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ABSTRACT

Finding the number of cubes in rectangular solids provides a cognitive framework for understanding the measurement of volume. This study reveals whether the activities provided in the context of equal investing and equal sharing that emphasize both numerical and spatial aspects of cube configurations cause any improvement in student strategies that lead them to use composite or iterable units in enumerating the cubes contained in rectangular buildings. This study also provides a review of literature and data collection methods. Contains 53 references. (ASK)

Children's Understanding of Rectangular Solids Made of Small Cubes

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INTRODUCTION

Mathematics concerns about systems and patterns in any forms such as numerical and figural. Finding patterns (invariants) in the relationships among and between elements of figures is in the domain of geometry (Goldenberg et al. 1998). There is a two-way interaction between numerical reasoning and geometry learning. Recent research (e.g., Battista & Clements, 1998, 1996; Bishop, 1989; Clements et al. 1997) suggest that such visual presentations as manipulative materials, concrete embodiments and intuitional devices offer a powerful introduction to the complex abstractions of mathematics. In fact, many concepts and processes in school mathematics can be tied to visual interpretations. That is, visual models can be build which reflect (a large part of) the underlying mathematical structure (Eisenberg & Dreyfus, 1989).

Although visual solutions are not considered finished work in mathematics they serve as an intuitional proof to stimulate further thought because they help people gain insights into calculations in arithmetic and algebra (Goldenberg et al., 1998). Piaget & Inhelder (1967) believe that intuition is the instrument of invention, whereas demonstration or geometrical reasoning in the strict sense is a matter of logical analysis. And indeed, by developing axiomatic procedures, modern geometry has attempted to segregate the two processes as completely as possible. The radical separation of intuition from logic or axiomatics has never been achieved in practice; and in fact, is unattainable in principle (Piaget & Inhelder, 1967). In short, in geometrical reasoning there always remain some links with intuitive experience whereby visual clues are used to stimulate mathematical thought (Nelsen, 1993) because visualization enhances a global and intuitive view and understanding (Hershkowitz, 1989).

For more than 100 years, mathematics educators have been interested in the visual and figural representations of mathematical ideas both in the work of individuals and in the process of teaching about these ideas (Bishop, 1989). There is a growing awareness that mathematics is primarily concerned with spatial, geometrical or configurational concepts (Smith, 1964). This awareness has led an increasing interest in visual models as instructional aids for teaching mathematics.

Cubes and cube configurations such as rectangular arrays of cubes are among the most commonly used instructional aids in school mathematics. For example, drawings of rectangular solids made of small cubes are included in textbooks to introduce the students with the concept of volume (Ben-Chaim et al. 1985). They are also included in large-scale educational assessment studies to measure students' knowledge of the concept of volume (Hirstein, 1981). Students are required to find the number of cubes contained in rectangular buildings. Battista & Clements (1998) argue that "the reasoning required to complete such tasks is important because it builds the cognitive framework for understanding the measurement of volume and the formulas for determining the volume" (p. 258). Such geometric representations as differently piled rectangular solids contribute effectively to students' understanding of number and measurement ideas such as length, area, and volume (Geddes & Fortunato, 1993). However, the development of students' understanding and their strategies for meaningfully enumerating cubes in rectangular solids is far more difficult than has previously been believed (Battista & Clements, 1998).

According to the results of the second National Assessment of Educational Progress, fewer than 40% of 17 year-olds could solve the problems related to the volumes of rectangular solids made of small cubes (Hirstein, 1981). The types of errors being made were related to either the number of visible cube faces or the number of visible cubes. Hirstein concluded that many of the errors made by students were due in part to their confusion between volume and surface area. Ben-Chaim et al., (1985) found almost the same patterns of performance with approximately 1,000 students from 5th-8th grades. They additionally found that some of the students double counted the cubes along the edges while some others did not account for the invisible cubes. Since the researchers presented students with pictorial representations only, they concluded "students of given age had difficulty relating isometric type drawings to the rectangular solids they represent" (p. 389).

The research by Battista & Clements (1998, 1996) revealed that it was neither a matter of confusion between volume and surface area nor a lack of understanding pictorial representations of rectangular buildings but a matter of correct spatial structuring. Battista & Clements (1996) defined spatial structuring as "the mental act of constructing an organization or form for an object or set of objects" (p.282). The process of spatial structuring includes establishing units, establishing relationships between units (such as how they are placed in relation to one another), and recognizing that a subset of the objects, if repeated properly, can generate the whole set (the repeating subset forming a composite unit) (Battista & Clements, 1996). In other words, forming some kind of composite units such as unit cubes, rows, columns, and layers in the whole and iterating them is a necessary mental construction for proper spatial structuring. How the students structured the building spatially can be inferred by examining their strategies.

Battista & Clements (1998, 1996) described student strategies into five main groups. First is the misapplication of volume formula. Students in this group multiply some numbers which are not relevant to the dimensions. Second group of students uses the formula rotely without any indication of the layer-structure of the buildings. Students in the third group conceptualize the set of cubes in terms of its faces and they account only for the visible or outside cubes. Fourth group of students conceptualizes the set of cubes as space filling but does not utilize layers yet. They start counting unsystematically and then start to form some iterable units such as rows or columns. Students of the last group conceptualize the set of cubes as forming a rectangular array organized into layers. They start counting sub-units of layers and continue with layer adding or iteration and finally layer multiplying.

Seeing the solids in terms of iterable units such as layers is a valuable skill that might be useful in later mathematical learning. For example, a more sophisticated application of layering strategies can be seen in calculus while reasoning about the volumes of revolutions. This seems a gradual development, however. An individual does not "read off" a structure from objects, but instead, creates a structure as a result of his or her mental actions concerning the objects (Battista & Clements, 1996, Cobb et al. 1992). Numerical reasoning plays an important role in this endeavor. In fact, there is a synergistic effect between numerical reasoning and geometry learning (Clements et al. 1997) such that one provides input for or constrains the other (Battista & Clements, 1996). Children attach meaning to the numbers when they learn one-to-one correspondence of numbers with the objects. Before that, numbers are nothing more than just words memorized and used for singing for example. In addition, children's initial understanding of geometry is mainly visual (van Hiele, 1986). In fact, "mathematical understanding is constructed to a large extent in images, many of which are spatial in nature" (Clements et al. 1997).

An individual gives meaning and structure to a spatial pattern based on his or her experiences that are influenced by available conceptual structures, intentions, and the ongoing social interaction in which he or she is involved (Wheatley & Cobb, 1990). The ways in which children think about spatial structures are influenced not only by the ways in which their own knowledge is structured, but also by the ways in which the context for thinking about and discussing these entities is structured. Children's prior knowledge that they constructed through experiences can be facilitated in similar contexts.

From the forgoing discussions, it seems that finding the number of cubes in rectangular solids is important for developing valuable mathematical skills, however students have difficulty with it. Major source of the difficulties is related to their use of inadequate strategies depending on their cognitive constructions. The fact that there are both numerical and spatial aspects of the problem of finding the number of cubes in rectangular buildings is an advantage because there is a synergistic effect between numerical reasoning and geometry learning. Therefore it appears that if we provide students with appropriate activities that emphasize both numerical and spatial reasoning in a socially desirable context we can affect their spatial structuring so that they use more viable strategies in a mathematically meaningful way.

This study assumes that learning arises as an independent contribution of the interacting student (Steffe, Thompson & Glasersfeld, 1998) and that instructional tasks are influential components of learning environments. Through instructional interactions, children gain access to resources and tools that both reorganize and transform their thinking, resulting in more powerful cognitive structures (Newman, Griffin & Cole, 1989). While solving problems using their prior knowledge (some of them are informal knowledge) children discover new ways of looking at things leading the development of that knowledge into more formal ones.

Children have intuitive knowledge of equal sharing because of their living experiences. Equal sharing is socially based and provides a semantically rich context (Empson, 1995) in which students' understanding about equal amounts can be facilitated. Equal investment may also provide a similar context in which thinking about and discussing spatial and numerical entities can be structured meaningfully. Through equal investment students can construct rectangular buildings while they deconstruct them through equal sharing. That way, the structural elements of cube configurations can be pinpointed through equal amounts of investment and sharing activities. In short, the main purpose of the activities is to lead students to attend to the iterable units in constructions by investing and sharing equally.

Purpose

Previous research has described students' solution strategies and errors in dealing with three-dimensional cube arrays and provided several cognitive constructions and operations that students required for enumerating the cubes in such arrays. This study is intended to reveal whether the activities provided in the context of equal investing and equal sharing that emphasize both numerical and spatial aspects of cube configurations do cause any improvement in students' strategies that lead them to use composite or iterable units in enumerating the cubes contained in rectangular buildings.

Significance

Previous research has not cleared how the instruction that takes both the numerical and spatial-structural elements of cube configurations into consideration and that is given in a socially desirable context would effect the students' spatial structuring of cube arrays. Therefore, the current study will show first, if the kind of activities that emphasize both numerical and spatial aspects of the rectangular cube buildings would cause any improvement in students' spatial structuring as revealed by their use of strategies. Second, the study will also highlight how students use numerical and spatial information interchangeably. Since the activities stress both numerical and spatial aspects of configurations it will be possible to see what kind of information students rely on first or more and if there is any change in this reliance.

The current study will also use a different paradigm to formulate the activities by embedding them in a socially desirable context, equal sharing and equal investing. Third, this study will reveal how this context as a whole would affect students' approach to rectangular solids made of small cubes. The information gained in this study is crucial for the development of effective instructional materials. It will also shed light on children's mathematical thinking in an area that intersects with both numerical and spatial domains.

REVIEW OF THE LITERATURE

The task "finding the number of cubes in rectangular buildings" is a three-dimensional, spatial-numerical mixed task. It is clearly related to some aspects of spatial visualization (Ben-Chaim et al., 1985). Spatial structuring is also a fundamental notion in understanding student's strategies for enumerating 3-D cube arrays (Battista & Clements, 1998). Therefore, the topics of discussions in this section are as follows: visualization in mathematics, the evolution of three-dimensional symbolization, stimulating spatial development, students' strategies in dealing with 3D cube arrays, and volume and spatial structuring.

Visualization in Teaching and Learning Mathematics

Mathematics educators are interested in spatial ability for at least two reasons. One is the strong correlation between mathematical achievement and spatial ability measured by standardized ability tests. The other is the belief that visual approaches can be a powerful introduction to the complex abstractions of mathematics by providing empirical and intuitional evidences to the learner. First, what is spatial ability?

Spatial ability, as it relates to mathematics, can be briefly defined as "the capacity to solve problems having primarily visual-spatial content by using at least some pictorial or geometric style of mental representation and/or operations that relate functionally to the visual or geometric content of the task" (Smith, 1998, p.10). As the definition implies, it is not always possible to separate spatial reasoning from the other types of reasoning such as analytic reasoning. The psychometric research conceived spatial ability to be of two types, orientation and visualization (Eliot, 1987; Friedman, 1995; McGee, 1979; Pellegrino et al., 1984; Wolf, 1988).

Spatial orientation involves mental rotations of rigid objects as a whole, either two or three-dimensional. These tasks, especially those in three dimensions, are the best test of "pure" spatial reasoning, precisely because they are the most resistant to solutions by sequential processing (Friedman, 1995). Spatial visualization comprises multi-step reasoning about parts of objects with multiple solution paths. It may involve many distinct steps in transforming an item or items, sometimes involving movement of only parts of the items.

In her meta-analysis, Friedman (1995) found that when space-math correlation(s) were grouped by spatial skill and averaged over studies, they ranged from about 0.30 for mathematics and two-dimensional orientations spatial tests to 0.45 for mathematics and three-dimensional visualization, the best -relating spatial skill. Three-dimensional orientation is also highly correlated with both computational ($r=0.38$) and higher-level mathematics ($r=0.43$) for younger students (9-14 year-olds). Correlations were generally higher for females, younger students, and visualization tasks. Ordinarily, visualization tasks can be broken down into sequences of steps involving orientation tasks. Therefore, spatial visualization has been the object of interest for mathematics educators.

Some additional aspects of spatial visualization make it important in many scientific and academic disciplines including mathematics (Smith, 1998). First, spatial visualization involves the integration of transformation of mental imagery with verbal and logical thinking. Second, the multi-step and complex nature of spatial visualization problems require multiple solution paths and search for a problem space. Similarly, Gutierrez (1996) considers visualization in mathematics as "the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, performed to solve problems or prove properties" (p.9).

In mathematics, learning to see beyond assumed details and make mental images that highlight only the essential features is important. In this respect, it is necessary to make a distinction between visualization and visual thinking (Goldenberg et al., 1998). Visualization comprises the imagining of objects in their absence, transforming them into other forms, or recreating the objects in the minds' eye, which are originally visible. Visual thinking, on the other hand, is the imagining of non-visual entities as objects or visual rendering of ideas. If we assume that visualization engender or transforms into visual thinking then it is reasonable to believe that work in the former increases the success in the latter. Therefore, it is wise to start from objects of greatest interest to the children. The world of the young children is three-dimensional, and so it makes sense to take three-dimensional geometry very seriously early (Goldenberg et al., 1998).

The Evolution of Three-dimensional Symbolization

In many areas of mathematics, it is of great value to be able to visualize and represent 3D configurations and to comprehend the geometrical relations among the various parts of a figure (Mitchelmore, 1980; Wilson & Osborne, 1992). Many people have quite a bit of trouble handling the 3D case visually, even if they find the 2D case very natural in a visual setting (Eisenberg & Dreyfus, 1989). The presentations of activities for children on solid shapes, using textbooks or worksheets, for example, meets with the well-known problem of representing three-dimensional solids or frames on flat two-dimensional paper (Orton & Frobisher, 1996).

Cooper (1990) argues that adult viewers construct mental representations as perspective-like embodying structural information about integrated 3D objects even when asked to reason about flat, disconnected orthographic projections. This may be because isometric projections present an approximate perspective view of objects and they appear more object-like than their orthographic equivalents. To support this, Roth & Kosslyn (1988) claim that imagery should be in three-dimensional form. This may be the case for adults but how about younger children?

Children's achievement in volumetric (three-dimensional) tasks is assessed through three different procedures: they are (1) given building blocks to build 3D geometrical objects, (2) required to read pictorial representations of objects, and/or (3) asked to draw the graphical representation of the object (Wolf, 1988). The challenge in blocks is to stop using just the volume of individual blocks in favor of learning to construct larger volumes from many blocks. In reading the pictorial representation of a real world object, one comes to realize that the picture represents a space-filling object, an object with volume. The challenge of drawing is to use the 2D-paper plane to represent the depth and volumes of the spatial world.

As one might think, drawing only is not an appropriate way to assess the presence or absence of a given type of information in a representational system (Mandler, 1988). Additionally, young children often lack the ability to understand 2D displays of 3D phenomena (Cohen, 1979) since it needs considerable conventionalizing (Bishop, 1979). A lack of motor coordination in drawing or child's inability to draw may also hide some part of the actual visualizing (Holloway, 1967).

The employment and awareness of perspectives is a gradual, developmental achievement (Morss, 1987). Wolf (1988) reports "as late as their third year, children use similar, rather than differentiated systems for depicting the features and locations of parts within a larger whole, independent of whether they are given markers and paper or building blocks" (p.232). In both her block model and her drawing of a house, for example, a child simply accords a block or a mark for each item she wishes to represent in a way that records rough location and number but nothing about external contour or exterior volume. In other words, her drawing lack the distinctive 2D, or graphic, strategies for representing volume with space around it.

At the same time, her block construction is only uni-dimensional (a pile of items) rather than a fully 3D construction. However, by the age of 5, children learn to distinguish between the two systems of spatial representation (Wolf, 1988). A five-year-old selectively uses 2D or pictorial strategies to depict the volume of a house. She uses a contour line to distinguish the figure of the house from the surrounding ground of the paper and depicts the several facets of volume in left-to-right arrangement on the page. Another 5 year old constructs the volume of a house: making use of a central cavity and construction along three axes to portray the shape of the form in space (Wolf, 1988).

In other words, between the ages of one and five years children gradually construct the distinctive rules of at least these two systems (2D graphical and 3D construction) of spatial representation (Wolf, 1988). In their drawing, five year-old children devise still other ways to portray the volume of individual objects, such as coloring them in or enlarging their outside contours. Viewed from an adult perspective, these are primitive strategies. Nevertheless, they are significant because they signal the child's awareness that when drawing, she must find 2D means to represent the 3D aspects of her spatial experience (Wolf, 1988). By the age of six, for example children may try to differ a disc from a sphere in their drawings by means of coloring (Wolf, 1988).

There is evidence that even young children can read graphic representations with a fair degree of sophistication (Wolf, 1988). Murphy & Wood (1981) measured the functional use of pictorial information in reconstruction tasks. Children starting from the age of four were able to use the pictorial information in 3D construction tasks. Performance significantly increased with age. Some of the 8 year-olds could do the task without even looking at the picture. They might have used the spatial information they constructed either through previous experiences or from relating the parts to each other or both. Whatever it was, it is clear that an average 8 years old can associate an object with its drawing.

To represent an object in perspective by means of a mental image or drawing necessitates an awareness of the point of view and the resultant apparent changes in shape of the object as seen from that viewpoint (Holloway, 1967). Perspective representation implies conscious co-ordination between object and subject, the recognition that both object and the subject occupy the same projective space extending beyond the object and including the observer. Seven year-olds are aware of the distinction between their successive views of the object but cannot imagine or draw the outcome of these changes of viewpoint. It is only when presented with a ready-made perspective drawing that the child is able to accept it as an adequate rendering of the object from a particular viewpoint (Holloway, 1967). Children of 8.5-9 years age can systematically apply the rules of perspective even in their drawings.

Mitchelmore (1976, 1980) describes school children's drawings of space figures into four main developmental stages. Stage 1 plane schematic: drawings show only a single face as if they are two-dimensional or taken orthogonally. Stage 2 solid schematic: drawings show several faces of the solid that may be visible or non-visible but this may or may not indicate the depth of the figure portrayed. Stage 3 pre-realistic: Only visible faces are included in the drawing and depth is apparent in the drawings. Stage 4 realistic: Parallel lines are drawn parallel and depth is represented properly.

Mithchelmore (1980) assessed children from 3rd, 5th, 7th, and 9th grades to see if the developmental stages are valid for a different culture. Since there was little variation among the first graders in a previous study, he did not include first graders in this study. He found very similar results. Third graders' mean was between first and second stage. Fifth graders mean was between the stage 2 and the stage 3A. Seventh and ninth graders' means were not very much different and between the stage 3A and the stage 3B. He concluded that the scores change in a linear fashion and "measure something that can be justifiably considered as general representational ability" (p. 90).

What is the relation, then, between a child's representational ability and level of spatial-perceptual development? It seems as if, at any point in time, a child has available one or more schemata (two-dimensional designs) for representing a particular class of space figures. Perception of a simple geometrical figure as representing a solid may be a function of two factors: the ability to perceive pictorial depth of the elements of a figure, and the ability to construct a correct mental representation of the entire figure (Deregowski & Bentley, 1987). The reason why older children's schemata appear more realistic than younger children's is that they are based in a more refined space (Euclidean as opposed to topological), for example, there is little variance in drawing stages at the first grade (Mitchelmore, 1980). We should expect the growth of representational ability to be highly dependent on spatial-perceptual development but always to lag behind it. The lag may be as much as three stages at some ages (Mitchelmore, 1980).

There are also marked differences in the difficulty of drawing different space figures (Mitchelmore, 1980). The stage of development of a child's schema for a given space figure depends not only on his or her general level of perceptual development but also on the particular representational problems presented at each stage (Mitchelmore, 1980). For example, young children find the cylinder the most difficult space figure to represent satisfactorily by a simple closed plane figure (possibly because of the curved face) (Mitchelmore, 1980).

Considerable differences exist between the subjects drawn from different cultures, and within cultures, between subjects differing in such attributes as age and education (Mitchelmore, 1980; Deregowski & Bentley, 1987). In sum, everyone draws using the schemata they know but those schemata develop overtime, coming to represent ever more complex properties as children get older or more experienced. Roth & Kosslyn (1988) argue that mental images of 3D objects are generated in a near-to-far sequence. That is, nearer surfaces of imaged objects are generated prior to distant surfaces. This finding may explain some aspects of why younger children can only attend to two dimensions of a 3D object. For example, they may say, "this is rectangle cause it looks like a door." Only some time later can they come to understand and represent the third dimension. Teaching may stimulate this understanding.

Stimulating Spatial Development

Some researchers believe that some basic principles of children's developing knowledge systems arise independently of the nature or degree of particular experiential confrontations with the physical world (Landau, 1988). Others argue that, consistent with a Piagetian view of cognitive development, when a young child is exposed to variety in the environment, both motivation and capacity to assimilate new information increase (Yarrow et al., 1975 cited in Poag, Goodnight & Cohen, 1985). For example, toys apparently reduce the young child's reliance on the mother in unfamiliar settings, and the variety of inanimate objects available to the infants was positively related to general mental, psychomotor, and cognitive/motivational development and experimental studies support these co-relational findings (Poag et al., 1985).

In a study by Lee & Kingdom (1996), the blind subjects who lost their sight later in life outperformed the congenitally blind subjects and showed patterns of performance that are similar to the sighted subjects' auditory results in visual imagery tasks. Although there are somewhat conflicting results in the literature regarding whether spatial ability can be improved, numerous studies (e.g., Ben-Chaim et al., 1988; 1985; Burnett & Lane, 1980; Lord, 1985; Smail, 1983) have indicated that it can be improved through training if appropriate materials are provided. It is not possible, however, to make a global improvement in all kinds of spatial abilities through a particular kind of training. Only partial and domain or task-specific improvements have been achieved so far, depending on the content of the training given.

Generally, construction activities with concrete materials in varying conditions and matching them with their pictorial representations (Battista & Clements, 1998; Ben-Chaim et al. 1985; Orton & Frobisher, 1996) are strongly recommended for stimulating spatial sense in the initial stages of learning 3D geometry. A general order of concrete-pictorial-abstract seems to be the most appropriate. Students are provided with learning experiences beginning with the concrete and pictorial stages, followed by the abstract stage to enable them to learn mathematics meaningfully. In fact, in order for intellectual growth to occur, the learner must be presented with novel situations which are one step further and challenge the learner's current thinking level (Cohen, 1979). There may be subjects who fail to perceive depth in the drawing used as a stimulus for example. In this case, pictures can be enriched by addition or augmentation of depth cues so that the subjects who originally saw the unmodified pictures as lacking in pictorial depth do perceive them three dimensionally (Deregowski & Bentley, 1987).

According to Gibson (1962), the shape of a substance is perhaps the main characteristic of an object. Planarity, curvature, slant, parallellity, span (distance), edges, and corners (vertex) may be conceived as variables of solid geometry. Active touch is an excellent channel of spatial information in that the arrangement of surfaces is readily picked up. Feeling them best gets the solid geometry of things. In other words, active involvement of the learner in the learning process is considered necessary that is, "one learns by doing." However, active control of the learner over the objects is not a panacea for improving spatial visualization skills. Instead, varying level of active control would be more beneficial (Smith, 1998). Research indicates that at the initial stages of spatial visualization learners need more active-control over the spatial objects. After having certain experience and level of sophistication they then benefit more from relatively low active-control conditions.

Visual materials become a necessary component for helping the inexperienced learner to bridge the gap between concrete experiences and symbolic representations of real-world phenomena (Miller & Burton, 1994). Concrete experiences with cubes- building, representing in 2-dimensional drawings, and reading such drawings- are helpful for improving students' performance in spatial tasks (Ben-Chaim et al. 1985). From an information processing approach, information is represented in short-term memory in a variety of modalities, such as auditory-verbally, visually, and semantically (Miller & Burton, 1994). Increasing the presentation modalities for any information will likely increase its efficiency to be processed. Such cognitive strategies as rehearsal, chunking, imagery, and connection to prior knowledge also increase the memory ability.

Two major theoretical perspectives namely developmental and the information processing models agree that the ability to use memory efficiently increases with age. However, developmentalists assume that adults have structurally larger working memories than children because of biological maturation, while the cognitive theorists believe that adults do not have larger capacities but developed better strategies within the structure of the cognitive system (Miller & Burton, 1994). In other words, children's apparent growth of short term memory capacity is the result of simple strategies becoming more automatic through use, and then being modified into more powerful strategies that allow for chunking of information held in short term memory. Since the information processing or cognitive-computational models assume that the memory structure is complete by the age of 4 the process of learning is the same throughout the life span. It is the more efficient use of strategies and increasing knowledge about cognitive structures that makes adults more efficient than children.

According to Anderson (1987), "the declarative knowledge encoded about the problem domain is again determined by the experiences of the learner such as instruction, reading of text, examples studied, and so on" (p.196). The role of experience in situated spatial visualization can lead to the use of strategies making use of chunking behavior (Smith, 1998). Chunking is the organizing of perception according to functional relationships to facilitate pattern recognition and recall in knowledge-intensive (or information-intensive) domains such as geometry (Koedinger & Anderson, 1995, 1990). Perceptual chunks are formed according to previously encountered experiences that can be either images to be repeated or procedures to guide iterations or the entire solutions. Forming units and units of units can be considered some kind of chunking such as perceptual chunking. Students' endeavors in dealing with 3D cube arrays can be seen as a struggle toward a more efficient strategy such as the use of chunking.

Students' Strategies in Dealing with 3-D Cube Arrays

Students use different strategies while attempting to find the number of cubes in rectangular arrays according to their mental structures available at the time of enumeration. This change is developmental and affected by learning on appropriate tasks. Battista & Clements (1996) classified students' strategies into several broad categories. Table 1 depicts the students' strategies in dealing with the enumeration of three-dimensional cube arrays in major categories (see Appendix A for a detailed description of each strategy).

Table 1: Students' strategies in dealing with cube enumeration tasks

Strategy	Description
A	Students conceptualize the set of cubes as organized sets into layers
B	Students conceptualize the set of cubes as space filling but does not utilize layers for enumeration
C	Students conceptualize the set of cubes in terms of its faces
D	Students use the formula L x W x H
E	Other, students do some multiplication but by irrelevant numbers

Except for D and E, these strategies are hierarchical in nature and develop from C to A. Strategy E seems to be a misapplication of the formula. Strategy D is a result of rote memorization of the volume formula without any indication of intuitive understanding of the measurement of volume. The hierarchical structure of the strategies aside some steps may not occur or be skipped very fast by the learner depending on the experiences such as instruction and learning environment. For example, experts use step skipping and abstract planning in developing geometric proof (Koedinger & Anderson, 1995). Similarly, as children gain more experience with cube arrays they skip counting schemas and just multiply the dimensions. They count the cubes along the edge just for finding the dimension. Indeed, people are natively inclined to use more efficient strategies, as they become more experienced in a domain (Siegler, 1987).

Battista, Wheatley & Talsma (1989) report, however, that people do not always use the strategies that they can use most efficiently in solving geometric problems. Some may need to be taught what strategies are available with developmentally appropriate activities to help them acquire efficient strategies in solving certain kinds of problems. Spatial structuring is a fundamental notion in understanding students' strategies in enumerating cube arrays (Battista & Clements, 1996)

Volume and Spatial Structuring

Conceptually, there are two kinds of volume: exterior or displacement volume and interior volume (Piaget et al., 1970). Exterior volume is the space occupied by an object in relation to the surrounding spatial medium. Interior volume is the amount of matter that is contained within a set of boundary for example the number of unit bricks in a construction. According to Piaget et al., (1970) children can first understand the interior volume but not the measurement of it.

Children should come to understand the logical operation of finding the volume before they are introduced with mathematical (numerical) calculation (Piaget et al. 1970). Logical operation here is the structuring of the cubes in rectangular buildings as iterable units for enumeration. To be meaningful, numerical reasoning should come out of this structuring. If the student for example, has structured the arrays as consisting of faces the volume formula does not make any sense for her or him. New structuring and new enumeration repeats until the layer iteration comes about. A schema for logical multiplication can be arrays of squares to be counted one by one first and then structured in rows or columns to be iterated. These schemas serve for understanding mathematical operations. Children of 4-5-6-7-8 years of age need empirical solutions since they cannot think deductively (Piaget et al. 1970). They cannot do the unit iteration right away. Gradually, they come to substitute mathematical multiplication for logical multiplication. Spatial visualization is one of the main bases in this cognition.

As mentioned earlier, spatial visualization is an aspect of spatial ability. It has been relatively well established that an overall improvement in spatial ability is not possible yet. However, it is claimed that the activities that are similar to the problem at hand improve the functional skills of individuals (Ben-Chaim et al., 1985). The problem here is the enumeration of cubes in rectangular buildings, a necessary step in understanding the measurement of volume at a conceptual level. Spatial structuring is a fundamental notion in understanding student's reasoning while enumerating 3-D cube arrays.

Spatial structuring is defined by Battista & Clements (1996) as "the mental act of constructing an organization or form for an object or set of objects" (p.282). Spatial structuring is a constructive process in which one comes to establish units, relationships between units (such as how they are placed in relation to each other), and recognize that a subset of the objects, if repeated properly, can generate the whole set (the repeating subset forming a composite unit). In their theory of spatial structuring, Battista & Clements (1996) defined four main cognitive milestones for structuring 3-D cube arrays spatially.

Students initially conceive of these 3-D rectangular arrays as uncoordinated sets, or *medley, of views*. Students at this stage only attend one face of a prism at a time. That is, their spatial structuring was local. Students who use C strategies (see Appendix A) in their attempts to solve the cube enumeration problems exemplify this conception.

At the second stage, students form *composite units* and use them in their mental operations. This structuring may or may not be appropriate for the solution of the problems depending on the reference it is based. For example, if the composite unit is based on the faces of the prism the solution will more likely produce wrong answer if it is based on layers, rows or columns as iterable units the solution will more likely to be correct. In other words, composites based on three-dimensional sub units can be considered proper structuring. Why some students count squares instead of cubes is still not clear. Although they know what a cube is and identify a single cube from its drawing they may not recognize the cubes in rectangular buildings. This needs to be further analyzed.

The third milestone is the *coordination*. At this stage students are able to coordinate orthogonal views moving beyond the local structuring of the medley of viewpoints conception of cube arrays. This is the recognition of the fact that different side views are somewhat spatially related to each other and interpreted in relation to each other. This coordination can be achieved on concrete and perspective like representations but not on different orthogonal views. This requires different kinds of experience and a higher level of structuring, *integration*.

To *integrate* views of a 3-D object is to construct a single coherent mental model of the object that possesses these views. A mental model is a perspective-like embodiment that possesses the attended spatial structures of an outside object that are used to simulate interactions with this object. The appearance of this model depends on how the student spatially structured it. That is, it can be individual cubes, an uncoordinated medley of faces, sets of columns or rows, or set of layers. In other words, student's mental model of an array determines how the student interprets his or her images or views of the array.

The difference between coordination and integration is that coordination implies an awareness of the relation between different views of the object while integration assures that those views have been put together and conceived as a coherent whole in the mind's eye. Any attempt to think different views in relation to each other can be considered coordination of the views. In order for integration to occur, one should put them together and interpret it as a whole, an isometric type mental picture. So the coordination is a necessary step for a proper integration.

Battista & Clements (1996) hypothesized two different processes to explain how the integration occurs operationally. One has to do with recall and the other is novel object generation. The existing conditions of the learner determine which process will take place. In order for recall to occur one should have similar prior experiences to generate a possible mental model of the whole object by recalling similar objects that s/he has previously perceived or conceived. Then the job of the learner is to fit the preexisting mental models to the new spatial information to create an appropriate mental model for the new phenomena.

If no recalled object seems adequate for a model, a second, alternative process could be activated. In this process, one might start to generate novel objects by performing transformations on images of objects that are available. For instance, one could create an appropriate holistic mental model of a 3-D cube array by visualizing an iterative translation of a single layer through a distance determined by the third dimension of the array. Both recall and object generation processes require the coordination operation. In both situations, one should create the whole from the units either previously encountered or newly generated and check it for appropriateness that requires coordination and integration of different views of the object at hand.

The theory also suggests that the storage and evocation of the objects be done originally that means as encountered previously. If for example, one does not have ample experience with different views of an object s/he cannot form an integrated whole from the orthogonal views of a 3-D object. According to Battista & Clements (1996) reasoning about an object with its different views and transformation between them is a necessary condition for understanding coordination and integration of different orthogonal views. The formation of the new image is a kind of transformation of the old images into the new object's shape and dimensions.

Summary and Research Questions

Finding the number of cubes in rectangular solids provides cognitive framework for understanding the measurement of volume. It is a three-dimensional, spatial-numerical mixed task that requires spatial visualization. Visualization inherently involves reasoning of different kinds such as numerical, analytic, and spatial. There is a synergistic effect between numerical reasoning and geometry learning. Consistent with the Piagetian theory, at around the age of 7-8, children have a global understanding of three dimensionality. When it comes to count the number of cubes contained in that three-dimensional object the problems occur.

Initially, children can only attend the cube arrays in terms of its faces. While enumerating cubes they count the cubes at the outer faces of the prism, some of them double-counting the cubes along the edges and triple-counting the cubes in the corners. At the second stage, children recognize the cubes as whole entities but not as systematic configurations. Therefore, they usually try to count the cubes one-by-one and lose track of their counting. They start to form some organization of the cubes but these structural entities are not systematic. At the third stage, they can form some kind of units that can be iterable somehow such as the whole prism consisting of layers and layers consisting of rows or columns.

Battista & Clements (1998, 1996) investigated students' understanding of rectangular solids made of small cubes and proposed guidelines for the instructional tasks that can help students develop more powerful ways of thinking about these problems. Two of their findings among others seem very crucial. First, they well documented that the most viable student strategies for the enumeration of cubes in buildings were the layering strategies and these take a long time to be established.

Second, they also suggested that students' attempts at enumeration engendered their spatial structuring of cube arrays while their spatial structuring led them develop better enumeration strategies. This lends support for the existence of a mutual interaction between numerical and spatial reasoning. In fact, it is assumed that there is a synergistic effect between numerical reasoning and geometry learning (Clements et al. 1997). Keeping this synergistic effect in mind, I hypothesize that the activities that emphasize both numerical and spatial reasoning in a socially meaningful context may help students develop more viable strategies by constraining the learner to attend to the structural elements of cube configurations.

Problem statement: What is the effect of equal investing and equal sharing activities on (4th grade) students' understanding of rectangular solids made of small cubes? Following sub-questions are posed to answer the main question:

1. What strategies do students use to find the number of unit cubes contained in rectangular solids before the instructional unit?

    a). How do they view the buildings?

    b). Do they use any layering strategies?

    c). Do they perform any unit iteration schemas?

    d). Is there any change in strategies with differing number of cubes?

2. How do students interact with equal investing and equal sharing activities during the course of the instruction?

    a). How do they approach to equal investing problem situations?

    b). How do they approach to equal sharing problem situations?

    c). How do they partition the buildings?

d). Do they form any composite unites?

e). What kind of information do they use first and more, spatial or numerical?

f). In what direction does this information use change?

g). Do they develop any unit iteration schemas?

h). Do they invent any layering strategies?

3. What strategies do students use to find the number of unit cubes contained in rectangular solids after the instructional unit?

a). How do they view the buildings?

b). Do they use any layering strategies?

c). Do they perform any unit iteration scheme?

d). Is there any change in strategies with differing number of cubes?

METHODS

This will be an experimental study with three phases. First phase will be clinical interviews with the participants to uncover their level of thinking about the cube enumeration tasks. Second is the intervention with selected and prepared activities intended to lead participants to form iterable units while enumerating cube configurations. Students' interactions with the activities will be the main concern in this phase. Last phase will be the same as the first one with clinical interviews intended to probe the students' level of reasoning about the cube enumeration tasks after the intervention. This three-phase procedure will be used to see if any change occurred during the intervention period and also to see if this change transfers to the cube enumeration tasks.

Participants

Measurement of volume is introduced around fifth grade. Since this study assumes that before the introduction of volume formula student should come to understand the logical operation behind the formula, participants will be chosen from the fourth graders. Participation will be on a voluntary basis. The criteria to select the participants are as follows. Total four participants will be slightly below (1), average (2) and slightly above (1) in mathematics achievement tests. They will be interacting in pairs (one pair at a time) during the instructional intervention. Since the participants' strategies for the cube enumeration tasks will be of interest for this study students who use different strategies will be accommodated so that any improvement could be detected.

Materials

Mainly three types of materials will be used: wooden centimeter cubes, rectangular buildings made by gluing individual wooden cubes together, and their drawings. There will be about a hundred wooden centimeter cubes and seven buildings with systematically varying number of cubes including 8, 9, 12, 16, 24, 36, and 48. Drawings of these buildings will be available during the study and used according to the aims of the study. Video camera, audio tape recorder, and paper and pencil will also be available for recording purposes.

Clinical interviews

Through clinical interviews, it is possible to collect and analyze the kind of data on mental processes at the level of subject's authentic ideas and meanings, and to expose hidden structures and processes in the subject's thinking (Clement, 1998). There will be pre- and post clinical interviews. These interviews will be intended to determine the students' ways of thinking about the cube configurations as presented both pictorially such as in textbooks and national examinations (see Appendix B) and concretely. Participants will be individually interviewed. Each clinical interview will be audio and videotaped to account all the actions of the participants while they attempt to enumerate the cubes in rectangular buildings. Think-aloud method will additionally be used if the strategy cannot be determined through observation.

They will be asked to find the number of cubes in rectangular buildings using first graphical and then concrete representations of rectangular buildings made of small cubes. In volume related tasks, although the differences between the modes are not statistically significant the 3rd and 5th graders' overall scores in problems with concrete presentations are slightly better than with the graphical representations (Battista & Clements, 1996; 1998; Phillips, 1972). Therefore, both modes of presentations will be used in the clinical interviews to detect if there is any variation in the strategies used by the students.

Intervention

Students will be audio- and videotaped during the intervention to account for their interaction with the problems. Intervention will consist of equal investment and equal sharing word problems that invite a pair of students to make equal investments for making buildings with different numbers of cubes or sharing the buildings equally (see Appendix C). Students have intuitive knowledge of equal sharing (Empson, 1995). Therefore, they might be able to initiate this intuitive knowledge while solving cube enumeration tasks. Equal investment will demand the use of this intuitive knowledge in the other direction. Shares will be different layers, rows, and columns that are the structural elements of buildings.

The main idea behind these activities is threefold: First, students systematically construct rectangular buildings made of small cubes using their structural elements such as cubes and some units of cubes (rows, columns, and layers). Second, they deconstruct them back into their constituent parts such as layers, rows, columns, and individual cubes in equal shares. Third, while they are trying to solve the problems they will have the chance of both numerical and spatial reasoning about investing and sharing situations. There is evidence that mathematical thought can be stimulated by providing visual clues to the observer (Nelsen, 1993) or the other way around (Battista & Clements, 1996). That is, a numerical demand may constrain the participants to attend to certain visual clues or visual clues may stimulate a more sophisticated numerical reasoning.

In the activities provided, there will be a numerical target for stimulating a spatial solution or a spatial target for a numerical solution. For example, while sharing eight cubes contained in a rectangular solid a student can either say "8 divided by 2 is 4" or he/she can see the two slices of 4 cubes and may claim their equality by placing them side-by-side. In either case, students can be encouraged to see the other solution, too. The purpose is to provide the students with both numerical and spatial information and have them feel the need for a solution that is satisfactory.

Battista & Clements (1998) proposed several principles for the development of effective tasks to help students construct an understanding of the spatial structure of cube buildings. They suggested that first, problems be presented in a way that allows students to construct their own personally meaningful solution strategies. Second, the teachers role should be facilitative by encouraging students to invent, reflect on, test, and publicly discuss strategies in a spirit of inquiry and problem solving. Third, for all activities, students first make predictions then check their predictions. They suggest students make predictions because students' predictions are based on their current mental models of cube buildings, which we are trying to develop. Additionally, the discrepancies, if any, between the predicted and the actual number will promote the cognitive conflict and the activity will focus the student attention on thinking about the numerical or spatial reasoning rather than physical activity.

Keeping these principles in mind, a three-step procedure will be used for the activities. The students will be provided first with drawings of buildings. That way, students will have a chance to mentally imagine the solution and predict an answer in their mind. If no attempt could be made then they will additionally be provided with concrete buildings. If still there is no attempt then they will be given individual cubes and encouraged to try to construct what is being asked with concrete cubes.

Analysis

Battista & Clements (1996) documented students' strategies they used while enumerating arrays of cubes with detailed descriptions (see Appendix A) from 3rd through 5th grade. These categories will be used to determine the students' level of reasoning about the cube enumeration tasks during the two clinical interview phases. A more open-ended format will be used to analyze the data generated during the intervention period to account for the newly emerging constructions made by the participant students.

In the following section, how answers to each particular research question will be sought is explained in detail.

Research Question 1: What strategies do students use to find the number of unit cubes contained in rectangular solids before the instructional unit?
a). How do they view the buildings?
b). Do they use any layering strategies?
c). Do they perform any unit iteration schemas?
d). Is there any change in strategies with differing number of cubes?
e). Do they use the same strategy in concrete and pictorial situations?

To assess this question, students' actions and verbalizations during the clinical interviews will be documented. Then, they will be classified according to the categories made by Battista & Clements (1996).

Research Question 2: How do students interact with equal investing and equal sharing activities during the course of the instruction?
a). How do they approach to equal investing problem situations?
b). How do they approach to equal sharing problem situations?
c). How do they partition the buildings?
d). Do they form any composite unites?
e). What kind of information do they use first and more, spatial or numerical?
f). In what direction does this information use change?
g). Do they develop any unit iteration schemas?
h). Do they invent any layering strategies?

This question is intended to document the participants' verbalizations and actions occurred during their interaction with the equal investment and equal sharing tasks to uncover the meaning participants hold about the problems and their solutions. The answer to this question will be sought in their actions and verbalizations during the intervention.

Research Question 3: What strategies do students use to find the number of unit cubes contained in rectangular solids after the instructional unit?

a). How do they view the buildings?

b). Do they use any layering strategies?

c). Do they perform any unit iteration scheme?

d). Is there any change in strategies with differing number of cubes?

e). Do they use the same strategy in concrete and pictorial situations?

The aim of this question is to account for any change in overall student's approach to the same cube enumeration tasks due to the instructional intervention. Similar to the first clinical interviews, students' actions and verbalizations during the clinical interviews will be documented. They then will be classified according to the categories made by Battista & Clements (1996).

Pilot study

A pilot study was carried out with three fourth graders (one above, one average, and one below average in mathematics) during a mathematics class hour but in a separate room. The participants happened to be all girls. The purpose of the pilot study was to check if the participants, questions, activities and the setting are all suitable for the purposes of this study. The pilot study lasted 90 minutes, in which 15 minutes were devoted to pre-interviews (approximately 5 minutes for each student), 60 minutes for the activities, and the last 15 minutes for the post interviews.

Although the students seemed to be bored because of the long period and similar activities two of them showed improvement in terms of the use of composite unites. Before the intervention, the average student used a C strategy for both small (16 unit) and large number (36 unit) buildings. After the intervention, she started to use B strategies for the small number building but continued to use a C strategy for the large number building. Above average student also made a little improvement. While she was using B strategies before the intervention she attempted to use a layering approach but she gave up and returned to her original strategy.

Some important decisions were also made during the pilot study regarding the participants, questions, setting, and procedures of the study. Participants will be interacting in pairs because it was hard and confusing to keep track of their conversations while they were in a group of three. Third student also seemed to be not involved at all. Therefore, pairs will be used in the actual study. Videotape will be used to account all of the participant actions such as gestures and hand movements. All the questions were revised and systematized in terms of language and the number of cubes in buildings to ascertain uniformity. Pre-interview, instructional intervention, and post interview will be conducted in three separate days in order to make each session shorter since the participants got bored in the 90-minute session during the pilot study. Finally, both concrete and pictorial representations of buildings will be used to account for the differences, if there is any.

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APPENDICES

APPENDIX A. Students' strategies while enumerating cubes in rectangular buildings:

A. "The student conceptualizes the set of cubes as forming a rectangular array organized into layers

1. Layer multiplying: Student computes or counts the number of cubes in one layer (vertical or horizontal) and multiplies by the number of layers.
2. Layer adding/ iteration: Student computes the number of cubes in one layer (vertical or horizontal) and uses addition or skip counting (pointing to successive layers) to get the total.
3. Counting sub units of layers: Student's counting of cubes is organized in layers, but the student counts by ones or skip counts by a number that does not equal to the number of cubes in a layer. For example, the student counts the top layer by ones, then counts on from the result, again pointing to each cube in the top layer, for each of the two remaining layers.

B. The student conceptualizes the set of cubes as space filling but does not utilize layers.

1. Column/row iteration: Student counts the number of cubes in one row or column and uses skip counting (pointing to successive rows or columns) to get the total.
2. Counting sub units of columns or rows: Student's counting of cubes is organized by row or column, but the student counts by ones or skip counts by a number that does not equal the number of cubes in a row or column. For example, the student counts by twos or ones, pointing successively to columns of four.
3. Systematic counting: Student counts cubes systematically, attempting to count both inside and outside cubes. He/she might, for instance, count the cubes on all the outside faces, then attempt to determine how many are in the center.*
4. Unsystematic counting: Student counts cubes in a random manner, often omitting or double counting cubes, but clearly tries to account for inside cubes.*

C. The student conceptualizes the set of cubes in terms of its faces.

1. Counting subset of visible cubes: Student counts all, or a subset of, cubes on the front, right side, and top- those that are visible in the picture.*
2. Counting all outside cubes: Student counts outside cubes on all six faces of the prism.*
3. Counting some outside cubes: Student counts outside cubes on some visible and some hidden faces but does not count cubes on all six faces of the prism.*
4. Counting front layer cubes: Student counts outside cubes in front layer.
5. Counting outside cubes, but not organized by faces.

D. The student uses the formula L x W x H.

Student explicitly says he/she is using formula, or implies it by saying, "Multiply this times this times this" (pointing to relevant dimensions). There is no indication of understanding in terms of layers. (If students used the formula, they were asked, "Why did you multiply these numbers together? Why does this work?")

E. Other.

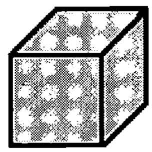
Student uses a strategy other than those described in A-D, such as multiplying the number of squares on one-face times the number on another face.

*This strategy was used, and cubes on some edges were double counted" (Battista & Clements, 1996, p. 263).

APPENDIX B. Cube enumeration task

CUBE ENUMERATION TASK

This is a small (unit) cube.

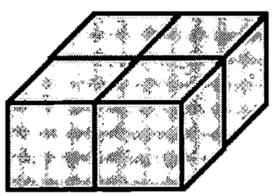


Below, you will see rectangular buildings made up of small cubes.

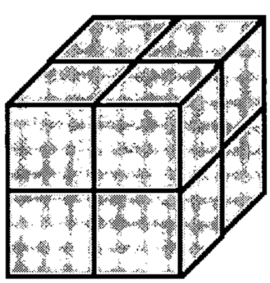
How many small (unit) cubes will it take to make each building below?

The buildings are completely filled with cubes, with no gaps inside.

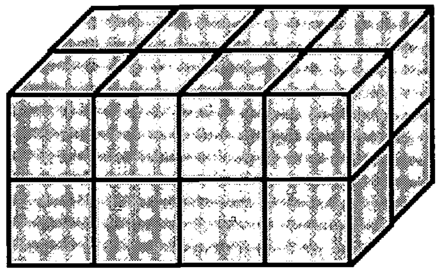
1.



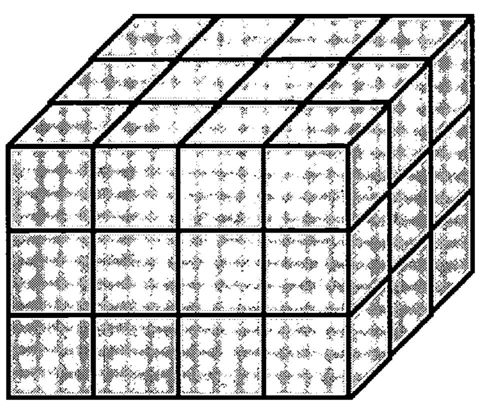
2.



3.



4.



APPENDIX C. Example activity

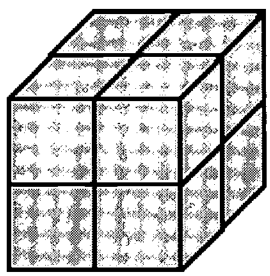
Sharing 1. You have this building. You want to share it equally between **two** people.

How would you do that?

Can you show me the equal shares on the picture by shading them in different colors?

How do you know that you shared equally?

Is there any other way to share it equally?



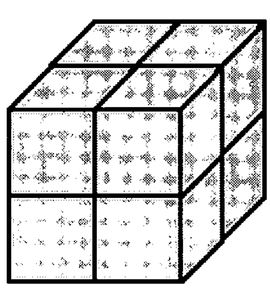
Investing 1. You are a team of **four** business people and you want to make this building by each of you investing equally.

How would you do that?

Can you show me the four equal amounts by using wooden cubes?

How do you know that you invested equally?

Is there any other way of doing this?



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