referenced in interview with JR*

JR:	Mmm, as a function?
Interviewer:	As a function.
JR:	No.
Interviewer:	What is your idea of function?
JR:	The graph of the function has to be continuous... or two segments do not make a sole function.

It seems that for this student the graphic representation has to be continuous and the function must be expressed by one algebraic expression. Most of the students develop a pseudo-structural conception (Sfard, 1992, pp. 75-76). Sfard states: "...some students would insist that a discontinuous curve represents several functions rather than one. ... The frequently observed inability to build a reasonable bridge between algebraic and graphic representations of functions." It seems we detected the same behavior with JR.

Discussion

The difficulties demonstrated by this student might not be a product of emphasizing an algebraic approach; instead the visual method used in teaching may emphasize the graphic representations of continuous functions expressed by one algebraic expression.

The student recognized the tables as representations of functions but failed to identify discrete graphs as representations of functions. For the student, the graphic representation of a function must be a continuous one and expressed by one formula. The student is competent in identifying the significant units in the construction of a graph: Domain, Range and $G_r = \{(x,r(x)) \mid x \in D\}$, but he has a concept image (in Vinner's sense, 1983, p. 293) related to continuous functions. During the teaching process carried out with JR since he was learning about functions for the first time using tables and graphing, the student was told to join the points; surely, the functions he was working with were continuous functions. Therefore, the teaching process has generated a cognitive obstacle in this student.

In studies carried out with teachers of mathematics, the teachers have the same concept image related to the idea of function-continuity as JR, and the books emphasize the same idea. Could the student be able to isolate the ideas of the function and the continuity by himself? It seems that we need to emphasize the teaching of functions showing more examples of discrete functions and their graphs. New tendencies in precalculus textbooks show there are more examples using discrete variables (see Connally, Hughes-Hallet, & Glkeason, 1997).

The results show that it is urgent to work on the construction of a learning environment where the students can construct functions with certain properties where they are implicitly obligated to find discontinuous functions and/or functions expressed by more than one formula. For example: Construct three different functions with the same property, that $|f(x)| = 2$ to all x real; or, construct three different functions with the same property, that $f(f(x)) = 1$ to all x real. The idea is that in this learning environment in a multiple systems of representation context, we would like to provoke a different thought from one related to the construction of a discontinuous function and by means of two or more algebraic expressions.

References

Connally , E., Hughes-Hallet, D., & Gleason A. (1997). *Precalculus: Functions, modeling change*. John Willey.

Hitt F. (1994). Teachers' difficulties with the construction of continuous and discontinuous functions. *Focus on Learning Problems in Mathematics, 16*(4), 10-20.

Hitt, F. (In press). Difficulties in the articulation of different representations linked to the concept of function. In *Representations and the Psychology of Mathematics Education*. Special Issue of the *Journal of Mathematical Behavior, 17* (1).

Markovits, Z., Eylon, B., & Bruckheimer, M. (1986). Functions today and yesterday. *For the Learning of Mathematics* 6(2), 18-28.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification - the case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy*. Mathematical Association of America, Vol. 25, pp. 59-84.

Steffe, L. (1991). The constructivist teaching experiment: Illustrations and implications. In E. von Glasersfeld (Ed.), *Radical Constructivism in Mathematics Education*. Netherlands: Kluwer.

Smith, K. (1993). *Precalculus with graphing and problem solving* (5th ed.). Pacific Grove, CA: Brooks/Cole.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematics Education in Science and Technology, 14*(3), 293-305.

PERCEPTION OF GRAPHS AND EQUATIONS OF FUNCTIONS AND THEIR RELATIONSHIPS IN A TECHNOLOGY-ENHANCED COLLEGE ALGEBRA COURSE

Armando M. Martinez-Cruz
Northern Arizona University
armando.martinez@nau.edu

We seek a better understanding of student integration of function ideas. We report college-algebra student perception of two function representations (graph and equation) and their relationships when graphing technology is available. Three students were selected for case studies of these ideas during summer 1997. We built an individual network of function ideas. Networks suggest that these students' function ideas were extremely weak, unrelated and contradictory. Affective factors and course nature might explain this. Terminal courses in mathematics greatly influence the way students retain and integrate the content. Also, a summer college algebra course provides few opportunities to integrate math ideas. Networks suggest student early integration of function ideas.

Curricular importance of functions has attracted the attention of the mathematics education community (Tall, 1992). Studies on function conceptualization report poor student (Dreyfus & Vinner, 1989) and prospective teacher (Even, 1989) understanding. They: (1) see functions as equations without an awareness of domain and range; or as graphs which are expected to be continuous, familiar or regular; and (2) use familiar procedures (the vertical line test) or formulas to identify functions. Students can grasp these ideas without relating them even though they represent the same object. Lack of relationships among function ideas may be useful to the learner for dealing with some problems, but it can be an obstacle to construct a formal function notion (Herscovics, 1989). Conversely integration of function ideas can help students in many ways. Students can gain a better understanding of the concept, move across its different representations and begin formalizing it. Graphing technology use in math courses provides alternative ways to teach the subject; visualizing algebraic variations with a graphing calculator favors the establishment of relationships between different function representations (Dunham & Dick, 1994).

Our research fits in this context. Previously we investigated students' knowledge of functions in a high-school precalculus course enhanced with graphing calculators. Three function conceptualizations appeared in those students (Martinez-Cruz, 1993). A network of function ideas for each student was built to represent their relationships. Understanding how students integrate knowledge can be a fruitful area in mathematics teaching. Perceiving understanding as interconnected knowledge (networks) suggests the critical importance of relating new knowledge that is the object of instruction to the existing knowledge that the learner brings to instruction (Carpenter & Fennema, 1991). We still know little about student understanding of these ideas and the way they relate them (Bright & Hoeffner, 1993).

This report is part of a larger project aimed to contribute to function teaching and learning with technology. In this study we aim to gain a better understanding of two particular representations (graphs and equations), the kinds of relationships used to connect them, and how such connections take place when graphing calculators enhance the mathematics content. We conducted this research at the college level during summer 1997.

Theoretical Framework

The 1993 theoretical framework incorporated historical (Kleiner, 1989) and psychological contributions (processes and objects) (Sfard, 1989) to function development; concept image and concept definition (Dreyfus & Vinner, 1989); and multiple representations. A result of that research was individual function networks for each of the seven participants. Networks were grouped as three models (graphs, equations, and unique correspondence) and helped us to improve our original framework, which we used here. We accept a constructivist view on mathematics knowledge.

The Study and Its Methodology

A college algebra course (taught by a mathematician) enhanced with graphing calculators in a Southwestern university was the context of this study. Three students were selected using purposive sampling (Lincoln & Guba, 1985) for case studies of their function knowledge and representations. Our research questions were:

(1) What is the knowledge that these students have of the graphic representation of functions (graphs)?

(2) What is the knowledge that these students have of the algebraic representation of functions (equations)?

(3) What kind of relationships do these students establish between these two representations?

Since students' ideas change over time, we relied on the interpretivist tradition of ethnographic research, for it provides methodologies for studying the evolution of change in math teaching and learning. Collection of data for each case study involved four interviews, daily classroom observations, practice tests developed by the author, in-class testing materials and office-hour interactions. Criteria related to the trustworthiness of the study (credibility, transferability, dependability, and confirmability) were also considered (Lincoln & Guba, 1985).

Interviews provided the most useful information to sketch students' thinking. Four protocols were developed for the study. Items emerged from the cascading design of the study. Pertinent function literature was consulted. Items asked about the relationship between equations and functions and between graphs and functions, to decide if a given graph was a function, or to provide function examples. Functions involved typically appear in a college algebra course. A domain analysis (Spradley, 1979) and a coding paradigm (Lincoln & Guba, 1985) of tests and materials identified relationships (or lack thereof) of function representations (graphs and equations) that students associated with the function concept. Those relationships were used to build a network of relationships between these two representations. Function ideas of one participant, Zafu, are given.

Zafu's Ideas About Functions

Zafu's function ideas in the beginning were mainly as a graph, which included the vertical line test (VLT). A one to one function is when you can pass a straight line through only one point of a line. He could not explain why the method worked. His familiarity with the VLT helped him to establish a relationship between functions and graphs, graphs of a line may be use[d] to determine if a line is a one to one function, two to one function, etc. He tended to mix the VLT with the horizontal line test (HLT). He was familiar with the phrases one to one function and two to one function but could not explain what they meant. Maybe his familiarity with them, the VLT and the HLT, created an obstacle to understanding the VLT. This confusion characterized his application of the VLT throughout the study. If given two parabolas (one horizontal and the other vertical) he could not decide which one was a function. Sometimes he applied both tests and other times he applied only one.

Zafu also held an algebraic image of functions that emerged to a lesser extent at the beginning. His first algebraic example of functions was $f(x) = Ax^3 + Bx^2 + Cx$ (A, B, and C the coefficients). His function ideas relied on familiarity and procedures:

> Well, I also remember that you need to perform like a vertical or horizontal line test and I don't remember whether it's one or the other or both. And that. You know. It can't be a parabola or a circle because of that, you have your vertical, all the vertical line test. But I can't really tell you what a function is....[Later he said] Uhmmmm. [thinking without writing anything]. I can think of examples. I can't really [give a definition, writing $f(x)$] words, like x can be, squared or it could be y [Writing $= (x^2 + y)$]. It's just a procedure that you decide, you know, or somebody makes up a math problem.

The action procedure appeared as a test for an algebraic expression to be a function:

> Just, something. [$(x)^2 + y$] is a procedure that you're doing to the, [thinking] any, any numbers I guess.

Zafu did not see that the expression $y = (x^2 + y)$ leads to $0 = y^2$ and relied on authority: $f(x) = (x)^2$ is an example of a function because [his] teacher said so last week[laughing].

Zafu's Ideas About Graphs

At the beginning of the study, Zafu thought of graphs as a visual representation of an equation of a line. This included many ideas. A graph is a visual representation, it doesn't need to be a graph. A visual representation included figures, while a graph was a curve (to which he referred as a line sometimes). He related graphs and equations: a graph is a (visual) representation of an equation. His ideas of graphs and function included the VLT as previously discussed. He showed several ideas about graphs (including recognition of familiar shapes, and vocabulary such as increasing and decreasing graphs, and maxima and minima). For a cubic-like graph he would say the following.

> Well it might be two connecting parabolas. It might be a sine curve. Maybe sine and parabolas are related. I have no idea. That's a guess on my part. Each one of those points has a vertex. Possibly [there are two parabolas here]. That's all I know.

He remembered the idea of a function being increasing, but could not tell if the function in that graph was increasing—he needed an equation. I can't tell you by looking at the graph. I can't. If you give the equation and ask me if this equation is increasing, I might be able to say. He identified maximum and minimum of the graph. He could not indicate the domain and it seems he was thinking instead of the interval solution to an inequality. He had difficulties reading information from graphs, particularly when solving equations such as $f(x) = \text{constant}$. He reversed x and y constantly but was able to find the answer later.

Zafu's Ideas About Equations

Zafu had a two-fold conceptualization of equations in the beginning. First, as an equality. A math problem containing an equal sign. Second, as a problem to be solved. He thought equations and functions were related for the function of a variable may be set equal to a math problem. This function of a variable referred to $f(x)$, function f of the variable x. Since all his function examples were given as function of a variable equal an equation (math problem), he saw a relationship between functions and equations. He knew functions were given by equations, like $f(x) = y$, function of x. His perception of a function as a shredding machine, communicated a procedural conception, since machines give you an output. The equation was this shredding. However, with this function idea he could not decide if $f(x) = x^5 + 7x = 3$ was a function. Perhaps he had not come to equate functions and equations. He knew that equations provided information about functions:

> The kind of information that I get, that equations tell about functions was that, their shape and relative size. [I mean by shape.] Well, like if you give me an equation that no matter what variables you plug in for x and y, you always get a parabola. So that's shape. [Other shapes?] Circle, yeah, uh, was there any other shapes? I don't remember. A line. [Relative size included location, domain and maybe scale.] Yeah, because we're plugging in. And also their relative location, because we're plugging in any variable that we want to choose, uh, for x and y, just that it'll be close to the origin, just so that we can graph it and make it easy for us to see it. We don't know like what values of the units are. It's just miles [laughing]. [It's just the numbers that you would plug in.]

Further description of the relationships between equations and graphs were found later. Zafu expressed equations and graphs were related since lines can be represented as equations. Equations may produce lines on graphs. He did not know if equations might not produce lines. He recognized y and f(x) are identical. This let him solve other problems. He also understood what the slope was and could find it from equations and graphs. He seemed to have mastered straight lines (even constant) but had difficulties with quadratic functions particularly when given the graph. He had an idea of what every coefficient meant but this knowledge was memorized and never understood. He could not represent graphically what he expressed verbally: he had to build a table to graph a parabola instead of using transformations of graphs (which he seemed to have mastered). At the end his knowledge of quadratic equations declined to the point of not remembering them. He did not have problems solving a system of two linear equations with the calculator, but had problems interpreting the solution graphically.

Zafu's Understanding of the Relationship Between Functions, Equations and Graphs

From this study data, Zafu perceived functions given as equations and therefore could be graphed. The vertical line test provided a connection between graphs and functions (see Figure 1). His function examples included familiar equations and graphs.

Conclusions

Informal data suggest that affective factors (such as whether the algebra course was a math terminal course) greatly influenced the way students retained and integrated the content. Students in the course appeared very little motivated to assimilate ideas; here the weakness of their function network was affected by their motivation. Additionally, a summer course provides few opportunities to integrate those ideas. However, the networks built provide an insight on how integration of function ideas occur. Graph-

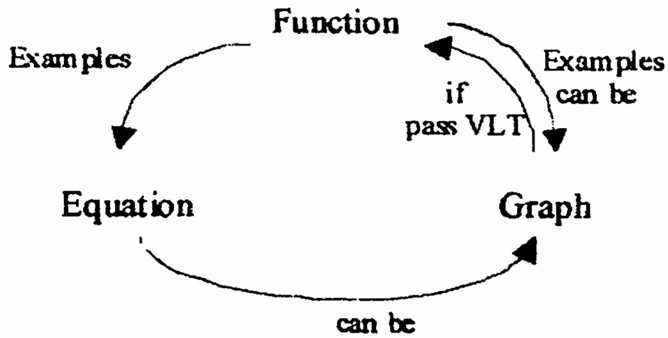


Figure 1. *A college algebra student's network of graphs, equations, and their relationships.*

ing technology might help students to establish more connections between different representations of function. For this to be accomplished mathematics courses need to be revised to use technology as a real tool.

References

Bright, G., & Hoeffner, K. (1993). Measurement, probability, statistics, and graphing. In D. T. Owens (Ed.), *Research ideas for the classroom: Middle grade mathematics.* pp. 78-98. Reston, VA: NCTM.

Carpenter, T., & Fennema, E. (1991). Research and cognitively guided instruction. In E. Fennema, T. Carpenter, & S. Lamon (Eds.), *Integrating research on teaching and learning mathematics.* Albany, NY: SUNY.

Dreyfus. T., & Vinner, S. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 20, 356-366.

Dunham, P. & Dick, T. (1994). Research on graphing calculators. *The Mathematics Teacher*, 87(6), 440-445.

Even, R. (1989). *Prospective secondary mathematics teachers' knowledge and understanding about mathematical functions*. Unpublished doctoral dissertation. Lansing, MI: Michigan State.

Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. In S. Wagner, & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra. Research agenda for mathematics education*, v. 4, pp. 60-86. Reston, VA: Lawrence Erlbaum and NCTM.

Kleiner, E. (1989). Evolution of the function concept: A brief survey. *The College Mathematics Journal*, 20(4), 282-300.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Newbury Park, CA: Sage.

Martinez-Cruz, A. M. (1993). *Knowledge and development of functions in a technology-enhanced high-school precalculus class. A case study.* Unpublished doctoral dissertation. Columbus, OH: The Ohio State University.

Sfard, A. (1989). Transition from operational to structural conception: The notion of function revisited. *Proceedings*. PME, v. 3, pp. 151-158. Paris, France.

Spradley, J. P. (1979). *The ethnographic interview*. New York: Holt, Rinehart and Winston.

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. pp. 495-511. New York: Macmillan.

UNDERSTANDING MATHEMATICAL EXPERIENCE

Tracy Noble, Ricardo Nemirovsky, Tracey Wright, and Cornelia Tierney
Education Research Collaborative at TERC
Tracy_Noble@TERC.edu, Ricardo_Nemirovsky@TERC.edu

We discuss the activities of two 5th grade boys working together during two days of mathematics class, on a problem of representing a motion along a linear path. Over the two days, the boys represent the motion in three different mathematical environments: at the blackboard using a table of positions and stepsizes over time; at the computer using a computer simulation of two people walking; and at their desks moving Cuisenaire rods along a meter stick. In this paper we ask the question: "How does one describe and grasp others' experiences". In attempting to answer this question, we find that it is essential to understand experience as simultaneously individual, social, and physical; and to be aware that what we see in students' experiences is necessarily related to what we come to see in ourselves.

Theoretical Framework

Theories in mathematics education often distinguish themselves by their focus of study. While often combining elements of several perspectives at once, some tend to highlight mental structures (Steffe, von Glasersfeld, Richards, & Cobb, 1983), others socio-cultural environments (Walkerdine, 1988), and others the interaction with representational objects (Kaput, 1991). Some of the current debates center on whether the field should make complementary use of differing points of view (Cobb, Yackel, & Wood, 1992; Bauersfeld, 1992) or make a choice among them (Lerman, 1996). Rather than supplementing one focus with another or making a choice among predefined possibilities, we attempt in this paper to describe students' experiences in a classroom, recognizing that: 1) Experiences are simultaneously individual, social, and physical (Goodwin, 1993; Ochs, Jacoby, & Gonzales, 1994; Meira, 1995) and that 2) What we see in students' experiences is necessarily related to what we come to see in ourselves (Confrey, 1991;

- Throughout these Investigations, the word "trip" was used to refer to a linear motion made by one or two actors, in which speed might vary.
- The tree marked the end of the trip.
- In the curriculum and in our conversations with students, we avoided using the language of racing or winning in reference to these simultaneous trips, but students often began using it on their own.

Ball, 1996). This raises the question: How does one describe and grasp others' experiences? In this paper, we do not provide general answers to this question, but instead grapple with it by attending closely to what two students, Norman and Luke, actually say and do, and how we come to understand it.

We will trace how, for example, the boys' individual ways of counting, short conversations among themselves and with the teacher, and the manipulation of physical objects all influence how they see, talk, and act within three different mathematical environments. We will also describe how trying to understand what the boys were doing affected our own understandings of the mathematics curriculum the boys were engaged with. In this paper, we will describe Norman and Luke's experiences in these mathematics class sessions by focusing on three aspects of their experience: 1) The sense of purpose, or the boys' diverse and changing understandings of what they are supposed to do; 2) The development of narratives, or the ways Norman and Luke tell stories about situations that change over time, and link these stories together; and 3) The sense of time, or how the boys constitute a sense of time in their work with situations that change over time. We propose these aspects not as general ways of understanding classroom experience, but as ways of grounding our analysis of these particular classroom experiences, and linking them to more general aspects of human experience.

The Study

This paper reports on results of a classroom study of 5th grade students learning about the mathematics of motion and growth in a public school classroom in East Boston, MA, where a four-week unit of the NSF-sponsored elementary math curriculum, Investigations in Number, Data and Space: Patterns of Change: Tables and Graphs©, written by three of the authors of this paper, was pilot-tested. The curriculum unit consists of three investigations of change over time in situations of motion and growth. In Investigation 1, students collected data about time and the position of a student walking along a straight line marked on the classroom floor. In Investigation 2, students used the Trips© software environment to enact walking trips for a boy and a girl character walking along parallel paths on the computer screen, and did similar trips with Cuisenaire Rods moved along the two sides of a meter stick. Finally, in Investigation 3, students learned about different modes of growth of two-dimensional patterns of colored tiles. During these Investigations, we videotaped daily in math class, and conducted individual and group clinical interviews with a small group of students, including the two boys described in this presentation. We met weekly to discuss the curriculum and data with the two teachers who piloted this curriculum.

After the pilot testing was completed and the curriculum revised several times, we continued the analysis of video-tapes, meeting weekly to discuss video episodes, and compiling detailed descriptions of every class session. Through discussions of episodes that were surprising and confusing to us, we found several classroom episodes which taught us a great deal, and for this analysis we decided to focus on episodes including the experience of two boys, Norman and Luke, during the second and third days of Investigation 2. Investigation 2 began with a session in which the students were introduced to two new environments: the Trips© software and the Cuisenaire rods and meter sticks activity. In the Trips© software, students chose the step size and start position of boy and girl characters, who then walked along a track on the computer screen, and whose trip was recorded in a table window and a graph window. In the Cuisenaire rods activity, students worked in pairs, each student with a Cuisenaire rod of a certain size, his step size. On the first day of this Investigation, the teacher introduced the two new environments, and students experimented with them. On the second day of this investigation, students were asked to make up a trip for a boy and a girl (by specifying starting positions and step sizes), to satisfy the following motion story: "The girl gets to the tree way ahead of the boy." Students were then asked to act out the trip using the Cuisenaire rods and meter stick, and to make a table of position over time for their trip. Later students used the Trips© software to make trips according to this and other Motion Stories.

While this analysis is done with the background of our experience of analyzing over eight weeks of classes in two different schools, we have done a detailed analysis of two days of Investigation 2, and we can describe in this paper a few moments out of this time, which are moments that help to illustrate larger issues about the experience of being a learner of mathematics. We have selected two moments from the class sessions to give the reader a sense of the work of the boys, and to illustrate the three aspects of the boys' experiences. Both moments occur about halfway through the third class of Investigation 2, after the boys have had two days of experience using the software and the Cuisenaire rods and meter stick for making trips. In the first moment, the boys discuss a trip that they have just made by moving the Cuisenaire rods along a meter stick, and where their conflicting senses of purpose are highlighted. In the second moment, which occurs about 10 minutes later, Tracey Wright and Luke tell stories about motion based on a table of numbers on the blackboard. In our analysis of this conversation, we will highlight the aspects of development of narratives and the sense of time. Through the analysis of both moments, we hope to provide a glimpse of the boys' experience in this mathematics classroom.

Moment 1: "I won."

Norman and Luke have just made a trip with the Cuisenaire rods, each taking a rod and flipping it over itself to take "steps" on each side of the meter stick. Norman, who has the role of the girl, has reached the end of the meter stick first, and the following conversation ensued:

Norman: I won.

Luke: How?

Norman: The girl gets way ahead of the boy. Do you remember number 1 [Motion Story 1]?

Norman and Luke had two different purposes for this activity with the Cuisenaire rods. Norman's purpose was to make a trip which would satisfy the criterion of Motion Story 1: "The girl gets to the tree way ahead of the boy.", which is what the boys were instructed to do. Luke, on the other hand, constantly checked the table of numbers on the blackboard while he took his trip, and moved his Cuisenaire rods in accord with it, which, if Norman had also followed the table, would have led to Luke reaching the end first. Norman and Luke, however, were each focused on their own motions: each moved his own Cuisenaire rod down the meter stick, engaged with his own goal and the physical objects, until Norman reached the end and Luke recognized that the trip he had envisioned had not occurred.

When Luke asks Norman how he won, Norman responds by the purpose of his activity, as opposed to describing the specific moves he made, such as taking a 7 cm. Cuisenaire rod when Luke took only a 4 cm. rod. If Luke is surprised that Norman reached the end first, it is because he does not share Norman's purpose of making a trip where Norman, playing the role of the girl, reaches the end first. Many different things contributed to Luke's sense of the purpose of this activity: the table of numbers on the blackboard, which was written as an example of a trip, but which became for Luke a guide to how to move; Luke's desire to "win" no matter what, which we saw throughout these class sessions; and no doubt other factors.

Moment 2: "Right now she's losing."

Luke and Norman are planning a trip to do with the Trips© software, and Luke says that the girl is supposed to win in this trip (to satisfy Motion Story 1: The girl gets to the tree way ahead of the boy.). Yet when Tracey asks him what his plan for the computer trip is, Luke points to the table written on the board, which describes a trip in which the girl would not win, and which is the same table he used in Moment 1. This leads to the following discussion. A portion of this table is reproduced below (See Table 1):

Tracey: Right now the girl's at 31 and the boy's at 33, so why is the girl gonna win? Right now she's losing.

Luke: No she ain't.

Tracey: She's not gonna win like this?

Luke: No, because from the 10 [points to the girl position 10] through the 25 she was leading [Tracey: yeah] but, and, and, from the 9

Table 1.
Partial table from blackboard.

Time	Boy Step Size	Boy Pos.	Girl Step Size	Girl Pos.
4	4	5	3	10
5	4	9	3	13
6	4	13	3	16
7	4	17	3	19
8	4	21	3	21
9	4	25	3	24
10	4	29	3	27
11	4	33	3	30

through the 25 [boy positions], I was, like, going slow. But, from the 25, like, from the bottom, I was, like, going faster, [Tracey: yeah] and she was goin' — and he was like goin' — slower [Tracey: right].

One of the first things that is surprising about this moment is that Luke can enter into this conversation with Tracey about an imaginary trip taken by a boy and a girl, based only on numbers written in a table. Luke's story-telling involved naming the actors in the story. He calls one of the actors "she", perhaps referring to the imaginary girl who moves according to the table, or to the girl represented on the computer software. He calls the other actor "I", placing himself in the story of the table, because he had moved according to the table, taking the role of the boy, when he moved his Cuisenaire rods down the meter stick. Another part of Luke's story is the time. When Tracey begins this conversation, she calls the last moment represented on the table "now", which allows her to create a story about the motions of the boy and the girl: " Right now, she's losing", in which the present, future, and past are defined. Luke takes on Tracey's use of time by using the past tense to refer to everything that came before 11 seconds: "she was leading", "I was, like, going slow". Without a sense of when "now" is in a story, it is difficult to describe events, as one must choose some verb tense in which to describe things that happened, are happening, or will happen.

Conclusions

In Moment 1, we see an example of an experience that has elements of the individual, social, and physical. The boys work individually on their trips along the meter stick, with completely different purposes in mind, but are forced to negotiate about their purposes when one of them reaches the end first, and "wins". It is possible for them to work toward separate goals, because each boy controls his own Cuisenaire rods, and can move independently down the meter stick, unlike in the software environment where the boy and girl characters move together. The individual, social, and physical elements are inextricable in this example and throughout these class sessions. We believe that seeing mathematical experience as necessarily individual, social, and physical all at once is a step toward grasping the complexity inherent in this and all human experience.

Our own emerging understanding of Norman and Luke's mathematical experience is grounded in our gradual noticing of how the same trip enacted in the different environments involves different questions and ways of acting. While using the Cuisenaire rods, one can ask, "Where are you now?", because at any moment both boys have reached a certain position on the ruler by flipping the rods. In Moment 2 we see that while using a table of numbers, one must define when "now" is, because the table represents in one place events that took place at many different times. One must also define "who" is moving according to the table, because the same table of numbers could be used to represent different actors, as we saw when Luke put himself into the story he told about the table. In this way Luke used his story about the table to link the Cuisenaire rods environment, where he was in charge of moving, to this static table of numbers describing the trip taken by an imaginary boy and girl. Thus, while we came to understand the differences between the different environments, and the mathematical value of these differences, which we had been led to "forget" by our own education, we watched the boys develop ways of linking the environments together.

References

Ball, D. (1996). What do student's know?: Facing challenges of distance, context, and desire in trying to hear children. In B. J. Biddle, T. L. Good, and I. F. Goodson (Eds.), *International Handbook of Teachers and Teaching*. Dordrecht, Netherlands: Kluwer.

Bauersfeld, H. (1992). Classroom culture from a social constructivistís perspective. *Educational Studies in Mathematics*. 23, 467-481.

Cobb, P., Yackel, E., & Wood, T. (1992) A constructivist alternative to the representational view of the mind in mathematics education. *Journal for Research in Mathematics Education*, 23(1). 2-33.

Confrey, J. (1991). Learning to listen: A student's understanding of powers of ten. In E. von Glasersfeld (Ed.), *Radical Constructivism in Mathematics Education*. Dordrecht, Netherlands: Kluwer Academic Publishers, 111-138.

Goodwin, C. (1993). *The blackness of black: Color categories as situated practice*. Paper presented at the conference on Discourse, Tools, and

Reasoning: Situated Cognition and Technologically Supported Environments, Luccia, Italy.
Kaput, J. J. (1991) Notations and representations as mediators of constructive processes. In E. von Glasersfeld (Ed.) *Constructivism in mathematics education*. Dordrecht, Netherlands: Kluwer, 53-74.
Lerman, S. (1996) Intersubjectivity in mathematics learning. *Journal for Research in Mathematics Education*, 27(2), 133-150.
Meira, L., (1995) Mediation by tools in the mathematics classroom. *Proceedings of the 19th International Conference for the Psychology of Mathematics Education*. Recife, Brazil: Universidade Federal de Pernambuco 1, 102-111.
Ochs, E., Jacoby, S., & Gonzales, P. (1994). *Interpretive journeys: How physicists talk and travel through graphic space. Configurations*, 2(1).
Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983) *Children's counting types: Philosophy, theory, and application*. New York: Praeger Scientific.
Walkerdine, V. (1988) *The mastery of reason*. London, UK: Routledge.

UNSPECIFIED THINGS, SIGNS, AND 'NATURAL OBJECTS': TOWARDS A PHENOMENOLOGICAL HERMENEUTIC OF GRAPHING

Wolff-Michael Roth
University of Victoria
mroth@uvic.ca

Over the past decade, there have been an increasing number of ethnographic studies that document the pivotal roles and functions of representation practices in science (e.g., Lynch & Woolgar, 1990). Among the representations, graphs are quintessential because they (a) constitute the best tools to represent covariation between continuous measures and (b) are useful to summarize large amounts of data in economical ways (Roth, 1996; Roth & McGinn, 1997, 1998). Although graphing is typically listed among the handful of skills biologists want university students to develop, graduates from university science programs seem to be ill-prepared to engage in scientific representation practices (Roth, McGinn, & Bowen, 1998). The present study was conducted to construct better understandings of the interpretation practices related to graphs among university students and professional scientists.

In the scientific community, the isomorphic relation between mathematics (including graphs) and the world appears to be taken for granted (Lynch, 1991). The psychological literature similarly treats graphs as signs that have unproblematic and unambiguous referents in the world (e.g., the literature review by Leinhardt et al., 1990) despite several studies that showed the considerable cognitive work involved constructing relations between marks on paper and natural phenomena (e.g., Roth & Bowen, in press). Our research findings fall in line with those studies that document the considerable work involved in interpretation. Even scientists do not treat most graphs as unproblematic indices to scientific phenomena. In fact, many of our scientists' interpretations and graph-related activities would have to be categorized as wrong and inappropriate in classical frameworks of analysis.

My study takes as its theoretical starting point a conception of cognition based in a theory of (social) practice which emphasizes the dialectic of phenomenological experience in a world that is always and already shot through with meaning. In the course of our individual experiences, we also develop unique interpretive horizons. In any situation, when researchers observe individuals in some task, such as interpreting graphs, we therefore have to ask at least two questions. First, to what extent has the individual participated in the practices that are used as (implicit) references for framing the analysis? Second, what is the individual's ontology relative to task and task elements? That is, rather than taking the nature of the task and task elements as given, we have to find out what the task and task elements look like to the individual. The fundamental attitude taken here is therefore, that respondents act in worlds that makes sense to *them*, and that they make moves that are allowable in the world as *they* perceive it. If the activities do not make sense to the observer, if a problem solving move appears "wrong," we should assume that the problem lies with the observer who does not have a good model of the problem solver and her world.

Data Sources

My database includes videotaped interpretations and use of graphs by 45 students enrolled in a second-year university ecology course, 10 students in their fifth year of a post-baccalaureate elementary education program who major in teaching science, and 15 practicing scientists from the domains of theoretical and field ecology, forest engineering, and physics. The graphs we asked our respondents to interpret were taken from the second-year ecology course; scientists were also asked to bring and interpret graphs from their own work. I also draw on materials from a 2-year ethnographic study of field ecologists' representation practices during data collection and analysis.

From Unspecified Things to Signs and to 'Natural Objects'

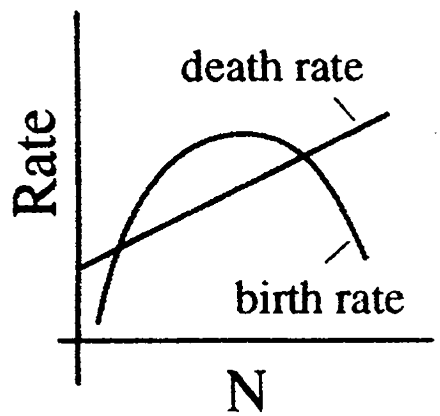


Figure 1. *One of the graph interpretation tasks in this study*

In the past, deficit models were used to account for the "errors" students made when interpreting graphs (Leinhardt et al., 1990). For example, students interpreted heights of graphs when they should have interpreted the slopes and vice versa. Interestingly enough, I found similar practices among many of our scientists who, for example, did not attend to the recursive nature of the relationship between density dependent birth and death rates (b=b[N]; d=d[N]) and the density of animal populations (N) (Figure 1). That is, some of our scientists interpreted $b - d < 0$ as a situation in which the population goes extinct (which is not true on the right end of Figure 1) rather than as one in which the population density is adjusted with a subsequent change in b - d. Furthermore, it was evident that even experienced scientists perceptually carved the graphs differently and for each aspect identified, they had to find out whether or not it had any relevance. Given the training (M.Sc. and Ph.D.) and experience (5+ years in research) of our scientists, cognitive deficit models are implausible.

Another important finding was that the concerns raised by the graphs and the practices scientists engaged in differed widely. For example, some field biologists found the population graph (Figure 1) unrealistic and suggested that no sensible interpretation could be given. Others were concerned that the graph did not indicate the variations which would be found in real data and therefore led to incorrect implications. There were differences between biologists with conservation concerns and those that had management concerns. The activities of field biologists also differed considerably from theoretical ecologists and physicists. The latter were more concerned with the modeling aspects and with finding other representations that better showed the 'real' contents of the graph. For example, there were representations of the (stable, unstable) equilibrium points (Figure 2.a) or population density over time (Figure 2.b) that they derived from the population graph.

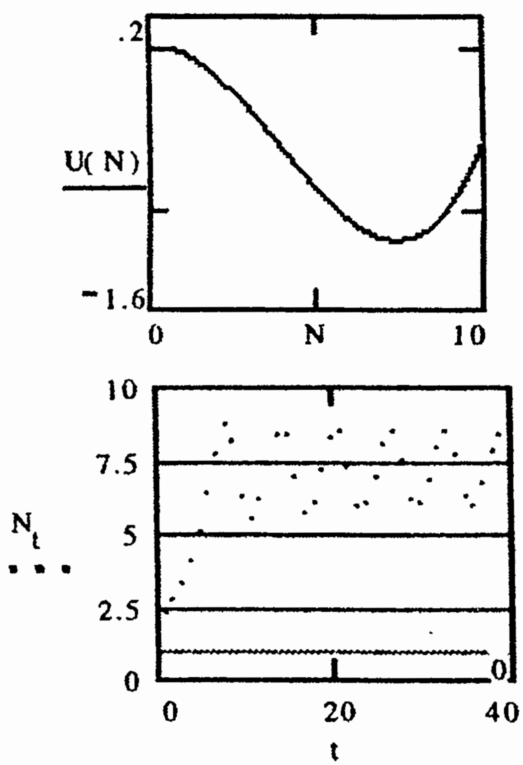


Figure 2. *Representations computer-generated by one scientist as he interpreted the population graph in Figure 1.*

Finally, rather than saying what a graph means, that is, elaborating possible 'natural phenomena' as referents {graph —> 'natural phenomenon'}, we in fact documented an ongoing movement from graphs to possible natural phenomena, and from familiar natural phenomena to their graphical representations {graph <—> 'natural phenomenon'}. In this dialectic movement, both graph and natural phenomenon were mutually constituted. As the interpretation unfolded, our respondents did not relate the graph 'en bloc' to some phenomenon, but related individual aspects of graphs to some phenomenon to see whether the aspect is in fact relevant. For example, they asked, Is the slope of the birth rate graph or its height the aspect to be interpreted? or Is the maximum of the birth rate $(b[N])_{max}$ or the maximum of the difference between birth rate and death $(b[N] - d[N])_{max}$ rate a relevant quantity?

Phenomenological Hermeneutic Model

Given the problems of earlier models of graphing in accounting for our data, we elaborated a new model which accounts for individual experience, common practices, familiarity with the referents. Our basic model (Figure 3) combines semiotic relations between sign (text, graph), refer-

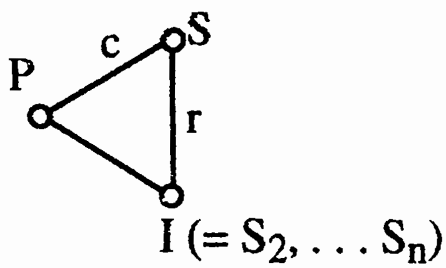


Figure 3. *Semiotic triangle with conventional and circumstantial constraints.*

ents (or sign content or state of the world), and interpretants (Eco, 1984) and situation semantics (Barwise, 1988). Accordingly, there is a relationship between a state in the world (propositional content P), the sign S standing for it, and the ways this relation can be elaborated (interpretant). These relations are constrained by the social conventions (r) currently in operation and by the circumstances (c). The propositional content is therefore constrained by a relation of four parameters $C_r(S,c) = P$. This content has to be elaborated in some sign system, so that we get into the familiar circular relationship of all sign systems and the interpretants include all other ways of expressing the content (S_2, S_3, ... S_n). Because of the conventional constraints, r, we can expect that the propositional content changes with the practices of the relevant community. Finally, the circumstances of the sign interpretation (e.g., the historical context of the interpretation, appearance of sign, source, familiarity of interpreter) affect the content. For example, when our participants talked us through the graphs from their own work, these were largely transparent. That is, rather than talking about the graph, and how its detail refer to and represent some natural phenomenon, some of our participants talked about the phenomenon and their personal experiences therein (see also Figure 5). In one instance, a forest engineer did not engage in explanations such as "This line means a decline in productivity with increasing logging distance," but elaborated extensive narratives about logging, the contexts of the operations, the machines and techniques involved, etc.

The model in Figure 3 is not sufficient, for in most cases, the graphs as complete sign objects only emerged from the interpretive activity. Our participants constructed the domain ontology in the course of their task. Thus, unlike with a word (e.g., "graph") where recognition is instant and where we do not have to wonder about its meaning, a considerable part of the interpretation sessions were taken up by the construction of the graph as a material sign that only in the second instance referred to something else (the natural object). Thus, the graphical interpretation is better described in an expanded model

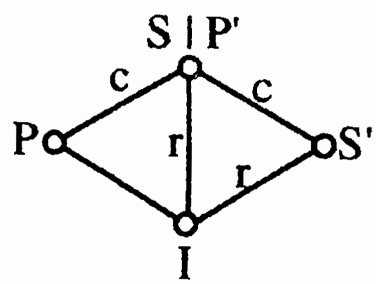


Figure 4. *During graph interpretation, there are two levels of (often overlapping) activity. The right semiotic triangle represents the construction of the graph as sign object, the left triangle the construction of the sign content.*

which includes the interpretation of an initially unspecified thing which becomes a graph, and therefore a sign object with the capacity to refer to some state (phenomenon) in the world (Figure 4). P' is the object, the graph itself; S' is the depth semantics disclosed by structural analysis; and the interpretant is the chain of interpretations producable by the community and incorporated into the dynamics of the graph (Ricœur, 1991). Our participants' interpretive activity is therefore described in the relation:

$$C_r(S_i,c) = C_r(S_i^1 + S_i^2 + \ldots S_i^n,c) = C_r(S_i^1,c) \circ C_r(S_i^2,c) \circ \ldots \circ C_r(S_i^n,c)$$

so that the "composite" interpretation arises from the interaction of the interpretation of the individual elements. In a particular interpretation, one element may not constrain the meaning of another (although it should). For example, in the population graph an element from the ontology may be $S_{pg}^1 \equiv \{b - d < 0\}$, which was frequently interpreted as "the population crashes"; this interpretation was often unconstrained by another element which embodies the recursive nature of the situation $S_{pg}^2 \equiv \{\Delta N \neq 0 \Rightarrow \Delta(b - d) \neq 0\}$. My research shows that to understand each scientist's interpretation, we had to reconstruct as far as possible all elements S^i of *their* domain ontologies and how these elements constrained each other.

Conclusions

Graphs are neither unequivocal, nor constitute sign structures that point to unique 'natural objects' or lead to a coherent set of interpretation practices. The work reported here provides rich details of the subtle changes in the ontologies (ensemble of elements perceptually available) of scientists and science students as they engage in graph interpretation tasks. In the course of the interpretation work, initially unspecified *its* are turned into objects with particular topologies that are said to correspond to specific features in the world. I theorize this interpretive work as a transition of graphs from *things* to *signs* which come to stand for *'natural objects'* and a corresponding double matrix of shifting referents in the semiotic relation of sign-referent-interpretant. Especially among physicists and theoretical ecologists, graphs often become *'natural objects'* in their own right.

This work has interesting implications for studying didactic situations where practitioners (professors, teaching assistants) "explain" the meaning of graphs. Here we find that on the instructor side, graphs do not appear in the explanations but are largely transparent. Instructors talk about phenomena without elaborating the mapping processes, based on a dialectic of familiar phenomena and representation at hand, which we observe when they themselves interpret novel graphs (Figure 5). From a student's perspective, the instructional situation frequently involves a double problem that they

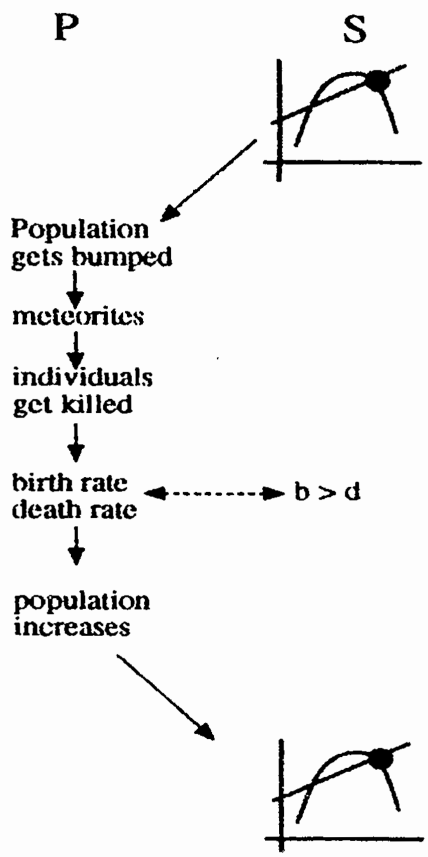


Figure 5. *Representation of a teaching assistant's "explanation" of the population graph to students in a second year university ecology course. The lacking references between the phenomenon P and its representation S show that he talks about states of the world, treating the graph as transparent.*

neither know the natural phenomenon, nor have they constructed the graph as sign object. In this way, we made sense of our previous results which showed that most students in an ecology class and even science graduates (B.Sc.) have considerable difficulties making sense of graphs or using them in appropriate situations.

References

Barwise, J. (1988). On the circumstantial relation between meaning and content. In U. Eco, M. Santambrogio, & P. Violi (Eds.), *Meaning and mental representations* (pp. 23-39). Bloomington, IN: Indiana University Press.

Eco, U. (1984). *Semiotics and the philosophy of language*. Bloomington: Indiana University Press.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*, 1-64.

Lynch, M. (1991). Method: measurement–ordinary and scientific measurement as ethnomethodological phenomena. In G. Button (Ed.), *Ethnomethodology and the human sciences* (pp. 77-108). Cambridge: Cambridge University Press.

Lynch, M., & Woolgar, S. (Eds.). (1990). *Representation in scientific practice*. Cambridge, MA: MIT Press.

Ricœur, P. (1991). *From text to action: Essays in hermeneutics, II*. Evanston, IL: Northwestern University Press.

Roth, W.-M. (1996). Where is the context in contextual word problems?: Mathematical practices and products in Grade 8 students' answers to story problems. *Cognition and Instruction, 14*, 487-527.

Roth, W. M. , & Bowen, G. M. (in press). Complexities of graphical representations during lectures: Aphenomenological approach. *Learning and Instruction*.

Roth, W. M., & McGinn, M. K. (1997). Graphing: A cognitive ability or cultural practice? *Science Education, 81*, 91-106.

Roth, W. M., & McGinn, M. K. (1998). Inscriptions: a social practice approach to "representations." *Review of Educational Research, 68*.

Roth, W. M., McGinn, M. K., & Bowen, G. M. (1998). How prepared are preservice teachers to teach scientific inquiry? Levels of performance in scientific representation practices. *Journal of Science Teacher Education, 9*, 25-48.

RE-THINKING COVARIATION FROM A QUANTITATIVE PERSPECTIVE: SIMULTANEOUS CONTINUOUS VARIATION

Luis A. Saldanha
Vanderbilt University
luis.a.saldanha@vanderbilt.edu

Patrick W. Thompson
Vanderbilt University
pat.thompson@vanderbilt.edu

We hypothesize that students' engagement in tasks which require them to track two sources of information simultaneously are propitious for their envisioning graphs as composed of points, each of which record the simultaneous state of two quantities that covary continuously. We investigated this hypothesis in a teaching experiment involving one 8th-grade student. Details of the student's experience and an analysis of his development are presented.

Confrey and Smith (1994, 1995) explicate a notion of covariation that entails moving between successive values of one variable and coordinating this with moving between corresponding successive values of another variable (1994, p.33). They also explain, in a covariation approach, a function is understood as the juxtaposition of two sequences, each of which is generated independently through a pattern of data values (1995, p. 67). Coulombe and Berenson build on these definitions, and on ideas discussed by Thompson and Thompson (1994b, 1996), to describe a concept of covariation that entails these properties: (a) the identification of two data sets, (b) the coordination of two data patterns to form associations between increasing, decreasing, and constant patterns, (c) the linking of two data patterns to establish specific connections between data values, and (d) the generalization of the link to predict unknown data values (p. 88).

Thinking of covariation as the coordination of sequences fits well with employing tables to present successive states of a variation. We find it useful to extend this idea, to consider possible imagistic foundations for someone's ability to see covariation. In this regard, our notion of covariation is of someone holding in mind a sustained image of two quantities values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value.

In our theory, images of covariation are developmental. In early development one coordinates two quantities values to think of one, then the other, then the first, then the second, and so on. Later images of covariation entail understanding time as a continuous quantity, so that, in one's image, the two quantities values persist. An operative image of covariation is one in which a person imagines both quantities having been tracked for some duration, with the entailing correspondence being an emergent property of the image (Thompson, 1994a). In the case of continuous covariation, one understands that if either quantity has different values at different times, it changed from one to another by assuming all intermediate values.

Purpose and Method of Inquiry

We asked the question, What conceptual operations are involved in students coming to envision and reason about continuous covariation of quantities? We hypothesized that students' engagement with tasks requiring them to track two sources of information simultaneously are propitious for their envisioning graphs as composed of points, each of which records the simultaneous state of two quantities that covary continuously. We elaborated this hypothesis in a teaching experiment involving one 8th-grade student, Shawn.

The teaching experiment covered three sessions. In these sessions Shawn engaged in a sequence of tasks centered around the activity of tracking and describing the behavior of the distances between a car and each of two cities as the car moves along a road (see right side of Figure 1). The bulk of this report is on the results of the first two sessions.

The activity employed a Geometer's Sketchpad sketch that allowed Shawn to simulate the car's movement by dragging a point with a computer mouse. This sketch allowed Shawn to display, or not, individually or simultaneously, the car's distances between it and each city. He could also chose to display those distances as perpendicular segments (Figure 1). Finally, the sketch allowed him to display a point of correspondence and its locus. These latter options, however, were made available to him as instruction proceeded (as were others, such as displaying axes or not). They were unavailable at the outset.

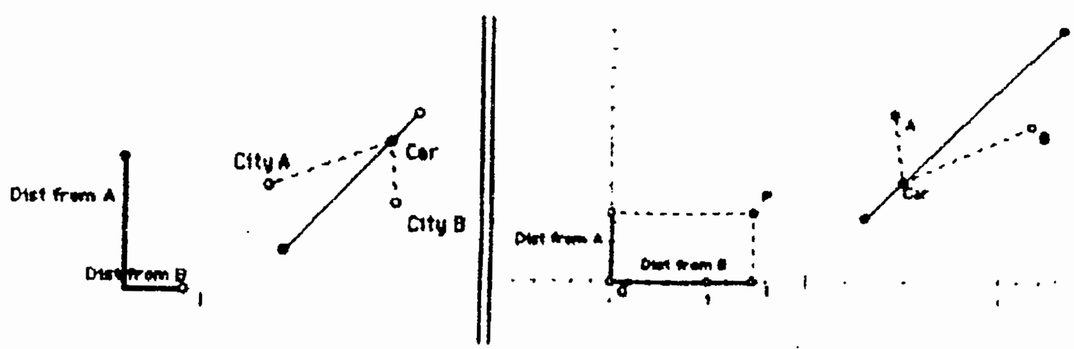


Figure 1. *Two snapshots of car positions. In each snapshot, Distance from A and Distance from B are each represented by a line segment's length. In the snapshot on the right, point P is displayed as the correspondence of the perpendicular segments representing AC and BC.*

The sequence of tasks was in three phases, each focusing on successive levels of operativity in images of covariation. We call the phases engagement, move to representation, and move to reflection. Engagement tasks focused on having Shawn come to understand the setting portrayed by the sketch and the basic task of tracking distances. Move to Representation tasks were intended to support Shawn's internalization of the covariation. Move to Reflection tasks were intended to have Shawn come to imagine completed covariation and its emergent properties.

Results and Analyses

Phase 1: Engagement. In the Engagement phase, Shawn was directed to move the car along the road while watching the distances between the cities and the car. He was also asked to describe each distance's behavior in relation to the car's position along the road. The vertical segment was visible while Shawn investigated the behavior of AC; the horizontal segment was visible while he investigated BC (Figure 1).

Shawn's observations of AC were at two levels. First, he immediately noticed the decrease in AC as he moved C (the car) away from one end of the road, and the ensuing increase in AC as the car passed the point where AC was smallest. He watched the vertical bar closely while moving C, referring to its height (and changes in it) interchangeably with the distance between C and City A. Shawn at first focused only on the distances, not the rates at which distances changed. After being asked whether AC changed faster in some places than in others, Shawn focused on the deceleration and acceleration of AC's length with respect to the changes in C's position. Shawn built up images of this accelerated change by noticing that the variation in the bar's height is almost imperceptible for positions of C near where AC is minimum and that at points farther away (e.g., endpoints) AC changes more with the same changes in C's position. His coordinations remained between changes in the bar's height and changes in C's position. This is as opposed to coordinating changes in AC with changes in C's distance from its start.

Shawn's observations of BC were at the first level; he gave an analogous description of BC's behavior in terms of the systematic variation of the horizontal segment.

Shawn then displayed both perpendicular segments simultaneously, together with a point, P, showing the correspondence of their lengths (see right side of Figure 1). He tracked the motion of point P as he moved the car along the road. His tasks were to describe the behavior of point P, and to say what P and its locus represent. Shawn said of P's motion, it moves with the two bars Ö. It moves along with the X and Y axis. He eventually wrote the following description:

It shows the car getting closer to the cities as both are decreasing. At one point the bar indicating City B pauses, meaning the closest point to City B, as the bar referring to City A keeps declining, be-

cause it is not yet at the closest point to City A. As I approach City A the bar pauses, telling me that I am closest to City A, while the bar referring to City B increases because the distance is increasing and City A is increasing.

Shawn gave this description with no recourse to the GSP sketch. It was as though he mentally moved the car, watching the variation in AC and BC. This is consistent with his having internalized the experience into a coherent set of actions and images which he could re-present (von Glasersfeld, 1995; Piaget, 1970).

Shawn struggled to understand the relationship between P and its locus, however. Underlying his difficulty was some uncertainty as to what P represented. At first Shawn explained that a location of P is a graphical representation of the position of the car on the road. He eventually began to develop a proto-multiplicative view of P, seeing that it's location combines the distance between the car and the two cities and represents how far or close you are to the two cities [...] 'cause you see both the bars.

Shawn's initial conception of the locus of P is revealed by his assertions: this [the graph] is where the car's traveling [...] the road must not be straight, it must be curved, and the car must be the correspondence point and the road must be the graph. After being asked if the graph tells the car's position on the road, Shawn eventually came to view the graph as the path of P which marks the distance between the two cities. He arrived at this by reflecting on the relationship between P's location and its locus, a process that involved having to explain what information was given by P's being in each of several specific locations.

Phase II: Move to representation. In this phase Shawn was presented with depictions of various road-city configurations. He was asked (a) to imagine and to describe the two distances behaviors, (b) to draw a prediction of P's locus as the car moved along the road. He used the sketch to test a prediction; we discussed each result.

Imagining and describing the behavior of P were difficult for Shawn. He required pencil and paper to reflect on the details of how changes in AC and BC would affect changes in P's location. He drew a hypothetical starting position of P and orthogonal arrows to indicate the change in P's position according to changes in AC and BC. By coordinating the values and changes in AC with those of BC he would deduce, and plot, new positions of P. He would then decide on the graph's shape and draw a rough sketch of it. In this way he was generally able to successfully predicted the locus' monotonic portions. The concavities he predicted were always opposite those generated by the sketch. For the first road-cities arrangement, Shawn was unperturbed by the discrepancy. He downplayed it, saying, So my prediction was pretty accurate but I forgot to leave out that little part there. When asked why he thought the graph should be arced one way instead of the other, he stated, All I knew is it was gonna go forward and down. Ö. I didn't know which way is the arc. In succeeding tasks Shawn became increasingly concerned about discrepancies between predicted and actual concavity. The question of how to know the concavity remained an open one.

Phase III: Move to reflection. In this phase Shawn was presented with various graphs plotting AC versus BC. His task for each was to explore and predict possible locations of the two cities relative to the road so that the car's movement would produce that graph. Figure 2 shows a graph examined by Shawn.

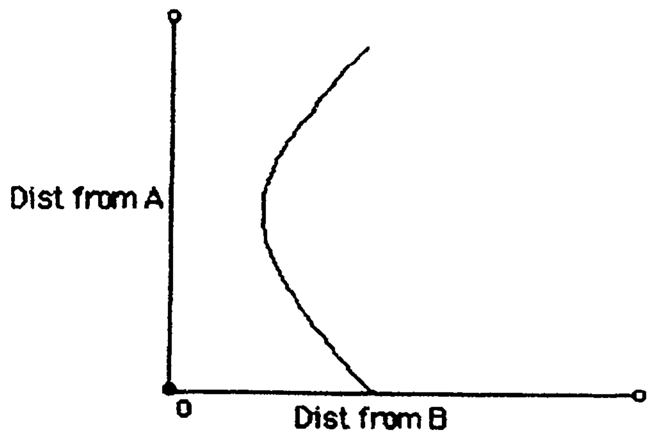


Figure 2. *A graph of AC versus BC corresponding to the completed motion of the car along the road relative to fixed locations of city A and city B*

Shawn identified those points on the graph corresponding to extreme values of AC and BC and he anchored his descriptions of the variations of the two distances around these points:

This is always going closer to A, 'cause it's always going down [...] This seems to be the closest point to B then it starts going right, back up [...] You're farthest from A and closest to A at the extremes of the graph. So figuring it's at the extremes of the road too, at the end of the road [...] The closest point to B must be in the middle of the closest point to A and the farthest one from A.

Thus, Shawn's images were of the completed variation of each of AC and BC individually. Coordinating each quantity's variation with its extreme values allowed him to deduce plausible locations for each city. In this way he was able to construct a corresponding road-cities arrangement, apparently without imagery of AC and BC explicitly covarying simultaneously.

Conclusions

The results of this study lead us to believe that understanding graphs as representing a continuum of states of covarying quantities is nontrivial and should not be taken for granted. Shawn's predominant imagery is consistent with his having developed a level of operativity at which he could intricately coordinate images of two individually varying quantities. There was also suggestion of his developing images of their sustained simultaneity, one that did not explicitly entail a conception of tight coupling so that one variation is not imagined without the other. Seeing graphs as intended here seems to require having tight coupling as a central feature of one's imagery.

References

Confrey, J. & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics*, 26, 135-164.

Confrey, J. & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education*, 26(1), 66-86.

Coulombe, W. N., Berenson, S. B. (unpublished manuscript). *First year students' development of the construction of covariation*. North Carolina State University.

Piaget, J. (1970). *Genetic epistemology*. New York: Columbia University Press

Thompson, P. W. (1994a). Students, functions, and the undergraduate mathematics curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education, 1,* (Vol. 4, pp. 21, 44). Providence, RI: American Mathematical Society.

Thompson, P. W. (1994b). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-236). Albany, NY: State University of New York Press.

Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27(1),* 2-24.

von Glasersfeld, E. (1995). *Radical Constructivism: A way of knowing and learning*. London: Falmer Press.

- The research reported here was supported by Le Fonds FCAR grant no. 972688. The first author gratefully acknowledges this financial support.

- For brevity, we use the symbols AC and BC to denote both the segments and their lengths.

Functions and Graphs
Short Orals

NOTATION AND LANGUAGE: OBSTACLES FOR UNDERGRADUATE STUDENTS' CONCEPT DEVELOPMENT

Marilyn P. Carlson
Arizona State University
carlson@math.la.asu.edu

A broad assessment of undergraduate students' function understandings was conducted by administrating a written exam and performing follow-up interviews with 60 precalculus students and 60 second semester calculus students. Results revealed that both groups of students had difficulty interpreting algebraic function notation. In fact, their inability to interpret the meaning of algebraic symbols was frequently the major obstacle in formulating their responses.

The precalculus students in this study were unable to accurately determine the solution(s) of $f(x) = g(x)$, given the graphs of f and g. They did not recognize that $f(x)$ refers to the "y-value", representing the vertical distance of the graph from the x-axis, and did not appear to recognize that "solving" an equation involves finding the x-value(s) where the y-values (heights) of the graphs are equal. Although some students were able to provide the correct answer to this question, even these were unable to provide a logical explanation for their solutions. They referenced $f(x)$ as "f times x", and indicated that solving $f(x) = g(x)$ involved "finding where the graphs cross".

When prompted to determine graphically what was represented by the expression F(EQ \f(x + y,2)), given a quadratic function F, most of the second semester calculus students had difficulty. They did not recognize _ EQ \f(x + y,2) as the average of two inputs. When asked to graphically represent EQ \f(F(x) + F(y),2), these students were unable to generate words to describe these symbols and were ineffective in discussing the graphic representation of this expression.

The students in this study had difficulty referencing and interpreting symbols in the context of a problem. Rather than struggle to "make sense" of mathematical notation, these students appeared content to work with superficial understandings. Only when confronted with more complex problems did many of their superficial understandings become apparent, suggesting that students need to encounter problems that promote their ability to verbalize using the language of functions.

SIXTH GRADERS EXPLORE LINEAR AND NONLINEAR FUNCTIONS

Mindy Kalchman
University of Toronto
mkalchman@oise.utoronto.ca

A group of sixth grade students (n=10) were introduced to the concept of function using a context-oriented, computer supported curriculum. Specific difficulties that older students have been reported to encounter with this topic were addressed. These included (1) understanding the notion of a variable (e.g., Collis, 1975), (2) understanding the notion of a contingency relationship between two quantitative variables (Piaget, Grize, Szeminska, & Bang, 1977), and (3) recognizing equivalency among the different representations of a function (e.g., Artigue, 1992).

The context of a walkathon was used to introduce students to functions. This context was chosen because children are familiar with money and distance as variable quantities, and understand that the money one earns in a walkathon *depends* on the distance traveled. Students invented different rules of sponsorship, and worked on identifying the characteristics of these rules in their graphic, tabular, and algebraic forms. Next, using spreadsheet technology, students explored how the different representations of a function are connected. This technology allowed students to see the algebraic, graphic, and tabular forms of a function simultaneously, and to witness the automatic effect that varying a parameter in one representation has on the others.

Pre- and post-instructional interviews were conducted. Students were asked a set of questions which were designed to measure (1) prefunction skills, (2) a basic understanding of the function concept, including the notion of a contingency relationship and the many ways of representing it, and (3) advanced functional reasoning. Overall, students improved significantly. The majority of children were successful on items related to prefunctional and basic functional understanding, but were not as successful on items requiring advanced functional reasoning.

A qualitative look at students' responses to individual items was also undertaken and showed that following instruction, students had at least a preliminary understanding of the concept of variable and of the notion of a contingency relationship. They also had developed flexibility in representing a function, and could recognize equivalency among the different representations.

PROCESS, STRUCTURAL, AND ENTITY COGNITION OF LINEAR FUNCTIONS IN A GRAPHING TECHNOLOGY ENVIRONMENT

Francine Cabral Roy
University of Wisconsin Madison
fmroy@students.wisc.edu

Jon Davis
University of Wisconsin Madison
jondavis@students.wisc.edu

Much of the research on the learning of functions addresses dual conceptions of functions as process (procedural) or object (structural). The purpose of this study was to analyze student cognition of functions in a technology based graphical environment through a process/object lens. Our choice to study the graphical representation is shaped by the explosive growth of graphing technology (Steen, 1990; Dugdale, 1993; Dunham & Dick, 1994) and the lack of empirical evidence to support the claim (Goldenberg, Lewis, & O'Keefe, 1992; Kieran, 1992; Sfard, 1991) that graphs encourage structural (object) understanding.

The participants in this study (n=12) were enrolled in a remedial college algebra course. After conducting a two hour lesson on the use of the CBL technology to create distance versus time linear graphs, students were interviewed about a set of tasks situated in the Hiker program context. To analyze structural understanding, the tasks required graphical translations based on (a) a change in starting position (vertical translation), (b) a change in speed (angular translation), and (c) a time delay in starting (horizontal translation).

All interviews were video-taped and transcribed. Using ethnographic techniques (Emerson, Fretz, & Shaw, 1995), the data was coded for general themes and then recoded for more local themes. The final recoding distinguished our definitions of process and structural graphical understanding as well as identified an alternative entity student understanding of the graph as an iconic figure. Only three students exhibited consistent structural understanding across tasks. The rest of the students' process/structural/entity notions were task dependent. In addition, the distribution of process, structural, and entity explanations differed among tasks.

Functions and Graphs Posters

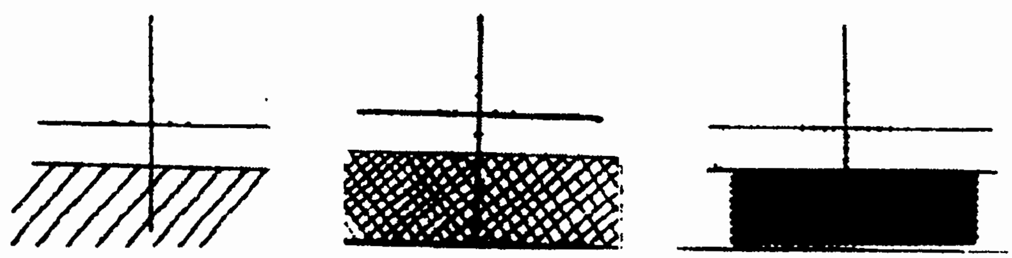
311

DIFFICULTY OF HIGH SCHOOL STUDENTS UNDERSTANDING THE DIFFERENCE BETWEEN DRAW-POINT AND PAIR-POINT

Claudia Acuna S.
Depto. Matematica Educativa
cacuna@mail.cinvestav.mx

Most high school students have already had some experience in plotting points to draw a straight line, usually with success. But these experiences do not make them aware of the differences when the point is on the plane like a pencil mark and when its main role is of an object linked to a pair of numbers. Goldenberg (1988) said that: "Young children have more difficulty coordinating two independent characteristics of an object than in centering their attention on one of them", in our case the points that belong to horizontal or vertical planes have powerful visual images. Our research examined the transformations between graphical, algebraic and natural language representations when students know how to plot in the plane. Three groups of high school students (n1= 87;n2=87; n3=77) were asked: What common characteristics do all the points have on the shadow zone? Students were shown the graphs pictured in Figure 1. Pair-point explanations were given in 1°, natural language indications in 2° and graphic representations in 3°. Table 1 presents the students' results

Reference



1° 2° 3°

Figure 1.

Goldenberg, E.P. (1988). Mathematics, metaphors and human factors: mathematical, technical and pedagogical challenges in the educational use of graphical representations of functions. *Journal of Mathematical Behavior.* 7, 135-173.

Table 1.
Student Results

	negatives positives	negatives negatives positives	negatives	others	y<-4 y<-n y<n
1°	21.8%	5.7%	5.8%	31%	19.5%
2°	31%	10.3%	8%	28.8%	16%
3°	5.5%	0	0	31.1%	36.3%

With 7.8% that take x like y in C6; Absent C2, 14.9%; C3, 4.5%; and C6, 16.9%

PRESERVICE ELEMENTARY TEACHERS' USE OF FUNCTION REPRESENTATIONS IN ANALYZING DATA SETS IN SCIENCE CONTEXTS

Anita H. Bowman
High Point University
abowman@acme.highpoint.edu

B. Gray Bowman
High Point University

The purpose of this study is to determine changes in function representations used by preservice elementary teachers when engaged in the task of writing conclusions based on tables of scientific data. Data set homework assignments were presented — at the beginning of the course, at approximately the midway point, and at the end of the course — to each of 26 preservice teachers enrolled in a sophomore-level physical science course. Each teacher analyzed, in randomly assigned order, data for the following functional relationships: (a) air pressure inside a soccer ball versus its rebound height; (b) angle of inclined plane versus time required for a toy car to travel the set distance; and (c) amount of hydrochloric acid added to a galvanic cell versus the voltage reading for the cell. In each case subjects were asked to write conclusions based on the given data set, explain fully their thinking in reaching that conclusion, and turn in all their written work.

Between the first two data set assignments, subjects completed 7 laboratory activities in which they were taught to analyze data by hand-graphing the data set, and, if approximately linear, (a) drawing the best-fit-by-eye straight line, (b) writing the equation in y = mx + b form, and (c) writing a *quantitative* conclusion based on the equation. That is, they were lead to consider each of the data sets in terms of five functional representations: situation, table, graph, equation, and verbal description. Between the second and third data set assignments, subjects were taught to analyze data sets by linear regression analysis using TI-85 calculators.

Analysis of subjects' responses for homework assignments was based on function representations used. Results for the initial data set assignment indicate that subjects began with a strong dependence on tabular representations. There were subsequent shifts to use of graphical (15%, 46%, 46%) and equation (0%, 42%, 38%) representations, but, surprisingly, no increase in quantitative (versus qualitative) conclusions (35%, 23%, 31%). The fairly nominal transfer of learning from laboratory situations to data set homework assignments might be serious cause for concern about how these subjects, in the future, will teach their elementary school students to make sense of functional data.

FIRST YEAR ALGEBRA STUDENTS' DEVELOPMENT OF THE CONSTRUCT OF COVARIATION

Wendy N. Coulombe
North Carolina State University
wendy.coulombe
@PenchantResearch.com

Sarah B. Berenson
North Carolina State University
berenson@unity.ncsu.edu

To better understand the ideas about functions that beginning algebra students bring to the classroom, eight first year algebra students were interviewed about their intuitive knowledge of covariation. The concept of covariation is based on the idea of change, that is, how changes in one quantity relate to changes in another. For example, when considering an increasing quantity, students use ideas of covariation to determine if another quantity is increasing, decreasing, or remaining constant. This study asserted that the concept of covariation entails (a) the identification of two data patterns, (b) the coordination of two data patterns to form associations between increasing, decreasing, and constant patterns, (c) the linking of two data patterns to establish specific connections between data values, and (d) the generalization of the link to predict unknown data values.

Given problem settings which require understanding of the relationship and change of two variables, we assert that students' ideas tend to evolve through three levels of thinking. We labeled these categories naive thinking, transitional thinking, and extended thinking. At first, students think in terms of procedures and formulas, and their view of function is primarily based on naive conceptions about slope. Next, as students' understanding of covariation begins to emerge, they are able to vary two quantities simultaneously and think in terms of patterns of covariation. Finally, when they extend their thinking, students are able to recognize the particular relationships that link covarying quantities.

The study of functions, appropriately introduced, can serve as a bridge from arithmetic thinking to algebraic thinking. The covariation perspective is an informal approach to the function concept that can leverage students' intuitive knowledge to help them make the connections necessary for meaningful learning of algebra.

SOME DIFFICULTIES FOUND IN THE LEARNING OF CONTINUOUS FUNCTIONS IN FIRST YEAR UNIVERSITY STUDENTS OF ENGINEERING

Hector Lara-Chavez
Unidad de Matematica Educativa
ICE-UAEM, ITZ

First year engineering students' notions of continuous function were explored during university instruction (n=45). According to Hitt (In Press) and others, the problems that students experience when learning a concept are related to difficulties, errors, obstacles, and the lack of articulation between representations of the concept. The obstacles, difficulties and errors may be related to the primitive conceptions of the students, pedagogy employed to teach them, to the perceptions of the students, or to the way in which they learn. This study of the difficulties was conducted during the semester of September 1997 to February 1998 with students from two universities in Mexico. The study was based on a series of two questionnaires about continuity of functions. We explored students' ideas in several semiotic systems of representation: in graphical, numerical, algebraic and natural (or verbal) representations. The students' difficulties appeared related to: a) concept of variable, b) perceiving no distinction between equation and function, c) concept of function, d) translations between semiotic systems of representation, e) operations with functions , f) notion of limit. It seems these difficulties were due to a mix of beliefs and poor teaching.

Reference

Hitt, F. (In Press). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior*.

GEOMETRIC THINKING RESEARCH REPORTS

ANALYZING CHILDREN'S LENGTH STRATEGIES WITH TWO-DIMENSIONAL TASKS: WHAT COUNTS FOR LENGTH?

Jeffrey E. Barrett
Illinois State University
jbarrett@math.ilstu.edu

Douglas H. Clements
State University of New York
at Buffalo
clements@acsu.buffalo.edu

Children coordinate their numerical and spatial knowledge whenever they measure space in two-dimensions. In this teaching experiment, fourth-graders measured and described general cases of polygons that satisfied specific constraints for perimeter or side length. Findings suggest that the transition between uni-dimensional and bi-dimensional geometric contexts requires extensive coordination operations involving numerical and spatial knowledge structures if children are to establish robust structures linking length, perimeter and area concepts. The progress of one child is examined in context to elaborate on the ways he restructured his strategic knowledge of length to incorporate measures of perimeter. Conclusions regarding instructional sequencing are given, along with suggestions for continuing investigation.

Curriculum in geometry typically assumes that children can learn to distinguish area from perimeter by fourth grade, although national tests have indicated otherwise (Lindquist & Kouba, 1989) . As noted by Kamii, many educators suggest that children usually confuse area and perimeter concepts, leading to erroneous responses on both types of task. Kamii counters that children make errors on tasks involving area because of their inability to abstract area from linear measures of space; children cannot understand area until they gain the abstract notion of area as a continuous 'band' or matrix of one-dimensional lines (1996) . Even the measure of linear quantity along a line segment is problematic for many children younger than twelve years; children who have not developed iterative strategies are inclined to count items rather than sub-segments, overlap or gap their iterative operations, and miscount the endpoints, especially with ruler markings (Boulton-Lewis, Wilss, & Mutch, 1996; Cannon, 1992) . Thus, *perimeter tasks*, like area tasks, may demand extensive coordination of linear operations (Barrett & Clements, 1996) . We argue here that operations for measuring perimeter cannot be interpreted as counting operations within a single dimension, but as a coordination of one-dimensional objects within two-dimensional space.

Methodology

This paper is part of a wider study of children's understanding of length concepts; we carried out a comprehensive teaching experiment with four children during the Spring of 1997 (Barrett, 1998) . These children represented distinct strategies for length and measurement. We sought to characterize the mathematics of the children themselves, attempting to make sense of their strategies and progress from within their own collection of concepts and practices (Steffe & Thompson, in press) . Our analysis was two-fold, involving an on-going negotiation between the mathematical thinking of the subjects and our developing models for their geometrical thinking. We employed a framework developed by Clements et. al. (1997) and a set of constructs that derived from a pilot study (Barrett & Clements, 1996) . We focused intensively on negotiations, conflicting meanings and growth as indicators of developing abstraction through increasingly connected representational structures.

Setting a Context for Alex's Story

Anna, Alex, Natasha and Paul exhibited four distinctive strategies for length. We employed a set of three strategy levels to analyze their work (Clements, 1997) . Alex moved from an early level 2 strategy to a more advanced level 2, verging on the use of level 3 strategies near the end of the study. We begin our account of Alex' developmental "growth spurt" by summarizing the changes in the other three. First, Anna, understood length by direct, visual, gross comparison, and sometimes by making inexact correspondence between her counting sequence and the number of visible markers along an object. Anna eventually came to depend less on direct comparison, reflecting instead on her motor activity. Paul, in contrast, understood length by a formalistic and abstract process: he projected conceptual units of units in one- and two-dimensional settings. He combined and decomposed linear figures by constructing and monitoring connections between one-dimensional and two-dimensional units of measure. Alex and Natasha tended to express length as the number of perceptual markers counted along an object. They counted *pseudo-units* for length. While Alex often employed visual guessing and did not establish consistent counts between parts and whole for polygons, Natasha was more consistent in her use of hash marks for length and perimeter. In particular, Natasha was able to establish a careful correspondence along a single dimension, but not along a two-dimensional path, whereas Alex usually failed to establish such correspondence. We sought to understand Alex' attempts to resolve perturbations he met while trying to coordinate his counting sequences along one-dimension with his active re-presentation of two-dimensional perimeter tasks.

Counting discrete items along an object: An inadequate strategy

In the first session Alex measured length along an object by pointing to the visible markers sub-dividing it; he counted along a plastic straw 48 cm in length, marked at 2 cm intervals with small notches cut from its surface. The interviewer suggested that the straw was 24 long, showing him a way of sectioning off first one part of the straw by bending it at the first notch, and then three, bending it at the third notch. Then he asked Alex to find the length:

Alex:	One, two, three, . . . , twenty-two, twenty-three.
Int.:	Okay. and you are touching what?
Alex:	The holes.
Int.:	And you are counting what?
Alex:	How much there are.
Int.:	How much what? 23 or 24 what?
Alex:	That there's a length.

Alex believed he had found the length by counting the number of holes along the edge of the plastic straw. Initially, he appeared to take the holes as components of length along the straw. Alex did not appear to operate on a conceptual image of units of length. His way of marking length did not include the generalization that would have led him to point finally to the end of the straw: he always stopped at the last hole, failing to count the last interval along the straw. However, later during that same interview, Alex drew a rectangular figure that he marked by making both hash marks and dots, placing a dot *in between* each set of hash marks, 24 dots in all around the perimeter of the rectangular figure. The dots corresponded exactly with the partitions created by the hash marks. Thus, by the end of the first interview Alex was able to discriminate between marks that delimited subdivisions along a continuous line segment (indicated by the hash marks) and the subdivided portions of the line segment (indicated by the dots). Nonetheless, Alex persisted in stating that the straw was 23 long, and not 24.

Apparently Alex allowed a disjointedness between his perceptual images for length and his conceptual notions for iterative counting operations on length tasks, much in the same way that the children in Fischbein's study could at once explain that a geometric 'point' might not constitute length or area in one context, yet attribute length to that point in a different figural context (1993) . Alex needed an operational model for connecting one-dimensional line segments into a perimeter.

Abstracting a 'wrapping' metaphor for perimeter: putting fringe on a rug

By now, Alex had used tiles in two different ways by this time during the teaching experiment, both to find length and distance across a band of tiles, and as a way of covering a two-dimensional region. During the third session, Alex combined his existing scheme for length with a 'wrapping' scheme: the interviewer asked him to consider how many "tiles" worth of fringe one would have if they were to wrap fringe around a rectangular-shaped rug in that room (the rug was roughly 2 tiles wide and three tiles long). He eventually found that it would require about 10 tiles worth of string. Later, the interviewer asked him to imagine placing fringe material around a rug that would be seven tiles by six tiles on its sides. Alex looked down at the pattern of floor tiles. He started to walk and count aloud as before, taking four steps in the first four tiles, but stopped:

Int.: Tell me about the perimeter of this rug. How much fringe do you need to buy for this rug?

Alex: [starts to walk around it, taking four steps inside the tiles but suddenly he stops, pauses, and begins talking:] 14, 12, ...[inaudible words here] 6 and 6 is 12 —— 7 and 7 is 14 —— 10 and 10 is 20 —— plus 4 plus 2 is 6, —— its 26.

Int.: You got 26?

Alex: Yeah.

He was beginning to depend on arithmetic schemes for numbers: he began trying to compose perimeter by summing two pair of equivalent values, based on symmetries (e.g. "5 and 5 is 10"). While this spontaneous reorganization led him to stop walking around the physical perimeter (the rectangular figure in the tile floor), as the interviewer, I could not yet be sure of what kind of length units he intended. I supposed his invention of an arithmetic solution was not yet integrated into his figurative perception of the floor tile pattern. So I asked him about his meaning for "units". He offered a tentative response that it was "26 tiles". At this point he used large sweeping motions of his arms to point out entire rows and columns of tiles in the floor along the four edges:

Int.: What does 26 mean here?

Alex: Twenty-six tiles?

Int.: Are there 26 square tiles? where are they?

Alex: Seven across here [sweeping his hands along one edge], and six across there, [sweeping his hands along the next edge] and 7 across there . . .

At the time, I believed he was referring to the literal tiles, both and columns of tiles (suggesting an area unit). His hand-sweeping motion toward rows and columns of tiles confirmed this belief.

As he stood looking at the tile floor and trying to describe how the 26 tiles could make up the perimeter of the rectangle pattern, I asked him once more to identify the 26 tiles. I expected that he would either count each corner tile once and find only 22 tiles, or count outside the figure and find 30 tiles. He began to reflect on his own actions and on his efforts to coordinate four sides:

Int.: So show me the 26 tiles. Where are they? Can you step in one at a time and show me all 26?

[Alex walked around two sides and counted aloud up to 13, but seemed concerned. He halted immediately after stepping through the second side of the figure. I asked him to start over for the camera:]

Alex: One, 2, 3, 4, 5, 6, 7, [7 is the corner tile. He turns the cor ner, and says:] 8, but, no, *it can't be,* (with unusual vocal emphasis) [pausing]

Int.: What do you mean?

Alex: Because you gotta add an extra tile, or use it again.

Int.: Tell me?

Alex: 'Cause I already used this tile and I gotta use it again.

Int.: Why?

Alex: Because I will not get six unless I do.

Alex paused. He was still counting the corner tile as one unit of length, and so by the time he reached the final tile on this second side (6 tiles long), he had only counted on by five, reaching twelve, but he said "thirteen". His hesitation was apparently based on reaching 13 unexpectedly.

Discussion

Alex was ready to re-organize his tile counting scheme for length, but only when his scheme kept him from counting around corners; he wanted to count the corner tile twice now, but lacked an effective justification for his new scheme. He needed a conceptual scheme to fit this context; Alex was still constrained by his one-dimensional concept of perimeter, following from his use of arithmetic operations as connectors to extend his one-dimensional measures of length into the two-dimensional world of perimeters and polygons. Kamii (1996) suggests the need for children to distinguish between discrete quantities of square tiles and continuous regions, and of distinguishing between uni-dimensional thinking and bi-dimensional thinking respecting area tasks. Such discrimination may also be required for perimeter tasks. Cannon (1992) distinguished between counting discrete symbols directly as *items in a collection* and counting symbols indirectly as *subdivisions along a continuous dimension* . We argue here that the operations necessary for measuring perimeter cannot be interpreted as *counting* along a given dimension, but as a *coordination* of one-dimensional objects through a second dimension.

All four children in the study were distracted by the perceptual salience of the notches on the straw, and by the square tile images; they struggled to isolate length along edges in these complex settings (Steffe, 1991) . For example, Alex learned, through the course of these sessions, to differentiate between abstract definitions (which Fischbein terms "conceptual geometric knowledge") and figural-graphic images (which Fischbein terms "figural geometric knowledge"). When he met crises, like when his arithmetic knowledge led him to question his counting of just 12 tiles along two edges that he expected to sum to 13, Alex's abstract numerical notions took precedence over his figural images, a finding consistent with work by

Ferrari (1992) . This account of Alex's work supports the generalization that children who integrate their conceptual and figural-imagistic knowledge resolve perturbations more easily than those who do not (Clements, 1997) .

How do children learn to distinguish between continuous and discrete quantity? In this study, Alex often tried to coordinate his experienced iterations along a linear object and the reified markers along that object symbolizing a previous movement through continuous space. Whenever these two sequences did not correspond exactly, there was a perturbation involving the symbolized *unit* (a continuous unit) and the symbolic *item* (a discrete item). Alex was forced to distinguish between parts of a continuous quantity (the un-segmented linear object) and the delimiters used to partition that quantity (the hash marks) (Steffe, 1991) . The question, "what are children counting for length?" still needs to be addressed in further detail. The question can be posed more generally: "how does one count a continuous quantity?" Ultimately, measurement operations appear to undergird children's knowledge of quantity and counting.

References

Barrett, J. E. (1998). *Representing, connecting and restructuring knowledge: The growth of children's understanding of length in two-dimensional space.* Unpublished Dissertation, State University of New York at Buffalo, Buffalo.

Barrett, J. E., & Clements, D. H. (1996, October 12-15). *Representing, connecting and restructuring knowledge: A microgenetic analysis of a child's learning in an open-ended task involving perimeter, paths, and polygons.* Paper presented at the Eighteenth Annual Meeting: North American Chapter of the International Group for the Psychology of Mathematics Education, Panama City, Florida, U.S.A.

Boulton-Lewis, G. M., Wilss, L. A., & Mutch, S. L. (1996). An analysis of Young Children's Strategies and Use of devices for length measurement. *Journal of Mathematical Behavior, 15*, 329-347.

Cannon, P. L. (1992). Middle grade students' representations of linear units. In W. Geeslin & K. Graham (Eds.), *Proceedings of the Sixteenth PME Conference (Vol. I)* (pp. 105-112). Durham, NH: Program Committee of the 16th PME Conference.

Clements, D. H., Battista, Michael T., Sarama, J., Swaminathan, S., McMillen, S. (1997). Students' development of length measurement concepts in a Logo-based unit on geometric paths. *Journal for Research in Mathematics Education, 28*(1), 49-70.

Ferrari, P. L. (1992). Problem-solving in geometrical setting: Interactions between figure and strategy. In W. Geeslin & K. Graham (Eds.), *Proceedings of the Sixteenth PME Conference (Vol. I)* (pp. 217-224).

Durham, NH: Program Committee of the 16th PME Conference.
Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*(2), 139-163.
Kamii, C. (1996, October 12-15). *Why can't fourth graders calculate the area of a rectangle?* Paper presented at the Eighteenth Annual Meeting: North American Chapter of the International Group for the Psychology of Mathematics Education, Panama City, Florida, U.S.A.
Lindquist, M. M., & Kouba, V. L. (1989). Geometry. In M. M. Lindquist (Ed.), *Results from the Fourth Mathematics Assessment of the National Assessment of Educational Progress* (pp. 44-54). Reston, VA: National Council of Teaching of Mathematics.
Steffe, L., & Thompson, P. (in press). Teaching Experiment Methodology: Underlying Principles and Essential Elements. In R. Lesh & E. Kelley (Eds.), *Research design* .
Steffe, L. P. (1991). Operations that generate quantity. *Learning and Individual Differences, 3*(1), 61-82.

HOW HAS MEASUREMENT BEEN TAUGHT IN MEXICO?

Mariana Saiz
Cinvestav IPN Mexico
msaiz@correo.ajusco.upn.mx

The structure, organization and contents related to measurement in the primary school curriculum have changed along time. An analysis of the textbooks used in school in Mexico at this level, in the last hundred years, offer us a view of the modifications and the coincidences of these models. The identification of seven stages that correspond to different models of teaching measurement is a result of this review.

Introduction

This work is inscribed in the larger research project named Teacher's Thinking About Volume and its Teaching.[1] The method selected for developing the research is the local theoretical models theory (Filloy, 1993). This is an observation theory that gives structure to a global project. For this purpose it integrates four theoretical components -formal competence models, cognitive processes models, communication processes models and teaching models -, this paper exposes the advances performed around an aspect of the latter one.

The teaching models component requires the proposal of a volume's teaching pattern, suitable to the goals of the global project. In order to attain this aim the execution of different tasks is necessary, one of them is reported in this paper. The analysis of other proposals for volume's teaching must be analyzed too. The results of the research presented here reflect the volume's educational tradition, an important aspect to consider when working with teacher's thinking and beliefs. Furthermore, in the same sense, it is important to take in account the teaching model the professor or future professor was taught with.

The construction of the different components is enriched by and gives place to conceptual nets. In them it is possible to distinguish the different elements that interfere in the observed mathematical ideas teaching. These nets take in account the results of the mentioned tasks, as well as the different aspects of the concept in question from a mathematical point of view (which has to do with the formal competence component). Also, they consider the cognitive processes that occur in the subject when learning volume (which can be obtained from the cognitive processes models). In the textbooks analysis, the formal competence models component has taken

[1] This project is directed to the attainment of the Ph. D. degree in the Educational Mathematics department of Cinvestav IPN MÈxico under the gudance of Ph. D. Olimpia Figueras.

an important role by means of the didactical phenomenology of volume proposed and performed by Hans Freudenthal (1983).

Teaching models in the Mexican textbooks (1898-1998)

The analysis of the textbooks used in Mexico gave as result the identification of seven stages, according to teaching patterns distinguished, the place occupied by measurement in the textbook, the proposed activities and the posed questions.

The changes observed from stage to stage are a consequence of educational reforms related to alterations in the paradigms of educational investigation as well as to political modifications in the history of Mexico.

It is important to comment that the Mexican Constitution contemplates education in its structure. The third constitutional article regulates education. The considered period limits in this work, coincide with moments in which changes to this article took place, some others correspond to modifications in the structure of the country's educational system established in educational reforms. Almost half of the period of time considered in this study has been an unstable one in Mexican's history. At each stage, a brief description of the historical moment is included.

From 1898 to 1929

In this stage the primary instruction was not official, the regulations of the time established the existence of something called "rudimentary instruction" that lasted two or three years (Ornelas, 1997). It is interesting to stand out that at the beginning of this stage the Decimal Metric System was recently adopted in Mexico as the official measurement system.

The textbooks for the primary instruction are all titled Arithmetic, many of them do not specify to which level they correspond. There exist textbooks called Geometry but they seem dedicated to older pupils, so the former ones are the only considered here.

In these books there is no chapter or section dedicated to measuring as an activity, in some of them there isn't even a chapter dedicated to measurement systems. In case there is one, it appears at the end of the book, with no previous work or definition. In most of the cases, this chapter reduces to a table presenting the measurement units and their equivalencies. Probably this is due to the fact that the new metric system was being introduced. Although some books have no reference to measurement, all of them include problems with data in denominated number forms, such as 5 liters, 20 kilometers and so on (Pape-Carpentier, 1883). Some authors present the measurement system as an application of arithmetic. Such is the case of a textbook which states "We have considered the numerical quantity as an abstract subject, now we are going to study it from a practical point of view or as refereed to another unity" (Echeagaray Allen, 1899, p. 54).

From 1930 to 1945

After the Cristiada, Mexico affronted a critical moment. An economical crisis affected the country, there were strikes all over the Mexican territory. Mexican historians consider that President Calles decided to offer more education for the people and, at the same time distract the attention from other requests (Ornelas, 1997). As a consequence, about 1934 the third Constitutional Article suffered an important modification: education is declared Socialist. The structure of basic school was also modified. It was stated that primary school would last six years divided into two blocks, the first four years for the elementary primary and the latter two for the secondary (Ornelas, 1997).

In some textbooks of the time, references about the convenience of encouraging children to measure distances, weight objects and similar ones are found. Some of them recommend that lessons should not be restricted to the classroom areas, others propose to make estimations. The variety of activities proposed is wide but some of them were never used before and will never be used again until the most recent stages. For instance, in Pichardo's (1930) mathematics textbook, children are asked to measure the schoolyard, a realistic situation is profited to introduce the decameter and a sequence of activities directed to the deduction of the rectangle area formula is included.

Spanish Professor Mart Alpera's book (1933), which includes Arithmetic, Geometry and Manual Labor, is used in this period. In the last section of the third grade book, to make a paving is proposed (an activity recommended by Hans Freudenthal about 1983 and never mentioned in the Mexican textbooks until 1994). It is interesting to point at the fact that, editions date of this book corresponds to the II Spanish Republic times, which instituted the Normal Spanish Schools in an advanced position with other similar centers in Europe, at the same height than the German innovations (Sierra and Rico, 1996, p.45).

A third example at this stage is an arithmetic textbook by Kempinsky (1938) which is a translation from German. As its contemporaries, the recommendation to look for situations in the daily experience of children is present. In these books measurement is not left to the end of the book. It is exposed in a section or group of lessons, dedicating some pages to each one (length, area, volume, capacity and weight).

From 1945 to 1960

This period coincides with one more rectification to the text of the third constitutional article. The statement that education is socialist disappears, instead there is a principles declaration in the sense that the aim of education is to develop all faculties of the human being, to encourage love to the homeland and the international solidarity's conscience of independence and justice (Ornelas 1997).

In comparison with the previous stages, the amount of text in each lesson is reduced. Didactical ideas such as making children measure and the recommendation to relate measuring activities to daily tasks continue to be used. The exercises proposed have the aim of giving sense to measurement. Regression in some aspects is perceived when comparing with the former stage. For example estimation, manual labors related to measurement and the usage of non-conventional units do not appear in these books. Measurement topics are presented one behind the other in a block of lessons, following the order length, area, volume, capacity and weight. Presenting all the multiples and submultiples of the principal unit in each case.

From 1960 to 1971

The fourth stage is a very important one in the Mexican history. The country goes through a period of stability. In 1960 the first edition of the Gratuitous Textbooks appeared. From that time on, the Ministry of Education (Secretarìa de EducaciÛn P'blica) has edited, and freely distributed, the textbooks to be used in all the country's primary schools.

The textbooks of the sixties show advances in the didactical approach. All the units and equivalencies, which formerly were given in a lesson or block of lessons, are now distributed throughout the six years of primary education. The approximation to measure is made gradually. In the first grade they work only with some of the principal units of measure, such as meter and liter. Area is approached until the third grade. In spite of this improvement, measuring is always performed and presented using the conventional measurement system. The books are not homogeneous, neither by grade nor the whole series from the first to the sixth level. In each textbook some topics are presented using affective situations that involve children in its usage and learning, while others are not. For instance, in the third grade book (Caballero y VillaseÒor, 1960) a section of lessons dedicated to measurement is included. For length and area an affective situation is posed at the beginning of the respective lesson. The case of volume, capacity and weight is different. For volume the lesson initiates defining the one meter side cube as the unity of volume measurements and then its multiples and submultiples.

From 1972-1980

The books edited by the government in 1972 were recommended to a commission of mathematicians. Although none of them had experience as primary level teaching, the result is a collection of books that reflect most of the paradigms on educational investigation of that time. In fact, this stage coincides with the institutionalization of educational mathematics research, and is a consequence of the failure of the so-called Modern Mathematics (Hitt, 1994). Mathematical ideas are presented in a practical context, like a mean to solve realistic problems. "They are not exposed as consummated facts but introduced with the intention that children will discover them" (Imaz, 1972, p. 8). Measurement's treatment reflects this aim very well. Before asking children to measure with standard units, the tasks proposed invite children to meditate about measurement, to make comparisons of length, area, volume, capacity and weight. All these activities may be performed with non standard units or using a direct comparison without numbers. The presentation of the standard measure system is subordinated to the proposal of problems and activities that show the necessity of counting with a conventional unit.

Besides these considerations, new elements appear such as using a balance to compare areas or the archimedean principle for measuring volumes. Estimation is recommended and used along the books for all kind of measurement.

From 1980 to 1993

In the previous period, textbooks were recommended to a group of specialists in mathematics. In fact, all the textbooks of the seventies were recommended to the different subjects' specialists. Teachers were not taken in account in this reform (Ornelas, 1997). Many of them were not prepared for the proposal included in the books. This situation caused a rejection from a sector in the Mexican Educational System. In order to diminish this antagonism, the Educational Ministry promotes, in 1980, a new reform. A group of basic level teachers initiate the task of developing a new series of books. The textbooks are integrated. Subjects are not separated in Natural Sciences, Language, Social Sciences, and Mathematics, as had always been. In these two textbooks the references to measuring are scarce. The third grade's textbook is modified in some lessons but it is essentially the same of the previous version. Textbooks from fourth to sixth levels are exactly the same used in the seventies. This situation produces a lack of continuity from the first two levels to the rest of them.

From 1993 to our times

In the scholar year 1993-1994 the new primary curriculum goes into effect. A new edition of the Gratuitous Textbook series is presented. The proposal of the new primary curriculum, shaped in the new plans and programs, is to develop the ability to learn permanently and with independence (SEP, 1994).

The Mathematics textbooks show important innovations. Problems solution is the mathematical learning motive (SEP, 1994). The idea of relating mathematical contents is deeply exploited. So it is the presentation of each subject in a realistic context. Lessons pretend children to extend their level of competence, in the mathematical structures included in the curriculum, as they advance in their way through school.

For measurement, most of Freudenthal's and other specialists' recommendations for the attainment of mental objects such as length, area and volume are present. Comparing the amount of material related, for example to volume, between these books and the former ones, a diminution is detected. The different units, multiples and submultiples appear only when they are needed. Many of them are never presented. This loss is compensated with the great variety of activities and tasks proposed.

While the amount of formulae diminishes (in opposition to the previous stages) every perimeter, area or volume formula that appears in a lesson has been deduced. The rectangle area formula is applied to obtain that of the triangle, and both of them are used to obtain those of a lot of quadrilaterals. In the sixth grade the formula for the circle area is intuitively obtained by approximations of the circumscribed polygons using triangles.

For volume, the only formulae that appear are those for the cube and rectangular prisms, in the sixth grade, and each of them is motivated by a lot of previous work that relates linear dimensions with cubic units.

In these books, measuring topics are not presented in the usual order (length, area, volume, capacity and weight) measurement in one or two dimensions is treated in parallel with the rest of the topics so a volume or weight activity may be encountered before some others for length or area. Measurement is spread all along each level textbook. There is not a series of lessons for these topics, in fact, lessons in the sixth grade book integrate various mathematical ideas from different axis such as fractions, measuring, probability or so.

Conclusions

Measuring teaching models vary along a century of Mexican education. These variations are grouped in seven stages. The amount of units and formulae presented diminish from stage to stage. The amount of activities proposed by the textbooks for each measurement topic increase from stage to stage. Although some ideas, such as inviting children to perform measurements, present realistic situations, work out of the classroom and some others are made explicit by some authors in all stages, not always they are reflected in the textbooks. With the exception of the period 1980-1992, this situation is improved from stage to stage.

References

Caballero, S. (1960). Mi libro de tercer aÒo. AritmÈtica y GeometrÌa. Conaliteg. SEP. Mexico.

Echeagaray, Francisco. (1899) AritmÈtica. Ed. Imp. Hijos de J.F. Jens. Mexico.

Filloy, E.. (1993). Tendencias cognitivas y procesos de abstracciÛn en el aprendizaje del ·lgebra y de la geometrÌa EnseÒanza de las ciencias , vol. 11, n˙m2. Pp 160-166. Valencia, EspaÒa.

Freudenthal, H.. (1983) Didactical Phenomenology of Mathematical Structures. The Netherlands: D. Reidel.
Hitt, F. (1996). Research Culture: Research orientation in mathematics education in Mexico. Proceedings of the 18th Annual Meeting of the PME-NA ERIC Columbus OH, USA.
Imaz, C. (1972). Matematicas, libro del maestro para el segundo grado Conaliteg SEP Mexico.
Kempinsky, Enrique. (1938) El primer aÒo de aritmÈtica por medio de juegos Ed. Rafael Donde Mexico
marta A., Felix (1933) AritmÈtica, GeometrÌa y Trabajo manual 1∫, 2∫ y 3er grados Ed. Publicaciones de la Revista de pedagogie Madrid, EspaÒa.
Ornelas, Carlos (1995) El Sistema Educativo Mexicano. La transiciÛn de fin de siglo. CIDE, Nacional Financiera, FCE. Mexico.
Pape-Carpentier Marie Olinde. (1883) AritmÈtica EdiciÛn especial para niÒos Imprenta Antonio M. Rebolledo Coatepec, Mexico
Pichardo, J. Arturo (1930) AritmÈtica del Hogar II Ed. Herrero Mexico.SEP (1994) Libro para el maestro Matematicas Tercer grado Mexico

UNDERSTANDING ANGLES FROM THE PERSPECTIVE OF A HIGH SCHOOL CEREBRAL PALSY STUDENT

Kenneth L. Shaw
Florida State University
kshaw@mail.pc.fsu.edu

Paul Durden
Mosley High School
bay4141@interoz.com

This paper examines how a high school cerebral student understands angles and angle properties. A case study during the one semester course was used to analyze her work. Results indicate that at least three factors impacted her learning and understanding of angles: orientation (layout of the angles themselves), dynamics (movement of angles or sides of angles), and the lengths of the sides of the angle. The use of technology, particularly *The Geometer's Sketchpad* was a great aide in assisting the student to understand the angle properties.

Participant

Amanda is 16 years old and has a moderate form of spastic cerebral palsy. Due to her muscles being permanently contracted, Amanda's movements are rigid and somewhat spasmodic. She has taken physical therapy treatments to improve her muscle control. As a result, she is capable of walking short distances with the aid of a walker. However, while at school, she travels about in her motorized wheelchair. Amanda has a delightful and positive attitude and cherished the time we spent with her. Her goal in the research project was to have people better understand her and her disability, "I just want people to be more educated about it, and I want people to realize that we are just like everybody else; we are normal people; we can learn. We are smart" (Shaw, Durden, & Baker, 1998, p. 204).

Seventy to eighty percent of all cerebral palsy cases fall into the category of spastic cerebral palsy (Borowitz, 1995). Close to a half million people in the United States have cerebral palsy and each year, about 5,000 babies and infants are diagnosed with cerebral palsy (United Cerebral Palsy Association, 1996). A third of these children will be moderately or severely impaired, a third mildly intellectually impaired, and a third will be intellectually normal. Amanda falls in the latter category, being intellectually normal.

Theoretical Framework

Due to the paucity of research on geometry learning by cerebral palsy students, we had to expand our examination of articles to those written about how cerebral palsy students learn mathematics in general. Magne (1994) made the connection that students with poor motor skills will often exhibit poor retention when it comes to mathematics. Since cerebral palsy affects every person differently, the retention problems will certainly be proportional to the extent of the brain damage of the student. Magne goes on to suggest that manipulatives, high visual tools, and individual instruction should be a high priority to maximize the learning of disabled students. Corneliussen, Lund, and Nilsen (1989) also provide helpful recommendations for coping with learning disabilities often seen in children with cerebral palsy: (a) Emphasize individualized instruction and pacing, (b) make the instructional environment as distraction free as possible, (c) subdivide the instructional activities into smaller activities, and (d) provide individualized instruction.

Thomson (1993) concludes from his study that computers can meet the needs of the cerebral palsy student. The computer would provide the student a way to be in control of their own work, express themselves adequately, and be independent.

Methods

Since a close examination of Amanda's understanding of angles was required, a case study method was chosen. During the semester geometry course, approximately 10 interviews were conducted with Amanda. Amanda always had her laptop computer with *The Geometer's Sketchpad* (Jackiw, 1995) software. The interviews were centered around geometry tasks in which she utilized the computer to assist her in her response. Amanda's occupational therapist was also interviewed to assist us in understanding Amanda and cerebral palsy. The therapist, who had worked with Amanda for several years and who has worked with more than 100 cerebral palsy students in her 21 years of being a licensed occupational therapist, sat with us during one of the interviews with Amanda and provided her expertise on why Amanda responded the way she did. Each interview was recorded and transcribed. At the beginning of the study, the interviews were very general, but as the study progressed, the interviews became focused on her understanding of angles. Throughout the study, her computer sketches were used to assist us in determining her understanding of angles. The sketches were ones that she created independently in class, on homework, and during the interviews.

Results

The results center around three main findings: orientation, dynamics, and side length of an angle. These three factors influence how Amanda views angles. Below are brief examples of how these categories impacted Amanda's learning.

Orientation

It's just that I have this picture of all those angles, and I can't see them very well because of my visual perception. And to me, if it's not straight up, if it's turned a different way, I can't tell a 90-degree angle. (March 5, 1997)

This quote is indicative of the problem Amanda had with orientation. If the sides of a right angle are oriented horizontally and vertically, then Amanda can easily identify the angle as a right angle. However, if one rotates the angle a quarter-turn around the vertex, Amanda has great difficulty identifying the angle measure by visual inspection. Where orientation for the two authors was not a problem, it was a tremendous challenge for us that recurred throughout the study. We were challenged and had a strong desire to see what Amanda saw. We metaphorically wanted to view the world through Amanda's eyes.

Amanda was able to use *The Geometer's Sketchpad* to assist her in measuring the angles. She realized that the way she viewed things was not always the way they were. This was frustrating to her. One example was when she estimated an angle measure to be close to 60 degrees, but when using *The Geometer's Sketchpad* she realized it was 90 degrees. As time went along, she became more confident that the measures found using *The Geometer's Sketchpad* were accurate, "When I measure it [an angle], I know that's how it has to be." When asked if she still saw a figure differently even after she measured the angles using the computer, she replied, "Yes, it still looks wrong, but measuring helps me believe what it's supposed to be." Amanda's visual skills are very poor and are not consistent. The technology built her confidence and allowed her to demonstrate with consistency concepts such as similarity and congruency.

Dynamics

Dynamics refer to moving an angle or part of an angle on the computer screen. *The Geometer's Sketchpad* is designed so students can easily modify figures by dragging points or segments. In probing Amanda's perspective of angles, we asked her to tell us when the line segment EF would be perpendicular to segment CD (see Figure 1). The instructor then moved point E from right to left rather slowly. As E moved from right to left, points G and H moved along segments CD and AB respectively while F remained fixed. Amanda found this task difficult.

I: Can you tell me when they look the same? (The instructor moved E from right to left, approaching and passing perpendicular.)

A: Which ones?

I: These . . . (Once again the instructor points to the angles EGD and EGC.)

A: I can't see it. . . . I don't know. . . . I'm sorry. (Amanda seems very frustrated.)

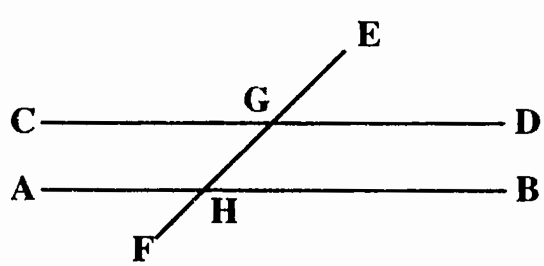


Figure 1. *Amanda's Difficulty With Dynamics*

I: No, that's fine; we need to know that you can't see it. . . . Does it help to change the colors of the lines?

A: A little I think.

I: Well what about these angles now? (The same linear pair at the top of the sketch, EGD and EGC.)

A: This one and this one . . . ? (Amanda indicates the angles but does not respond)

I: Can you drag the point so they look the same to you?

A: (Amanda tries) I can't tell.

We soon realized that Amanda had great difficulty with the movement of the figure. It was only after we removed segments AB and FG and moved E from right to left slowly, stopping intermittently for about five seconds, that she was able to visually process the information and be able to state with some degree of confidence and accuracy when EG became perpendicular to CD (see Figure 2). Two points we learned. One is that the figure must be simple. The segment AB added a level of complexity and caused the visual processing to be overwhelming. With the software, it was easy in the future for Amanda to hide the extraneous data so she could focus on the desired task. Additionally, we learned that the dynamics cause visual processing problems for Amanda. She needed time to process each action in still-mode to grasp the information.

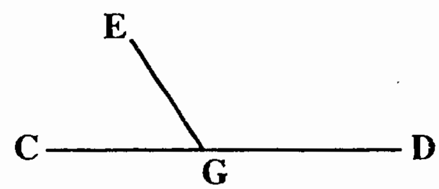


Figure 2. *Minimizing the Visual Distraction*

Side Length of an Angle

Another problem surfaced with how Amanda viewed angles. It had to do with the angle side length.

I: Is there something about those longer lines that make the figure [see Figure 3] look different to you?

A: Yes, it's hard when this one is long [pointing to BC] and this one is short [pointing to AC]. . . . It doesn't look like all of the angle is there. It looks like this one is bigger [angle DCB] but I know it's not.

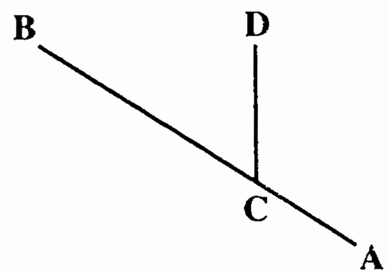


Figure 3. *Angle Side Lenghts*

On several occasions she mentioned the angle where the side lengths were longer made the angle bigger. Amanda's visualization of this and similar figures was so influential in her thinking that it appeared to overpower the facts of the figure. Labeled an "uncontrollable image" (Presmeg, 1992), the image was so vivid it caused Amanda to be blinded by other important elements of the problem.

Discussion

Although our goal was to see what Amanda saw, view geometry through Amanda's eyes, we believe we have only received a glimpse of how she viewed geometry. We now better understand how multiple visual components cause undo distractions, how dynamics blur understanding while proper processing time aides in understanding, and how side lengths of angles affected Amanda's estimations of angle measures. Amanda remarked how she enjoyed using the technology, "I like using the computer in geometry because the figures I construct can look pretty, like other students." The technology was a necessary and valuable aid in contributing to Amanda's success in high school geometry.

References

Borowitz, K. (1995). *Cerebral palsy: A multimedia tutorial for children and parents.* Available: http://avery.med.virginia.edu/~smb4v/tutorials/cp/cp.htm.

Corneliussen, G., Lund, M., & Nilsen, E. (1989). Coping with common learning disabilities in children with cerebral palsy. In H. Rye & M. Donath (Eds.), *Children with severe cerebral palsy: An educational guide.* (ERIC Document Reproduction Service No. ED 331 219).

Jackiw, N. (1995). *The geometer's sketchpad (Version 3.02)* [Computer software]. Berkeley, CA: Key Curriculum Press.

Magne, O. (1994). *Mathematics and quality of life: A new theme in special teacher education.* (ERIC Document Reproduction Service No. ED 377 667).

Presmeg, N.C. (1992). Prototypes, metaphors, metonymies, and imaginative rationality in high school mathematics. *Educational Studies in Mathematics, 23,* 595-610.

Shaw, K. L., Durden, P., & Baker, A. (1998). Learning how Amanda, a high school cerebral palsy student, understands angles. *School Science and Mathematics, 98*(4), 198-204.

Thomson, R. (1993). A case study of a cerebral palsy student's use of a computer. *Masters Abstracts International, 33* (02), 0333. (University Microfilms No. MM90900).

United Cerebral Palsy Association. (1996, February). *Cerebral palsy: Facts & figures.* (Available from United Cerebral Palsy Association, 1660 L Street, NW, Suite 700, Washington, DC 20036-5602).

ANALYZING STUDENTS' LEARNING IN SOCIAL CONTEXT: A STUDENT LEARNS TO MEASURE

Michelle Stephan
Vanderbilt University
michelle.l.stephan@vanderbilt.edu

The purpose of this paper is to provide an account for one student's learning as it occurred in the social context of the classroom. This paper provides examples taken from a four-month classroom teaching experiment in order to illustrate the view that students learn as they participate in and contribute to emerging mathematical practices. The focus of the classroom teaching experiment was on the development of an instructional sequence intended to support students' construction of personally-meaningful ways to reason about measurement.

In recent years, researchers have shown that learning involves social and cultural aspects (cf. Lave, 1988; Rogoff, 1990; Saxe, 1991). Thus, the view that learning occurs in a socially-situated context is increasingly common. Despite different perspectives on the role of social and cultural processes in learning, most researchers in this area agree that students' development cannot be adequately explained in cognitive terms alone. In our view, learning is both an individual construction *and* a process of enculturation (Cobb and Yackel, 1996). This perspective, called the emergent perspective, allows us to view individual students' mathematical development as participation in the taken-as-shared mathematical practices of a classroom community. The purpose of this paper then is to use the findings from a case study to describe the mathematical development of one student as she participated in the mathematical practices of a classroom community.

Setting

The first-grade classroom that was the subject of this study was one of four first-grade classrooms at a private school in Nashville. The class consisted of 16 children, 7 girls and 9 boys. The teacher was an active member of the research team[1] and continually worked at developing a teaching practice consistent with the reform guidelines of the NCTM *Professional Standards for Teaching Mathematics* (1991). The data for this study was collected during a teaching experiment which lasted from February to June

---

[1] Throughout this paper I will speak in third person when referring to specific details concerning the first-grade teaching experiment. In these cases it is important to acknowledge the research: Paul Cobb, Beth Estes, Kay McClain, Maggie McGatha, Beth Petty and myself. Erna Yackel and Koeno Gravemeijer also collaborated on this project but did not participate on a daily basis.

1996. One objective of this teaching experiment was to design and enact an instructional sequence focusing on measuring. The instructional intent of the measurement sequence was formulated such that measuring was an activity in which students were mentally acting on space. Further, we hoped that the number that results from iterating a measurement unit for the last time would signify not simply the last iteration itself but rather the result of the accumulation of the distances iterated (cf. Thompson & Thompson, 1996). The two students who appear in this paper were followed throughout the teaching experiment on a daily basis. Every whole-class and *targeted* small-group discussion was video-recorded.

Classroom Mathematical Practices

The notion of a classroom mathematical practice has been discussed in several papers (e.g., Bowers, Cobb, & McClain, in press; Cobb and Yackel, 1996; Yackel, 1997). Classroom mathematical practices are the taken-as-shared ways in which a classroom community comes to reason and communicate mathematically. These communal practices are established as students explain and justify their solution methods and often involve symbolizing. One indication that a mathematical practice has become taken-as-shared in the classroom community is that certain mathematical interpretations have become beyond justification (Yackel, 1997). The relationship between mathematical practices and individual students' mathematical interpretations is seen as reflexive. Individual students' development is analyzed in terms of their participation in the emerging, communal mathematical practices. Further, students are seen to contribute to the evolution of the classroom mathematical practices as they reorganize their activity. In the remainder of this paper, I draw one example from a larger case study to illustrate an analysis of one student's learning that is cast in terms of participation in mathematical practices.

Results

On the 11th day of the instructional sequence[2], the teacher posed problems in the context of a smurf village. Students were asked to pretend they were smurfs and use a bag of food cans (i.e., Unifix cubes) to find how long various items around the smurf village (classroom) were. The taken-

---

[2] The first 10 days of the instructional sequence consisted of an introduction to the notion of measuring by engaging students in pacing activities. Students paced the lengths of items to find their measure and subsequently traced five feet on paper to form a footstrip. Two mathematical practices emerged as students engaged in these activities. Due to the shortness of this paper, I draw my example from episodes that occurred 11 days after the sequence began. For a more detailed analysis see Stephan (1998) and Stephan, Cobb, Gravemeijer, & McClain (1998).

as-shared interpretation of measuring items in the classroom involved filling the length of an item with a bar or rod of cubes that stretched the length of the item and then counting the cubes. The teacher suggested using a more efficient measuring device: a bar of 10 cubes/food cans which they named a smurf bar. Instructional activities with the smurf bar included having students find the lengths of various items around the room and cutting pieces of paper signifying different-sized wooden boards for building a smurf house.

The mathematical practice that became interactively constituted as the students engaged in the instructional activities above concerned measuring by iterating a bar of 10. The teacher and researchers typically focused whole-class discussions on the results of measuring with the smurf bar. For example, if a student was finding the length of a table by iterating the smurf bar end to end and counting "10, 20, 30, 33," we asked the student "Where is 33?" or "Can you show how long something 33 cans is?" We posed these types of question for two reasons. First, we were trying to encourage explanations that focused on students' interpretations of their measuring activity with the smurf bar/food cans rather than focusing on simply an observable method of iterating. Second, we were trying to support students' interpretations of measuring as an accumulation of distance. In other words, when students had completed a third iteration and uttered "30," some students may have reasoned that 30 signified the space filled by the 30th cube rather than the space filled by 30 cubes. It was therefore important to us that this issue become an explicit topic of whole-class discussion. As students participated in these types of discussions, it became taken-as-shared that "the whole 33," for instance, signified the space extending from the beginning of the 1st cube to the end of the 33rd cube (i.e., an accumulation of distance interpretation).

It is important to note that this taken-as-shared understanding was explicitly negotiated during whole-class discussions. Initially, several students reasoned that "20" signified the *10* cans comprising the second iteration and that the 10 cans in the 2nd iteration would be counted "21, 22, 23, . . .". However, as students discussed alternative mathematical interpretations, it became taken-as-shared that the result of iterating a bar of 10 signified an accumulation of distance. Further, it became taken-as-shared that physically iterating along the spatial extent of an item created a partitioned, whole space. The account of the taken-as-shared mathematical meanings given above serves to document the immediate social situation in which students were acting. No claims are being made in the above analysis about how any individual is reasoning; rather, the analysis documents the taken-as-shared mathematical interpretations of the community. In the remainder of the analysis, I use the mathematical practice as backdrop against which to explain one student's, Meagan, mathematical learning.

As Meagan *initially* participated in these conversations, measuring, for her, appeared to be dependent on the act of placing the bar of 10. For example, on the 14th day of the instructional sequence, the students were asked to measure with a bar of 10 cubes for the first time. The teacher asked the students to work in pairs and to measure items in the classroom. Using a smurf bar for the first time, Meagan began measuring the height of an animal cage. She placed one end of the smurf bar at the bottom of the cage and said, "10." She iterated the bar end to end along the height of the cage and counted "10, 20." She placed the bar a third time and counted "30" even though the third iteration of the smurf bar extended past the top of the cage. Then, she counted the cubes within this placement "31, 32, 33." It seemed that for Meagan, measuring was dependent upon the placement of the bar of 10. For Meagan, counting "30" as she placed the smurf bar for a third time meant that the cubes within that iteration should be counted "31, 32, 33, . . ." This way of participating in the third mathematical practice indicated that Meagan was not coordinating measuring with the bar of 10 with measuring with individual cubes of which the bar was composed. In other words, iterating with the bar of 10 did not signify a curtailment of measuring with individual cubes.

Her partner, Nancy, indicated that she disagreed with Meagan's measurement and remeasured the height of the cage by counting as follows: (iterates the bar once) "10," (iterates the bar a second time) "11, 12, 13,. . ., 20," (iterates the bar a third time) "21, 22, 23." It appeared that for Nancy, iterating the bar of 10 signified the space filled by cubes thus far as can be seen when she counted by single cubes to justify her method of measuring. Further, when asked where "20" was, for example, Nancy indicated the space from the beginning of the 1st cube to the end of the 20th cube. This suggests that the result of measuring signified an accumulation of distance for Nancy. Although Meagan accepted Nancy's measurement, she continued to measure in the manner she had before.

During the subsequent whole-class discussion, Nancy and Meagan measured the length of the white board at the front of the classroom as the teacher used a pen to mark the end of each iteration and wrote numerals to record how many cans they had iterated thus far. Further, as they measured, a researcher asked questions such as "Where is the 20?" Nancy answered these questions by pointing to the space beginning from the edge of the board to the numeral 20. The mathematically significant issue that became the topic of whole-class discussion involved describing what each number word meant in terms of the amount of space/individual cubes that had been filled/iterated. We conjectured that this type of discussion was important in that the conversation dealt with what measuring signified, the quantities specified while measuring, rather than with only how to measure (i.e., how to count cubes). Although Meagan did not overtly participate in this exchange, she seemed to reorganize her prior participation as indicated by her activity on the following day.

The next day, measuring seemed to signify the accumulation of distance for both Nancy and Meagan. During a whole-class episode, Alice and Chris demonstrated where they would cut a piece of adding machine tape (signifying the length of a wooden board to be used to build a smurf house) so that it measured 23 cans. They iterated the smurf bar once and said "10," iterated it a second time and said "20," and then iterated it a third time and counted individual cubes "21, 22, 23" from the end of the second iteration. Edward challenged their explanation by arguing that they had really only measured 13 [*sic*]. He then described his method by iterating the bar three times saying "10, 20, 30" and then counting "31, 32, 33" within the third iteration (as Meagan had done the day before). A discussion then ensued in which Alice counted individual cubes by 1s to show Edward and others that two iterations signified the distance covered by 20 single cubes. Then, she counted three more cubes from 20 to show 23. Thus, the mathematically significant issue that had emerged as a topic of conversation involved counting the number of individual cubes that were accumulating as the bar of 10 was iterated (coordinating measuring with a bar of 10 with measuring with single cubes). Meagan explained that she did it a different way than the others but said that she needed the teacher's help. She iterated a smurf bar while the teacher marked the beginning and end of each iteration with a piece of masking tape. The researcher stopped her after two iterations and asked her how many cans would fit in the spaces she had marked. She answered that 20 cans would fit in the total space marked by the two iterations thus far and counted up three more cans to mark 23.

This episode is significant for two reasons. First, Meagan seemed to have reorganized her prior activity as she participated in the conversations with her partner and in the whole-class discussions during the last two class periods. In the context of the preceding whole-class discussion, iterating the rod of 10 twice signified the accumulation of space covered by 20 cubes. Another way to say this is that the number word "20" signified a composite unit, an entity or an amount of space covered by 20 cubes rather than the space covered by the 20th cube. A second reason this episode is significant is that it brings to the fore the role of symbolizing in Meagan's activity. Meagan asked for the teacher's help in making a record of her measuring activity. As she reasoned with these symbols, Meagan interpreted the result of each iteration as an accumulation of space filled by cans. Further, when reasoning with such symbols, she coordinated measuring by iterating a bar of 10 and measuring by iterating single cubes. Thus, as she participated in the prior conversations in which mathematically significant issues arose, she developed relatively powerful ways of reasoning with symbols. When she *did* reason with symbols, Meagan coordinated measuring with a bar of 10 and measuring by iterating single cubes. However, when Meagan did *not* reason with symbols, she did not make such a coordination. This points to the significant role that reasoning with symbols played in Meagan's measurement activity. Further, in participating in these and other whole-class discussions in this way, Meagan contributed to the constitution of the third mathematical practice. In other words, her reasoning with symbols was an act of participation in the emerging mathematical practice and constituted not only her learning but also a contribution to the taken-as-shared practices of the community.

Conclusion

In this paper, I presented one example to illustrate the view that students' learn as they participate in and contribute to emerging mathematical practices. Although the example mainly focused on Meagan, it is clear that both Meagan and Nancy participated in and contributed to the math practice in different ways. The mathematical practice, as well as whole class discussions were described in order to document the social context in which Meagan and others were participating. An analysis that focuses *only* on the development of students, as opposed to one that locates students' learning in social context, would have described only the cognitive reorganizations Meagan made (merely as a consequence of social interactions). In contrast, the analysis in this paper gives equal importance to social and psychological processes by describing students' learning as participation in mathematical practices.

References

Bowers, J. S., Cobb, P., & McClain, K. (in press). The evolution of mathematical practices: A case study. *Cognition and instruction*.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31*, 175-190.

Lave, J. (1988). *Cognition in practice: Mind, mathematics, and culture in everyday life*. Cambridge, England: Cambridge University Press.

National Council of Teachers of Mathematics (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

Rogoff, B. (1990). *Apprenticeship in thinking: Cognitive development in social context*. Oxford, England: Oxford University Press.

Saxe, G. (1991). *Culture and cognitive development: Studies in mathematical understanding*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Stephan, M. (1998). *Supporting the development of one first-grade classroom's conceptions of measurement: Analyzing students' learning in social context.* Unpublished doctoral dissertation, Vanderbilt University.

Stephan, M., Cobb, P., Gravemeijer, K., & McClain, K. (1998). *The evolution of one first-grade classrooms' development of measurement conceptions* Paper presented at the annual meeting of the American Educational Research Association, San Diego, CA.

Thompson, A., & Thompson, P. (1996). Talking about rates conceptually, part II: Mathematical knowledge for teaching. *Journal for Research in*

Mathematics Education, 27, 2-24.
Yackel, E. (1997, March). *Explanation as an interactive accomplishment: A case study of one second-grade mathematics classroom*. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL.

Geometric Thinking
Short Orals

REEXAMINING THE VAN HIELE MODEL OF GEOMETRIC THOUGHT THROUGH A VYGOTSKIAN LENS

Laura Brader-Araje
University of North Carolina, Chapel Hill
lbrader@mindspring.com

Tracie Yarbrough
University of North Carolina, Chapel Hill
tyarbrou@email.unc.edu

Because the van Hiele model and Vygotskian theory are still relatively new to American educators, an examination of the two theories will enhance the understanding of each. The purpose of this presentation is to examine the van Hiele Model of Geometric Thought through a Vygotskian lens. Doing so facilitates a deeper understanding of the van Hiele Model and reveals opportunities to refine the model's structure.

Analysis of the van Hiele Model is grounded in Vygotskian theory; specifically, the Vygotskian concepts to be applied include the zone of proximal development, mediated learning through both tool and sign use, and both inter/intrapsychological planes. Vygotskian theory holds the notion that learning is social. Just as the zo-ped emphasizes the importance of more capable peers, mediated learning refers to utilizing other forms of assistance. Knowledge that is constructed within oneself occurs on the intrapyschological plane and is enhanced by social interaction, whereas knowledge exchanges between individuals take place on the interpsychological plane.

In the examination of the van Hiele Model of Geometric Thought, these components of Vygotskian theory add to the understanding of the psychological processes used to learn and grapple with geometry. By applying Vygotskian theory to the van Hiele model, strengths as well as deficiencies of the model are revealed. This in turn improves current understandings of learning and teaching geometry in particular, and mathematics more generally.

One goal of the International Group for the Psychology of Mathematics Education is to address psychological aspects of learning and teaching mathematics. This presentation addresses this goal in that Vygotskian theory contributes to an improved understanding of the psychology of social learning, the van Hiele Model of Geometric Thought provides specific direction (for both teachers and students) in the teaching of mathematics, and the marriage of the two provides an impetus for a better mathematics education for all.

ASSESSING GEOMETRIC UNDERSTANDING IN MATHEMATICALLY TALENTED MIDDLE SCHOOL STUDENTS

Marguerite M. Mason
College of William and Mary
mmmaso@facstaff.wm.edu

Sara Delano Moore
University of Kentucky
sdmoor1@ukcc.uky.edu

The population in this study consists of 120 mathematically talented rising 7th, 8th, and 9th graders who had not taken a formal course in geometry. All 120 subjects completed the CDASSGP van Hiele test (Usiskin, 1982). In addition, 64 randomly selected students were interviewed individually for 30-45 minutes using an abbreviated version of Mayberry's (1981) tasks. Based on the CDASSGP test data, 35.8% of these students did not "fit the model", in contrast to the 12% which Usiskin (1982) found. Even though they were younger than Senk's high school students, they exhibited higher van Hiele levels. Using the probabilities developed by Senk (1989), 70% of the mathematically talented students had van Hiele levels 2 or greater and so have a probability greater than .75 of proof writing success after a year long course.

Analysis of clinical interviews confirmed that, like regular students, these individuals did not demonstrate the same level of thinking in all areas of geometry included in the school program. Many of the subjects had not been exposed to or did not remember what the critical defining attributes of various figures were, and they tended to look for similarities and differences in figures to deduce these attributes. Once they had established a definition, correct or incorrect, most reasoned consistently from it. In general, the students were capable of handling inclusion relationships if they had suitable definitions of the elements involved. The subjects showed strength in deductive reasoning, but not formal proof constructions.

It appears that geometric understanding in gifted students depends on a student's van Hiele level, logical reasoning ability, and amount and quality of basic geometric knowledge. A thorough assessment of geometry readiness should include an assessments of the student's van Hiele level, logical reasoning ability (e.g., as demonstrated by SAT scores), and geometric knowledge.

References

Mayberry, J. (1981). An investigation in the van Hiele levels of geometric thought in undergraduate preservice teachers. *Dissertations Abstract International, 42*, 2008A, (University Microfilms No. 80-23078).
Senk, S. (1989). Van Hiele levels and achievement in writing geometry

proofs. *Journal for Research in Mathematics Education, 20*, 309-321.
Usiskin, Z. (1982). *Van Hiele levels and achievement in secondary school geometry.* (Final Report of the Cognitive Development and Achievement in Secondary).

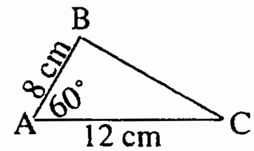
SIMILARITY, CONGRUENCE, AND ANGLES WITHIN CLOSED FIGURES

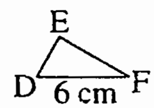
Sara Delano Moore
University of Kentucky
sdmoor1@ukcc.uky.edu

Marguerite M. Mason
College of William and Mary
mmmaso@facstaff.wm.edu

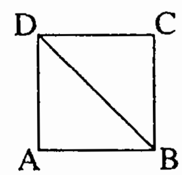
The purpose of this study is to examine 80 middle school students' understanding of similarity and congruence in open and closed figures. The closed figure questions were:

Δ ABC is similar to ΔDEF.
How long is $\overline{DE}$?
How do you know?
What is the size of <EDF?
How do you know?





ABCD is a square. $\overline{BD}$ is a diagonal.
Name an angle congruent to <ABD.
How do you know?



The open figure questions included showing two parallel lines cut by a transversal with the students naming 2 angles congruent to a given interior angle and justifying their choices. Students were asked to estimate the size of 3 angles which were the same size and orientation of <ABC in #1 and <BCD and <ABD in #2. They were also asked to define congruent, similar, and angle.

How long is DE? Sixty-five of the students reasoned between the 2 figures, most using multiplicative thinking (4 students used additive thinking, 2 used visual estimation). The remaining 7 students who responded just estimated the length of DE.

What is the size of <EDF? Only 42.5% gave the size of <EDF in degrees (40% of the students reported the size of <EDF in centimeters). When estimating the size of an angle not in a closed figure, only 6 of these students gave answers in inches or centimeters. Thirty-three percent of the students answering 60° appeared to be using the mathematical properties of similar triangles. Students' responses varied in method (e.g., estimation) and solution. Their methods can be summarized as follows:

Comparisons

Between Figures: 60° | 30° | < estimation

Within ADEF: < estimation | perimeter | < sides | other

Congruency. Of the 47 students who defined congruent correctly, 13 used the fact that the two angles were the same. Twenty students, defining congruent as "the same", used a definition of "opposite" or "mirror image" in the square problem. Among students providing an incorrect definition, 9 defined congruent as "opposite" or "mirror image". Six of the 13 students providing no definition of congruence also suggested a notion of opposite or mirror image.

Geometric Thinking
Posters

350

SOLVING GEOMETRIC PROBLEMS WHOSE SOLUTION IS NOT UNIQUE: THE CASE OF PROSPECTIVE SECONDARY TEACHERS AND THE CONCEPTS OF TANGENTS AND CIRCLES

José N. Contreras
The University of Southern Mississippi
jcontrer@ocean.st.usm.edu

Problem solving is a multidimensional process. On one dimension, problems can be categorized as problems that can be solved using one, two, or multiple strategies. On another dimension, problems can be categorized as routine problems, problems similar to others, or novelty problems. On a still another dimension, problems can be categorized as problems whose solution is unique, problems with no solution, and problems whose solution is not unique. Some of these categorizations depend on the problem solver's perspective and they are not mutually exclusive or exhaustive.

In this paper I examine 14 prospective secondary mathematics teachers' solutions to geometric problems whose solution is not unique in the context of tangents and circles. The participants were enrolled in a college geometry course and were asked to provide the complete solution to two problems. The first problem asked them to draw two tangent circles, and the second problem was: *Given two circles, construct as many common tangents to the circles as possible.* All the participants had access to compass and straightedge. They were also asked not to erase any failed attempt. Regarding the first problem, 13 students drew only the case of externally tangent circles. Only one student failed to draw two tangent circles. Regarding the second problem, only two students considered the cases in which it is possible to draw 0, 2, 3, 4 or an infinite number of common tangents. However, one of these students answered the question of how many tangents can be constructed but he did not construct or draw the tangents for each separate case. The other cases considered were: 4 common tangents (6 students), 2 common tangents (four students), four and three common tangents (1 student). Only one student drew an incorrect diagram and none of the students considered the case of one common tangent to the two circles. It is worth to note that none of the students considered the meaning of *constructing* a geometric object. In addition, some students made drawings that were somewhat sketchy and lacked precision.

Probability and Statistics Research Reports

THE LEARNING OF CONCEPTS OF STOCHASTIC IMPLIED IN THE BINOMIAL DISTRIBUTION BY MEANS OF THE USE OF DIFFERENT REPRESENTATIONS AND CONTEXTS

Jesús Colín Miranda
Escuela Superior de Ingeniería Mecánica y Eléctrica del I.P.N. –México

This study refers how students of probability use and interpret representations (as conceptual models, Steinbring, 1989) and contexts (in teaching situations: urns, coins, Galton board), in the learning of concepts of stochastics implied in the binomial distribution.

Introduction

At the Escuela Superior de Ingeniería Mecánica y Eléctrica del I.P.N. – México, a high rate of irregularity has been observed in the learning of stochastics, particularly in the learning of binomial distribution and its concepts involved.

Traditional teaching of school mathematics (including stochastics) generally begins with the definition of the concept expressed with a mathematical sign. The above-mentioned can be observed in some text books (Miller, I. and Freund, J. E., 1985; Meyer, P. L., 1973; Esparza, S. 1987; among others). Starting from such a definition, in most of the texts, it is exemplified and it is put operatively in game in particular situations.

However, from an epistemological view, Steinbring (1984, 1989, 1991) outlines that the concept is not identical to the sign nor to the object, but rather is gradually constituted in the subject as a result of recurrent interaction among them in progressive levels of abstraction of such interaction, and by contrasting these two characterizations, the students will be aware of the fact that concept is not identical to its definition. That is, it can only have implicit definitions of concepts, definitions that represent the relationship between the sign level, and the object level as an open relationship to development. The above-mentioned, is represented by means of an epistemological triangle: object - sign - concept.

Steinbring (1989) distinguishes three epistemological levels of stochastic knowledge. For the structure of the content of probability and statistics, he considers concepts, methods and diagrams. For students' context of learning, he takes into consideration representation means, activities and tasks. For example, student uses pie charts, histograms, column graphs, stem and leaf displays, etc., in classroom and at home to visualize empirical and theoretical distributions. For teaching process, he refers to the planning, organization, guidance, modification, improvement, support and development of the process by the professor. He adds that "...if one intends to teach stochastic concepts, methods and diagrams as mathematical techniques for build up a coherent theory, then the random character and specific nature of probability is very quickly lost. Stochastics degenerates to a collection of rules and recipes with no explanation" (p. 204). He also affirms that since "... stochastic is not only the subject matter new to the curriculum, but it is a type that is completely different from school mathematics. ... it is essential to develop different perspectives and interpretations of mathematics when teaching probability and statistic" (p. 205).

Considering the epresentation aspect, concepts are not directly accessible to perception, or to an immediate intuitive experience (Duval, 1993). This is, the acquisition of the concept is not achieved directly, but rather it goes by intermediary non discursive representations (p. 63). Thus, Duval (1994) points out the importance of the semiotic character of the representations, as well as, the diversity of representation registrations for a given situation or the coordination among them. These include figures, Cartesian graphics, symbolic notations, and inevitably the natural language. The resource to several registrations seems a necessary condition so that 1) the mathematical objects are not confused with their representations and 2) so that they are recognized in each one of them. This way, by means of the comprehension of the representations (semiosis) the comprehension of the concept is achieved (noésis).

Heitele (1975) made an analysis that takes into account results of psychological investigations of the development regarding stochastic ideas, adults' failures in stochastic situations, and the history of the probability. He proposed a spiral curriculum of the stochastic that goes from a intuitive plane to a formal plane (of iconic-active activities to symbolic representations), whose organizing principles are the fundamental ideas (from an epistemological and pragmatic point of view). "... those ideas which provide the individual on each level of his development, with explanatory models which are as efficient as possible and which differ on the various cognitive levels, not in a structural way, but only by their linguistic form and their levels of elaboration" (Heitele, 1975, p. 188). And he adds "The usefulness of such a model can only be shown by using it in the teaching at all levels". (p. 203)

Design

An exploratory study was carried out with two groups of students' in their fifth engineering semester, 52 in total, of the shift vespertine (21 year olds) at ESIME - IPN.

Didactic strategy. Twenty-eight students of the GROUP 5C2V were taught binomial distribution, according to traditional teaching method, that is to say, following presentation in text books. The solved examples were taken from the same books. The other group (24 students of the GROUP 5C5V) were taught binomial distribution by means of a didactic activity designed in four stages: Galton board (vertical and tilted) and the urn model; Pascal triangle, rule product-sum and triangle of possibilities; and binomial distribution.

Procedure. Questionnaire 1 with 16 problems was applied to students of both groups before teaching process, and a Questionnaire 2 with 10 problems was applied after the process. The Questionnaire 1 contains problems that concern to binomial probability distribution as: sample space, stochastic variable, combinatorics (combinations and permutations with and without substitution), as well as isomorphic problems (in none of them is requested to calculate probabilities). Questionnaire 2 contains some problems (five of combinatorics and two more are asked to calculate probabilities) of Questionnaire 1 and refers to concepts like measure of probability, sample space, stochastic variable, combinatorics, the addition rule of probabilities, independence, equidistribution and symmetry. The last of them corresponds to the binomial distribution.

Examples of problems. (1. C1). We have five men and four women. How many different couples can be formed a) of men?, b) of women?, c) mixed?, d) in general? Explain (use diagrams, some representation) and justify the answer. (2. C1). The are five green marbles and four blue in a bag. Two of them are extracted at the same time. How many different cases may occur in which: a) both are green?, b) both are blue?, c) one of each color comes out? Explain and justify the answer (use diagrams, some representation). (3. C2). There are five green marbles and four blue marbles in a bag. Two of them are extracted at the same time, at random. Which is it the probability that a green one and a blue one come out? Explain and justify the answer (use diagrams, some representation).

Analysis. Problems were selected taking into account concepts, the way to present them in different contexts (isomorphism) and the type of problem. For answers, solution strategies, correction of answer, errors (conceptual, operative and symbolic) and types of representations used by students were taken into account. In this first exploration natural language was mainly used as representation register in each enunciated problem.

Results

The understanding involved of concepts in binomial by students was investigated through answers given in Questionnaires 1 and 2. In Table 1 (total and percentages) shows a summary results obtained from the application of both Questionnaires. As for the operative aspect, students whom the binomial distribution in traditional form was taught, used symbolic representation register, and obtained a higher percentage (17.85%) of correct answers in the problem that involves binomial (problem 10), that those of the other group (16.66%). However, these last ones showed a vast diversity of strategies and use of correct representations (tree diagrams, Pascal triangles) (50%) that the first ones (32.11%). Approximately the same number of isomorphic problems were identified by both groups of students,. This suggests that if one poses activities which allows teaching articulates a variation of representation registers (natural language, figure, symbolic notation: triangle of Pascal, triangle of possibilities) and contexts (Galton board, urns, for example), this will lead to a major possibility in the conceptual construction.

Table 1
Questionnaire Results by Group

	Group 5C2V				Group 5C5V			
	Before		After		Before		After	
Problems	Form.	Repr.	Form.	Repr.	Form.	Repr.	Form.	Repr.
8. - 1.	3	1	2	2	4	1	6	6
11. - 3.	0	0	2	1	0	0	2	2
9. - 4.	0	15	16	4	0	4	14	7
12. - 6.	0	3	11	4	0	1	8	4
10. - 2.	0	0	9	1	0	0	6	2
13. - 5.	1	0	8	11	0	0	10	7
15. - 7.	0	0	16	7	1	0	9	9
16. - 8.	0	0	15	7	0	0	6	4
9.			0	1			0	1
10.			5	9			4	12

Note. FORM. = Correct Formula, REPR. = Correct Representaton

Table 2
Questionnaire Results (Percentages)

	Group 5C2V (28 Students)				Group 5C5V (24 Students)			
	Before		After		Before		After	
Problems	Form.	Repr.	Form.	Repr.	Form.	Repr.	Form.	Repr.
8. - 1.	10.71	3.57	7.14	7.14	16.66	4.16	25.0	25.0
11. - 3.	0	0	7.14	3.57	0	0	8.33	8.33
9. - 4.	0	53.57	57.14	14.28	0	16.66	58.33	29.16
12. - 6.	0	10.71	39.28	14.28	0	4.16	33.33	16.66
10. - 2.	0	0	32.14	3.57	0	0	25.0	8.33
13. - 5.	3.57	0	28.57	39.28	0	0	41.16	29.16
15. - 7.	0	0	57.14	25.0	4.16	0	37.5	37.5
16. - 8.	0	0	53.57	25.0	0	0	25.0	16.66
9.			0	3.57			0	4.16
10.			17.86	32.14			16.66	50.0

References

Duval, R. (1993). *Semiosis and noesis.* Conference A.P.M.E.P., I.R.E.M. pp. 118-137.

Duval, R. (1993). Registres de représentation sémiotique et fonctionnement cognitif de la pensée *Annales de Didactique et de Sciencies Cognitives,* 5, pp. 37-65, IREM of Strasbourg.

Duval, R. (1994). *Representations graphique et symbolique de la Maternelle à L'Université.* CIEAEM 46, Vol. 1, Toulouse, France.

Esparza, S. (1987). *Elementos de probabilidad,* Editorial I.P.N., Mexico.

Fischbein, E. 1975, *The intuitive sources of probabilistic thinking in children.* Reidel Publishing Company, Holland.

Heitele, D. (1975). An epistemological view on fundamental stochastic ideas. *Educational Studies in Mathematics,* Vol. 6, 187-205.

Meyer, P. L. (1973). *Probabilidad y aplicaciones estadísticas.* Fondo Educativo Interamericano, S.A. México.

Miller, I., & Freund, J. E. (1985). *Probabilidad y estadística para ingenieros.* Prentice Hall Hispanoamericana, S.A.

Steinbring, H. (1984). Mathematical concepts in didactical situations as complex systems: The case of probability. In Steiner H. G., Balacheff N. *Theory of mathematics education.* Adeleide, Austria, Occasional Paper Not. 54, IDM Bielefeld, 56-88.

Steinbring, H. (1989). The interaction between teaching practice and theoretical conceptions. Studies in Mathematics Education. *The Teaching of Statistics.* R. Morris (ed.), pp. 202-214, vol. 7, UNESCO.

Steinbring, H. (1989). Mathematics in teaching process: The disparity between teacher and student knowledge. *Researches en Didactique des Mathématiques,* Vol. II, Not. 1, 65-108.

Steinbring, H. (1991). The concept of chance in everyday teaching: Aspects of a social epistemology of mathematical knowledge. (Trad. Ana Ma. Ojeda S.) In *Educational Studies in Mathematics* 22, 503-522.

Steinbring, H. (1991). The Theoretical Nature of Probability in the Classroom. In R. Kapadia and M. Borovnick, (Eds.), *Chance encounters: Probability in education,* chapter 5, pp. 135-167. Kluwer Academic Publishers.

COMPARING DATA SETS: HOW DO STUDENTS INTERPRET INFORMATION DISPLAYED USING BOX PLOTS?

Susan N. Friel
University of North Carolina at Chapel Hill
sfriel@email.unc.edu

The need to develop an understanding of statistical inference is an important component of statistics education (NCTM, 1989). In the middle grades, exploring students' ability to compare two or more data sets of the same or different sizes provides beginning work with statistical inference. Different kinds of graphs may be used as tools for data comparison. The purpose of this study was to look at what sense eighth grade students' made with respect to comparing two or more data sets using box plots as a representational tool.

Purpose of Study

The need to develop an understanding of statistical inference is an important component of statistics education (NCTM, 1989). There are a number of places in the statistics curriculum where the idea of statistical inference may be introduced, for example by using summary statistics such as the mean and range to compare data sets or looking at relationships between two variables using correlation. For middle grades students (NCTM, 1989), it seems sensible that comparing two or more data sets engages them in a kind of statistical reasoning that is more sophisticated than that involved in simply describing a distribution. Lehrer and Romberg (1996), among others state that it may be natural for students want to make comparisons of two or more groups (e.g., boys with girls, one grade level with another).

Children's understanding of the processes involved in comparing groups of data appears to have received little attention. Gal and Wagner (1992) found that when students were asked to visually compare two data sets that are displayed using side-by-side picture or bar graphs, they used a variety of strategies to make comparisons including determining summaries of the data in each group (e.g., identifying means), finding totals of the data values in each group, and other strategies such as arguing that the data are more spread out in one group than the other. When such comparisons involve data sets that do not have the same numbers of data values and are displayed as side-by-side picture or bar graphs, students may argue that it is not possible to make comparisons because of differing sample sizes.

Both box plots (box-and-whisker plots) and stem plots (stem-and-leaf plots) also may be used to compare more than one data set (Graham, 1987; Landwehr & Watkins, 1986). Using a back-to-back stem plot permits comparison of two data sets while any number of parallel box plots may be used to compare multiple data sets. Box plots highlight only a few important features of the data; because most of the data "disappear," both the shape of the distribution and the actual data values cannot be clearly identified. When the number of data values in data sets to be compared are different, box plots "hide" this difference, showing a five-number summary for each set of data.

The purpose of this study was to look at what sense eighth grade students' made of comparing two or more data sets using box plots as a representational tool. This paper looks at one particular problem that involved in making comparisons which among Ms. Choy's three class periods and their respective scores on the same quiz (Figure 1) as displayed using parallel box plots.

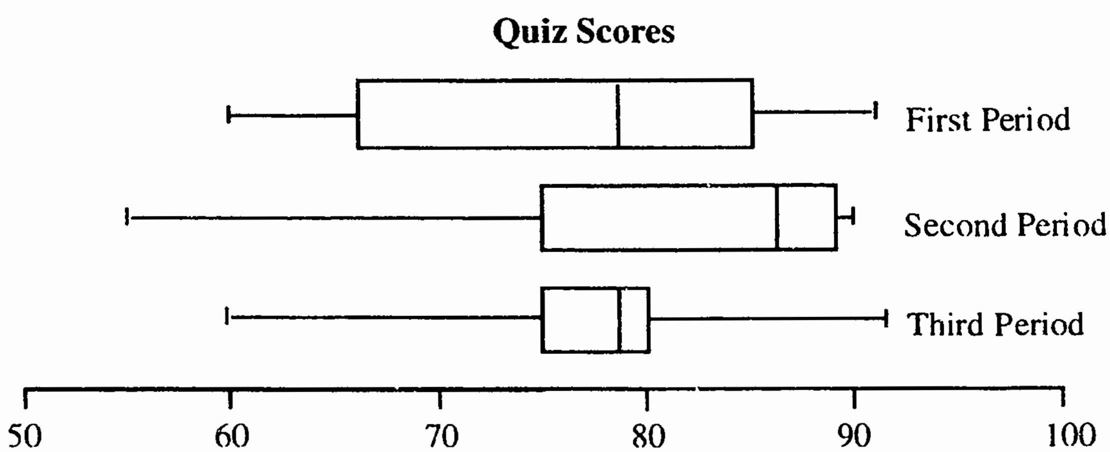


Figure 1. (Lappan, et. al., 1998, p. 76) Ms. Choy wants to analyze the achievement of her eighth grade classes on a quiz. These box plots represent the quiz scores of Ms. Choy's first-period, second-period, and third-period classes. Questions asked are included in text of paper.

Research

During Fall, 1997, a study was conducted to look at the ways students reason about comparisons of data sets and about topics related to sampling; the former is the focus of this paper. Fifty students were involved in the study, distributed between two eighth grades in a private middle school located in a southern urban area. The students were generally higher achieving, with little cultural or ethnic diversity present.

Pre- and post-tests which included looking as students' understanding of concepts related to visual comparison of data sets using bar graphs were administered. An instructional unit (Lappan, et. al., 1998) focused on comparing data using box plots and on a variety of activities exploring sampling was taught by each the two teachers over a period of eight weeks, beginning in mid-October and ending in December. Selected written assignments were collected throughout the instructional unit, and several lessons were observed for purposes of understanding student thinking. A sample of students (3 from each class) also participated in interviews focused on content currently being studied at various times throughout the unit. These students were selected based on teacher recommendations concerning ability to be verbal, on willingness to be interviewed, and on responses made on the pre-test.

The results reported in this paper focus on the final interview with the six students. In that interview, an end-of-unit task (Figure 1) provided in the curriculum materials was used in an interview format to probe student reasoning about making comparisons of data sets using box plots.

Curcio's (1987) three components of graph comprehension were used as an organizing framework for designing the interview:

1. **Reading the data** involves "lifting" the information from the printed page to answer explicit questions for which the obvious answer is right there in the graph.
2. **Reading between the data** includes the interpretation and integration of information that is presented in a graph.
3. **Reading beyond the data** involves extending, predicting, or inferring from the representation to answer implicit questions. The reader gives an answer that requires prior knowledge about a question that is at least related to the graph.

The first two components focus on elementary levels of questioning that involve data extraction. The latter component is tied to questioning that involves not only interpreting a graph but utilizing the graph to assess realistic implications from the data (Pereira-Mendoza, 1995).

This study was not designed to assess a particular instructional model or curriculum. Rather, the author reasoned that taking a "snapshot" of what students knew about this topic without instruction would not be as productive as trying to assess what students knew after having an opportunity to gain some experience with the process of statistical investigation and with some of the key concepts related to comparing data sets. These students had not been exposed to the use of box plots prior to the instructional unit.

Results

The results are organized using the questioning framework (Curcio, 1987) as it was designed to help students attend first to the data as displayed through the representations and then to interpretation of these data.

Read the Data Questions

- What are the minimum and maximum values and the median for first period?

All six students had no difficulty with this question; some also identified the lower and upper quartiles:

For the first period class, the minimum seems lie at 59 or 60. The first quartile is about 66. The median is...78. The fourth quartile [She was identifying the third quartile value but referring to the fourth quartile interval] is about 85 and then the maximum is about 91 or 92.

Read Between the Data Questions

- Suppose there are 24 students in the class. How many had scores in the interval of the minimum value to the median for first period?
- How many had scores that fell in the box (including the quartiles) of the box plot for first period?
- Do you think there are any outliers in the period three scores? Explain your thinking.

All students had little difficulty with the first two questions. They either discussed their reasoning in light of the fact that the box plot is *divided up into four sections evenly*, or into sections of *25% here, 25% here,...* and so on. One student referred to 1/4 of the data being in each interval.

With respect to outliers, all but one student agreed that there would be outliers in the data for period three. They recalled that *there was some formula we were given* using 1 1/2 times the difference between the first and third quartiles. Four of the six students were able to estimate possible outliers by visualizing movement of 1 1/2 times the box length above or below the quartiles; one student indicated that he needed to know the values for the first and third quartiles. One student did not recall outliers as a concept when questioned.

Read Beyond the Data Questions

- Which group do you think did better on the quiz (comparing period one with period three; comparing period two with period three)? Explain your thinking.
- If you were Ms. Choy, which of your classes would you think was most successful on the quiz? Explain your thinking.

Comparing the first and third periods involves acknowledging that the minimum, maximum, and median values are the same. The first and third quartiles differ in their interval widths; students needed to make sense of this in terms of which class did better. Three students said that third period did better. They offered reasons such as third period being *compact, more consistent*. This means that *like you have several of one number or several of a couple numbers*. With first period, *the box is a lot wider which means you could have numbers that ranged any where between those two spots* (i.e., quartiles). The remaining students felt that the two periods did about the same (2 students) or that first period did better (1 student). These students focused on pairing the 5 numbers across box plots; the discrepancies in terms of the first and third quartiles not both being higher or lower or the same was perplexing. They had to reason about how the distribution of the data might look in the box. The possibilities of higher and lower scores in first period were offered as reasons for why either first period did better or both did about the same.

Comparing the second and third periods may be easier in that the median for second period is quite a bit higher than that for third period. All six students agreed that second period did better than third period, although their reasoning was different. For one student, her only reason was that *the median was higher*. Other students (2) compared percent intervals:

> *The second period* [did better] *due to the fact that...I'm going to make a general statement here which is the whole reason box plots are useful...is because 50% of the second period did better than 75% of the third period.*

Others focused on other aspects of the displays:

> *Second period because...the third quartile's way ahead, the median's way ahead ...the median is a lot higher and the third quartile is sort of almost where the maximum is because they have so many of their scores placed on the higher end of the range of scores.*

Comparing the three periods, students built on their earlier reasoning; all identified second period as having done better, each using an extension of earlier reasoning for comparing second and third periods. So, the one student again said, *Because the median is higher.* Students who addressed percents noted, *But 50% of the second period has done better than 75% of the first and third periods.* Students who looked at the spread of scores continued with this focus,

> *I think that second period did the best than the third and first period just because they have so many of their scores placed on the higher end of the range of scores.*

Only one student focused on the scale of the plots, adding to her evaluation,

> *And then I'd say, on the whole, the classes did fairly well because they all...it definitely...even the lowest point...was above 50 and most of them...50% in fact...50% to 75% were above 75* [the score].

Conclusions

The purpose of this study was to look at what sense eighth grade students' made with respect to comparing two or more data sets using box plots as a representational tool. Of particular interest is the third category of questions involving reading beyond the data. In making comparisons using box plots, it appears that some students focus on reading the data as they attempt to pair the respective five numbers (maximum, minimum, median, and first and third quartiles) across box plots and use these comparisons to make decisions about which class did better. When this strategy did not work, these students (with the exception of one student) were able to modify their reasoning to look between the data and discuss the possible distributions of data within intervals of the box plot. The other students seemed to focus on between the data information, either adapting their language to the usefulness of box plots by noting 25%, 50% or 75% intervals of the data or to commenting on the spread of the scores and the clustering of scores at the higher end of a plot when compared to other plots.

With the increasing inclusion of statistics content across the K-12 curriculum, it is possible to begin to explore the development of thinking in this area. What has been discussed in this paper begins to focus this discussion on statistical inference and the role of box plots as a tool for making comparisons as one of the reasons for conducting statistical investigations.

References

Curcio, F. R. (1987). Comprehension of mathematical relationships expressed in graphs. *Journal for Research in Mathematics Education, 18*, 382-393.

Gal, I., & Wagner, D. A. (1992). *Project STARC: Statistical reasoning in the classroom.* (Annual Report: Year 2 NSF Grant No. MDR90-50006). Philadelphia, PA: Literacy Research Center, University of Pennsylvania.

Graham, A. (1987). *Statistical investigations in the secondary school.* New York: Cambridge University Press.

Lappan, G., Fey, J., Fitzgerald, W., Friel, S., & Phillips, E. (1998). *Samples and populations.* (Teacher's Edition) Palo Alto, CA: Dale Seymour Publications.

Landwehr, J. M., & Watkins, A. E. (1986). *Exploring data.* Palo Alto, CA: Dale Seymour Publications.

Lehrer, R., & Romberg, T. (1966). Exploring children's data modeling. *Cognition and Instruction, 14* , 69-108.

National Council of Teachers of Mathematics (NCTM). (1989). *The curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Pereira-Mendoza, L. (1995). Graphing in the primary school: Algorithm versus comprehension. *Teaching Statistics, 17*, 2-6.

Wainer, H. (1992). Understanding graphs and tables. *Educational Researcher, 21*, 14-23.

STUDENTS' STATISTICAL THINKING

Graham Jones, Carol Thornton,
Cynthia Langrall, & Edward Mooney
Illinois State University
jones@math.ilstu.edu

Bob Perry
University of Western Sydney
Australia

Ian Putt
James Cook University
Australia

Based on a synthesis of the literature and observations of young students over a year, a framework for assessing statistical thinking was formulated and validated. For each of the four major constructs of this framework — reading data displays, organizing and reducing data, displaying data, and analyzing and interpreting data — four levels of thinking were identified which reflected a continuum from idiosyncratic to analytical reasoning. The framework was validated through case study data obtained from 12 students, four from each of grades 1 through 3. Results suggest that while the framework produces a unified picture of children's statistical thinking, there is static in the system which generates inconsistencies among construct level, especially reading data displays. The framework has implications for classroom instruction.

Reform documents such as those of the National Council of Teachers of Mathematics (1989) reflect the increasing importance of data handling at all levels of the school curriculum. This increased emphasis has created the need for research on statistical learning, especially in the elementary grades, where there has been a tendency to focus merely on graphing rather than on broader aspects of data handling (Shaughnessy, Garfield & Greer, 1996). Although some elements of students' statistical learning have been investigated in areas such as data organization (Mokros & Russell, 1995) and graph comprehension (Curcio, 1987; Friel, Bright, & Curcio 1997), the research on students' statistical learning is emergent rather than well established. This is evident in the fact that research has not generated a framework for describing students' statistical thinking as has been the case in mathematical topics such as whole numbers (Moser & Carpenter, 1984).

Through observation and insights into students' thinking in various statistical tasks, this study seeks to (a) develop an initial framework for describing and predicting how students think in statistical situations, (b) generate assessment protocols based on the initial framework, and (c) validate the framework using the assessment protocol.

Theoretical Perspectives

The initial framework (available from the authors) is based on our year-long observations of students' statistical thinking and previous research (Curcio, 1987; Friel, Bright, & Curcio, 1997; Mokros & Russell, 1995). It incorporates four key constructs: reading data displays, organizing and reducing data, displaying data, and analyzing and interpreting data that have been adapted from four data handling concepts identified by Shaughnessy, Garfield, & Greer (1996). Reading data displays involves describing representations of data, identifying information stated in a display, and recognizing connections between different displays of the same data . Organizing and reducing data incorporates mental actions such as ordering, grouping, and summarizing data. It also involves describing data by representative or typical measures such as mean, median, mode, and range. Displaying data incorporates constructing representations that exhibit different organizations of the data. Analyzing data involves recognizing trends and patterns in the data, and making inferences, interpretations, and predictions from the data. This construct includes what Curcio (1987) refers to as reading between the data and reading beyond the data (p. 384).

Our framework hypothesizes that students' thinking can be described across four levels for each of the four constructs. Level 1 is associated with idiosyncratic thinking; level 2 is seen to be transitional between qualitative and quantitative thinking; level 3 involves the use of informal quantitative thinking; and level 4 incorporates analytical and numerical reasoning about data. These levels are consistent with cognitive research (e.g., Biggs & Collis, 1991) that recognizes different levels in the complexity of students' thinking.

Method

Students in grades 1 through 3 from a midwestern school formed the population for this study. Twelve target students, four from each of the three grades, were purposefully selected from this population. Based on teacher assessment of student achievement records, two students were selected from the middle 50% and one from both the lower and upper quartiles of each grade level.

The process used to validate the framework was similar to that used in an earlier study (Jones, Langrall, Thornton, & Mogill, 1997). It involved four components: (a) interviewing and analyzing target students' responses to the Statistical Thinking Protocol; (b) examining the stability of target student's thinking over the four constructs; and (c) illuminating the distinguishing characteristics of each thinking level. Qualitative analysis was used to address all three parts of the validation.

The major source of data was children's responses to the Statistical Thinking Protocol. This protocol was administered to each target student near the beginning of the school year by a member of the research team.

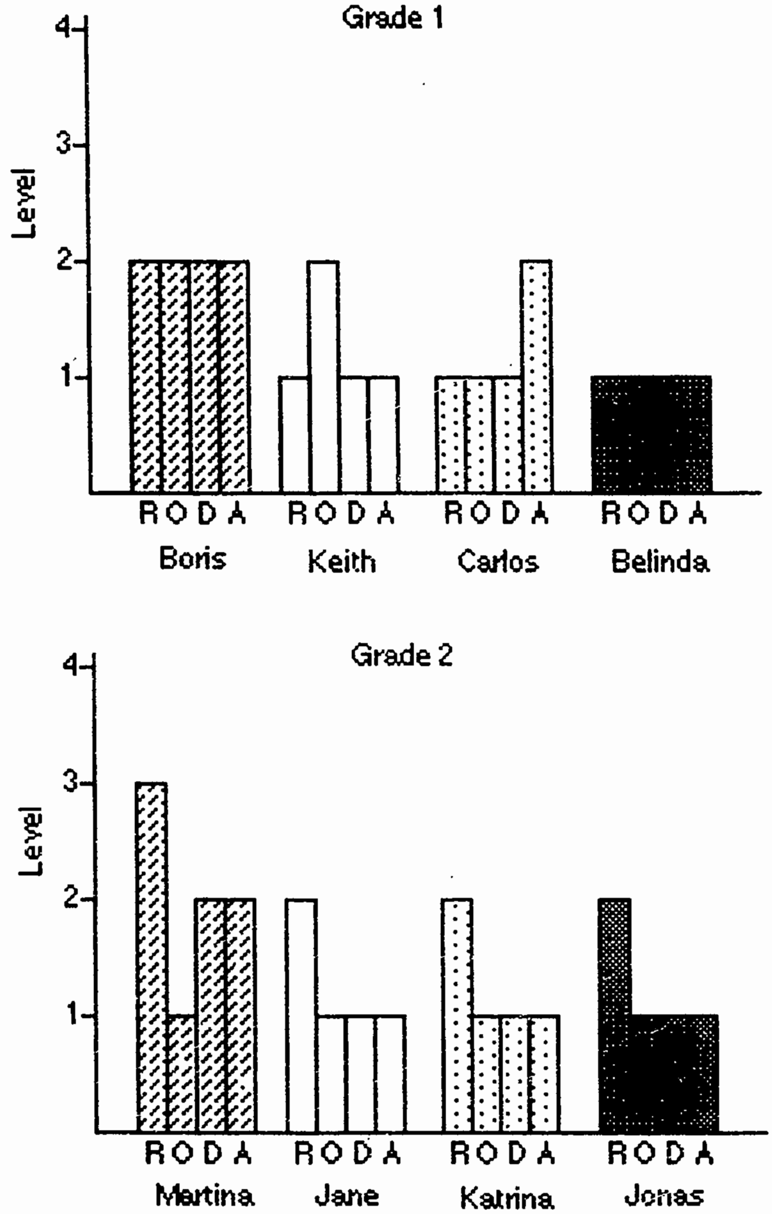
The protocol was based on the framework and comprised three major tasks, each of which contained open-ended questions followed by series of probes. Eight questions were associated with reading data displays (R), nine with organizing and reducing data (O), three with displaying data (D), and 15 with analyzing and interpreting data (A). Sample questions for one of these major tasks, Sam's Friends, are presented in Figure 1. Each question is labeled R, O, D, or A according to which framework construct it assesses. Students' responses were audiotaped and transcribed, and student artifacts such as drawings and graphs were collected.

The double coding procedure described by Miles and Huberman (1994) was used to code the transcripts. Initially, two of the researchers independently coded all items of each student's interview protocol. Using the framework descriptors as criteria, items were coded according to construct and level of thinking exhibited by the student. These codings classified by construct were then used to determine the dominant (modal) statistical thinking level for each student on each of the four constructs. Agreement was reached on the coding of 40 levels out of 48, that is 83 percent. Variations were clarified until agreement was reached for each student on all four constructs. During the coding process described above, the researchers used a data reduction approach (Miles & Huberman, 1994) to discern key thinking patterns exhibited by students at each level of the framework and across all four constructs.

Results and Discussion

The profiles in Figure 2 show the levels of statistical thinking for each of the 12 target students by grade level. Level 1 thinking was exhibited by three of the four students at each of the first and second grade levels, while all of the grade 3 students showed thinking that was indicative of level 2. Although there is growth in children's statistical thinking across the three grades, this is largely a feature of the higher levels of thinking shown by grade 3 students. Even so, none of the grade 3 students demonstrated thinking that went beyond level 2. It is also clear that students' thinking across the four constructs was not completely stable. For example, grade 2 students' levels of thinking on reading data displays were generally higher than those on the other three constructs. These observations suggest that while the framework produces a unified picture of students' statistical thinking, there is static in the system which generates inconsistencies with respect to thinking levels across constructs. Some of this static may be eliminated as constructs like reading data displays are refined. However, our observations suggest that these students often regress to idiosyncratic or even mystical reasoning when they are capable of higher level statistical thinking.

With respect to reading data displays, level 1 students often gave idiosyncratic responses that had little relevance. For example, in response to question D1 (Figure 1b), Carlos perceived the bars as representing the heights of Sam's friends rather than the number of friends who came to



R = Reading Data Displays, O = Organizing Data
D = Displaying Data, A = Analyzing & Interpreting Data

Figure 1. *Profiles of grade 1 and 2 students' statistical thinking*

visit. Students at level 2 were able to make more sense of the data and, in particular, were able to recognize when two visual displays represented the same data set. In response to questions on organizing and reducing data, neither level 1 nor level 2 students were able to deal with the notions of average or spread in any meaningful way. Boris' response to question O1 (Figure 1a) illustrates how one child tried to make sense of the concept of average. He reasoned *4 because 2 is close to 4, and 3 is close to 4 and 1 is*

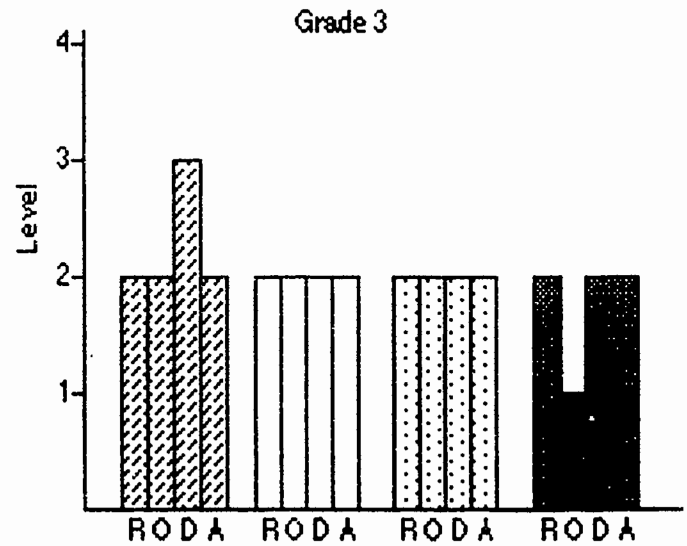


Figure 2 *Profiles of Students' Statistical Thinking*

close to 4 and 2 is close to 4 and 7 is close to 4. While the reasoning is inchoate, Boris appeared to look for a balancing number like the mean or median.

With respect to displaying and analyzing data, students exhibiting level 1 thinking were not able to construct valid visual representations of data, but did show some ability to read between the data (Curcio, 1987, p. 384). Jane's response to question R1 (Figure 1b) on constructing visual representations is typical of level 1 students' thinking. She drew a pictograph of *baby snakes on top of each other* that bore no relation to the original graphs or the data beyond showing that [some are] *more* [and some are] *less, and there is one lesser.* However, in response to A4 (Figure 1a), Jane appeared to read between the data: *I counted all the Xs and came up with 19.*

By way of contrast level 2 students showed some ability to complete data displays and even to "read beyond the data" (Curcio, p. 384). For example on question R1 (Figure 1b), Keith's response produced a visual representation that maintained the integrity of the data but used "doors" instead of Xs. Pointing to each door, Keith said, "That's where people can walk in." In response to A5 (Figure 1a), Candy showed evidence of thinking beyond the data. She reasoned that 10 friends would visit Sam each week because *usually 1 or 2 come each day.* This comment referred to her previous interpretation of average as *mostly at least 2* [friends] *came on each day.*

One implication arising from these results is that student learning in data handling may be promoted if teachers were more aware of the range of students' statistical thinking. Such awareness might be enhanced through open-ended questions and probes like those in this study.

References

Biggs, J. B., & Collis, K. F. (1991). Multimodal learning and intelligent behavior. In H. Rowe (Ed.), *Intelligence: Reconceptualization and measurement* (pp. 57-76). Hillsdale, NJ: Erlbaum.

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades 1 through 3. *Journal for Research in Mathematics Education, 15*, 179-202.

Curcio, F. R. (1987). Comprehension of mathematical relationships expressed in graph. *Journal for Research in Mathematics Education, 18*, 382-393.

Fennema, E., Franke, M. L., Carpenter, T. P., & Carey, D. A. (1993). Using children's mathematical knowledge in instruction. *American Educational Research Journal, 30*. 555-584.

Friel, S. N., Bright, G. W., & Curcio, F. R. (1997). Understanding students' understanding of graphs. *Mathematics Teaching in the Middle School, 3*, 224-227.

Jones, G. A., Langrall, C. W., Thornton, C. A., & Mogill, A. T. (1997). A framework for assessing children's thinking in probability. *Educational Studies in Mathematics, 32*, 101-125.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis*. Thousand Oakes, CA: Sage.

Mokros, J. & Russell, S. J. (1995). Children's concepts of average and representativeness. *Journal for Research in Mathematics Education, 26*, 20-39.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: The Council.

Shaughnessy, J. M., Garfield, J. & Greer, B. (1996). Data handling. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education, 18*, 382-393.

TRACING THE ORIGINS AND EXTENSIONS OF MATHEMATICAL IDEAS

Regina D. Kiczek
Rutgers University
rdkiczek@eden.rutgers

Carolyn A. Maher
Rutgers University
cmaher@rci.rutgers.edu

Representations of mathematical ideas and the development of those representations have been traced for a group of five students who have had the opportunity to engage in carefully designed investigations over several years. As tenth-graders, the students worked together on a combinatorial task that they first encountered in the fifth grade. The study revealed that these students retrieved earlier ideas and images, built upon them, and used their new representations to solve an extension of the task.

Objective

This research is one component of an on-going longitudinal study of students' building of mathematical ideas as they engage in problem-solving tasks with other students. The study in progress focuses on the development of combinatorial and probabilistic reasoning in children, and continues through high school. A particular group of students who have worked thoughtfully and collaboratively on meaningful mathematical problems since grade one were encouraged to make conjectures, develop theories, and justify their solutions for a variety of mathematical problems.

These students, now in tenth grade, are participating in a Friday after-school mathematics program. Our interest is to examine whether students make connections between ideas explored in earlier years and those currently being considered. Particular questions that are being investigated include: (1) What is the nature of the original representations of the students' mathematical ideas? (2) Does a student's representation change over time, and (3) if so, how?

Theoretical Framework

The longitudinal study has made possible a rich collection of data on children's thinking in the form of videotapes and student work. This enables us to examine the outcomes of a particular activity within the broader domain of outcomes of both earlier and later events. By studying a student's thinking at various points in time and tracing the development of that thinking, it is possible to understand how original ideas were modified. It is also possible to study how these ideas were retrieved to solve new problems. The work of Pirie and Kieren (1992) suggests that students revisit ideas and refine their conjectures; also, when faced with a new situation, they "fold back" to an inner level of understanding, reflecting upon and reorganizing their earlier ideas in light of new information.

Problems involving combinatorics have been shown to be useful for studying mathematical thinking (Maher & Martino, 1998, 1997, 1996). These problems afford children the freedom to create their own strategies and rules. In addition, students benefit from their direct experience with data and the manipulation of that data as they look for patterns and relationships. This personal building is an important foundation for the later study of probability (Kapur, 1970).

Methodology

Subjects: An intense case study of a focus group of five students from the original study is being conducted (Kiczek, in progress). The students, Ankur, Brian, Jeff, Michael and Romina, revisit problems they investigated in elementary school, followed by appropriate new extensions.

Data: The data for this report come from videotapes of problem solving sessions in grades five and ten. A task that they originally explored in the fifth grade (the *Pizza Problem*) was the first of a series of related investigations involving combinatorics. The task asked students to determine the number of different pizzas that could be made when there is the option of selecting from among four toppings, and then to find a way to convince each other that they had accounted for all possible choices. This task was investigated by the students again in grade ten. The current data come from after-school small group sessions of several hours duration each, while the earlier data were collected in a classroom setting in which the task and extensions were given over several days.

Procedures: At least two cameras were used to record each session Afterwards, a team of researchers studied the tapes, making notes, transcribing, and verifying each transcript. The collection of data includes: (1) videotapes of students working together; (2) follow-up interviews of individual students or groups of students; (3) videotape transcripts; (4) analyses of the transcripts; (5) students' written work; and (6) researcher notes. The data are organized as a "video portfolio" (Maher & Martino, 1996), a detailed record of what the student says and writes about a problem, the notation and strategies that are used, the pupil-to-pupil and pupil-to-teacher talk, and particular successes and frustrations. This "portfolio" provides a collection of many events that, when viewed chronologically, create a dynamic moving picture which tells the story of the development of a particular idea.

Results

Before the first tenth grade session, the high school students, graduate students and the teacher/researcher conversed informally while eating pizzas brought in as an after-school snack. Some students recalled solving problems involving pizzas in earlier years. The teacher/researcher asked if they remembered what they had done and how many pizzas they had found.

After some discussion, the students reconstructed the original Pizza Problem and extended it by adding a fifth topping. They then proceeded to work on the problem.

Solving the Pizza Problem in Grade 5 - Spring 1993. Romina, Michael, Jeff, Brian and Ankur worked as a group on the Pizza Problem as fifth graders, although they used a variety of strategies and representations to produce the sixteen combinations. These included a partial tree diagram, lists, and an organization that systematically controlled for variables. Michael drew circles to represent the various pizzas, labeling each with its toppings. They created codes using letters or abbreviations to represent the four toppings (for example, pepperoni = pep; m = mushrooms) and also decided to code for a pizza with no toppings (plain = pl; c = cheese (plain)). They distinguished between the cases of "whole" (plain or one topping) pizzas and "mixed" (two or more toppings) pizzas.

Solving the Pizza Problem in Grade 10 - Fall 1997. In grade ten, Michael was the first to begin working on the problem. While the other four students worked collaboratively, talking aloud about combinations of toppings and patterns that they were observing, Michael spent at least fifteen minutes quietly developing his own solution. Romina, Jeff, Brian and Ankur began by using a code of letters to represent the different toppings, similar to the notation used in grade five; however, as the students began to list the combinations, they switched their notation to the numerals one through four. Michael, on the other hand, invented a symbolic representation based on a binary coding scheme. The others decided that if five toppings were available, thirty different pizzas could be made with at least one topping, plus one plain cheese pizza, for a total of thirty-one. Michael disagreed and Ankur challenged him to produce the missing pizza.

Michael: I think it's thirty-two - with that cheese. And without the cheese, it would be thirty-one. I'll tell you why.

Ankur: Mike, tell us the one we're missing then.

Michael responded by explaining what the zeros and ones meant in his representation and how they are used to write base ten numbers in base two.

Michael: Okay, here's what I think. You know like a binary system we learned a while ago? Like with the ones and zeros - binary, right? The ones would mean a topping; zero means no topping. So if you had a four-topping pizza, you have four different places; in the binary system, you'd have - the first one would be just one. The second one would be that (writes 10); that's the second number up. You remember what that was? This was like two, and this was three (writes 11).

Jeff recalled where they had seen this before.

Jeff: I know exactly what you're talking about. It's the thing we looked at in Mr. Poe's [their 8th grade teacher] class; it was with computers.

Michael continued to relate his coding scheme to the pizza problem.

Michael: Well, you get, I think — I have a thing in my head. It works out in my head. You've got four toppings. This is like four places of the binary system. It all equals up to fifteen. That's the answer for four toppings.

Romina sought clarification about the assignment of meaning for the zeros and ones. It was Jeff who responded to her question.

Romina: So is the one — is that your topping?

Jeff: Yeah. Each one is a topping. The zeros are no toppings. The ones are toppings.

Michael then summarized his conclusions; Brian's "Wow!" indicated his enthusiasm for Michael's solution.

Michael: So you go from this number (0001), which is in the binary system, it's one, to this number (1111), which is fifteen, and that's how many toppings you have. There's fifteen different combinations of ones and zeros if you have four different places.

Brian: Wow!

Michael: I don't know how to explain it, but it works out. That's in my head — these weird things going on in my head. And if you have an extra topping, you just add an extra place and that would be sixteen, that would be thirty-one.

Michael omitted the representation for a plain cheese pizza. However, by adding one to the fifteen combinations (four toppings) and to the thirty-one combinations (five toppings), he accounted for all possibilities.

Jeff: And then you add the cheese?

Michael: Plus the cheese would be thirty-two.

With the assistance of the other students, Michael presented his binary coding scheme to the teacher/researcher, saying, "This is the way I interpret it into the pizza problem." When the teacher asked questions about Michael's solution, the other students were able to respond.

Teacher: Whatís the difference between 1-0-0-0 and 0-1-0-0?

Jeff: Well, that would be the difference between an onion pizza and a pepperoni pizza.

Jeff then suggested they label each column with the name of a topping. Michael agreed, noting that a one in a column indicated that pizza had that particular topping. As an extension, the students were asked to consider the case where ten toppings were available. While investigating this extension, the group discovered that a string containing all zeros represented a plain cheese pizza. Finally, they were asked to generalize to the case of n toppings. Eventually they proposed the generalization of 2 n, where n is the number of topping choices.

Teacher: Okay. So what do you think of this way to do it?

Jeff: I'm impressed.

Brian: Yeah.

Conclusions/Significance

In the fifth grade problem solving session, the original representations displayed by the students made use of notation that enabled them to keep track of their ideas and account for all possibilities to reach a solution. In the tenth grade session, representations displayed by Romina, Jeff, Brian and Ankur were similar to those they used earlier. Michael's representation, however, was different, drawing from an image he retrieved from his eighth grade mathematics class. What is significant is how quickly and easily the students were able to map their ideas into Michael's representation and then generalize to solve extensions of the problem.

The interest and enthusiasm of the group of tenth graders was evidenced by their willingness to immediately engage in thoughtful mathematics, even before the first tenth grade session formally began. Throughout the episode it can be seen that the students were comfortable with each other and enjoyed working together. They carefully attended to Michael's explanation and they listened and responded to each other's questions. It is apparent that they were pleased with the results of the session and with their success in solving the problem for the general case. In a subsequent session, Michael's binary notation resurfaced as the students applied it to solve an isomorphic combinatorics problem. The findings support the importance of introducing rich problems to young children and providing opportunities to revisit these problems as students grow older and have more tools available to build upon their earlier ideas.

References

Davis, R.B. & Maher, C.A. (1997). How students think: The Role of Representations. In L.D. English (Ed.), *Mathematical reasoning: Analogies, metaphors and images* (pp. 93-115). NJ: Lawrence Erlbaum.

Davis, R.B. (1992). Understanding "understanding". *Journal of Mathematical Behavior. 11*, 225-241.

Davis, R.B., Maher, C.A. & A.M. Martino (1992). Using videotapes to study the construction of mathematical knowledge by individual children working in groups. *Journal of Science Education and Technology, 1*(3), 177-189.

Davis, R.B. (1984). *Learning mathematics: The cognitive science approach to mathematics education.* Norwood, NJ: Ablex Publishing Co.

Kapur, J.N. (1970). Combinatorial analysis and school mathematics.

Educational Studies in Mathematics, 3, 111-127.

Kiczek, R.D. (in progress). *Tracing the development of probabilistic thinking: Profiles from a longitudinal study.*

Kieren, T. & Pirie, S. (1994). Mathematical understanding: Always under construction. In *Proceedings of the 18th Annual Conference of the International Group for the Psychology of Mathematics Education*. Lisbon, Portugal.

Maher, C.A. & Martino, A.M. (1998). *Brandon's proof and isomorphism. Can teachers help children make convincing arguments?* (pp. 77-101). Rio de Janeiro, Brazil: Universidade Santa Ursula.

Maher, C.A. & Martino, A.M. (1997). Conditions for conceptual change: From pattern recognition to theory posing. In H. Mansfield & N.H. Pateman (Eds.), *Young children and mathematics: Concepts and their representation.* Sydney, Australia: Australian Association of Mathematics Teachers.

Maher, C.A. & Martino, A.M. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27*(2), 194-214.

Maher, C.A. & Martino, A.M. (1992). Individual thinking and the integration of the ideas of others in problem solving situations. In *Proceedings of the 16th Annual Conference of the International Group for the Psychology of Mathematics Education.* Durham, NH.

Pirie, S. & Kieren, T. (1992). Creating constructivist environments and constructing creative mathematics. *Educational Studies in Mathematics, 23*, 505-528.

This work was supported in part by National Science Foundation grant #MDR-9053597 to Rutgers, the State University of New Jersey. Any opinion, findings, conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation.

See the paper by Muter & Maher in this volume: Recognizing Isomorphism and Building Proof: Revisiting Earlier Ideas.

JUSTIFICATIONS TO 5-8 YEARS-OLD STUDENTS' RESPONSES TO DECISION PROBLEMS

M.en C. Araceli Limón Segovia
CINVESTAV-IPS, México
eramon@mail.cinvestav.mx

Results from individual clinical interviews given to 17 students between 5 and 8 years old are presented in this report. These individual interviews refer to problems regarding decision making. They included the use of materials of a discrete nature (chips in urns) and of a continuous nature (spinners divided into sections). The findings allow us to characterize some kinds of response and observe that before proportional compositions, decision making is very complex to students, even when the nature of the materials is different. It is not the same with other compositions before which decision making with materials of a continuous nature is easier for students.

Introduction

The immediate antecedent of this study was an investigation that considered the liability of the possible curricular insertion of stochastic thought processes absent in preschool and elementary school programs (Limón, 1995). Particularly, the report presented here, is part of a project that has the purpose of analyzing the liability of teaching probability to students 5-8 years old. The ideas of probability are conceived here as a model that permits us to study chance assigned to numbers between zero and one. This analysis will be in qualitative terms in the case of preschool children or young pupils.

We first pursued infoı mation about the students: Does the nature of the aleatory devices (discrete, continuous) have an influence in solving decision problems? What strategies are used to resolve decision-making problems? What difficulties are detected in the estimation of probabilities of events?

Theoretic Elements of Stochastics in Educational Mathematics

Commonly, the 5-8 year old pupil attend our National Educational System in a preparatory grade in preschool, or, in the first or second grades in elementary school. The study of the axis of prediction and chance begins in the third grade, since, according to Piaget, when a child is 8 years or older he is in the concrete operational stage. The insertion in the curriculum seems to be supported by the thesis of this epistomologist, who, from his psycogenetic view, came to the conclusion that the idea of chance is constructed gradually and corresponds to other mental operations (Piaget & Inhelder, 1951). It is notable that those who developed the study plans and programs for preschool and elementary school, considered these relevant contributions of Piaget. However, they did not take into account that he carried out his research without considering the role of teaching. Accordingly, some researchers indicate that, through teaching, children can develop their intuition with respect to concepts of probability (Fischbein, 1975), even as early as the first grade (Falk & Levin, 1980). This development can be accomplished by starting from the beginning of prefiguration, coined by Bruner, which demonstrates the need to promote intuitive understanding starting in the elemental levels of teaching when the child cannot apprehend them in a more elaborated, analytical manner. Heitele (1975) also defended this argument.

Ahlgren and Garfield (1988) assure us that difficulties concerning the constitution of the concept of a rational number and the proportional reasoning contribute to the difficulty that is present in the development of correct intuition about the fundamental ideas of probability. This affirmation leaves by the side a qualitative approach towards the chance situations.

Methodology

The group included 15 elementary school students and 2 preschool students. We classified the clinical interviews by the type of materials that were used, of a discrete nature (urns), or of a continuous nature (spinners). The decision making technique was used in the individual interviews and this consisted of asking the students to choose one of two urns which in each case, the proportions of the urns' contents favored either an impossible outcome or a desired outcome (Fischbein, 1975). Each child was allowed to observe the contents and afterwards to answer the question *What box is it easier to get blue out of without looking*? For the spinners they were asked, *In which spinner is it easier for blue to fall* ? The justification of his response was always solicited.

Analysis of Results

Double impossibility. Given two red chips in the left urn and three red chips in the right urn; and in case of continuous material, two spinners of the same color (red); 10 of the 17 students were incorrect from the beginning when a discrete material was used. Upon using the continuous material, the answers were correct and immediate. Analyzing the justifications of the children; these are characterized in two ways: Conservation of the initial double impossibility and centering toward the secure event. The impossible event is related to the two urns and spinners, since students justified their response by arguing that blue-colored chips did not exist. Upon facing an impossible event, some children centered their attention towards a sure event that was different from the one being asked. They justified their responses by arguing that it was just as hard to get a blue chip out of one urn as it was out of the other, because there were only red ones.

Double certainty. Given two blue chips in the left urn and three blue ones in the right urn and the continuous material, two spinners were blue-colored produced the following results. After carrying out a few extractions, 10 students decided that they would draw a blue chip. Their answers were immediate in the continuous material and according to their justifications, these are characterized in three ways. Conserving certainty was shown by children who justified that it was just as easy to choose a blue chip from either because there were only blue ones in both. Other students supported their answers by arguing that there were blue chips in the urns. Others centered on what remained after selecting a chip. Jéssica (7,1) argued that it is easier in 3/3 than in 2/2 because, when one is taken out, there are two left in both urns. This type of answer did not show up in the continuous material. Others centered on the number of chips in the urns with emphasis on the greater or lesser amounts of chips. This did not show up in the continuous composition.

Certainty-impossibility. Given the left urn has two blue chips, and the right one has two red chips and one red spinner and one blue one, all of those interviewed choose correctly. We have classified the justifications in the following three ways. Consideration of certainty without impossibility was evidenced by children who included certainty as well as impossibility into their argument. This response was observed with both continuous and discrete materials. Centering on the impossibility was noted by one of the girls who justified it in this manner:

Ariadne (6,10)

A: [The blue chip is] In the left one.

E: Why?

A: Because you can't take any out of the right one because there's red.

Possibility-certainty. Given that the left urn has one red chip and one blue chip (possible event), in the right urn there are two blue chips (sure event), and one spinner has one quarter part blue (possible event), in the right urn there are two blue chips (sure event), all of the children responded correctly. They affirmed that it was easier to get blue in the right urn that the left urn. The majority of the pupils (9 of the 17) demonstrated that they centered on certainty to justify their decisions, and demonstrated that they considered that the event was sure:

Isaim (6,9)

I: [Choose] the right one.

E: Why?

I: Because everything [in the urn] is blue.

In the previous dialogues the justification of the assignment of greater possibility for the sure composition over the possible. This was used with both materials. Some students centered on the cardinality. They justified comparing the cardinality of the blue chips or that of the blue areas. Only three of those students interviewed considered both compositions to sustain their decisions. There would be classified as taking consideration of the certainty and of the possibility of the event. There were two justifications in particular that explain the obtainment of possible results with equivocations.

Antonio (6, 10)

A: In the right one.

E: Why?

A: Because there are more. No. Here there is a red one and a blue one. What if I make a mistake?

Proportionality. In the left urn there is one blue chip and one red chip; in the right urn, there are two blue chips and two red chips. The spinners used in this composition were divided into halves and fourths: the spinner on the left had two quarters colored blue, and the other two colored red, alternately. The spinner on the right was half red and half blue. The least frequency of correct answers was obtained on this composition; out of seventeen pupils, five in the discrete case and seven in the continuous selected the two urns in the initial interviews. Upon analyzing their justifications, it was found that only two students considered the proportionality of the chips in their initial interviews. Although the answers given by Ornela and Araceli are correct, their arguments reveal that for them the existence of chips of the solicited color, blue, is sufficient reason to assign equal facility for extraction. Their responses indicate a centering on the existence of the blue color while they responded correctly they did not consider the proportionality of the chips. Some students centered their attention on the quantity of blue chips or in the magnitude of the blue area, without considering its proportion in respect to the whole. They assigned greater facility to the urn that had more blue chips or to the spinner divided in half, just because it had blue. For these students, it appears they were centering on cardinality. We detected that only two children (Jéssica and Luis Miguel) justified their thinking by alluding to the proportionality in the discrete case. To do this, they transformed the two blue quarters into a half, and in this way identified that proportionality existed.

Inequality of the favorable cases and equality of the possible cases. Given the left urn has one chip and three red chips, and the right urn has two red ones and two blue ones, we detected the following types of arguments that support students' responses: Centering towards the existence of the blue color, Ariadne claimed that for her the simple existence of the blue color was reason enough to assign equal chance of extraction. Students who centered on cardinality assigned greater facility where there were more chips or to the blue area, without bothering with the amount of red ones. When students compared the red and blue areas in the left roulette, some observed that the areas of each color of the roulette was more probable on the left [alternating quarters].

Ma. José (7,4)

E: Why?

MJ: Because it has a little piece of red and a lot of blue.

Conclusions

This study addresses an interesting problem and presents conclusions which raise future questions as to why students had more difficulty with discrete materials and which materials may support student thinking. Of note was the finding that children had greater difficulty in making decisions when they used discrete materials rather than continuous. As far as various compositions, it was observed that the double impossibility represented more difficulty that the double certainty decision [In the compositions of proportionality and inequality in favorable cases with equality in possible cases]. When some students considered the favorable results, without establishing a relationship between those with the sum of possible results. This type of justification becomes more evident in dealing with materials of a discrete nature. We noticed that the proportional composition is the most difficult one for children independent of the type of material utilized.

Finally, we need to know if in the development of the lesson, the physical apparatus used can be considered as that which anchors or immobilizes the knowledge of the subject. We will have to analyze which type of anchor serves with the use of diverse aleatory devices or activities and how these influence the determination of probabilities on behalf of the participating subjects.

References

Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children*. Dordech Holland: Reidel

Falk, R., & Levin, I. (1980). A potential for learning probability in young children. *Educational Studies in Mathematics, 2*, 181-203.

Heitele, D. (1975). An epistemological view on fundamental stochastic ideas. *Educational Studies in Mathematics, 6*, 187-205.

Limón, A. (1995). *Critical analysis of the possible curricular insertion of stochastic thought processes absent in preschool and elementary school programs*. Master Thesis, CINVESTAV-IPN, Mexico

Piaget, J., & Inhelder, D. (1975). *The origin of the idea of chance in children*. New York: W.W. Norton.

SUPPORTING STUDENTS' REASONING ABOUT DATA[1]

Kay McClain
Vanderbilt University
Kay.Mcclain@vanderbilt.edu

Paul Cobb
Vanderbilt University
cobbp@ctrvax.vanderbilt.edu

The purpose of this paper is to describe how one group of students came to reason about data while developing statistical understandings related to exploratory data analysis. In doing so, we will present episodes taken from a seventh-grade classroom in which we conducted a twelve-week teaching experiment. One of the goals of the teaching experiment was to investigate ways to proactively support middle school students' development of statistical reasoning. As part of our efforts we developed an instructional sequence designed to focus on the "big ideas" in statistics. This effort also included the development of two computer minitools which we viewed as integral aspects of the sequence. As such, the computer minitools were intended to support students' emerging mathematical notions while providing tools for data analysis. The purpose of the instructional sequence was then to support students' ability to reason logically about ways to structure data in order to make an argument.

Our purpose in this paper is to describe how one group of students came to reason about data while developing statistical understandings related to exploratory data analysis. In doing so, we will present episodes taken from a seventh-grade classroom in which we conducted a twelve-week teaching experiment. One of the goals of the teaching experiment was to investigate ways to proactively support middle school students' development of statistical reasoning. In doing so we developed an instructional sequence designed to focus on the notion of distribution as one of the "big ideas" in statistics. As part of our development efforts, we viewed computer minitools as an integral aspect of the sequence, not technological "add-ons." As such, the computer minitools were intended to support students' emerging mathematical notions while providing tools for data analysis.

Instructional Sequence

As we began to design the instructional sequence to be used in the seventh-grade classroom, we attempted to identify the "big ideas" in statistics (cf., Hancock, Kaput, & Goldsmith, 1992; Konold, Pollatsek, Well, & Gagnon, in press; Mokros & Russell, 1995). It was hoped that these themes would guide the emergence of the sequence and allow us to avoid creating

[1] The research in this paper was supported by the National Science Foundation under Grant No. RED-9353587 and by the Office of Educational Research and Improvement under Grant No. R305A6007.

a listing of separate, loosely related topics that are typically characteristic of middle school curricula. In doing so we came to focus on the notion of distribution. This enabled us to treat notions such as mean, mode, median, and frequency as well as others such as skewness and spread as characteristics of distributions. It also allowed us to view various conventional inscriptions such as histograms and box-and-whiskers plots as different ways of structuring distributions.

As we began mapping out the instructional sequence, we were guided by the premise that the integration of computer minitools was critical in supporting our mathematical goals. Students would need ways to organize, describe, and compare data sets. This could be best facilitated by the use of the computer. However, in our efforts we tried to avoid creating tools for analysis that would offer either too much or too little support. This quandary can be best described by the current debate over the role of technologies in supporting students' understandings of data and data analysis. This debate is often cast in terms of what has been defined as expressive and exploratory computer models (cf. Doerr, 1995). In one of these approaches, expressive, students are expected to recreate conventional representations and inscriptions with only an occasional nudging from the teacher. The approach that we took to designing computer-based tools for data analysis offers a middle ground between these two extremes. It introduces particular tools and inscriptions that are designed to fit with students' current ways of understanding, but are intended to build toward conventional representations (Gravemeijer, Cobb, Bowers, & Whitenack, in press).

The instructional sequence developed in the course of the seventh-grade teaching experiment involved two computer-based minitools. The first computer minitool was explicitly designed for this instructional phase and provided a means for students to manipulate, order, partition, and otherwise organize small sets of data. Its use in the classroom made it possible for students to act on data in a relatively direct way that would not have been possible had we used commercially available tools that offered students a palate of conventional ways of structuring and organizing data. The first minitool also contained a value bar that could be dragged into the data and used to estimate the mean or mark the median. There was also a tool that could be used to determine the number of data points within a fixed range. It is interesting to note that the students used both of these tools in ways that we did not anticipate.

The second computer tool can be viewed as an immediate successor of the first. The tool offered a range of ways to structure data. Two of the options can be viewed as precursors to standard ways of structuring and inscribing data. These are organizing the data into four equal groups so that each group contained one-fourth of the data (precursor to the box-and-whiskers plot) and organizing data into groups of a fixed interval width so that each interval spanned the same range on the scale (precursor to the histogram). However, three other options available to students do not correspond to standard inscriptions. These involve structuring the data by (1) making your own groups, (2) partitioning the data into groups of a fixed size, and (3) partitioning the data into two equal groups. The key point is that this tool was designed to fit students' ways of reasoning while taking big mathematical ideas seriously.

As we worked to outline the sequence, we reasoned that students would need to encounter situations where they had to develop arguments based on the issue for which the data was generated. They would therefore need to develop ways to analyze and summarize the data in order to substantiate their recommendations. As a result, instructional tasks typically involved students being given one or two sets of data and asked to make a recommendation to a particular person concerning their analysis. However, this followed an extensive discussion in which students talked through the data creation process. The data then had a history and could be investigated.

Classroom Episode

As part of the classroom participation structure, students were expected to explain and justify their solutions. This was facilitated by the use of a projection system that allowed students to display their data sets from their computer and discuss their ways of organizing the data to support their decision. In addition, students were asked to produce written arguments that then served as the basis for classroom discussions. Students often changed their initial judgments based on the whole-class discussions. In this way, the students' ways of reasoning were constantly being challenged and modified in light of others' arguments.

As the sequence progressed, students were also asked to create inscriptions of the data that would serve to support their recommendations. In doing so they had to reason about ways to provide enough information to support their argument without reproducing data sets in their entirety. This initially proved problematic in that students had trouble distancing themselves from the classroom and understanding what someone outside their own investigations would need to know. Their initial attempts were cryptic and lacked detail. Further, their lack of experience in engaging in mathematical argumentation made their discussions about their recommendations problematic. They typically assumed a great deal on the part of the listener. The teacher worked to support their development of mathematical argumentation by often posing as an "outsider" and asking questions from an uninformed point of view. As the sequence progressed, students began to develop ways of reasoning and arguing that supported their ability to engage in data analysis.

Battery Analysis

Early in the sequence, students were asked to analyze the results from tests on two separate brands of batteries. Before engaging in the analysis of the data, the teacher and students talked about the data creation process. The students discussed ways that batteries might be tested, focusing on the data that would result from such tests. After an exhaustive discussion, students were then asked to compare the test results on ten batteries from each of two brands to determine which was the "better" battery. Students began by using the minitool to organize the data. Afterwards, they presented their arguments in whole-class discussion. Cara was the first student to share her argument. She began by explaining that she used the range tool to identify the top ten batteries out of the twenty that were tested. In doing so, she found that seven of the top ten were Always Ready.

Cara: See, like there is 7 Always Ready, there is like 7 of the Tough Cell are way back. Seven out of 10 of the Always Ready are the longest.

Kip: Good point.

Teacher: Okay. Jane, I'm not sure I understood so could you say it for me.

Jane: She is saying that seven out of ten of the batteries that lasted the longest are Always Ready. So these are better 'cause more of them lasted longer.

At this point, Ben raised his hand to say that he did it a different way.

Ben: Can you put the representative value on 80? (The teacher moves the vertical value bar to 80 on the scale.) Now, see there's still [Always Ready batteries] behind 80 but all the Tough Cell is above 80 and I'd rather have a consistent battery that is going to give me above 80 hours instead of one I just have to guess.

Teacher: Questions for Ben? Jane?

Jane: Why wouldn't the Always Ready battery be consistent?

Ben: All your Tough Cells is above 80 but you still have two behind 80 in the Always Ready.

Jane: Yeah, but that's only two out of ten.

Ben: Yeah, but they only did ten batteries and the two or three will add up. It will add up to more bad batteries and all that.

Jane: Only wouldn't that happen with the Tough Cell batteries?

Ben: The Tough Cell batteries show on the chart that they are all over 80 so it seems to me they would all be better.

Jane: (nods okay).

Ben appeared to be making an argument based on the fact that all of the Always Ready batteries lasted at least 80 hours. He had used the value bar to partition the data by placing the bar at 80. He then reasoned about the parts of each data set that were greater than 80. In doing so, he had concluded that Always Ready was the better brand of battery.

At this point, Jen seemed to try to make an argument based on what both Cara and Ben had offered.

Jen: I was just gonna say that, well, even though 7 of the 10 longest lasting batteries are Always Ready that the two lowest are also Always Ready and if you were using them for something important you might end up with one of the bad batteries.

Jen had noticed that while Tough Cell had 7 of their 10 batteries in the top ten, they also had the lowest two batteries. She then reasoned that with such a wide range you could not depend on Always Ready batteries for *something important.*

The students continued to engage in similar investigations using the first minitool for several weeks. As they worked, many of them frequently used the value bar to partition the data. They would place the bar at a particular location along the scale and then reason about the number of data points that appeared above or below. It is important to note that the position on the scale they used in partitioning the data did not appear to be arbitrary. For instance, in a task about health care, many placed the bar at age 65, arguing that they had noted which data were senior citizens. In this way, they used the bar to create quantitative descriptions of the data.

As the students worked, we continually analyzed the nature of their activity as we developed and modified tasks. As a result, once students seemed to be developing ways to organize the data and an understanding of what is involved in a mathematical argument, we introduced the second minitool. As previously noted, this tool was designed in a manner similar to an axis plot. It not only allowed us to eliminate the magnitude bars, but it also allowed us to increase the number of data points in the sets. These features supported a shift in the nature of students' arguments — they began to focus on characteristics of the distributions instead of the individual data points. For example, students argued about the location of the *hill* in comparing two sets of data, noting a shift in the cluster of data points. They also began to use the language of a five-point summary (extremes, quartiles, and median) as they made arguments such as: *one-half of this data set is in the range of three-fourths of the other data set.*

Conclusion

We would stress that the purpose of the instructional sequence was not that students might come to create certain graphs in particular situations or calculate measures of central tendency correctly. Instead, it was that they might reason logically about ways to structure data in order to make an argument. This is a radically different approach to statistics than is typically introduced in middle schools. As the goal was *not* to ensure that all students could create certain types of graphs, the teacher continued to support the development of arguments that could be justified to other members of the classroom community in terms of reasoning about distributions. Thus, the focus in discussions was on the meaning that students' records of their analysis activity had for them. In addition, students seemed to reconceptualize their understanding of what it means to know and do mathematics as they compared and contrasted solutions. The crucial norm that became established was that of explaining and justifying solutions in the context of the problem being explored.

References

Doerr, H. M. (1995, April). *An integrated approach to mathematical modeling: A classroom study*. Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Gravemeijer, K., Cobb, P., Bowers, J., & Whitenack, J. (in press). Symbolizing, modeling, and instructional design. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design*. Mahwah, NJ: Lawrence Erlbaum Associates.

Hancock, C., Kaput, J., & Goldsmith, L. (1992). Authentic inquiry with data: Critical barriers to classroom implementation. *Educational Psychologist*, 27, 337-364.

Konold, C., Pollatsek, A., Well, A., & Gagnon, A. (in press). Students analyzing data: Research of critical barriers. *Journal of Research in Mathematics Education*.

Mokros, J., & Russell, S. (1995). Children's concepts of average and representativeness. *Journal for Research in Mathematics Education*, *26*(1), 20-39.

AN ANALYSIS OF STUDENTS' STATISTICAL UNDERSTANDINGS[1]

Maggie McGatha
Vanderbilt University
mmcgatha@aol.com

Paul Cobb
Vanderbilt University
cobbp@ctrvax.vanderbilt.edu

Kay McClain
Vanderbilt University
kay.mcclain@vanderbilt.edu

This paper documents the analysis of performance assessments conducted in a seventh-grade classroom. The purpose of the assessments was to obtain baseline data on students' current statistical understandings that would then inform future instructional design decisions in a current research project. The tasks were designed to provide information about students' current understandings of (a) the mean and (b) graphical representations. Students typically viewed the mean as a procedure that was to be used to summarize a group of numbers regardless of the task situation. Data analysis for these students meant "doing something with the numbers."

In our current work, we are refining our approach to instructional design in the context of students' development of statistical thinking in seventh and eighth grade. As part of the pilot work, we reviewed the literature on statistics teaching and learning in order for us to clarify what the "big ideas" should be in statistics at the middle-school level. There are actually only a handful of studies available that focus on students' statistical understandings. These studies fall into two categories: (1) studies that examine students' understanding of the mean and (2) studies that examine students' statistical understandings in the context of data analysis. All of the early research focuses on students' misunderstandings and misconceptions of the mean (e.g. Mevarech, 1983; Pollatsek, Lima & Well, 1981, Strauss & Bichler, 1988). More recently, researchers have studied how students use the mean to summarize and compare data sets (e.g., Gal, I., Rothschild, K., & Wagner, D. A., 1990; Mokros and Russell, 1995) These studies emphasize that traditional instruction may provide students with the appropriate algorithm for the mean, but leave them with an incomplete conceptual understanding. An emerging research trend is focusing on studies where students are involved in the more complex activity of data analysis (e.g., de Lange, van Reeuwijk, Burrill, & Romberg, 1993; Hancock, Kaput, & Goldsmith, 1992; Konold, Pollatsek, Well, & Gagnon, in press; Lehrer & Romberg, 1996). Typically, these studies outline the process by which students

[1] The analysis reported in this paper was supported by the Office of Educational Research and Improvement (OERI) under grant number R305A60007. The opinions expressed do not necessarily reflect the views of OERI.

analyzed and reasoned about data in more innovative instructional approaches. Clearly, the two categories of studies highlight different aspects of statistics instruction. The first set of studies emphasize the mathematical *content* (the mean) and the second set of studies document the mathematical *process* involved in data analysis. We believe it is crucial to transcend this dichotomy between content and process by developing an instructional approach that focuses simultaneously on data analysis and on mathematical content.

In the context of data analysis, one alternative for the mathematical content in statistics is to move beyond simply understanding the mean to more unifying big ideas. For example, several authors stress the importance of students coming to view data as an entity as opposed to a collection of individual data points (Hancock et al., 1992; Konold et al., in press; Mokros & Russell, 1995). One idea that helps identify what might be involved in viewing data as entity is that of a space of potential data values. In particular, we conjecture that students who view data as entity see the individual data points as located within a space of possible values. As an example, Hancock et al. document that the students in their study rarely used the axis plot option of TableTop even though they had used the software to conduct data analyses for a year and could explain the meaning of the icons when shown on axis plots. They suggest that the very thing that made the axis plot powerful – the fact that it corresponded to a space of possible values, rather than to a single value – also made it harder to understand. In other words, since the students did not conceptualize the individual data points as located in the space of all possible values, the possibility of using an axis plot did not occur to them. A similar analysis holds in the case of Konold et al.'s observation that the students in their study rarely used the histogram option of the DataScope software. A histogram involves structuring the space of all possible data values into equal intervals. These examples illustrate the importance of a space of potential values as a big idea in statistics instruction.

A second big idea, closely related to the first, that came to the fore in our reading of the literature is that of group propensity (Konold et al., in press). In defining group propensity, Konold et al. refer to the rate of occurrence of some data value within a group that varies across a range of data values. For example, the data value in question might be that of being a boy rather than a girl. Unless individual data points are located within a space of possible data values in which they can take on the values of boy or girl, the propensity of being a boy cannot be formalized as, say, 65%. As Konold et al. observe, the possibility of comparing groups in terms of means or relative frequencies did not occur to the majority of students in their study when they conducted data analysis. This can be accounted for in terms of their lack of understanding of the big ideas of a space of potential values and of group propensity. The development of these two big ideas together constitute major steps towards understanding the statistical concept of distribution and are, therefore, mathematically significant. We have also pointed out the crucial role these two big ideas play in data analysis. For us, these two big ideas would allow classroom research to focus on students' mathematical development as they participate in data analysis activities.

In order to support students' development of an understanding of data analysis along with an understanding of the big ideas in statistics, it is important to develop instructional sequences which (1) build on students' current understandings and (2) support shifts in their current ways of reasoning. As part of our pilot work for our current project, we conducted classroom performance assessments in order to obtain baseline data on students' current statistical understandings. The assessments were conducted during the fall semester of 1996 in three sessions of a seventh-grade class. During the sessions, a former middle-school teacher who was a member of the research team posed tasks to the students as they worked together in groups. The tasks were designed to provide information about students' current understandings of (1) the mean and (2) graphical representations of data (inscriptions) because these two topics were the focus of the statistics chapter in the textbook series used by the students in their previous instruction. By focusing on students' current ways of reasoning, the subsequent instructional materials could build from their current knowledge. The purpose of this paper, then, is to document the analysis of these performance tasks which will then serve to inform subsequent decisions concerning instructional development.

Results of Analysis

The general format for the three mathematics class sessions in which the performance tasks were conducted was a whole-class introduction to the task, student collaboration in small groups, followed by a whole-class discussion of their solutions. Students worked in groups composed of from 3 to 6 students with the number of groups varying from task to task. In the following sections of this paper, we will begin by describing the context of the task[2], the design decisions underlying the task, and our anticipation of how students would respond to the task. Second, the small-group work is analyzed in order to highlight the various solution methods. Finally, the whole-class discussions are analyzed to clarify students' understandings.

Task 3: Basketball All-Star

In the Basketball All-Star task, students were asked to make a decision based on a given set of data. We designed this task in an attempt to gain information on how the students would deal with the issue of variability as it related to the mean. We anticipated that some groups would reason that

[2] Due to space restrictions, only one performance assessment is presented. For a detailed analysis of all tasks see McGatha, Cobb, and McClain, 1998.

the data set with the larger mean was the better choice without going back to the task situation and considering the impact of variability on the decision in this particular instance. We therefore designed the task so that the data set with the larger mean also had the greater variability. For us, then, the data set with the larger mean would not be the better choice because in this situation (scoring in basketball) consistency would be more important when making a decision.

Basketball All-Star. In the Basketball All-Star task shown below, students were given a listing of the number of points scored by each of two basketball players in each of eight games. They were then asked to decide which player should be selected to play in the all-star tournament based on these scores.

> *One player will be selected from the Meigs basketball team to play in the all-star tournament. Below is a listing of the points scored by the top two candidates for the last eight games of the season. Based on this information, present an argument to support the selection of one of the players.*
>
> *Player A: 11 31 16 28 27 14 26 15*
> *Player B: 21 17 22 19 18 21 22 20*

Figure 1. *Basketball All-Star Task*

Group work. This task was approached in two distinct ways. The majority of students, five of eight groups, solved this task by calculating the total or the mean. The remaining three groups also initially found the mean or the total, but then they reconsidered the task situation and decided the player with the higher mean was not the better player for the tournament. For the five groups who calculated the total or the mean, this task was about totaling or averaging the points of each player and determining the winner by comparing the outcome. Each of these groups selected Player A because he had the higher total and/or the higher average number of points. For these groups, the mean provided the best summary of the data regardless of the situation. If we consider the prior school experiences of these students, reasoning about the problem in this way seems reasonable. In traditional school mathematics a group of numbers is often summarized by calculating the mean.

The other three groups of students initially began the task in the same way as the groups described above. However, they subsequently selected Player B even though his total points were lower than Player A. After calculating the total or the mean, these groups went back to the task situation and decided that the player with the higher mean was not necessarily the player that should be sent to the tournament. These discussions will be elaborated in the next section.

Whole-class discussion. The teacher began the whole-class discussion by asking students to defend their choice of Player A or Player B as the one to send to the all-star tournament. The first group to share their thinking argued for sending Player B.

Student: Our group said that you should send Player B to the tournament because he has a...even though Player A's average [sic] is higher than his, which was only by eight points, he has a more steady...in his games...his points...they're more...they're not so up and down like Player A's are...where one day... he goes from 11 points to 31 points so that's why we said Player B.

Only one other group shared their thinking and they argued for sending Player A to the tournament. However, their argument went beyond their reasoning about the mean. This group had been challenged by another group that was sitting at the same table and as a result, expanded their argument. They created a situation to explain the variability in Player A's scores and argued that Player A, in addition to having the higher average, was probably a "team player." His low scores (11, 16, 14, 15) indicated that he was giving the ball to other players and he probably earned high assists in those games rather than high points. His high scores (31, 28, 27, 26) indicated that he helped the team out when they were falling behind. This group continued their support of Player A by creating a situation to downplay Player B's consistent scoring. They argued that Player B had scores that were about the same which indicated that he had a "personal goal" to achieve in each game and was not thinking of the team. This group believed the mean was the best summary of the data sets and the counterargument about consistency in scoring did not alter their position. In order to defend their choice of selecting the player with the higher mean, this group appeared to rely on their knowledge of basketball to create a narrative that would support their choice of Player A.

Conclusion

In our analysis we found that students typically viewed the mean as a procedure that was to be used to summarize a group of numbers regardless of the task situation. Data analysis for these students meant "doing something with the numbers" which, we conjecture, was grounded in their prior school mathematics experiences. Similarly, we found that students' conversations about graphical representations highlighted the procedures for constructing graphs with no attention to what the graph signified and how that related to the task situation. In order for students to participate in genuine data analysis, it will be necessary to support a shift in students' current ways of reasoning towards data analysis as inquiry rather than procedure.

References

de Lange, J., van Reeuwijk, M., Burrill, G., & Romberg, T. (1993). *Learning and testing mathematics in context. The case: Data visualization.* Madison, WI: University of Wisconsin, National Center for Research in Mathematical Sciences Education.

Gal, I., Rothschild, K., & Wagner, D. A. (1990, April). *Statistical concepts and statistical reasoning in school children: Convergence or divergence?* Paper presented at the annual meeting of the American Educational Research Association, Boston, MA.

Hancock, C., Kaput, J. J., & Goldsmith, L. T. (1992). Authentic inquiry with data: Critical barriers to classroom implementation. *Educational Psychologist, 27* (3), 337-364.

Konold, C., Pollatsek, A., Well, A., & Gagnon, A. (in press). Students analyzing data: Research of critical barriers. *Journal of Research in Mathematics Education.*

Lehrer, R., & Romberg, T. (1996). Exploring children's data modeling. *Cognition and Instruction, 14* (1), 69-108.

Mevarech, Z. (1983). A deep structure model of students' statistical misconceptions. *Educational Studies in Mathematics, 14*, 415-429.

Mokros, J., & Russell, S. (1995). Children's concepts of average and representativeness. *Journal for Research in Mathematics Education, 26* (1), 20-39.

Pollatsek, A., Lima, S., & Well, A. D. (1981). Concept of computation: Students' understanding of the mean. *Educational Studies in Mathematics, 12*, 191-204.

Strauss, S., & Bichler, E. (1988). The development of children's concepts of the arithmetic mean. *Journal for Research in Mathematics Education, 19* (1), 64-80.

MIDDLE SCHOOL STUDENTS' MISUSE OF THE PHRASE *50-50 CHANCE* IN PROBABILITY INSTRUCTION

James E. Tarr
Middle Tennessee State University
jetarr@frank.mtsu.edu

Purpose of the Study

The study was part of a larger investigation (Tarr, 1997) which sought to examine the impact of probability instruction on students' thinking during-, immediately following-, and seven weeks after the instructional program. The primary objective of the present study was to determine students' strategies for describing conditional probabilities, and how these strategies — which included the misuse of *50-50 chance* — changed as the result of probability instruction.

Theoretical Framework

The instructional program used in the larger study was based on two theoretical positions. The first is a pedagogical orientation that acknowledges the importance of teachers' knowledge of student cognitions (e. g., Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Jones, Langrall, Thornton, & Mogill, 1996; Shulman, 1986) and views teaching as helping students to construct knowledge through problem solving and engaging in mathematical discourse (e. g., Cobb, Yackel, & Wood, 1993; Lester, 1989; Simon, 1995). The second is a cognitive framework that describes and predicts middle school students' thinking in conditional probability and independence (Tarr & Jones, 1997). This framework was used to generate assessment protocols and to inform all phases of instruction.

Methodology and Data Sources

Twenty-six students from a fifth-grade class were randomly assigned to one of two instructional groups, and 13 students from a second fifth-grade class served as a control group. The teacher-researcher used a cognitive framework (Tarr & Jones, 1997) to develop an eight-day instructional program that comprised a series of problem tasks and key questions, diversified journal prompts, and parallel and extension activities. In using the problem tasks, the teacher-researcher encouraged classroom discourse and attempted to foster students' understanding by listening to their individual responses, assessing their thinking in relation to the cognitive framework, and adapting instruction accordingly.

Three mathematically-equivalent forms of an interview protocol were used to assess students' cognitions at three points in time: prior to-, immediately following- and seven weeks after the instructional program. Using multiple sources of data – audiotaped interview assessments, videotapes, journals and worksheets of the case study students, and the teacher-researcher's journal observations – case study analysis was undertaken to identify patterns and changes in students' cognitions during the instructional program and to determine the catalyst for any such changes in probabilistic thinking.

Results of the Study

The instructional program impacted students' understanding in a substantial way. In particular, qualitative analyses discerned numerous key learning patterns among case study students. Compared to baseline assessments in conditional probability, students following instruction were more likely to make appropriate use of the phrase *50-50 chance* in describing conditional probabilities. In particular, prior to instruction students used the phrase 50-50 chance inappropriately in two ways. First, they applied the phrase to probability situations in which all events in the sample space were equally likely to occur, and concluded that each event had a 50-50 chance. In addition, when the sample space contained two elements, they often assumed each outcome had a 50-50 chance, even when the two events were not equally likely. This latter invalid use of numbers was especially troublesome when students considered conditional probabilities in without-replacement situations.

Inappropriate use of the 50-50 chance phrase was particularly evident at the initial assessment as demonstrated by five of six case study students. Sandra exemplified the problematic use of this phrase in the two aforementioned ways. Her multiple misuse of the phrase 50-50 chance was apparent as she predicted the outcome of random experiment in which a class president and vice president were to be selected, one after another, from a bag containing five names: Rick, Maria, Beth, Steve, and YOU. The following excerpt illustrates Sandra's two misuses of the 50-50 chance phrase:

I: After school the principal decides to draw the names out of a bag. Is it more likely that the class president will be a boy or a girl, or is it the same chance for–

S: [Interrupts interviewer's question] –Same chance.

I: Why is it the same chance?

S: Because... well... it's a 50-50 chance because you could be a boy or a girl...

I: What about for you, specifically for you?

S: Well, anybody– you could pick anybody, a boy or a girl, it would be the same chance [In this case, the sample space comprised two events, boy and girl, and each, according to Sandra, have the same chance].

I: Is it more likely that your name will be read, is it more likely that your name will not be read, or is it the same chance?

S: Knowing my luck, it probably won't be read!

I: [Laughs] ...Can you use numbers to work it out?

S: It would be a 50-50 chance.

I: What do you mean by that?

S: Well, Steve could be picked, Beth could be picked, Maria, or Rick, or maybe me, so... same chance that everyone else would have [She noted that each element in the sample space has the same chance].

I: Let's say that Maria's name was announced as the class president. [After Maria's name was removed, four names (two boys and two girls) remained in the bag] Now the name of the vice president is going to be announced next. Is it more likely the vice president will be a boy or a girl, or is it the same chance for a boy or a girl?

S: Same chance because there's two girls and two boys. It's like a 50-50 chance because a boy could be picked or a girl could be picked [In this case, she seemed to correctly identify that the probability of each event was a 50-50 chance].

I: Compared to the first time, has the chance that your name will be read changed or is it the same chance as it was before?

S: Well, it's the same as before because it's still the same chance because anybody could be picked and you still have a 50-50 chance that anyone could be picked.

I: Compared to the first time, has the chance that a boy's name will be read changed or is it the same as it was before?

S: It's the same as before because a girl or a boy could be picked. It's a 50-50 chance.

I: Compared to the first time, has the chance that a girl's name will be read changed or is it the same as it was before?

S: Same... for the same reason.

Sandra's proclivity to regard practically any event as having a 50-50 chance impeded her from realizing that the probabilities of events changed in the without-replacement situations. In essence, because events always have a 50-50 chance, then the probabilities of such events essentially never change because, according to her, they are always 50-50.

Growth in Sandra's thinking was clearly evident on the post-instruction assessment as she seemed to carefully monitor the composition of the sample space. In a parallel item from the post-instruction assessment, two group leaders were selected without replacement from a bag containing the names Colleen, Jack, Roberto, Teresa, and YOU:

I: Is it more likely the leader of the first group will be a boy or a girl, or is it the same chance?

S: Well, since I'm a girl and there's two other girls, I'd probably say that it's more likely to be a girl because there's only 2 boys and there's 3 girls because I'm a girl.

I: Is it more likely your name will be read, it won't be read, or is it the same chance?

S: It would be the same chance out of five people. That's what I would think, because it's not like it's always going to be you that's going to be picked out, and it's the same chance for everybody including you. It could be Colleen, or maybe Teresa.

I: Is it a 50-50 chance?

S: Not really because there's five people so it would be... see, a 50-50 chance means there's two people, so it would be, like, a 20-20 chance.

I: Let's say that your teacher draws a name from the bag and announces, *It's a girl!* Has your chance of being named as the first group leader changed or is it the same as before?

S: Well, I'd probably say that it's the same chance for all three of us because we all have the same chance of getting it. It'd probably be, like, a 33 and 1/3 chance.

I: I'm going to draw the name of the second group leader next. Do you predict the next group leader will be a boy or a girl?

S: It's the same chance now because there's 2 boys and 2 girls.

I: Has the chance that your name will be read changed or is it the same chance as before?

S: It's changed because now there was 3 girls, but now there's two girls so it's a 50-50 chance between the boys and the girls.

I: Has the chance that a girl's name will be read changed or is it the same chance as it was before?

S: It has changed because now there's 2 boys and 2 girls and so it's a 50-50 chance between both of them and before it was like a... I don't know what kind of chance, but (it was) not that. [She seemed unable to mentally compute the probabilities for the events, girl and boy, when the sample space contained 3 girls and 2 boys]

Sandra kept track of the composition of the sample space after each trial as evidenced by her use of the words, before and now. Moreover, she identified that it was not possible for five different students to each have a 50-50 chance of being named leader of the first group, and determined the corresponding probability to be 20%, not 50%.

The change in Sandra's thinking may have been attributed to an instructional episode which focused on the probability of drawing a Snickers from a bag containing 1 Snickers, 2 Butterfinger, and 3 Milky Way candy bars. More specifically, after candy bars were sampled without replacement the question was posed, *At what point does it become a '50-50 chance'* *for drawing a Snickers?* While some students correctly identified that the bag would have to contain 1 Snickers and 1 other candy bar, many others asserted a 50-50 chance was represented when the sample space comprised 1 Milky Way, 1 Butterfinger, and 1 Snickers candy bar. Classroom discourse ensued and confronted the misconception that three events can each have a probability of 50%.

Discussion

The instructional program successfully challenged students' inappropriate use of the phrase *50-50 chance* by developing the concept of the *probability of an event* and a fundamental principle that the entire sample space comprises 100% of the probabilities. Indeed, analysis of students' learning revealed that both concepts were critical to empowering students to discern when the probabilities of events change. Moreover, by having students determine the probability of equally-likely events, they were able to learn the correct meaning of the phrase, 50-50 chance. Finally, evidence of case study students' learning during instruction determined that whole-class discussions were likely catalysts for resolving students' misuse of the phrase 50-50 chance. Such problematic use of the phrase was resolved after students who held these misconceptions were selected to share their thinking in order to stimulate whole-class discussions.

Conclusion

The pervasive misuse of the phrase 50-50 chance prior to-, and during probability instruction is a significant finding of the study. Heretofore, research has not documented the inappropriate use of this heuristic in two distinct ways. Moreover, this study is the first to report how such a misconception can be resolved through instruction. The rich documentation generated in this study on case study students' learning is intended to provide helpful knowledge of student cognition for both researchers and mathematics teachers.

References

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Education Research Journal, 26*, 499-532.

Cobb, P., Yackel, E., & Wood, T. (1993). Learning mathematics: Multiple perspectives, theoretical orientation. In T. Wood, P. Cobb, E. Yackel, & D. Dillon (Eds.), *Rethinking elementary school mathematics: Insights and issues. Journal for Research in Mathematics Education Monograph Series*, Number 6 (pp. 21-32). Reston, VA: The Council.

Jones, G. A., Langrall, C. W., Thornton, C. A., & Mogill, A. T. (1996). Using children's probabilistic thinking in instruction. In L. Puig, & A.

Gutierrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education*, Volume 3 (pp. 137-144). Valencia, Spain: University of Valencia.

Lester, F. K. (1989). Reflections about mathematical problem-solving research. In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 115-124). Reston, VA: The Council.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26* (2), 114-145.

Tarr, J. E. (1997). *Using middle school students' thinking in conditional probability and independence to inform instruction*. Unpublished doctoral dissertation, Illinois State University.

Tarr, J. E. & Jones, G. A. (1997). A framework for assessing middle school students' thinking in conditional probability and independence. *Mathematics Education Research Journal, 9,* 39-59.

Probability and Statistics Short Orals

INVESTIGATING KNOWLEDGE STRUCTURES IN INFERENTIAL STATISTICS

Cengiz Alacaci
University of Pittsburgh
ceast2+@pitt.edu

The purpose of this study was to investigate the nature of knowledge base that can support the ability to choose an appropriate statistical technique for applied research situations. Two types of tasks involving research scenarios and statistical techniques were used to elicit knowledge from subjects with different levels of experience and with different specializations. Subjects were chosen from four groups: statistical consultants (experts in applied tasks), mathematical statisticians (experts in theoretical tasks), graduate students in research methods, and in mathematics education (novices). Individuals from each group were interviewed using the repertory grid technique (Kelly, 1955). Subjects were asked to indicate the similarities and differences (constructs) among groups of research scenarios and statistical techniques. The themes represented in the constructs were coded and classified into four types of knowledge; research design, theory, procedures and nontechnical aspects.

Results showed that although there was no statistically significant difference between the extensiveness of knowledge used by experts and novices in any of the task environments. However, differences in the use of specific types of knowledge were observed. Compared to novices, a bigger chunk of the knowledge used by experts was related to features of research design. Also, knowledge used by statistical consultants was found to be richer in terms of elements related to research design compared to mathematical statisticians. These results indicate that knowledge of research design is the most prominent component of a well-developed knowledge base that can support the ability to select an appropriate statistical technique for applied research situations. For statistics education, it can be suggested that explicit connections should be made between statistical techniques and the types of research design for which they can be used while teaching inferential statistics.

Reference

Kelly, G. (1955). *The psychology of personal constructs.* New York: Norton.

ELEMENTARY SCHOOL TEACHERS' PROCEDURAL AND CONCEPTUAL KNOWLEDGE OF MEDIAN

Lynne M. Gregorio
North Carolina State University
JCGregorio@juno.com

Sarah B. Berenson
North Carolina State University
berenson@unity.ncsu.edu

This study examined elementary school teachers' content knowledge of median. The constructivist framework for this study focused on teacher content knowledge, specifically procedural and conceptual knowledge of median (Hiebert & Lefevre, 1986). An open-ended assessment of statistical content was given to 55 elementary school teachers as part of Teach-Stat, a professional development workshop (Friel & Bright, 1996). Teacher responses to three questions relating to median were examined and teacher profiles were created.

Results indicated that many teachers viewed median as middle. These data suggested that relating median to middle is the vital first step when developing procedural and conceptual knowledge of median. Only those teachers who defined median as middle were able to demonstrate procedural and conceptual knowledge of median on the remaining two assessment questions.

Other views of median included median as midpoint, mean, and mode. It appeared that teachers were more likely to *conceptually equate* median with midpoint than with mean and mode. However, teachers often confused the terms median, mean, and mode. None of the teachers that defined median as midpoint, mean, or mode were able to demonstrate procedural and conceptual knowledge of median in the remaining two questions on the assessment.

References

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics.* Hillsdale, NJ: Lawrence Erlbaum.

Friel, S., & Bright, G. (199?). Teach-Stat Final Report. Chapel Hill, NC: Mathematics and Science Education Network.

THE PEDAGOGICAL PERSUASIVENESS OF SIMULATION IN SITUATIONS OF UNCERTAINTY

Jesse L. M. Wilkins
Virginia Polytechnic Institute and State University
jwilkins@vt.edu

George C. Reese
University of Illinois at Urbana-Champaign
g-reese@uiuc.edu

Students of all ages have been found to have misconceptions and lack sound intuition in situations of uncertainty (e.g., Kahneman, Slovic, & Tversky, 1982). The NCTM *Curriculum and Evaluation Standards* state that in order to overcome these misunderstandings and build sound probabilistic understandings students should be involved in hands-on activities that model situations of uncertainty and use simulations to determine probabilities and solve problems (NCTM, 1989). However, few studies have been conducted to investigate the effectiveness of computer simulations in overcoming probabilistic misconceptions.

This study involved 107 eighth graders and investigated the persuasiveness of computer-based Monte Carlo simulations on students' choice of strategy for the problem known as Monty's Dilemma. Each student was presented with the problem as a mock game show. Students were then either taught the best strategy for winning Monty's Dilemma using traditional instructional methods or directed to investigate the problem using an interactive computer-based simulation.

Students investigating Monty's Dilemma through simulation were subsequently more likely to choose the more statistically sound strategy for winning ($c^2(1) = 4.05$, $p < .05$). Although little evidence in the study suggests that students understood the concepts underlying Monty's Dilemma, results do suggest that computer-based simulations are more persuasive than traditional instructional techniques in convincing students to make statistically sound decisions.

References

Kahneman, D., Slovic, P., & Tversky, A. (Eds.). (1982). *Judgment under uncertainty*. Cambridge: Cambridge University Press.

NCTM. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

PROBABILTY AND STATISTICS POSTERS

404

CONDITIONAL PROBABILITY AND PRE-SERVICE TEACHERS: CHARACTERISTICS OF REASONING

Lisa J. Carnell
High Point University
lcarnell@acme.highpoint.edu

The purpose of the study was to determine characteristics of reasoning about conditional probability. The study is grounded in Falk's (1989) analysis of major difficulties: (1) difficulty defining the conditioning event; (2) difficulty with the temporal order of the conditioning event and the target event; and (3) confusing conditionality and causality.

Subjects were 13 pre-service middle grades teachers who had (a) mathematics or science as a certification area and (b) correctly identified the conditioning event and the target event in two screening conditional probability problems. Each subject was interviewed on 6 conditional probability problems, 2 for each misconception. Problems were presented in random order. Interviews were taped and transcribed. Transcripts were analyzed by categorizing subjects' responses and comparing responses between problem versions for individual subjects and across problem types.

Over all, no subject was misconception-free, and no subject showed evidence of a misconception on every problem. Findings from this study indicate that pre-service middle grades teachers' reasoning can be generally characterized as follows:

1. Use of inferred events as conditioning events rather than use of specified event in the problem.
2. Disregard of the conditioning event when it occurs after the target event in real time.
3. Inappropriate use of independence.
4. Inappropriate application of prior knowledge from other content areas.
5. Use of causal reasoning rather than conditional reasoning.
6. Use of procedures for computation of probabilities in inappropriate situations.
7. Oversimplifying the problem by failing to use all relevant information in the problem.

References

Falk, R. (1989). Inference under uncertainty via conditional probabilities. In R. Morris (Ed.), *Studies in mathematics education: The teaching of statistics, volume 7* (pp. 175-184) Paris: UNESCO

USE OF COMPUTER-BASE MINITOOLS IN SUPPORTING THE EMERGENCE OF STATISTICAL MEANING IN CLASSROOM DISCOURSE

Jose Luis Cortina
Vanderbilt University
jose.l.cortina@vanderbilt.edu

In this poster session we present two computer-base minitools that were developed for a 7th grade classroom teaching experiment in the context of statistics.[1] The general purpose of the experiment was to bring students to view data sets as entities that are distributed within spaces of possible values (Cobb, In Press). The minitools were used in 27 of the 34 sessions of the classroom teaching experiment. In a regular session a problem was presented to the students that usually related to social or scientific issues. The class would discuss the relevance of the problem, as well as the different procedures that were involved in gathering the data. In small groups, the students then explored the data with one of the minitools and tried to reach a conclusion. Finally the teacher and the students, using a computer projecting system, discussed the different approaches to the problem.

The classroom teaching experiment was designed over a hypothetical learning trajectory. This was supported by the minitools, through offering ways of inscribing and organizing data that fitted the evolving taken-as-shared ways of reasoning of the community. For this reason, the minitools included options of structuring data that did not necessarily correspond to conventional inscriptions. The first minitool helped the students conceive data inscriptions as representations of measured qualities of objects. Further, it helped them explore qualitative characteristics of collections of data points. Building from the first minitool, the second helped the students analyze data distributions multiplicatively, identify global patterns in the data, and describe these patterns in quantitative terms (Cobb, In Press).

References

Cobb, P. (in press). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical thinking and learning*.

[1] The classroom teaching experiment was conducted in fall 1997 by Paul Cobb, Kay McClain, Koeno Gravemeijer, Jose Cortina, Lynn Hodge, Maggie McGatha, Beth Petty, Carla Richards, and Michelle Stephan. The minitools were developed by Koeno Gravemeijer, Paul Cobb, Michael Doorman, and Janet Bowers.

INTERPRETING DATA IN *REAL-LIFE* CONTEXTS

Patricia M. Lamphere
Oklahoma State University
lampher@okway.okstate.edu

Making mathematics meaningful for middle school students involves incorporating activities which integrate mathematics into real-life applications. The interpretation of data from a variety of representations reinforces the connections of mathematics within other disciplines. One area of cross-curricular content is the study of population density which occurs in middle school social studies and can be reinforced in mathematics studies of data analysis and interpretation.

In this study, 40 eighth-grade students responded to a set of questions which reflected the idea of population density and its representations in graphical and rule form. One question asked students to draw a graph to represent the "population density" in one square mile by first identifying a scale which would be compatible with each of three examples. Analysis of responses indicate that less than 13% of the students were able to correctly identify and apply a common scale to each of the drawings. Approximately 18% of the students drew diagrams which correctly utilized the same scale factor across the examples, but did not identify the scale used. Almost 70% of the students gave incorrect responses with no consistent scale apparent, gave no response, or choose a different scale for each of the examples. A misunderstanding seems to exist in students' concept of what scale means and how it can be applied to represent data in a pictorial form.

For the second question, students were to use information given as a percentage of the population residing in apartments or single-family dwellings, to identify the community as having a high, medium, or low population density, and then arrange in order on a number line the letters representing the communities from least to greatest amount of land area. Only 5% of the students responded correctly to both questions. Over 50% of the students responded incorrectly to both questions. Twenty percent of the students answered the first question correctly, but were unable to complete the number line portion. Generally, students had difficulties translating one form of representation to another and making and supporting inferences about sets of data. These two skills are critical for making decisions in real-life, everyday situations.

STUDENT WAYS OF THINKING ABOUT DISTRIBUTIONS

Melissa Mellissinos
San Diego State University
mmelliss@crmse.sdsu.edu

I am conducting a study about college student conceptions of data distribution using a grounded theory approach (Glaser & Strauss, 1967). Grounded theory is an empirical approach to generating theory in which data collection and analysis are intertwined in a single process. So far, the data (from semi-structured interviews) reveal some interesting patterns in students ways of thinking about distributions. A few examples are that students tend to:

- relate what they know about graphs in rectangular coordinates to graphs of distributions. For example, they tend to believe that a distribution involves two variables. They also tend to interpret the slopes of distribution curves.
- believe that representations of distributions are not vague; distributions show actual data values, frequencies and categories explicitly. Representations that hide or collapse information are often not considered distributions.
- think about distributions in terms of their surface features. For example, pie charts and histograms are distributions, even when what they represent is not understood.
- indicate that in a distribution, something is distributed. (e.g., numbers, people, test scores).
- apply a non-statistical (everyday or mathematical) meaning to words that have a specific and different meaning in statistics (e.g., parameter, range, distribution).

Although these findings are preliminary, I expect them to appear as properties of categories in the grounded theory about student conceptions of distribution that I aim to generate.

Reference

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research.* Chicago: Aldine.