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ABSTRACT

The fourth volume of this proceedings contains 29 full research reports continuing on from Volume 3, 84 short oral communications (abstracts only) and 34 poster presentations (abstracts only). The full papers include: (1) "Beliefs, Teacher Education and the History of Mathematics" (George N. Philippou and Constantinos Christou); (2) "Working Class Students and the Culture of Mathematics Classrooms in the UK" (Hilary Povey and Mark Boylan); (3) "The Co-ordination of Meanings for Randomness" (Dave Pratt and Richard Noss); (4) "Learning Obstacles in Differential Equations" (Chris L. Rasmussen); (5) "Images and Definitions for the Concept of Increasing/Decreasing Function" (Shakre Rasslan and Shlomo Vinner); (6) "Why Is Proof by Contradiction Difficult?" (David A. Reid and Joann Dobbin); (7) "Self-Efficacy Beliefs as Mediators in Math Learning: A Structural Model" (Martin Risnes); (8) "Teachers' Pedagogical Content Knowledge of Geometry" (Lynn Rossouw and Eddie Smith); (9) "Conviction, Explanation and Generic Examples" (Tim Rowland); (10) "Children's Multiplicative Problem-Solving Strategies in Real-World Situations" (Silke Ruwisch); (11) "National Tests: Educating Teachers about Their Children's Mathematical Thinking" (J.T. Ryan, J.S. Williams, and B.A. Doig); (12) "Patterns of Mathematical Misunderstanding Exhibited by Calculus Students in a Problem Solving Course" (Manuel Santos-Trigo); (13) "The Evolution of Mathematical Practices: How One First-Grade Classroom learned to measure." (Michelle Stephan and Paul Cobb); (14) "Conceptualizing Mathematics Teaching: The Role of Autonomy in Stimulating Teacher Reflection" (Peter Sullivan and Judith Mousley); (15) "Learning through Reflection with Mature, Low Attaining Students" (Malcolm Swan); (16) "Using Research into Children's Understanding of the Symmetry of Dice in Order to Develop a Model of How They Perceive the Concept of a Random ZGenerator" (John M. Truran); (17) "Students' Solutions of Inequalities" (Pessia Tsamir, Nava Almog, and Dina Tirosh); (18) "Do Equilateral Polygons Have Equal Angles?" (Pessia Tsamir, Dina Tirosh, and Ruth Stavy); (19)

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"Meaningfully Assembling Mathematical Pieces: An Account of a Teacher in Transition" (Ron Tzur, Martin A. Simon, Karen Heinz, and Margaret Kinzel); (20) "Research Methods of the 'North' Revisited from the 'South'" (Paola Valero and Renuka Vithal); (21) "An Experiment in Developing Proof through Pattern" (Sue Waring, Anthony Orton, and Tom Roper); (22) "What Makes a Mathematical Performance Noteworthy in Informal Teacher Assessment?" (Anne Watson); (23) "Participating in Learning Mathematics through Shared Local Practices" (Peter Winbourne and Anne Watson); (24) "Intuitive Counting Strategies of 5-6 year old Children within a Transformational Arithmetic Framework" (David Womack and Julian Williams); (25) "Differences in Teaching for Conceptual Understanding of Mathematics" (Terry Wood); (26) "Children's Beginning Knowledge of Numerals and Its Relationship to Their Knowledge of Number Words: An Exploratory, Observational Study" (Bob Wright); (27) "A Study of Argumentation in a Second-Grade Mathematics Classroom" (Erna Yackel); (28) "Teacher Perceptions, Learned Helplessness and Mathematics Achievement" (Shirley M. Yates); (29) "Students' Understandings of the Role of Counter-Examples" (Orit Zaslavsky and Gila Ron). (ASK)

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BELIEFS, TEACHER EDUCATION AND THE HISTORY OF MATHEMATICS

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Department of Education, University of Cyprus

This paper is a short report of the design, implementation and evaluation of a teacher preparation program based on the history of mathematics. The project was carried out in two different universities during the past eight years. The results provide sufficient support to the initial assumption that the program could be effective in improving prospective teachers' attitudes toward mathematics.

Most students fail to learn mathematics mainly because of teachers' inadequate knowledge of and beliefs about mathematics (Fennema & Franke, 1992). The new vision of the mathematics curriculum implies extended demands on the part of the teacher, who is expected to select worthwhile tasks, orchestrate classroom discourse, seek connections that deepen mathematical understandings, help students use technology, and assess progress (Swafford, 1995). The growing awareness of the teacher's role increased attention on preservice education, and hence the task of mathematics educators to design new programs, appropriate for the changing mathematics curriculum, became a prerequisite to any reform effort.

Teachers need a well organized mathematical background; they have to be acquainted with the ever-growing content and structure of mathematics, its capacity to represent the world, its use in communication and solving problems, and they need an understanding of how mathematics is acquired, structured, and retrieved by the learner. Brown and Borko (1992) advocate that preservice programs should provide for growth in content knowledge and pedagogical content knowledge. Content knowledge is the amount and organization of the knowledge "per se" in the mind of the teacher. Pedagogical knowledge refers to the ways of formulating and representing the subject that makes it easy to students; it might be considered as a subset of content knowledge, including forms of representation, illustrations, explanations and demonstrations, unifying ideas, clarifying examples, powerful analogies, relationships and connections among ideas, and students' conceptions.

A crucial factor in designing preservice programs is the level of students' knowledge and beliefs. Teacher educators should overcome students' lifetime of experience in traditional classrooms, in a culture that holds as valid a number of assumptions about mathematics and its teaching (Richardson, 1996). A progressive teacher education program should provide for changing widely held myths that mathematics is a fixed body of knowledge rather than a field of inquiry, that knowing mathematics means remembering and applying rules, and that to each problem there is always one best answer which should be found in a few minutes.

Teachers' beliefs and teacher education

Beliefs and attitudes are mental states organized through experience, predisposing one to respond in a certain way. Beliefs constitute the individual's subjective knowledge and could be overlapping with knowledge in the sense that knowledge is a personalized conception of understanding, which does not separate the knower from the known, it is idiosyncratic and contextual, and emerges during action (Richardson, 1995). Attitudes are more specific and relate to one's tendency to react toward a certain event in a favorable or unfavorable way. In general, beliefs and attitudes are thought to drive action, but experience and reflection on action may change them.

Since a teacher's knowledge is translated into practice through the filter of his/her beliefs about mathematics and its pedagogy (Swafford, 1995), the student-teacher is expected to develop understandings, which consist of knowledge and beliefs, skills and abilities that are directly related to the teaching task, together with some personality characteristics such as interests, temperaments, and moral standards. Hence, a major goal of teacher education is to develop students' beliefs and attitudes about mathematics and its learning. To change existing beliefs involves to get students engage in personal explorations, experimentation, and reflection, resulting in modified images as part of personal knowledge, a new perspective of teaching and learning that would lead to changes in classroom practice (Clarke, 1994). Such a change, is expected to improve teaching, though changes in beliefs do not necessarily translate into changes in practices.

History of mathematics and mathematics learning

What the students learn in teacher education is directly connected with their future role as organizers of learning activities. In this context, the way mathematics is taught is more important than the topics covered. Thus, mathematical experience of prospective teachers should challenge old and foster new dispositions, and develop self-confidence, ability to apply mathematical methods and symbolism, and a perspective on the nature of mathematics through historical and cultural approach.

The potential of the history of mathematics to enhance mathematics teaching, to motivate learners make the necessary connections and realize the continuity of human culture through the centuries, has been advocated by teachers, historians, and educators alike. Most of the mathematical concepts and methods taught in today's classrooms were developed by mathematical geniuses. Yet, they are taught within a short period, as finished and polished products. It should not be of surprise that many students face learning difficulties similar to those encountered in the history of mathematics. History of mathematics could be a means to facilitate learning in different ways, though mathematical development need not necessarily retrace history itself. One can easily produce many good reasons for using history in mathematics teaching. Avital (1995) maintains that exposing students to some of the developments of the subject in the historical and social context in which they were originated has the potential to enliven and humanize the subject, it can teach us about possible learning difficulties, help us improve teaching by following the process of creation in mathematics, induce us create a climate of search and investigation, and it can provide us with exercises and problems in which there is a search progressing to a goal. A diachronic outlook would give real meaning to Zeno's arrow i.e., velocity, a place in the present, a bow and a target somewhere in the future, otherwise "each one of us becomes like in mid-air, frozen, unmoving, and dead, because the past is over and done with and might never have been, and the future ... may never come" (Heiede, 1995, p.231).

Several possible ways have been proposed to incorporate mathematics in teaching. Avital (1995) proposes that it is possible to break the image of mathematics as a boring and difficult subject, add some color and enliven the subject by considering its human side and exposing students to anecdotes and exciting stories from the lives of great mathematicians. Swetz (1995) sees an analogy between a student of problems of history and an art student who visits a museum to study a masterpiece of a genius. Both students have similar cognitive and affective gains, though the pedagogical and intellectual influence of a good history problem could be greater, because in mathematics the student is expected to search for solutions to questions originated hundreds of years ago. Fauvel (1991) suggests that work is still needed in this direction and proposes to explore misconceptions, errors, and alternative views to help in resolving difficulties for today's learners.

The teacher education program based on the history of mathematics

We assumed that the mathematical background of primary school teachers could rely on an overall grasp of the nature of mathematics, an "advanced literacy" in fundamental concepts and methods, and a competence in mathematical thinking. Building upon existing views, we designed a program following the historical development of fundamental ideas, taking advantage of the cultural environment. The program was based on selected topics and paradigms from the history of mathematics studied in the context of their genesis. It was expected that following a historic evolutionary process, studying some of the big problems that intrigued and inspired top mathematical minds, would motivate prospective teachers. Coming to know some of the successes, and understanding some of the failures of well known mathematicians could improve students' conceptions about the nature and the significance of mathematics and liberate students of misconceptions, fears, and negative attitudes. Such a journey would facilitate students' constructing meanings and support their self-image of mathematics and its learning.

The content of the program. At the University of the Aegean (UA) the program consisted of one content and one method course, while at the University of Cyprus (UC) one more content course was added giving the program a luxury to take care of a wider set of ideas at a deeper level. The journey began with prehellenic mathematics, included a lot of Greek mathematics, a little of the post-hellenic contribution (Hindu, Moslem, medieval and enlightenment mathematics), and concluded with six topics from contemporary mathematics (calculus, liberation of geometry, liberation of algebra, set theory, logic, and Boolean algebra).

The prehellenic mathematics (number systems, arithmetic operations, simple problems, and geometry) aimed at two goals: To draw attention to the empirical approach, which served the needs of those societies, and to let students realize the variety of possible approaches to solve everyday problems. The number systems were completely treated in this unit. The selection from Greek mathematics occupied the major part of the first course and consisted of challenging problems and concepts associated with big names. At the opening, the focus was on the contrast between the Babylonian-Egyptian empiricism and the deductive method developed and perfected by Greeks, who laid down the foundations of civilization by asking general questions, seeking for principles, and drawing conclusions based on reason.

From Pythagorean mathematics we included proportions, figurative numbers, the discovery of irrationals, and certainly the Great theorem. Students tried many proofs and extensions of this theorem (including Pappus extension and cosine-rule) as a means to overcome misconceptions that each problem has one best solution. Some nice "solutions" of the three famous problems of antiquity were discussed to help students realize the actual meaning of "solution to a problem", and view mathematics as a human creation.

Euclid's postulate system is used to introduce the idea of axiomatization and deductive reasoning; the proof of his proposition draws attention to the "failure" of a great master and opens the way to a respectful but intense critique of authority. Some geometric proofs of algebraic identities are included and Euclid's classic proof of the infinitude of primes (example of an existence theorem) is thoroughly discussed. We start with two primes, proceed to Euclid's actual theorem IX21: "Let that A, B, and C are prime numbers, I say that that there are more prime numbers", and finally the modern formulation is constructed and proved, i.e., "Let P be the set of all prime numbers $P \equiv \{p_1, p_2, p_3, ..., p_n\}$, show that there exists a prime number p_k that does not belong to P, i.e., there is a prime p_k, such that $p_k \notin P$".

Using Archimedes <u>helix</u> the student is lead to "solve" both the quadrature and the trisection problems. Squaring of a parabolic segment and applying the polygon method to find an approximate value of <u>pi</u>, the student gets in touch with a pioneer work on the concept of limit. The sieve of Eratosthenes is used to search for prime numbers and his method to find the circumference of the globe is discussed. We spent time on Ptolemy's theorem leading relations between chords-equivalent to trigonometric formulas. Diophantus proposition II 8, to divide a square number into two rational numbers is taken as an opportunity to mention Fermat's last theorem.

The aim of the chapter on calculus is to help students understand fundamental concepts, simple methods and applications, and its significance in our technological society. It aims at understanding basic concepts and definitions and some simple applications (maxima and minima, Newton-Raphson's method and Simpson's rule).

The efforts to prove the Fifth Postulate form Ptolemy and Proclus up to Saccheri and Lobachevsky serve as an example of miss-faith to authority and the labors humanity put forth to "liberate geometry". Students are initially shocked and resist to accept the possibility to draw more than one parallels from a point outside a line, they are impressed to learn that in hyperbolic geometry there are defective triangles whereas there are neither rectangles nor similar triangles. The liberation of the students minds continues with the liberation of algebra. Non-commutative algebra is introduced through Hamilton's quaternia and Cayley's matrices, contrasting properties of the traditional algebra and the matrix algebra.

The unit on set theory is mainly devoted to the properties of the two binary operations and the one unary (the complement), ordered pairs, triples and n-tables, cross-multiplication, binary relations and equivalence relations, leading to the definition of function. Concentration on properties continues in the unit on mathematical logic and the algebra of propositions. In the final part we attempt an introduction to Boolean Algebra and some applications to logic circuits, the ultimate goal being to let students have a taste of what formal mathematics is really like, realize how the two previous units could be unified under an abstract system, and understand how the computer depends on mathematics.

The courses were taught by the experimenters in two single hour lecture/discussion sessions and one activity/problem-solving session of one and half hour duration; the latter was mostly conducted by a teaching assistant. Lecture notes and handouts were used, though students were expected to follow many sources.

The program evaluation

At the UA, the program was assessed in terms of its effectiveness to improve students attitudes, by concurrently administering a questionnaire to newcomers (E1) and students exposed to the program (E2). At the UC, students' attitudes were measured before they attended the first course (P1), after the first course (P2) and when they completed the program (P3).

<u>The participants</u> at the UA were 231 prospective primary teachers (130 at E1 and 101 at E2), while at the UC the subjects were all first year prospective primary teachers enrolled in 1992 (N=162), those completed the first course in 1993 (N=137), and finally those who completed all three courses in 1995 (N=128).

<u>The Scales</u>. The questionnaire consisted of three complementary scales: The Dutton Scale, with eighteen items ranging from extremely negative attitudes to extremely positive (factors from 1.0 to 10.5). The Justification scales with two ten-item scales, one for liking and one for disliking mathematics. The Self-evaluation scale, a linear scale, on which students located their feelings on a number-line, from 1-extremely negative, to 11-extremely positive attitudes.

<u>Statistical analysis</u>. The t-test and the Chi Square-test were applied to analyze the responses, separately on each item, and to test for differences that might have occurred during the implementation period. The points of the Self-rating scale were grouped into five levels: 1-2 meaning extremely negative attitudes, 3-5 negative, 6 neutral, 7-9 positive attitudes, and 10-11 love for mathematics.

Results and Discussion

Table 1 shows that that a high proportion of students bring to Teacher Education extremely negative attitudes (E1, P1). Note, for instance, the proportions of students who endorsed the statement "I detest mathematics and avoid using it at all times". Similar proportions of students endorsed the statements "I never liked mathematics","I have always been afraid of mathematics" and "I do not feel confident of myself in mathematics". The same pattern of responses also appeared in the Self-rating scale; Indeed 36.9% and 33.5% of the subjects in UA and UC, respectively, located themselves in the range 1-5. Concerning the reasons, the students stated that they like mathematics primarily because: "it develops mental abilities", "it is practical and useful" (48%, 39%), "it is interesting and challenging", and "it is necessary for modern life", while they dislike mathematics because of "lack of understanding" and "lack of teacher enthusiasm". The overall picture at the entry stage is a warning that the situation could develop into a catastrophe, unless something drastic would change during preservice education.

Table

Selected responses from the tree scales used

	Dutton Scale	**Univ. of Aegean**			**University of Cyprus**				
	Attitude Statements	E1 %	E2 %	p	P1 %	P2 %	P3 %	x^2	p
1[1] 1.0	I detest mathematics and avoid using it at all times	26	16	.00	24	12	12	10	.00
3 2,0	I am afraid of doing word problems	17	24	.01	15	31	24	9.8	.02
10 5.9	Mathematics is as important as any other subject	41	46	.19	32	60	64	37	.00
13 7.7	I like mathematics because it is practical	22	23	.90	34	55	43	13	.00
	Justification Scale	**Un. of Aegean**			**University of Cyprus**				
	I like mathematics because...	E1	E2	p	P1	P2	P3	x^2	p
	it is interesting and challenging	26	29	.39	35	57	53	16	.00
	it is necessary for modern life	48	61	.01	35	49	76	47	.00
	it develops mental abilities	58	62	.23	47	68	72	23	.00
	I dislike mathematics because ...								
	of lack of understanding	31	26	.05	24	17	19	2.5	.28
	of lack of teacher enthusiasm	32	44	.00	25	18	39	14	.00
	it is never related to real life	14	7	.00	15	7	8	6.9	.03

Self-rating Scale	**University of Aegean**		**University of Cyprus**		
	E1	E2	P1	P2	P3
Attitudes	%	%	%	%	%
Detest mathematics (scale points 1-2)	14.6	5.9	14.3	7.3	3.1
Negative feelings (scale points 3-4-5)	22.3	25.7	19.2	13.8	18.0
Neutral attitudes (scale point 6)	20.0	16.8	7.5	9.5	7.8

[1] Numbers in first row indicate the order, and in the second the weighing factors of the items.

The results showed a positive change of prospective teachers' attitudes in both Universities, though more striking in the second case. Items in the lower third of the Dutton scale (1-6) reflect negative attitudes, in the middle third (7-12) neutral, and in the upper third (13-18) reflect positive attitudes. At the UA, the t-test showed significant change ($p \leq 0.01$) in six items, three negative and three positive, five of which indicated improvement of attitudes. At the UC, the x^2-test revealed significant change ($p \leq 0\ .05$) in 14 items, 12 of which indicated positive change. For instance, the proportion of students who "detest mathematics", or "never liked mathematics" dropped significantly in both Universities, while conversely, the proportion of students who "enjoy working and thinking about mathematics outside school" raised significantly. The increase on the statement "I am afraid of doing word problems" is a change of attitudes in the negative direction.

Table 1 shows that improvement in attitudes was affirmed by responses on the Self-rating scale. The proportion of students who detest mathematics dropped during the project ($p \leq 0.01$), the proportion of neutral subjects remained rather constant, and naturally the proportion of positive attitudes raised significantly. Change was also observed on the Justification Scale. For instance, the proportion of subjects who liked mathematics because "it is necessary for modern life" or "because it develops mental abilities" raised significantly ($p \leq 0.01$). More students were convinced about their teachers' "lack of enthusiasm" at the end of the program rather than at the beginning, and fewer students continued to believe that "mathematics is not related to everyday life".

Conclusions

The results of this study affirmed the assumption that prospective teachers bring to teacher education misconceptions and negative attitudes towards mathematics, and supported the hypothesis that the mathematics preparatory program incorporating the history of mathematics would prove effective in changing students' attitudes. A substantial proportion of students were found to bring with them negative feelings about mathematics, a subject they will soon have to teach, mostly due to improper teaching, repeated failures, and misconceptions. The situation calls for special attention, because the teaching profession is not popular, and this trend is not likely to change in the foreseeable future. Teachers will continue to view mathematics as a fixed and finished discipline, teach along the traditional lines, and thus negatively influence students' attitudes. One of the tasks of preservice education is to break down this vicious circle.

Change in attitudes was sought as a second goal of the teacher mathematics preparation program. History of mathematics was one of two factors which proved to be quite decisive, the other one being the cultural environment. Improvement of attitudes was evidenced by three complementary scales. In both universities, significant changes were found on the Dutton scale, on the Justification scale, and on the self-rating scale. Results were particularly encouraging at the UC due to the additional course and the experience gained from the first trial of the program. It should be noted, however, that the present study did not disentangle several factors that might have been operative. One of these factors relates to the mental models that the program created in students' minds about mathematics; a second factor is the presence of instructors themselves and the way they presented the models in the classroom. Yet another factor, which is currently under examination, concerns the permanence of this change and its effect on actual teaching behavior.

In one item (two at the UC) attitudes were found to develop in the negative direction (see item 3). This should not be surprising, since most of students' attitudes were formed over their entire school life and were the outcome of long prejudices of the social environment. It seems that some emotions in the minds of students are so persistent to change that it would take additional time and more challenging experiences to override them. Given the significance of the goal, however, no effort is too much, particularly since the findings seemed to offer a light at the end of the tunnel. Change was found to be non-correlated to any of the subjects' characteristics tested: gender, type of high school, mathematics performance, and family sociocultural conditions. This was a rather surprising finding and it needs further investigation. It would be very encouraging, if the program is really so powerful as to affect invariably the attitudes of students, irrespective of individual characteristics.

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WORKING CLASS STUDENTS AND THE CULTURE OF MATHEMATICS CLASSROOMS IN THE UK

Hilary Povey, Sheffield Hallam University
Mark Boylan, Northcliffe School

We describe some mathematical work we undertook with a group of working class school students outside the context of their school mathematics classroom at a University mathematics education centre. We then consider their reflective responses to that experience and highlight two key contrasts they volunteered between the culture of their school and the culture they found at the University. We suggest that these reflections from the school students call into question some of the current 'common sense' of schooling within the UK and in particular question whether or not the culture it generates in mathematics classrooms enhances the achievement and motivation of working class students.

We begin by describing briefly[1] some work with dynamic geometry software undertaken at a University mathematics education centre in the UK by a group of working class school students. We attempt, by offering examples[2] from the students' writing and speaking about the mathematics, to include enough indicative detail to convey what the experience was like for them. We then consider the students' subsequent reflective writing. We note their stated preference for speaking and discussing rather than 'writing everything down' and report their observations about productive relationships between teachers and learners. On the basis of the reflective writing, we speculate that some current ways of conceptualising good practice for mathematics classrooms may be unhelpful to such students.

Background to the work

One of us worked at a centre for mathematics education at a University and the other taught in an inner city school which had a predominantly white and almost wholly working class intake. We arranged for a class of students to pay two visits to the centre. They were a 'top set' of fifteen year olds that the teacher, Mark, had been working with for nearly two years. He had pushed them (and himself) to experience teaching and learning styles that positioned the students as the makers of mathematics (Cobb *et al*, 1992, Povey, 1996). He was experimenting with the idea of 'hinge moments of learning' (Boylan, 1996), times when there seem to be qualitative shifts in our thinking. It seems that these hinge moments - when we swing around, find a new direction for our thinking, see things from a different angle - are likely to provoked by being placed in novel situations and we conjectured that

[1] A more detailed account is given in Boylan and Povey, forthcoming.
[2] Pseudonyms are used for the school and for the students' names.

coming to the University to do some mathematics might provide some such moments for these students.

Currently, new technology also seems to be a source of such opportunities. In addition, it can support epistemological pluralism (Turkle and Papert, 1990) and, as we have argued elsewhere (Povey, 1997), using computers for mathematics opens up the possibility of a different and more democratic relationship to knowledge. Such considerations are particularly pertinent for those learners who are not part of the hegemonic group and who therefore do not suppose easily that their knowledge is authoritative (Povey *et al*, forthcoming). We therefore decided that, on the visit, we would offer them the opportunity to work with some software unfamiliar to them; we chose to use a dynamic geometry package.

The work with the students

We began with a brief but explicit introduction to the nature of dynamic geometry software, mentioning the objects that the software made available to the user - points, lines and circles - and invoked a simple example to describe the difference between drawing and construction. We used the idea and vocabulary of mathematical construction and constraint as we interacted with the students working on the computers and some were later also able thus to conceptualise and articulate their work. They spoke, for example, of having to 'connect the shapes mathematically'.

During the visits we set them a number of tasks. We asked them first to explore some pattern making with transformations; this task included little that was new to the students mathematically but gave them the opportunity to become familiar with the syntax of the software. Next, we showed them a diagram (Figure 1) of two intersecting circles and an equilateral triangle and asked them to *construct* the figure so that it did not come apart when they dragged it about.

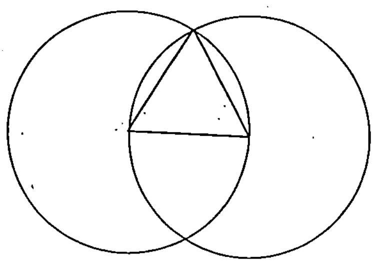

Figure 1

As expected, this was challenging for all the students. However, none of the pairs of students was completely lost and several worked through to a solution. The activity appeared to be engaging and to provoke mathematical discussion, argument and reflection. Hilary visited the school after this session and spoke with a group of four students about their experiences.

Lucy: The first time we did it, it didn't stay did it? You could drag it and it would make all different triangles, it would make isosceles and everything but then, what did we do? We thought it had to be at the middle, on the middle of the other so when we'd done that it were all right

Peter: We did the same as them and it all fell to bits [laughs]

Lucy: [laughs] That's what we did first time

Peter: And then we couldn't work out how to do it so we asked them and they showed us how to do it

Lucy: We showed them that the point had to be the middle with the same radius and then the other point had to be the other radius

Hilary: Why do you get an *equilateral* triangle when you do that?

Lucy: Because it's the radius and they've all got the same, it's the radius

Those who found a solution were asked if they could *prove* that the triangle must be equilateral; we asked, 'How can you successfully argue that it must be what you say it is?'. Hoyles *et al* (1995) have written about the connection between geometric proof and construction and the need for such proof to have 'communicatory, exploratory and explanatory functions alongside those of justification and verification' (p101). Jones (1997) has suggested that using dynamic geometry software may help students 'develop their ability to use mathematical language effectively in presenting a convincing reasoned argument' (p127): this, for us, is an essential element of 'mathemacy' if it is to support the democratisation of knowledge (Skovsmose, 1994, Giroux, 1992). Here we were inviting oral argument about a mathematical object operating as a generic example. We pointed out that the position, orientation and size of the construction could be varied by dragging but that the chosen mathematical construction of 'equilateralness' remained unchanged. Then we asked, 'Could you build an isosceles triangle? Or a right angle triangle?' A variety of ideas were tried but time constraints meant that many were interrupted before they came to fruition. Lucy and Tina had decided to approach the isosceles triangle problem by drawing a pair of circles not constrained to be the same size.

Tina: We had to make the circle wider, like there had to be a bigger space like ...

Lucy: ... one circle were bigger weren't it, one circle were bigger than the other one

Hilary: Do you know why it turned out to be an isosceles triangle? Why wasn't it just any old triangle?

Lucy: ... because these two are the same distance because they are both going from the middle so they are the same distance apart

Peter and Darren described having tried a different approach. Talking about it afterwards, they were able to draw on an idea that had come up later in the session, even though the vocabulary escaped them, the idea of the perpendicular bisector of a line.

Darren: We tried using just one circle instead of two ... what we was trying to do was seeing if we could make an isosceles triangle by, we drew a circle then we put two points down near the bottom equally apart from each other and then put another point at the top above the points and then connected the lines together but it fell apart. Perhaps we could have tried it like them

Hilary: Let's stick with your idea for a moment. If you draw these two points and join them up, where has this point got to be?

Darren: Directly in the middle of the other two

Hilary: So how could you do it?

Lucy: Oh that other thing the, the those lines, oh oh (sketching in the air)

Peter: That line point connected oh oh (nodding)

Hilary: I'll tell you the name 'cos it's the first time you'd ever heard it. It's called a perpendicular bisector

Peter: Yeah!

Lucy: That's it!

Hilary: Why would that work?

Lucy: 'Cos you could fix it directly at the point (points to line crossing circle) and then spread it up to wherever you wanted it

Peter: Yeah!

Darren: We could have done with knowing about that perpendicular bisector earlier!

Next we worked away from the computers asking the students to choose from a supply of tissue paper circles, geostrips, plain paper, pencils and pairs of compasses and to use the materials in any way they chose to make a square. Various ideas emerged and were shared, with discussion focussing on what property of the square the particular solution was exploiting. The students were then invited to return to the computer room and choose any polygon they liked to construct. All the pairs set to work with a will and had no difficulty in setting themselves a task and beginning to tackle it before, again, we ran out of time. Naturally, this was substantially as a result of patterns of working which Mark had set up with the group over time. Nevertheless, we also felt that the dynamic geometry environment had been motivating and had encouraged them to think and to discuss mathematically.

Talking

Following the visits, Mark asked the students to write about their expectations of the visit, their impressions and what they thought they had learnt. From these writings we want to draw out two themes. We speculate that both are connected to social class and school culture and their mismatch. First, a number reported on the liberation of not having to write everything down but instead being required to discuss, to invoke memory, to create and manipulate images. For example:

[At school] you are given a text book and given some notes and told to get on with it with the minimum amount of fuss. At Hallam I found it was different in that you discuss something first ... it is useful because you are looking at the problem from a different point of view ... I learnt about the University that they discuss things more than writing things down (Joseph)

In school we do all our work on paper (Roger)

... the work that we did involved more thinking and remembering what we had done, at school we usually write down everything that we learn or have learnt in the past (Zoe)

The work we did there was quite challenging and I enjoyed it a lot. I enjoyed puzzling things out and trying my ideas. I also enjoyed being part of the 'group' and knowing that I was there to not just work on my own but to work with someone who I could talk to, work with and relate to. It also felt good to be able to talk to other people about my work ... (Joanne)

A big difference between Hallam University and Byron School is the atmosphere. At Byron you are usually told to get on with your work quietly, mostly on your own. At Hallam, you were allowed to discuss a lot of your work, both with the teachers and other students.... but the best thing was that we didn't have to do a lot of writing! (Matthew)

These responses were thought provoking for us. It has been suggested that working class school students spend a significantly greater amount of time than others writing and less time on pupil discussion tasks (Duffield *et al*) with 'more urging pupils along to 'do' their work' (p10). The approaches to learning mathematics that the students had described and valued had been an important part of the way Mark had tried to work with this class. Nevertheless, the unspoken realities and culture of working class school life nudged him in the direction of 'write it down'. When reflecting on this, Mark identified a strong fear in himself that, if there is not a written record of work done, then the work will be less valid. He also recognised how this displays a lack of confidence that students will really learn more through discussion: they had better have a written record to help them 'revise' in case the content is not learnt. We conjecture that the current emphasis in the National Curriculum in the UK on teachers' record keeping, evidence, inspection and testing is a pressure away from the oral and group work these students so enjoyed.

Informality

The students found the reality of the university atmosphere very different from what they had expected: this reminded us of the cultural differences between working class school students and teachers in schools. All but one of the students was very positive about the visits. In particular, over half of the students commented, all but one favourably, upon the relationships between teachers and learners experienced in the University context. For example:

Another way that the atmosphere is different to Byron is in the way that people speak to each other. At Hallam, when you speak to a teacher you use their first name instead of calling them Mr or Mrs. We were even allowed to call Mr Boylan Mark. It was a bit awkward at first, but you soon got used to it ... I learnt a bit about life in a University, how people refer to each other by their first names (teachers and students) and how there is a much more relaxed approach to work (there was for me anyway). (Matthew)

I expected that the centre would be very like Byron and would have a lot harsher relationship between student and teacher ... I really enjoyed the visit to the centre because it was very different compared to the school. The atmosphere in the centre was very calm and that of a easy working environment. I enjoyed it most because the relationship between student and teacher was very friendly, the teachers treated you more equal to them and more like an adult. (David)

The relationship between students and teachers was different from school because the atmosphere seemed more friendly and you could call the teachers by their first names instead of using sir or miss. You couldn't mess about but then again it wasn't as strict as school. (Joseph)

... I was a little apprehensive, I thought that the university would be like a large office block with big rooms and strict lecturers ... but when I got there I noticed that it was nothing like I had imagined. The atmosphere at the university was completely different from the atmosphere at school because of the way you treat the lecturers. You didn't have to call them 'miss' or 'sir' you just treat them like ordinary people ... all the lecturers that we spoke to were really friendly. (Tessa)

The [teacher education] students talked to Hilary about what they was doing more like as a friend than as a teacher. Hilary didn't really treat them as if she taught them. She treated them as friends (Zoe)

Another thing which I noticed was different at the university was the relationship between the teacher and the students. At the university all the teachers and students called each other by their first names and if you hear a group of students talking to a teacher it sounds just like a group of friends ... it was much more quieter than a school, less busier ... and instead of having break times like at school when you go to the toilet, you can go when you want and take a break when the teacher feels you need one (Patrick)

I thought it would be like school but a bit stricter but it was a nice change from school. It was a laid back atmosphere which enabled me to work better ... you could call the teachers by their first name. This helped the atmosphere and made it easier to talk to them. (Carl)

For many of these students, just as secondary school was harsher, stricter and more formal than primary school, so in turn University was expected to be more so than secondary school.

Discussion

It might be tempting to dismiss all this: naturally, relationships in the privileged and unusual setting of a visit to a University are going to be less formal and more easygoing than those typical in school and a more experimental and open pedagogy involving discussion with peers sits easily in time taken out of normal schooling. However, here are students who see themselves as being capable of self-regulation and able and willing to discuss pedagogy.

I also learnt a lot about myself. I learnt that I can work with a partner and in groups to solve problems, and I can work on a puzzle until it is solved, correcting any mistakes I make and learning from them. (Matthew)

Pedagogic rules are mostly kept implicit in school classrooms (Edwards and Mercer, 1987) and, generally, do not support the development of the reflective knowing which 'has to be developed to provide mathemacy with an element of empowerment' (Skovsmose 1994, p117). 'This ... is not accidental: it is a consequence of the structure of power in the classroom and a particular view of how pupils learn' (Quicke and Winter, 1994, p444). The students wanted more informal relationships and more opportunity to discuss and work together. One reading of Jo Boaler's research (1997) is that a relaxed and informal atmosphere, the expectation that students work jointly with their peers and the requirement of self-regulation were key elements in creating the mathematical success of working class students. This is not usually what is on offer: schools with working class students see more 'teacher anxiety about control and wariness of pupil autonomy' (Duffield *et al*, 1996, p11) than other schools. Current measures of 'effective' schools and classrooms lay considerable emphasis on highly structured approaches to learning, on students being 'on task' and on tight teacher control. But we need to ask

... what is it about control which makes us so scared of losing it. Who really needs control? Children seem to live comfortably in confused chaos from an early age. They cope with multiple disordered stimuli by ignoring most of the chaos and stressing only a small section which they then choose to work on ... If we accept that tightly controlled classrooms have been inhibiting learning, should we not also consider the possibility that learning cannot take place unless there is confusion and conflict? (Breen, 1990, p39)

No-one can be happy to see school students wasting their time and making nothing of the learning opportunities available to them in mathematics classrooms. However, it may be that current demands on teachers for order, structure, discipline and control mitigate against creating learning environments in which working class students can develop both a sense of themselves as authoritative learners and a thirst for further study. Corinne Angier, teaching in a working class school, writes

I find that the children in my classroom are desperate for dialogue on all sorts of levels. The challenge for me is to provide them with the space in which to develop their mathematical voices and not to drown out their efforts with a cacophony of discordant demands. As a friend, a parent, a sibling, I find it much easier to allow dialogue on somebody else's terms. I happily participate in hundreds of conversations with my own children that lead into blind ends; a luxury I rarely afford the children I teach. In the classroom there is always the curriculum, the lesson plan, the implications for classroom management, most of all their is the fear of anarchy ... (Povey *et al*, forthcoming)

In this paper we have tried to unravel some of the connections between social class, learning mathematics and current models in the UK of how effectively to structure the curriculum and the classroom. We hope, by sharing some of the reflections on learning from a group of working class students, to render problematic some of the current 'common sense'. The implications for practice are far from clear; but unless we pay attention to what such students are saying, we are unlikely even to be asking ourselves the right questions.

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THE CO-ORDINATION OF MEANINGS FOR RANDOMNESS

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We are aiming to build a computational environment for exploring stochastic systems, with which we can observe children's meaning-making for ideas such as fairness, randomness and chance, and how these meanings evolve during interaction with the tools in that domain. We discuss the work of two children in order to illustrate how the tools re-shaped their sense of randomness and chance.

Our starting point, following Lave (1988) and others broadly based within the situated cognition tradition, is that mathematical knowledge is constituted within a situation and is therefore deeply contingent upon it. Thus, a situation will incorporate resources in the form of tools and structures, which shape — and in turn are shaped by — the knowledge constructed by the tool-user. Our long-term aim is to study the use of tools for the expression of meanings for stochastic phenomena. In order to do so, we have been busy developing an expressive computational medium which can act as a window on learners' thinking as they articulate the evolution of their fundamental ideas of chance and probability.

This notion of 'window' is elaborated in *Windows on Mathematical Meanings* (Noss & Hoyles, 1996). The central idea is that a computer-based environment offers some methodological advantages over the sorts of workplace or everyday environments that have been the main focus for the situated cognitionists (and some disadvantages too which we do not have space here to consider). In such an environment the researcher can have systematic control over a subset of those resources which are external to the learner, controlling them quite finely in ways which may be difficult to attain in more conventional environments. In addition, a computational medium can afford a trace of the outward manifestation of the learner's thinking through his or her actions, in the form of key presses, mouse-clicks, and choice of the environment's primitive structures.

Much depends, of course, on the choice of medium: ideally the researcher needs to be able to implement easily changes in design during the research endeavour. In this iterative scenario, the researcher simultaneously supports and investigates the child in his or her desire to express their mathematical thinking (see diSessa, 1995, for more on the question of iterative design). We have chosen to work with Boxer, a computational medium which is particularly suited to an iterative design methodology.

We are working with young children (aged between 10 and 11 years) and have chosen to focus on their construction of meaning for stochastic phenomena. This is not an arbitrary choice. It stems from the fact that our appreciations of stochastic phenomena live in two almost distinct worlds: the one 'everyday' and the other 'mathematical'. We are aware that we are using these descriptors in a rather loose way, but we wish to distinguish between those kinds of activities in which mathematical activity is, at most, incidental (like Lave's, 1988, supermarket shopping) and that of mathematical activity in which there is an explicit goal to separate — some might say abstract — formal concepts from concrete context-bound activity. The stochastic is formalised in just this way in the form of the Laws of Probability and this formalisation is generally regarded as elusive for many children and adults (see, for example, Kahneman, Tversky & Slovic, 1982). In particular, Konold (1989) has observed that adults often fail to recognise a situation as essentially stochastic and attend to what actually happened rather than what might have occurred, a tendency he has named the *outcome approach.*

It is at the intersection of the everyday and mathematical ways of thinking that glimpses of mathematical meaning-making can often be found. In particular, we expect that by watching learners trying to express themselves mathematically/ computationally, we (and they) might learn more about their intuitive, everyday thinking — *and vice versa.* In this paper, we look at the activity of two children (aged 11 years) who worked with a Boxer-based microworld that we are designing iteratively (see Pratt & Noss, 1996, for further discussion of the iterative design methodology). The central objects of the microworld are a series of 'gadgets', computational tools which behave in many identifiable respects like their everyday counterparts. For example, in the DICE gadget (see Figure 1), the learner can 'throw' a dice with varying strengths, as well as inspect and change the 'workings' which produce the behaviour of the dice. Other gadgets included, inter alia, a coin, spinners, coin rolling and Frisbee throwing.

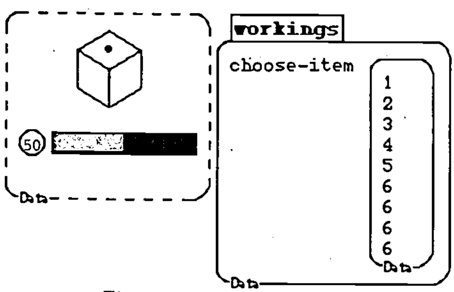

Figure 1 : The DICE gadget.

The DICE gadget is activated using the strength bar, depicted in Figure 1 as a solid black bar with a circular switch at one end. We can imagine the child controlling the strength by allowing a tube (the black bar) to fill with a red fluid until the switch is clicked. The strength of the throw, 50% in this case, is represented by the amount of red fluid. Clicking directly on the dice itself causes the dice to be thrown with the same strength as last time. When the dice is clicked it rolls 'dice fashion' and indicates the outcome on its top surface. The outcome is controlled by the workings box, which can be edited by the child. In its default (as shown), the dice is biased towards sixes; the probability distribution in this default form defines Pr(scoring a 6) = $\frac{4}{9}$.

A gadget therefore embodies quite explicitly and accessibly a mathematical representation of how it works. This representation is both instructive, in the sense that the children need to make sense of it, and constructive, in the sense that it can be modified and used as a building block for extended ideas (see O'Reilly, Pratt and Winbourne, 1997, for further discussion of these notions).

An illustrative episode

Sophie and Jackie are two 11-year-old children, who were exploring a range of gadgets provided as resources. Their challenge was to identify which gadgets they felt were "working properly" and to "mend" those gadgets which appeared to be working incorrectly. The incident began with the two children choosing to explore the DICE gadget shown in Figure 1.

We shall present and analyse the episode in four slices, each illustrating a salient issue supported by excerpts from the children's discussions, and postpone discussion until the subsequent section. The first slice is taken from the children's work as they first began to explore the DICE gadget. Sophie and Jackie were trying to make sense of how the dice behaved. They were activating the dice using various strengths and sometimes repeating the last strength to gain a feel for the outcomes. At this stage, the children were working at top level, not yet having been introduced to the various tools such as the workings box. (*Researcher* refers to the first author of this paper.)

Slice 1. "You can't tell what comes next"

1. J: Shall we try it a few times? ......Three
2. Both: Six
3. S: Shall we do it again. There it goes.
4. J: Six. Is it Six again? First it was Three and now it was...
5. S: It might be Three now.
   They click again and the dice lands on Three.
6. *Simultaneously,* J: It is. S: Yeah, It is!
7. J: Three again. *To Sophie:* It might have heard you!
   Clicks again and the dice lands on number four
8. J: Four. Something different almost all the time.
   They click again and the dice lands on number one
9. S: One
10. J: A bit strange, that.
11. S: It'll probably be another One then a Four. I doubt it though.
    They click and the dice lands on number one
12. S: It is actually *Click again* I wonder if it's going to be a Four again. It's a bit of a coincidence if it is.

So Sophie and Jackie expected not to be able to predict the next outcome (although they had some success - line 6!). And they have some sense of the distribution, however murky, as evidenced by Sophie's comment on line 11. An important criterion for Sophie and Jackie of "working correctly" was that the outcome could not be regularly predicted. We take this as evidence that one meaning for randomness held by Sophie and Jackie is unpredictability.

Slice 2: "They should all be the same really"

Slice 2 took place shortly after slice 1 and began by an observation that no fives nor threes had been generated and that sixes were quite frequent. At this point, the girls' attention was drawn to the *workings* box, which had until then been ignored. The children noted that there were indeed more sixes in the workings box and continued with **repeat 50 [click dice]** in order to view the result of fifty dice throws. The graphing tool, which displays in pictogram form the outcomes in the results box (see Figure 2), brought the predominance of sixes into focus.

13. S: *Points at the graph (see Figure 2)* Woa. There are more Sixes. Yeah. There definitely are more Sixes. I don't think the dice is working properly.
14. J: Yeah. It should be around the same for each of them. The Fives and the Threes are the same but ..
15. Researcher: Right. Right. And what about the others? Do you feel they're OK. - the One, Two, Three, Four, Five?

16. S: Don't think they are. I think they should all be the same really.
17. J: Yeah. I think, Threes, Ones and Fives are OK.
18. S: Yeah
19. J: ...and the Twos are sort of OK., yeah, but it's the Fours and the Sixes that are a little bit over the top.
20. S: Well, the Fours - the Sixes have got too many and there's ones that haven't got enough.

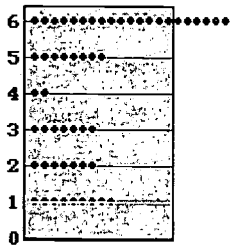

Figure 2 : The graphing tool shows a predominance of sixes

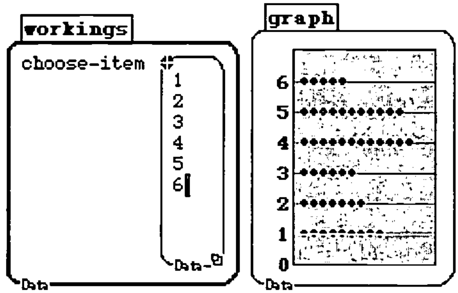

Figure 3: A fair dice? Sophie and Jackie have amended the workings box

The graph confirmed for them, even though the evidence was quite limited, that the dice was not working 'properly'. However, they not only saw problems in the number of sixes but also in the frequencies of some other scores. Sophie and Jackie decided at this point to set about 'mending' the DICE gadget. They edited the workings box and, after 50 trials, generated another graph (Figure 3).

From our point of view, we see this situation representing a fair dice because we focus on the workings. However, Sophie and Jackie's reaction to the graph indicates how differently they understood the situation.

They point to the row of Fives and Fours

21. J: It's a bit better there.
22. S: But we haven't got enough Sixes now.
23. J: Yeah.
24. S: And also there's too many Fours.
25. J: If we added one more Six on to the em ...
26. S: Yeah. But still, Fives and Fours are too many, though, aren't they?
27. J: Yeah. They need to be a bit less. But you can't make them less, can you! (Sophie: No) ... cos then there'd be nothing.
28. S: Unless you take one away No, because there isn't any more to take away.

They return to the workings box, add another six.

Their focus was on the results, which were more salient for them than the workings as a means for understanding what we would call the distribution of the outcomes. The girls' intuition was to correct for the discrepancy between the number of times different scores had occurred by adjusting the workings accordingly. The girls then iterated, three times in all, through a process in which they amended the workings box, repeated fifty clicks, looked at the graph and changed the workings box accordingly. Sophie's and Jackie's intuitions suggested to them that fairness should result in equal, or very nearly equal, frequencies for each possible outcome (e.g. line 26). So the workings were used as an *input* tool; a change in the workings caused a change in the graph but it was the graph which validated the fairness or otherwise of the dice.

The first two slices through this episode illustrate how the expressive domain enabled the children to hold their intuitions open to scrutiny through the window of the workings box, how the children's intuitions shaped their tool use. However, the next slice illustrates the reverse process: how the children's intuitions were themselves being shaped by the structuring resources in the domain.

Slice 3: "It's not necessarily not working properly"

Slice 3 took place shortly after the above three attempts to modify the workings. The girls had just been discussing the meaning of **choose-item**. The researcher decided to make explicit connections between an everyday dice and the DICE gadget.

29. Researcher: OK. If you threw a real dice, fifty times - OK. -like you're doing - What do you think might happen? - If it was an ordinary, fair dice - What do think would happen, if you did it that number of times?
30. S: Well, I think - I don't even know - I don't think it would turn out so that it's all the same because you can't say, well, - If it's fair - that it's gonna turn out all the same. Because if it's fair, then it's gonna be random, so ...
31. J: I think it would be like umm, quite fair, but it wouldn't be exact.
32. Researcher: Right...
33. S: You wouldn't be able to say well, it's gonna be so many Sixes - There's gonna be eight Sixes, there's gonna be eight Fives, eight Fours, eight Threes.
34. Researcher: Right...
35. J: You might have like, ten of ...
36. Researcher: Right. So in your workings at the moment that you've got, do you think that's a fair situation that you've - the way you've written the workings? You've written 'choose-item One, Two, Two, Three, Three, Four, Five Six, Six.'
37. S: No.
38. J: No. Maybe we should do like two of everything.
39. S: Or one of everything. Because otherwise it's not gonna be fair, is it?

Sophie and Jackie modified the workings box (Figure 4) in accordance with Jackie's suggestion in line 38. However, the above discussion had emphasised a connection between the workings box and fairness, as opposed to a connection between the graph and fairness. An immediate reaction was again to be concerned that there were too many fours and twos. However, such concerns were cut short by Sophie's pivotal insight.

40. S: I've just thought of something. It's not necessarily not working properly, because it's got to be random, hasn't it?

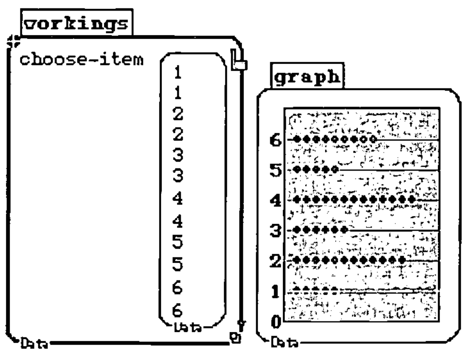

Figure 4 : Implementing Jackie's fair dice.

Sophie recognised that fairness in the workings box did not necessarily imply fairness in the data. Random behaviour may result in variation between the frequencies of different results even when the dice is fair. Sophie's assertion needed cking as we see in the next slice.

Slice 4: "No patterns in outcomes"

Slice 4 took place towards the end of the episode with the dice gadget. By this stage, and after much experimentation, Sophie and Jackie had mended the DICE gadget by editing the workings to read: **choose-item [1 2 3 4 5 6]**. They were now checking whether the dice really was working properly. They referred to the results box, which simply contains the history of their interactions with the DICE gadget in the form of a list of outcomes.

41. J: We could have a look at this (points to the list of numbers in the 'results' box).
42. S: The results
    The fifty results were listed in a box called RESULTS. Sophie and Jackie scrolled through the list.
43. J: It looks about normal.
44. Researcher: What do you mean when you say that?
45. J: It looks quite random - because *they're not* all in groups or anything.
46. S: In quite a bit there's like so many Twos together or so many Fours or whatever, but it definitely looks random. It doesn't look as if they've got all the Ones at the top or the Twos or the Threes ......
47. Researcher: Right. There's no obvious pattern to it.
48. J: No.

What Sophie (line 46) and Jackie (line 43) seem to be saying is that there is no obvious pattern in the results, and that this makes it "random" (line 45) or "normal" (line 43). This assertion (that a lack of pattern is an aspect of random behaviour) is seen as consistent with their earlier assertion that fairness in the workings box does not necessarily imply fairness in the data, another attribute of random behaviour.

Discussion

For Sophie and Jackie, fairness and randomness were closely related, although, as we have seen, these notions became disaggregated as they worked with the microworld. We were able to observe three aspects of Sophie's and Jackie's intuitions about fairness which underpinned their thinking:

- if the dice is fair, you can't tell what comes next;
- if the dice is fair, there are no obvious patterns in the sequence of results;
- if the dice is fair, the different outcomes should be equally, or nearly equally, distributed.

If we are to make sense of how these meanings were re-shaped during the activities, we need to ask how Sophie and Jackie knew these things. It seems helpful to view these kinds of intuitions in terms of basic sense-making devices for interpreting stochastic phenomena — what diSessa (1983) has called phenomenological primitives, or p-prims, abstracted directly from everyday experience. One feature of p-prims is that they have attached to them coefficients or priorities, which determine which p-prims are activated under any particular set of circumstances. The reliability priority is particularly important here; p-prims, which prove to be consistent with other p-prims and with further incoming data, have their reliability priorities increased.

Viewed in this light, Sophie and Jackie make sense of the dice situations with a loosely connected set of p-prims around the notion of *fairness*, activated by associated p-prims concerning unpredictability, a lack of pattern in results, and an expectation that results would appear roughly equally distributed. These, we might imagine, are abstracted directly from experience with dice and other kinds of random generators (like cards) during informal games playing.

These meanings are insufficient as ways of making sense of randomness. They allow a data-oriented view of the world, where Kahneman and Tversky's heuristics and Konold's outcome approach can flourish. Such everyday experiences do not normally afford opportunities for the construction of p-prims relating to the distributional features of stochastic behaviour. We have charted how Sophie and Jackie developed new meanings for fairness through their interactions within the microworld. The four slices taken as a whole illustrate how Sophie's and Jackie's existing intuitions about *fairness*, often based on actual outcomes, are co-ordinated with new meanings, derived from interacting with the workings box, representing the theoretical distribution. Far from seeing data-oriented intuitions of fairness as some sort of misconception, we see these primitive meanings as foundational knowledge for the emergence of new, perhaps more sophisticated, meanings.

We speculate that the interactions within the microworld provided exactly the sorts of control and feedback which everyday experience omits. The key point is that the children could explicitly manipulate formal, textual representations of the distribution (in the form of the workings box) and connect the information so generated with informal pieces of knowledge generated by direct manipulation of the system, and existing intuitions. Fragile connections were quickly supported (or refuted) by evidence allowing a re-structuring of the system of p-prims. It is worth acknowledging that the collection of p-prims is still likely to be weakly structured; Sophie and Jackie will need many such carefully designed experiences before we can hope for the emergence of expert-like knowledge. This incident with Sophie and Jackie offers though a glimpse of what sorts of experiences are likely to be effective.

We are struck by the ways in which the detailed structure of the tools available (the grapher, the dice, the results box etc.) and the connections forged and used between them, became important in their own right to the children's sense of the situation. Here the specificities of the environment are crucial: the objects and relationships of the microworld were carefully planted precisely so that they could support the forging of mathematical meanings in use. Elsewhere (see Noss and Hoyles, 1996) we have called this process *webbing*, the ways in which the internal and external resources support the learner in forging new connections, new meanings during activity. In Sophie and Jackie's case, the resources of the microworld which proved crucial were, among others:

- the **repeat** primitive, which supported a move towards aggregation of results;
- the *graphing tool* which supported the visualisation of the results in terms of qualitative variations (e.g. there were too many sixes);
- **choose-item**, the primitive we designed to articulate explicitly the notion of random choice among a set of numbers;

- the *workings box*, which provides a living representation of the distribution (or, at least, a way of thinking about the distribution).

So what did the two children learn? It would be presumptuous to argue that they abstracted the mathematical relationships between randomness, fairness, and the underlying distribution. What we *can* assert is that Sophie and Jackie articulated some of these relationships *within* the context and with the expressive power which the tools afforded. These *situated* abstractions — 'you can't tell what comes next' (slice 1); 'It looks quite random - because *they're not* all in groups or anything' (slice 4); 'If it's fair - that it's gonna turn out all the same' (slice 3). In each case, the student is saying something about the structure of the situation, something which simultaneously points to its underlying relationships but situated within the setting of the results table, the graph. They are abstractions which are at once powerful expressions of the mathematics they see, but firmly embedded within the system at hand for expressing it.

Sophie and Jackie evolved their thinking to incorporate and co-ordinate both data and distributional oriented meanings for fairness and randomness. Note how we are expressing this shift: we stop short of saying that they saw the mathematical structures differently (they may, of course have done so), but we can be sure that they had different, new ways of expressing what they thought, and that this evolved during the activities. In some ways, this is analogous to what happens when we become sufficiently inculcated into mathematical discourse to appreciate the Laws of Probability. It is then that we recognise a formal expression, to complement our informal intuitions of the stochastic. It seems that many people never achieve this fusion. We do not, of course, claim that we have observed Sophie and Jackie doing any such thing: the abstraction that Sophie and Jackie have made was forged through interaction with the very particular tools available in the microworld and so carries the hallmark of those tools. Theirs is a situated abstraction, powerful within its limited domain of application, but also in its unrealised potential as an internal resource capable of further webbing in the future.

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LEARNING OBSTACLES IN DIFFERENTIAL EQUATIONS

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This study investigated six students' understandings of and difficulties with qualitative and numerical methods for analyzing differential equations. From an individual cognitive perspective, the following obstacles were found to influence the development of students' understandings: the action-process-object dilemma, the tendency to overgeneralize, interference from informal or intuitive notions, and the complexity involved with graphical interpretations. From a sociocultural perspective, students' understandings were influenced by instruction that did not seek out students' explanations, classroom interactions that implicitly established procedure-based mathematical justifications, and the use of technology that was disconnected from the learning process.

The typical engineering or physical science student in the United States begins his or her university studies in mathematics with a year of calculus, followed by differential equations in the second year. In the past decade there has been a nationwide effort to revitalize the calculus curriculum and its instruction, but less well-publicized and far less well-researched are the changes occurring in content, pedagogy, and learning of differential equations. Several recent curriculum reform efforts in differential equations are decreasing the traditional emphasis on specialized techniques for finding exact solutions to differential equations and increasing the use of computing technology to incorporate qualitative and numerical methods of analysis. Prior to these reform efforts, the typical differential equations course did not use technology for instruction or problem solving and mainly emphasized analytic techniques for finding exact solutions to well-posed problems.

Research on student understanding of differential equations, however, has lagged behind these reform efforts. This is problematic as curricula developed without the link to research into student thinking is likely to less effective than curricula that take student cognition into consideration. To date, there is very little published research on student understanding in differential equations. Artigue (1992) investigated first year French university students' understandings and difficulties with qualitative analyses of first order differential equations. Solís (1996) described the framework and theoretical background for a future study that examines how students might use visual and analytic modes of thinking about linear differential equations. Except for these two studies, the vast majority of published reports in the United States have been expository or descriptive in nature. The purpose of the study reported here was to examine students' understandings of and difficulties with qualitative and numerical methods for analyzing differential equations and to identify the factors that shape these understandings and difficulties in one such approach to reform. For new areas of interest, such as differential equations, the identification and description of the difficulties students have is an important first step for research into student learning.

Theoretical Orientation

The theoretical orientation employed in this study is based on the emergent perspective (Cobb & Bauersfeld, 1995). This framework strives to coordinate the individual cognitive perspective of constructivism (von Glasersfeld, 1995) and the sociocultural perspective based on symbolic interactionism (Blumer, 1969). From this viewpoint, mathematical learning is considered to be "both a process of active individual construction and a process of enculturation into the mathematical practices of the wider society" (Cobb, 1996, p. 35). The study also draws on the work of Herscovics (1989) and his description of the various cognitive obstacles associated with learning algebra. Herscovics described three kinds of obstacles: epistemological obstacles, obstacles associated with the learner's process of accommodation, and obstacles induced by instruction. The later two he termed "cognitive obstacles" and he noted that "just as epistemological obstacles are considered normal and inherent to the development of science, so should cognitive obstacles be considered normal and inherent to the learner's construction of knowledge" (p. 61).

Herscovics used Piaget's theory of equilibration to frame and relate his discussion of the cognitive obstacles associated with the process of knowledge construction and those induced by instruction. Although this is a good start, different theoretical perspectives are needed when considering these two types of cognitive obstacles. This is where the emergent perspective comes into full view. From the individual cognitive perspective, a constructivist theory of learning is used as a lens to examine the difficulties encountered by individual learners. However, when viewing the obstacles induced by instruction, the sociocultural perspective is more appropriate. As Cobb (1996) notes, the particular perspective "which comes to the fore at any point in an empirical analysis can then be seen to be relative to the problems and issues at hand" (p. 35).

In order to clarify these two cognitive obstacles described by Herscovics and the different theoretical perspectives used in the analysis, the terms "individual cognitive obstacles" and "sociocultural cognitive obstacles" are used. The difficulties that students have can be viewed with either of these lenses. In fact, it is most likely the case that a combination of these obstacles comes into play at any one time and no one lens is entirely appropriate. However, for the purposes of discussion, individual and sociocultural cognitive obstacles are discussed separately.

Method

Qualitative research methods were used to investigate the understandings and difficulties of six students in an introductory course in differential equations for scientists and engineers at a large mid-Atlantic university in the United States. The six students who participated in the study were part of an intact class of approximately 18. The class met for three 50-minute lectures per week in a room with no computer resources. *Elementary Differential Equations* by Boyce & DiPrima and a computer supplemental text *Differential Equations with Mathematica* by Coombes, Hunt, Lipsman, Osborn, & Stuck were the course texts. This was not an experimental section, but rather the typical configuration for this course.

Data included transcripts from four semi-structured individual interviews with the six students, instructor and other mathematics department faculty interviews, copies of students' quizzes, exams, computer assignments, and an end-of-the-semester questionnaire administered to all students in the target class as well as to students in six similar sections of the course. I also attended and audio-taped every class session. Audio-tapes were used to supplement my fieldnotes. As Schoenfeld (1992) points out, "if we are to understand how people develop their mathematical perspectives, we must look at the issue in terms of the mathematical communities in which students live and the practices that underlie those communities" (p. 363). A modified version of analytic induction (Bogdan & Biklen, 1992) was used to analyze the data.

Each of the four interviews consisted of three to five problems, all of which were reviewed by two mathematicians for validity and appropriateness. Students were asked to "think aloud" as they worked through the problems. Since the purpose of the interview was to explore students' understandings and difficulties, a semi-structured interview format was used. Students were frequently asked, "Why did you do ...?" or "How would you explain to another student why ...?" None of the problems required the use of Mathematica, but most of the problems made use of previously generated output from Mathematica. In this way, I was able to gain insight regarding students' understandings of the computer output without the frustrations students often encounter with syntax.

Sample Results and Discussion

Two of the goals of the computer supplement were to guide students into a more interpretive mode of thinking and to enhance students' ability to perform qualitative and numerical analyses of differential equations. The instructor's numerous comments in lecture about the positive role of graphical and numerical methods of analysis reveals that he views these as worthy goals. The student interviews suggest, however, that these goals were less than adequately achieved. For example, students had difficulty making important conceptual, symbolic, and contextual connections to the various graphical representations used to visualize solutions to differential equations. Some qualitative and numerical methods of analysis were learned in isolation from other aspects of the problem and the concept of stability and its connection to the reliability of numerical methods was not well understood. These difficulties, among others, are interpreted in terms of individual and sociocultural cognitive obstacles.

Individual Cognitive Obstacles

The following individual cognitive obstacles help to frame and make sense of students' difficulties: the action-process-object dilemma, the tendency to overgeneralize, interference from informal or intuitive notions, and the complexity involved with graphical interpretations. The first three of these are subsequently discussed. For a more complete discussion, see Rasmussen (1997).

The action-process-object dilemma refers to an individual's struggle in moving toward more complex ways of thinking about a particular concept. Roughly speaking, an action is an external transformation of mathematical objects that is performed by an individual according to some explicit algorithm. A process conception comes about after an internal construction is made that performs the same action, and an object conception is held if it is possible to perform actions on processes, in particular actions that transform them (Dubinsky & Harel, 1992).

In the graphical setting, students tended to think of a differential equation as something for which you find constant solutions for by calculating when the derivative is zero. More specifically, students could find the equilibrium solutions and even classify them as stable or unstable by sketching in "test" curves, but they did not think of these additional test curves as solutions to differential equations. In order to conceptualize all of these curves as solutions to differential equations, one needs to be able to conceive of the solutions without actually computing them. In other words, students lacked a process conception of differential equations.

Students' tendency to overgeneralize algebraic operations like the distributive property and operations like differentiation and integration are well documented in the literature. The interviews with students in differential equations revealed several specific instances of this tendency to inappropriately generalize as well. For example, in a direction field-differential equation matching activity, students tended to overgeneralize the notion of an equilibrium solution by looking for vectors pointing along the curve for which the derivative was equal to zero.

Another instance of an inappropriate generalization occurred when students were asked to use a given direction field to sketch what they saw as the exact solution to an initial value problem as well as the Euler method approximation. The approximate solution curves they sketched tended to simply follow the same overall pattern of their exact solution with no regard to the direction field. On the one hand, this suggests that they learned to carry out the algorithm without fully understanding what the symbols referred to. This may have roots in instruction that emphasized procedural competency over conceptual understanding. On the other hand, the Euler method sketches may be a result of an overgeneralization notion that approximate solutions always just follow the exact solutions. This inappropriate overgeneralization has important implications when the reliability of a numerical method is considered.

Regarding interference from informal or intuitive notions, several of the interview problems revealed that students had an inappropriate notion of stability. In particular, students tended to associate stability with regularity and predictability: much like that of a stable home environment. For example, one student incorrectly thought that the direction field for $dy/dx = (2y - 5x + 3)/(x + 1)$ typified stability because "all the solutions are going in the same direction. They are all increasing and then at some point starting to decrease, so you'd say they are stable." In contrast, instability was associated with wildly different and unpredictable behavior. This inappropriate (but reasonable from an informal viewpoint) concept of stability also has serious ramifications when considering the reliability of numerical methods.

In some ways, students' notions of stability are similar to the well-documented misconception that graphs of functions are necessarily smooth and without abrupt changes or other irregular behavior. Students tended to think that solutions to differential equations (which are functions, after all) behave in ways that are not too strange. This is most definitely not the case for nonlinear differential equations. For example, in the first problem of the second interview, students were provided with the direction field for the nonlinear differential equation $dy/dx = y(x - y)$ and asked to describe how the limiting behavior depends on the initial point in the (x, y) plane. It is intuitively appealing, yet incorrect, to think that solutions that hint at asymptotic behavior continue that behavior. This intuitively appealing idea even led the famous mathematician Smale astray. According to Gleick (1987), Smale "proposed that practically all dynamical systems tended to settle, most of the time, into behavior that was not too strange. As he soon learned, things were not so simple" (p. 45).

Sociocultural Cognitive Obstacles

In the preceding section, individual cognitive obstacles (or factors) that helped shape students' difficulties were summarized. These obstacles provided insight into the development of students' understandings and helped frame specific difficulties. The picture is incomplete, however, without an analysis and discussion regarding the role of instruction in the learning process.

Regarding the role of technology, most students felt that Mathematica was disconnected from the course and, although they liked the aspects of visualization, they felt it did not contribute much to their overall understanding about differential equations. One of the students in the study expressed a common sentiment when he referred to the Mathematica projects as a "one time thing." Like many of the other students, he just wanted to get the syntax down and move on and do his learning elsewhere. This disconnect between the Mathematica assignments and the rest of class has instructional roots. Despite the fact that the computer assignments counted for 20% of the grade, Mathematica outputs were rarely discussed in class and, although the instructor made references to Mathematica, it wasn't used in any direct way in lecture to assist learning. Mathematica was peripheral to the course and, in most cases, peripheral to students' understandings. In this regard, we have to see the use of technology as problematic.

It is a bit naive, however, to simply cite the instructor for not integrating Mathematica more thoroughly into the learning process. Besides the typical barriers that instructors face when attempting to incorporate technology into the curriculum (e.g., time, availability, experience, beliefs and values, etc.) there are influences outside the classroom that led to the current state of implementation, or lack thereof. Although it was policy that Mathematica be used in all sections of the course, additional support to consider and develop effective means to integrate technology into instruction was not provided. Partly as a result of this lukewarm embrace, the computer supplement was instituted in a way that required little or no faculty investment. One may argue that this path of least resistance was a wise choice, for otherwise it may not have been implemented at all. That may be so, but this path does not appear to have had the intended effect on students' understandings. In a sense, the preceding discussion can be framed in terms of external cultural influences on the microculture of the classroom, which, in turn, had ramifications on students' mathematical understandings.

Just as it is too simplistic to look solely towards the instructor for the manner in which technology is or is not integrated into the classroom, it is naive to think that all modes of integrating technology in the classroom will significantly affect students' understandings. How would, or rather could, the integration of technology affect students' learning? To begin to address this, the mathematical discussions that went on in the target classroom *without* technology are considered. The assumption is that understanding current instructional practices without technology will help inform future instructional practices *with* technology.

From the sociocultural perspective, classroom interactions and the discourse students participate in constrain and enable the development of mathematical understandings. In the class under investigation, interactions were typically one directional--the instructor talked and the students (presumably) listened. Schoenfeld (1988) describes how this type of instruction typically leads to the belief that only the very bright are capable of creating mathematics or really understanding mathematics. As a corollary, students must then accept what is passed down from above with the expectation that they can make sense of it for themselves (p. 151). The students in this study demonstrated the belief that mathematics must just be "accepted" on several occasions.

It is also informative to consider examples of classroom interactions that involved exchanges between the instructor and students. The vast majority of such exchanges can be characterized as "question asked-question answered." The instructor was respectful of students' questions, but they were not used as a springboard for further discussion or mathematical development. The point is not to denigrate the instructor' s intentions, but rather to shed some light on the mathematical activity and discussion that is a factor in the development of students' understandings.

For example, when covering the linearization of the system of differential equations that model the behavior of the pendulum, the instructor drew a picture of the situation, noted the variables involved, wrote down the nonlinear differential equation, calculated the Jacobian, and concluded that the linearization is $x' = y$ and $y' = -x$. One of the students in the study then asked, "How did you get the $x' = y$ and $y' = -x$?" The instructor's response was,

Instr: Because that' s the rule. Remember that the linearized system is $x' = Ax + By$ and $y' = Cx + Dy$ and the partial of G with respect to y is 0 and the x' is already linear so you don't have to touch it.

Unfortunately, we never get to find out what this student' s real question is. Is it a conceptual question? Is he just having trouble following the procedure? We don' t know. The results from the fourth interview do suggest, however, that students don' t understand the notion of linearization much beyond being able to do it. We also know that the instructor' s response was accepted as sufficient justification and sheds some light on what it means to understand mathematics in this course. Besides capturing this procedure-based approach, the preceding exchange illustrates the implicit acceptance of those involved regarding what counts as an acceptable mathematical explanation and justification. The response, "Because that' s the rule," to Robert' s question was accepted as a mathematically sufficient justification. Such implicitly established norms are referred to as "sociomathematical norms" (Yackel & Cobb, 1996).

Conclusions

Educational research in the teaching and learning of undergraduate mathematics is a relatively recent phenomenon and the topic of differential equations is even newer. As with the start of many new areas of research, a firm grounding regarding students' conceptions is an essential component for future curricula and instructional approaches. The information gained on students' understandings is one of the strengths of this study. This study did more than simply document students' difficulties, however. The emergent theoretical perspective was used as a means to understand and interpret these difficulties. Understanding the nature of students' difficulties is the first step to thoughtful instruction and curricula designed to help students through the difficult process of constructing new knowledge. Lastly, technological advances continue to offer opportunities to rethink and reorganize what and how we teach.

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IMAGES AND DEFINITIONS FOR THE CONCEPT OF INCREASING / DECREASING FUNCTION

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ABSTRACT

Definitions and images, as well as the relation between them of the increasing / decreasing function concept, were examined in 180 Israeli Arab high school students. A questionnaire was designed to explore the cognitive schemes for the increasing / decreasing function concept that becomes active in identification problems. One of the research questions aimed to check whether the students knew to define the concept of increasing / decreasing function. Another question was whether the students knew how to apply the definition of the concept for specific functions. A third question investigated whether the students knew that the increasing / decreasing property was local and not global. The results show that 68% of our sample knew the definition, but only between 36% to 64% of the students knew how to apply it.

This study examins several aspects of the images and definitions that junior high school students have regarding the increasing / decreasing function. Concept images and concept definitions (henceforth called *images* and *definitions*) have been discussed in detail in several papers (Vinner & Hershkowits, 1980; Tall & Vinner, 1981; Vinner, 1983; Vinner & Dreyfus, 1989: Vinner, 1991; Rasslan & Vinner, 1997). We will therefore introduce them here very briefly. All mathematical concepts except the primitive ones have formal definitions. Many of these definitions have been introduced to high school or college students at one time or another. The student, on the other hand, does not necessarily use the definition when deciding whether a given mathematical object is an example or a nonexample of the concept. In most cases, he or she decides on the basis of a concept image, that is, the set of all the mental pictures associated in his / her mind with the name of the concept, together with all the properties characterizing them.

The concepts of the increasing as well as the decreasing function are central in the chapter about functions and their graphs. In many countries, including Israel, the chapter on functions and their graphs is taught in the ninth grade. The topic is mentioned again and again in high school courses and elementary college courses (pre-calculus and calculus). In most mathematical textbooks one can find definitions such as the following: *A function f is said to be increasing on the interval I if for every two numbers x_1, x_2 in I: $x_1 < x_2$ implies $f(x_1) < f(x_2)$. A function is said to be decreasing on the interval if for every two numbers x_1, x_2 in I: $x_1 < x_2$ implies $f(x_1) > f(x_2)$.* (Salas and Hille, 1974, p. 122).These definitions are algebraic, formal, rigorous and general.

Sometimes, in order to present a new concept, authors of mathematics textbooks limit themselves first to a "special case" which is supposed to illustrate the rigorous definition.Simple examples of concepts can be found in these textbooks such as derivative, even / odd functions, and increasing / decreasing functions as well. The "special case" in our instance was the power function of the form $f(x) = x^n$ where n is natural. The definitions of the increasing function or the decreasing function concepts were examined rigorously by observing the even power function of the form $f(x) = x^n$ where n is even and the odd power functions of the form $f(x) = x^n$ where n is odd. The "special case" approach frequently causes serious difficulties in the formulation and the application of concept definitions (Rasslan, 1996; ; Rasslan and Vinner, 1997; Vinner and Dreyfus, 1989). In a previous study (Ben David, 1986), where a close observation was made in a 10^{th} grade class, it was found that in the mathematical textbook the universal quantifier "for every x_1, x_2" was missing from the definition of the increasing / decreasing function concept. It was also found that there seemed to be a tendency among high school students to use as a definition of the concept increasing / decreasing function specially the verbal definition "If x increases y increases then the function increases" and "If x increases y decreases then the function decreases". It was found also that there seemed to be a tendency among high school students to link the increasing / decreasing function concept with positive / negative elements. Six students out of 26 showed this tendency. They tended to think that a function increases in a certain domain where the y values are positive ($f(x) > 0$) and a function decreases in a certain domain where the y values are negative ($f(x) < 0$). We were curious to see whether the " positive for increasing" or / and " negative for decreasing" misconceptions would be found in a larger and more heterogeneous sample.

In another study (Vinner, 1991) it was found that for a considerable percentage of students (29%) at certain university, defined the increasing function concept by using the above literal definition. It was found also that 30% of the students at that university had defined the increasing function concept by using the formal definition but the universal quantifiers "for every x_1 , x_2 " were missing.

Many of the difficulties students have with mathematics are a result of communication failure. The *pseudo-conceptual behavior* and the *concept substitute phenomenon* discussed in detail in several papers (Vinner, 1997; Vinner, 1994; Rasslan and Vinner, 1997; Rasslan & Vinner, 1995) are examples of such a failure. The *pseudo-conceptual* behavior might give the impression that such behavior is based on conceptual thinking but, in fact, it is not.

The ***criterion*** as a concept substitute is a special case of the concept substitute phenomenon. The concept substitute is a common tendency in many students to avoid concepts. It is a typical class situation in which students have to face concepts. This is seen, when the teacher asks them "What is ...?". This study investigated the following:

1. What are the common definitions of the increasing / decreasing function in a certain domain concept given by high school students?
2. What are the main images of the increasing / decreasing function in a certain domain concept that these students use in identification tasks?
3. What are the main misconceptions that these students have according to the increasing / decreasing function concept?

METHOD

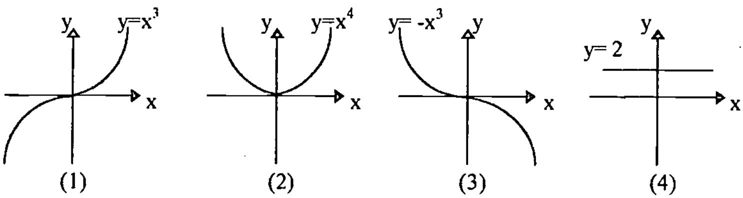
Sample

Our sample comprised six classes of Israeli Arab students, three classes of 10^{th} graders and three classes of 11^{th} graders. The total number of students was 180.

The Questionnaire

The Questionnaire in figure 1 was administered to all subjects in the sample. Questions 1 through 3 were designed to examine some aspects of the increasing / decreasing function images of the respondents, whereas Question 4 was designed to examine their definitions. Questions 1 and 2 were designed to examine the ability to reason and the ability to apply the definition of the increasing / decreasing function concept. Question 3 was designed to examine whether the students realized that "increasing / decreasing" is a local property of a function and not a global one. We believe that no special preparation is required in order to answer these questions; only an understanding of the basic mathematical language is needed. Such an understanding is a necessary condition to any mathematics lesson.

1. In the following figures, 4 functions and their graphs are given. Which one of these functions increases everywhere? Explain your answer.



2. Which one of the functions in question 1 decreases everywhere ? Explain your answer.
3. The following function is given: $f: R \rightarrow R$

   $f(x) = x^6$

   Yussef said: the given function is increasing.
   Munir disagreed and said: Yussef's answer is wrong. The given function is decreasing.
   What is your opinion? Who is right, Yussef or Munir? Explain your answer.
4. In your opinion, what is an "increasing function in a certain domain"?

Figure 1. The questionnaire

Procedure

The questionnaire was administered to the students in their classes. They were not asked to fill in their names, only their background information. It took them 25 minutes at most to complete the questionnaire. About 40 randomly chosen questionnaires were analyzed in detail by the two authors in order to determin the answer categories. On the bases of this analysis the rest of the questionnaires was analyzed by the first author.

RESULTS

The Definition Categories

We categorized the students answers according to methods described elsewhere (Vinner, 1983; Vinner, 1989; Vinner & Dreyfus, 1989; Rasslan & Vinner, 1995; 1997) when dealing with other concepts (function, slope, even / odd function). We illustrated each category using a number of sample responses.

Question 4

Category I: An algebraic definition (with the use of the universal quantifiers) (1%)
Example: *An increasing function is one which satisfies: for every $b > a$, $f(b) > f(a)$.*
Category II: A condition about the function derivative (3%)

Example: *An increasing function is a function whose derivative is positive.*
The students used here the derivative as a concept substitute. These students have the tendency to substitute the concept by working criterion (or mathematically by a sufficient condition).

Category III: Increasing means a positive slope (2%)
Example: *An increasing function is one whose slope is positive.*
This category actually is the visual version of the previous category (II). Here, the notion of a positive derivative was replaced by the notion of a positive slope. One of the meanings associated with the derivative concept is the slope of the tangent to the function graph. Some students spoke about the slope of the function instead of the slope of the tangent. It is not clear whether this is a result of tendency to use short expressions or is an indication of fuzzy ideas.

Category IV: If x increases y increases (41%)
Examples: *1. It is a function which satisfies the condition: If x increases then y increases, and if x decreases then y decreases. 2. Whenever the domain increases the range also increases, and whenever the domain decreases the range decreases.*
The use of short expressions instead of the formal definition of concepts has been discussed in detail in several studies (Rasslan, 1996; Vinner, 1991). These formulation present a dilemma to us as mathematics teachers. On one hand, we would like to use short expressions because it is a nuisance to repeat the full definition every time there is a need to clarify or to remind the concept. On the other hand, this might lead to a meaningless repetitions of short expressions which students easily pick up and use for the sake of their survival.

Category V: A combination of a definition with a concept substitute (21%)
Examples: *1. It is a function which satisfies that if x increases then y increases and its slope is greater than 0. 2. It is a function in which its slope and its derivative are positive* $a > 0$ *and satisfies* $f(A) > f(B)$, $A > B$.
These formulations present to us as mathematics teachers, a dilemma in addition to the previous one (Category IV). On one hand, it is possible that students who use dovetailed formulations want to demonstrate their knowledge. On the other hand, the students do not distinguish between the definition and its implications (More about this can be found in Vinner, Linchevsky and Karsenti, 1993).

Category VI: Incorrect answers based on pseudo-conceptual mode of thinking (26%)
Examples: *1. It is a function under 0 or over 0. 2. It is a function in which the value of its element is greater than the value of its previous element* $f(x+1) > f(x)$. *3. It is a function which is under x axis. 4. It is a function in which its slope is positive and always under x axis. 5. It is an even function.* 7% of the students did not answer the question at all.

From the above categorization it turns out that at least 68% of the students (categories I - V) knew the definition of the increasing function concept. This might be considered as an extremely good result. However, if one takes into consideration the most striking fact that only 1% of our sample knew the formal definition and that students in categories II and III confused definitions with criterions for these definitions, and that students in category IV used short expressions which do not necessarily indicate satisfactory knowledge of the concept, then the entire picture is not so encouraging. About the rest (category VI, and those who did not answer the question), which are 32% of the students, we can claim that they did not know the definition of the increasing function concept.

The Concept Images. Questions 1-3

Various aspects of the increasing / decreasing function concept, as conceived by the students, were expressed in their answers to questions 1 to 3. Some of these aspects are given below:

Question 1

Category I: An explanation based on the right algebraic definition (with the use of the universal quantifiers) (1%)

Example: *The function which increasing everywhere is the function $y = x^3$ because it is satisfies the condition: for all $b > a$, $f(b) > f(a)$.*

Category II: An explanation based on the condition about the function derivative (5%)

Example: *1. $y = x^3$ is always increasing because its derivative is positive. 2. $y = x^3$ is always increasing because its derivative $y' = 3x^2$ is positive and the power is even.*

Category III: Increasing means a positive slope (3%)

Example: *The function $y = x^3$ is a function in which its slope is positive.*

Category IV: If x increases y increases (27%)

Examples: *1. $y = x^3$ is the function which satisfies the condition: if x increases then y increases and if x decreases then y decreases. 2. $y = x^3$ is the function which satisfies the condition: if the domain increases the range also increases, and if the domain decreases, the range decreases.*

Category V: A combination of a definition with concept substitute (21%)

Examples: *1. The function $y = x^3$ is the function which satisfies the condition: If x increases y increases and its slope is positive. 2. The function $y = x^3$ is the function in which its slope and its derivative are positive ($a > 0$) and satisfies: $f(A) > f(B)$, $A > B$. 3. The function $y = x^3$ is the function which satisfies the condition: If x increases then y increases and its derivative is always positive.*

Category VI: An explanation based on a well known family of increasing functions - The odd power functions (3%)

Example: *The function $y = x^3$ is increasing everywhere because it is an odd power function.*

Category VII: An explanation based on the visual arguments. (4%)

Examples: *1. The function is $y = x^3$ because it increases from down to up. 2. The function is $y = x^3$ because it increases from left to right. 3. The function is $y = x^3$ because it passes from the third quarter to the first one.*

Category VIII: Right answers without explanations (12%)

Example: *The function which increases everywhere is $y = x^3$.*

Category IX: A wrong answer or a non sensical explanation (22%)

Examples: *1. The function which increases everywhere is $y = x^4$ because it is under the x axis. 2. The function which increases everywhere is $y = x^3$. When $x > 0$ the function increases. 3. The function which increases everywhere is $y = x^3$, because x is less than 0. 4. The function $y = x^3$ increases everywhere according to the law $f(x) = -f(x)$ as well as y axis intersects the function.*

32% of the students in this category (13 students out of 40) linked between increasing and being positive. 2% of the students did not answer the question.

From the above it follows that about 64% of the students (categories I-VII) know to apply the definition of increasing function concept. As to other 12% of the students (category

VIII) we cannot claim this, but we also cannot claim the opposite. About the rest (category IX, and those who did not answer), 24% of our sample, we can claim that the increasing function concept is not clear to them.

Question 2

The categories of this question were supposed to be the same as in question 1. This was true in the majority of the cases. However, 7% of the students who gave a right answer without reasoning (category VIII) in question 1 gave a wrong answer in question 2. It is another indication that a right answer without reasoning is not an indication of a true understanding of the concept as described elsewhere (Rasslan, 1996; Rasslan & Vinner, 1997; Rasslan & Vinner, 1995; Vinner & Dreyfus, 1989) when dealing with other concepts. Moreover, 1% of the students of our sample who answered question 1 correctly gave a wrong answer to question 2. This happened when these two students used once again the short expression "*If x increases then y increases*" in order to explain why a function is increasing. The second time they added: *The function $y = -x^3$ decreases everywhere because $x > 0$*. From the above, it follows that about 31% of our sample we can claim that the concept of decreasing function was not clear.

Question 3

Category I: The students understand that a function can increase and decrease in different domains (36%)

Category I_a: An argument related to the function derivative (8%)

Example: *Both are wrong. The function increases when $x > 0$, $f'(x) = 6x^5$ and decreases when $x < 0$.*

Category I_b: The reasoning of the students is based on the general property functions of the type $y = x^n$ (where n is natural and even) increase when $x > 0$ and they decrease when $x < 0$ (7%)

Example: *This function increases in one domain and decreases in another domain because it is an even power function. Yussef's answer as well as Munir's is wrong.*

Category I_c: An argument based on the visual aspect of increasing / decreasing concept (21%)

Examples: *1. The function increases when $x > 0$ and decreases when $x < 0$. 2. The function increases in R^+ and decreases in R^-.*

Category II: The student draws a "parabola" or mentions the word "parabola" and refers to the parabola properties (19%)

Examples: *1. Both are wrong, because the function $f(x) = x^6$ is a parabola, the graph of $f(x) = x^4$ similar to it and it is known that this function increases in $x > 0$ and decreases in $x < 0$. 2. This function increases in the domain $x \geq 0$ and decreases in the domain $x \leq 0$ because its graph is a parabola.*

Category III: Answers which can be considered as right but they lack specifications or explanations (3%)

Example: *Yussef is right, the function increases and Munir is right; the function decreases.*

Category IV: A correct reference to a specific aspect of an increasing function with a failure to link it to the task (8%)

Examples: *1. Yussef's answer is the right, $y' = 6x^5$, it means that $a > 0$. 2. The function increases because if x increases then y increases. 3. Yussef's answer is right because the slope of the function $a > 0$, it means that the function increases in the domain $f: R \to R$.*

Category V: A confusion between increasing and being positive (x^6 positive, x positive, y positive) (10%)
Examples: *1. The function increases because x is positive. 2. I agree with Yussef because the exponent is always positive. 3. Yussef's answer is right because $x^6 > 0$.*
Category VI: Incorrect answers based on erroneous understanding or an incorrect explanation or meaningless answers (16%)
Examples: *1. In my opinion Yussef's answer is right. If we substitute x and multiply by 6 the result increases. 2. Yes, Yussef, because R is all the integer numbers. 3. Yussef's answer is right because if we substitute x by any number the number increases. 4. Yussef's answer is right because if we solve the function, it will be x = 6, then it increases, it does not decrease.*
7% of the students did not answer the question.

From the above, it turns out that about 60% of the students in our sample (categories II, IV, V, IV and those who did not answer), we can claim that they did not know that the increasing / decreasing of a function is a local property and not a global one. About other 3% of the students (category III) we cannot claim it, but we also cannot claim the opposite. As to the rest (category I), 36% of our sample we can claim with certainty that they knew that increasing / decreasing is a local characteristic and not a global one.

When we discussed category IV in question 4 we claimed that the use of short expressions in the definition of increasing function in a certain domain is not necessarily an indication of convincing knowledge of the concept. Therefore, we examined the answers to questions 1 and 3 of these students who used short expressions in thier definitions of a increasing function (Question 4). We found out that 15% and 30% of them gave incorrect answers to questions 1 and 3, respectively.

One of the goals of this research was to examine the tendency to link the increasing / decreasing function concept with being positive / negative. In the analysis of the answers to Questions 1 and 3 above, it turned out that between 7% - 10% of our sample had this misconception.

DISCUSSION

One of the goals of this study was to expose some common images of the increasing / decreasing function held by high school students. This has a direct implication for teaching. If one wants to teach the increasing / decreasing function to a group similar to our sample, it is important to know the starting point of the students (Rasslan & Vinner, 1997, Vinner & Dreyfus, 1989). Taking into account the difficulties mentioned in this study, at least some doubts should be raised whether the "special case" approach to the increasing / decreasing function in a certain domain concept is the most effective way for teaching such a concept. The pool of examples introduced to the students should include many different examples. Only this may increase the chance that one example will not become a prototype and as such also a concept substitute. A similar conclusion was mentioned also by Vinner and Dreyfus (1989) according to Dirichlet-Bourbaki's approach to the function concept, and in Rasslan (1997, 1996) according to other concepts, such as the even / odd function concept.

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WHY IS PROOF BY CONTRADICTION DIFFICULT?

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Abstract: The reasoning underlying proofs by contradiction is less difficult than it is often thought to be. Children reason using contradictions in game playing and in checking conjectures. This suggests that the difficulties students have with standard proofs by contradiction in mathematics may arise from issues of emotioning, especially the need from which their reasoning arises.

Barnard & Tall (1997) remind us that students often have difficulty with the standard proof that $\sqrt{2}$ is irrational. That students have difficulty with this proof is something that most teachers of upper level mathematics have seen. Barnard and Tall suggest a model of the reasoning involved in the proof which emphasizes the importance of collapsing concepts such as "evenness" into procepts. They conjecture that inability to reason at a proceptual level is the key to students' difficulty with this proof.

This may be so. It does not, however, address one assertion they make, that proof by contradiction is *itself* difficult and that students are "unfamiliar with the possibility of proving something true by initially supposing it to be false" (p. 43). In this paper we would like to argue that proof by contradiction is not all that difficult in terms of the reasoning involved, but that many standard proofs (such as the $\sqrt{2}$ proof) *are* difficult because of the emotioning involved.

Emotioning

Antonio and Hanna Damasio (Damasio, 1994) examined cases of patients who suffered damage to their brains that left their intellectual capacities unchanged, yet kept them from functioning in a normal way. The Damasios conjecture that it is damage to these patients' emotioning capacity, the capacity to care about the decisions they make, that is disabling. They point out that any decision we make is a choice between a huge number of possibilities, most of which we do not even consciously consider because they have already been rejected by a pre-conscious emotioning process. Options which feel wrong are not examined to see if they are reasonable. The Damasios' work provides a neurological basis for a process which is familiar from the work done by Hadamard and Maturana. Hadamard notes that the process of mathematical invention is primarily one of making decisions — choosing between the numerous combinations of ideas our minds are able to produce. And Hadamard notes that "an affective element is an essential part in every discovery or invention." (1945, p. 31) It is also integral to Maturana's arguments, especially when he asserts that scientific investigations involve reasoning which *comes about* as a result of emotions. "Reason drives us only through the emotions that arise in us" (1988, p. 50).

Reasoning with contradiction.

Recently we had the opportunity to play a card game called Set with small groups of 7 year old children. The object of Set is to find sets of three cards which are either all the same or all different in terms of three "parameters": Colour, Shape and Number. (There is a fourth parameter, Shading, which was left out when playing with the children. In the figures for this paper it is used to represent colour. Purple is solid, green is striped and red is hollow.) For example, in Figure 1 the three cards form a set in which all parameters are different:

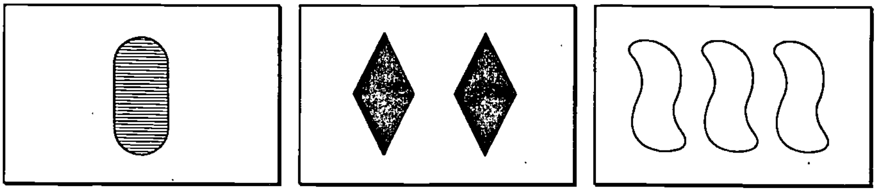

Figure 1: One green bandaid, two purple diamonds, three red worms*.

Figure 2 shows another set in which the Number parameters are all the same, but the Colour and Shape parameters are all different:

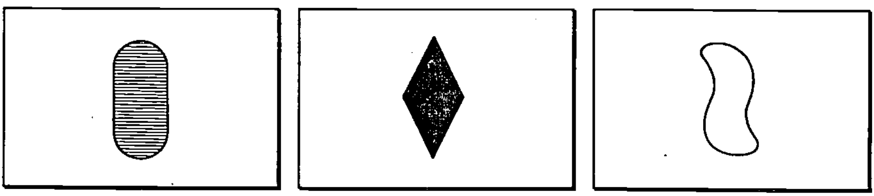

Figure 2: One green bandaid, one purple diamond, one red worm.

The three cards in Figure 3 do not form a set, because the Colour parameters are not all different or all the same. There are two Green and one Purple.

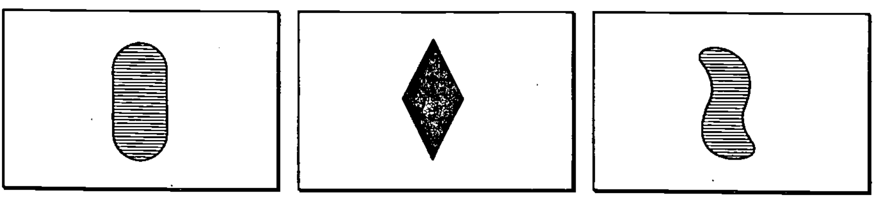

Figure 3: One green bandaid, one purple diamond, one green worm.

There is a lot of scope for mathematical activity in playing Set, including opportunities to construct proofs by contradiction. For example, consider the six cards in Figure 4.

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* The names "Bandaids," "Diamonds" and "Worms" were chosen by the children.

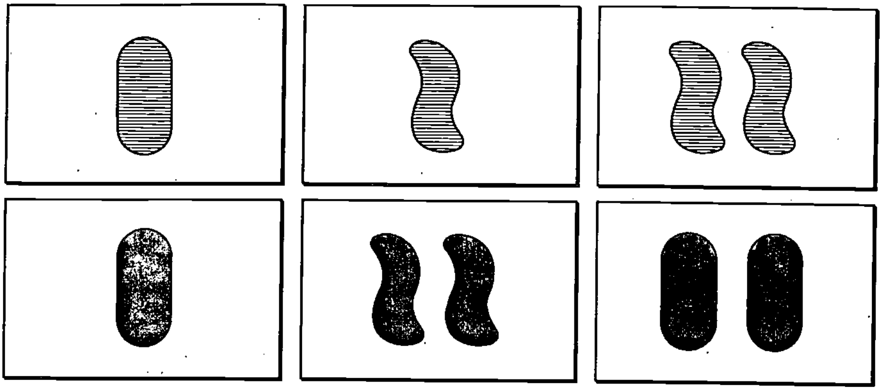

Figure 4: The six cards remaining at the end of a game

These were the six cards remaining after one game of Set we played with the children. (Normally there are eight cards on the table, but at the end there are only six left.) Several deductions can be made. For example, because there are no cards with three shapes we can conclude that any set must be all the same number. This leaves two possibilities (two green worms, two purple bandaids, and two purple worms or one green bandaid, one purple bandaid and one green worm). Neither of these are sets, so we could conclude that there is no set, by the following simple proof by contradiction:

1. Assume a set exists.
2. There are no cards with three shapes.
3. From 2 we can conclude that a set with all three numbers is impossible.
4. From 3 we can conclude that the set contains three cards with the same number.
5. Assume that the set contains only cards with two shapes.
6. From 5 we can conclude the set is two green worms, two purple bandaids, and two purple worms which is not a set, therefore one of our assumptions (1 or 5) was wrong.
7. Assume assumption 5 was wrong.
8. From 7, 4 and 3 we can conclude the set must contain only cards with one shape.
9. From 8 we can conclude that the set is one green bandaid, one purple bandaid and one green worm which is not a set, therefore one of our assumptions (7 or 1) was wrong.
10. If assumption 7 was wrong, then 6 implies that 1 must be wrong, so either way our initial assumption that there is a set leads to a contradiction.
11. Conclusion: There is no set (by contradiction).

This proof by contradiction may seem unnecessarily detailed. We have written it in this way to assist the reader in interpreting the following transcript of the discussions between four girls, Kelley, Lynn, Arial and Meg, and the two authors (DR and JD) concerning the existence of a set in the cards shown in Figure 4. The transcript has been edited for clarity, with short omissions marked with ellipses (...) and longer omissions marked with asterices (***). A long dash (—) indicates a pause. References to the steps of the argument listed above are inserted where appropriate, in parentheses and italicized; e.g., (*Step 1*). The reader should also keep in mind that no transcript can capture the complexity of four children and two adults interacting. For example, no attempt has been made here to indicate cases where more than one person was speaking at a time.

1.	DR:	Now we've only got four cards left so this is going to be hard folks because there is only going to be six cards not eight.
2.	Arial:	No set.
3.	DR:	Do you think there's not a set Arial? Do you have a reason for that or do you think that just because you can't see—
4.	Lynn:	Because there are no threes, look. (*Step 2*)
5.	DR:	Because there's no threes
6.	Kelley:	I'll show you. But can we do it by twos?
7.	DR:	If they were all twos that would be okay.
	***	
8.	DR:	... Does it matter that there's no threes?
9.	Arial:	No
10.	DR:	Why is it O.K.?
11.	Arial:	Because you could get three of the same number.
12.	DR:	Okay
13.	Meg:	I know two sets.
14.	DR:	Okay Meg, can you tell us what your two sets are? [Meg arranges the cards into the 'sets' {two green worms, two purple bandaids, two purple worms} and {one green bandaid, one purple bandaid and one green worm}.]
15.	DR:	Okay now, does everybody agree that those are sets? Oh, Kelley's got a problem. Kelley
16.	Kelley:	Because these two are the same shape and that one's different and these two colors are the same.
17.	DR:	Do you understand Meg? [Meg folds her arms together and makes a displeased face.]
18.	JD:	Does that mean yes?
19.	DR:	... So that's not a set. So is there a set in here?
20.	Lynn:	No
21.	Kelley:	Yes. There somewhere is.
22.	DR:	Okay let's keep looking then

At this point they had enough information to conclude that there is no set, but no one has come to a definite conclusion. It is clear however that they understand that the absence of cards with three shapes is important, and that they can explain why Meg's "sets" were not properly formed. After they all looked again for sets, Kelley suggested {one green bandaid, one purple bandaid, two purple bandaids} and Lynn explained that it was not a set "Because there are two ones and one two. And you don't get one-one-two." Then Arial repeated her claim (originally made at line 2) that there was no set possible.

23. DR: ...Arial is about to explain why there are no sets so we can go on and play again.

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24. Arial: If there's two here and two here and you get all ones. (*Step 5*)
25. Kelley: Then there's no sets there because they're all different things (*Steps 6-7*)
26. DR: So if you put all the twos together, it's not a set?
27. Kelley: Can we do two sets? Since there's only six?
28. DR: No what we'll do is we'll start over.
[Lynn and Meg are playing with their own little collections of cards and are not participating in the conversation about why the remaining cards do not generate a set.]
29. Arial: You see a set, then you would just put these together and then you would have to put like two ones together, it would have to be like six cards together but you could only have three. So then you put this one in front of this one but there's- you'll need one more of these. (*Steps 8-9*)
30. DR: One more green one.
31. Arial: Yeah but. Oh no.

In line 24, Arial is taking two of the cards with two shapes, and noting that the third card with two shapes would not form a set, and that the remaining cards could not form a set with the two cards because they have only one shape. Her observation at line 11, "you could get three of the same number," seems not to have gone as far as Step 3 which says you *must* get three of the same number. In line 25 Kelley seems to have understood Arial's argument as showing that there is no set possible using the cards with two shapes.

Line 29 contains several deductions in the form of a proof by contradiction. Our interpretation of her first sentence is: "You see, [if there was] a set, then you would just put these [two worm cards] together [and then add the one green worm card to go with the two green worm card] and then you would have to put like two [green] ones together, [and finally] it would have to be like six cards together [to make sure every card was with another card that matched two parameters] but you could only have three [which contradicts the rule that a set is made up of three cards]. Note that Kelley proposed changing this rule (in lines 6 and 27) to make it possible to make "two sets", with two cards in each set. Arial's last sentence in line 29 concludes the proof by contradiction as listed above, by showing that there is no set composed only of cards with one shape.

This is the most extended case of an argument involving contradiction we observed when playing Set with these children, but in every game the children made use of simple arguments based on proposing a hypothesis and then showing it led to a contradiction. Game playing is not, however, the only context in which reasoning by contradiction is used.

Students doing mathematics use reasoning by contradiction spontaneously whenever they check a calculation and find they have made an error. For example, Bill, a 15 year old, used reasoning by contradiction when checking a conjecture he made that $27 \div 18 \times 11 \approx 17$ provides a model for a general solution to the Arithmagon problem* (see Figure 5).

The numbers on the sides of this triangle are the sums of the numbers at the corners. Find the secret numbers.

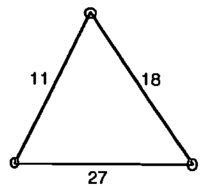

Figure 5: The Arithmagon Problem prompt given to Bill.

Bill: Let's say you had 3, 6, 17. 17 divided into 6, times 3, equals 8.5. But would it be 9? — No, how could it be 9? Because you, you get 6. Unless this was, um, negative three. But then you get 3 from here. You would have to have, um. To get a negative, you have to have 6. But then 9 and 6 is 15. It would not work.

Bill assumed his method was general, and then tried it on a triangle with the numbers 3, 6, and 17. When this didn't produce a solution he concluded that his original assumption was false.

Formal proofs by contradiction are difficult for many students, however, the ease with which quite young students use contradiction in arguments suggests that it is not the reasoning itself which causes the problem. Barnard & Tall's suggestion that inability to reason at a proceptual level interferes with students' ability to successfully use proof by contradiction in some contexts in quite plausible, but we believe that a factor which is more important generally is that of *need*.

Needs to prove.

So far we have considered the logical form of the reasoning used in Set and by Bill. But the important issue when trying to understand what makes proof by contradiction difficult is not the form of the reasoning, but rather the emotioning that goes with it. In talking about the emotioning involved in proving, we use the

* A more complete description of Bill's reasoning can be found in Reid, 1995a.

term "needs." Others have also written about the needs which proving can satisfy, especially the need to explain (e.g., Hanna 1989, de Villiers 1991, Reid 1995b).

How is the emotioning involved in Bill's reasoning and in playing Set different from that in the standard proof of the irrationality of $\sqrt{2}$? What needs are involved in each case? In Bill's case he had made a conjecture, but he was not certain of it. He had a need to *verify* . Note that this need would have been satisfied no matter what the result of his checking was. If it turned out that his method did result in a right answer in this case he might have concluded that his method was generally valid. In this case inductive reasoning would satisfy his need to verify. As it happened, however, he disproved his conjecture, using the deductive reasoning involved in a proof by contradiction. The reasoning in the two cases differs, but the need underlying that reasoning is the same.

In Set there is usually a fair amount of inductive evidence that there is no set before effort is expended on deducing a contradiction. Individual cards would have been checked and eliminated. The need here is a need to verify. Inductive reasoning has provided only partial verification. In this case however the distinction between inductive reasoning providing probable verification, and deductive reasoning providing definite verification is clear. Reasoning by contradiction satisfies the need to verify almost completely. Only the inevitable uncertainty that an error in reasoning has been made remains.

When a student is asked to read a proof or to prove by contradiction that $\sqrt{2}$ is irrational, what need drives that proving? It is very rare that a need to verify comes into play. To feel a need to verify one must be uncertain of the result. In the case of $\sqrt{2}$ it is unlikely that there is any uncertainty at all. In fact, embedded in the students' concept image of irrationality is likely to be the fact that $\sqrt{2}$ is irrational. It is a canonical example (along with π) of an irrational number. For many students "irrational" *means* "numbers like $\sqrt{2}$ and π." The need that drives students to prove that $\sqrt{2}$ is irrational is something other than a need to verify. It is likely to be a need to function in a social context, like a classroom in which proving is an activity done primarily to ensure good marks, not to verify or explain mathematical statements.

Conclusion

We do not mean to claim that the conceptual difficulties noted by Barnard and Tall are not real. We do wish however to emphasize that when we examine students' reasoning we must not do so in isolation from their emotioning. Students who have flexible procepts of the concepts involved in the standard proof of the irrationality of $\sqrt{2}$ may be better able to produce the proof, but they will be doing so only because they have been asked to do so by their teacher. It is perhaps those students who do not *know* that $\sqrt{2}$ is irrational who are in a better position to prove that it is so.

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SELF-EFFICACY BELIEFS AS MEDIATORS IN MATH LEARNING : A STRUCTURAL MODEL

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This paper reports an attempt to examine and describe students' mathematical beliefs at undergraduate level. Our findings indicate that self-efficacy beliefs play a mediating role in the influence of prior mathematical experiences on motivational beliefs and mathematical achievement.

During the last ten years researchers have stressed the role of students' beliefs in the teaching and learning of mathematics. This is part of a renewed interest in the affective aspects of mathematical education, as summarized in McLeod & Adams (1989), McLeod (1992), Pehkonen & Torner (1996). McLeod (1992) presents an overview of studies relating to the affective domain and describes four categories of beliefs studied in the literature, beliefs about mathematics, beliefs about oneself as a learner, beliefs about mathematics teaching and beliefs about the social context. The beliefs students bring into the learning situation, determine both how they approach mathematics and the mathematical skills they are motivated to acquire and use.

Purpose of the study

The purpose of this study is to examine relations between student beliefs about oneself as learners of mathematics and beliefs about mathematics. We hypothesize that beliefs related to self-efficacy judgments of own capacity will play a mediating role in the influence of mathematical experience on motivational beliefs and achievement. We also want to examine the interplay between belief constructs used by researchers coming from different theoretical positions.

Theoretical framework

Beliefs about mathematics are closely related to motivational issues. Eccles and her colleagues have developed a model of academic choice and achievement based on expectancy-value theories of behavioral choice as developed by Atkinson as his colleagues (Eccles (Parson), 1983). This motivational model links students achievement to their expectations for success on the task and the value they attach to the task. The expectancy component may be considered to include students beliefs about their ability to perform the task, and the achievement value part may be considered to be composed of intrinsic value, utility value, attainment value and cost, (Wigfield, 1994, Pintrich & DeGroot,1990). The intrinsic or interest value is related to the enjoyment in performing the activity and the

interest in pursuing the task. The cost value may be compared to negative emotional aspects like anxiety relating to engaging in the task.
The other main component of beliefs in this study relates to beliefs about oneself as learner of mathematics. Expectancy judgments are conceptually related to estimates of self-confidence and self-efficacy in achievement situations. Measures of self-confidence or self-concept have a long history in studies on math learning (McLeod,1992). The term self-efficacy is used in social cognitive theories to describe how a person think he is able to accomplish on a given task (Bandura, 1997). Following Bandura we could say that how people behave can often be better predicted by their beliefs about their capabilities than by what they are actually capable of accomplishing, for these beliefs determine what individuals do with the knowledge and skills they have. The term self-concept may be seen as a global estimate of efficacy, compared to self-efficacy giving a more domain and task-specific measure of efficacy. Another aspect of self-efficacy considered to play an important role in academic self-motivation are students' self-beliefs of efficacy to strategically regulate the learning process (Zimmerman et al 1992). Historically many of the studies on affective variables like confidence and anxiety have been related to the question on gender differences in mathematical learning (Leder, 1992) . We also include in our study some preliminary results on the issue of gender.

Sample

The target group for this investigation was students at age 19-20 starting a study program in economics and business administration at a college in Norway. As part of their program the students have to take a compulsory course in calculus emphasizing applications in economics. In the first week of the fall semester of 1996 the students were asked in an ordinary class session to fill in a self-report questionnaire. Of 290 students present in class, 266 completed the form.

Instruments

The questionnaire consisted of three parts, one part for items relating to students' beliefs, one part for information on math courses in upper secondary school and one part including a test on mathematical performance.
In the affective part, the students were asked to indicate if they agreed or disagreed to statements on a four-point Likert scale ranging from strongly agree to strongly disagree. We decided to use a four-point scale as this is supposed to force a discrimination between the first two and the second two choices leaving the middle option open. On the items of self-regulation we used the original five-point scale with answers ranging from not well at all to very well. For the analysis the answers were recoded such that a high score indicated beliefs considered to be beneficial to math learning. Using an exploratory factor analysis, we identified eight factors describing the structure of student mathematical beliefs (Risnes,

1997). In this study we will concentrate on the following five belief constructs each loading on three items:

Regulation. Measures self-efficacy for self-regulated learning adapted from Zimmerman, Bandura & Martinez-Pons (1992).
Sample item: "How well can you concentrate on school subjects?"
Motivation. Measures self-efficacy as part of motivational beliefs adapted from Pintrich & De Groot (1990).
Sample item: "I'm certain I can understand the ideas taught in this course".
Ability. Variable named 'self-perceived ability to learn' in Skaalvik & Rankin (1995). This variable is analog to 'self-concept of mathematics ability' (Pokay,1996) and 'confidence in learning math' (Fennema & Sherman, 1977).
Sample item: "I can learn mathematics if I work hard".
Interest. Variable for mathematics as an interesting and enjoyable subject analog to 'intrinsic motivation' in Skaalvik & Rankin (1995).
Sample item: "I like mathematics".
Anxiety . Variable in the tradition of studies on mathematics test anxiety analog to the Fennema-Sherman math anxiety scale.
Sample item: "I feel anxious at mathematics tests".

Students prior exposure to mathematics are measured by year giving the number of years (1,2 or 3) taking a math course in upper secondary school and grade giving the grade (2 lowest, 3, 4, 5 highest) in their final math course. The variable program indicate if the students have followed a general/ academic oriented "gymnasium" track or a vocational oriented commercial/ business track in upper secondary school.
Student performance is measured by a test developed by Norwegian Mathematics Council in the 1980'ties to assess student qualifications at entering university level. The test measures basic mathematical knowledge from comprehensive school at age level 16-17. The students were to provide the answers themselves on 38 items covering the areas: arithmetic, everyday life, algebra, problem solving and geometry. The test result is measured by using as indicators the sum of the scores in each of the five subareas.
We also include in our analysis the variable sex coded female=0 and male=1.

Method

To study the relationships between variables, we apply structural equation modeling (SEM) techniques (Bollen,1989). SEM may be used to test hypothesis about relations among observed and latent variables. One advantage of SEM compared to conventional regression analysis is that it adjusts for measurement errors in the variables. We used the implementation LISREL8.14 developed by Joreskog & Sorbom (1996). The results are based on studying the covariance matrix for the indicators involved, treating the indicators as continuos variables. For estimation of the models we use estimation by the method of maximum likelihood. The effective sample size in our SEM analysis is 58 females and 170 males.

Confirmatory factor analysis

We start by presenting a measurement model for the five belief constructs identified by the previous factor analysis. In this measurement model, Model 1, each of the beliefs are treated as a latent variable loading on the three indicators given by the items in the questionnaire. Evaluating the model by the chi-square goodness-of-fit test, we find a chi-square of 131 with 80 degrees of freedom. The Root Mean Square Error of Approximation (RMSEA) measure used to assess the degree of lack of fit of the model is .053. Browne & Cudeck (1992) suggest that RMSEA values less than .08 indicates an acceptable fit and values less than .05 indicates a good fit. Based on the fit statistics we conclude that our measurement model gives an acceptable fit to the data.

Model l may be seen as a confirmatory factor analysis indicating that our constructs will give an adequate description of students' belief variables.
This makes it feasible to present a scale for each of the belief variables by taking the average of the scores on the three indicators involved. By analyzing the mean values for the five belief variables for each level of the class variables sex, program, years and grade , we find the following pattern:
The values for all the beliefs variables, except regulation, are becoming monotonicly more positive with more years of math and with better grades in mathematics. Students coming from the academic program are scoring higher on all the variables except for regulation. These findings are in agreement with what we would expect and we could say that the results are adding some content validity to our belief constructs.
A t-test gives sex differences with females expressing a more positive view than males on interest, motivation and regulation. Contrary to many other studies we do not by this test find sex differences in ability (self-concept) and anxiety.

Structural model.

The focus in this paper is to use structural equation modeling to study structural relations between our belief constructs. We hypothesize that the variable for sex and the latent variable for school background with indicators year, grade and program, will influence the belief variables and the test score. Based on self-efficacy theory we start with the assumption that the self-efficacy beliefs regulation, ability and motivation will have a mediating role on interest, anxiety and test result. By allowing for modification of the model based on deleting non-significant paths and using modification indices to include new paths, we present our structural model, Model 2. Figure 1 shows the structural part of Model 2 giving estimated path coefficients between the latent variables, omitting the indicators for each latent variable. In estimating Model 2 we have included covariances between the error terms of two of the five test indicators and between the indicators for school background. Estimation of the model produced goodness of fit statistics like chi-square of 356 with 236 degrees of freedom and RMSEA= .047. We conclude that Model 2 gives a reasonable good fit to our data.

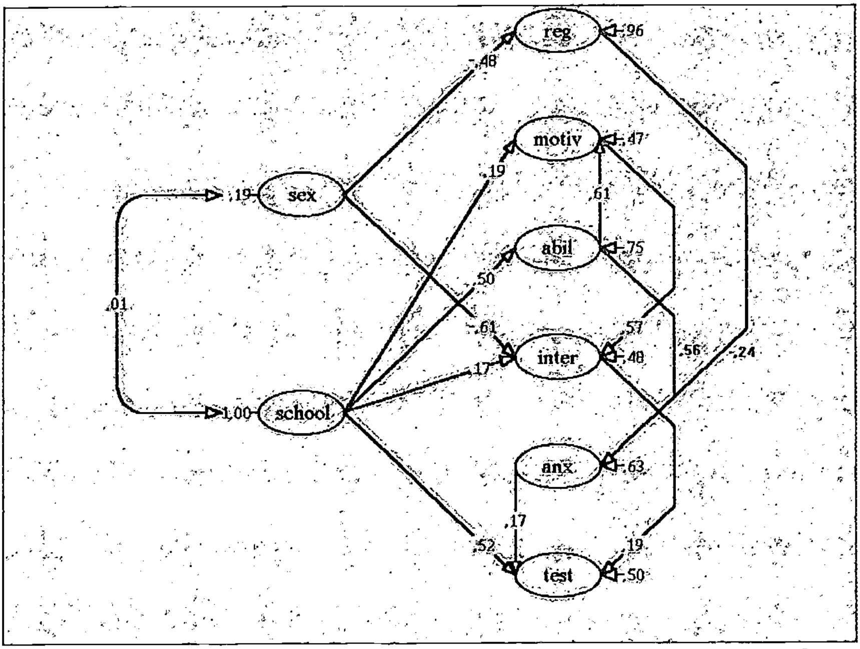

Figure 1. Stuctural model 2 with estimated path coefficients and error terms for latent variables.

In the previous analysis we found a significant sex difference in regulation and interest and in Model 2 both these variables play an intermediating role in the influence of sex on the test result. The variable school for prior exposure to mathematics, has a direct influence on test and the belief variables ability, motivation and interest, with paths from ability via motivation to interest and from interest to test. The signs of the effects are as would be expected with better background in mathematics giving more positive beliefs and better test score. Our findings are in some contrast to Stage & Kloosterman (1995) who found no significant paths from high-school exposure to belief variables. Model 2 indicate

that interest and anxiety mediates the role of self-efficacy beliefs on achievement even though this was not confirmed for anxiety in Meece et al (1990).
The variable regulation is measuring self-efficacy of regulation in a general context without any reference to mathematics and it seems reasonable that this variable is unrelated to the specific background in mathematics. In our analysis we did not find a significant path from regulation to self-efficacy measures of ability or motivation analog to findings in Zimmerman et al (1992).
We find the beliefs ability, interest and anxiety to have a significant total effect on the achievement variable test with coefficients .16, .19, .17 respectively. This may be compared to Malmviuori & Pehkonen (1996) who identifies self-confidence as the most important predictor of math achievement with a standardized regression coefficient of .265. The total effect of school background on achievement is as high as .66. In this structural model we now find that sex has a total effect on anxiety, with males showing less anxiety.
The total variance explained within Model 2 as measured by squared multiple correlation, is for test .50, interest .52, motivation .53, anxiety .37, ability .25 and regulation .04.
The correlation values between the latent variables as estimated in Model 2 are given in table 1.

Table 1. Correlation between latent variables in Model 2

	reg	motiv	abil	inter	anx	test	sex
motiv	0.00						
abil	0.00	0.71					
inter	0.05	0.66	0.49				
anx	-0.24	0.40	0.56	0.26			
test	-0.04	0.45	0.45	0.47	0.37		
sex	-0.21	0.02	0.02	-0.25	0.06	-0.02	
school	-0.01	0.50	0.50	0.45	0.28	0.66	0.03

The variable motivation for self-efficacy of motivation from Pintrich & De Groot (1990), has a nonsignificant correlation with the variable regulation for self-efficacy of self-regulated learning in Zimmerman et al (1992), indicating these variables to be rather independents traits of self-efficacy. The variable ability measuring self-concept of math ability correlates strongly (.71) with the variable motivation and it looks like these variables are highly related as measures of self-efficacy. These variables are trying to measure rather global aspect of efficacy judgments and do not fully satisfy the recommendations made by self-efficacy theorists to relate self-efficacy measures to specific tasks. The rather high correlation of .56 between ability (self-concept) and anxiety is still below the value .87 reported in Pajares & Miller (1994).

Conclusion

Based on the confirmatory factor analysis in Model 1, we conclude that the five belief variables give valid information on students' beliefs relating to the learning of mathematics. The analysis indicate that the structural Model 2 gives an adequate representation of possible relations between our variables. In this model self-efficacy beliefs are influenced by sex and prior exposure to mathematics and these beliefs will through their paths to the variables for interest and anxiety, have an indirect influence on achievement. The variable for achievement is directly influenced by school background and the motivational belief variables interest and anxiety.

We notice that judgments of personal efficacy influence the level of interest and anxiety as indicated by self-efficacy theory. It may be argued that generalized judgments related to outcome expectations like confidence and self-concept, are different from particularized self-efficacy beliefs relating to specific tasks (Bandura, 1997). We still find our self-efficacy beliefs to be rather good predictors of mathematical achievement even though our measure of achievement is not specifically related to the constructs of self-efficacy.

It would be possible to specify several models giving reasonable good fit to the data and the data themselves do not give any clear preference for which of numerous equivalent models to choose. Model 2 is based on the hypothesis that variables of self-efficacy will play a mediating role in the influence of prior mathematical experiences on the test result. We take our empirical results to give some evidence to the significance of self-efficacy theory (Bandura, 1997) in describing and understanding the influence of mathematical beliefs in the learning of mathematics.

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TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE OF GEOMETRY

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This paper reports on research done on teachers' pedagogical content knowledge (PCK) in geometry, two years after they have completed an INSET-course offered by the Primary Mathematics Project at the University of the Western Cape, South Africa. It describes two main sources of PCK, the teachers' knowledge of geometry and the teachers' knowledge of pupil learning. This study is interpreted within the context of the teachers' classroom actions and the interaction that takes place.

Introduction

The research report explores teachers' pedagogical content knowledge in the field of geometry two years after completing the Certificate in Primary Mathematics Education (CPME), an INSET course offered by the Primary Mathematics Project (PMP) at the University of the Western Cape (UWC). A research programme was established to study the impact that the CPME course had on teachers' perceptions of their practices; the teachers' knowledge of geometry teaching and learning, and the beliefs that teachers' have about the teaching and learning of geometry.

Shulman (1987), in his conceptualization of the teacher's knowledge base, places special emphasis on the pedagogical content knowledge (PCK) as a means to identify teaching expertise which is local, part of the teacher's personal knowledge and experience. Several studies points to the significance of PCK on teachers' professional growth (Feiman-Nemsher, 1991; Evan et.al., 1996). These studies show that the relationship between the teachers' knowledge of a subject and classroom practices have a significant bearing on the professional knowledge and practices of teachers.

The aims of this study were to explore:

1. The teachers' knowledge of geometry as a source of PCK.
2. The teachers' knowledge of pupil learning as a source of PCK.

The scope of the research included issues such as what content-specific knowledge of geometry teachers displayed; what knowledge teachers possessed about the understandings and strategies that learners bring to learning and the teachers' awareness of preconceptions and misconceptions that learners bring to class.

Theoretical Considerations

The teacher's pedagogical content knowledge in this study is interpreted within the context of the teachers' classroom actions. In this sense it is knowledge-in-[illegible] (Schön, 1983). Pedagogical content knowledge essentially refers to what the [illegible]r knows about teaching a specific content field. Shulman (1987) describes this knows about teaching a specific content field. Shulman (1987) describes this kind of knowledge as (a) the different ways of representing and formulating the subject matter to make it comprehensible to others, (b) understanding what makes the teaching of specific topics easy or difficult and (c) knowing the conceptions and pre-conceptions that learners bring to the learning situation. This description suggests that PCK has to be interpreted from a particular content field, in this case geometry. PCK integrates both content and pedagogy, in this instance the teacher's knowledge of geometrical ideas/concepts and their relationships, integrating it with what the teacher knows about the teaching and learning of these geometrical ideas and concepts.

Several studies have proposed a further categorization of PCK. Askew et. al. (1997) have studied teachers' knowledge of teaching styles, teaching resources, pupil responses and classroom management. After analysing the PCK of eight fifth grade teachers, Marks (1990) has painted a portrait of PCK as composed of four major areas: (a) knowledge of subject matter, (b) knowledge of student understanding, (c) knowledge of the instructional processes and (d) knowledge of the media for instruction. Cochran et. al.,(1993) have offered an expanded view of PCK which is based on a constructivist view of learning and suggested that it should include knowledge of the subject matter, knowledge of pedagogy, knowledge of the students and knowledge of the environment.

The working model that was constructed by the researchers to study teacher's PCK of geometry teaching emerged from the data collected. It is articulated in this report under the categories stated below, followed by a description:

- **Knowledge of geometry**

The teachers' knowledge of geometry consists of knowledge of content and knowledge of connections in geometry. Knowledge of content refers to the facts, skills and concepts of the geometry curriculum, for example knowing what an angle is. Knowledge of connections can be described in terms of how the particular content is connected to other topics in geometry, in mathematics and in other subjects (cross-curricular)

- **Knowledge of learning geometry**

The teachers' knowledge of pupil learning includes the knowledge of the pupils abilities and strategies, their developmental levels, the prior conceptions that they bring to class, as well as misconceptions that they may develop. It also includes an understanding of which aspects are easy or difficult to learn.

- **Knowledge of teaching representations**

The teachers' knowledge of teaching representations refer to the selection of suitable activities, the type of explanations, illustrations or analogies used, the sequencing of activities and awareness of a level structure. It also refers to the teachers' management style.

- **Knowledge of the environmental context of teaching**

The teachers' knowledge of the environment is understood as the teachers' understanding of the social, political, cultural and physical environment that shape the teaching and learning process (Cochran et. al., 1993). It includes an understanding of the ethos of the school and the teachers' role in the reformed classroom. It also includes knowledge of the socio-economic environment, the societal values which are pursued and especially how materials are adapted for the local context.

These categories should not be seen in isolation but are highly integrated. For example, the use of a co-operative setting may represent knowledge about pupil learning, knowledge of the environmental context and/or knowledge of teaching representations.

Although research thus far shows no direct link between teachers' understanding of content knowledge and learner performance, there is evidence that teachers whose mathematical knowledge is more connected, are also more conceptual in their teaching (Fennema & Franke, 1992). A lack of content knowledge in teachers not only leads to an inability to determine the correctness of answers but also to teacher responses which are either ad-hoc or mathematically inadequate (Evan et.al., 1996).

The Design of the Study

The research data comes from six of the cohort of 24 CPME teachers with between 7 and 20 years of teaching experience. An ethnographic research design was chosen as its qualitative methods provide us with sufficient flexible alternatives for describing, interpreting, exploring and explaining the process of teaching and learning within the classroom, while working closely with the participants.

The research design has two principal features: interviews and classroom observations. The data obtained from the semi-structured interviews, discussing the teachers' perceptions of their practices has been reported elsewhere (Rossouw et.al.,1997). The second principal feature involved observing the six teachers, selected through opportunity sampling (Miles & Huberman,1984), depending on whether the teachers' were teaching geometry at that particular time. Each teacher agreed to teach three geometry lessons of their choice, which they believed to be good examples of geometry teaching. The duration of these lessons ranged from 40 to 50 minutes, with an average of 47 minutes per lesson.

With the commencement of each observed lesson, each participant completed a pre-instruction schedule which included probes of: (a) the teachers' expectations for the lesson, (b) awareness of pupils intuitions and misconceptions, (c) which aspects they believed pupils will find easy or difficult to learn (d) why the particular teaching strategy was selected. The teachers' lessons were video recorded and the two researchers independently wrote field notes of their observations of what the teacher was saying and doing. From the video recordings and field notes it was possible to construct "thick descriptions" which encapsulated both the action (field notes) and the talk (transcripts). The observations were followed by post-instruction interviews which were conducted mainly to clarify teachers' actions for some of the issues that had emerged during the lesson. For example, a teacher would be interviewed to illuminate a particular sequence of representations or to clarify what was perceived to be an inconsistency or misconception in the teachers' content knowledge.

In total our database consisted of 18 completed pre-instruction questionnaires, 18 video recordings of geometry lessons together with field notes on each, and 18 post-instruction observation interviews. Analysis of these descriptions was done by open coding (Strauss & Corbin, 1990). Memo writing (Miles & Huberman,1994) was used to document the issues emerging from the classroom observations.

The remainder of this paper discusses, two sources of teachers' pedagogical content knowledge, namely the teachers' knowledge of geometry and the teachers' knowledge of pupil learning.

The teachers' knowledge of geometry

The teachers' knowledge of geometry was taken to be knowledge of geometric facts, concepts and principles, as well as the relationship among these facts, concepts and principles. Hiebert & Carpenter (1992) define the degree of understanding in terms of the number and strength of connections one is able to make. The extent of the teachers' knowledge of geometry was therefore interpreted in terms of the type of connections being made. A further distinction was made between vertical and horizontal connections. Vertical connections refer to the links established within geometry theory through the linking of different geometrical procedures and concepts. Horizontal connections are the connections made with other subjects as well as with the physical world and are identified either from explicit reference or through the use of physical and/or visual models.

Ken, a seventh grade teacher, places far more emphasis on making vertical connections by referring to the future importance of a particular concept or procedure that was taught. Pupils had to master the properties of parallel lines because they would require it in Grade 9. References to horizontal connections were mostly fleeting, like **"you see angles where the streets meet."** This example shows Ken's low awareness of the complexity of the relationship between spatial thinking in the real world and geometry. This confusion between 3D- and 2D-space was found to be quite common among elementary school children (Berthelot, 1994).

On the other hand, Tessa, a sixth grade teacher made more horizontal connections, often referring to concepts being used in other subjects like needlework or technology. She favours tasks which are set in the context of real world experiences. This is how she describes getting her ideas for teaching a lesson on angles :" **There are two men with different neck ties ... it has stripes which form angles .. I then decided to use it as a basis for my lesson.**"

Their goals for teaching geometry also differ markedly. For Tessa a lesson on perimeter was taught in a particular way "**because the learner needs to be equipped for society to make his/her own purchases.**" Ken on the other hand, frames his goals for teaching geometry in terms of "**understanding the concepts, ...identifying the important relationships... .**" This is how Yola, a fourth grade teacher, describes the goals of one of her lessons "**...those matchboxes and the old milk box that they see... they must see it as I can do some mathematics with it... .**" For her mathematics has to be fun for the pupils which will ensure that they will be keen to learn it.

Tessa relies on her own real life experiences as the main source of information for her lessons. For example, a lesson on perimeter becomes a lesson on buying cooking utensils and clothes according to neck or waist size. Yola also relies on her own experiences but as a participant in in-service workshops and makes extensive use of materials obtained from these workshops. Ken's ideas for lessons comes mainly from text-books.

The teachers' personal knowledge of geometry appears to be fairly stable, except for those critical moments when the unexpected happens. Such a critical moment occurred in one of Yola's lessons:

Yola	**: You take a square and turn it around and it's a kite.**
Learner	**: No, Miss .. a diamond.**
Yola	**: a kite and a diamond is one, it's the same thing. It all depends on how the diamond is drawn. Some draw it with the top lines shorter, and in the other diamond all four sides are the same."**

When queried afterwards on the relationship between a kite and a diamond (rhombus), she explained that she could vaguely recall from her high school days that there was a relationship although she could not explain exactly what it was.

KNOWLEDGE OF PUPIL LEARNING

Ken and Yola's awareness of the pre-conceptions that pupils bring to class is rooted in what they know was done in previous classes. They therefore assume the knowledge that was supposed to have been mastered in the previous school year, as the starting point of their lessons. Tessa on the other hand identifies the preconceptions from the real life experiences that the pupils bring to class. The fact that the pupils knew about clothing sizes was utilized as a starting point for a lesson on perimeter.

All the teachers professed that they were unaware of misconceptions that pupils might develop during the lesson. However, what is significant is the teachers' choice of manipulatives and how this reflects the teachers implicit knowledge on how this contributes or prevents the development of misconceptions in pupils. For example, in Ken's lesson on the properties of parallel lines, the pupils used protractors to measure angles in order to establish relationships between corresponding and alternate angles. The pupils inability to use protractors contributed towards a situation where at the end of the lesson pupils could not identify corresponding or alternate angles with confidence. Tessa on the other hand used informal measuring devices, like string and tape, to measure the perimeter of objects. The concept of perimeter was more firmly established through this.

The teachers' knowledge on what is easy or difficult to learn is also better understood from the context of the lessons. Teachers relied mainly on their past experiences to guide them in either spending more or less time on those aspects that they perceive the pupils will find difficulty in learning. Ken spent a lot of time on the use of a protractor and far less on the classification of angles. Yola spent less time on the recognition of shapes and more time on the properties of shapes. When queried on the time management during their lessons, both Ken and Yola responded that from past experience they knew the areas where their pupils needed more time so that they would not struggle unnecessarily.

The teachers' awareness of the developmental level of the pupils is reflected in the sequencing of the activities. Some of Tessa's lessons show progress from a "hands-on" activity with real world objects, to a worksheet with visual representations and finally to application. Ken's lessons almost always begin with a worksheet of visual representations of the concept/idea to be mastered. This is then followed by checking with manipulatives and finally there is the application. Nancy, a fifth grade teacher, introduces each component of the concept taught, separately. For example she uses paper-folding to establish the symmetry line and mirrors to focus on the mirror image. When prompted to explain the sequencing of activities she replied : **"It is easier to help them to understand each part, then to put the parts together and then finally the whole picture would be very clear to them."**

All the teachers use a co-operative setting to teach, though with varying reasons for doing this. For Tessa, learner interaction is a valuable enterprise for shared knowledge. This is how she describes her reasons for the classroom setting " ...**in groups the weaker child sometimes understands better from his/her friends than from the teacher. That is why I am in favour of group work."** Ken also sets up his pupils in groups. However, he tends to take up most of the time expanding on concepts and giving instructions in the class. He keeps a tight reign on the pupils and only allows pupil-pupil interaction after he has sanctioned it. Yola's presentations, through its emphasis on manipulative work, not only emphasizes peer group learning but also learning from interaction with the materials.

DISCUSSION

The variations described above in the teachers' pedagogical knowledge of geometry and knowledge of pupil learning, led to the identification of three orientations - life skills, investigative and mastery. It should be emphasized that not all lessons taught by the teachers' can neatly fit into the three categories. However, it can be used to classify the predominant orientations of the teachers in this study.

We characterize the life skills orientation as concerned with teaching geometry as a means to equip pupils with life skills. These teachers' emphasize the links between the geometry in school and real life situations, where that knowledge becomes functional. The teachers knowledge of teaching goals plays a pervasive role in the type of teaching representations they use. Associated with this kind of orientation is the view that pupils learn better if the knowledge has practical value for them.

The investigative teacher tends to treat all geometry topics as opportunities to engage pupils in manipulative activities. The activities are not selected on the basis of utilitarian value but simply becomes a means to an end. The discovery of patterns through the use of manipulatives, for example paper-folding, is highly valued. Associated with this view is the knowledge that pupils learn through their interaction with the materials.

The mastery teacher is pre-occupied with understanding. The facts, concepts and procedures are uppermost in this teacher's mind. The focus is on what kind of mathematics pupils will need to build on in future. Consequently the selection of activities and the classroom organisation are but means to help the pupils to attain mastery for future purposes. The main source of information is the textbook and learning is perceived as mainly from the teacher, and far less from peer interaction.

CONCLUSION

In this paper we discussed some initial findings regarding primary school teachers' pedagogical content knowledge of geometry. Our data shows that, based on the teachers' knowledge of geometry and pupils learning, distinct orientations can be discerned in these teachers. Our paper identified three such orientations which we have described as "life skills", "investigative" and "mastery". Even though these teachers had the same learning experience through their attendance of the CPME course two years earlier, their PCK of geometry showed marked differences. In the light of these findings it seems that teachers eventually develop their own pedagogical content knowledge which is shaped by their own experiences and perceptions.

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CONVICTION, EXPLANATION AND GENERIC EXAMPLES

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In this paper, I distinguish between the purposes of proof in terms of (i) community-of-practice certification, and (ii) enlightenment and explanation of inductive discoveries and standard theorems. I argue the case for wider acceptance of the appropriateness and validity of generic arguments for the second purpose, and for more attention to the deliberate deployment of generic examples as didactic tools.

PROOF AND RESEARCH

Proof is central to mathematics, yet the purposes of proof need careful consideration in order to achieve some consensus on what might count as a proof in a given context. I find it useful to begin from an observation of Reuben Hersh (1993) that the role of proof in the classroom is different from that in research.

For research mathematicians, claims Hersh, the purpose of proof is conviction; the formal proof is the guarantee of 'truth'. Indeed, mathematics is uniquely characterised by a deductive mode of reasoning which builds on truth in order to attain further truth. Support for such a view was provided in a recent BBC TV documentary celebrating Andrew Wiles' proof of Fermat's Last Theorem[1]. The programme begins with the historical background to the theorem, and proceeds to explain for a 'lay' audience what might be required to demonstrate that it was true - a 'mathematical proof'. Mathematicians Peter Sarnak, Nick Katz and Ken Ribet elaborate:

Sarnak:	A mathematician is not happy until the proof is complete and considered complete by the standards of mathematics.
Katz:	In mathematics there's the concept of proving something, of knowing it with absolute certainty.
Sarnak:	Which, well it's called rigorous proof.
Ribet:	Well rigorous proof is a series of arguments ...
Sarnak:	... based on logical deductions.
Ribet:	... which just build one upon another.
Sarnak:	Step by step.
Ribet:	Until you get to ...
Sarnak:	A complete proof.
Katz:	*That's what mathematics is about.*

[1] BBC2 15th January 1996: See also http://www.bbc.co.uk/horizon/96-97/bafta/fermatrn.html and Singh (1997). Fermat's Last Theorem claims that $x^n + y^n = z^n$ has no integer solutions for $n>2$

In the practice of mathematical research, as Hersh points out, mathematical proofs are not atomised into sequences of sentences in formal logic, with mechanical checks against rules of inference, but are submitted instead to the scrutiny of qualified (human) judges. It is an interesting case of peer-review within a particular community of practice. The rules for mathematical argument are sufficiently circumscribed, and the objects under consideration defined with such precision, that any two qualified judges are expected to be in agreement as to the correctness (or otherwise) of a given proof. Approval by such expert gate-keepers amounts to the achievement of conviction for the community as a whole, most of whom are unlikely ever to read the proof for themselves, although some may be aware of the conclusion, the theorem which has been added to the corpus of mathematical knowledge. The effectiveness and rigour of the process was in fact highlighted by the progress of Wiles' 200-page proof once it had been submitted to *Inventiones Mathematicae* for publication. Nick Katz was one of the referees:

Katz: So for two months, July and August, I literally did nothing but go through this manuscript, line by line. What this meant concretely was that essentially every day, sometimes twice a day, I would E-mail Andrew with a question: I don't understand what you say on this page on this line. It seems to be wrong or I just don't understand.

Wiles: So Nick was sending me E-mails and at the end of the summer he sent one that seemed innocent at first. I tried to resolve it.

Katz: It's a little bit complicated so he sends me a fax, but the fax doesn't seem to answer the question, so I E-mail him back and I get another fax which I'm still not satisfied with. This, in fact, turned into the error that turned out to be a fundamental error and that we had completely missed when he was lecturing in the spring.

It had taken Wiles seven years to construct the proof; he needed a further year to remedy the flaw that Katz had detected. At last the referees were convinced.

PROOF IN THE CLASSROOM

The situation in the classroom is very different.

> In the classroom, convincing is no problem. Students are all too easily convinced. Two special cases will do it. (Hersh, 1993, p. 396)

In the teaching context, says Hersh, the primary purpose of proof is to *explain*, to illuminate why something is the case rather than to be assured that it is the case. The Pythagorean proposition is a cornerstone of mathematics, and no student should doubt its truth; but a proof serves to shed light on its inevitability.

By way of example of classroom practice, consider one rich mathematics task, adapted from Foxman *et al.* (1982, pp. 102-111)

> **Partitions**. The number 3 can be 'partitioned' into an ordered sum of (one or more) positive numbers in the following four ways: 3, 2+1, 1+2, 1+1+1. Find all such ordered partitions of 4. In how many ways can other positive numbers be partitioned?

The mathematics teacher introduces this task to the class, and organises discussion in pairs. Work on this activity soon produces some data: as well as the 4 given partitions of 3, they find that there are 2 partitions of 2, and 8 possible partitions of 4. Emma (say) notices that as the number to be partitioned increases from 2 to 3 to 4, so the number of partitions doubles from 2 to 4 to 8. Her prediction that there will be 16 partitions of 5 is subjected to empirical confirmation. Cathy goes on to make the conjecture that 'this always happens'. The conjecture is arrived at by process of *inductive* inference.

The teacher calls the class together for plenary discussion of the problem. There is consensus about the universal validity of the doubling pattern. The epistemological issue at this point is not one of conviction, but of insight. *Why* is it that the number of partitions doubles at each stage? The teacher develops an explanation of the doubling - a *deductive* argument such as the following. Consider any partition of n. If I increase by one the size of the last part, I have produced a partition of $n+1$. If instead, I adjoin an additional part of size 1, I have produced a different partition of $n+1$. So there are at least twice as many partitions of $n+1$ as there are of n. Further argument establishes that this process accounts for every partition of $n+1$. This constructive proof explains the observed doubling phenomenon.

GENERIC EXAMPLES

The argument above (relating partitions of $n+1$ back to those of n) is effectively presented, from the point of view of both explanation *and* conviction, by assigning a particular value to n, say 3. The exposition then describes how each partition of 3 begets two partitions of 4. Indeed, my experience with students indicates that careful scrutiny and comparison of the 4 partitions of 3 alongside the 8 partitions of 4 can trigger explanatory insight concerning the way that each partition of 3 is related to two partitions of 4. Such an argument amounts to proof by generic example (Pimm and Mason, 1984; Balacheff, 1988).

> The generic proof, although given in terms of a particular number, nowhere relies on any specific properties of that number. (Pimm and Mason, 1984, p. 284)
>
> A generic example is an actual example, but one *presented in such a way* as to bring out its intended role as the carrier of the general. (*op. cit.* p. 287, emphasis added)

The story (probably apocryphal, but see Polya, 1962, pp. 60-62 for one version) is told about the child C F Gauss, who astounded his village schoolmaster by his rapid calculation of the sum of the integers from 1 to 100. Whilst the other pupils performed laborious column addition, Gauss added 1 to 100, 2 to 99, 3 to 88, and so on, and finally computed fifty 101s with ease. The power of the story is that it offers the listener a means to add, say, the integers from 1 to 200. Gauss's method demonstrates, by generic example, that the sum of the first $2k$ positive integers is $k(2k+1)$. Nobody who could follow Gauss' method in the case $k=50$ could possibly doubt the general case. It is important to emphasise that it is not simply the fact that the proposition that the sum $1+2+3+ \dots + 2k = k(2k+1)$ has been verified as true in the case $k=50$. It is the *manner* in which it is verified, the *form of presentation* of the confirmation.

By contrast, consider a possible inductive approach to Lagrange's Theorem[2] on groups. We might observe that the order of every subgroup of D_4 is a factor of 8, that the order of every subgroup of $\mathbf{Z}_{15}$ is a factor of 15, and so on. But these confirming instances lack explanatory power. The usual proof of the theorem enumerates cosets, and we could indeed demonstrate that (for example) $\mathbf{Z}_{15}$ is partitioned by the five cosets of the subgroup {0, 5, 10} - but without gaining any insight as to why the cosets of subgroups of other finite groups should (a) be equinumerous and (b) partition the group. Any possibility of analogy with other groups would depend on structuring the presentation of the example in an appropriate manner.

> The generic example involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class. (Balacheff, 1988, p. 219)

In this way the generic example serves not only to present a confirming instance of a proposition - which it certainly is - but to provide insight as to *why* the proposition holds true for that single instance. The transparent presentation of the example is such that analogy with other instances is readily achieved, and their truth is thereby made manifest. Ultimately the audience can conceive of no possible instance in which the analogy could not be achieved.

A video[3] published by the British *Office for Standards in Education*, intended to exemplify and promote 'direct teaching' of mathematics in schools, features one teacher, Kate, with a class of 10- and 11-year-olds. In the middle phase of the 'Numeracy Hour' lesson, the children investigate the 'Jailer Problem' in small groups, before being brought together by Kate (presumably for some direct teaching). The details of the problem are not important here, but the solution turns out to hinge on the fact that every square number has an odd number of factors. In fact, Kate explains this to the class by reference to (what we recognise as) a generic example. She points out that every factor of 36 has a distinct co-factor, with the exception of 6, and so 36 has an odd number of factors. She then generalises, "One of the factors of a square number is a number times itself; that's why it's a square number, isn't it?". Her choice of 36 is interesting – small enough to be accessible with mental arithmetic but with sufficient factors to be non-trivial. No reference is made in the commentary to this aspect of her teaching and proof strategy.

Liz Bills and I have recently considered ways in which exposure to examples may lead students to make both appropriate and inappropriate inductive generalisations (Bills and Rowland, in press) The second category is illustrated by the case of Trevor, a student who witnesses the derivation (by his teacher) of $(x-3)^2 + (y-5)^2 = 4$ as the equation of the circle with centre (3, 5) and radius 2, yet who perceives the final '4' as the diameter of

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[2] The order of (number of elements in) every subgroup of a finite group divides the order of the group. D_4 is the symmetry group of a square (order 8) and $\mathbf{Z}_{15}$ the set {0, 1, 2, ... 14} under addition modulo 15

[3] *Teachers Count.* Ofsted, Crown Copyright, 1997

the circle. Bills and I distinguish between *empirical* generalisations and *structural* generalisations. The former derive only from the form of 'results' (usually numerical) and observed relationships. Like Cathy and Emma's doubling generalisation, empirical generalisations may possess predictive potential but lack explanatory power. Of course, inappropriate empirical generalisations such as Trevor's lack both. Structural generalisations, on the other hand, are based on underlying meanings, structures or procedures. They go beneath the form of results to achieve explanatory insight.

We argue that inductive reasoning transcends empirical speculation when explanation is available to the student as a structural generalisation of some kind. A generic example which successfully "speaks the generality" (Mason and Pimm, 1984, p. 284) for the audience has the quality of such a structural generalisation. We describe two small-scale empirical studies in evidence. Both involve offering college mathematics students generic examples to account for mathematical generalisations. The second study, for example, concerns proof of the number-theoretic theorem that every prime number p has a primitive root,[4] by reference to the generic case $p=19$. The students completed questionnaires which asked (*inter alia*):

- Does the explanation for $p=19$ convince you that $p=29$ has a primitive root?
- Does the explanation for $p=19$ convince you that every prime has a primitive root?

Some two-thirds of the students responded in the affirmative. Their comments were characterised by the student who wrote:

> It is easy to follow the logical progression of the proof for $p=19$ with any other prime in mind, and I can see no area of the proof which gives me any doubt that it wouldn't work for any prime.

This student and others have made a judgement as to which aspects of the generic proof are transferable. Semadeni identifies this as an issue, but one that is not peculiar to generic arguments:

> How can one know whether the child is concerned by the validity of the proof by inner understanding and not just by being prompted by the authority of the teacher? Without dismissing this criticism, we note that it applies to any proof in a textbook: if the author finds his proof correct and complete, this does not automatically imply that students understand it. (Semadeni, 1984, p. 34)

GENERIC EXAMPLES AND DIDACTIC STRATEGY

I had, in fact, shifted from a conventional-algebraic to a generic presentation of the primitive root proof some years ago, within a 48-hour unit on Number Theory, out of conviction that most students found the 'proper' proof impenetrable. The standard general proof (see, for example, Baker, 1984, p. 23) is surprisingly indirect and laden with notational complexity. I had already taken for myself the stance which Hersh (1993, p. 397) subsequently labelled Humanist (in contrast to Absolutist) in my approach

[4] p has a primitive root if the group $\{1, 2, 3, \ldots, p\text{-}1\}$ under multiplication modulo p is cyclic

to proof. Whereas the Absolutist teacher chooses the shortest and most general proofs, for the purpose of certification, the Humanist teacher uses the most enlightening proofs for the purpose of explanation. Initially this choice may be at the expense of succinctness and even elegance.

I believe that learners of mathematics at all levels, including university students, should be assisted to perceive and value that which is generic in their own particular insights, explanations and arguments. The barrier between such a level of knowing and the writing of 'proper' proofs is then seen for what it is - a lack of fluency not with ideas, but with notation. For this reason, I question Balacheff's inclusion of proof by generic example in an inferior category of "pragmatic proofs", along with naïve empiricism and crucial experiments. I would regard generic proofs as being of a very different order of generality from these.

In their study of college students' understanding, production and appreciation of proofs, Harel and Sowder (1996) also present something of a deficit view of generic proof schemes.

> In a generic proof scheme, conjectures are interpreted in general terms but their proof is expressed in a particular context. This scheme reflects students' *inability to express their justification in general terms*. (p. 43, emphasis added)

In some instances, this may indeed be the case. In others, it may be that the generic example adequately bears the intended generality, and is fully sufficient for purposes of conviction and explanation. There is an obvious sense in which proof by generic example might be a half-way house between empirical generalisation and generalised formal proof (Bills, 1996, p. 84). In my view, there is a great deal more scope for recognition of well-constructed generic examples *as didactic devices*, as potentially-sufficient justifications or as 'stages' towards arguments presented with conventional generality.

I provide one further example from my own teaching to exemplify the latter. The context is a one-to-one Number Theory supervision with Jonathan, who is trying to prove his conjecture that the congruence $x^2 + y^2 \equiv 0 \bmod p$ has $2p+1$ solutions for primes $p \equiv 1 \bmod 4$. He realises from examples that a possible key to proof is the fact that for the primes under consideration, every quadratic residue [$x^2 \bmod p$] can be paired with another to give a multiple of p. [For example, with $p=13$ and $x=6$, $6^2+4^2=52$]. Nevertheless, Jonathan is stuck.

Jonathan: Which ... that's the bit I can't, I'm not ... able to explain. I can't, I'm not, I can't say why they pair off, like that [...] They add up to give p each time, these two ... these pairs of squares ...

My concern is to assist Jonathan to "say why", and to give him some role in the construction of an argument.

Tim: OK. I mean, can you take it any further than there. You're absolutely right. How can you take it any further than "there always happens to be"? [...]

Jonathan: ... they really do separate, but I can't explain, why they separate.

By this stage, I have made a spontaneous but *conscious pedagogic decision* to construct the argument as a generic example, and I think aloud as I decide which prime will best "speak the generality". I consider 13, and reject it in favour of 17, because minus 1 is then congruent to 16, and so can be shown to be a quadratic residue of 17 in a particularly direct fashion.

Tim: OK. Well, I'd like to take you a bit further down that road [...] I'm just wondering whether to talk about thirteen, or something that's less obvious, because an argument can be more forceful when you can't just - other than the numerical calculations - say "Well, obviously". Yes?

Jonathan: Yes.

Tim: So suppose ... [pause] Right, OK, [..] suppose we took seventeen.

Jonathan: Yes, OK.

Tim: And the first thing to note is, you should know that minus one is a quadratic residue, and that's a particularly easy one because four squared is sixteen, which is minus one.

Here, I am drawing attention to a feature[5] of the particular prime [17] that holds for every prime $p \equiv 1 \bmod 4$. The concreteness of 17 nevertheless facilitates the construction of the argument.

Jonathan: Oh, yes.

Tim: So just bear in mind if you will that four squared is minus one, yes?

Jonathan: Right.

Tim: Now pick - *you* pick, anything from nought to sixteen.

Jonathan: Ten.

Further discussion establishes that the quadratic residue 15 [i.e. 10^2 mod 17] must have additive inverse $(4 \times 10)^2$. Whilst this can be verified numerically, the conviction lies in the structure of the argument, not in the numerical calculation. Moreover, Jonathan's choice of 10 could be replaced by any other residue mod 17 in the same argument.

Tim: Now, can you see that in principle you can do that with absolutely ... I mean, you took ten, but you *could* do that with anything that you took.

Jonathan: Yeah.

Tim: And it's because, minus one is a quadratic residue.

Jonathan: Oh ... right [chuckles].

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[5] This standard result of number theory amounts to the fact that the odd prime divisors of x^2+1 (where x is an integer) are all of the form $4k+1$; moreover, every such odd prime divides some integer of the form x^2+1

Tim: Well, it takes us beyond this kind of level of "there always happens to be one". I mean, that's *why* there always has to be one.

Jonathan: Right, OK.

Our discussion of that problem went no further; Jonathan went away and wrote out a conventional, generalised proof, making no reference to particular values of p, x or y.

CONCLUSION

Whereas generic examples as proof schemes are reasonably well-established in the literature, their status is typically presented as inferior to proofs presented in formal and conventional generality. In classroom situations – especially for purposes of explanation, but also for conviction - I question whether that pejorative view of the status of generic arguments is justifiable.

I also urge that more conscious attention be given in mathematics teacher education – at all levels – to the deployment of generic examples, as didactic tools, for purposes of explanation and conviction.

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CHILDREN'S MULTIPLICATIVE PROBLEM-SOLVING STRATEGIES IN REAL-WORLD SITUATIONS

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This paper describes an empirical investigation about arithmetical problem solving strategies and action patterns of primary school children dealing with multiplicative real-world situations. Wee chose three multiplicative real-world settings, which were similar to each other in their arithmetical structure but differed in the situational contexts. 67 second-graders – age 7 to 8 – and 55 third-graders – age 8 to 9 - were combined into pairs and confronted with two of these contexts in a simulating situation. The results of the qualitative analysis show that the arithmetical strategies observed are mainly influenced by the possible action patterns instead of learnt algorithms. The acceptance of the solutions and the heuristics seem to be much more determined by school experiences and routines.

Theoretical Backround

Within the last two decades the body of literature related to our topic has grown quite voluminous. On the one hand a lot of research has been devoted to the development of multiplicative concepts (eg., Greer 1988, 1992; Nesher 1988; Schmidt & Weiser 1995; Vergnaud 1983, 1994) and the problem-solving strategies of primary school children dealing with multiplicative problems (eg. Anghileri 1989; Bönig 1995; Brown 1992; Burton 1992; Kouba 1989, Selter 1994; Steffe 1988).

On the other hand you find numerous studies on the use and learning of mathematics outside school in comparison to classroom routines inside (eg., Carraher, Carraher & Schliemann 1985, 1987; Lave 1988; Lave, Smith, and Butler 1989; Lave & Wenger 1991; Masingila, Davidenko & Prus-Wisniowska 1996; Saxe 1991; Scibner 1984).

While the work of the first group of studies concentrate on primary-school children's strategies dealing with one-step word problems the second group of investigations is interested in mathematical strategies in real-world situations with a focus on older children and adults.

Now, the main concern of our work is the combination of real world situations and primary school children. Therefore we investigate the multiplicative problem-solving strategies by German second- and third-graders (age 7-9) in real world situations. The main focus of this paper will be on the observed arithmetical strategies which were used in the underlying contexts.

Design of the Study

To serve the purpose of our study we looked for real-world settings which are similar

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to each other in their arithmetical structure but differ in the situational contexts. Thus, they had to meet the following criteria (see Ruwisch 1995):

- They have to belong to the children's experience.
- They have to represent the whole complexity of the real situation.
- They have to be realistic concerning the used materials.
- They have to be open for different solutions and different problem-solving strategies.
- They have to open different possibilities for the handling of materials.

We constructed three multiplicative settings that satisfy these conditions: All the three deal with shopping lists, which have to be filled in by the children (see Ruwisch 1998).

In the first situation, called „*classroom party*", the children were given a list of seven goods. They were asked to buy each of these goods for a party with 18 children. They could do so in a fictitious supermarket where all the goods were presented in their genuine packaging. The children's task was to determine the number of packages needed.

In the second setting, called „*juice punch*", the children were given the instructions for the mixture of a fruit punch. The instructions contained the number of glasses of juice necessary for the punch. The children were offered bottles of juice in three different sizes. Their task was to determine the number of bottles that had to be bought for the punch.

In the third situation, called „*doll's house*", the children were asked for their help in tiling the three rooms of that house. Packs of small tiles were offered in three different sizes, and the children should again determine the number of packs needed.

While all three situations include multiplicative structures and are constructed in a very similar fashion, they also show some striking differences (see table 1): They do not only differ in their situational context but also in the underlying mathematical content: numbers, volume and area.

Although all three settings belong to the situational model of equal measures, the first may be seen as equal grouping, because the material given is countable. Furthermore the „doll's house" also requires the interpretation of the given rooms as a rectangular array of square tiles. Referring to the arithmetical structure of the tasks, all three situations contain tasks of the structure $x \cdot b = a$ and also the more difficult variant $x \cdot b > a$. If the children combine packs with different numbers of elements, they are faced with tasks in the form of $x \cdot b + y \cdot c = a$ and $x \cdot b + y \cdot c > a$ respectively.

But there are also slight differences in the arithmetical demands between the three settings: While in the first situation „classroom party", the total number a of goods needed is given by the constant '18', the total number of glasses for the „juice punch" is also given but differs with the different juices. The solution to the third problem, „doll's house", requires the determination of the total number of tiles needed by measuring the three rooms.

situational context	given materials	situational model of multiplication	arithmetical structure
numbers „classroom party“	goods with different numbers of elements (*b*) per pack	*equal groups* number of packs number of goods per pack total number of goods	$x \cdot b \geq 18$ packs with *b* given as 2, 3, 4, 5, 6, 7 or 9
volume „juice punch“	bottles of juice with different volumes (*b*)	*equal measures* number of bottles number of glasses per bottle total number of glasses	$x \cdot b \geq a$ *a* given in the instructions as 15, 12, 8, 5 or 20 $b \in \{2,5,7\}$
area „doll's house“	three rooms of different area (*a*)	*rectangular array* number of rows number of columns number of tiles	$x \cdot b \geq a$ *a* to be detemined $b \in \{3,6,8\}$
	packs with a different number of tiles (*b*)	*equal groups* number of packs number of tiles per pack total number of tiles	

Tab. 1: Similarities and differences of the designed problems

Data and Methodology

Subjects. The subjects of the study were 122 children from 7 different German public primary schoolclasses. If possible, all students of these 7 classes were included. Thus, 67 second-graders – age 7 to 8 – and 55 third-graders – age 8 to 9 – participated in our investigation. At the time of inquiry the second-graders had not been introduced to multiplication or division yet. The third-graders had already been instructed with the multiplication and division facts of problems with small numbers up to $10 \times 10 = 100$.

Procedure. The children were withdrawn from the classroom and confronted with the material in a separate room. They were given a short introduction into one of the situations described above. Then they had to work by themselves until they indicated to us that they had finished. On average the working-phase - which was videotaped - lasted 20 to 30 minutes. This working-phase was followed by a short re-interview about the actions we had observed. Since we were also interested in symmetrical interactions between peers in handling real-world problems, all children worked in pairs. Within two weeks, the children were confronted with two of the three situations in order to gain information about the stability of their strategies independent of the situational context. We also varied the sequence of those two situations for the reason that we expected to observe transfers from one problem to the next. Tab. 2 shows the design of the procedure.

		situation 1	situation 2
grade 2	group 1 (8 pairs)	classroom party 8 pairs	doll's house 9 pairs
	group 2 (5 pairs)	classroom party 9 pairs	doll's house 7 pairs
	group 3 (6 pairs)	doll's house 7 pairs	juice punch 7 pairs
	group 4 (5 pairs)	juice punch 5 pairs	classroom party 6 pairs
grade 3	group 1 (7 pairs)	classroom party 8 pairs	doll's house 8 pairs
	group 2 (8 pairs)	doll's house 8 pairs	juice punch 8 pairs
	group 3 (10 pairs)	juice punch 10 pairs	classroom party 11 pairs

Tab. 2: Procedure of the investigation[1]

Results

Analysis. The videotapes were transcribed in various different forms. We mainly worked with categories directly on the tapes to analyse the used arithmetic strategies. A first transcription of the verbal comments was necessary to analyse the used heuristics and shown actions. Additionally, some videotapes were transcribed very specifically and carefully in the form of interactional case studies.

Children's strategies. For the remainder of this paper I will mainly concentrate on the arithmetical strategies and the accompanying action patterns in the third situation, the „doll's house".[2] Table 1 shows that the children were required to connect the given packs of three, six, or eight tiles with the three floors of the doll's house to solve the problem. There were different ways to come up with this connection. On the one hand the connection could be produced directly. We call this „one-step-patterns". On the other hand the children could determine the total number of needed tiles for every room first, before connecting this information to the given packs. We call this „two-step-patterns". Table 3 shows the observed strategies and action patterns.

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[1] The number given in the first column indicates the number of pairs which worked together in both situations.

[2] For the other results see eg. Ruwisch 1995, 1996, 1998.

<table>
<tr><th colspan="6">two-step-patterns (13/13)</th></tr>
<tr><th colspan="6">step one: determination of the total number of tiles (13/13)</th></tr>
<tr><td colspan="3">tiling (10/9)</td><td colspan="2">measuring (3/3)</td><td>estimating (0/1)</td></tr>
<tr><td>counting (7/1)</td><td>mixed strategy (3/0)</td><td>multiplicative facts (0/7)</td><td>counting (3/1)</td><td>multiplicative facts (0/2)</td><td>multiplicative facts (0/1)</td></tr>
</table>

<table>
<tr><th colspan="2">one-step-patterns (5/3)</th><th colspan="5">step two: determination of the number of packs (12/12)</th></tr>
<tr><td rowspan="2">double counting (2/1)</td><td rowspan="2">multiplicative relations (3/2)</td><td rowspan="2">reconstruction of the used packs by counting (3/4)</td><td colspan="4">construction of the necessary packs (9/8)</td></tr>
<tr><td>double counting (6/1)</td><td>addition facts (0/1)</td><td>multiplicative relations (2/4)</td><td>estimating (0/1)</td></tr>
</table>

Tab. 3: observed arithmetical strategies and action patterns[3]

ONE-STEP-PATTERNS

8 pairs showed one-step-patterns. We could distinguish two groups. The first group of 3 pairs determined the needed number of packs by measuring out the area of the rooms with the help of one pack and double counting processes: They counted the number of tiles per pack as well as the number of the packs required. The other group of 5 pairs showed one-step-patterns without counting processes. They measured the length and width of the rooms expressed in numbers of tiles and assigned packs with the corresponding number of tiles to the area: E.g., if one room had the length of eight tiles, the pairs chose packs with eight tiles. Thus, one pack of the tiles corresponded to one column of the area. Among this group 2 pairs showed more complicated assignments: E.g., they measured the widths of the kitchen as four tiles and assigned one pack of eight tiles for two rows.

TWO-STEP-PATTERNS

Step one: determination of the total number of tiles. We observed three different patterns: 1) opening the given packs to tile the rooms, 2) measuring the needed number of tiles for each room, and 3) estimating the required number. Most students (19 of 26) *tiled the rooms* with the given materials. Afterwards, the second-graders among these students (10 pairs) determined the total number by counting the tiles one by one, pointing at the tiles at the same time. 3 of these pairs showed already advanced strategies: they used the first three or four numbers of the number patterns, before they changed to rhythmic counting. The majority of the third-graders (7 of 9) counted the number of tiles per row and column to multiply them. 4 of the 6 *measuring pairs* counted while measuring the whole rooms, whereas 2 pairs from third grade measured only the length and the width of the rooms to multiply those numbers. The same did

3 The given numbers in the table tell you, how many second-graders (first number in every bracket) and third-graders (second number in every bracket) showed the pattern mentioned. Some pairs are missing, because they stopped their work at a specific point of workprogress.

the *estimating pair* with the difference of estimating instead of measuring the length and the width.

Step two: determination of the number of packs. Before analysing the second step of these children's working-phase in more detail it has to be mentioned that only 1 pair carried out this step completely by itself. All other 23 pairs had to be reasked, how many packs the interviewer had to buy. Nonetheless, again two different pattern occurred. On the one hand 7 pairs of students tried to reconstruct the packs they had used during the first step: They packed the single tiles into packs again by combining the wrapping with the corresponding number of elements or they tried to remember correctly. On the other hand 17 pairs showed a new working-phase mostly independent of the used materials. While the second-graders among them used double counting predominantly, 4 out of the 7 third-grader-pairs showed multiplicative assignments.

Interpretation and Conclusions

In correspondence with the results of Anghileri (1989), Kouba (1989) and many others, the children in our investigation showed a great variety of strategies and patterns. They could not only use these for solving one-step word problems but also for solving non-standard problems in real-world contexts. It was very striking for us that very few students used addition facts: The second-graders mainly counted, whereas the third-graders used multiplicative facts or - in dealing with the other two situations - number patterns. This observation may be caused by the German way of instructing multiplication, which is based more on number patterns than on relations between additive and multiplicative facts. But in this case we would have expected the results of the more advanced second-graders to show additive strategies in contrast to number patterns and rhythmic counting. The reason is that the second-graders hadn't been given any instruction into multiplication before the time of our inquiry. Therefore, we think that a different explanation might be more useful: Under the given settings the children had the possibility of handling the materials within the real-world contexts while solving the mathematical problem. This possibility leads to strategies which allow parallel actions: The children took one pack while speaking out the following number of the pattern. Thereby they could come up with the required number of packs directly. It would be interesting for further investigation, whether the children use the same strategies we observed when they are faced with situations, in which the use of the materials like in our settings is not possible.

Concerning the stability of the used strategies within the two different problems the children were confronted with we can note the following quite remarkable observations: Most of the children were not influenced by the situational context. Instead they solved both problems in the same manner: The estimating pair in the „doll's house" also used estimating reasoning in the „classroom party" situation. Children measuring out the bottles solving the „juice punch" also used measuring patterns in the „doll's house." At a first glance these results didn't meet my expectations, but I think that although the situations differ in their context, they were very similar in their structure. Therefore, they could all be solved by the same strategies and action patterns. Thus, children may tend to use one specific strategy as far as it is possible.

This interpretation is strengthened by the observation of the children's behaviour, when they reached a point where their strategy failed. In these situations the real context of the problems became more important. For example, most children preferred solutions without remainders. Thus, they tried to avoid them or - in the end - they argued situationally bounded: „There will also be other packs in a real supermarket" or „Let's take some more for the teachers." This observation might be explained by the unknown experimental situation the children were in. Since our investigation took place inside the school, the children mainly used learned routines which they thought we were expecting from them. Although we didn't tell them anything about our interest in mathematics, they knew we were interested in school knowledge. Further investigations must show, if classes used to different routines in their daily mathematics courses argue in a different way.

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National tests: educating teachers about their children's mathematical thinking

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The conflict between formative and summative motivations of assessment leads many educators to reject formal National tests as diagnostic instruments: they are regarded as at best insensitive and at worst instruments of state control. School management and teachers do however pay a great deal of attention to them, and they are increasingly used as public indicators of pupils' progress and teachers' effectiveness. This paper describes an analysis of performance, misconceptions and errors made in the 1997 tests by 7 and 14 year olds, which has been reported to all Primary and Secondary schools in England and Wales, and is intended to support teacher education. Opportunities for exploiting this approach and the practical and political problems involved are discussed.

Introduction

This paper reports the results of an 'error analysis' of samples of scripts from the 1997 national tests in mathematics taken by all 7 year old and 14 year old pupils in England and Wales. The analysis involved 396 scripts from the 7 year olds on a 33 mark test and 835 scripts from the 14 year olds on (about 240) marks (including null responses due to the tiering of entries). As such this is the largest published UK study of children's misconceptions in mathematics tests since the 1980s (APU, 1983; Hart, 1981), and as far as we know is the first on this scale for seven year olds.

Children's errors and misconceptions are the starting point for effective diagnostically designed teaching (see the seminal work on this in the UK by Bell et al, 1983). Experimental teaching studies certainly have shown some remarkable successes: the strategy is usually conceived of in the neo-Piagetian tradition of inducing cognitive conflict, but has been developed in recent decades into the theoretical framework of 'conceptual change', especially within science and mathematics education (e.g. Posner et al, 1982). However, many teachers do not use these methods, and seem to be unaware of their potential for improving classroom practice.

The aim was to identify the common errors made on these high profile tests. We expect any recommendations or implications, because they relate to tests which teachers see as determining the public view of their effectiveness in the system, to be more likely to attract attention.

On the other hand a strict limitation on the results is imposed by the nature of the tests, which were designed to test children's success on the National Curriculum and not to diagnose misconceptions as such. How significant a limitation this is will become clear following the report of results, after which we will discuss the problems and make some recommendations for ameliorating these.

Method

The two study samples (of 396 seven year olds and 835 fourteen year olds) were selected from a national sample of children in England and Wales which was a random sample of pupils drawn from the relevant year group sitting the tests in a stratified sample of schools. We were able to make use of a wider sample of 1489 seven year olds results on their test, which was essentially a 33 mark, 31 question, approximately one hour long, mainly written (with four introductory orally presented questions) test of mainly number (with a few questions on data handling and shape).

This data set was used to build a Rasch scale for the sample, mapping all pupils' ability estimates and the item difficulties in the usual way (Wright and Masters, 1982). The item to model fit was within bounds normally considered acceptable (infit mean square 0.8 to 1.2), with one unusual item on the borderline (infit mean square 1.3). A similar analysis using Masters (1988) partial credit model for the 14 year olds was performed on the full sample of 835 scripts (of two written papers including about 60 marks each, covering the four targets of the National Curriculum for this age). The modified Rasch scale produced by the computer program QUEST which takes account of null data is discussed in Adams and Khoo (1996). The result is a single difficulty estimate for each item and an ability estimate for each child consistent with the Rasch measurement assumptions, (only 4 mark points fell outside the model fit tolerance of mean square 0.7 to 1.3).

The identification and analysis of errors was essentially the same for the 7 year olds and the 14 year olds and followed three stages. The first stage involved the research team and markers in anticipating likely errors at face value. The second stage involved a scrutiny of the first 70 scripts available, listing all errors we could identify, and drawing up a coding sheet of all those which occurred more than once. Thirdly the coders recorded the errors for the entire data set, about ten per cent were second coded for verification and the results keyed into EXCEL direct.

Finally any error occurring on less than 3% of scripts was regarded as 'null', on the grounds that one might not expect to see one occurrence in a random sample of 30 children of the sort one might find in a classroom. An average ability of those making each error was calculated. This average is the mean of the Rasch estimates of the abilities of the children who made the error, and must be interpreted carefully together with the number of children making the error. This average was finally used to map the error response on the same scale as the estimated difficulty of the correct response, indicating the behaviours to be expected of children of a given ability. We therefore interpret the scale as indicating the responses to be found in groups of children at each test score or 'level'.

A final critical stage in the process is the interpretation of the scale as a description of characteristic thinking and specific behaviours (correct answers and errors) expected at various levels of attainment. This formed the bulk of the research report to the government agency who are editing and printing this for distribution to all schools in England and Wales, i.e. about 6000 schools.

Results for children age 7

A striking feature is the large number of omissions and the scarcity of any written methods shown. This would seem to be a comment on the difficulty of setting a written test for children at this age. Further, there is a surprisingly large number of questions (9 out of 31) whose errors are incoherent or are 'null', i.e. too rare to reach our 3% criterion. Of the errors which we could make sense of, many were associated with children's comprehension problems: these can be either explained by poor test design or by children's poor understanding. At least two questions were deeply flawed; in other cases it is hard to be sure of the explanation.

Many children had difficulty with the division sign, and many cannot cope with the equals sign as other than an instruction to calculate (e.g. 83 - 37 = ? makes sense as a command but 38 - ? = 11 does not). These suggest conceptual problems: either missing language, missing concepts or misconceptions. Other significant conceptual problems include the use of a smaller-from-larger error in subtraction (83 - 37 = 54), errors in scale readings (preference for a unit of one) and money notation. At the lower ability level, we see problems with simple language, such as 'right-angle' and 'lighter' and the counting down error: 18 - 4 = 15. (We suspect that the latter is an error which many teachers see as a 'slip' rather than a significant misconception, see Fuson, 1992.)

Level	Typical performance at each level	Typical common errors
Level 3	Give the solutions to problems in common-sense terms; calculate simple products; add prices of goods to exactly £1; solve addition and subtraction problems involving regrouping; read unlabelled intermediate markings on scales; recognise the ÷ sign and solve simple division problems; find numbers to complete a number sequence; understand = sign as an equality.	£1.05 is written as £1.5 or £1.5p £1.05 is written as 105 or £105 83 - 37 = ? , smaller digit from larger gives 54 Unfamiliarity with the division sign (e.g. 45 ÷ 5 , 20 ÷ 2)
Level 2a	Recognise a correct method for solving simple multiplication problems; calculate simple products or quotients using times tables; order a set of 3-digit numbers; show a simple fraction pictorially; interpret data in a table.	Misunderstanding the remainder in context (no bus for 30 children) Scale readings 705 or 700 instead of 750
Level 2b	Draw reflected images of given shapes; identify the units for measuring given attributes of an object; solve simplest addition problems involving regrouping; calculate simple sums and differences; add coins to make a specified total less than £1.	Misunderstanding = sign in 32 + 6 = 40 - □ Misinterpreting the ÷ sign (e.g. as multiplication) Shading the snake symmetrically
Level 2c	Calculate half of a 2-digit number; find a number between 10 and 20 in a given set of numbers; calculate remainders from simple division.	Using 4 coins for 23p rather than 3 coins: not reading the two parts of the question 38 - □ =11 gives 29 "3 pots of 4 pencils" = 3+4 pencils
Level 1	Recognise a given geometric figure; draw geometric figures of different sizes; recognise a right-angle; read data from a table; write a spoken 2-digit number in digits; find the largest number in a given set of 2-digit numbers; use = and + symbols correctly; interpret a diagram representing a 2-digit number.	Labelling the lemonade bottle as 2 Kilograms capacity Read scale as 700 (nearest whole mark on scale) 18-4=15: counting down error Misunderstanding the term 'lighter' Reading 'right-angle' as triangle or rectangle
	Performance of children at higher levels includes those indicated for lower levels.	**The errors listed are most likely to be made by children at the level adjacent.**

Figure 1: Performance descriptions and errors by level, 1997

In general of course the average and below average attainers on the test had most problems with the language and presentation in the test, caused by lack of understanding of terms or by poor, complex or artificial questions. In this sense the test can be viewed as a measure of the schools' success in induction of the children into the formal academic language games required (Walkerdine, 1988). One has to be concerned about the impact of this on the practice (literally) of teachers who must prepare the children for these tests. It seems that there is a level of ability or at least social apprenticeship required to break through the test barrier and begin to show what the children know rather than what they don't know: Cockcroft made this point strongly about UK examinations at age 16 as long ago as 1982.

In the report to teachers on the project we were able to provide the map in Figure 1 and interpret it in such a way as to paint a picture of progression in mathematical thinking found across the ability range likely to be found in their class. We said, for instance (Doig, et al, 1997, page 4):

> *... there was evidence that many children were puzzled by number sentences in arithmetic. For most children, other than those achieving level 3 on the test, the equation 32 + 6 = 40 - ? made no sense... Only around half of the level 3 children correctly answered this question. Children often respond to the = sign as a demand for an immediate action or calculation (a process conception of the sign where = is read as 'makes'). But here an understanding of the sentence as an equation is demanded (a structural conception, where = is read as 'is the same as').*

Results at age 14

The number of items and errors for the 14 year olds required that the report contained a number of maps like that above for the different topic areas, or 'attainment targets' of number, algebra, geometry and data-handling. For instance in the report to teachers about children's progression in thinking about number (Fox et al, 1997, page 7-8):

> *The weakest children in this study ... sometimes ignored the minus or percent sign, or ordered decimals in the form 7.9, 7.10, 7.11 etc. They avoided fractions in various ways, for instance by assigning a whole number value to a probability...*
>
> *The over generalisation of whole number ideas to the extended numbers led to errors in fractions, negative numbers and percentages. The children at level 4 were likely to see the 3 parts shaded out of ten parts as 3/7 or even one third. They added -7 and 4 to get -11, and even added 15 to -10 to get 25.*

At levels 5 and 6 there were still signs of intuitions based on whole number being extended. Thus children may think that zero is the lowest number. A well-known "largest is smallest" conception found in children's ordering of decimals (i.e. 0.155 is smaller than 0.15 because 155 is larger than 15) was found in a new form: some children appeared to think that adding a positive number to a minus number would make it smaller. Errors with subtraction were common: -2 - (-8) = -10 or +10 for instance, and may be based on combinations of rules for whole number (2 plus 8 is 10) and rules learnt for negatives (minus a minus is a plus). The processing strategy of 'detachment of signs, manipulate numbers as whole numbers, and reattach signs' seems to prefigure similar strategies for algebra....

The detail available from the 14 year olds work should be relatively more useful to teachers: there is a larger collection of errors, many significant diagnostically. Although there were still many questions which appeared to yield minimal diagnostic information, and certainly still problems with the children's misreading and misinterpretation of questions, in general the 121 common errors recorded indicated erroneous *mathematical thinking* rather than a mere communication problem.

The general trend in what children can and cannot do should be of value in itself, and some of the specific errors identified could be of practical use in teaching. Thus for instance we were able to describe progression in children's understanding of negative numbers (Fox et al, ibid, page 12) :

In the children's responses on the test we see this in negative numbers when children treat the two parts of the number, the minus and the 'number' separately. Thus in 'number lines' the scale may be marked -20, -30, 0, 10, 20,..(because the ordering is 20 then 30, and the minus sign is attached afterwards) and later the sequence gets -4 inserted thus: -7, -4, 1,... (because the sequence is read 1,4,7 and the minus sign is afterwards attached). Similarly we can explain why children answer -4 + 7 = -11.

More able children carry similar approaches into their subtraction of integers, ... when asked to choose integer-cards to give the lowest possible answer, 26% of the children said -2 - (-8) and gave the answer -10. In another part of the question some gave the same subtraction -2 - (-8) to give the largest value 10. The separate treatment of the two parts of the integer is a reflection of the origins of the concept in whole number experiences where -2 refers to the process of subtracting the whole number two. Perceiving the two parts as a single entity, in fact a number, is a requirement for the completion of the concept ...

Conclusions and discussion

In conclusion, although most of the errors we found were not substantially 'new' to researchers, and maybe to many sensitive mathematics teachers, we have been able to begin to map the errors children made on an ability scale alongside their test scores and the items they succeeded on, and we have been able to inform teachers about specific errors of significance and to hint at diagnostic teaching related to them which might improve their practice. This has helped to provide a fuller picture of the mathematical thinking to be expected of the children in typical classes of 7 and 14 year olds, and we hope that this will help to enrich teachers' mental models of their learners. Apart from subject knowledge this is believed to be the most essential element required for expert teaching (Fennema and Franke, 1992).

The significance and limitation of the quality of this work is, of course, circumscribed by the quality of the questions in the test, which was beyond our control. This was especially unfortunate in the case of the seven year olds, and we do not plan to develop this approach further with this age group at this time.

In future we propose to make more effective use of the research literature in designing items for the older children, and we plan a deliberate process of evolutionary selection of effective items. The extensive use of pre-testing and a plan for development and reporting extending over 3 to 4 years will support this.

We argue that the criterion for the design of items which are diagnostic should be made as important as the criterion of 'fairness' and political accountability. We expect the significance of the tests for teachers to be strong enough to encourage them to opt in to this project, but it will be important that any recommendations for practice are soundly based and researched to avoid demoralising teachers.

The vision set out argues for the integration of the classroom diagnosis and teaching concerns into the purposes of national assessment. The tensions between two conflicting roles of assessment should here be recognised. In the past national testing has been widely perceived to be about external accountability of teachers and schools. There has been an assumption that publication of tables and results will exert pressure and that this would provide motivation to raise achievement and 'standards', forced from outside as it were. The logic of error analysis points in a different direction, towards informing teachers and schools about their pupils' thinking. As a consequence testing may have an educational role, helping teachers to improve their classroom practice.

Finally there is the motivation of the government agencies: they face a contradiction between their desire to ensure that the tests are 'fair' and reliable summative assessments of the curriculum (responding to the political community), and the need to justify the tests as instruments which can support improvement in teaching practices through our diagnostic programme (responding to the research and teaching community). This is a reflection of the political contradictions in the new UK government's standpoint on schools and teachers: declarations of 'zero tolerance' of failure alternate with promises of support for 'education, education and education'.

We are aware of this tension and the nature of the conflict this may induce. A final and essential element of our plan therefore will be to try to involve the widest possible community of researchers, schools and teachers in our project, through their professional associations, journals and the school improvement movement.

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PATTERNS OF MATHEMATICAL MISUNDERSTANDING EXHIBITED BY CALCULUS STUDENTS IN A PROBLEM SOLVING COURSE

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Why do students often experience difficulties in understanding and dealing with mathematical ideas? This is an important question that need to be investigated in terms of explaining basic categories of analysis that include the presence or lack of different type of knowledge in the students approaches to different mathematical problems. This study documents the use of an integrative model of analysis that involves modes of thinking related to the discipline as a way to explain difficulties shown by students. An epistemic knowledge in which students need to search for mathematical arguments to apply their resources could be categorized differently from the use of problem solving strategies. The presence of metacognitive strategies appears as transversal knowledge that influence students decisions and application to understand and think of diverse forms of solutions. Students' approaches to the problems go hand in hand to what is seen as important in problem solving instruction.

Introduction

Lakatos (1978) presented a view of mathematics in which conjectures or problems play a central role in the development of mathematical ideas. This view is essential for developing a problem solving epistemology in which the learner is actively engaged in understanding and developing mathematical content. Lerman (1990) referred to Lakatos' view of mathematics as the "quasi-empirical program" in which the growth of mathematical knowledge is seen as a process which involves conjectures, proofs, and refutations of mathematical ideas. In this view, there is room for uncertainty of mathematical knowledge as part of the nature of mathematics. Schoenfeld (1994) stated that:

> ...learning to think mathematically means (a) developing a mathematical point of view -valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade and using those tools in the service of goal of understanding structure- mathematical sense making (p. 60).

Schoenfeld also suggested that the learning environment in which the students solve mathematical problems should become a culture of sense making. That is, solving mathematical problems is not simply getting answers but looking for connections, approaching problems in different ways, generalizing solutions, and extending problems into other domains. From this perspective, mathematics is not a fixed knowledge of rules and procedures already developed and polished by experts, but an open area with room for negotiation and discussion of mathematical ideas or conjectures.

The approach adopted for teaching calculus in the study focused on the discussion of problems as a means to developing understanding of calculus. The course also included a metacognitive component in which the students were asked to monitor their processes when solving problems. For example, the students would explain what they were doing, what they were expecting to accomplish, and why they were using specific strategies. This type of activity seemed to be useful in monitoring the students' processes while solving the problems. Criteria for judging whether or not a mathematical statement is true are part of the epistemic component of mathematics and play an important role in the use of problem solving. For example, extending problems into other domains often requires analysis of critical points that need to be considered in the process of transforming the problems. As a consequence, discussion of these criteria should also be incorporated into mathematical problem solving instruction.

The purpose of the study is to analyze the work shown by college students who took a first calculus course with emphasis on problem solving. Although the main part of the analysis will focus on documenting difficulties that students encountered while dealing with two mathematical problems, it was also important to explain the consistency and implementation of problem solving activities used by the instructor during the development of the course. That is, some of the difficulties shown by the students might be also explained in terms of what the instructor actually valued end asked the students to do consistently in the classroom.

Subjects, Frame of Analysis, and Procedures

Seven college students were interviewed towards the end of the course. The purpose of the interviews was to obtain information concerning the main ideas that the students use when solving problems and to probe the difficulties that they encounter when dealing with specific problems. Problems which could be solved by using different means were used as a vehicle to elicit information about the students' understanding of a particular problem and their ideas about the way that they plan to solve the problem. The selection of strategies, the reasons for changing strategies while solving the problem, and the understanding of basic mathematical concepts involved in the problem were aspects to be explored during the interaction. The students were asked to think aloud when solving the problems; the interviewer asked clarification questions whenever required. The information collected from the interviews was contrasted with the information gathered from the assignments and class observations in order to identify basic categories in the ways that the students approached the problems.

The research questions that are discussed in this study are:

(i) What difficulties did the students encounter in the use of problem solving strategies when solving mathematical problems?

(ii) Were those difficulties related to the class instruction?

A main objective of the study was to explore in detail the difficulties that the students experienced in solving the problems. For this purpose, a model presented by Perkins and Simmons (1988) was adopted for the analysis of the student's difficulties. Perkins and Simmons introduced this model as an integrative model for explaining some difficulties that students normally experience in science, mathematics, and computer science. The model included four frames of knowledge (categories that distinguish kinds of knowledge), that is, i) content, ii) problem solving, iii) epistemic, and iv) inquiry. This model considers heuristics and metacognitive strategies within any of the four frames and what Schoenfeld identified as beliefs within the problem solving or epistemic frames. Perkins and Simmons suggested that heuristics, beliefs, and self-monitoring practices are orthogonal to each of the frames of knowledge. They stated:

> [I]n general, one set of contrasts addresses the form of the knowledge in question - strategic, background beliefs, autoregulative - whereas the four frames address what the knowledge in question concerns - matters of content, problem solving, epistemology, or inquiry (p. 314).

Although the models discussed by Schoenfeld (1992) and by Perkins and Simmons (1988) share the need to include different types of knowledge for analyzing students' understanding, there are some distinctions regarding the exclusiveness of the frames. For example, the students' difficulties in using a systematic list to identify patterns could be related to the problem solving frame, while the students' difficulties in supporting the validity of a statement could be related to the epistemic frame.

The main characteristics of the frames (content, problem solving, epistemic, and inquiry) are useful in order to follow the corresponding analysis.

The content frame includes the terminology, definitions, and algorithms or rules related to the content. For example, algebra might include a variable, expression, equation, solution, and graph as important components of the content frame. It also includes the corresponding metacognitive strategy associated with the use of the content. Ways to recall information or to use the notation are associated with this frame.

The problem solving frame includes specific and general problem strategies, managerial strategies, and beliefs about problem solving. This involves the solving of routine (textbook) and nonroutine problems.

The epistemic frame includes a set of criteria used to validate the use or acceptance of a particular result, that is, the evidence or explanation that clarifies the use of a particular concept, rule, or procedure.

The inquiry frame includes specific and general beliefs and strategies that are used to extend or challenge the knowledge of specific content.

Students may exhibit naive or limited notions regarding the four frames, they may show hardly any notions at all, or they may have no idea what the epistemic foundations of a particular subject matter are, or how to explore the domain completely. Therefore, it is necessary to analyze the students' understanding and ways of solving mathematical problems by focusing on these frames. Such focusing could help provide information about what types of activities should be included in mathematical instruction.

Analysis of the Students' Work

The students showed difficulties in the use of symbolic language to represent concepts, such as functions, limits, and derivatives. Although they were able to recognize and use notation that was similar to the examples discussed during the class, that is, f(x) = 3x and g(x) = 3, they failed to apply the notation to cases, such as $S(x^2 - 3)$, and composition functions. The students discussed several examples in which some functions were graphed; however, they often failed to use the elements discussed during class that helped determine the graphs. They strongly relied on assigning some values to calculate some points of the graph. Perkins and Simmons (1988) stated that "...as has been widely recognized, students do not approach subjects new to them with empty minds. They bring preconceptions that often rival and override those of the topic itself" (p. 308).

The most common approach used by the students to calculate lim f(x) when x goes to **a** was to evaluate f(a) and check whether or not the result was a number. For example, when the students were asked to justify the existence or non-existence of **a** so that $\lim\left(\frac{2x^2 - 3ax + x - a - 1}{x^2 - 2x - 3}\right)$ when x approaches 3, they relied on replacing the variable x by 3 in the given expression. Only 5% of the students recognized that the numerator and denominator could have (x - 3) as a factor. That may indicate that the students' understanding of the content and ways to use it is limited to the types of experience that have been successful for them in a specific context; they failed to extend the problem or to examine other possibilities.

It was observed that the students adopted several tactics that they used to apply some concepts. For example, the concept of derivative was associated with the terms "tangent" or "maximum". As a consequence, in any problem that involved the term "tangent", they rushed to get the derivative of the expression and to try to use it in the problem. For example, one of the problems of the assignments involved *approximating* the rectangle with maximum area inscribed in a triangle; the students' response was to get the derivative of the algebraic representation that defined the area and to examine the critical points. This suggests that the students developed or adopted a response to the applications of some concepts and used it as a mechanism for solving the problems.

Students may become proficient in the use of rules or algorithms but fail to recognize the principles that support them. The epistemic frame is important because it may help students to judge and examine the mathematical ideas studied during mathematics instruction. For example, the students experienced difficulties discussing the validity of statements, such as, "if two functions have the same derivative, then they are equal"; "a rational function is continuous except when

$x = 0$"; "if limit f(x) + limit g(x) exists when x approaches **a**, then lim f(x) and lim g(x) both exist when x approaches **a**"; and other similar statements. This may suggest that the students' examinations of these types of statements are based on "naive" intuitions and are not often based on the search for examples or counterexamples that could test those statements. It was also observed that the students intended to generalize results based on the exploration of a few cases. For example, since $(f(x) + g(x))' = f'(x) + g(x)'$, then they thought that $(f(x)g(x))'$ was $(f'(x)\ g'(x))$. That is, in general, the students lacked the strategies for criticizing the validity of mathematical statements.

The students struggled when attempting to extend the problems into other more general contexts. It was observed that the students showed resistance when they were asked to explore other applications or linkages of the problems. Perkins and Simmons (1988) stated that "students, however, show little tendency to engage in problem finding and, indeed, conventional schooling offers few opportunities for such activity" (p. 314). Indeed, the students indicated that the ultimate and most important stage when solving problems was to get the solution. They did not show interest in pursuing the problems after they were able to solve them. They were aware that the type of practice that often is required during instruction to solve the problems, but not to go beyond getting the solutions. It is suggested that the students could consider the exploration of other applications of the problems or content, if such activities become a regular part of the instruction and are components of the students' evaluation. For example, the problems should explicitly incorporate questions regarding extensions, applications, or transformations of the problems.

The students often relied on the use of an "equation cranking approach" for solving the problems. They pursued some calculations or some data transformations in order to identify or relate an equation that they could fit to the problem. This approach often impeded the students' search for possible qualitative solutions to the problems.

The students also showed awareness of strategies that have been useful for tackling problems from textbooks. Perkins and Simmons (1988) suggested that students evolve "naive" notions or straightforward and pragmatic generalizations for solving problems. For example, they often recognize a pattern of solutions embedded in the textbook problems. The textbook problems are organized in such a way that they are similar to the examples given for that section and one can recognize the material and the methods for solving them. In addition, the students' ways for checking the solutions of the problems were based on matching the solutions provided at the end of the textbook with their solutions. They did not use other strategies that could help them test their solutions. Therefore, the students' final target was to get the response given in the textbook. As a consequence, any intention to reflect on the sense of the solutions, to look for other applications of the problem, or to search for possible extensions was often reduced to finding the given solution.

This phenomenon also influenced the confidence level of the students about the solution of the problems. For example, the students often asked for the solutions of the problems on the assignments. When working on the assignments that did not include the solutions, they hesitated to discuss the likelihood of their solutions. Koplowitz (1978) stated that "if we are to help the students become better problem solvers we must first help them develop a better sense of when they have solved a problem and a better sense of when a particular method is appropriate to use in a given problem" (p. 307).

Developing the students' confidence involves encouraging them to consider ways to evaluate their work and to approach the problems in several ways.. Santos (1996) pointed out that at the beginning of the course the students might use the solutions of the problems given by the instructor as a means to verify their solutions. However, it is necessary to show the students that verifying the solution is part of the process of solving the problem and is, therefore, one of the responsibilities of the problem solver.

Discussion of Results

The students relied on matching their mathematical resources, mainly the ones recently studied, to the information given in the problem. They were also reluctant to consider a new frame for solving the problem. Although during class instruction the importance of considering several possibilities or ways for solving the problem before getting involved in a specific approach was emphasized, the students seemed to lack this kind of flexibility, and they showed consistency in considering only the approach that appeared to be "safe" in solving the problem. They failed to derive a possible set of strategies that could lead to the solution of the problems. There was no clear evaluation of what they were doing or where they were going while working on the problems; however, if their initial intent did not help them make any progress, then they often examined other alternatives. That is, total failure in solving the problem was normally the only outcome that motivated the students to search for other alternatives.

New considerations for approaching the problems often came after having worked on the calculations that were related to some terms involved in the statement of the problem and not being able to use them to solve the problem. For example, "tangent to the graph" suggested to them to get the derivative of the function; "at the point 3" suggested to the students to evaluate at 3; "the straight line tangent to" suggested the same slope. However, they failed to recognize the role of the parameters **a** and **b** and the relationship with the information that they obtained from the statement. It was evident that they knew the content involved in the problem but were unable to make the necessary connections for using it.

It is suggested that the students made sense of the statement of the problem in parts without considering the problem as a whole. For the rectangle problem, the students also explored the familiar terms until they got an expression that related one side of the rectangle as a function of the other side. The students at this stage did not know how to deal with this expression and struggled to relate it to the statement of the problem.

Originally, a main purpose of the course was to implement a problem solving approach in which the students would have the opportunity to participate in discussions involving the process of solving problems. The students could thereby speculate on possible solutions, test examples and counterexamples, and present their ideas about the content involved in the problems. It was observed that the instructor initially encouraged and motivated his students to participate during the class; however, the students' participation decreased notably as the course advanced. Two explanations are related to this phenomenon. One is the extent of the material that needed to be covered during the course, and the second is related to the difficulty of managing a new classroom situation in which there are more variables to consider in order to make classroom decisions. For example, to make the students' participation meaningful required spending time reflecting on the students' ideas and ways to link them to the content being studied. The instructor intended to direct the class discussions towards this goal initially, but, at the same time, he wanted to make sure that the students could identify sequential reasoning while discussing the content or problem. Byers (1984) stated that "there is usually the desire to keep ideas as simple and straightforward as possible within the framework of the course" (p. 35). Unfortunately, this direction was the most popular and desired among the students.

Although the instructor fully intended to provide a problem solving environment in his class, he ended up giving more emphasis to finding the solutions of several problems and showing the students the content involved in the problems than in letting the students express their ideas and discuss the problems on their own. This suggests that it is difficult to overcome a way of teaching that has permeated the mathematics education environment for many years. Even when the instructor showed willingness to incorporate other activities into the classroom, he sometimes presented the solution of a problem or the proof of a relationship on the blackboard to the class without determining whether or not the students were following his presentation.

Concluding Remarks

It is suggested that it takes time to overcome some of the students' practices for working on problems. For example, it was observed that the students' first approach to the problems was to get involved in calculations but often without having a clear understanding of the problems; this strategy was inconsistent with the activities developed during class instruction. Although during the problem solving instruction the students were encouraged to explore various alternatives and to monitor their progress when using them, it was observed that the students mainly reflected on what they were doing when they reached an impasse while trying to solve the problems. They did not recognize the need to check their solutions and it was only when the interviewer suggested that it was important to verify and to understand their solutions that they did so.

Results show that in order to help students to improve their ways of solving mathematical problems, it is necessary to pay attention to the mathematical content, cognitive and metacognitive strategies, ways of validating and using mathematical arguments, and ways of extending the problems. When all these ingredients consistently become part of mathematical instruction, then the students may develop their own frames for solving mathematical problems. These frames should be similar to what people in the field of mathematics use while working on mathematical problems or developing mathematical ideas.

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Appendix: Examples of problems used during the students' interviews

1. Is there any <u>a</u> so that

$\lim \left(\frac{2x^2 - 3ax + x - a - 1}{x^2 - 2x - 3}\right)$ exists when x approaches to three?

2. Discuss the following questions: (i) If lim f(x) and lim g(x) do not exist when x approaches a, can lim [f(x) + g(x)] exist when x approaches a? (ii) If lim f(x) exists and lim[f(x) + g(x)] exists when x approaches a, must lim g(x) exist when x approaches a? (iii) If lim f(x) exists and lim f(x) g(x) exists when x approaches a, does it follow that lim g(x) exists when x approaches a.

3. Find values of **a** and **b** so that the line 2x + 3y = a is tangent to the graph of $f(x) = bx^2$ at the point where x = 3.

4. Find all rectangles with integer sides whose area and perimeter are numerically equal.

THE EVOLUTION OF MATHEMATICAL PRACTICES: HOW ONE FIRST-GRADE CLASSROOM LEARNED TO MEASURE

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This paper provides an analysis of one first-grade classroom's mathematical activity as it engaged in a measurement sequence that was designed to support students' development of increasingly sophisticated measuring conceptions. We have used the interpretive framework developed by Cobb and Yackel (1996) to analyze the evolution of the classroom mathematical practices that became taken-as-shared as students engaged in measuring activities. We found that the taken-as-shared measuring practices of this classroom community evolved over time and became relatively sophisticated as the class engaged in measuring situations.

The intent of this paper is to document the mathematical practices that emerged as one first-grade community engaged in an instructional sequence designed to support their development of increasingly sophisticated measuring conceptions. A large body of literature on children's conceptions of measurement has amassed over the past three decades with undoubtedly the most influential work being that of Jean Piaget and his colleagues (Piaget, Inhelder, Szeminska, 1960). As a result of Piaget et al.'s analyses, many researchers have tried to isolate the ages at which children develop certain measurement concepts (e.g., Smith, 1968). Other researchers devised training programs in order to increase the acquisition rate of measurement concepts (see Beilin, 1971). However, few studies have been done which address social and cultural influences on children's development of measuring abilities. Further studies might describe individual students' development as participation in the local mathematical practices of the classroom community. The purpose of this paper is to document the collective mathematical/measuring practices that were interactively constituted by one first-grade community. This documentation can then be used as the background against which to describe individual students' development (such a report is in progress). This paper is significant in that by documenting the evolution of the collective mathematical practices, an account of the mathematical content that emerged during the course of the measurement sequence is detailed.

Data Corpus

The data for the study was collected during a first-grade teaching experiment in the second semester of 1996 and consisted of 62 video-recorded mathematics lessons and three sets of field notes. There were 16 children in the classroom, 7 girls and 9 boys. The teacher was an active member of the research team and was working to establish a practice that is consistent with the reform guidelines of the NCTM Professional Standards for Teaching Mathematics (1991).

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The Evolution of the Classroom Mathematical Practices

The framework that guided the analysis was developed by Cobb and Yackel (1996). The framework coordinates a social and a psychological perspective. The social perspective consists of three aspects: classroom social norms, sociomathematical norms, and classroom mathematical practices. These have been explained in detail elsewhere (Cobb and Yackel, 1996). The most underdeveloped aspect of the framework is classroom mathematical practices. Classroom mathematical practices involve the taken-as-shared ways in which the classroom community comes to act, communicate, and symbolize mathematically. In describing the mathematical practices that became taken-as-shared, it is important to note that no claims are made about how any one individual student is reasoning. Rather, the patterns in the ways the community is acting and describing as they act with tools are analyzed. One way to document the mathematical practices of a community is to analyze the public discourse (i.e., the argumentation in whole-class discussions). Mathematical practices involve ways of acting and communicating that become taken-as-shared after a prolonged period of negotiation. One indication that a mathematical practice has become established is when a particular way of acting or symbolizing is beyond justification (Yackel, 1997). Thus, analyzing the collective argumentation over an extended period of time was the method that was used in this paper to document the evolution of the classroom mathematical practices.

Analysis

The objective of this teaching experiment was to design and enact two closely related instructional sequences on (1) measurement and (2) mental estimation and computation with numbers up to 100. The intent of the measurement sequence was to proactively support students' measurement conceptions, not to teach them the correct measuring procedures per se. These conceptions would then support students' development of strategies for mental estimation and computation.

The emergence of math practice 1 - Measuring by pacing (3 days)

At the beginning of the measurement sequence, the teacher told the class a story about a king who lived in a kingdom and sometimes needed to measure for various purposes. The teacher asked the students to imagine that they were servants in the kingdom and help the king determine how he might use his feet to find how long various items were. They decided that he could measure by placing his feet heel-to-toe and counting each pace as he walked alongside the item he was measuring. Subsequent instructional activities included asking students to pretend to be the king and find how long various items around the classroom (kingdom) were using their own feet. As students worked in pairs measuring items, we[1] observed their activity and asked clarifying questions when we were unclear about their solution method. The teacher typically initiated whole-class discussions that drew on her and our observations of students' activity. As students participated in these discussions, the

[1] We use the word "we" throughout the remainder of this paper to refer to the research team which included Paul Cobb, Beth Estes, Koeno Gravemiejer, Kay McClain, Beth Petty, Michelle Stephan, and Erna Yackel. It is paramount to keep in mind that decisions were made on a daily basis throughout the teaching experiment and that the classroom teacher was integral to these decisions.

practice of measuring by pacing emerged. Initially, students paced the length of an item and counted each step that was taken. It appeared to be taken-as-shared that measuring involved counting paces, and the result of measuring signified a sequence or chain of individual paces. This claim was supported by the observation that students' explanations of their pacing activity concerned how many steps had been taken rather than the amount of space or the amount of an item that had been covered. For instance, consider an activity in which two students measured the same item and obtained different measures. The taken-as-shared way of explaining the contrasting results involved explaining that the student with bigger feet paced *faster* (less *paces*) than the student with smaller feet rather than the student with bigger feet taking up more space as she paced. In order to support a conceptual discussion of students' pacing activity, the teacher made a record of each footstep by placing masking tape at the beginning and end of each pace. Once this record was made, some students began to argue about the spaces covered by each pace rather than simply the paces per se. Thus, the taken-as-shared goal of measuring seemed to involve covering the physical extent of an item with a sequence of paces.

As pacing activities continued, whole-class discussions began to focus on how students counted their last few paces when measuring an item. Initially, it seemed to be taken-as-shared that measuring was grounded in the *activity* of pacing; in other words, the placement of each foot defined the space that was being measured. For instance, as one student paced along the edge of a carpet, her last pace extended past the end of the carpet. Another student argued that the rug could *not be measured* since the last pace extended past the end of the rug. He argued that "the carpet can't move," as though the *act* of measuring took priority over the physical extent of the rug. This kind of contribution was significant in that no one challenged this interpretation. It seemed to be taken-as-shared that measuring was dependent upon the act of pacing. To account for parts of a pace that extended past the endpoint of an item, some students removed their last pace and said "14 and a little space," for example. Other students turned their last pace sideways so that it fit exactly within the end of the rug and counted "15." This suggests that as students paced the length of an item, some began to view pacing as a way of filling a space. Eventually, students' explanations began to focus on the amount of space that was filled as the result of pacing. Thus, the mathematical practice that became interactively constituted involved filling space by pacing the physical extension of an item (space was filled in the sense that students did not extend their paces past the endpoint of an item).

As students continued to engage in pacing activities, it became taken-as-shared that the result of pacing appeared to signify a sequence or chain of individual paces. For example, if students paced the carpet and obtained 15, they argued that 15 was the last foot (or the space covered by the last foot) that was paced rather than the whole span of the rug. For us, this way of reasoning about space was mathematically significant in that we were trying to support students coming to act in a spatial environment in which measuring signified an accumulation of distances (i.e., that "four" signified the space covered by four paces, and the space covered by four paces was included in or nested in the space covered by five paces, etc.). Thus, we felt that it would be productive if students had an opportunity to discuss in whole class what the results of pacing signified.

The emergence of math practice 2-Measuring by iterating a strip of 5 paces (7 days)

After measuring by pacing had become relatively routine, the teacher told the students that the king was spending all his time measuring rather than doing things that were necessary to run a kingdom. She asked if they could think of another way to measure so that the king himself would not always have to measure things for other people. After several suggestions, the teacher related that one of the king's advisors suggested putting a picture of the king's foot on a piece of paper. The teacher asked students to draw five feet on a strip of paper and then asked how the king might use such a strip. One student offered, "This would be faster. . . if he had lots more feet, he could just take a big step with all the feet together. . ." This contribution indicates that some students were drawing on their participation in the first practice of pacing to anticipate how using the footstrip would be more efficient. The students created strips that consisted of five feet drawn together heel-to-toe (see Figure 1) and iterated the *footstrip* alongside an item counting "5, 10, 15. . ."

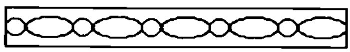


Figure 1. The footstrip.

The second mathematical practice that emerged as students engaged in measuring activities with the footstrip involved measuring by iterating a strip of five paces. The emergence of the second mathematical practice can be seen to grow out of the first mathematical practice of measuring by pacing. It became taken-as-shared that the placement of one footstrip signified five paced feet. Also, it was taken-as-shared that something 15 long signified the result of taking 15 paces without actually going through the pacing activity itself.

A second critical issue re-emerged as students participated in the second mathematical practice. Recall that the previous mathematical practice involved filling the extent of items with individual paces and that this activity was integrally tied to the activity of pacing. As students constituted the *second* mathematical practice, the idea that the activity of measuring took priority over the physical extent of the item became an explicit focus of conversation. Two students were using a footstrip to measure the length of a cabinet in the classroom which ended against a wall. The cabinet actually measured a little over 17 king's feet, hence part of the footstrip from the fourth iteration extended past the cabinet and up the wall. Most students argued that since the footstrip extended past the endpoint of the cabinet, the cabinet could not be measured. Recall that this issue had emerged as students paced in the previous practice. Again, the *activity* of measuring took priority over the physical extent of the item being measured. One student, however, began to argue that this method of measuring would work if they pretended to cut the footstrip where it met the end of the cabinet. After a prolonged discussion, students' contributions indicated that they accepted this new way of measuring with the footstrip. In fact, the day after this discussion, no student appeared to have any difficulty when part of the footstrip extended past the endpoints of an item. This discussion seemed to be a pivotal point in the negotiation of the second mathematical practice. As the result of this negotiation, mentally cutting the footstrip was constituted as taken-as-shared. In other words, there seemed to be a shift in students' explanations of their measuring activity in that now mentally cutting the footstrip was beyond justification. The space to be filled was now independent of activity, independent of the *placement* of the footstrip. Now, the space to be filled took priority over the measurement activity rather than the act of placing the footstrip defining what was being measured. Measuring was no longer tied to the physical act of placing a footstrip; rather it had become taken-as-shared that the footstrip could be mentally cut as the need arose.

A transition also seemed to begin as students continued to participate in conversations in which they measured with the footstrip. The transition involved moving from filling space by iterating to imagining and describing amounts of space that had been measured or could be measured. In other words, some students were beginning to talk about "whole spaces" rather than paces or footstrips. In fact, many students used their hands to show other students how much space was covered by five paces, for instance. This was significant because it indicated the beginning of a shift from describing the results of measuring as a sequence of *individual* paces to describing the result as whole measured spaces. Although this way of explaining became taken-as-shared as students participated in the *next* practice, a transition seemed to have started as students iterated the footstrip.

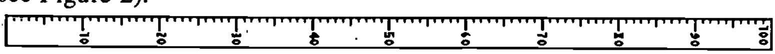
<u>The emergence of math practice 3-Measuring by iterating a bar of ten</u> (11 days)

The teacher introduced a new scenario about smurfs who lived in a village and sometimes needed to know how long things were. To accomplish this, the smurfs used their food cans (unifix cubes) by placing them end to end and counting them. Our intent was that students' activity of measuring with the unifix cubes would in turn serve as a basis for a more sophisticated tool to be developed later. Initially students were given a bag of unifix cubes and asked to find how long various items around the smurf village (classroom) were. Subsequent activities included giving students a piece of adding machine tape and telling them that it signified the length of an animal pen. They were also supplied several other pieces of adding machine tape that signified the lengths of different animals in the kingdom (e.g., dog, cat, horse, etc.). Students were then asked to see not only if each animal would fit inside the pen but also how much extra room there was for the animal to walk around in the pen. The taken-as-shared method of measuring items in the classroom involved making a bar or rod of cubes that stretched the length of an item and then counting the cubes. The teacher suggested that there may be a more efficient way of measuring with the food cans and asked students if they could think of a way that the smurfs could measure without carrying around a whole bag of cubes. Students gave several suggestions including carrying only a bar of ten cubes which they named a *smurf bar*. Activities with the smurf bar included having students find the lengths of various items around the room and cutting pieces of paper signifying different-sized wooden boards for building a smurf house. The instructional activities here were different than before in that now the students were taking an external unit as the measuring unit as opposed to measuring with a part of their body (or a record of their bodily activity, i.e., the footstrip). The mathematical practice that emerged as students participated in the activities described above involved measuring by iterating a bar of ten. The emergence of the third mathematical practice of iterating bars of ten can be seen to grow out of the second mathematical practice. As a result of participating in the second mathematical practice of measuring by iterating a strip of five paces, taking the results of measuring as a given was taken-as-shared. I.e., taking 23 cans, for example, as signifying an object's measure and immediately iterating collections of ten cubes along a strip of paper so that the paper measured 23 cans long was beyond justification.

Further, the transition from talking about individual units to describing "whole spaces" that had begun in the previous practice seemed to become taken-as-shared as students measured by iterating the smurf bar. Now instead of talking about the result of measuring with the smurf bar as a sequence of individual cubes, students' explanations indicated that the result of iterating signified whole measured spaces. In other words, if a student had measured an item and obtained 35 cans as the result, it was beyond justification that 35 signified the whole space covered by 35 cubes rather than the space filled by the 35th cube. It seemed that the transition from talking about a sequence of individual units to describing the result of measuring as whole spaces that had begun in the second practice, had now become taken-as-shared. At the very least, it had become taken-as-shared that physically iterating along an item created a partitioned space. In other words, measuring was tied to the act of iterating and the result of this action was a whole space structured (partitioned) into collections of tens and ones.

<u>The emergence of math practice 4-Measuring with a strip of 100</u> (5 days)

Next, the teacher built on students' participation in the practice of iterating bars of ten to support the emergence of a new measuring tool that was to be 100 cubes long (see Figure 2).



<u>Figure 2</u>. The measurement strip.

First, the students made their own 10-strips by cutting adding machine tape that was ten cans long and used them to find lengths of items around the room. After the students had used the new 10-strip, the teacher taped several of the ten-strips together. In doing so, she attempted to ensure that the 100-long measurement strip grew out of the students' prior activity of iterating strips of ten; that students would build imagery of the 100-long strip as a curtailment of measuring with bars/strips of ten. For example, we hoped that the numeral "30" on the 100-long strip would signify not only three iterations of the smurf bar but also the space filled by 30 cans. Hence, we hoped that activities which involved the use of the measurement strip would build on students' participation in the third mathematical practice. Our instructional intent in introducing the measurement strip was so that students would no longer carry out the physical act of iterating. As students measured items around the room with the measurement strip, the mathematical practice that was constituted involved measuring with a strip of 100.

Initially, many students measured by laying the strip down alongside the item and counting by tens and ones until they reached the endpoint of the item rather than finding where the farthest endpoint corresponded with a numeral on the measurement strip. For example, when measuring the length of a table, many students laid the measurement strip next to the edge of the table and counted each collection of ten from the beginning of the table, "10, 20, 30" rather than simply read off the numeral "30." Thus, it appeared that many students' measuring activity with the new measurement strip evolved out of their participation in the last mathematical practice of iterating with a bar or strip of ten. Again, many students' initial use of the measurement strip involved the physical activity of iterating. However, it soon became taken-as-shared that the length of an item could be measured by laying down the measurement strip alongside the item and simply reading off the numeral corresponding to the position of the farthest endpoint. It was also taken-as-shared that when students measured the length of an item and read off a number, that number signified the space that extended from the beginning of the measurement strip to the line signifying the end of the item. This mathematical practice became established fairly quickly. Now it seemed to be taken-as-shared that students were acting in a spatial environment in which space was already partitioned. Space no longer had to be partitioned in activity by physically iterating. Rather, laying down the measurement strip simply specified the measure of the spatial extent. It was as if now students were acting in an environment in which an item was already partitioned, it already had a measure, and the measurement strip simply specified the measure.

<u>The emergence of math practice 5-Reasoning with a strip of 100</u> (5 days)

As the nature of the instructional activities changed, reasoning with the measurement strip to solve various problems became the fifth mathematical practice. These new problem situations were different in that students were no longer given a physical item to measure; rather they were told the measure of items, say 41 and 59 cans long, asked to specify the spatial extension of those items on the measurement strip, and then reason about the relationships between these spatial extensions. When students specified the spatial extent on the measurement strip, it was taken-as-shared that the extent specified an already-partitioned spatial extent whose measure was 45. It was as if the 45 cans signified an objective property of a partitioned spatial extent. In other words, the spatial extent did not have to be measured, it *was* 45 cans long and its length could be easily specified on the measurement strip.

In some tasks, students were asked to compare the lengths/heights of two items on the measurement strip, and in others they were asked to use the strip to add or find the difference between the lengths of items. In either case, it became taken-as-shared that a measure signified an objective property of an item. In other words, there had been a reversal between the spatial extent and its measure. Before, the spatial extent of an item was physically present and the goal was to find its measure by laying the measurement strip beside it. Now, the measure was given and the goal was to use the measurement strip to specify the spatial extent signified by its measure. In this way, the measure of an item signified an objective property of an item because the measure was not being found, it was being used (taken as a given) to specify an item's spatial extent. The spatial extent now signified the height of a stool or the height of a chair and these heights took on a life of their own so to speak. They could be used to compare with other heights and could be added to other heights. As can be seen, the taken-as-shared way of reasoning with the measurement strip went beyond simply specifying already-partitioned spatial extents on the strip. The practice that was constituted involved working from someone else's measure instead of actually finding the measure itself.

Conclusion

The collective mathematical learning of this first-grade class became more sophisticated over the course of the measurement sequence. At a more general level, we hope that this analysis serves as paradigmatic of the use the classroom mathematical practice as a construct for analyzing the collective mathemical activity of a classroom culture. We have illustrated that the mathematical practices did not occur in isolation but rather each grew out of the prior one, i.e., there was a history of development. Also, it is important to note that this community's learning did not occur apart from using tools such as the footstrip, the smurf bar, and the measurement strip. Thus, a further analysis would involve tracing the signifying function of the tools throughout the measurement sequence.

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CONCEPTUALISING MATHEMATICS TEACHING: THE ROLE OF AUTONOMY IN STIMULATING TEACHER REFLECTION

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Opportunities for autonomous learning are seen as key elements of the development of an orientation to reflective teaching, which in turn is essential to career long learning. Data are presented from one case from a study of teacher education students' use of an interactive multimedia resource. The resource seems to have contributed to the development of reflective skills.

Engaging teacher education students in learning about teaching mathematics

A challenge for mathematics teachers is to encourage students to see mathematics as useful, integrated, and interesting, and to foster environments where students can learn the necessary processes and attitudes to allow them to continue as learners of mathematics long after they have finished formal studies in mathematics. This double challenge exists quadrupally for mathematics teacher educators. Not only must they respond to the above challenge themselves, but they also must find ways to encourage their own students to see the *pedagogy* of mathematics as useful, integrated and interesting, and to develop processes for career long learning about pedagogy. We see the latter quarter of this challenge as integrally connected to teacher reflectiveness. This aspect of teacher education is the focus of this paper.

We have argued the following points are various times (Mousley & Sullivan, 1997; Sullivan & Mousley, 1994):

- there is a need to ensure that the methods and the message of mathematics teacher education are compatible;
- there is a need to challenge, rather than necessarily reproduce, teacher education students' prior experience of the teaching/learning process in mathematics;
- any change in beliefs and practice requires the development both of a sense of ownership of the change by the students and some stimulus for that change;
- teachers balance multiple competing perspectives while making decisions progressively over the course of lessons;
- teacher education programs need to acknowledge and prepare teacher education students for this complexity;
- the study of classroom dilemmas, especially with peers, can challenge prior conceptions and create awareness of the existence of alternative approaches.

We see, in particular, that the development of orientations to study and reflection on aspects of teaching are both the tools and the outcomes of these processes.

The emphasis on development of thinking, reflective practitioners rather than merely transmitting a body of knowledge and skills is widely recognised (e.g. Ashcroft & Foreman-Peck, 1994; Jaworski, 1995; Lerman, 1997; Mason, 1986; Schifter & Fosnot, 1993; Sch^n, 1987). Skovsmose (1994) argued for mathematics pedagogy that encourages a shift from technical knowing to reflective knowing: the latter involving the "competence needed to be able to take a justified stand in a discussion of (technical) questions ... the general competence needed to be able to react as a critical citizen in today's societies" (p. 101). It would seen that reflective knowing about the teaching of mathematics is vital for the same reason.

At least three key elements of this development of reflectiveness are:
- awareness of the complexity of aspects of mathematics teaching and of the possibility of different teacher responses in different situations;
- development of an orientation to reflective approaches; and
- emphasis on the language necessary to study and communicate about aspects of mathematics teaching.

The development of language for communicating about teaching forms a key element of our research. The framework that underpins our analysis of language usage by the students in the case studies summarised below has three levels:

- *Identification and description:* Comments are restricted to terms which are used to report on what is happening and to describing events, with little attempt to analyse or explain the purpose, rationale, or implications, of teaching actions.
- *Analysis:* Users go beyond merely describing, and analyse the implications of particular actions and events for broader teaching and learning goals.
- *Explanation:* Students describe and analyse teaching actions and events, and also explain the implications and particular actions and suggest alternatives.

These level are directly comparable to the unistructural, multistructural, and relational levels, respectively, of the SOLO taxonomy (Biggs & Collis, 1982).

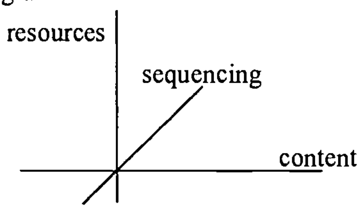
All of the above can be achieved within conventional lecturer directed teacher education programs. However, we argue that there are significant advantages in the incorporation of some measure of student autonomy particularly for programs focussing on teacher reflectiveness.

The term *autonomy* refers to a broad range of concepts. The following are some terms used to describe approaches that can refer to autonomous learning:

Learning modes	*Teaching modes*	*Metacognitive*	*Attitudinal*
self-managed	self-teaching	self assessment	self-actualisation
independent	flexible/alternative delivery	critical self awareness	self-motivation
self-directed	open learning		life-long learning
student-centred			empowerment

There is increasing recognition that autonomous learning modes can contribute to effective professional education (Bruce, 1995; Candy, Crebert & O'Leary, 1994; Bridges, 1997; Charleton, 1996; Jackson, 1997; Jones & Jones, 1996). It has also been argued that programs that incorporate opportunities for autonomous learning contribute to a shift from dependency to students taking responsibility for their own learning (Hoberman & Mailick, 1994; Popovich & VanVeldhuizen-Scott, 1994). Further, developments in technology increase options and expectations for more flexible and autonomous learning opportunities (Anderson & Alagumalai, 1996).

A key issue in the extent to which opportunities for autonomous learning contribute to the development of reflectiveness is students' motivation. A study of the motivation of secondary mathematics students identified three aspects which can contribute to student learning—interest, control, and arousal (Middleton, 1995)—and it is likely that these apply to mathematics teacher education as well. Clearly both interest and arousal are important characteristics of any educational program; but we see the sense of control as central, particularly with adult learners and most especially in programs where an orientation to continued learning is desirable.



Control itself is multidimensional. This figure illustrates three of its dimensions (i.e. programs can increase the student sense of control over resources used in the program, over sequencing of experiences, and over the content itself).

Of course, a further element is the sense in which students accept the control offered. Programs position themselves in various ways at different times, but decisions can be made that actively promote students' sense of control.

This is the report of one aspect of a larger research project. This aspect sought to examine the effect of increasing students' sense of control over the sequencing, and the content of a teacher education course focussing on mathematics.

Methodology

A group of undergraduate teacher education students used an interactive multimedia resource that allows the intensive study of a mathematics lesson, as part of a work requirement of a course on the study of mathematics teaching. Interaction with the resource was in small groups, for approximately 20 hours. (See Mousley & Sullivan, 1997, for a description of the development and structure of the resource.) The students had control of the emphasis and direction of their learning about teaching mathematics in that they could choose and sequence aspects of teaching to focus on. The research sought to examine the extent to which the students reflected on the teaching incidents presented, and the impact of our attempt to transfer some pedagogical control to the students.

Structured case study methods were used. Each subject contributed to survey items, the writing of lesson critiques before and after use of the resource, responses to formulated scenarios, interviews, journals maintained throughout the process, and reflective essays written after the completion of the program. Each student's interactions with others and with the resource were observed by research associates (taking structured field notes), and some groups' interactions were videoed. The subjects were also interviewed following their next practicum experience.

Data presented are from the responses of one case only—"Karrie". Some other data from this case have already been reported in Mousley and Sullivan (1997). There are two reasons from not using a different case: space limits the full development of a new case; also Karrie was selected from among a larger set of cases by research associates as representative of the data overall.

The development of reflectiveness

Space permits selections of a few elements from only three types of data to be presented here: critiques of the lesson presented in the resource, responses to a scenario, and a ranking of factors—each of which was sought before and after using the resource. As with all of the data in the project, the changes noted are subtle. Nevertheless, it is clear that there is a more sophisticated analysis of teaching after the use of the resource; and this seems to evidence the development of reflective thinking.

Prior to the use of the resource, the students watched the lesson that formed the basis of the resource and wrote a critique of the lesson. The following is a selection of representative comments from Karrie's critique:

> Good the way she goes around the class while they are discussing.
> Good the way she alternates who she asks questions ie; girl, boy, girl, boy.
> Tells the children a lot of information in a round about sort of way ie; Woolworths.
> Clear step by step instructions.
> Good way she uses blackboard to show flat shapes–3D.
> Sets the children to work well. They appear to understand the task well.
> Constant interaction with the children.

The critique is thorough and comprehensive, but is restricted to comments which *describe* actions and events.

The students also wrote a critique of the same lesson after using the resource, but prior to the next teaching practicum. Karrie's next critique indicated quite a different level of conceptualising mathematics teaching. She went beyond merely *describing* what was happening to *analysing* the events, with critique, justification, explanation and consideration of alternative approaches. Here comments included:

> Waits before moving around room. Could move around earlier to show speaking to whole class.
> Good the way she questions people to gain more answers, gets a clearer idea of what they are talking about.
> I think the practical application is not relevant to students (building for Woolworths). Need something children can relate to like new school building etc.
> Good way constantly asking question to provoke answers and get the class thinking.
> Good having a challenge to get children to think
> Good not to feed the children answers, instead we get an insight into what children already know. .

It is clear, here, that Karrie is more confident in her ability to critique this lesson, and has developed more sophisticated language for describing teaching. It can be inferred that Karrie is thinking about teaching at a more advanced level, which we see as the essence of increased reflectiveness.

During the interviews, students were invited to comment on classroom teaching scenarios (eg. Hoyles et al., 1984). The following in the response to one of the scenarios prior to the use of the resource:

Interviewer: Imagine that you are now a teacher and you have your own class. Yesterday you noticed that many children in your class were restless and uninterested in the mathematics lesson and they have been restless so far today. What will you do about it in this lesson? What might you do about this in the longer term?

Karrie: I don't know, probably make them all run around for a bit at first to get them all settled down. And then sit them down on the floor, and, or maybe go through some concrete tasks with them so that they all get a chance to fiddle around and have a go.

Interviewer: How about in the longer term?

Karrie: Try and make them more interesting so that they're not so restless. Try and, I don't know, build their interest level, so they enjoy it a bit.

These initial responses was superficial in that the restlessness was taken to suggest an excess of energy and lack of interest.

After using the resource, in response to the same scenario, Karrie responded:

> Get them more involved in the lesson so that they can come up with their own ideas for learning. So that they feel as though they are their lessons ...
> And in the longer term, probably the same thing, make sure that the lessons involve them all of the time.

Here, the earlier suggestions of exercise and interest were replaced with notions like "come up with their own ideas" which suggest acceptance of desire to engage the students in their own learning. This is indicative of considerable development in both the level of language used and the reflection on the act of teaching.

Another source of data from within the interview was to ask the students to rank factors of influence, presented on cards. The users were asked to arrange the cards in order, from most important to least important, and were told it was possible to have equal placings, and to explain the ones they felt to be most significant. After they had arranged the cards in order, subjects were asked to indicate a point below which the cards were of little or no significance to them. (The procedure used was an adaptation of the Q-sort technique [Sax, 1979; Best & Kahn, 1986] which had been structured for intensive work with few persons [see Sullivan & Mousley, 1996, for a full discussion of this procedure].)

Prior to the use of the resource, the cards that were identified by Karrie as being significant were as follows, in order. (Cards of the same rank are marked with a "/").

> Lesson fun for students / Variety / Good organisation / Creativity
> Good communication skills / Confident, caring but strict / Communicate with student on their own level / Catering for all abilities
> Honesty / Encouragement / Providing clear worked examples for the children to copy

Karrie's explanation of what she interpreted to be the meaning of the first five cards was as follows:

> Okay, a 'lesson fun for students' so that they've got interesting lessons.
> 'Variety' so that they're not doing the same old boring lesson everyday like for maths, not doing tables or multiplication or that.
> Ah, 'good organisation', so that I'm organised, the teacher's organised and knows exactly how it's going to shape out so she answers any questions and she's able to work through it.
> 'Creativity', making the lesson fun and exciting.
> And, 'catering for all abilities' not just going after the high level ability in your class, catering for everybody, of every level.

After using the resource, Karrie ranked the cards in the following way:

> Catering for all abilities / Meaningful and relevant / Getting the children to think for themselves
> Variety / Flexibility / Lesson fun for students / Good organisation / Good communication skills / Communicate with students on their own level
> Encouragement / Monitoring student understanding / Inspire students to create and succeed

Her description of the meaning of the first five of these was as follows:

> Catering for all abilities'. Provide the lesson so that you've got the 'slow' ones and the 'strong' ones.
> 'Meaningful and relevant', again, so that they think it has purpose. That you are not just saying it and then forget it.
> 'Getting the children to think for themselves' so that they feel that they are involved in the lesson, so that they have got to provide the answers to the questions.
> 'Communication on their own level' so that you are not talking above them, so that they are able to understand.
> 'Good organisation', so that when you get in the class, you don't want to have to be running backwards and forwards, and losing all the children.

Before her use of the resource, fun and interest were again seen by Karrie as most important, along with organisation. Afterwards, the children's thinking, their understanding and meaningful learning were ranked higher. This again is evidence of a more sophisticated understanding of teaching mathematics. The development in thinking is compatible with the stage suggested by Fuller (1969), which is notable in that this change occurred within a relatively short period of time.

In summary, it seemed that Karrie expanded her vocabulary for talking about teaching, has more insights into teaching processes and was more thoughtful after using the resource. There was a change from a view of teaching as entertaining to teaching as a process of engaging children in development of understandings. On balance, it seems that Karrie developed a more reflective and sophisticated conceptualisation of mathematics teaching and an extended ability to analyse and discuss classroom interaction.

A sense of control over the learning

Autonomy is essentially about learners taking control over their own learning. Our second research focus was the extent to which the sense of control offered by the resource was adopted by its users.

In data collected after using the resource, Karrie referred in particular to two aspects; the contribution which working in groups of peers makes to the learning process, and the sense of control offered. In a written response she used phrases like:

... work at my own pace ...
... no pressure ...
... groups provided an opportunity to discuss other interpretations and experiences ...
... enabled us to bring our own experiences ...

Such responses were reinforced by a journal reflection on the use of the resource:

... to work at our own pace was a bonus in learning, as it enabled us to go back over anything we didn't understand and watch it again
... working in groups added insight
... this is a key in learning, experiencing for ourselves and learning from the mistakes made from us and others.

An issue is whether, and in what way, more student control contributes to reflectiveness. The following is an extract from an interview, when Karrie was asked to comment about whether she was conscious of the resource during her practicum:

Interviewer: Which aspects of the... program were you conscious of during (the practicum)?
Karrie: Group work that they did in class and the way that they worked
Interviewer: In what way did the use of the program affect what you observed... ?
Karrie: I saw a lot of open ended maths questions and stuff like that and that made me think straight away of the program ...
Interviewer: Which one would you compare it to?
Karrie: Well it could have gone the same way. (The teacher) just gave me a task. They went about it their own ways, but what they came up with was incredible. Mainly the different levels they achieved.
Interviewer: In what way was your use of the ... program affect what you did on the teaching round?
Karrie: I'd let them sort of describe the answers ... tell me the answers they'd given so it sort of lengthened the lesson.

While many aspects of the resource may have contributed to this connection and transfer from the formal course to the practical situation, the evident sense of ownership of the process suggests that control may have been important factor—an element we need to research further.

Summary and conclusion

Education for the complexity of mathematics teaching requires both active engagement of teacher education students and development of processes required for on-going learning, of which reflectiveness is a key element. We argue that the sense of control offered by autonomous approaches to learning can contribute to developing aspects of reflectiveness.

Data from one case study were presented which indicated a noticeable development in the quality of the language and thinking associated with interpretation of teaching incidents, and a recognition of sense of control over earning. Other cases demonstrated similar shifts in student teachers' conceptualisations of mathematics teaching.

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LEARNING THROUGH REFLECTION WITH MATURE, LOW ATTAINING STUDENTS [1]

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This two-year investigation evaluated strategies for learning mathematics with low-achieving students in Further Education (FE) colleges in England. A substantial collection of discussion activities were developed to enhance metacognitive activity where conceptual obstacles were identified and targeted. A pre-test-post-test design was combined with detailed classroom observation. Two 'connectionist' teachers, sympathetic to metacognitive teaching approaches are compared with two' transmission' teachers who preferred exposition and practice. In the first year, without intervention, students showed modest gains with no clear difference between the two pairs of teachers. In the second year, the discussion activities contributed to statistically significant gains among the 'connectionist' teachers, but little difference was observed with the 'transmission' teachers.

Introduction

Each year in England, one half of the student cohort at the age of 16 fail to attain grade C or better in the GCSE (General Certificate of Secondary Education) examination in mathematics. This grade constitutes a minimum requirement for many careers and for entry into higher education. Many students therefore decide to retake the examination in a Further Education (FE) institution. The retake course is typically one year long, and often involves a rapid 'coverage' of the entire syllabus. Evidence of learning within such an environment is unimpressive (Audit Commission/HMI investigations(1993)). In addition, recent expansions in this sector, coupled with reductions in capital funding have resulted in less time being devoted to such courses and an increasing reliance on part-time teaching staff. This context provides a bleak and challenging environment for educational research.

This project attempted to create and implement a learning programme intended to foster an increased amount of reflective activity among students aged 16-19. We first attempted to develop a collection of activities that would encourage students to identify, confront and over come common conceptual obstacles through supportive social discussion. We then sought to measure the impact of these activities on teaching styles and learning outcomes.

Background

In the early 1980s, following the widespread use of diagnostic assessments, there was a considerable growth in understanding the extent and nature of common conceptual obstacles in learning mathematics and attempts were made to overcome these by recognising and building on children's existing conceptual frameworks (Johnson (1985)). Our own work with secondary school students explored ways of stimulating "diagnostic teaching approaches" in several content areas within mathematics (Swan (1983), Bell et al (1985)). Comparative experiments

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2. *This Centre incorporates the Shell Centre for Mathematical Education*

demonstrated in several areas of mathematics the greater effectiveness of teaching which: uses pre-testing to clearly identify students' prior states of conceptual understanding; targets specific conceptual obstacles and misunderstandings; employs sharply focused discussion material, intended to provoke "cognitive conflict"; and engages students in cognitive and metacognitive activity aimed at describing and explaining strategies and errors.

Our work confirmed that both teachers and learners have to modify considerably their conceptions of what constitutes appropriate mathematical activity. An orientation towards 'working through exercises' and 'getting answers' has to give way to 'working on ideas' and 'constructing well-knit concepts and methods'. We note with Baird et al (1986, 1992) the resistance to change generated by such conflicts with students' existing conceptions of learning.

In a further research project (Bell and Swan (1993); Bell et al (1997)) we worked with teachers to develop a number of classroom interventions which involved students in adopting novel classroom roles. These included, for example, students constructing tests and mark schemes, students teaching students, students designing teaching materials and students evaluating the purposes of lessons. Although observational evidence suggested that these approaches were effective in promoting reflection and learning, their overall effects on mathematical attainment were not measured.

In the preliminary work on the current project, it became clear that the need for introducing more reflective approaches to learning was, if anything, greater in FE than in secondary schools. The repeat GCSE lessons that we observed were longer than those found in schools - they lasted from one to three hours - and it was not uncommon to see students spending this time either passively listening to a teacher explanation or silently working through exercises. (During one, not atypical, two hour lesson observed, the teacher spoke 2046 words (average length of utterance: 93 words) while the students spoke a total of 32 words (average length of utterance:1.7 words).

> Under pressure to stay busy, students who have become accustomed to meaningless tasks are unlikely to see themselves as needing to learn how to guide their own learning. In other words, the cognitively rich get cognitively richer, while the cognitively poor get metacognitively poorer." (Kilpatrick (1985)).

In short, the project attempted to explore how 'transmission' methods of teaching which create a passive learner dependency may evolve into 'constructivist' approaches in which students are given opportunities to interpret ideas, negotiate meanings, to be challenged and thereby construct some new understanding of their own. This leads to a view of the syllabus as possible learning outcomes rather than a list of content that must be explained at a predetermined pace.

> A constructivist view of the syllabus requires a shift from a knowledge-based view of mathematics to an interpretative base of which reflection is a necessary part. (Burton (1993))

The structure of the project and some difficulties encountered

During the first year of the project, eight teachers were observed working normally, and their classes were pre- and post-tested on five different mathematical topics using diagnostic tests developed from earlier research (e.g. Hart (1985)). In parallel with this, the teachers and I collaborated in developing a collection of reflective classroom activities. During the second year, four teachers implemented the activities with a fresh cohort of students and qualitative and quantitative comparisons were drawn between the experiences and learning of the two cohorts with these four teachers.

The selection of the four teachers proved unexpectedly difficult. Initially, eight FE colleges were contacted. Three colleges were deemed unsuitable because their courses were being restructured, drop out rates among students were too great, and staffing was undergoing change. One further college contained no identifiable 'groups' of students at all. (In this college, students were allocated individual timetables and their work was completed entirely within drop-in workshops.) From the remaining colleges, eight lecturers were selected. These lecturers planned to teach the same, one year course to two similar samples of at least 15 students using a range of approaches, including whole-class textbook-based and more individualised methods. During the two years of the project, three lecturers withdrew from the project because student attendance became so low that their classes had to be cancelled or reorganised and one teacher withdrew with illness. The sporadic attendance of students also meant that the pre- and post-testing data was based on smaller samples than were originally envisaged.

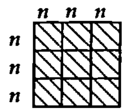
Learning activities

The mathematical activities designed for this project were intended to encourage students to pause and reflect on the meanings of mathematical concepts, notations and strategies and to distinguish between different modes of learning; notably practising for fluency and sharing meanings through discussion. Five types of reflective activity were developed:

1. Collecting together equivalent mathematical representations

Mathematical concepts are inextricably bound up with graphic and symbolic representations. Most concepts have many representations; from conventionally accepted notations to informal mental representations. The type of activity suggested here is intended to allow these representations to be shared, interpreted, compared and classified so that students construct an understanding of the underlying concepts.

One approach involved offering groups of students a number of mathematical representations on cards and asking the students to sort the cards into sets so that each set contains different representations with an equivalent meaning. The focus of the activity was thus on interpretation rather than on the production of representations.

Examples of alternative representations for sorting			$3n^2$
$9n^2$	Square n then multiply the answer by 3	Multiply n by 3 then square your answer.	$(3n)^2$

Teachers initially tried presenting these activities on large printed sheets. This approach did not generate sufficient student discussion, so we decided to use a 'card sorting' approach. This proved more effective because it required a collaborative rather than individual, output, allowed a range of approaches, encouraged translations in multiple directions, allowed teachers to monitor the discussions and adjust the nature of the challenge appropriately (by removing or adding cards).

In general, the students appeared to find sorting activities engaging and enjoyable. Students became aware of each others' difficulties, and they were often observed explaining notation to one another. Disagreements over interpretation cannot always be resolved by rational argument. Notations are, in essence, arbitrary yet convenient conventions. The discussions did, however, raise an awareness of the *need* for agreed interpretations of notational conventions and the teachers were able facilitate the sharing of these interpretations in a meaningful context.

2. *Evaluating the validity of mathematical statements and generalisations*

These activities were intended to encourage students to focus on common convictions concerning mathematical concepts. A number of commonly made statements and generalisations were provided and students were asked to examine each one in turn and decide upon its validity. This typically involved deciding whether a statement was *always, sometimes or never true* , then justifying this decision with examples and explanations. In addition, students were asked to add conditions or otherwise revise generalisations so that they would become 'always true'. In practice, these activities met with mixed success. They resulted in learning only when the students were encouraged to give detailed explanations and generate a range of examples.

Examples of generalisations for evaluation.	If you multiply 12 by a number, the answer will be greater than 12.	The square root of a number is smaller than the number.	The more sides a shape has, the more right angles it can have.
If you double the radius of a circle, you double its area.	The shape with the smaller area must also have the smaller perimeter.	$3 + 2y = 5y$	$2(x+3) = 2x + 3$

3. Correcting and diagnosing common mistakes

Students were presented with common mistakes or misinterpretations which they were asked to correct and comment on. (This type of activity differed from 'evaluating generalisations' because the causes of the mistakes were not explicitly stated - that was left for students to uncover.) For example:

The other day I was in a department store, when I saw a shirt that I liked in a sale. It was marked "20% off". I decided to buy two. The person behind the till did some mental calculations. He said, " 20% off each shirt, that will be 40% off the total price." I replied, " I've changed my mind, I think I will take five shirts." Explain the cause of the mistake.

We observed that mistakes shifted the students' attention from *obtaining* answers to one of finding *reasons* for answers. They provided a *planned agenda* for classroom discussion of common difficulties which otherwise would have only occurred on an 'ad hoc' basis. The approach also permitted the discussion of common difficulties in a non-threatening environment.

4. Resolving problems which generate cognitive conflict.

In this type of activity, students were asked to tackle problems which were designed to make students aware of their own inconsistencies in understanding. After tackling a problem intuitively, students were invited to revisit the problem using a different, given, method. Students were asked to compare the results obtained by the different methods and reflect on the inconsistencies in the answers obtained.

In practice conflict problems were successful in exposing the difficulties that students have with understanding mathematical concepts. They were observed discussing common beliefs, such as 'letters in algebra stand for objects', ' graphs look like pictures of situations' and 'a shape is enlarged in proportion if the same amount is added on to each side'. Some teachers, however, appeared so anxious to 'cover' the syllabus that they were unwilling to generalise and consolidate the learning. They appeared to equate 'obtaining an answer' with 'learning' and, when a correct answer was obtained, they wanted the student to 'move on' rather than spend time redoing the task in a different way, or reflecting on the causes of errors. The result was that students who were profitably engaged in learning were interrupted and moved to a fresh activity before they had been allowed sufficient time to reformulate their understanding.

5. Creating problems and connections between concepts and representations.

These activities fell into two categories; students were either asked to construct concept maps to illustrate links within topics or construct fresh examples to illustrate an idea or to satisfy given constraints. Our previous research (Bell, Swan et al (1993)) showed that lessons involving concept maps can easily become derailed into learning *how to draw* concept maps rather than using them to enhance reflection. We decided therefore to offer students a concept map in each topic to modify and extend, with the encouragement to 'write down all you know about' a given mathematical statement or object. Students were also invited to create their own mathematical problems to satisfy given constraints. It was hoped that by engaging with such tasks, students would engage with the structure of problems rather than just their solution.

The concept mapping activities were not used frequently by teachers. They appeared to see these only as useful 'lesson fillers' or 'lesson enders'. Only once was a teacher observed building a whole lesson around the activity. The value of concept mapping was therefore not fully appreciated or realised. Creating problems was also seen to be something of a 'luxury' activity by some teachers. Where the activity was used, however, there was evidence of reflection and learning. Students appeared to enjoy taking on the role of a teacher as their partners tried to solve the problems they themselves had created.

These activities show considerable potential to improve the quality of reflection and analysis of concepts and problem structures, but their 'face validity' is not immediately evident to teachers and students in preparation for an examination which rewards fluency rather than creativity.

Teacher beliefs and learning outcomes.

The final sample for pre- and post-testing were the classes of four teachers; Alan, Chris, Denise and Ellen. In year 1, each teacher taught in their normal style; in year 2 they used the teaching interventions described above. This design made it possible to compare the effectiveness of the two modes of working.

The four teachers had different 'belief systems' that affected their teaching styles. Alan and Chris were clearly more sympathetic to a 'connectionist' belief system (Askew et al (1997)); that is they saw the importance of emphasising the links between different aspects of the curriculum, were more amenable to the view that mathematics is best learned through challenging, interpersonal activity and that students have misunderstandings which need to be recognised, made explicit and worked on through class and small group discussion. Denise and Ellen, both appeared to prefer 'transmission' modes of working, where they emphasised the importance of procedures and routines and saw their task as offering clear explanations followed by intensive practice.

In year 1, Alan and Chris felt unable to behave in a way which was fully consistent with their beliefs because of the lack of time and appropriate resources. In year 2, however, the resources provided by this project allowed both Alan and Chris to implement their preferred teaching strategies more fully.

In year 1, Ellen preferred long periods of whole class exposition with shorter periods of practice, while Denise preferred shorter periods of exposition, followed by longer periods of practice. In year 2, though expressing enthusiasm for the project activities, both teachers failed to implement the activities in 'connectionist' ways. Denise tended to issue activities without suggesting why or how students should work on them. Students thus often treated the printed resources as they would a textbook and worked alone or with a partner, often refusing to discuss concepts or review answers. Ellen, however, did modify her approach to encourage more discussion, but she continued to dominate and channel classroom discussion through herself.

The pre- and post-test results show that in the first year of the project, there was little difference in the effectiveness of the four teachers. Most of the learning gains were slight (<10%). In the second year, however, the classes belonging to Alan and Chris made greater (>10%) and more consistent gains (across tests) than the classes belonging to Denise and Ellen.

Mean pre-post-test scores for each test during each year of the project for paired teachers.

Year 1	Number (Decimals and Fractions)			Number (Operations and Rates)			Algebra (Functions and Graphs)			Algebra (Expressions and Equations)			Space & Shape (Length, area, volume +)		
Alan & Chris	42	45	+3 n.s.	31	38	+7 *	28	33	+5 ?	24	33	+9 **	29	36	+7 n.s.
	n = 19			n = 17			n = 16			n=9†			n=11		
Denise & Ellen	34	40	+6 ?	34	41	+7 ?	21	25	+4 n.s.	41	49	+8 n.s.	20	33	+13 ?
	n = 13			n = 7			n=11			n=10			n=6		
Year 2															
Alan & Chris	46	60	+14 **	42	56	+14 **	26	39	+13 **	29	40	+11 **	33	46	+13 *
	n = 24			n = 25			n = 19			n = 22			n = 18		
Denise & Ellen	55	65	+10 **	55	59	+4 n.s.	32	33	+1 n.s.	43	45	+2 n.s.	30	37	+7 *
	n = 15			n = 13			n = 18			n = 19			n = 18		

Key: ? represents $p<0.1$; * represents $p<0.05$; ** represents $p<0.01$. (dependent t-test).
(† Note that Alan did not take the year 1 algebra test)

This appears to show that the teaching activities did have some success where the teachers already had a predisposition to work in ways which were sympathetic to the reflective approach; but they did not where the teachers were unable to adopt more connectionist approaches.

The results from the individual tests, aggregated across the four teachers in the final sample, present a sobering view of how difficult it is to achieve substantial learning gains in the FE environment. We have shown that students enter FE with many profound gaps in their understanding of basic mathematical concepts and that lecturers' 'normal' approaches to teaching make little impact on this state of affairs. The reasons for this appear to be due to poor attendance, motivation and passivity among students and an overwhelming emphasis on rapid syllabus 'coverage' using transmission methods of teaching, even among teachers who hold beliefs about learning which conflict with this practice.
Shifting this state of affairs has proved difficult in FE. Significant, (yet modest) gains have only been observed with teachers who are sympathetic with 'connectionist' beliefs about learning when they are also provided with resources consistent with

these beliefs. 'Transmission' belief systems appear robust even in the face of evidence of their ineffectiveness. This appears equally true for both teachers' and students' views of learning; many students *expect* to perform imitatively and we have witnessed several negative reactions when teachers attempt to modify their approach. We suspect that more significant gains will only prove possible over a longer time scale, when the belief systems of all participants (teachers and students) are addressed more explicitly. The FE sector has proved a difficult and unfashionable environment in which to conduct research, yet its importance is such that further work is urgently needed, particularly with students who have been failed by our secondary education system.

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USING RESEARCH INTO CHILDREN'S UNDERSTANDING OF THE SYMMETRY OF DICE IN ORDER TO DEVELOP A MODEL OF HOW THEY PERCEIVE THE CONCEPT OF A RANDOM GENERATOR

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In order to develop a model of children's understanding of random generators, this paper reviews research on children's responses to dice and other embodiments of similar generators. It questions the received view that many children consider dice to be biased against "6", and argues that they may use a representativeness heuristic to interpret dice behaviour.The research findings are used as a basis for proposing a "Handbook Model" for integrating research research in stochastics learning into a readily accessible form which recognises both the structure of the subject and the complexities of individuals' responses to it.

Is "6" Really an Unpopular Number?

Six-sided dice are very common random generators (RGs). Although most adults see them as symmetrical, they are not seen as such by many school-age children, some of whom believe that they are biased against "6". Some writers consider that this view arises because many dice games require players to toss a "6" before starting to move, and that it is these relatively infrequent but long waits which are recalled most readily. This view matches the "availability heuristic" of Tversky & Kahneman (1974) in which subjective probabilities are based primarily on the ease with which previous experiences may be recalled. In my experience such an interpretation is the "received wisdom" in educational circles and probably also within the wider community.

My paper questions this "wisdom". In doing so it also contributes to the "model-building" stage of Romberg's (1983) steps in "normal science" and shows how this model-building might provide an integrating structure for reporting research.

Model building within "Normal Science"

For Romberg there are three major steps in normal science:

- fact collection;
- confrontation, confusion and reflection;
- model building.

So far stochastics education research has, quite naturally, tended to concentrate on the first two stages as researchers accumulate data and paid less attention to the final stage. Even today relevant data is limited, but there is sufficient to start building a model which is urgently needed now that stochastics is taught to students of all ages.

So in this paper a very small, simple area of stochastics research—one which has produced counter-intuitive findings not mentioned in standard summaries of stochastics learning such as Shaughnessy (1992) and Borovcnik & Peard (1996)—is taken as a basis for starting to build a structure to assist both teachers and researchers. The structure is a prose summary of the findings set within a hierarchy of contexts. The hierarchy makes the material readily accessible to someone with a specific need, and the prose is an effective medium for expressing the complexities being communicated. In a short paper one can only hint at the extent of the hierarchy, but the example is broad enough to indicate how a hierarchy might be constructed.

The model is called a "Handbook Model" because the style and structure conform with those in handbooks within the hard sciences, e.g., Higgins & Davies (1996), where each individual part of the text is directed specifically to a specific aspect of the subject under consideration. References are cited, but they are subordinate to the summary. This contrasts with the more discursive style of Bishop, Clements, Keitel, Kilpatrick & Laborde (1996) where summaries of the references tend to form the basic units in the articles. The structure specifically aims to interpret research findings in a way which may be readily received by those who primary concern is teaching rather than research and to act as a complement to standard curriculum documents and text-books. It is inspired by Balacheff's (1997) comment that

> [p]sychology is only part of the relevant approaches to the problems raised by mathematics learning and teaching, and for example one must be able to take teaching processes as an object of study as such, as well as the epistemology of mathematics from a teaching/learning perspective.

Field of Research Being Surveyed

The research examined concerns children's understanding of RGs with a small number of equi-probable, distinct outcomes, called here "Dice-Family Random Generators" (D-FRGs), which may be embodied in a number of forms, usually dice, spinners, and urns. The view taken here follows Henry (1997) in seeing an RG as an abstract mathematical concept with various real-world models. This view is analagous to the idea that Euclidean geometry is an abstract idea which may be modelled by various marks on paper. Sometimes this distinction is pedantic, but this paper examines understanding of the mathematical idea as reflected by experiences with various embodiments, so it is worth maintaining here.

Because of their dichotomous nature, I do not consider here the special case of RGs with only two possible outcomes. Nor RGs with asymmetric outcomes or urn models containing coloured balls, where the elementary outcomes are not easily distinguishable. Nor, finally, cases where embodiments of different RGs are compared with each other. A simple case is the best to choose for illustrating the construction process.

Kerslake's Research

Kerslake (1974) asked about 700 English children from Years 3 to 6: "When throwing a dice (die), do you think that some numbers are easier to get than others?" Between 59% and 68% of each Year group answered "yes". She claimed that the percentage answering "yes" in Year 6 was less than in the earlier years. The wording of the question has a distinct "expecting answer yes" ring to it, so the results may be biased towards "yes". But, assuming the question to be reasonable, and estimating the proportion of the children in each Year group, a χ^2 test suggests that the reported "improvement" may merely reflect chance variation. Kerslake also asked the children who believed that dice were biased to state which number they thought was the most difficult to get and found strong evidence for antipathy to "6", but cited no evidence that children were using an availability heuristic.

Interestingsly when this question was replicated by Kempster (1982) as part of a teaching experiment, it was found that the percentage of 24 Year 6 children in New South Wales who considered some nuimbers to be harder *increased* from 29% to 67% for reasons which Kempster suspected were due to some aspect of the teaching.

About ten years later, Kerslake's pioneering, but limited, research was supplemented and questioned by the independent work of two researchers, one from within mathematics education and one within psychology. We shall consider them in that order.

Green's Research

Green (1982) asked a similar question to Kerslake's as a part of a written test to about 3000 English children, using a much more precise wording: "When an ordinary 6 sided dice is thrown which number or numbers is it hardest to throw, or are they all the same? The percentage who considered all numbers to be equally likely increased significantly from 67% in Year 7 to 86% in Year 11. At all ages almost all of those who believed that there was a hardest number stated that it was "6", and "1" was the only other number nominated.

These results are strikingly different from Kerslake's and the differences between Green's Year 7 results and Kerslake's Year 6 results are far too great to be attributable to development alone. We cannot be sure why. However, Green's form of questioning and administration is more thorough and mathematically sound than Kerslake's, so his results probably have more general validity.

Both authors agree that if children consider a die to be biased then it is most likely to be biased against "6" and Green provides statistically significant evidence to suggest that older children are more likely to believe that dice are symmetrical, but for unknown causes which might be "logical maturation, practical experience, education, indoctrination ... ?" (Green, 1983, p. 31).

This result is the one most widely quoted (e.g., Watson, Collis & Moritz, 1994), but Green (1983) reports much more about children's perception of dice. This preliminary investigation with 139 secondary school children involved clinical interviews as well as pencil and paper tests. Some questions were particularly revealing. Of the sample 33% had no favourite number between 1 and 6, and for those who had one, "6" was moderately popular, "1" was markedly less preferred and "3" was decidedly most preferred. Interestingly, Green's interviews revealed that some of the children chose "6" more as a social response than as a statement of strong belief.

Furthermore, when the children had to choose a number to give them the best chance of winning a prize in a simple competition dependent on the throw of a die, 23% chose "3" and 21% chose "4". There were 26% who regarded "any number" as ap–

propriate, but only 5% who chose "1" and 9% who chose "6" in spite of their earlier statements of preference for these numbers. Green then tested the constancy of these views with other questions which asked about both "best number on a die" and "worst number on a die", and found a consistent trend to prefer the more central numbers. So, when pushed to make a choice, even those children who believed a die to be symmetrical tended to choose central numbers. The more mature responses tended to be from children whose measures of reasoning ability were relatively high.

Teigen's Research

These results match those of Teigen (1983), who presented 73 Norwegian tertiary students with "a box containing 12 tickets, numbered from 1 to 12. The numbers were written on the blackboard, and the students were told that as there was only one ticket of each kind, all numbers had an equal chance to be drawn. Still, they had to make a guess, and write down which number they thought would turn up" (p. 14). The four central values ("5", "6", "7", "8") were chosen by 59% of students. When a different 42 students were promised a small financial reward for correct predictions, 76% opted for the four central values and the percentage opting for the outer values ("1", "2", "11", "12") decreased. That the prospect of reward may influence such choices has been known since at least the work of Cohen (1964, ch. 3).

Teigen then replicated his first question with a new group of 201 students, of whom 56% chose central values. But when he presented the same group with a new set of 12 raffle tickets, with six blue tickets numbered from "1" to "6" and six red ones numbered from "7" to "12" he found that the most popular numbers were "3", "5" and "9". He argued, partly on the basis of other experiments not reported here, that the students were using the representativeness heuristic, not the availability heuristic, and that their decisions were not influenced by "favourite" or "unpopular" numbers.

Teigen's question looks at an "outcomes approach" to probability (discussed in Konold *et al.*, 1991, p. 301), where students are more concerned to specify what will happen in a particular case rather than to present a more long-term perspective. Falk & Konold (1992, p. 154) see this dichotomy as arising because probability involves an understanding of both proportion and chance. Teigen's research, which has received little attention within mathematics education, emphasises the chance component.

Replications

When we consider both Teigen's and Green's work it is clear that the availability heuristic is an inadequate model of the complex thinking which even simple chance processes engender. There is only space to summarise some of the research they have inspired. I have chosen mainly some verifications of Teigen's work and some unpublished or poorly disseminated findings. The research is used to construct a Handbook Model summary about responses to D-FRGs which sets the received wisdom about children's dislike of "6" into a richer and pedagogically more useful context.

K. Truran (1996) replicated Teigen's questions with 43 Year 5 and 36 Year 7 children, some of whom were interviewed. At both levels the central numbers were more likely to be chosen and some children were definitely using a strategy of choosing a middle number. I have found similar results with a group of 26 primary teachers at an in-service course (20%, 56%, 24% for lowest, middle, and highest 4 numbers respectively), and with a group of 68 pre-service primary teachers (22%, 51%, 26%).

K. Truran (1994) also asked "if you toss this dice is there a number or numbers that are hardest to throw or are they all the same?", and found that 60% of the Year 5 and 88% of the Year 7 students considered all numbers were the same. The latter is significantly higher than Green's results for Year 7. However, when she repeated her question with "hardest" replaced by "easiest", 74% of the Year 5s (a significant increase) and 83% of the Year 7s (a small drop) considered that all numbers were equally likely. Clearly the wording of a question may be important and a single holistic question is unlikely to provide adequate evidence of understanding. Even so, questioning procedures which do not make allowance for this may still be of value, e.g., K. Truran's work cited below, and the confirmation by Watson et al. (1994) that older children are more likely to provide correct responses to questions of this type.

Preliminary results from K. Truran's current research provide another important insight. She posed the same question as above, using the "hardest" wording, to the same children with respect to dice, spinners, and urns, each with 6 or 12 outcomes numbered from "1" to "6" or "12". For all three forms and for both sizes a minority of the Year 5 children considered that the largest number would be the hardest and very few chose any other number. The percentage was much higher for the dice (27% & 18%) than for the other RGs (4% – 16%). The figure for the 6-die (27%) was higher than that for the 12-die (18%), but this ordering was reversed for the other two RGs. None of the Year 7s considered any number to be hardest for the spinners or the urns, and only a very small percentage for the dice. These figures make it clear that different embodiments of the same RG are viewed quite differently by a significant minority of upper primary children.

My own clinical interviews with 32 children in Years 4, 6, 8 & 10 (the basis of J. Truran (1992)) have shown that while some children do recall "6" as being difficult to throw, others recall long waits for other numbers:

> [No numbers are easier to get because] say you want a number and you just throw it and you don't get it, and sometimes when I do that I get really mad when I don't get it. Say you're on 99 in Snakes and Ladders and you need the number one and you can't get it so you just stay there and the next person has their turn and they catch up.(DR, Year 4, female, age 8: 7)

And students may have unusually fortunate experiences with "6" which lead them to considering it to be an easy number to throw: "Once I got it 6 times and the seventh time I didn't get any more sixes" (DL, Year 6, male, age 11:4)

So children may use an availability heuristic, but this need not lead them to conclude that a die is asymmetric or that "6" is the least likely outcome. And what none of the research tells us is what process leads from the complex, indecisive and inconsistent answers given by young children to the confident, matter-of-fact answers provided once the concept is understood, e.g., "Because the dice is made evenly and all numbers are on there only once" (LH, Year 8, male, age 13:1)

Constructing a Handbook Model for Summarising These Findings

Clearly students' responses to D-FRGs are variable, depending, *inter alia,* on the embodiments, the context, the rewards, and the size of the possibility space. Similar findings permeate the literature on the understanding of probability. And, as Konold, Pollatsek, Well, Lohmeier & Lipson (1993, p. 393) have pointed out, "a subject can switch from correct to incorrect reasoning while reasoning about what an expert would consider to be the same situation." So it is particularly difficult to provide a unifying structure into which individual findings may be embedded and it is clear that neither the availability nor the representative heuristic are sufficient in themselves.

The view is taken here that the fragmented pattern formed by current research findings may be reduced by using the concept of an RG as a unifying feature, principally because it is a fundamental probabilistic concept, far more fundamental than stochastic outcomes. So the following summary uses a hierarchical structure—RGs, D-FRGs, and Symmetry of D-FRGs—to indicate how the hierarchy might be extended. It also brings together the holistic and outcome approaches to RGs, thus clarifying the circumstances where inconsistent reasoning appears.

The example below presents a model text closely related to the main topic. Citations would normally be included as footnotes or within parentheses, but are not included here to save space. Most may be easily deduced from the discussion above.

An Example of the Handbook Model

Random Generators

Children's responses to RGs of all sorts depend on a number of features which are unrelated to the RGs themselves. The more closely a child is involved with the operation of the RG, the more likely it is that misconceptions will be revealed. Promising rewards for succesful operation of an RG is particularly effective for highlighting misconceptions.

RGs may be viewed holistically or as a means of generating a specific outcome which is of immediate interest. Asking children to predict the next outcome from an RG (an "outcomes approach") is mathematically unreasonable. However, such situations do arise in life and children's predictions can reveal understandings about RGs which holistically oriented questions do not bring out. So it is desirable that both approaches are used when teaching or diagnosing.

Dice-Family Random Generators

D-FRGs are RGs with between 4 and 12 discrete equi-probable outcomes. RGs with 2 or 3 outcomes tend to produce different types of responses because of their simplicity. More than 12 outcomes are difficult for children to conceptualise, so it is likely that they will use different ways of interpreting them. Mature understanding about the behaviour of D-FRGs is probably positively correlated with thinking ability.

Symmetry of Dice-Family Random Generators

Many children do not believe that these RGs are symmetrical; this belief often seems to decline with age, for reasons which are not well understood. The misconception may be revealed by asking questions like "which number is easiest to get or are they all equally easy", but minor changes in wording, such as replacing "easiest" by "hardest", can lead to quite different responses. It is wise to ask several questions of different forms.

Many children and adults whose understanding of the holistic approach to dice is sound will still prefer a specific number when asked to use an "outcomes approach", and many of these tend to choose central numbers (a representativeness heuristic). It is not know how such people respond when the outcomes concerned are not able to be placed easily in order in the way that numbers are.

It cannot be assumed that children who seem to have a clear understanding of one embodiment of a D-FRG will necessarily have the same clear understanding of others which are mathematically identical. The larger the number of outcomes the more likely children are to reveal misconceptions. Children need experience with a variety of forms.

Conclusion

This paper has shown that the received wisdom about some children's belief that dice are asymmetric must be called into question and shows how using different approaches is able to elucidate different aspects of understanding. It then provides a structure within which the research findings might be effectively conveyed to those for whom they might be of most practical use, thus suggesting that the setting in which psychological findings are reported is one aspect of the construction of their meaning. Within the theme of this Conference, it may be seen as a suggestion about how reporting components could *Change* in order to integrate the very *Diverse* results generating by *Mathematics Education* research.

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STUDENTS' SOLUTIONS OF INEQUALITIES

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This article describes the first phase of a project related to secondary school students' ways of thinking about linear, quadratic, rational and square-root inequalities. Our findings show that using graphical representations of parabolas when solving rational and quadratic inequalities usually yielded correct solutions. Difficulties arose when students failed to reject the excluded values or chose inappropriate, logical connectives. The most prevalent source of difficulties was inappropriate analogies between equations and inequalities.

Inequalities play an important role in mathematics. They interweave in various mathematical topics including algebra, trigonometry, linear planning and the investigation of functions. They also provide a complementary perspective to equations. Accordingly, the *Curriculum and Evaluation Standards for School Mathematics* specifies that all students in grades 9-12 should learn to "represent situations that involve variable quantities with expressions, equations, inequalities and matrices" (NCTM, 1989, p. 150).

In spite of these recommendations, inequalities receive relatively little attention and are usually discussed only by mathematics majors in the upper grades of the secondary school. Even at this stage, discussions are usually concise; emphasising the "practical" algorithmic perspective. Generally, attention is mainly paid to "How to solve" and not to "Why to solve it this way?"

To implement the above mentioned recommendations, it is crucial to consider students' ways of thinking about inequalities. However, so far research in mathematics education has paid only limited attention to students' conceptions of inequalities and to suggestions for instructional tools (e.g., Linchevski & Sfard, 1991; Dreyfus & Eisenberg, 1985).

This paper describes the first phase of a project aimed at promoting teachers' knowledge of students' ways of thinking about inequalities. In phase 1, which took place at the beginning of 1997, high-school mathematics majors students were asked to solve equations and inequalities and justify their answers. In phase 2, which took place in March, 1997, the eight teachers of these students were asked to solve the same equations and inequalities in (correct and incorrect) ways they presumed their students would. In phase 3, which took place during the summer vacation of 1997, the participating teachers were given a course to promote their awareness of students' thinking about inequalities. In phase 4, which will take place in April 1998, phases 1 and 2 will be repeated with the same teachers and their new classes.

This paper focuses on phase 1 of the project, namely, students' ways of thinking about inequalities. The main research questions were: (1) What common (correct and incorrect) approaches are used by secondary school students to solve linear, quadratic, rational and square-root inequalities? (2) What are the possible causes for incorrect solutions to such inequalities?

Methodology

An "Equation and Inequalities Questionnaire", consisting of five equations and 10 inequalities, was administered to 160 high-school mathematics majors, during a 90-minute mathematics lesson. These students had studied and applied inequalities for about three months. They had also previously studied how to draw the graphs of functions related to all the expressions given: linear functions (e.g., $y=-8x$), rational functions (e.g., $y=\frac{x-5}{x+2}$), quadratic functions (e.g., $y=x^2-25$), square-root functions (e.g., $y=\sqrt{x}$). In addition, they had learnt the investigation of the interrelated behaviour of a combined system of two functions (e.g., drawing the two graphs: $f(x)=\frac{2x-2}{x+1}$; $g(x)=1$ and examining for which values of x $f(x)<g(x)$).

In the present research, the students were asked to solve the equations and the inequalities and to explain each step of their solutions. Twelve students, who provided many incorrect solutions were requested to explain their answers in individual interviews. This paper relates to students' solutions to the inequalities (the 10 inequalities are listed in the first column of Table 1).

Results

Table 1 shows that the highest rates of correct solutions were given to the linear inequality (drill 6) and to the quadratic inequalities where the related equations had two square roots (drills 4, 8, 9). The lowest rates of correct solutions were given to the compound, rational inequality (drill 15), and to rational and square-root inequalities with non-zero numbers on the right side of the inequality (drills 3, 5). Typically, the subjects' responses to the questionnaire were elaborated, and thus provided substantial information related to their ways of thinking.

We shall first relate to main strategies that the subjects used to solve the inequalities and then to the main difficulties they encountered while solving these tasks.

A. Strategies for solving inequalities

The following three main strategies were used by the students to solve the given inequalities (see Table 1):

(1) Algebraic manipulations. This was the most prevalent approach. It consisted of (i) adding or subtracting identical expressions to both sides of the inequalities; multiplying both sides by the square of the denominator, or, multiplying both sides by a negative factor and changing the direction of the inequality sign accordingly. (ii) examining the quadratic inequalities (i.e., $ax^2+bx+c>0$) by first relating to the quadratic roots, or by investigating the sign of 'a' and the sign of $\Delta=b^2-4ac$. (iii) relating to an inequality of the type $ab>0$ as a compound system of {a>0 "and" b>0} "or" {a<0 "and" b<0}.

Table 1: *Correct solutions and strategies for solving inequalities (in %)*

Strategies:		Algebraic Manipulations		Drawing of Graph		Number-Line		TOTAL
Solution:		Correct	Incorrect	Correct	Incorrect	Correct	Incorrect	Correct
Inequalities								
Linear								
(6)	$-8x>0$	76	18	--	--	--	--	76
Rational								
(1)	$\frac{x-5}{x+2}<0$	16	45	28	4	1	2	45
(5)	$\frac{2x-2}{x+1}<1$	7	66	15	3	1	3	23
(15)	$-3<\frac{x-3}{x+1}\leq 3$	1	40	9*	6	6	16	16
Quadratic								
(4)	$x^2-25>0$	27	33	31	2	1	5	59
(8)	$x^2<16$	29	38	24	4	1	2	54
(9)	$(x-1)(x-2)>0$	13	34	44	2	2	4	59
(11)	$x^2-x+1>0$	24	50	8	2	--	--	32
Square-Root								
(3)	$\sqrt{x+5}<4$	18	73	--	--	4	1	22
(7)	$\sqrt{x}\leq 0$	39	53	1	--	3	--	43

* These students also used the number-line in their solutions

(2) Drawing a Graph-- Students drew the graph of the relevant function and used it to attain the solution. Only the rational and quadratic inequalities were solved in this way. All those who used this approach reached the stage where the inequality had a quadratic expression on the left side and zero on the right. Accordingly, the only type of graph used was that of parabolas. Finally, students examined the values of x for which the parabolas were either positive (e.g., drill 4) or negative (e.g., drill 1).

(3) Using the Number-Line-- A substantial number of students applied algebraic manipulations and used the graph of the number-line for final conclusions when solving the compound, rational inequality and several students used it to solve all drills

involving the logical connectives "and" or "or". About half of those who solved the compound, rational inequality by number-line, actually applied both graphs and number-lines (a detailed description of the last two strategies could be found in Dreyfus & Eisenberg, 1985).

Table 1 indicates that using the drawing of the relevant graph usually yielded correct solutions (except for the rational, compound inequality). Applying only algebraic manipulations yielded the highest rate of incorrect solutions.

B. Difficulties in solving inequalities

Here we briefly present various difficulties that students exhibited while solving inequalities.

B.1. Difficulties with excluded values

Two types of "range restrictions" should have been taken into consideration-- non-zero denominators in rational expressions and non-negative values under the square-roots.

Table 2: *Relating to the excluded values when correctly/incorrectly solving inequalities (in %)*

	No of Drill	1	5	15	7	3	
	The Drill:	$\frac{x-5}{x+2}<0$	$\frac{2x-2}{x+1}<1$	$-3<\frac{x-3}{x+1}\leq 3$	$\sqrt{x}\leq 0$	$\sqrt{x+5}<4$	AVERAGE
Excluded Values	**The solution**						
Found*	Correct	24	11	8	33	24	20
Found	Incorrect	24	24	20	9	13	18
Neglected	Correct	22	13	9	12	--	11
Neglected	Incorrect	29	48	42	40	62	44
No Answer							

* The majority of the students found the correct, excluded values

Table 2 shows that only about 38% (in average) of the participants explicitly mentioned the restrictions of excluded values (about half of them incorrectly solved these inequalities). Despite not mentioning excluded values, a number of students did get correct solutions to the rational inequalities, since the excluded values did not affect their solutions. In drill 3, however, students who failed to reject the excluded values were easily identified, as it led to erroneous solutions. Drill 7 did not allow a clear distinction between students who found the excluded values and those who did not-- Some students briefly wrote x=0, and it was impossible to determine whether they considered the excluded values or alternatively, incorrectly solved the related equation.

In some cases, students presented incorrect sets of excluded values, usually, confusing the restrictions imposed by a denominator with those imposed by a square-root; that is, demanding non-negative values for a denominator and non-zero values in cases of square roots.

B.2. Inappropriate choice of logical connectives

Students' difficulties with logical connectives are widely documented (e.g., Dreyfus & Eisenberg, 1985; Parish, 1992). In our case, the highest rate of such confusion was found in the solution of the compound inequality -- about 13% of the students used the connective "and" instead of "or" and visa versa. About 5%-10% of the incorrect solutions to the rational and quadratic expressions involved inappropriate choices of connectives (drills 1, 5, 15, 4, 8 and 9). In a number of interviews students exhibited a profound lack of understanding regarding when and how to apply "and" and "or". For instance, Gill (Grade 10) solved drill 1 by writing:

$$\begin{array}{ll} x+2<0 & x-5>0 \\ x<-2 & x>5 \end{array}$$

Like a number of other students, he just left the two options without combining them to attain a solution. In an interview he was asked to go further into the matter.

Interviewer:	What is one supposed to do with the two results?
Gill:	I think that the solution is "or".
Interviewer:	How did you get x>5?
Gill:	By imposing a positive numerator.
Interviewer:	And how did you get x<-2?
Gill:	By imposing a negative denominator.
Interviewer:	So, what should the connection between the two be?
	"and"... sorry,.. ah.. "or" (thinks) I would like it to be "and" but it is impossible, thus it is probably "or"...

B.3. Incorrectly deducing signs of factors from sign of product / quotient

About 5% of the solutions to the three rational inequality and about 15% of the solutions of each quadratic inequality (except for drill 11), consisted of the following insufficient claims: $a/b>0 \Rightarrow a>0$ and $b>0$; $a^2>b^2 \Rightarrow a>b$; $a^2<b^2 \Rightarrow a<b$. Surprisingly, a small number of participants used erroneous, though visually similar considerations, claiming that $\frac{x-5}{x+2}<0 \Rightarrow x-5<0$ and $x+2<0$. (Note that Gill in his interview related only to positive numerator and negative denominator neglecting the cases involving negative numerator and positive denominator).

B.4. Solving equation instead of inequality

About 5% of the participants changed the given, inequality symbol to an equal-sign in almost each of the non-linear inequalities, and solved the problem as an equation instead of an inequality. In two cases was the rate of such solutions significantly higher: (1) in drill 7, ($\sqrt{x} \le 0$) where by solving the equation 12% of the students provided a correct, yet unjustified solution; and (2) in drill 11 ($x^2-x+1>0$) 63% of the students found no quadratic roots to the relevant equation and incorrectly concluded that the inequality, too, has "no solutions".

B.5. Multiplying / dividing by factors that are not necessarily positive

A nonnegligible number of students inappropriately multiplied both sides of the rational inequalities by the denominator, taking no account of non-positive cases (29%, 44% and 47% in drills 1, 5, and 15 respectively). Another facet of this error was exhibited in the linear inequality (drill 6) -- 17% of the students divided both sides by (-8) without changing the direction of the inequality sign.

In the interviews, some students explicitly referred to the analogy they drew between equations and inequalities, assuming that the same solving procedure holds for both. Dan and Ron, for instance, both solved $-8x>0$ by dividing both sides by (-8) and concluding that $x>0$. Dan explained:

Dan:	I know I solved correctly since I used methods that I have already successfully used many times before when solving equations.

Ron, however, understood during the interview the limitations of the analogy between the two mathematical entities (equations and inequalities):

Ron:	I can substitute values for x. So, if I choose 2, then.. sorry, I was wrong
Interviewer:	What do you mean?
Ron:	The solution should be $x<0$... I solved it as an equation...

Sara multiplied both sides of $\frac{2x-2}{x+1}<1$ by $x+1$, getting $2x-2<x+1$. She concluded that $x<3$. In her interview she related to equations as a possible source for her inappropriate solution:

Interviewer:	In your opinion, is your solution correct?
Sara:	I can check by substituting a number for x. (She writes if $x=2$ then $\frac{2}{3}<1$). It's OK.
Interviewer:	What if I choose to substitute $x=(-5)$?
Sara:	Then I would get ... $3<1$.. It is wrong! Then it has to be smaller than something and bigger than something else.
Interviewer:	Can you figure out what's wrong with your solution?
Sara:	I don't know...
Interviewer:	What did you do at the beginning?
Sara:	I multiplied by $x+1$
Interviewer:	And what happens when one multiplies?
Sara:	Oh... When multiplying by a negative factor there is a need to "change the direction" of the inequality.. "<" changes to ">"... But I did not multiply by a negative number.. Actually I don't know whether it is negative or positive..

Interviewer:	What did you have in mind when you solved the inequality?
Sara:	I used the rules of solving equations... I thought about equations.

B.6. Forming meaningless connections with quadratic roots

About 15% of the solutions to drills 4 and 8 ($x^2-25>0$ and $x^2<16$) were the following: $x^2>25 \Rightarrow x>\pm5$ or $x^2<16 \Rightarrow x<\pm4$. When asked to explain these solutions, the students resorted to the procedures they used to solve equations. A typical such response was provided by Tom:

Interviewer:	You wrote $x>\pm5$, actually I don't understand what you meant by that. Can you, please, explain it?
Tom:	Eh... all the numbers that are bigger than.. As a matter of fact, I myself am not quite sure what it means.
Interviewer:	What did you mean by writing it?
Tom:	I don't know...
Interviewer:	Why did you write it in this way?
Tom:	I did it the way I am used to solving such equations.

B.7. Solving the square of the given inequality

Several students computed the square of each side of the rational inequalities (Drills 1, 5). Anat, for instance, provided such a solution to Drill 5. Her written explanation, much like those of the other students who used this same procedure to solve these inequalities, related to equations.

Anat: I remember that there is something strange about denominators in inequalities... It has something to do with the square of.. something.. Well, I know that also in equations if $a=b$ then $a^2=b^2$... So I believe that this should work here too.

Talli used the same idea, when neglecting the excluded values in solving the square-root inequality -- Drill 3.

Talli: It is a rule that the result of a square root is considered positive, therefore I solved the inequality by first computing the square of each side. Then I got $x+5<4^2$, which is easily solved and thus $x<11$.

Conclusions and Educational Implications

Three approaches to the solutions of inequalities were exhibited by the participating, high school students: The graphical drawing of the function, the use of number-line, and algebraic manipulations. Our findings indicate that the students used the graphical drawings approach to solve rational and quadratic inequalities, and that its use usually yielded correct solutions. It seems that this graphical approach, and other, similar graphical approaches could provide the students with visual images of the solutions and thus are likely to facilitate the interpretations of the results (see, for instance, Dreyfus & Eisenberg, 1985). Interestingly, the participating students applied this approach only to rational and quadratic inequalities, and not to other types of inequalities. The reasons for and implications of this observed behaviour should further be studied.

Our findings indicate that many students draw inappropriate analogies between the solutions processes of equations and those of inequalities. Notably, the high school students who participated in our study were mathematically-majors who solved various types of inequalities only a few months before they responded to the questionnaire. Yet, the striking, structural similarities between these two mathematical entities (equations and inequalities) create a strong, intuitive feeling that the strategies that hold for solving equations should hold for inequalities as well. Our results indicate that in this case, much like in many other well documented cases, intuitive beliefs successfully compete with the formally acquired knowledge (e.g., Fischbein, 1987). It is, therefore, crucial to thoroughly discuss with the students, the similarities and differences between equations and inequalities and to raise their awareness of the role of intuitive beliefs in their thinking.

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DO EQUILATERAL POLYGONS HAVE EQUAL ANGLES?

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Research in mathematics education strives for theories that have predictive powers (e.g., theories that enable the prediction of how students are likely to respond to given tasks). In a series of papers, we describe the theory of intuitive rules, arguing that this theory has a strong, predictive power. This paper tests the prediction that students at various grades and with different level of achievement in mathematics will argue, in line with the intuitive rule "Same A-same B", that the equality of the sides of a polygon implies the equality of its angles and vice versa. By and large, the results confirm that students' reactions to such tasks could indeed be interpreted as evolving from this intuitive rule.

In recent years a considerable amount of research has been dedicated to the analysis of students' responses to comparison tasks in both science and mathematics. When speaking of comparison tasks, we have in mind scientific, mathematical as well as everyday situations when we compare two entities in terms of a given characteristic. Children start making comparisons at a very young age, for instance, in relation to toys, birthday presents, school grades, allowances and hours serfing the Internet.

However, people have different ways for making comparisons, and they are not necessarily conscious of how they do it. In many cases their comparisons are based on irrelevant properties of the compared entities. An example of such a comparison is: "The more you eat the stronger you get", an idea which is not necessarily correct. These claims follow a certain intuitive rule, namely, *More of A- more of B* (Stavy & Tirosh, 1994), where B is the characteristic in question (i.e., strength) and A is the characteristic which one tends to compare by (i.e., amount of food eaten).

Research in mathematics and science education studies students' understanding of specific notions. When we give these studies a close look it appears that often, reported responses are in line with use of the intuitive rule *More of A-more of B* (e.g., regarding the concepts of *actual infinity*: Tirosh & Tsamir 1996; *point:* Tsamir, 1997; *angle*: Noss, 1987; etc.). It has, thus, been suggested that awareness of the role of the intuitive rule *More of A-more of B* could serve as a means to both *explain* and *predict* students' responses to different tasks.

The present study considered a logical variation of this intuitive rule, that is, *Same of A-same of B* or *Equal A-equal B*. This rule underlies instances like, "Boys of the same age study in the same grade level" and "Polygons with the same number of sides have the same number of diagonals". Here, again, B is the characteristic in question (i.e., the grade level / the number of diagonals) and A is the characteristic which one compares by (i.e., the boys' ages / the number of sides). We investigated students' ways of comparing various characteristics of polygons. In this paper we would like to focus on students' tendency to deduce, for both triangles and quadrilaterals, the equality of angles from the equality of sides, and vice versa. It was predicted that students would tend to make valid connections (in the cases of triangles) and invalid ones (in the cases of quadrilaterals) between the equality of the sides of both polygons and the equality of their angles.

This study aimed to investigate whether, indeed, students viewed the equality of sides as determining the equality of angles of a given polygon, and whether differences in their level of mathematics instruction affected the extent to which they applied the rule *Same of A-same of B.*

Method

Subjects

Two hundred and thirty-one students in grades 4 to 11 participated in this study: 50 fourth graders and 44 sixth graders from elementary school; 35 ninth graders and 35 eleventh graders, who studied at the intermediate level (9th-R and 11th-R); 36 9th graders and 31 eleventh graders, who were mathematics majors (9th-H and 11th-H). There were three levels of mathematics classes, R stands for the intermediate level and H for mathematics majors.

Materials and Procedure

A questionnaire dealing with different characteristics of polygons was distributed during geometry lessons and the students were given about 90 minutes to answer it. Here we present four sample problems, related to the connection between the equality of the sides of any triangle or quadrilateral, and the equality of its angles.

Questionnaire

A triangle is a polygon with three sides, and a quadrilateral is a polygon with four sides.

Problem 1

In a triangle MLC all sides are equal, i.e., ML=LC=CM.
The statement: All the angles in the triangle are also equal, i.e., $\sphericalangle M=\sphericalangle L=\sphericalangle C$ is
true / false / another answer. Explain your answer ____________

Problem 2

In a triangle MLC all angles are equal, i.e., $\sphericalangle M=\sphericalangle L=\sphericalangle C$.
The statement: All the sides in the triangle are also equal, i.e., ML=LC=CM is
true / false / another answer. Explain your answer ____________

Problem 3

In a quadrilateral MLCD all sides are equal, i.e., ML=LC=CD =DM.
The statement: All the angles in this quadrilateral are also equal, i.e., $\sphericalangle M=\sphericalangle L=\sphericalangle C=\sphericalangle D$ is
true / false / another answer. Explain your answer ____________

Problem 4

In a quadrilateral MLCD all angles are equal, i.e., $\sphericalangle M=\sphericalangle L=\sphericalangle C=\sphericalangle D$.
The statement: All the sides in this quadrilateral are also equal, i.e., ML=LC=CD=DM is
true / false / another answer. Explain your answer ____________

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Basically, there were two kinds of tasks:

1. Tasks in which the responses are consistent with the intuitive rule *Same of A-same of B* - problems 1, 2: Here the sides of a triangle are equal if and only if its angles are equal as well. Both problems relate to equilateral triangles. Thus, in the triangular cases, the correct mathematical answer is in line with the intuitive rule *Same of A-same of B.* In problem 1, 'A' stands for the length of the sides and 'B' stands for the sizes of the angles; whereas in problem 2, the roles of 'A' and 'B' are reversed.

2. Tasks in which the responses run counter to the intuitive rule *Same of A-same of B* - problems 3, 4: Here we deal with the angles and sides of quadrilaterals. The solution is undecidable and thus, in these problems the use of the rule *Same of A-same of B* leads to incorrect responses. Only in the specific instance of the square are the statements of problems 3 and 4 true. The equality of the sides of a quadrilateral (*same of A)* does not necessarily determine the equality of its angles *(same of B)* (e.g., the rhombus) and vice versa (e.g., the rectangle).

Results

The results are presented in two sections: the first deals with problems 1 and 2, where the intuitive rule *Same of A-same of B* is applicable, and the second with problems 3 and 4, where it is inapplicable.

I. The Intuitive Rule *Same of A - Same of B* is Applicable

Most students from all grade levels and almost all mathematics majors, argued correctly that **the equality of the angles of a triangle results from the equality of its sides** (Table 1). A somewhat lower percentage of students, but still a substantial number of participants from all grade levels and most of the mathematics majors, argued correctly that **the equality of the sides of a triangle results from the equality of its angles** (Table 2). In all cases, those who claimed "another answer" expressed the view that the answer is undecidable.

Table 1: *Students' Responses, by Grade and Academic Level, Deducing the Equality of Angles from the Equality of Sides in Triangles (in %)*

Grade-	4th (n=50)	6th (n=44)	9th-R (n=35)	9th-H (n=36)	11th-R (n=35)	11th-H (n=31)
The Three Angles are Equal						
* True	65	77	77	94	82	96
False	22	16	12	-	14	-
Another answer	3	5	-	-	-	-
No answer	10	2	11	6	4	4

* correct answer
9th-R / 11th-R : 9th and 11th Graders in the intermediate level
9th-H / 11th-H : 9th and 11th Graders who are mathematics majors

Table 2: *Students' Responses, by Grade and Academic Level, Deducing the Equality of Sides from the Equality of Angles in Triangles (in %)*

Grade-	4th (n=50)	6th (n=44)	9th-R (n=35)	9th-H (n=36)	11th-R (n=35)	11th-H (n=31)
The Three Sides are Equal						
* True	56	59	66	92	72	88
False	30	25	31	3	14	3
Another answer	-	2	3	5	-	9
No answer	14	14	-	-	14	-

* correct answer

About half of the students did not justify their judgments. Among those who did, three lines of reasoning accompanied the correct answers to problems 1 and 2, **linking the equality of the angles and the sides in triangular figures**. The first expressed valid ideas, while the other two types of explanations, invalid ones.

1. *It is actually an equilateral triangle and this kind of triangle has equal angles, 60° each* (9th-R grader); or *In any triangle, equal sides are situated opposite to equal angles, so the sides are equal* (11th-H grader). This justification, which drew upon geometrical theorems were mainly provided by 9th and 11th graders (both R and H).

2. *Equal sides make equal angles* (4th grader); *Angles and sides are connected and are equal at the same time* (6th grader); or *Sides and angles influence one another.* (9th-R grader). These justifications totally lacked any reference to triangles, which were always referred to in the previously mentioned line of reasoning. Instead, the correlation between the equality of the sides and the angles was presented as a general attribute of any polygon. Such an approach was found in all grade levels except among the mathematics majors; and it suggested use of the intuitive rule *Same of A - same of B.*

3. *Same number of sides and angles* (6th grader); *The sides are equal, the number of sides is the same as that of the angles, thus the angles are equal too* (4th grader). This type of justification was quite rare and presented only by the young participants (4th and 6th). However, these students explicitly used the intuitive rule *Same of A-same of B.*

A few 4th and 6th graders explained their view that **the equality of the angles / sides is not derived from the given equality of the sides / angles.** Most of these students gave no explanations to their incorrect judgments. Those who did explain: *Angles and sides are different geometrical entities, thus, there is no connection between their sizes* (4th grader).

As mentioned before, a number of students marked "Another answer" as their response to either problem 1 or 2, usually explaining that *It's indecisive* or *it depends* with no specifications what it would depend on. For instance, one 6th grader explained his response to problem 1: *It all depends whether the triangle becomes an equilateral one.*

It is also notable that the tendency to make invalid connections decreased with grade level and almost disappeared among the mathematics majors.

II. The Intuitive Rule *Same of A – Same of B* is Not Applicable

Tables 3 and 4 show that on the average, only about 25% of the non mathematics majors, and an average of about 55% of the mathematics majors, correctly noticed that **the equality of the angles in a quadrilateral is not determined by the equality of its sides** in problem 3; and that **the equality of the sides in a quadrilateral is not determined by the equality of its angles** in problem 4.

Here, too, about half of the participants did not provide any justification. Among those who did justify their claims, four types of justifications were given to the correct judgment:

1. *If one considers the rhombus, then it is clear that in quadrilaterals, the sizes of the angles are not determined by the sizes of the sides* (9th-H grader); or *The claim [of equal angles] is false and the rhombus can serve as a suitable [counter] example for this claim* (11th-H grader). This kind of justification, by providing a valid counter example, was mainly and much used by 9th and 11th graders, majoring in mathematics. It is notable that using the rectangle as a counter example in problem 4 was significantly less popular.

2. *I have never heard about such a theorem regarding quadrilaterals* (9th-R grader); or *There is no theorem about such a connection between the sides and the angles of a quadrilateral* (9th-H grader); *No geometrical theorem links the sides and the angles of a quadrilateral. If there is such a connection, we would have studied it* (11th-H grader). This line of reasoning, expressed the limited view that *Geometrical theorems which I am familiar with are the only ones to be considered. Those I have not heard about, probably do not exist.* Surprisingly, this view was presented only by 9th and 11th graders, majoring in mathematics.

Table 3: *Students' Responses, by Grade and Academic Level, Deducing the Equality of Angles from the Equality of Sides in Quadrilaterals (in %)*

Grade-	4th (n=50)	6th (n=44)	9th-R (n=35)	9th-H (n=36)	11th-R (n=35)	11th-H (n=31)
The Four Angles are Equal						
# True	68	79	74	33	72	29
* False	22	10	26	52	24	57
* Another answer	-	-	-	8	-	4
No answer	10	11	-	7	4	10

* correct answer   # answer in line with the intuitive rule *same-same*

Table 4: *Students' Responses, by Grade and Academic Level, Deducing the Equality of Sides from the Equality of Angles in Quadrilaterals (in %)*

Grade-	4th (n=50)	6th (n=44)	9th-R (n=35)	9th-H (n=36)	11th-R (n=35)	11th-H (n=31)
The Four Sides are Equal						
# True	62	66	66	39	67	26
* False	22	14	31	50	25	57
* Another answer	-	2	3	8	-	3
No answer	16	18	-	3	8	3

* correct answer   # answer in line with the intuitive rule *same-same*

3. *Angles and sides are distinct entities and there is no way to infer from one to the other* (4th grader). This kind of unsatisfactory justification was presented only by young (4th and 6th grade) participants.

4. *Quadrilaterals with equal sides are just similar* (11th-H grader). This invalid claim was used by an 11th grader to justify the claim that equal sides do not imply equal angles, and by a number of 9th and 11th grade, mathematics majors when justifying the claim that the equal angles of a quadrilateral can not be derived from the equality of the sides.

Another response given by some students in each grade level was that in the given problems, it was **impossible to determine** the equality. Most of these students did not elaborate, claiming for instance that: *No one can determine whether or not the angles are equal. It depends on the given quadrilateral* (9th-R grader).

Most of the students wrongly argued that **the claim that the angles of a quadrilateral are equal when its sides are equal, is true; or that the claim that the sides of a quadrilateral are equal when its angles are equal, is true**. The justifications to this assertion included:

1. *One has just to consider the square* (9th-R grader); *The question (No 3) deals with a square, because no matter how much one extends the length of its sides its angles still remain the same* (6th grader). This kind of justification, using the square as a paradigmatic model, was provided by all but mathematics majors.

2. *According to a theorem, the angles opposite equal sides are equal* (9th-R grader); or *It is actually an enlargement of the theorem about the triangle, linking the equality of the sides and the equality of the angles which are situated opposite each other* (9th-H grader); or *By drawing a diagonal in the quadrilateral we get two triangles, then we twice apply the theorem linking equal sides to equal angles* (9th-H grader). This line of justification, provided by 9th and 11th graders, was based on the overgeneralization of theorems related to triangles.

However, there was a distinct difference between the sophisticated, complex, "formal looking" presentation of the mathematics majors, and the simple, vague statements of the others.

3. *The number of the sides equals the number of the angles* (4th grader). A small number of 4th and 6th graders explicitly expressed, in this strange invalid way, their use of the rule *Same of A-same of B.*

4. *The equality of angles influences the equality of the sides* (6th grader); or *Equal angles mean that the sides are equal too* (9th-R grader). This line of reasoning was expressed by all but mathematics majors to both problems 3 and 4, regardless of the figure used. This was consistent with the intuitive rule *Same of A-same of B.*

Discussion and Educational Implications

The findings of this study firmly support our expectations that students at various grade levels would argue that the equality of the sides and the equality of the angles in any polygon are linked. *Equal* angles mean *equal* sides and vice versa. This line of reasoning is in accordance with the intuitive rule *same of A-same of B.* The application of this rule to the quadrilateral led a substantial number of students to erroneous conclusions-- (a) the angles in an equilateral quadrilateral were incorrectly declared to be *equal*; and (b) the sides, in a quadrilateral with equal angles, were incorrectly regarded as being *equal* as well. This phenomenon was quite prevalent among young students and non-mathematics majoring students, but found also among more than a third of the mathematics majors, most of whom explicitly argued *the same-the same*. The older, mathematics majors supported their judgments that *the same* (sides)-*the same* (angles) by formal arguments, overgeneralizing mathematical theorems they had learnt. Other students presented their claims in a rather general manner. The result was use of invalid, formal-looking justifications for the answers which were probably determined by the intuitive rule *Same of A-same of B.*

Mathematics majors tried to (mis)adjust their mathematical knowledge to the new problems. Being aware of the need to formally validate every mathematical claim, they constructed some invalid "proofs" in order to support their responses, which were probably based on intuitive ideas. It seems that they related to proof merely as 'something one needs to present, when knowing the answer' (see also Tsamir, Tirosh and Stavy, 1997).

It should be noted that the correct answers to problems 1 and 2 could be reached not only via the relevant mathematical theorems, but also by applying the intuitive rule *Same sides-same angles.* Hence, even when students did get the correct answers to the problems and validated them with claims such as, *Equal sides correspond to equal angles* (9th-R grader), we cannot be certain that their responses were entirely based upon formal knowledge. It is also possible that these students reached the correct conclusion by applying the intuitive idea of *Same sides-same angles.* We must conclude that correct answers which are consistent with an intuitive rule do not necessarily reflect students' correct understanding. Moreover, answers in line with the intuitive rule *Same of A-same of B* were given indiscriminately to all four problems by a similar rate of participants from each grade level (non-mathematics majors). They exhibited a tendency to simultaneously grasp the equality of the sides and the angles in any polygon.

The intuitive rule *"Same of A-same of B"* was found to be remarkably influential in directing students' reasoning. Therefore, we recommend that teachers be aware of the role which this intuitive rule plays in students' analysis of problems and their solutions. Furthermore, when presenting problems, teachers should consider whether their solutions may be in line with an intuitive rule or counter to it. When presenting problems which lend themselves to an intuitive solution, teachers should not be satisfied with the correct answers alone, but probe further to be certain that the students are not just applying the intuitive rule. One method of probing is to assign the same problem in a different representation (e.g., Stavy, Tirosh and Tsamir, 1997). However, where these are not available, as in our study, we suggest that related, counter-intuitive problems, which are an extension of the original, be presented to probe the validity of students' answers. An example of this is using the progression from the "intuitive" case of the triangle to the "counter-intuitive" case of the quadrilateral.

To sum up, by relying on our experience regarding students' ways of applying this rule, we can propose a reliable prediction of their problem-dependent, correct as well as erroneous, responses. Such an ability to foresee possible intuitive triggers and obstacles, should serve as a tool for meaningful instruction. The oral presentation will refer to additional geometrical cases of the same kind and to suggestions of implications for the design and construction of instructional tools.

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MEANINGFULLY ASSEMBLING MATHEMATICAL PIECES: AN ACCOUNT OF A TEACHER IN TRANSITION

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Abstract: *This is an account of a fifth-grade teacher's (Nevil) practice as he responds to the challenges of mathematics education reform. We considered 4 sets of classroom observations and interviews, focusing on how Nevil teaches new mathematics content to his students. Nevil decomposes his understandings of the mathematics into pieces that he would be able to elicit from the students, and creates activities with physical objects (manipulatives) in a way he believes would enable them to "see" the connection among the pieces similar to his understanding. Such an account can contribute to understanding the nature of teachers' solutions to the problems posed by the reform and the processes by which mathematics teacher development can occur.*

This paper addresses teachers' attempts to change their practice in light of a significant reform in the content and process of mathematics teaching currently envisioned in the US (cf. National Council of Teachers of Mathematics, 1989, 1991, 1995). By using "practice" we refer to what the teacher does (e.g., planning, assessing, interacting with students) and to everything the teacher feels and thinks about what he or she does. Because the reform is not about implementing a monolithic way of teaching, teachers struggle to meet the challenge of reform by creating useful alternatives that reflect their beliefs, knowledge, motivations, and view of professional development opportunities. This paper presents an account based on Nevil's practice, a teacher in the Mathematics Teacher Development (MTD) Project. The MTD Project is a 4.5-year research project studying elementary mathematics teacher development, including a 3-year instructional program designed to promote the development of practicing and prospective teachers (grouped together). Nevil participates in our research as a case study while participating in communities committed to mathematics education reform (e.g., the MTD program, his school district).

Theoretical Framework

We generate theoretical accounts of the teachers' practice to organize our understandings of teachers in transition (Simon & Tzur, 1997). Each account represents our commitment to understand how the teacher organizes her or his experience of teaching. Our approach stresses *explaining the teacher's perspective from the researcher's perspective*, a stance that involves a subtle, but important distinction. As researchers we attempt to articulate the teacher's approach to the problems of practice: What and how does the teacher perceive? How does the teacher interpret, think about, and react to situations as she or he perceives them? However, our account may differ greatly from what teachers would articulate about their own practice. We structure our accounts using particular conceptual lenses, often not shared by the teacher, that define our focus and guide our interpretations. In this sense, our work is distinguished from studies in which teachers articulate their own perspectives (cf. Schifter, 1996).

Simon (1995) described the particular conceptual lenses we use in thinking about mathematics teaching and learning. In using "mathematics learning" we stress the constructivist sense of transformations in the learner's conceptions through reflecting on their activities and the results of those activities (von Glasersfeld, 1989). Thus, we place at the core of mathematics teaching a process of generating and continually modifying hypothetical learning trajectories for students based on the teacher's evolving understandings of the mathematical terrain and of students' conceptions. We recognize the paradox within which we work. To conceptualize the teacher's practice we must set aside our current view of practice. Yet, our current view of practice is fundamental to what we notice.

Methodology

We use Simon's (1996) teacher development experiment methodology, that combines case studies of teachers with whole class teaching experiments (Cobb, in press) in teacher education classes. Our case studies focus on the question: "How does a teacher teach her or his students mathematics that is beyond what they already know?" We chose this focus because it is comprehensive, reasonably well-defined, and critical to understanding mathematics teacher development. The focus is comprehensive in that it requires inquiry into each phase of teaching: planning, interaction with students, and assessment. It is reasonably well-defined in that it guides our decisions as to significant data. It is critical in that it: (a) emphasizes teachers' alternatives to traditional (tell-and-show) teaching and their knowledge of and about mathematics, learning, and teaching, (b) provides lenses for articulating the teacher's development over time and for looking across case studies, and (c) pushes us to articulate and expand our thinking about mathematics teaching.

Our accounts are based on data sets. A data set consists of videotaped classroom observations of two or more consecutive and related mathematics lessons, and audio-taped interviews. In our analysis we focus on sections of data that pertain to the central research question. We create either tentative inferences about the teacher's practice or questions about possible interpretations of the data. Those inferences and questions serve as the basis for hypotheses we generate and continually revise to explain subsequent data.

This account considered four sets of data. The first set was generated prior to the beginning of the teacher education program (Spring, 1995). The next three sets were generated during the first semester of the program (Fall, 1996). In this paper we focus on the fourth set, which includes four consecutive observations and interviews on Nevil's teaching of division, because this set is representative of his practice.

Accounting for Nevil's Practice

Nevil is a fifth grade teacher (second year teaching) in a district that provides a list of desired grade-level outcomes in specific content areas, leaving the teacher to create learning activities for the students. As an enthusiastic teacher who implements many reform ideas, Nevil is pleased to work under these guidelines. He is committed to minimizing his telling and showing students the mathematics they need to learn. He strives to create a collaborative climate and encourages his students to participate in all activities.

He poses tasks to his students, requires them at times to use manipulatives, and encourages communication and reasoning about the mathematics. Nevil intends for students to make connections between ideas and procedures. He assesses the success of his lessons by attending to students' expressions of their mathematical ideas and interests.

In the interview prior to the first lesson we observed in the unit on division, Nevil described the previous lesson in which the students worked on an introductory problem: "The kids started out with 16 Unifix cubes and I asked them... to show their partner what 16 divided by 2 would look like." In response to the researcher's questions about the purpose of that task, Nevil said:

> We have looked at division conceptually. ... I am trying to get the students to understand that division has to do with beginning with a number of things and then dividing it into sets. ... Some of the students have a really good understanding of what division is all about, but I feel like that there are still some that might not quite get it.

Nevil conducted a three-part lesson that proceeded along his plan. First, he engaged pairs of students in working with unifix cubes on the problem 20÷4, so that they re-visit, or figure out, both meanings of division (quotitive and partitive). Nevil interacted with the pairs to: (a) assess their meaning for division and (b) promote their ability to use both. When Nevil concluded that most students were able to "see" division as breaking a set into smaller equal sets (after about 25 minutes), he collected the Unifix cubes and moved to the second part. He asked the students to solve three long division problems that he wrote on the board (140÷7, 568÷8, 5454÷6), and interacted with them to assess how efficiently they were executing the procedure. When he concluded that most students knew how to carry out the procedure, he moved to the third part—a 30-minute whole-class discussion about the meaning of each step in the algorithm as performed in the problem 140÷7.

From the outset, Nevil's plan and implementation indicated to us his intention that the students first give meaning to the operation, and then, after recalling the procedure, use this meaning to interpret the steps in the algorithm. He emphasized that the students were to show and tell, not the teacher. However, we noticed several salient points in his work:

1. Nevil asked the students to show 16÷4 and 20÷4. In order to do so the students already needed to know what division means—ostensibly one goal of the lesson.
2. Nevil chose <u>not</u> to make the cubes available to students when the focus turned to the long division algorithm.
3. Nevil did not blame his students for not making sense of the subtraction step, and proposed and implemented essentially the same plan for subsequent lessons.
4. Nevil began with open-ended tasks and became more and more directive.

The three parts in Nevil's lesson provide a first glance into his practice. One key feature of his practice is using physical objects (unifix cubes) to promote the students' ability to see and explain the meaning(s) of division. He wanted the students to establish a view of division that can support both their learning of related topics (e.g., multiplication, fractions) and their making sense of the steps in the division algorithm. He expected that through seeing each other's arrangements of cubes and hearing the corresponding explanations, the students would use division meaningfully. Another key feature is the connection among the lesson's three parts. Nevil seemed to expect that once the students demonstrated division of 20 cubes into smaller, equal sets and executed the algorithm, they would be able to make sense of each step in the long division algorithm based on their image of having worked with the cubes.

We take a closer look at data pertaining to the third part, in which Nevil taught the meaning of steps (pieces) in the algorithm. The segment of whole class discussion below begins after Nevil had written on the board the algorithmic solution of 140÷7 contributed by a student (N=Nevil, S=Student):

N: Okay, I want to make sure that we are understanding what's going on with this long division stuff. It sounds like we are saying that we are dividing. But then it looks like we are multiplying and shifting stuff, subtracting stuff and I always thought division was putting things into sets. Writing a big number and either finding how many sets of a small number we have or how many is in each of the smaller sets. So what [is] subtracting all about? [and] Why did we put the 2 here?

S: Umm, you got it because 14 divided by 7 equals 2. And if you would put the 2 over the 1, someone would think it would be in the 100 digit, so it would be 200.

N: What does the 2 mean guys?

S: Umm, it means the 10 is [inaudible] 2 means 10.

N: Okay so by putting the 2 above the 4?

S: It's in the 10's column.

N: [A little later, frustrated because many students do not follow the issue about the location of the 2] I want you to understand what this algorithm is. ... If you understand what the numbers mean you will be able to do the algorithm. ... Are we really trying to figure out how many times 7 goes into 1? Yes or no?

S: Yeah.

N: Does that make sense? If I gave you 140 unifix cubes and I said divide 140 by 7. Would you be trying to figure out how many times 7 goes into 1?

S: No.

N: I am just trying to see what you are doing. You have a hundred, you have a basket full of unifix cubes and I say divide it by 7; what would you do?

S: Just take them all and divide them into 7 parts.

N: Okay, so I would make seven smaller sets out of all of those things in the basket. Okay. Am I dividing 7 into 1? Not really.

[Here, after exploring another child's reasoning for not having a digit in the 100's column, Nevil becomes more frustrated.]

N: Maybe I am not helping you think. Umm, long division is a representation of what we would do with the unifix cubes. It's an abstraction. ... so you would get the same number on top that you would get with the unifix cubes. ... if I gave you 140 unifix cubes and I said divide this by 7 you would all get 20. Some of you would find 20 sets of 7 and some of you would find 7 sets of 20, but your answer is 20. ... So we are trying to find a way that shows what we would do with the unifix cubes. ... And let's use the example of making seven groups out of the 140 unifix cubes. ... Would you have 100 and more in each of the groups?

S: No.

N: No. So how could we represent that in long division? How many zeros---how many hundreds do we have in each of the groups?

S: None.

N: Zero. Put a zero here (writes "0" above the "1" of the 140). How many tens?

S: Two.

N: Two. All right. We are showing that that is how we are doing this. Sort of showing by doing these numbers.

N: [A little later] Okay, or 20 times 7 is 140 if we are thinking of place value. Correct? That's really what we are saying. ... Okay, now when we look at it, why do we subtract? What is subtraction doing for us here?

The last excerpt highlights three key points that underlie our postulation of a major aspect of Nevil's practice, namely, breaking the mathematical terrain into connected pieces. We noticed his: (a) frustration and the subsequent intensive work on the piece (algorithmic step) of digit location, (b) attempt to direct the students to connect that piece with a previous piece—division of cubes, and (c) decision to move on to the next piece (subtraction). Nevil saw a direct connection between grouping physical objects and the location of digits in the quotient. His first attempt was to make that connection apparent to the students by leading them to recall their image of dividing cubes and the corresponding meaning of division, and to relate it to the digits in the quotient. He could not understand why they did not see the connection that was so clear to him. Interestingly, and typical of Nevil, he did not blame the students for not making the connection. Instead, he said that he sees a direct connection between the algorithm (an abstract representation with numbers) and the cubes (7 groups of 20, or 20 groups of 7). Then, he asked leading questions about the proper location of each digit to which the students contributed the proper answer. We inferred that Nevil's decision to move on to the next piece (the meaning of the subtraction step) reflected his satisfaction about the students' contributions, that is, a match with his understanding.

Working on the next piece, "Why subtraction?" lasted more than three lessons. Nevil repeatedly asked that question, searching for students' contributions that would match his view of "accounting for everything that was in that original set. ... comparing the original set to the first number that we come up with." Students contributed various reasons such as "finding what is left over," or "making the numbers smaller," or "addition would make the number bigger." In response to each contribution he attempted to understand its meaning and to find a way to elicit another (more suitable) response, frequently redirecting the students to visualize division of the cubes. Moreover, during the second lesson he decided to change the object visualized, from unifix cubes to base-10 blocks, because he realized that the blocks made a much clearer connection to the partitioning of groups larger than one in the base-10 algorithm. During the interview that followed the second lesson he realized their inability to visualize division with base-10 blocks and that the problem they had explored did not present a good case of "accounting for the difference." Thus, the third lesson consisted of essentially the same three parts: showing 100÷6 with base-10 blocks, procedurally executing the algorithm for that problem, and finally discussing the meaning of each step.

To account for Nevil's practice, we specified his perspective about mathematics, knowing mathematics, coming to know mathematics (i.e., learning), and teaching mathematics. (We do not claim that any of these is an explicit assumption of Nevil's.) Nevil seems to think about mathematics as a body of connected pieces. He seems to conclude that what he can see exists out there, hence other people—particularly his students—can also perceive it. Nevil's meaning for knowing mathematics focuses on a person's ability to perceive each piece and connect it to other pieces. Coming to know mathematics means coming to see and connect (assemble) relevant pieces. In particular, a learner appropriates each piece in the form needed for the assembly by participating in activities such as arranging arrays of physical objects, listening to other learners' ideas, or answering the teacher's questions. Moreover, assembly is possible <u>if and only if</u> the pieces are "perceived" properly, that is, they correspond to the way they exist in Nevil's mathematics. (Note: "If and only if" is used here in a dual logical sense: if a learner "sees" the pieces properly he or she would make the apparent connection; if a learner presents the proper assembly then he or she must "see" all of the pieces in their proper form.) Nevil's view of teaching seems to focus on enabling the students to: (a) "see" for <u>themselves</u> each piece in a form suitable for the assembly and (b) explicate connections among pieces. The focus on students' ability to see for themselves the conceptual piece in the form suitable for assembly is crucial, because it implies the students' active role in arranging objects and discussing the mathematical pieces as a means to assure assembly for themselves. It is here that Nevil's practice markedly deviates from the traditional view of teaching and learning—that what one teaches leads directly to what the capable students learn.

The account above helps to explain Nevil's understanding of his role as a teacher. He engages the students in activities that can: (a) make apparent each and every piece required for the assembly and (b) make apparent the connections among pieces. His responsibility is to create situations in which the students can "see" and assemble the pieces for themselves, and he assesses the teaching as successful, hence move on, if their behavior indicates to him that they "see" the connected pieces similar to how he "sees" them.

This account helps to explain the four salient points we listed above. The first two points (using the cubes; putting them away) can be explained by his expectation that students can see what he can see. The word "divide" means for him, and hence for students who arrange the cubes properly, to break a set into smaller, equal sets. He expects that the students should be able to execute and understand division of a set of cubes, and to use this understanding in the absence of the cubes while making sense of the long-division computation (140÷7). Because Nevil's goals included computing the result of division, the first task was for the students to show a simple division computation with cubes. Because his goals included the use of a visualized division with cubes, he puts away the cubes before beginning discussion of long division. The third point referred to Nevil's refraining from blaming the students when they did not contribute the response he expected and to repeatedly asking the same questions and implementing the same plan. We explain this point as an implication of his assumption that the connected pieces exist out there and thus can be perceived by students. Because he does not question this assumption, he concludes from the students' inability to assemble the pieces or to independently see the connection between the three parts of his lesson that he failed to present one or more of the pieces in the form needed for assembly. Thus, he strives to create situations and ask questions that would lead students to "see" what is obvious to him, and fosters small-group or whole-class interactions between those who "see" and those who do not, so that students show and tell, not the teacher. The fourth point—becoming more and more directive—can be explained by Nevil's perception-based view and his ultimate goal that students see for themselves, for which he continually assesses their contributions. His goal focuses on the product of seeing the connected pieces, not on the process of conceptual transformation that brings about such "seeing." Thus, at times when his activities fail to bring about students' contributions of the expected pieces or connections (his preferred option), he regards his leading questions as a reasonable teaching strategy that enables to accomplish his goal without telling or showing the answer himself.

Discussion

This paper presented a theoretical account of a teacher in transition, who strives to avoid a traditional show-and-tell approach by implementing what he regards as reform-recommended teaching strategies. Our account proposes an interrelated perspective of the teacher's sense making of reform ideas such as active learning (e.g., using manipulatives, communicating in small groups) and of her or his own mathematical understanding. In particular, we speculate that the teacher's own focus on seeing mathematical pieces and connections leads to believing that they "exist out there," hence can be perceived by the students. This view of learning is a fundamental aspect of the teacher's practice, and contributes to comprehending key activities of the teacher: decomposing the mathematical terrain into pieces, finding activities that can help elicit from the students the pieces in the form expected for assembly, highlighting pieces in such a way that connections among them become apparent, assessing the extent to which students' contributions match his or her decomposed mathematics, and repeating pieces and/or connections if the assessment reveals gaps between the expected and the actual assembly. In this sense, the teacher's perspective about learning is perception-based in that it does not consider the conceptions underlying one's "seeing," nor the work needed for constructing each piece and connection. Consequently, the teacher focuses on students' acquisition of desired mathematical pieces and connections, not on the nature of students' thinking and how they construct new conceptions by transforming their current conceptions.

Such a perspective is a challenge to reform-oriented teacher education. Specifically, it highlights the interrelated issues of how teacher educators can: (a) understand teachers' interpretations of their own teacher education experience and (b) focus their work on facilitating teachers' construction of a new perspective in which mathematical ideas are not "there" (hence perceivable) but rather constructed. To illustrate how this account can be useful in teacher education, we present two examples from our work with teachers in the program that represent our evolving ways of noticing significant aspects of teachers' participation in the reform. First, we find this account useful in analyzing other teachers' work on mathematics. We now notice that many teachers with whom we have worked as teacher educators, including MTD participants, seem to think about the mathematical ideas they learn in terms of coming to see mathematical pieces and connections among those pieces in a meaningful way they did not see before. Correspondingly, we have heard teachers express their sense that reform-oriented professional development opportunities improved their ability to help students see for themselves. Using this account, we notice that such expressions indicate the intimate link between two aspects of a perception-based perspective: the teacher's mathematical thinking and her or his thinking about mathematics as perceivable to others. Second, we assume that teachers, as we engage them in learning about students' mathematical thinking and development, can learn to consider the students' current conceptions and plan for teaching that builds on, and promotes transformations in those conceptions. Nevertheless, promoting such learning in the teachers appears to be very difficult because attending to students' thinking requires that teachers decenter, that is, set aside their own view of the mathematics in order to conjecture about possible ways of thinking underlying the students' behavior. However, teachers' practices which include a perception-based view of knowing mathematics and a focus on decomposing mathematics into connected pieces impede decentering or the development of a conception-based view. From our constructivist perspective, we explain this in terms of the teacher's assimilation of reform-oriented experiences into their current perception-based perspective. Therefore, we continue struggling with: (a) better understanding different aspects of perception-based teaching and (b) hypothesizing learning trajectories that have the potential to initiate a transformation to conception-based teaching.

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Research methods of the "north" revisited from the "south"

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This paper discusses issues arising in so-called "southern" or "developing" countries, such as Colombia and South Africa, when researchers in mathematics education try to follow research methods that have been created mainly in the "northern" or "developed" world. These issues have to do with the inefficiency of those methods to approach several key issues of research in mathematics education in developing societies. We claim that the instability of these social contexts should lead to questioning of the assumption of stability that underlies scientific production in the developed world, and, therefore, to engender research methods that grasp the real complexity of the phenomena of mathematics teaching and learning in developing contexts. The knowledge produced by those means should also be considered as an important part of the international knowledge on mathematics education.

Introduction

Most of the papers presented in PME show exemplary pieces of research, carried out aseptically and whose clear methodology warrants validity, relevance and all other research criteria ascribed to quality findings. Although the methodological framework of these studies is stated as clearly as possible, few reflections, criticisms and actual discussion is devoted, both in the papers and in the presentations during the conference, to methodological issues and the difficulties emerging from them in particular contexts, i.e. the socio-political circumstances, the history and the material conditions within which the research process has taken place. Our main purpose in this paper is to present reflections about this meta-level of research, by considering the methodological difficulties that researchers in mathematics education in social contexts, different from the developed world, may face when trying to carry out their studies.

This reflection comes to be relevant in PME for several reasons. First of all, the general current international discussion about what is research in mathematics education (e.g. ICMI Study 1994 *What is Research in Mathematics Education and What Are its Results*?; or Clements and Ellerton, 1996) does not only question the acceptance of the positivist research paradigm and its quality standards as the only way of approaching research in the discipline, but also regards the necessity and possibility of exploring diverse research perspectives that involve and consider the social, political and cultural needs for doing research in both developed (Cotton and Gates, 1996) and, mainly, developing societies. Second, as PME becomes a really international group, its members should question whether the group's always given or taken for granted view of research contributes to the inclusion of research that may not follow the "standard" format, but that constitutes relevant knowledge for the realities of under-represented countries and researchers. Third, facing the challenges of doing research in rapidly and constantly changing societies as those in developing nations, will also contribute to mathematics education research in all other contexts, including the places that currently dominate in the production and dissemination of it. This is because the former can bring into focus new relevant problems and insights that may remain hidden in "normal" research situations. This has implications also within PME, in how relations between researchers, their research agendas and frameworks are defined in terms of their importance for well-represented and under-represented countries in the group. And finally, it is crucial to reflect on the implications of importing research methods and intending to use them without being aware of the very dynamic nature that the research processes acquire in developing societies. We should not keep on being blind to the challenges that the specificities of these societies impose on the act of researching and producing knowledge about the teaching and learning of mathematics.

The main thesis of this paper is that there is a need to question research methods in mathematics education, created in developed social contexts and applied in developing countries, because their importation can be shown to be inefficient in grasping the nature and complexity of research processes in those unstable, developing social contexts. We begin with a conceptual note on some of the terms used; then present two cases illustrating this inefficiency; we discuss assumptions built into research methods used in mathematics education; and finally highlight some challenges and key issues that face researchers in mathematics education, especially from developing countries.

Some conceptual considerations

The terms research, method and methodology may be interpreted in different ways and, in fact, in the literature in mathematics education research they are sometimes used differently. Usually research in mathematics education is viewed as a systematic, disciplined inquiry, typically, on an issue related to the teaching and learning of mathematics (Kilpatrick, 1992). The systematism refers to the coherence, organization and reflection that characterizes the way of tackling the issue to be investigated; and the discipline also suggests that the process is open to be examined and verified. These systematism and discipline are achieved through a research method which groups together: the definition of the research problem or selection of a research object; the general epistemological approach from which the object is viewed and which determines its nature and the nature of the very same act of studying it; and the methodology which states the stages followed to gain knowledge about the issue. This includes activities like formulating theoretical tools to approach a research question, designing a strategy to obtain information about it, selecting specific techniques and instruments, analyzing the information, interpreting it, and drawing conclusions and findings in respect to the issue that was the center of the whole process. And finally, the whole method guarantees the quality of both the process and its results.

Two cases of "disturbed" research in developing societies

Let us consider two cases of research that, for us, exemplify the keystone of our claim. These two projects fit into the category stated in PME as mathematics teacher education and professional development.

A case in teachers and administrators' professional development in Colombia

The PRIME I Project was carried out by a research team from "una empresa docente" (Universidad de los Andes, Bogota - Colombia). It completed the second cycle of a long-term action research project exploring the issues of low quality in secondary mathematics education, from an institutional point of view (Perry et al., 1996a). In this study 15 schools participated from Bogota, represented by 2 administrators and 2 secondary mathematics teachers from each. It intended to involve teachers and administrators in action research activities concerning their practices in the school, as a way to promote their professional development (Perry et al., 1997, Valero et al., 1997a). The methodology to study their process of involvement in the professional development activities and its effect in the functioning of the teaching of mathematics in the school, was previously planned. It included a series of qualitative and quantitative instruments to be applied at the beginning, in the middle and at the end of the process, in order to find the possible changes happening in the schools.

The first set of data was collected as expected. While lots of efforts were made to collect the second and specially the third sets, towards the end of the project, more or less 40% of the participants had abandoned it (Perry et al., 1996b). Reasons for this related to the instability of staff in public schools; the internal fights and conflicts in the schools which created obstacles to their participation in the project (Valero et al., 1997b); and the impossibility of both teachers and administrators to manage their time due to their excessive academic and administrative charge. But also researchers were unable to collect some information as planned because some events like an unexpected general teacher strike that took place in the middle of the project and altered the possibilities for participation for some teachers. Because of this lack of information, the implementation of the initially designed methodology raised several questions about the validity of the comparisons that had to be established, the accuracy of the conclusions and the possibility of giving them a certain degree of generality.

Nevertheless, the very failure in the application of the previously planned methodological strategy and the disruptions in the data showed the existence of relevant problems that came to be more interesting in revealing the nature of the processes taking place in the teaching of mathematics in some Colombian schools. The problem that the research team faced then was, what meaning should be given to the data collected and how to re-articulate the whole experience in order to reveal the cornerstone of the disruptions and their effect in the teaching of mathematics.

A theory-practice study in initial teacher education in South Africa

This study attempted to investigate how and why primary school student teachers would implement a "social, cultural, political approach" to a mathematics curriculum during their teaching practice (Vithal, 1997). The research design was developed to allow the student teachers to jointly negotiate an opportunity to implement a radical new approach to the mathematics curriculum with the resident-teacher in the school and the researcher/teacher-educator. The initial teaching practice session would be followed by a period of curriculum design and, then, actual implementation of projects in schools would occur in the second teaching practice session. All this was to be captured as data through interviews, classroom observations, journals, students' work and so on.

A series of disruptions and changes in important aspects of the context led to continual modifications in the research strategies (Vithal, 1998). Along the study, the notion of teaching practice itself was undergoing change in response to pressures within the faculty; as well as in response to new national policies, norms and standards in teacher education. The researcher/teacher educator had to simultaneously make research and curriculum decisions, which could potentially conflict. The first preparation phase in schools had to be replaced with campus-based sessions only, because the university closed as a result of student and staff protests. This meant that student teachers had to create the opportunity to implement their ideas on their own. The second phase where students actually implemented their projects was also disrupted when schools closed due to a teacher strike. The impact of such disruptions on the student teachers' attempt to implement their innovation differed from school to school and from student teacher to student teacher (Vithal et al., 1997).

Several questions emerged: How are the disruptions to be managed in the methodology? What should the researcher do with the disruptive data produced? How can it be taken seriously as contributing to the research endeavour instead of only as something negative that must be thrown out, or confessed to and apologized for as poor methodology? And finally, if brought into the focus of the study, what are its implications for the research question, analysis and the knowledge produced as well as for rigor and quality in research? Focussing on the disruptions themselves has meant that some other insights have emerged both relevant to the study and not, but still pertinent to the context. It has also meant that several aspects of the research methodology had to be reconsidered especially if the changes and disruptions are to be made visible and brought to the center of the methodological debate (Vithal, 1998).

Problematizing research methods from the developing world

What is glaringly similar in these two cases is the changing and unstable nature of the context in which the research was conducted. We do not suggest that researchers in other conditions may not face similar difficulties; however, we argue that the scale of the problem is different. The chronic nature, depth and extensiveness of the instability, as well as their being beyond the control of the researcher, require us to fundamentally re-think and re-create research methods that allow relevant research to be undertaken in such situations, and that still preserve notions of rigor, scholarship and quality. The cases presented draw attention to issues like what is the nature of the whole research process; which are the assumptions adopted when approaching a problem; what findings are eventually produced and how is the research to be evaluated.

Revisiting the assumptions of research in mathematics education

Generally, researchers in developing countries import and apply methods created in developed countries. Such methods have as a strong underlying assumption, the stability and "normality" of the setting in which research occurs. The functioning of a developed society is viewed as continuous, and this comes from the actual steady growth and unfluctuating life conditions that modernity has built. The whole view of science in developed countries is based on this characteristic. And even though a "post-modern" world, with all its predicaments, begins to irrupt in science and research, stability is not yet questioned.

The stability assumption, then, is present in the components of the research method. Research objects and problems are viewed as stable, not in the sense that they may remain the same all along the research process, but in the sense that they are always meant to be available for the researcher to study or to interact with. The processes of teaching and learning of mathematics are therefore, considered to be stable and available to be researched. The approaches are also stable, as fundamental epistemological positions in respect to the object and the process of inquiry allows stating of the existence of the research issue and presupposes that it will be maintained as a part of the reality that can be studied. And methodologies are stable because, although understood as iterative processes, the changes to the original designs are minor in terms of their distance from the initial research focus. Besides, the researcher should make sure in previously designed methodological plans how to control the possible disturbing variables, and how to assure the collection of all the data needed. Even in the case where some data is missing in a later application of research instruments, the analytic methods allow a percentage of error (e.g., consider the case of statistical tools). This also implies that there is a relative stability and continuity assumed in the relations between the research objects, the procedures, the analysis and the eventual findings. Therefore, criteria for assessing the quality of the research process and its results, like validity, objectivity rigor and precision, predictability and reproducibility, among others (see Kilpatrick, 1993 or Sierpinska, 1993), are also associated with the stability assumption precisely because they look upon the steadiness and linearity of the research process.

But in contrast to this stability, developing societies are characterized by instability, given by the constant and abrupt reorganization of political, social and economic forces. The big predicament in research arises when it tries to be conducted following methods that rely on this stability assumption and require stable conditions to produce the expected scientific results. When the research process is obstructed by uncontrollable disruptions emerging from the very same unstable nature of the social context and of the research objects that are considered, then the whole process of research has to be reconceived to allow the disruptions themselves to reveal key problems that should be addressed in order to understand, interpret or transform the real issues of the teaching and learning of mathematics in developing societies.

Challenging researchers in mathematics education

Questioning the stability assumption and being aware of it challenges researchers in mathematics education from both developed and, especially, developing societies. Rapidly and constantly changing research environments force a problematizing of the whole research method, because the smooth correspondence assumed between the research object, the methodology developed to investigate it and its outcomes cannot be taken for granted. In both the cases described, the researchers point to the instability raising other research questions that were not in the direct focus of the study but, nevertheless, are important and relevant to the context. This implies that researchers need to be open to the possibility of changes in the research process, but also to be more radically responsive and flexible in actually modifying their research objects, as disruptions in the context and, therefore, in the methodology, appear. The whole method itself also has to respond to the changes in the research focus. A more ·c understanding of the relationship between the question, methodology, analysis and outcomes has to underpin the research process, where relevance to context becomes a crucial consideration in the research process. The analysis and findings should not discard or attempt to "correct" the disruptive data, but to focus on it as the "authentic" or "actual" data in the research process. The production of the data itself may need to take different forms and require researchers to re-examine criteria for quality research.

For instance, the concept of validity (see for e.g. Kilpatrick, 1993) should be carefully reconceived. In the second case described, student teachers had to be relied on for data collection. This had implications for the kind of data collected and the relationships set up between the researcher and the participants in the process; and this could be considered a problem for the internal validity of the study. Therefore, it may be necessary to argue for stronger involvement of research participants in unstable situations, not only on ethical but also on methodological grounds. This could mean that different criteria, for e.g. "a democratic-participatory validity" may need to be developed to indicate the nature and extent of research participants' involvement in a study (Vithal, 1998).

Instead of evaluating a study only in terms of its generalizability, which is connected to external validity, we may consider its generative capacity as an important criterion (Vithal, 1998). Generativity can be taken as the extent to which a study originates new research objects for study and alternative research methodologies, as well as produces new outcomes. In other words, a generative study "unseat[s] conventional thought and thereby opens new and desirable alternatives for thought and action" (Kvale, 1996, p.234). In the first case mentioned, the research team had to abandon the idea of describing the possible changes happening in the schools as a consequence of the impact of the professional development strategy implemented. Instead, they had to direct their attention, for example, to the analysis of the structural conditions that promote or constrain the establishment of relationships between the administrators and the group of mathematics teachers, for building the relationships needed for professional interaction and development (Valero et al., 1997b). In this sense, the study had a generative capacity.

All the above implies that a creative attitude toward research methods in mathematics education needs to be encouraged, especially in developing world contexts where the prevailing culture is one of being consumers of theories and practices produced elsewhere, rather than of viewing themselves as producers. Recent challenges to research methods, such as the provocative question of whether research epistemologies, which are largely a product of the developed world, may be racially biased (Scheurich and Young 1997), should inspire researchers to critique and suggest new directions in mathematics education research that are more relevant and authentic to their contexts, however tentative. Such an approach must surely enhance our overall research endeavors in mathematics education. Unless researchers in these situations begin to challenge "mainstream" mathematics education research and face the dominant contexts in which mathematics education occurs in their countries, they could continue to be trapped in a double bind. That is, they do "good quality" research, in carefully chosen, less problematic research environments, about marginally relevant topics to the general status of mathematics education in their country; or risk having their research considered methodologically poor, but nevertheless relevant to the situation of the majority of mathematics learners in their context.

Issues in importation of research: equilibrating relationships

Questions of what and how importation in research in mathematics education occurs are as important as who is doing the importation and for what purposes. Much of the importation of research is from the "north" to the "south" through the complex network of international literature, agencies that fund research, Ph.D. students from developing countries who study in "northern" institutions, and other mechanisms. Perhaps the most overt and predominant form of importation is that of research findings in mathematics education. How children learn or should learn mathematics, how teachers teach or should teach, etc. are ideas imported by researchers who live and work in developing world contexts. These ideas are also brought by those who have particular expertise and may be funded to work there. It is not surprising, therefore, that research questions explored in developing countries are often closely related to those same findings and are investigated within their corresponding research frameworks as theory, methodology and practice (in this sense, our case descriptions show how we too are implicated in this importation). It is not a coincidence, for example, that research related to technology in mathematics education dominate in international conferences and journals, rather than issues such as teaching and learning mathematics in large, under-resourced multilingual classrooms, which are relevant concerns in developing countries.

With these types of research come particular ways of posing research questions and their corresponding ways of investigating them. Research objects are constructed within particular research processes based on particular kinds of research relationships. The criteria for quality and relevance that emerged and are applied in the original context in which these questions and methods arose, also come to be applied in all other contexts. The point in this is not in completely neglecting importation, but mainly in problematizing it and seeking ways of building more equitable reciprocal relations.

Can there also be exportation? First, the question is not only what countries with strong and dominant research traditions could and actually do offer countries struggling to develop their research capacities; but equally, what these developed countries can gain and learn about the phenomena in mathematics education in the developing world. For instance, what relevant insights can be gained by investigating "socio-constructivist approaches" to mathematics education in large multi-lingual, multi-grade classrooms found in the poorest parts of poorer countries, and how these insights could produce advances in theory and practice for all contexts. Secondly, and this is the main thesis of this paper, there is a need for developing-world contexts to raise new questions and focus on new research objects, develop different methods and produce a more relevant knowledge base to their world, but also with other corresponding criteria for rigor, scholarship and quality. The tension that must be kept in check is how we do live in the global mathematics education village in which our research and understanding of *all* mathematics education learners is enhanced.

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AN EXPERIMENT IN DEVELOPING PROOF THROUGH PATTERN

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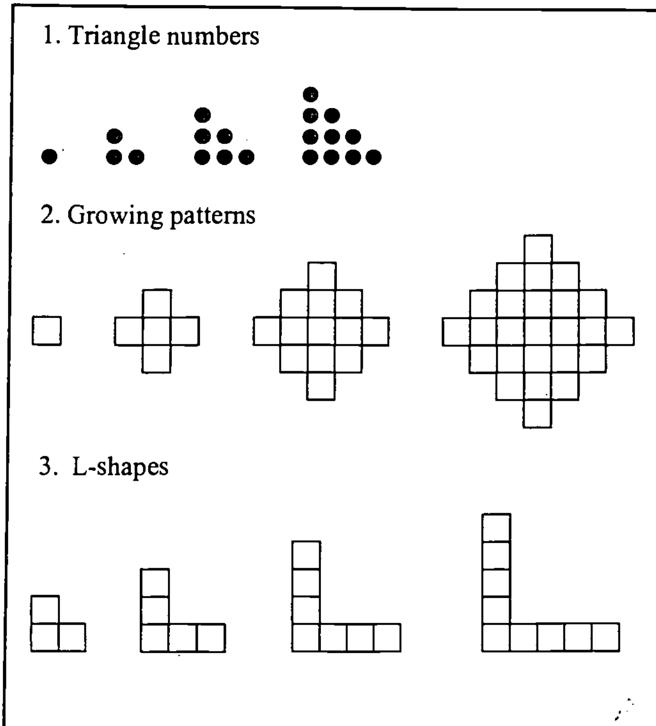
Three classes totalling 71 students aged 14-15 years were involved in an experiment to see if the understanding and use of proof could be enhanced and improved through pattern. One class formed an experimental group, taught in a carefully planned environment which included considerable extra emphasis on both pattern and proof, and the other two classes served as control groups taught according to their respective teacher's interpretations of the demands of the national curriculum. Although differences were not large, and transfer to less familiar tasks was limited, the experimental group did achieve greater success on post-test questions and revealed greater development than the control groups. There was evidence of a strong case for the use of supporting diagrams.

Design of the experiment

During recent years, a number of studies of various aspects of children's patterning abilities have been carried out, for example Stacey (1989), Pegg (1992), Redden (1994) and Orton and Orton (1996). The work described here represents a new venture, one which considered whether appropriate curriculum activities involving pattern might lead to improved skills in proving. Three parallel classes of Year 10 (aged 14-15 years) students took part in the experiment (Waring, 1997), all the students being of higher ability and from within the tenth to twenty-fifth percentiles. Careful statistical analysis detected no significant differences between the three classes, and the students had all followed the same curriculum prior to entering Year 10. One of the three classes was termed 'experimental' (group E) because their curriculum was enhanced, and the other two 'control' classes (groups C1 and C2) were taught according to the scheme of work normally used in the school and based on the national curriculum. None of the students knew that they were taking part in an experiment. The teaching programme for the experimental group included both special materials designed to guide students through investigations leading to patterns which necessitated analysis, and other aspects of the standard curriculum which involved a pattern-based approach. Group E was taught by the researcher, who was also their normal mathematics teacher, group C1 was taught by a teacher who was known not to favour investigative approaches to mathematics, and who basically followed the textbook, and group C2 was taught by a teacher who was known to employ investigative approaches in interpreting the demands of the mathematics curriculum but who was denied access to the materials and teaching approaches used with group E. The fact that the researcher was also the teacher of the experimental group was not considered inappropriate, because the aim was to investigate whether pupils could be led to develop their capabilities in proving within an enhanced environment. Piloting of all tasks was carried out using a class of very able Year 8 students (aged 12-13 years), and a class of Year 11 students (aged 15-16 years) of similar abilities to those in groups E, C1 and C2. The test instruments consisted of a post-test and an end of year examination, both common to all three groups, and selected students were also interviewed, using their responses to the post-test and some new tasks as a focus for further questioning.

The teaching programme

The teaching programme for Group E was a vital component of the overall research project, and therefore must be described. The aims of this teaching programme included first to raise levels of awareness of the need for proof, then to learn something of what might be considered acceptable as proofs, and finally to attempt to create proofs. Teaching strategies included exposition, informal class discussion, small group work, written work based on structured questions, written work based on open questions, investigations based on diagrams, and investigations which involved practical activity. Variety in mathematical content was achieved through, for example, investigations based on number patterns which could be explained by use of algebra and/or diagrams, diagrams which could be explained by use of algebra, spatial problems which led to number patterns, and problems which



This typical triangle numbers question was based on dot patterns.

There were two tasks on growth, one based on triangles on isometric paper, and the other based on squares on squared paper.

The L-shapes task was based on using matches to assemble the shapes. Formulae were required for the number of small squares, the perimeter of the L-shape, and the total number of matches.

Figure 1: Diagrams for some of the training tasks

led to solutions which could only be explained in words. The diagrams for three of the many tasks are shown in Figure 1, simply as examples of typical pattern-based question-types. Generally, such tasks included developing a number pattern, seeking a formula, and being challenged to prove that the formula was always true. There were, however, other tasks which were not so typical, and these remained more problematical for the pupils, as the results will reveal.

The post-test and examination results

Three scores are available from the post-test, namely on pattern, on proof, and on 'basic' mathematics (representing the rest of the curriculum for the year), and the mean percentage scores for the groups are shown in Table 1. The sizes of groups were clearly comparatively small for significance testing, but nevertheless the Basic

	Basic	Pattern	Proof
E (n=25)	82	70	39
C1 (n=22)	75	59	18
C2 (n=24)	81	66	19

Table 1 : Mean percentage test scores

scores tally with the claimed comparability of the three groups, referred to earlier and established by other means, and the Pattern and Proof scores indicate differences between the groups. For completeness, similarities and differences were investigated using t-tests, and the levels of significance are shown in Table 2, with levels of significance of greater than five per cent regarded as not significant (NS). These results suggest that the work on pattern and proof throughout the year had made a difference to group E. The results concerning pattern are particularly interesting, remembering that group C1 was not taught using pattern, but that the curriculum for C2 was likely to have included some pattern work.

	Basic	Pattern	Proof
E / C1	NS	0.025	0.0005
E / C2	NS	NS	0.0005
C1 / C2	NS	NS	NS

Table 2 : Levels of significance of differences between group means

Six weeks later, all the students sat a common end of year examination which represented the work completed throughout the year by all three groups, so two pattern and proof questions ('Pattern 1' and 'Pattern 2') were included. The relevant mean scores obtained by the students in the three teaching groups are shown in

	Basic	Pattern 1	Pattern 2
E (n=24)	67	86 (98, 55)	61 (78, 30)
C1 (n=22)	66	63 (88, 5)	52 (70, 15)
C2 (n=24)	60	62 (84, 10)	59 (80, 20)

Table 3 : Mean percentage scores on end of year examination

Table 3. Both overall performances and separate (pattern, proof) scores are shown here. The results from this end of year examination basically corroborate the findings of the post-test for the experiment. Space does not allow these two pattern questions to be shown here, but Pattern 1 was of a very familiar type to the students in group E, whereas Pattern 2 would have been a novelty to all students. Detailed scrutiny of where students were successful and unsuccessful on these two questions clearly indicates that, on the one hand, students in group E were very likely to be able to recognize and to some extent analyze patterns of a type similar to those they had met before but, on the other hand, they were no more able than groups C1 and C2 to recognize and analyze patterns which were unlike those met previously and which did not have an obvious visual structure. In other words, transfer was limited.

Performance on pattern and proof elements of the post-test

The mean percentage scores on the elements which were concerned with pattern in the four post-test tasks are shown in Table 4. All groups found the pattern elements of Task 1 easy. In Task 2 there was a significant difference ($p < 0.025$) between group E and each of groups C1 and C2. It seems that the experiences with pattern in C2 had not enhanced their performance on this question, which was perhaps surprising. It was clear from the responses of a number of students in C2 on this task that they were so intent on using differencing that they lost track of their objectives. Differencing was widely used in Task 3(c) where, for example, one student wrote, "If you find the differences of the differences, the differences of these seem to be 6", but was unable to continue. The results of this task were interesting in that group C2 performed better than group E (though the results are not statistically significant) and that group C1 performed very badly and produced a mean score which was significantly different from the other two. In Task 4 the means for groups E and C2 were identical, while that for group C1 was lower, and significantly different from the other two, again suggesting group C2 had benefited.

TASK 1

In the diagram showing the calendar for the month of May a group of four numbers in a square has been highlighted. Adding the diagonally opposite numbers gives $15 + 23 = 38$

and $16 + 22 = 38$

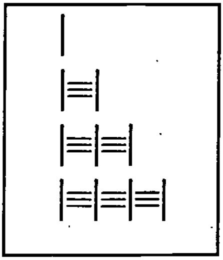
a) Repeat for other squares and describe what you notice about these totals.

b) Explain why this happens.

M A Y						
Su	**Mo**	**Tu**	**We**	**Th**	**Fr**	**Sa**
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31			

TASK 2

a) Complete the table to show the number of rails (r) needed for each number of posts (p) in a fence.



p	r
1	0
2	3
3	
4	
5	
6	
20	

b) Describe any patterns you see.

c) Explain how to get the number in the second column from the first.

d) Write down a formula for the number of rails (r) needed for p posts.

e) Explain why the formula always works.

TASK 3

{9, 10, 11, 12, 13} is a set of five consecutive integers.

a) How many multiples of i) 2, ii) 3, iii) 4, iv) 5 are there?

b) What is the maximum number of multiples of i) 2, ii) 3, iii) 4, iv) 5 in any set of five consecutive integers?

c) 2 x 3 x 4 = 24 is a product (multiplication) of three consecutive integers.

i) Work out the product of three other consecutive integers.

ii) Repeat until you see a pattern.

iii) What appears to be true every time?

iv) Explain why this will always happen.

TASK 4

a) Find $2^2 - 1^2$, $3^2 - 2^2$, $4^2 - 3^2$.

b) Continue up to $9^2 - 8^2$ and describe any patterns.

c) If n is the smaller of two consecutive numbers write down an expression for the larger.

d) Explain, using algebra and/or diagrams, why the result in b) will always happen for the difference between the squares of any two consecutive numbers.

Task	1	2	3	4
E	100	95	30	52
C1	95	80	9	34
C2	91	78	38	52

Table 4 : Mean percentage scores on pattern elements

Task	1	2	3	4
E	84	64	8	16
C1	61	28	5	5
C2	65	17	2	2

Table 5 : Mean percentage scores on proof elements

The mean percentage scores on the elements of the tasks which were concerned with proof are shown in Table 5. From this we see that the proof for Task 1 was successfully completed by at least 60 per cent of the students in the control groups, and by 84 per cent of the experimental group. Most of the correct proofs were written informally, for example, "The totals for the diagonals will always be the same because the top two numbers and the bottom two numbers have a difference of 1. The smaller number on top goes with the larger number on the bottom, and vice versa". Just one student in group E included the algebraic explanation, "$x + y = (x + 1) + (y - 1)$". Group E scored 64 per cent on Task 2, much higher than the other groups. Here, two examples of correct explanations were: "For every post you add three rails but you don't have three rails where there is no post at the end and so you take away three rails" (for $r = 3p - 3$), and, "Get the number of posts minus 1, then times the number by 3 because that is how many rails fit a post; the reason one is minused (*sic*) is because that is the first post which doesn't form any rails" (for $r = 3(p - 1)$). There were only three correct proofs for Task 2 from group C1, and only four from group C2, but there were some partially correct answers from these two groups. There were no complete proofs of Task 3 from any group, and only eight students (four in group E, two in group C1 and two in group C2) achieved partial explanations. All but one of these realized that a set of three consecutive numbers would contain at least one even number, making the product inevitably even. Algebraic representation was not used by any student. For Task 4, only two students from Group E drew diagrams which showed they understood the structure of the pattern. Seven other students (four in group E, two in group C1 and one in group C2) gave partial answers. Given the original hope that a programme of study on pattern and proof in this age and ability range would lead to increased performance on proof as well as pattern, the overall results can only be described as somewhat disappointing. Clearly, the limited success of this experiment indicates that the search for intervention strategies which will enhance the performance of students in the area of pattern and proof should continue.

Interviews

Eight students from group E were interviewed in order to pursue the idea of proof in a one-to-one environment in which the spoken word could be used to question and explain. All but one of the students had found Task 1 easy, but one important point did arise, namely the difficulty of explaining in writing, ("It's easier to talk"). Six of the students had completed Task 2 successfully. Task 3 was more unusual, and generated much more discussion. Six of the students had made little progress in the written test, and still required very considerable steering to appreciate the idea that multiples of 6 must feature in 3(c). Typical comments were, "I spent a long time on this; I didn't really understand the way it read", and, "I wasn't really sure what I was doing". It seems that these students were not able to cope with the language of what was in no way a 'standard' pattern task. Some of the students had not realized there was a connection between the first two parts of the question. The other two students had made good progress in the written test, but in one case only by spending so long on the question that no other questions were attempted. Nevertheless, both of them were very easily steered to a full understanding of the structure and all the ramifications of the task.

In Task 4, six of the students had correct numerical results in the test, and had noticed either that the differences were odd numbers or that they increased by 2, but only two students had recognized that they represented a sum of consecutive integers. None had drawn a diagram in the test, but five opted to use one in the interview, appreciated its structure for a particular case, and were readily moved on to an understanding of the general case. One student had not been able to begin to prove anything in the test, but had no difficulty in understanding a proof based on a diagram. Three students opted for an algebraic approach, but all of them at some stage seemed to lose the sense of what they were trying to do. Eventually, they established that the difference was $2n + 1$, and was therefore an odd number. For one student, the discussion became so laboured that the problem was left unfinished. Two students had an incomplete appreciation of the situation after some discussion based on an algebraic approach, but the subsequent introduction of a diagram clarified the situation. One said, "Just looking at the diagram it's easier to see. You can see it both ways, but it's easier looking at it with a diagram". All the students interviewed appreciated the nature of proof, and in most cases either eventually constructed a proof or understood the creation of a proof. Lack of appreciation of proof was not a major cause of the inability of group E students to construct proofs.

Some general conclusions

In evaluating the successes and failures of the teaching programme, it is fair to claim first that, although successes were limited, group E did have more success than groups C1 and C2. The awareness of the need for proof had certainly been raised in group E, where students appeared to be more prepared to look for proofs than they were when the programme started, and than students in groups C1 and C2, and this was particularly clear in the interviews. In fact, group E students were likely to want to try to provide a proof and, even if they could not complete one on their own, they were receptive to the teacher then 'piloting' them to a conclusive proof. There was certainly enough evidence that it was appropriate to use pattern as a motivator for the study of proof.

Just as with younger children (Orton, 1998), these Year 10 students were inclined to rely too much on differencing, which sometimes inhibited progress towards finding a generalization because it needed to be derived from the mathematical structure of the task. Thus, progress towards finding a proof could be prevented because the focus of analyzing the task was not directed towards the underlying structure. It should be recorded that on some occasions students were able to relate their sequence to a known sequence like the square numbers, and were able to proceed to a generalization on that basis. The issue of the place of diagrams arose on many occasions. There were some students who lost their way completely when they attempted to use an entirely algebraic approach, and there were many students who not only benefited from the availability of a diagram, but also commented on how helpful they found it. This raises other questions. Should students always be encouraged to refer to a diagram and attempt to analyze the structure of the pattern from that? Would a diagram be likely to help all students equally (Krutetskii, 1976)? The case for the support of diagrams under all circumstances is a strong one on the basis of our evidence.

Finally, it was clear that transfer was limited. Students were more likely to succeed with analyzing new patterns if they were of a type similar to ones encountered previously, and conversely were not likely to succeed with patterns arising from completely new contexts. The extent to which transfer of learning is likely to take place has been the subject of debate throughout the twentieth century (see for example Cormier and Hagman, 1987). What our evidence suggests is that, even within the fairly limited area of mathematics under consideration here, little transfer of competence in analyzing patterns and providing proofs developed within the one year available for the experiment.

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WHAT MAKES A MATHEMATICAL PERFORMANCE NOTEWORTHY IN INFORMAL TEACHER ASSESSMENT?

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Abstract. *Some brief accounts of classroom incidents involving exceptional mathematical performance are analysed. Similarity of some of the incidents allows comparison of teachers' responses to the performances. It is found that the influence single incidents can have on a teacher's view of a pupil's mathematics might be outweighed by the teacher's existing view, or might itself outweigh future incidents. The skills of the pupil in attracting the attention of the teacher, and the teacher's view of mathematics, are found by counter-example to contribute only partially to such judgements.*

Background and rationale

Teachers' judgements about their pupils' mathematics affect future learning in a variety of ways.

Firstly, pupils' self-images as mathematicians may be partly formed by perceiving how the teacher sees them, this image being projected as a measure of psychological support through verbal and non-verbal signals [Eisenberg,1991]. Pupils may attempt to portray themselves as mathematicians by getting a sense of what is acceptable in the classroom and acting in ways which they expect to be interpreted positively [Argyle, 1969; Goffman,1959]. The classroom therefore becomes an arena for the giving and interpreting of social signs, according to the expectations of the teacher and pupil participants. The teacher's view of mathematics and what it means to be a strong mathematician dominates, so success in mathematics can be influenced by the teacher's expectations [Nash,1976]. The teacher's style, reflecting her beliefs about mathematics, can, through classroom rules and rituals, affect children's beliefs about mathematics and hence how they respond to the teaching [Brown,1995].

Secondly, during one-to-one interactions, teachers make judgements about how to respond to individuals: whom to support and whom to challenge, whom to tell what to do and whom to encourage to take risks and find their own ways to do things, whom to praise and of whom to demand more [Nash,op cit.]. These responses would be influenced by the mathematical sense the teacher makes of what is offered to her by the child, either orally or in writing. As Morgan [1996] says:

what is being assessed: is it the student's mathematical understanding or competence or is it his or her competence in creating a text that will be judged to be appropriate within the genre anticipated by the teacher? (p. 24)

Thirdly, it is common in UK for pupils to be split into groups studying different mathematics, or for pupils to be working on an individual pathway unrelated to what others in the room are doing.

Decisions about suitable curricula are based on teachers' judgements. It is also usual to split secondary pupils into separate tracks for different educational treatment [Boaler,1997], this splitting sometimes being based on test results and sometimes based, partly or wholly, on teachers' judgements.

Fourthly, in UK teachers' assessment decisions are now a formal part of high stakes assessments leading to further education, employment, and other social opportunities.

For these four reasons the pupils' mathematical futures therefore depend heavily on teachers' judgements about their possible future success in mathematics. It is therefore important to know how judgements are formed. It has been shown experimentally that interpersonal judgements are strongly influenced by first impressions, to the extent that subsequent evidence which contradicts these impressions may be ignored, and the incident leading to the first impression may become exaggerated in the perceiver's memory [Nisbett and Ross,1980]. Watson [1997] shows how, in classroom-based mathematics assessment, the accumulation of evidence of achievement is subject to interpretation by the teacher, and this interpretation is influenced by early or outstanding impressions. Teacher assessment might therefore be overly influenced by some incidents, while other incidents might be ignored. Jones and Thrumpston [1994] report that teachers on an assessment course 'had been led to early assumptions about a child from a single response' but in a devised situation involving group discussion about their judgements were 'enabled to revise assumptions when further evidence (was) available'.

Bright et al [1997] offer frameworks for understanding children's mathematical thinking, emphasising that incidents have to be interpreted for their mathematical importance to be recognised. However, the teachers observed were, like those of Jones and Thrumpston [op cit.] engaged in a project and consciously working on these issues. In normal school life teachers do not meet for the sole purpose of discussing their judgements.

Teachers and data

In this paper, the teachers referenced were in their normal settings, aware that they were the focus for research on assessment but were not aware of how the detail was to be used. They will have been more aware of assessment issues than usual because they were being observed or questioned about them. They had not deviated far from their normal practices of informal assessment which depended on the accumulation and development of impressions, complemented in general by accumulation of written artefacts or test scores. These practices are typical for UK teachers and have to be seen in the light of recent national training in assessment techniques within a national framework which includes mathematical thinking and processes among its descriptors.

The incidents used in this paper are all selected from a larger body of data consisting of pre-interview observation notes, thirty unstructured interviews about informal assessment practices (during which many teachers gave illustrative stories) and longer term observation of ten pupils, aged 10 to 12, in two teachers' classrooms. Teachers were from upper primary and lower secondary classes, but the illustrative stories came from a wider range of their experience. The purpose of the selection and discussion of stories is to identify, describe and contrast some classroom incidents during which, in some way, exceptional mathematical performance is displayed which the teacher notices and talks about. Features of these incidents are compared to each other and related to outcomes in terms of teachers' judgements, in order to indicate answers to the questions:

> What factors influence the teacher to accept a single mathematical performance as noteworthy evidence to add to her picture of the student's general achievement, and what lead her to dismiss the performance as adding nothing of worth to the picture?

Performance is taken to mean oral, written or physical communication which indicates the results of some mathematical thinking, such as giving answers, making suggestions or indicating something with a diagram. *Exceptional* means different to some norm, either for the class, or for one pupil, or relative to the teacher's expectations. *Noteworthy* means that it was recorded in some way, either mentally or in a formal way, and hence might contribute to the teacher's view of the pupil.

Incident 1

Teacher 1 had taught 11 year-old Pupil 1 for one lesson which had consisted of a whole-class discussion during which the pupil had said nothing. In the next lesson pupils were finding how many shapes could be made from five squares joined at edges. Duplicates which were rotations of each other were supposed to be eliminated, but reflections were not. Pupil 1 correctly retained two pieces which were reflections of each other while the majority of the class wanted to eliminate them. This was described by the teacher as evidence that she was, compared to the rest of the class, a strong mathematician, especially as others who made the same decision subsequently demonstrated other strengths in mathematics. The pupil showed no further exceptional performance for the whole term but the teacher persisted in expecting it, still referring to her as 'strong' in conversations with the researcher who pointed out the lack of evidence. This continued until the end of term when she was unable to express a spatial concept in algebra, or understand the expression given to her, unlike many others in the class. He then changed his mind about her.

Features of this incident include:

- the teacher was trying to get to know the whole class;
- an early mathematical performance was very good compared to the rest of the class, and compared to other 'strong' mathematicians;
- the teacher clung to his early impression in spite of lack of further evidence;
- he changed his opinion eventually after she showed unusually weak performance compared to others.

Incident 2

10-year-old Pupil 2, thought of as 'average', had found a question about flowcharts which used the words 'change' and 'inverse change'. She asked what 'inverse' meant and the Teacher 2 suggested she should look it up in a mathematical dictionary, which said words to the effect that 'if you turn something upside down or back to front in maths you have its inverse[1]. The inverse of 2/3 is 3/2.' The ordinary dictionary said 'reversed or opposite'. From these definitions the pupil managed to

[1]This is the definition as reported by the teacher, who believed it to be misleading.

construct a meaning for 'inverse' which allowed her to work correctly reversing several flowcharts. From this incident the teacher formed the view that the pupil was a strong mathematician because she could think logically and had strategies to use when she did not understand something.

Features of this incident include:

- student performs in a way which is recognised and valued by the teacher;
- this single incident changes the teacher's opinion of the student, or forms a first impression of a student who had otherwise not made an impression;

Incident 3

Pupil 3 was 8 years old and doing a shopping activity in her classroom. Purchases were to be made, change was to be given, and transactions had to be recorded appropriately. Teacher 3 had not given any guidelines about methods of recording, and was surprised when Pupil 3, who was calculating the cost of 'two lots of beans' each costing 12p, used the multiplication sign in her written work without prompting. Subsequent discussion revealed that Teacher 3 had a very high regard for pupils who could use formally taught mathematics in other situations in school, saying:

.... if they already know something they can demonstrate it by answering the question, or extrapolating, or solving the problems, or thinking laterally ....I had taught them at different levels and they could obviously do it in the lesson, but I knew this one girl REALLY understood it, even though her numeracy wasn't that brilliant, because later on when she was doing some shopping exercises she actually used the multiplication sign in her work.

The teacher persists in her view that the student was 'not especially numerate'.

Features of this incident include:

- a pre-existing view of the student;
- a particular view of maths which valued the kind of performance shown by this student;
- comparison with the student's own previous performance;
- the teacher was reluctant to change her opinion, or include this performance in her ideas about numeracy, in spite of the difficulties of transferring formal mathematics to a practical situation [Lave,1988] and the voluntary introduction of symbols, shown by Hughes [1986] to be a difficult hurdle from informal to school mathematics.

Incident 4

Pupil 4, aged 8, had been asked to produce any calculation he liked which gave the answer 20. Teacher 4 had been expecting simple sums such as 10+10 but he was using much larger numbers, such as 350-320. Other pupils produced many more calculations, such as simple number bonds systematically produced, but his were fewer, less systematic, and more unusual. The teacher commented:

> *he doesn't talk much but there's obviously something going on down there.*

Features of this incident include:

- a new student treats an open-ended task in unexpected manner, different to others, and not expected of him;
- the performance matches something in the teacher's view of mathematics;
- the teacher's expectations may have been altered by the incident.

Incident 5

Teacher 5 had started a lesson by writing 'b x b = b' on the chalkboard for her 10 year-olds to discuss. She thought the discussion would cover a range of mathematics: the meaning of the symbols, square numbers, possible ways to explore the equation and so on. Pupil 5 called out immediately that the answer had to be one. The teacher asked why and he replied that it is the only number which 'timesed by itself gives itself'.

In discussion the teacher said that the pupil had a range of learning difficulties, had his own personal learning support assistant and was weak in mathematics, an opinion which had not changed as a result of this incident, which was described as a 'fluke'.

Features of this incident include:

- the student's performance was strong compared to the rest of the group and the age-group in general;
- the teacher keeps to her original opinion of weakness in spite of a mathematical performance which she did not expect immediately of anyone in the group;
- the choice of problem suggests that the teacher should have valued the student's answer;

Incident 6

Pupil 6, aged 14, was unable to read or write adequately and hence very weak in most school subjects including mathematics, which had been taught using textbooks at his previous school. During an early lesson in a new school there was a discussion about how many ways there would be to get from the origin to particular points on a coordinate grid if one was only allowed to move vertically and horizontally away from the origin. All pupils were looking at a diagram on the whiteboard for the discussion. Pupil 6 was, as usual, saying nothing until the teacher asked how many ways there would be to get to the point (5,6). No one else replied. Eventually Pupil 6 said that it would be the number it took to get to (5,5) and the number it took to get to (4,6) added together. The teacher realised that the pupil had a very highly-developed sense of spatial relationships, and was possibly a much stronger mathematician than had been thought while he was dependent on textbooks to learn.

Features of this incident include:

- the teacher did not expect this performance from this student, and it was unusual for the class;
- the performance was valued by the teacher;
- the teacher's opinion and expectations changed as a result of one incident.

Incident 7

Pupil 7 was not in the 'top group' for mathematics. Her SATs scores were nearly two levels higher than the teacher had estimated. Subsequent conversation with the pupil revealed that, according to the teacher:

> *she had been very good at picking up linguistic cues in the tests*

It was assumed that her language ability had helped her to score highly, higher than many in the top group. There was no intention to move her into the stronger group.

Features of this incident include:

- a pre-existing view of the student, expressed in a level estimate;
- the student performs well for her age-group according to external expectations, and better than her own previous performance;
- the teacher's view is not changed;
- the performance is credited to the form of assessment and non-mathematical features, thus showing a teacher's view of mathematics at odds with the testing procedure.

Analysis

In cases 1 and 2 the teacher had no clear pre-existing view of each student's mathematical capabilities. Single incidents when the student displayed strengths relative to the whole class (case 1) or the teacher's expectations (case 2) led to the formation of views that the students might be strong mathematicians.

In the other 5 cases pre-existing views about some weaknesses existed and the students' performances were strong compared to their own previous performance as well as compared to the class as a whole. In case 3, in spite of the performance fitting the teacher's values and displaying strengths which are supported by research, the teacher's mind was not changed by the single incident. In case 4 there was an indication that expectations had changed slightly as a result of the incident: 'there's obviously something going on down there'. In cases 3 and 4 the perceived weaknesses were within an average range for the age-group.

In cases 5 and 6 the students were seen as having significant academic weaknesses outside a normal range for the age-group. The particular performances discussed were exceptionally strong for the individual and the group. Nevertheless only one of the incidents seemed to be incorporated into a changed view of the student.

Case 7 illustrates a pre-existing view of a student which had been institutionalised by setting. The student's performance was exceptional for herself and for the age-group, but the 'incident' of a national test was not incorporated into the teacher's view of the student's mathematical capabilities. In fact, other attributes were given as the cause of the success.

Cases 1 and 2 illustrate the power of first impressions. In case 1 it is interesting to note that, as predicted by Nisbett and Ross [op cit.], the teacher refuses to change his mind in the light of contradictory evidence until such evidence became extreme. Cases 3 and 4 show two comparable situations with contrasting teacher responses. Cases 5 and 6 show incidents which strongly contradicted established views yet still produce contrasting responses. Case 7 shows dismissal not just of a noteworthy performance but of a whole incident. In all cases, apart from 7, the performances were of mathematical actions which were valued by the teachers.

Discussion

The incidents above illustrated the social skills of pupils able to draw the teacher's attention to their thinking and were *a priori* exceptional when compared to the performance of the rest of the class, the teacher's expectations for the class or the teacher's prior impressions of the pupil. Whether the teacher recognises or dismisses a single exceptional incident may depend on the teacher's own view of mathematics, priorities within that view and the teacher's previous opinion of the pupil. However case 5, and possibly 3, above show that performance of mathematical thinking valued by the teacher is not noteworthy in itself. Some teachers may be willing to change their minds as a result of one incident, while others who seem not to have been influenced immediately may accumulate information over time and their views change gradually as a result. The teachers in cases 5 and 7 showed no inclination to change their views as a result of exceptional performance, even as part of an accumulative process; the teacher in case 1 made up his mind as a result of one incident but would not change it for subsequent, less startling, incidents. The contrasting responses above show that although gaining the teacher's attention, fitting in with her view of mathematics and being exceptional are important, further factors also contribute to noteworthiness. The teacher's own self-esteem may depend on preserving her confidence in making judgements; her professional functioning (such as discriminatory curriculum practices) might depend on making hasty judgements and ignoring contrary evidence. She may therefore need a willingness to change, a willingness to admit that previous judgements may be wrong or the vision to handle the pedagogic consequences of flexibility.

Conclusion

The teachers in the incidents above are forming personal judgements, basing their expectations upon these, and not necessarily changing their views when different evidence is apparent. Potentially good mathematicians may not be recognised. Conversely it is possible that pupils needing more support are being overlooked because their social skills, or occasional successes, convince the teacher they are stronger than they are. Social skills, exceptional performance and adherence to the teacher's mathematical values are seen to be important but not the only factors deciding noteworthiness of a pupil's performance. Other factors, requiring further research, are also influencing the judgement process.

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PARTICIPATING IN LEARNING MATHEMATICS THROUGH SHARED LOCAL PRACTICES

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In this paper we develop a theoretical perspective from which to identify and describe local communities of practice which we suggest are useful in thinking about mathematics teaching and learning. We exemplify this perspective with descriptions of individual mathematics lessons and consider briefly the implications of its application to wider notions of schooling.

Introduction

Theories of situated cognition provide us with tools for analysing apprenticeship models of learning (Lave 1988, 1993, 1996, Lave and Wenger, 1991). What is more they suggest that learning only takes place within communities of practice. Seeing schools and classrooms as learning communities has encouraged some writers to explore the application of the apprenticeship model to school learning (Adler 1996, 1997, Lerman 1997). Learning, for Lave, has no necessary connection with deliberate teaching; the learning to which she refers is directly related to the practices of the community whereas the knowledge taught in school bears little relation to the practices and, indeed, the actual function of the school as a community.

There have also been useful applications and adaptations of elements of this theoretical perspective. Boaler for example (1997, 1998) uses Lave's ideas about learning transfer to develop a convincing critique of traditional, transmission models of mathematics teaching and learning and assessment in particular. Theories of situated cognition have also been deeply influential on the ideas of other key figures engaged in mathematics education research; Noss and Hoyles' idea of 'situated abstraction' (1996), for example, could be understood in this context.

We will return briefly in the conclusion to the important task of somehow coming to see schooling in general in terms of practice. In this paper we want to focus more narrowly on what will be called local shared practices, or local communities of practice. So, whilst we will put off the solution of the larger problem of whether theories of situated cognition apply to schooling, we will nevertheless attempt to make some useful statements about teaching and learning.

Community of practice

Terms like 'communities of practice' seem very much a part of current discourse and, perhaps for this reason, and in spite of Lave's writing, their meanings may not always be clear. For this reason, too, we want to make clear what we understand by the term. A community of practice, in the sense in which we use it here, must have certain necessary features[1] :

1. participants create/find their identity within the practice;
2. there has to be some social structure which allows participants to be positioned on an apprentice/master[2] scale;
3. the community has a purpose;
4. there are shared ways of behaving, language, habits, values, and tool-use;
5. the practice is constituted by the participants;
6. all participants see themselves as essentially engaged in the same activity.

The first four features could be interpreted to be true of schools; it is less obvious that the last two could be seen in (the formal instruction side of) schools. In most mathematics lessons the teacher is not engaged in learning mathematics, although both teacher and pupils could be, and sometimes are, engaged in doing mathematics. Whilst we might say that pupils, as well as teachers, together constitute the practice within all classrooms, pupils' participation is often passive, and therefore if they can be said to constitute the practice this would only be through their acquiescence.

Local (communities of) practice.

We believe that a way forward might be to describe schools in terms of multiple intersections of practices and trajectories. Within schools, however, we believe that it is possible to talk sensibly of local communities of practice (Lave 1993). Such communities may be local in terms of time as well as space; they might 'appear' in a classroom only for a lesson and much time might elapse before they are reconstituted (although it may be possible to detect the subtle effects of the echo that remains after their passing in the trace of learners' trajectories or the development of other practices.) Apart from these spatial and temporal constraints, local communities of practice (LCP) display all the elements we have listed.

We find the construct of local community of practice to be both useful and usable: it is possible to identify LCPs through observation; setting out to initiate the creation of LCPs would be, we suggest, a support to planning for the effective mathematics learning of students. The model should be accessible to colleagues, both fellow teachers and beginning teachers, and so provide a focused mechanism for the study of pedagogy. We could, for example, ask our students how they proposed to initiate the construction of local communities of practice within their lessons.

We will now give one example of local community of (mathematical) practice and a second example where we detect no such local practice.

[1]Listed in this order for convenience of reference only.

[2]We use these terms because we want to emphasise the social nature of such judgements. The expert/novice distinction has the attraction of gender neutrality, but it suggests a kind of cognitive-psychological activity in which we are not engaged here.

Example 1: Exploring the graphical calculator

Each member of a class of 13 year-old girls was given a graphical calculator to use as her own. These calculators (in 1996) were very powerful[3], but the feature that was central to the establishment of the practice described here was that any machine could be connected to an overhead projector display screen.

For their first activity using the calculators the students were asked to work individually and in groups to explore the machines. No explicit mathematical agenda was set, but students were asked to respond to these questions:

- How is the calculator similar to other calculators you have used?
- How is it different?
- What is there, if anything, which surprises you?

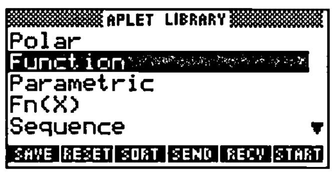
Students did their initial exploration by themselves at home. The next day they presented their personal responses, observations and ideas to the others in their group and together the group planned a joint presentation to be made to the whole class in the following lesson. In preparatory discussion with the teachers (the researcher acted as a teacher) the focus was on the skills needed to be able to work effectively as a group, in particular skills such as listening, helping, asking useful or helpful questions.

Sara's group decided that they would include her report on her exploration of polar co-ordinates.

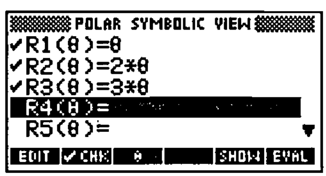
Sara refers to her mathematical activity using the situated symbolism provided by the calculator; she does not otherwise label her mathematical activity as such. This may be an important aspect both of calculator use and of setting up local practices, but we do not go into that in detail here. Here is part of Sara's presentation to the whole class:

Sara starts by saying she has 'just messed around'.
She turns on her machine and displays something like this:

It represents a set of possible domains in which she might explore.



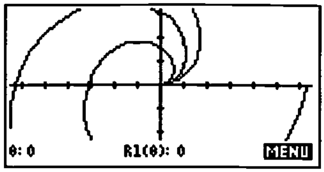
Sara selects *Polar* and so displays a set of polar functions she has already defined:



[3]Hewlett Packard HP38G

She says:

After you put in any, you know, you want...then you just press PLOT...and, I just love this one...it comes out so lovely...



Class: Ooh, wow...

someone in the class says: My one doesn't work.
Sara: Your scales aren't the same as mine...must be the axes...I've actually zoomed out.
Teacher: How could Sara show us what the axes are?
Sara: All I did on mine was just zoom out.

Through a combination of the setting and the technology Sara, the others in her group and the rest of the class together validated the investigation of polar equations as an appropriate mathematical activity - albeit one that was outside the curriculum. Similar willingness to explore and explain was demonstrated by all groups and is taken to be a defining characteristic of this local community of practice. It is possible that Sara's central role in establishing it may prove to have been pivotal in her developing identity both within and beyond the classroom. She had always been seen as 'good at maths' - she was now seen to be becoming a master of this calculator-mediated maths.

Example 2: Four 4s

A class of 11 year-old girls and boys enter the classroom and settle themselves into their seats. The children have just begun their studies at their new secondary school.

The teacher gives the children an example to introduce the task he wants them to do. He explains that you can use four 4s to make the number 9:

$$4 + 4 + 4 + 4$$

He responds during his explanation to questions that some children ask.

Linda: Why isn't that the same as $4 + 12$?

Teacher: *(words to the effect that)* You do the dividing first to get the same as

$$1 + 4 + 4$$

A few more examples follow and the children then work on these by themselves.

The teacher sets the children their homework. He asks them to show how to construct as many as they can (he does not say 'all') of the numbers from 1 to 25 using exactly four 4s. The children leave.

....

At home, Evelyn has completed as much of the homework as she can. She can't yet do 21 and 25 and she wants to do them. She telephones her uncle. They talk for a long time on the 'phone. He helps here with ideas of indices and 4^0, but suggests to Evelyn that, perhaps, she should only make use of these ideas if she can explain them back to him. Evelyn explains convincingly.

Evelyn completes the homework and hands it in.

....

About two weeks later, Evelyn and her uncle are talking over a meal. He asks how the homework went. Evelyn explains that the teacher took the books in but didn't get a chance to mark them yet.

About three weeks later, Evelyn's book is collected again. The work is marked and returned. She reads that she has done well. The class has moved on to another topic by now.

From the point of view of the teacher, this second example might appear to be two lessons and a homework. He might well have thought (though not in these terms) that the children's responses came out of the practice that had been established in his classroom. He probably did not see the children's work as a product of the multiplicity of practices whose intersection only he observed in his classroom. Some of his children were participants in practices which included not only the telephoning of mathematically-inclined uncles, but also the expectation that this was an appropriate and natural thing to do. To be sure the success of all lessons must depend to some extent on such multiple participation. We think that this second example is typical of many lessons whose success effectively depends wholly on what children bring with them: no local shared practice had been initiated.

Discussion

As we have said, we think that there is a distinction between lessons we choose to call local communities of practice and others where we make no such identification. It is helpful to think of any classroom as an intersection of a multiplicity of practices and trajectories. In our first example we want to say that, from this rich layering of practices and becomings, local practices emerged which were defined by and required the active participation of those who together constituted those practices; within such practices there is, by definition, a strong social pull on all - including the more peripheral - to participate. In the second example, whilst there may have been pockets of active participation, we want to say that the learning which happened was much more a product of the complex identities in practice that teachers and learners brought with them when they stepped into the classroom than anything that happened to them once they were there.

Local communities of practice can therefore be said to be at least an indicator of effective teaching. For consider the learning that takes place outside of such a practice; this will owe much (everything?) to those larger practices which together constitute the learner's identity - as scholar, perhaps, as a good student of average ability, as classroom clown, as truant; we suspect, too, that even the best teachers in effect rely on their students' unarticulated, unrecognised participation in those other practices for whatever success they might achieve.

Telos

The local communities of practice we have described support a common direction of learning which, from the perspective described here, is a defining feature of those communities. Lave and Packer (Lave 1996) have found it useful to emphasise such direction - telos - as an essential stipulation of any theory of learning. In our reading we understand telos to refer to the way that an individual becomes what they are going to be within a community of practice. The learning they do is both a determinant of this direction and in part determined by the complex paths which students have taken to get where they are. Thus people can appear, superficially, to be learning the same thing but the knowledge they gain, and the effect it has on them, can be very different. For instance, several pupils can learn that $3/16 + 13/16 = 1$: for some this may be obvious and uninteresting; for others it may be an important realisation; for others again it may provide a moment when they begin to feel powerful mathematically, and begin to see themselves as valuable participants in the lesson, classroom, or community of practice.

In a broad sense telos is an unfulfilled potential to move or change in many different ways; telos could be conceptualised as a set of constraints in some sense inherent in situations and in the individual's predispositions to respond to situations as she does. Whilst at school learners are also preparing to become the socially functioning beings they are going to be as well as productive adult members of society; their experiences at school are also mediated by the images of themselves that learners bring with them. Lave's contribution here is distinctive in that it looks at the learner's experience, rather than the teacher's view, school organisation or curriculum.

From the point of view of the learner in a mathematics classroom, for three times a week, 38 weeks a year, it is seldom clear how one's experiences affect the process of becoming the person one is going to be; possible exceptions are those few who see themselves as joining a community of mathematicians. However, if we accept a notion of local practice, we can identify smaller-scale "becomings" in which many more learners do participate. As we saw in example 1 above, many learners can become originators of mathematical questions by participating in the practice of asking questions; they can become masters in the use of certain tools (the calculator); they can become masters in operating within a particular set of social constraints. By constraining the foci for attention, and by recognising and working with predispositions, rather than ignoring them, a teacher is more likely to be able to initiate local practices which enable learners to see themselves as members of a mathematical community.

To develop the idea of telos mathematically, in a local practice the telos of individual students could be, for a short while, similarly aligned[4] just as individual functions may share local approximations. We believe that this alignment of factors is made more likely if the lesson is planned to encourage the development of a local community of mathematical practice.

Conclusion

We can now summarise the features that we believe to be necessary in a classroom if those within it are to constitute a local community of practice:

1. pupils see themselves as functioning mathematically within the lesson;
2. within the lesson there is public recognition of competence;
3. learners see themselves as working together towards the achievement of a common understanding;
4. there are shared ways of behaving, language, habits, values, and tool-use;
5. the shape of the lesson is dependent upon the active participation of the students;
6. learners and teachers see themselves as engaged in the same activity.

Our discussion of local communities of practice has practical objectives which include, importantly, providing an aid to beginning and practising teachers as they work to improve the mathematical experiences of their students. However, we think that there are broader issues here which the notion of LCP helps us to raise.

It would clearly be absurd to claim that only mathematics lessons which have seen the initiation of local communities of practice can be productive. We suspect that most successful learners actually experience few such lessons. From our perspective we take this to mean that the success of individual learners will be associated with their positioning within communities of practice not as yet described. The initiation of LCPs for which we argue in this paper, represents a small practical step suggested by our theoretical perspective; the next, much larger

[4]The alignment we have in mind resembles that to which the Cognition and Technology Group at Vanderbilt University (CGTV) refer (1996). They suggest (1996) that in order for children's competencies to reveal themselves a number of elements have to be properly aligned. For CGTV the computer can be seen as an element of a physical and social context which affords or enables "early" competencies" in young children's number. This provides a link between our notion of LCP and the situated abstraction of Noss and Hoyles (1996). Just as they claim the computer provides domains which support students' abstraction, so we claim LCPs support students' growing image of themselves as someone who is legitimately engaged in mathematical practice, as someone, in other words, who is becoming a mathematician.

step might be to map the complex processes by which some students (far too few) come to value and experience participation in those practices most valued by schools and society, and most do not.

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INTUITIVE COUNTING STRATEGIES OF 5-6 YEAR OLD CHILDREN WITHIN A TRANSFORMATIONAL ARITHMETIC FRAMEWORK.

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Abstract
This research describes a 14 week study of a small group of 5 and 6 year old children inventing their own symbols for mathematical operations and using them to ask and answer questions about numbers in a 'tranformationally rich' subculture. Within an initial 'stepping-stone' situation, the symbols referred to children's actions rather than objects, but subsequently through a series of graded group activities, their overt stepping actions were replaced by covert counting actions. The aim was to develop a pedagogy through which the children's intuitive transformational arithmetic could prepare them for a firm and confident understanding of the conventional system of arithmetic.

Introduction and rationale

The following broad presuppositions motivated the investigation:-

i) All children begin school with a significant and valid core of intuitive notions about number, derived from their pre-school experience. They also possess a latent mental number ability which is considerably undervalued in schools. (Durkin, 1993; Klein & Starkey, 1987)

ii) From informal observation of African children learning in the absence of structured educational materials, it seemed that 5 and 6 year old children's views about numbers might have a coherence and validity which is ignored in the traditional school environment. Sinclair & Sinclair (1986) argue, from a Piagetian standpoint that children construct concepts on their own, without receiving formal instruction, and so build up a numerical world of their own - one which is often incompatible with that presented in a formal educational setting. This seems to be the case in children's well documented difficulty in understanding the commutativity of the additive operation.

iii) In view of the undoubted ability of young children to reason rationally and our belief that the foundation of mathematical knowledge rests on the co-ordination of the action schemes of the young child, it seems almost paradoxical that children should find mathematics difficult. Piaget and others have argued that a powerful reason for this difficulty is the premature introduction of written mathematical symbols.

iv) Children's understanding of *conventional* mathematical symbols is greatly overestimated - but children can invent and use their *own* symbols with great facility. (Gardner & Wolf, 1983; Hughes, 1986). Hughes and others have shown that many if not most of the difficulties children face in arithmetic lie not with algorithmic ...res, but in understanding the precise nature of the relationship between the operation symbols of mathematics and the reality which they represent. Young children (and some up to 9 years old), are exceedingly reluctant to use the conventional +/- symbols to represent existing relationship between numbers; this is commonly referred to as a *transformational view* of number operations. However, Atkinson (1992) and others in the 'emergent mathematics' movement have shown that young children are capable of inventing and using signs to stand for mathematical procedures which they carry out. Samples of writing by the 5 and 6 year old children in the present study also reveal a sophisticated grasp of one of the purposes of writing - that of conveying ('telling') information to another person.

v) Sinclair (1990) has argued that the almost exclusive focus in schools on numbers as *cardinalities* ignores children's extensive and sound everyday knowledge of numbers as *ordinals*, 'concretized' in the familiar recitation of the number sequence 'one, two, three, ...'.

Specific aims of the study

i) To familiarize children with the mental techniques of *counting-on*, *counting-back* (counting-on in the reverse direction), and *counting up* (determining how many numbers lie between two given numbers). These three mental strategies were found to have been extensively used in a mini-study carried out with 70 of the writer's teacher-trainees (Womack, 1997a).

ii) To enable children to compute mentally with numbers (up to 100) with a high degree of proficiency.

iii) To bring children to an understanding of the purpose and nature of arithmetic signs - particularly the role of the operator signs in algebraic formulations of 'missing number' expressions.

In view of the above considerations, it was proposed to create 'transformational' scenarios in which children would be encouraged to give each other instructions to carry out actions. *These actions were designed to have a natural correspondence with the internal actions associated with counting strategies*. Children would be given opportunities to invent and use their own symbols for these actions. It was hypothesized that these conditions might alleviate many if not all of the above constraints on children's use and understanding of arithmetic symbols.

Approximately 60 separate activities were carried out by the children in addition to various drawing and story-reading activities, of which only a small illustrative sample can be given here.

Experimental methodology and results

If mental counting actions were to be based on 'external' physical actions (and ultimately both represented by the same signs), then a natural model for both would need to involve a sequence of some kind. Therefore the initial activity chosen to familiarize children with the properties of the number sequence, was a series of stepping stones, unevenly distributed across a stream - a familiar scenario to the rural Lake District children. Such a 'number trail' introduced no extraneous 'culturally pregnant' concepts such as rectilinearity or equidistance, and engendered only the idea of *succession* from one known number to the next.

There were several stages of abstraction: moving from the stepping activity to a mental counting action; and within each of these stages, the medium of presentation was varied from oral to written words, and from children's hand-written signs to researcher's signs on cards. Questions were asked either by the researcher or by the children to one another and took the form of instructions to do something or questions such as, 'what would happen if ...?'

We highlight six typical scenarios:-

(1) Initially, the stones were unnumbered and children were asked to 'walk or step on' or 'step back' a given number of paces along the stones.

(2) The stones were numbered and the activities moved indoors to a table-top. Children could move coins or a doll from number to number; an action intended to further the identification of the stepping action as *a movement from one number to another*.

(3) The numbers (1 to 20) were written in a 'vertical' arrangement on a narrow roll of paper, which was held up from the table to give practical meaning to the words 'up' and 'down', which would be used in future sessions.

(4) The numbers were written in a conventional left to right format on a strip of paper which was raised slightly at the higher number end.

(5) A conventional 1 to 100 number roll was used (on which number positions were marked relatively unobtrusively). At this stage, movements up and down the roll were synonymous with counting-on and counting-back strategies (See examples below).

(6) A 'missing number board' was constructed to allow cards representing each of the 3 terms in a transformational number sentence to be displayed. When one of these was turned over, it presented a number sentence problem of the form 'What is on the back of the card?' (see Session 12).

The signs created by the children

The linguistic function of the signs used, whether words or symbols (oral or written), was an *instruction* to carry out an action or determine a quantity; signs were rarely if ever used as *descriptions* of a state or existing relationship. This was in line with a 'transformational' philosophy which determined the character of the various activities. The signs used, instructed children to answer a question from a given position on the 'number trail'; either 'Where will you reach if you move up/down so many steps?', or the more difficult and later type of question, 'How many steps must ...ke to reach ... (a given number)?' Since the number sequence has direction, the children used the expressions 'up 1', 'down 2' etc. to indicate movement along it. These were initially written as *words* by the children but eventually, it was decided (by the whole group) that *symbols* would be quicker to write (and read). The idea of signs was introduced in the following way (tape transcription):-

Session 3 activity: Inventing abbreviated 'up/down' signs (arrows)
Researcher sat among the group of children (P, Pk, J, G & T):
Res: *Who can think of some signs we can use instead of writing these words which take you so long?*
P: *Use arrows ...... arrows.*
Res. to P: *Arrows, alright. You try. Show us what you would do; how you would do 'up 3'. P's going to show us.* (P produces first card - three small vertical arrows.)
J: *One arrow then put the 3.*
Res. to Pk: *OK He's got 3 arrows. What do you think Pk? How would you show it? You show us on your card. ... 'up 3' with an arrow.* (Pk draws on a card - three arrows and the symbol 3.)

Cards with arrow signs in the vertical plane ('up' and 'down') were made by both children and researcher and these were used in various game situations at different times. These were distinguishable from other printed green cards which indicated only numbers along the sequence. The arrow cards therefore indicated the number of steps between any two numbers.
The following activity was concerned with *interpreting* the arrow sign cards.

Session 4 Activity (i) Checking meaning of the arrow signs
Res: *This time I'll show you a green card to say what number to start at.* (One green card is distributed to each child.) *Remember your up/ down cards which tell you how many to count-on.* All the up/down cards (1 to 5) are checked with each of the children. All were remembered from the previous week and all children could read them accurately.
Activity (ii) Carrying out the instructions of arrow signs
Res: *So if I was to give that card to you. So if you started at 10 and did that* (up-pointing arrow 2), *what number would you get to?*
J: *Twelve.*
Comment: The cards presented to the children in turn were: 14, arrow-down 2; 7, arrow-up 1; 5, arrow-up 2; 16, arrow-up 5; 10, arrow-up 3.
It was found that all children could carry out the exercise successfully for the above numbers. (Child T showed some uncertainty - or preferred to count-on in an almost inaudible whisper).

Fuson has drawn attention to the different meanings for the +, - and = signs, presented by different problem situations and notes that these are rarely provided in textbooks (Fuson, 1992, Womack, 1995) In this 'stepping-stone' scenario, children needed to use different counting strategies in order to answer the two types of question:- *Where ill you reach ?* (answer, an ordinal), and *How many steps?* (answer, a cardinal).

These questions cut across the traditional categories of 'addition' and 'subtraction' problems and necessitate appropriately different signs. For example, to answer the second type of question involves counting the numbers of steps taken or needing to be taken. This involves a *counting* strategy, whereas count-on and count-back require a *counting-out* strategy. An example of answering this type of question is given below:-

Session 5: Introducing 'Puzzle-questions' (finding the transform).
The aim was twofold: to assess children's ability *to guess which up/down arrow card has been taken,* from an observation of the movements of a coin along a sequence of numbers written on a strip of paper; also to check whether children could write this transformation down in a number sentence format.
Activity (i): A Game - 'Watch me jump - then draw it!'.
Researcher places his coin on the number 5 on the vertical number-sequence board, selects an up/down arrow card from the pack and carries out the card's instruction in discrete jumps. The card is not seen by any of the children enabling all children to participate at the same time as the child who answers. Researcher places the card face down in front of one child and asks; '*What is on the other side of the card?*' The child is then asked to 'draw the card' on paper - by means of an arrow and a number.
Each child was asked one question and all the children's answers were correct.
Activity (ii) Watch me move - then draw it.
The game above was repeated for another two 'rounds' but the researcher's transforming action was shown not in jumps but *in one complete movement of the coin.* This meant that children could no longer count any jumping movements of the researcher's hand, but would be required to use their imagination.
Comment: Out of 10 questions (2 questions per child), only 2 errors were made.

The children were also able to mentally supply missing numbers in transformational 'equations', which used a number sentence format with arrows but no 'equals' sign. For example, in the transformation A, $\uparrow$B; C , children could find either A, B or C when two of the terms were given.
In the previous game, children were required to find the transformation; in the following game, they had to find the *initial term*, given the transformation term and the final term (i.e. position).

Session 12: Finding the Missing Number. (Initial number only).
A board was constructed on which could be placed any of the number cards (green position-number cards and transformation-number arrow cards) When a card is turned over, it shows a question mark (?). This folding display board allowed children to test their 'transpositional skills' to find any one of the three terms.
Finding the Initial term (positive transforms only - see Vergnaud, 1982)
The researcher arranged the three cards as shown (in first example, the 3 cards were ?, 2 and 7), and asked each of the children in turn. The most able child was

asked first, to 'cue' the other children into the activity. Children's comments are included:-

? ↑2 7 G's answer is 5 *I knew it was 5 because I thinked it.*
? ↑3 9 P's answer is 6
T: *I want to do one ...*
? ↑4 10 T's answer ..... *Bit hard that one ...* (too difficult)
? ↑3 9 Pk's answer: 6
J: *I'm playing it!* *(All children express delight).*
? ↑1 7 J's answer: 6

The missing-final-term problems solved by the children were equivalent to finding y in problems of the form:- position A, count-on B = position y. (y is unknown, A, B are known) Problems to find a missing *initial*-term or *transform*-term with a positive transform (e.g. position y, count-on A = position B; position A, count-on y = position B), were found to be much easier than the negative transform problems (i.e.. position y, count-back A = position B; position A, count-back y = position B), particularly for the 5 year olds. The 6 year olds were significantly more successful in finding the missing number than the 5 year olds in this activity.

The research attempted to provide a small, distinct 'sub-culture' in which children's learning was driven by a need to make sense of an immediate situation in terms of their prior and general understanding of the 'way the world works'. It was believed that continuous interaction of learner and researcher in a socially enjoyable setting would optimize the opportunities to explore children's intuitive approaches to questions about number. This was a small, pilot teaching experiment in which children were learning maths concurrently in their normal classroom setting and therefore there was no attempt to prevent 'contamination' from conventional formats for 'doing sums'. No attempt was made to formally 'test' the children and each session was enjoyed by the children as a social 'out-of-school' experience involving reading, drawing and guessing the answers to 'puzzle questions'. When the researcher sensed that the questions were becoming too difficult, questioning was stopped, ensuring that most of the answers which children gave were 'correct'!
Note that no use or reference was made of traditional mathematical symbols for operations or the = sign but the invented signs were used in contexts which cut across these traditional categories. On no occasion were the questions referred to as 'sums' although one child in writing did refer to the activities as 'sum pitchs'!

Conclusions

In a comprehensive review of research on whole number addition and subtraction, Fuson (1992) outlines several stages of cognitive development. For example, initially children's counting lacks cardinal meaning and only gradually do they begin to relate the last counted word to cardinal meanings for the group of counted objects - what Fuson calls a 'count-to-cardinal' transition in word meaning (p248). However, later when required to *count-out* a given number of objects, children must shift from the given cardinal meaning of the number to its 'counting meaning' as the last counted word. We regard this as the *ordinal* meaning. With this knowledge, and given two number words, a child can add them by first counting out objects for one number word, then counting out objects for the other number word, and finally *counting all* the objects. Subsequently, sequence words and not objects are used to count with, together with a 'keeping track' method for numbers counted out. This is referred to as a *counting-on* strategy. Count-back and count-up are both inverses of count-on, although count-back is described by Gray & Tall, (1994) as the 'natural' reverse process, albeit of enormous complexity (p125).

However, in contrast to previous research, a transformationally rich arithmetic environment provided an alternative route for children to develop counting strategies (Womack, 1997b). Two reasons for this difference are:
(a) the questions we asked children cut across the traditional categories of 'addition' and 'subtraction', necessitating two different 'subtraction' signs. This avoids ambiguity and confusion between different meanings for the same sign.
(b) There are no unstructured objects or sets involved, (only sequenced actions and positions), and so there is no shift from count-all to count-on to be observed.
Numerical solution procedures carried out by children appear to be heavily dependent on their previous experience and the pedagogical approach through which they learn. Fuson herself stresses this point (ibid p. 259) and questions the applicability of developmental sequences identified in the United States to other cultures.

In conclusion, we claim that the children in this study found a meaningful environment for developing mental counting strategies based on

- the arithmetic of transformation problems,
- an ordinal number sequence model, and
- invention of their own operation signs.

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DIFFERENCES IN TEACHING FOR CONCEPTUAL UNDERSTANDING OF MATHEMATICS[1]

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Teaching in ways that promote students' thinking and reasoning about mathematics requires more complex and different teaching practices than those found in conventional classes. Empirical analyses of interaction patterns found in classrooms that emphasize conceptual understanding reveal variability among teachers in their practices. These findings were generalized to create descriptive categories of the increased demands for teaching. The categories were then linked to concepts drawn from developmental psychology and sociology. From this a theoretical framework was created that integrates teaching and learning.

The findings from the past 25 years of research in cognitive psychology on children's ways of knowing have profoundly effected the ways in which mathematics is being learned in schools currently. It is now widely accepted among educators that as children learn they impose their own interpretations and create theories about the experience that makes sense to them. The ways in which children learn mathematics with conceptual understanding is reasonably understood. Moreover, in addition to the advancements in cognition to mathematics education, there has also been an increasing awareness among mathematics educators of children's need to adapt to a social existence and to develop a system of shared meanings in order to participate in their culture.

Although this research has influenced our understanding of and changes in approaches to learning mathematics in schools, relatively little is known about the teaching practices that coincide with these revisions in learning. Researchers, such as Ball (1993), acting as teachers, provide rich descriptions of their mathematics classes. However, only a few studies exist which empirically examine teaching in these classes with the same detail and attention to theory building as found in the investigations of learning. Therefore, the purpose of this paper is first to present findings from empirical investigations into the forms of mathematics teaching which correspond with the revised perspectives on children's learning; and, second, to offer a theoretical framework for integrating teaching practices with types of learning.

Previous attempts to describe the teaching practices that occur in mathematics classes that emphasize conceptual understandings are found in Wood (1996). In this study, a teacher was found to be actively involved in the creation of a notably different context for learning mathematics which enabled children to engage in argumentation. In addition, quantitative analyses of children's learning in these classes indicate that pupils develop better conceptual understanding of mathematics than do students in

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traditional mathematics classes (e.g., Wood & Sellers, 1996). Furthermore, qualitative analyses of children's learning indicate that in such settings children anticipate and engage in far more thinking and reasoning about their own and others mathematical ideas (Wood, 1991).

However, findings from an investigation of teaching practices found in several such classes revealed that teaching does not consist of a singular practice, but rather varies on two dimensions--expectations for class members' participation and the breadth of pupils' thought. Although, the distinctions in practice create different constraints and opportunities for students' learning, of more significant interest for this paper are the increased demands for teaching.

THEORETICAL ORIENTATION

The theoretical orientation for the research draws on constructs from developmental psychology and micro-sociology generally well known by the mathematics education community. In addition, the study draws on anthropology and corresponding research that acknowledges the centrality of teaching to human development. Based on their research with humans and primates, Kruger and Tomasello (1996) contend that teaching is a distinguishing human behavior which emerges with the ability to recognize intentionality in the mind of others. It is this aspect that enables adults to become aware of those situations when children do not know what they as adults know. When this occurs, adults are seen to take steps to alleviate this situation by 'teaching'; telling and showing the child what it is they do not know. This coupled with the fact that "schooling" is universal and sustained by those adults with the best of these tendencies has led to the argument that "knowledge of subject matter in the company of these tendencies" is all one needs to be a teacher. However, Murray contends that this view is no longer sufficient in a culture that demands students do more than simply "learn the material," instead they "must understand." In order for this to occur, Murray asserts teaching in modern schooling needs new and different forms of teaching practices. Teachers can no longer simply tell students information instead they must find ways to enable pupils to develop understanding of mathematics through their own thinking and reasoning.

DATA RESOURCE, METHODOLOGY AND ANALYSIS

Data Resource

The examination of teaching in these classes has evolved over the past ten years. The classes were established as part of an earlier project which embraced constructivist theory to develop instructional materials and to describe children's learning of arithmetical concepts within the context of everyday school settings (cf., Cobb, Yackel, & Wood, 1989). The current project goes beyond the first year of teaching in those classes to examine the nature of the teachers' practice that has evolved since the initial year.

The lessons frequently consisted of children solving project developed activities in pairs for 25 minutes followed by class discussions of their solutions for 20 minutes.

The aspect of the lesson of interest for the investigation of teaching is the class discussion. This is the most complex lesson event for both teachers and students and therefore provides the greatest opportunity to examine demands of teaching in such situations.

Lessons were collected on a daily basis during the first four weeks of school. These *sequential lessons* were primarily used to examine the manner in which each teacher initiates and establishes the social norms that underlie the interaction that occurs during the mathematics class. Additional lessons were collected twice a month for 2 consecutive days thereafter throughout the remainder of the school year. Another set of lessons were drawn from this latter data source and consisted of those discussions in which the same instructional activity was used by all the teachers. These *anchor lessons* were used as contrast lessons.

Methodology and Analysis

The methodology and analysis follows a qualitative research paradigm in which observations of lessons were the data source for the interpretative procedures used. Initially, the methodology and analysis used were similar to that of Glaser and Strauss (1969) in which categories were developed and coded. In this approach, the process of coding is seen to underlie analysis. That is, interpretation is an integral part of the coding process. Following this analysis, microethnographic interpretative procedures similar to those described by Erickson (1986) and Voigt (1990) were used to analyzed selected discussions.

Lesson Logs

Each lesson was videotaped and fieldnotes taken. Following the videotaping, a log of each lesson was made which served as a detailed record to be used for later analysis. The technique of logging captured the nature of the interaction that occurred during the lesson but it did not contain the precise detail of a transcript.

Coding Scheme

A set of coding categories were developed for coding the discussions in a line-by-line manner. The coding categories were applied to pilot logs and subject to refinement in the process until the events could be reliably and consistently identified. Each line of the log was coded individually by three members of the research team and then discussed as a group, and any points of differences in coding were further deliberated among the members until consensus was reached.

The coding scheme consisted of categories used to record the verbal comments of the teachers and their students. One set of coding categories was used to analyze the norm statements made by teachers. These norm statements addressed a range of topics from students' motivation to mathematics. To illustrate, the norm statements made by the teachers about their expectations for children's participation as listeners were coded for three types of expectations: *"pay attention to what Amy says," "listen and try to understand,"* and *"listen to see if you agree or disagree."*

Another set of coding categories was used to analyze the teachers' questions or statements made during class discussion. Separate coding categories were developed for analysis of students' discourse, although these are still undergoing revision. The codes then served as the starting point for describing the patterns of interaction and the discourse that occurred during the class discussions. Initially, the codes served as a common notation for reacting to each sentence in the log which allowed not only for the identification of patterns in the interaction and discourse but also for the initial interpretative conjectures used to guide the microethnographic process.

Microethnography

Following the coding, the typical patterns in the interaction and discourse were identified and used to establish the episodic sections of the discussion. Further examination was conducted using microethnographic interpretive procedures. These procedures continued the line-by-line examination of the dialogues but provided indepth analysis of the events that occurred in the episodic sections. Interpretative memos were written which contained detailed descriptions of each event. As a team, the memos were discussed and a final summary written following recommendations of Krummheuer (personal communication, March, 1996).

RESULTS

Empirical Findings

Teachers' Activity During Discussion

The findings from the empirical analysis of the class discussions for the second half of the year, including the anchor lessons, identified patterns in the interaction and discourse. The results were then condensed into generalizable patterns of interactive and discursive exchanges which were used to create descriptive settings within which the teachers' activity was categorized. In the exchanges that occurred between the teacher and the student, three patterns of interaction were found to exist.

Within the first pattern of interaction the teachers' activity during the exchange consisted of asking the student explaining such questions as: How did you do it?; What did you do?; and, How did you add 20 to 32? The students' explanations consisted of providing descriptions of their thinking and solutions to the problems. This situation can be depicted as students "telling how they solved the problem".

In the second pattern of interaction, the students not only told how they solved the problem but in addition were frequently asked to give reasons for and to clarify their thinking. The teacher asked the questions listed above, but also inquired:What do you mean? I don't understand; and, Why did you add 20 to 32? The teachers' "meaning" probing questions required children to clarify their thinking in order to be understood, while the "why" questions solicited from students' the reasons for their ideas.

In the third pattern of interaction, teachers not only questioned pupils about how they solved the problem, or to clarify their thinking, but the teachers also asked questions such as: How do you know that?; and, Can you prove that? which were challenges and required responses which provided justification.

Teacher Initiated Norms and Patterned Interactions

The findings from empirical analysis of the sequential lessons collected during the first weeks of school and the remainder of the first semester provided information about the norms each teacher established with the children. A comparison among the teachers of the types of norms that were established revealed differences among the teachers in the expectations for students participation. In particular these differences in expectations for participation were most notable for students as listeners. Identification of the norms constituted by the teachers were then linked to the routine patterns of interaction identified in the earlier analysis described above. Differences among the norms established by the teacher were found to differentiate among the classes and thus to account for the variations in the patterned interactions and discourse identified in the analysis of the discussions.

THEORETICAL FRAMEWORK

With these results in mind, generalizable patterns of interactive and discursive exchanges were distilled and connected to specific normative expectations. Conceptualizing the data in this manner allowed for the abstraction of the empirical results into categories and for connections to be made to theoretical constructs in developmental psychology and sociology. From this a theoretical framework was generated through a process similar to that described by Glaser and Strauss (1967). The following section introduces the dimensions of the theoretical framework moving from the relatively static categories produced from empirical analysis to the dynamic axes that represent to theory.

Responsibility for Thinking Axis

Taking into consideration the three patterns of interaction and comparing them to theoretical constructs drawn from developmental psychology, the opportunities for students' to engage in reflective thinking can be theorized as shown in Figure 1.

Mathematical Thinking	Explainer student		Listeners teacher
comparing contrasting	• tell different solutions	Responsibility for Thinking ↓	• accept solutions • elaborate solutions
reasoning questioning	• clarify solutions • give reasons		• ask questions • provide reasons
justifying challenging	• justify • defend solutions		• make challenges

Figure 1

The differences in thinking among the categories indicate an increase in the complexity of the reflective activity of the child in accordance with findings of research in developmental psychology. Students in the first category are involved in reflective thinking while comparing and/or contrasting their ideas with others. Rogoff (1990) describes knowledge constructed in such settings as the ways children develop"skills and understanding." The second category is one in which confusion, complexity or ambiguity arise. In these situations, individuals are involved in thinking which involves questioning and reasoning in order to make sense of the situation. The research of Entwistle (1995) indicates that knowledge construction occurs as students reduce complexity/confusion by integrating ideas. The third category is one in which conflict or disagreement arises which is resolved through reasoning which involves critically examining and justifying existing conceptions. In her research, Kuhn (1992) found that knowledge transformation occurs when individuals reflect on contradictions in their thought. Kitchener has also shown that knowledge construction occurs when individuals evaluate the thinking of others.

Further, the nature of teaching is also seen to be shifting from a position of telling students information to one of asking questions in order that the children become the providers of mathematical knowledge. Thus, the inclusion of the Responsibility for Thinking axis enables links to be made from the categorical changes in children's thinking to differences in teachers' questioning. Furthermore, moving down the categories from top to bottom, each interaction creates increased demands for students' thinking. Teachers' questioning is directly related to demands on students' to develop mathematical thought and become responsible for their learning. Thus, one axis theorized is Responsibility for Thinking (represented by the black arrow).

Responsibility for Participation Axis

The second dimension on which teaching differs is the extent to which teachers' actively involve pupils in the interaction and discourse. These differences are linked to the norms for listeners' participation in the discussion as shown in Figure 2.

Responsibility for Participation

Responsibility for Thinking

Mathematical Thinking	Explainers student	Listeners teacher	Listeners students
comparing contrasting	• tell different solutions	• accept solutions • elaborate solutions	• compare/contrast solutions
reasoning questioning	• clarify solutions • give reasons	• ask questions • provide reasons	• ask questions
justifying challenging	• justify • defend solutions	• make challenges	• disagree • make challenges

Figure 2

Garfinkel, H. (1967). *Studies in ethnomethodology.* Englewood Cliffs, NJ: Prentice-Hall.

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The teachers in the first category typically constituted with the student-listeners' expectations for participation that consisted of listening to "compare ways" with the speaker. In other words, the expectation for listeners was to listen to see if their way was different from the explainer so they knew whether to volunteer to contribute a different way. Teachers in the second category established expectations for student-listeners that included not only compare their ways, but also to try to understand what the explainer said. Moreover, if they did not understand, the listeners in this situation were expected to ask questions. Finally, in the third category, listeners were to compare their way, try to understand, and to decide if they agreed or disagreed with the student explainer. This meant they were expected to follow the reasoning of the speaker. Furthermore, if they agreed or disagreed, they were to say so.

As the expectations for students to participate as listeners increased, the opportunities to engage in complex reflective thinking also magnified. Therefore, a second dimension on which the interaction varied was Responsibility for Participation (represented by the black arrow).

These categories can be connected to theoretical constructs of sociology with regard to the social structures that exist in every day life and consist of normative patterns of interaction and discourse. These patterns are repeatable and reliable such that "much goes without saying" (Garfinkel, 1967). In this case, the norms that the teachers initiated with their students and the routines of participation that evolved were found to be directly related to pupils opportunities to participate in the discussion. Teachers' normative expectations for participation were found to be directly related to increasing demands for a more active role from the students as listeners. As the expectations for students to maintain an active role increased, so did the responsibility for students to participate in their learning.

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CHILDREN'S BEGINNING KNOWLEDGE OF NUMERALS AND ITS RELATIONSHIP TO THEIR KNOWLEDGE OF NUMBER WORDS: AN EXPLORATORY, OBSERVATIONAL STUDY

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This report focuses on detailed descriptions and discussion of eight observations relating to the numeral knowledge of 5- to 7-year-olds. The observations are drawn from videotapes of individualised assessment and teaching sessions and include: Identifying two-digit numerals by saying number words from one or using the names of the constituent digits; saying number words from one in order to write a numeral; writing the right hand digit of a teen numeral first; "reversing" when identifying two-digit numerals ('72' as '27'); difficulty in identifying numerals to '10'; difficulties with '12'; and strategies relating to numeral sequences. An overview of two studies by A. Sinclair, H. Sinclair et al. is included to exemplify earlier research.

Particularly in the last 20 years, there has been much productive research activity in arithmetic in the early years (ie from birth to 7 or 8 years of age) (eg Fuson, 1992). One important aspect of early arithmetic is children's developing knowledge of written arithmetic and its relationship to their verbal arithmetic (eg Hughes, 1986). With rare exceptions (eg Sinclair & Sinclair, 1986; Sinclair, 1991; Thompson, 1982) research in early arithmetic does not focus on how children acquire knowledge of numerals, for example, the numerals in the range '1' to '100'. "[W]e do not have very much information about how or at what ages children learn the mathematical marks for numbers" (Fuson, 1992, p. 72). By and large, research endeavours in the area of written arithmetic have focused on learning and using the symbols for operations in conjunction with numerals rather than what might be viewed as an earlier topic, ie learning and using numerals per se. As well, there is a general acceptance that, in some sense, children's verbal arithmetic develops prior to their written arithmetic (eg Hughes, 1986, pp. 77-8; Steffe & Cobb,.1988, pp. 321-2) and that this parallels the prior development in an historical sense, of verbal number systems over written number systems. In his classic work on the history of number words and number symbols, Menninger (1958/1969), wrote: "Just as a child learns to count one, two, three long before it learns to write or calculate with the numerals 1, 2, 3, the spoken number language precedes the written language" (p. v).

Some earlier research on early numeral knowledge. Studies in Switzerland of two different aspects of young children's numeral knowledge by A. Sinclair, H. Sinclair and collaborators are discussed in order to exemplify earlier research. Sinclair, Siegrist and Sinclair (1983; see also Sinclair & Sinclair, 1986) report an investigation of young children's ways of notating quantities. This involved interviews with 4-, 5-

and 6-year-olds, 15 of each, none of whom had attended formal schooling. The children were asked to represent collections of identical objects, eg three pencils, or verbally stated quantities, eg 'write down four houses'. Six categories of responses are described: (a) global representation of quantity, eg '111111' is written for six balls; (b) representation of the object kind, eg a drawing or a 'B" for three balls ; (c) one-to-one correspondence with non-numerals, eg 'TT' for two houses or 'III' for three balls; (d) one-to-one correspondence with numerals, eg '123' or '333' for three houses; (e) cardinal value alone, eg 5 for five houses; and (f) cardinal value and object kind, eg '5B'. Sinclair and Sinclair (1984, see also Sinclair & Sinclair, 1986; Sinclair 1991) report an investigation of children's interpretations of numerals in everyday contexts. This involved presenting pictures in which numerals featured, and asking children about the numerals. Examples of the kinds of pictures they used are: (a) a birthday cake with five candles and the numeral '5' written on the cake; (b) a bus with '22' on it; (c) houses with numerals on them; (d) runners with large numerals on their T-shirts ; and (e) a sales person and a cash register showing '12.50'. Children's responses on a total of ten tasks were classified into five categories: (a) no response; (b) description of the numeral, eg a vague description such as referring to the numeral as writing, or correctly naming the numeral; (c) global function, ie a reference to the function of the numeral in the context but not related to quantification; (d) specific function, eg the numeral on the cake tells age of the child whose birthday it is; (e) tag, eg the 'five' on the cake says 'cake'.

Focus and methodology. This report focuses on the results of some exploratory investigations into children's beginning knowledge of numerals and its relationship to their number word knowledge. Videotaped records of individualised, assessment and teaching sessions with 5- to 7-year-olds (mostly first-graders but also some Kindergarteners) and extensive field notes written by the researcher constitute the primary data sources for this report. The assessment and teaching sessions and the field notes arose from a three-year applied research and development project (eg Wright, 1994; Wright, Cowper, Stafford, Stanger & Stewart, 1996) focusing on the professional development of teachers in advancing the arithmetical knowledge of low-attainers. The method involves a simple, observational approach, is compatible with that detailed by Cobb and Whitenack (1996), and includes exploratory analyses and writing detailed descriptions and plausible explanations of exemplary events.

Some key terms. In this report, "numeral identification" refers to naming a displayed numeral, eg a numeral card for '7' is displayed and the child is asked a question such as, "What number is this?". "Numeral recognition" refers to selecting the appropriate numeral card from a randomly arranged collection of numeral cards, eg "Which card has the number seven?". "Numeral sequence" refers to a written sequence of numerals without omission, arranged horizontally, eg the numeral sequence from '11' to '20'. "Forward Number Word Sequence (FNWS)" refers to a sequence without omission of spoken or heard number words, eg the FNWS from "eleven" to "twenty". Similarly, "Backward Number Word Sequence (BNWS)" refers to a backward sequence.

Research questions. This report addresses the following research questions: What strategies do children use to identify, recognise and write numerals and what difficulties do they encounter? What explanations can be advanced for these strategies and difficulties? What are some implications for teaching? Because of space limitations, the questions are addressed incidentally in the following section.

Observations and Discussion

Described below are nine observations related to young children's knowledge and use of numerals. The observations are organised into four categories and are drawn from the videotaped records referred to earlier. It is beyond the scope of this report to address the important issue of the frequencies of these kinds of observations. Nevertheless virtually all of these are considered to be interesting, educationally significant, worthy of further investigation and not documented elsewhere.

Category A: Relating to Numeral Sequences.

Type A1: Able to sequence numerals from '1' to '10' but unable to identify them. A project teacher was assessing a child's ability to identify numerals in the range from '1' to '10'. The numerals were presented in random order and the child named almost all of them incorrectly. The teacher then asked the child to put the numerals from '1' to '5' in order and the child completed this correctly. The teacher then similarly asked the child to order the numerals from '6' to '10'. This was completed with only one error, ie '8' was placed after '9' rather than before it. Following this, the child named the cards from '1' to '10' pointing at each in turn, and saying 'eight' for '9' and 'nine' for '8'. Further observation of this child revealed: (a) a numeral identification strategy which relied on a visible numeral sequence from '1' to '10'; and (b) incorporation of this numeral identification strategy into a strategy for doing written addition.

Type A2: Identifies and sequences numerals from '1' to '10' but does not say FNWS from 'one' to 'ten'. A project teacher described a child who had difficulty in saying the number word sequence from 'one' to 'ten', and at the same time could identify numerals in the range '1' to '10' when presented in random order. As well, this child could arrange in numerical sequence, a random arrangement of the numerals from '1' to '10'. This challenged our beliefs that children's knowledge of number words is, in a developmental sense, primary to their knowledge of numerals. Thus although typically a child's knowledge of number words develops in advance of their knowledge of the corresponding numerals, it appears that from a phylogenetic perspective this need not necessarily be the case.

Category B: Pre-eminence of individual digits in two-digit numerals.

Type B1: Writes right hand digit first when writing '13' to '19'. There were several observations of children who, when writing a numeral in the range '13' to '19', consistently would write the right hand digit first. But when writing numerals beyond '19' they would write the left hand digit first. This lead to the hypothesis of two different strategies for writing numerals, ie in situations where the child is asked to write a numeral or perhaps, arrives at an answer, eg 'sixteen' via a verbally-based strategy and then attempts to write a numeral as the answer: (a) The child uses the sound image of the number, eg 'sixteen' to generate a visualised image of the numeral, ie '16' and then proceeds to write in temporal sequence, the digits as they appear from left to right, ie writes the '1' first. (b) Using the sound image of the number word, the child generates a sound image in the range 'one' to 'ten' corresponding to the first part of the sound image, eg the child uses 'sixteen' to generate 'six'. The child then writes '6'. The child now focuses on the 'teen' part of 'sixteen', realises that they must write '1', and also realises that it must be written to the left of '6'. The strategy labelled (b) might be referred to as one of "splitting-the-sound-image". Clearly, use of (b) would not be restricted to numerals in the teens. Very likely, the child who uses (b), uses it consistently for teen numerals and numerals beyond the teens. But, when (b) is applied to any numeral beyond the teens, from an observer's perspective, the child appears to be writing the numeral in the normal way. Only in the case of numerals in the teens is it revealed that the child is using the generalised strategy of splitting-the-sound-image. A plausible explanation is: The child uses the strategy to write most or all two-digit numerals, not just the teens. When the child 'hears', ie imagines or re-presents the sound image eg 'forty-five' the child cannot or at least does not then visualise '45'. Rather the child visualises '4', writes '4', visualises '5' and finally writes '5'. This strategy could also explain why some children have difficulty writing two-digit numerals correctly in columns, particularly when numerals in the teens are involved.

Type B2: Uses digit names to identify two-digit numerals. This strategy can be seen as complementary to the splitting-the-sound-image strategy for writing numerals. When required to identify a numeral, eg '28', the child first uses the left hand digit to generate the sound image 'twenty', then uses the right hand digit to generate the sound image 'eight'. This is labelled the digit-by-digit strategy. On first consideration this explanation may appear to be trivial and obvious. Nevertheless, we could distinguish this strategy from one of, in one step as it were, using '28' to generate the sound image 'twenty-eight'. Presumably, this is the method used by adults. In most cases the digit-by-digit strategy works fine and appears no different from the adult strategy. However, in special cases eg '12' the digit-by-digit strategy fails spectacularly. It can be seen now that this child is in the classic bind. Trapped in a well that merely gets deeper and deeper so to speak. The strategy succeeds in a vast majority of cases and there is little likelihood that the child will become fully and explicitly aware of the inadequacy of the strategy. To do so would appear to be far beyond the child. There were frequent observations of children who had persistent difficulty in learning to correctly name the numeral '12'. This explanation accounts for such difficulties. The digit-by-digit strategy parallels the strategy of sounding out letters in order to pronounce a word. It seems reasonable to suggest that children may not be fully aware of the discreteness of the system of digits making up numerals from the system of letters making up words. Thus the digit-by-digit strategy is likely to be further strengthened because of its similarity to the strategy of sounding the letters to pronounce a word.

Type B3: Identifying a two-digit numeral as its reverse numeral (eg '72' as '27'). Kindergarteners (ie 5- or 6-year-olds) who could identify numerals in the teens were observed identifying two-digit numerals as if the digits of the numeral were in the reverse order from left to right, eg '72' is identified as 'twenty-seven'. In doing so the child would not immediately respond. Typically there was a period of a few seconds during which it was clear the child was thinking. It is reasonable to assume that these children had not been explicitly taught to identify two-digit numerals, and particularly the numerals from '21' onward. A plausible account for this observation is: The child uses what she knows about identifying a numeral such as '17' to identify '72'. When identifying '17' the child focuses on the right hand digit to say 'seven', and then on seeing the left hand digit, the child says 'teen'. The child attempts to apply this strategy to the unknown '27'. Looking at the right hand digit, the child says 'seventy', the child now looks at the left hand digit and then says 'two'.

Category C: Strategies involving Forward Number Word Sequences (FNWSs).

Type C1: Saying a FNWS in order to identify a numeral. An example of this strategy is when confronted with the task of naming '8' the child says "one, two, ... eight!". Several children were observed to use this in the case of naming '12' as well as numerals in the range '1' to '10'. In the case of Jack a comparison was made of his performance on two tasks: (a) given a collection of plastic digits he was asked to make the number twelve; and (b) he was asked to identify '12'. Jack could solve (a) reasonably easily but invariably had difficulty with (b). A plausible account is: Jack's difficulty is that of generating the sound image 'twelve'. What Jack finds easy is to generate a visualised image for '12' on hearing 'twelve'. What he finds difficult is to generate the sound image 'twelve' on seeing '12'. Jack had a general strategy for identifying numerals but, as discussed earlier, his strategy failed in the case of '12'. We suppose that Jack's difficulty might well go unnoticed in a classroom teaching situation. Firstly, Jack may not have an awareness that he has a difficulty. Even if he is aware, he may develop ways of concealing his difficulty. His difficulty may be misinterpreted by the teacher as an error of calculation rather than an error of numeral identification. Consider a classroom approach involving a quick transition from instructional settings involving materials to ones involving written arithmetic. This difficulty could seriously impede Jack's progress on written arithmetic. Consider how Jack might attempt a written problem such as 12+3. If Jack's first step is to translate the numerals into sound images of number words, and then count-on, then

Jack has a major difficulty because he can't produce the sound image "twelve". Alternatively, Jack might proceed by visualising a numeral sequence, "12 13 14 15", then Jack could write the answer "15" without using the sound image "twelve".

Type C2: Saying a FNWS in order to write a numeral. This observation is complementary to the immediately previous one. On being asked "to write the number seven" the child begins by saying the FNWS from 'one' to 'seven', then the child writes '7'. This was observed in the case of numerals in the teens and for '20' as well. A plausible explanation is that, for each number word in turn, the child produces a visualised image of the associated numeral, and thereby is able to produce a visualised image for the numeral of the requested number. If this explanation is valid, a question that arises is whether the child is visualising the numerals in temporal sequence or alternatively, visualises a spatially organised sequence containing several numerals.

Category D: Other Observations.

Type D1: Difficulty in identifying numerals in the range '1' to '10'. A minority of project children had significant and ongoing difficulties with identification of numerals in the range '1' to '10'. There were instances of children who had been at school for seven terms (ie quarters) and could not name these numerals. At the same time there was no obvious explanation for this difficulty, eg by and large the children had not been identified as having disabilities in learning. Project teachers' deliberate and concentrated efforts to directly teach the children to identify numerals from '1' to '10' were only partially successful and progress was quite slow. As well, these children had similar difficulties learning to identify numerals in the range '11' to '20'. We set out to understand better, the nature of the children's difficulties. We investigated if the difficulty could be explained in terms of inability to discriminate digits. That is, was it the case that, for example, the child had difficulty discriminating between the symbols '6' and '8'. In order to investigate this we presented the children with tasks involving the sorting of numeral cards. For example, the child is given a stack of 15 cards — five '3' cards, five '6' cards and five '8' cards. By and large the children completed these tasks without difficulty and thus we concluded that their difficulties with numeral identification were not attributable to difficulties in visual discrimination. What seemed apparent was that these children had difficulty in generating the sound image associated with a displayed numeral, ie they could not produce the name of the numeral. The view that, for some of these children the source of this difficulty is organic seems reasonable. On the other hand, the difficulty may be significantly influenced by contextual factors that largely determine the kinds of strategies the children develop for naming numerals. The project teachers were of the view that by and large the children who had these relatively severe difficulties with numeral identification, similarly had difficulties in early literacy, eg with letter identification.

Type D2: Difficulties associated with '12'. Difficulties with identification of '12' have been alluded to above and are now discussed specifically. Teachers of young children are well aware that learning to use the numeral and number word for the number 'twelve' presents particular difficulties for children. The following episode provided some insight into this difficulty. Sue, had the goal of teaching Brendon to correctly name '12'. Brendon typically identified '12' as 'twenty'. Sue presented a task in which Brendon was given a stack of numeral cards on which either '12' or '20' appeared. Brendon's task was to say the name of the numeral on the card at the top of the stack, and then to place the card into one of two stacks, ie a '12' stack and a '20' stack. Brendon could not easily complete this task and in response, Sue modified the task by replacing the '20' cards with '11' cards. The modified task, which as before, required Brendon to name the numeral prior to sorting it, was significantly easier for Brendon. We speculated that, on the original task, the source of Brendon's difficulty was that of having to name the numerals, rather than sorting per se. Accordingly, Sue presented Brendon with a task involving sorting but not naming, using a stack of cards for '12' and '20'. This task was much easier for Brendon. Our conclusions from this series of observations with Brendon were consistent with our thinking as described above, ie that the source of difficulty is the requirement to generate the sound image 'twelve'. The evidence supports the belief that the child's generalised strategies for generating names of numerals do not work well for the numeral '12'.

Type D3: Arithmetical strategies involving numerals rather than number words. There were several observations where in solving arithmetical tasks, children gave indications that they were thinking in terms of visualised numerals when they might have been expected to be thinking in terms of number words. Two examples are: (a) A child completes a task, and is able to indicate the answer on the Hundreds Chart (ie the 10X10 grid of numerals) but not say the number word; and (b) a child completes an task, and is able to state the two digits that make up the answer but not say the number word. Cases such as these challenged our assumptions that for young children, mental arithmetic involving thinking in terms of sound images of number words is always primary to mental arithmetic involving visualised numerals. Thus in settings were children are frequently reading and naming numerals and which involve numeral sequences, children's mental activity may mainly involve visualised numerals. Under such circumstances children are likely to become facile at using visualised numerals when doing mental arithmetic, and to have a preference for using visualised numerals rather than sound images of number words.

Conclusion. As indicated earlier, there is a paucity of research into young children's beginning numeral knowledge. The observations reported in this study highlight the importance of this topic in pedagogical terms — eg, difficulties in numeral identification or recognition can lead to computational errors, the sources of which are likely to be difficult to detect in a classroom teaching situation. Thus this exploratory study indicates several areas which could be a focus of future systematic research studies.

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A Study of Argumentation in a Second-Grade Mathematics Classroom

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The analysis presented in this paper focuses on argumentation as a means to analyze both the interactive constitution of explanation and its evolution as an instructional sequence progresses. This extends our previous work on more general criteria for mathematical explanation, which we called sociomathematical norms, to specific types of explanations that are constituted as aspects of taken-as-shared classroom mathematical practices.

The purpose of this paper is to show how an analysis of argumentation in classroom discourse can be used to explicate the reflexive relationship between individual student's mathematical explanations and the taken-as-shared classroom mathematical practices that emerge collectively. In the course of doing so, we show how it is possible to use the evolution of what is taken as a normative explanation, as an instructional sequence progresses, to account for mathematical learning.

Method and Data

This research report is based on a detailed analysis of the argumentation in two three- to six-week long teaching experiments conducted in a second grade classroom in an American school in which mathematics instruction followed an inquiry tradition (Cobb, Wood, Yackel, & McNeal 1992). The first teaching experiment took place at the beginning and the second at the end of the school year. The class consisted of 20 students of average ability and was taught by an experienced teacher with whom we have collaborated for several years. The classroom teacher, who provided all instruction during the teaching experiments, was supported throughout by the research team which was present during every lesson and which met with her several times a week to discuss the students' activity during mathematics instruction and to use that as a basis to plan for future lessons. The data used for the analysis consisted of video recordings of every lesson, field notes, and transcripts of selected classroom episodes. Additional data sources are video recordings of individual interviews conducted with each student before and after each teaching experiment and copies of the students' written work from each lesson.

In this paper, I first describe the theoretical framework that was used in the analysis of argumentation. Next, I explain how the study of argumentation clarifies the relationship between individual and collective argumentation, using examples from the first teaching experiment for illustrative and clarifying purposes. Finally, I indicate a second aspect of the research which involved documenting changes in normative understandings of explanation.

Theoretical Framework

Mathematical explanation and justification can be studied from a number of different perspectives. In this paper, our interest is in mathematical explanation and justification as an interactional accomplishment and not as logical argument. The focus is on what the participants take as acceptable, individually and collectively, and not on whether an argument might be considered valid from a mathematical point of view. Krummheuer's work on argumentation (1995) and Cobb and Yackel's (1996) interpretive framework for analyzing classroom mathematical activity provide the theoretical framework for the approach taken in this paper.

In his study of the ethnology of argumentation, Krummheuer (1995) analyzes argumentation using Toulmin's (1969) scheme of conclusion, data, warrant, and backing. According to this scheme, a conclusion is a statement that is made as though it is certain, data is the support one might give for the conclusion, warrant refers to the rationale that might be given to explain why the data are considered to provide support for the conclusion, and backing is further support for the warrant, indicating why the warrant should be accepted as having authority. To clarify, Krummheuer uses the example of two boys who are working collaboratively to solve the problem 4 x 4 = __. The boys decide that the answer is 16 because 8+ 8 = 16. According to Toulmin's scheme, 8 + 8 = 16 is the data. As Krummheuer points out, it is legitimate to ask what 8 + 8 has to do with the product 4 x 4. The rationale that is given is the warrant for the data. The two boys provide some insight into what serves as a warrant for them when they say, "It's 4 sets of fours ... 'Cause 4 sets, um 4, 2 sets make 8." As they continue, one boy holds up his fingers as he says, "You have 2 more sets. Like it's 2 and 2 make 4." Krummheuer interprets this action as indicating that for these boys, 2 and 2 make 4, provides further explanation of why it is appropriate to figure out the value of two sets of four and two more sets of four, that is, the action functions as backing for the warrant.

In summary, Krummheuer uses the notions of data, conclusions, warrants, and backing as a means to analyze argumentation and to explicate how it is an interactive constitution by the participants. For him, argumentation in any given situation "contains several statements that are related to each other in a specific way and that by this take over certain functions for their interactional effectiveness" (Krummheuer, 1995, p. 247). Statements do not have a function apart from the interaction in which they are situated. Thus, what constitutes data, warrants, and backing is not predetermined but is negotiated by the participants as they interact.

Krummheuer's approach to argumentation is useful here for two reasons. First, it clarifies the relationship between the individual and the collective, in this case between the explanations and justifications that individual children give in specific instances and the classroom mathematical practices that become taken-as-shared. As mathematical practices become taken-as-shared in the classroom, they are beyond justification and, hence, what is required as warrant and backing evolve. Similarly, the types of rationales that are given as data, warrants and backing for explanations and justifications contribute to the development of what is taken-as-shared by the classroom community, that is, to the mathematical practices of the classroom. Second, it provides a way to demonstrate the changes that take place over time. In particular, we have been able to demonstrate that what constitutes warrant and backing for students changes as an instructional sequence progresses.

The complexity of classroom research presents a challenge for the researcher, not only for data analysis, but also for conceptualizing the analysis process itself. Here we follow the interpretive framework developed by Cobb and Yackel (1996) (see Figure 1). The framework coordinates a psychological perspective with a sociological perspective at the level of the classroom. In this paper we focus on

SOCIOLOGICAL PERSPECTIVE	PSYCHOLOGICAL PERSPECTIVE
Classroom social norms	Beliefs about own role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions

Figure 1. Interpretive Framework for Analyzing the Classroom Microculture

classroom mathematical practices and individual students' mathematical conceptions. The specific aspect of each of these that is of interest here is mathematical explanation. A key feature of the framework is that it posits a reflexive relationship between the sociological constructs and their psychological correlates. For example, to say that the relationship between individual children's mathematical conceptions and classroom mathematical practices is reflexive means more than that they are mutually enabling and constraining. It means that one literally does not exist without the other. In terms of explanations, we would say that as children give explanations that they deem viable, they are both acting in accordance with the taken-as-shared normative understanding of what constitutes acceptability and also contributing to the ongoing negotiation of what is taken as normative. Thus, the explanations that individual children give and the normative understandings of what constitutes an acceptable explanation are mutually constitutive.

Previously we have referred to normative understanding of what constitutes an acceptable mathematical explanation and justification as a sociomathematical norm and argued that such normative understanding is negotiated as the teacher and students interact in the classroom (Yackel & Cobb, 1996). For example, in the inquiry classrooms we have studied, including the classroom that is the subject of this paper, it became normative that explanations and justifications increasingly involved describing actions on what had become experientially-real mathematical objects for the students. Consequently, explanations that did not carry the significance of actions on taken-as-shared mathematical objects that were experientially real for the students were frequently challenged. This criterion for mathematical explanation applies across instructional sequences and tasks and in this sense is a general criterion, and thus a sociomathematical norm. In this paper we extend this notion by analyzing the explanations students give for specific tasks within an instructional sequence and the corresponding understandings of what constitutes acceptable explanations that become normative for the class collectively.

In this sense we are dealing with an aspect of classroom mathematical practices rather than with sociomathematical norms.

Results and Discussion

In the following we discuss the analysis of argumentation in the teaching experiment data from two perspectives. First, we use an example to show how argumentation can be used to analyze what becomes taken-as-shared in the classroom. In doing so, we demonstrate what we mean by the reflexive relationship between the individual and the collective. Second, we illustrate how what is taken as warrants and backing change over time and in doing so demonstrate that this evolution can be used to account for the constitution of classroom mathematical practices and hence mathematical learning.

Argumentation as a Means to Analyze the Interactive Constitution of Explanation

Several of the instructional activities used at the beginning of the school year involved dot patterns. In one activity, dot patterns were flashed on an overhead screen several times, each time for only a few seconds. The children's task was to figure out how many dots there were in the pattern and to describe (explain) how they "saw it." The instructional intent of the task was to foster development of visual imagery for, and to generate discussions about, number relationships for small numbers. For example, on the second day of the school year, the teacher displayed the dot pattern shown in Figure 2. The children's responses included both counting and non-counting explanations. For example, several students reported that they saw seven *because* they saw it as three and one and another three. Interestingly, none of the children who gave such a response explained how he or she arrived at a total of seven based on the three, one, and three. These children seemed to take it for granted that their explanation was sufficient, that is, that the other students would immediately know that $3 + 1 + 3 = 7$. Yet we know from individual interviews conducted with the pupils at the beginning of the school year and from their mathematical activity in the classroom that some of them would have to count by ones to solve this addition problem. Nevertheless, all of the children in the class were able to readily see that there were "three dots in the first row, one in the second row, and three in the third row" (at least one child referred explicitly to the number of dots in each row of the pattern in this way).

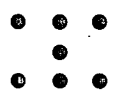

Figure 2. Dot pattern for seven.

Thus, we might assume that, in Toulmin's scheme, the statement that $3 + 1 + 3 = 7$ would serve the function of data for the conclusion that there were 7 dots in the pattern. No warrant would be necessary. A warrant would be needed if children had no way to understand why the sum of $3 + 1 + 3$ was relevant to answering the question of how many dots there were. Following this assumption, we might presume that all of the children had a way of making sense of the relevance of the explanation. This would be significant because it would mean that even those children who would have to count by ones to solve 3 + 1 + 3 could understand that solving this addition problem was a viable alternative to counting the dots by ones.

On the other hand, such responses had the potential to contribute to children's interpretation of the task as having two components: reporting the arrangements of the dots and figuring out the totality of dots. However, the instructional intent of the task was not that they engage in these two separate tasks, but that they use the visual imagery to think about numbers in various ways, including thinking of a quantity as composed of other smaller quantities. Thus, the expectation was that the configuration of dots a child would report would indicate something about how the child figured out the totality. Expectations such as these have to be established. For example, in this and future lessons some children reported how they viewed the arrangement of dots without relating it in any way to the total.

A case in point is Nina's solution given later in the same lesson when the teacher flashed the dot pattern shown in Figure 3. Nina reported only that she saw 4 at the top, 1 in the middle, and 4 at the bottom. Her response implies that she assumed that this was an adequate response. From this information we are unable to determine if Nina knew that there were 9 dots in all and, if so, how she figured it out. She may have been simply reporting how she "saw" the pattern. Furthermore, since her response was accepted by the teacher with no further probing, we do not know if, for Nina, there was a relationship between the four, one, and four that she reported and the totality of dots. Thus, we cannot assume that for Nina, four and one and four served as data for the conclusion that there are 9 dots. Furthermore, we cannot assume that the statement, there are nine dots *because* 4 + 1 + 4 = 9, would have explanatory relevance for Nina. For her, a warrant may be needed. Thus, the earlier explanation that there are 7 dots (in Figure 2) *because* there are 3 dots, one dot, and three more, may not have been adequate for Nina. In this example, we have pointed to a critical feature of explanation. That is, the adequacy of an explanation depends on all of the participants in the interaction, not only on the person giving the explanation. In this sense, we are treating explanation as a communal, not an individual, activity (cf. Cobb et al., 1992).

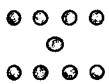
Figure 3. Dot pattern for nine.

Several of the responses that children offered for the task posed next in the lesson were particularly useful to clarify the interrelationship between the arrangement of dots reported and ways of figuring out the totality. In this case the dot pattern shown was for the number ten. It was identical to the pattern for nine except that it had two dots in the middle row. The first child to respond to the dot pattern for ten reported that she counted them by ones. The next two responses were given by Anita and by Denzel.

Anita: *I knew that 4 plus 4 is 8 and I counted 2 more and that made 10.*
Denzel: *Put 2 from the middle at the top. That makes 6 and the other 4 makes 10.*

In these responses, both Anita and Denzel explain *how* they used the configuration as they "saw" it to figure out how many dots there were in the pattern. That is, they provide the explanatory relevance of the configurations they used, 4, 4, and 2 for Anita, and 4, 2 and 4 for Denzel. In fact, we might infer that their interpretations of the pattern were based, in part, on number combinations that they were able to use to determine the sum. As Labinowicz (1985) points out, what we "see" is based on what we understand, not the other way around.

In this instance the teacher did not capitalize on these responses. The students were left to make sense of these explanations for themselves. Voigt (1995) notes that such incidents serve an important function in supporting students' mathematical learning by making it possible for them to become aware of more conceptually advanced forms of mathematical activity while, at the same time, leaving it to them to decide whether to take up the intellectual challenge. In this case, some of the children would still need to count the dots by ones to determine the total. However, we would argue that, because these explanations include explicit information about how the configurations were used in determining the sum, even these children might make sense of the relevance of these more sophisticated solution methods and might begin to reconceptualize their own counting activity in terms of the quantities in the individual rows of the configuration.

Despite the fact that the teacher did not call attention to these solutions in this instance, such explanations became more common. Within a few weeks it was normative for students to give explanations of that type in both the dot pattern activity and other visual imagery activities. Describing visual patterns in terms of smaller quantities and giving explanations of how to find the total using those quantities had become a taken-as-shared classroom mathematical practice. The explanatory relevance of statements such as, "There are 7 dots because there's three in the top row, one in the middle, and three in the bottom row," was apparent to an increasing number of students. That is, a warrant was not needed to support this as data for the conclusion that there are seven dots.

Thus far, we have taken a close look at the emergence of one type of explanation that became normative in the classroom. The explanations that were given by some of the children in the class were central to the constitution of the practice. As the classroom practice became taken-as-shared, normative understandings of what constitutes an acceptable explanation in this situation also evolved.

Argumentation as a Means to Analyze the Evolution of Explanation

As the first teaching experiment progressed, other classroom mathematical practices were constituted. These are: describing solutions by relating problems to previously solved problems; describing visual configurations in terms of fives, tens, and doubles; using the imagery of visual configurations to explain number relationships and thinking strategies; and acting with known number relations to solve problems. In each case, the constitution of the practice was facilitated by explanations given by individual children. To use Krummheuer's terms, we would say that the evolution of classroom mathematical practices parallels (is enabled and constrained by) an evolution of what children take as data, warrants and backing in argumentation. This, in turn, is closely related to the individual children's conceptual understandings and possibilities. In the following we present an example to illustrate what we mean by this evolution.

One practice that emerged was describing solutions by relating problems to previously solved problems. Some children's use of thinking strategy explanations for double tens frame tasks contributed significantly to the negotiation of this classroom mathematical practice. By the end of the second week of the school year, a number of children were describing visual patterns in terms of smaller quantities and giving explanations of how to find the total using those quantities. The teacher then introduced the double tens frame to provide opportunities for children to relate visual patterns to fives, tens, and doubles. One instructional activity that she used was to place some chips in each frame of a double tens frame and to flash the image briefly, several times, on the overhead screen. The task was to figure out how many chips there were and to explain how you figured it out. The teacher used chips of two colors, one color for the left frame and a different color for the right frame. In doing so, she created a situation where children might interpret the task as finding the sum of two quantities—the quantity in the left frame and the quantity in the right frame. An important feature of the instructional activity was that the tasks were sequenced so that students might relate the configurations in subsequent tasks to those in prior tasks. Thus, the teacher created opportunities for students to relate tasks to each other, without requiring them to do so. For example, in the ninth lesson of the school year the teacher posed the following sequence:

Task 1: five chips in the left frame and five in the right frame
Task 2: six chips in the left frame and five in the right frame
Task 3: six chips in the left frame and four in the right frame

Several children gave responses in which they related tasks. For example, when called upon to explain his solution to Task 2, Hakeem replied that "Five and five is ten and ... it was one more dot on the six. One more dot on the five and that made 6 and I put them together and that made 11." Later, when questioned by another student, he elaborated, "Because watch. I got 10 and then she added one more and look, then it went to the 6 and this was 5 and then this goes to this (demonstrating with his hands, first holding up five fingers and then changing to six) and that made, that made 11." This elaboration was particularly informative in the class discussion since Hakeem had previously explained his solution to Task 1 by holding up both hands and remarking that "then I looked at all of them. That's how I know it was ten." Thus, his two explanations taken together could be interpreted by others as clarifying both *that* he saw the tasks as related and also *how* he used the relationship to solve the second task. Similarly, he solved Task 3 by relating it to Task 2. Such direct references to previous problems became important in the class discussion as other students took note and subsequently attempted to use similar strategies. In this lesson, one boy specifically commented on Hakeem's solution to Task 2 as insightful, describing it as a "genius" solution.

Over the next few weeks, increasingly many children began to give solutions of this type. In addition, the way they explained their thinking became more cryptic. For example, in contrast to Hakeem's explanation in which he explicitly said that there was "one more *dot on the five and that made 6,*" explanations (for other problems) such as, "It's just one more", or "You had 11 and all you did was add one more", or "One more, so 16", became common. These slight differences may seem insignificant. Yet, following Krummheuer's analysis of argumentation, we infer that these increasingly cryptic statements served as data for the conclusion without the need of a warrant or of additional backing. This evolution in what constituted sufficiency for the children is precisely what makes it possible for us to refer to the classroom mathematical practices that emerged as taken-as-shared. We also note that while these cryptic explanations were based on tasks that were posed with visual material, explicit reference to that material was not made, but increasingly, the explanations were about number and number relations.

Concluding Remarks

In this paper, we have used an analysis of argumentation to demonstrate how explanation was interactively constituted in one mathematics instructional sequence at the beginning of the second grade school year and to illustrate the relationship between children's explanations and the classroom mathematical practices that emerged. We have also demonstrated that what children took for granted evolved as the instructional sequence progressed. Classroom mathematical practices themselves emerge as what is taken for granted evolves. This is significant because it helps us account for how children learn as they participate in the communal activity of the classroom.

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TEACHER PERCEPTIONS, LEARNED HELPLESSNESS AND MATHEMATICS ACHIEVEMENT

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As part of a longitudinal study of mathematics achievement, teachers rated the classroom behaviour and mathematics achievement of 258 primary and lower secondary students in an Australian city. The ***Student Behavior Checklist****, used by the teachers, measured learned helplessness and mastery behaviours in the classroom. Teachers' perceptions were compared with students' achievement in mathematics and self-reported motivational indices one year later. Teachers' ratings of achievement were predictive of students' subsequent achievement. Their ratings of academic behaviour significantly predicted students' depression, which in turn was related to explanatory style and task involvement in mathematics. The implications of these results for the psychology of mathematics education are discussed.*

Introduction

To what extent can teachers identify students with the disposition to exhibit learned helplessness? Although the concept of learned helplessness has a long history in psychology, there appears to be no recognised measure of this trait in terms of teachers' perceptions and judgements. Helplessness was described by Peterson, Maier and Seligman (1992) as loss of motivation, changes in cognition and emotion, and a reduction in behavioural agency results in passivity. Among the changes in cognition was the perception of non-contingency or belief that important outcomes were uncontrollable. In classroom contexts it is likely that helplessness is observed through the way students respond to situations of actual or conceivable failure. Teachers should therefore be in a position to assess at least some of the recognised dimensions of helplessness as they surface in classroom life. Mathematics is an area of the curriculum in which success and failure are highly salient (McLeod, 1992), but the extent to which teachers' perceptions of students' classroom behaviour influence their assessments of achievement and are predictive of students' subsequent motivation and achievement in mathematics is not known.

Teacher ratings of classroom behaviour have been correlated with students' general achievement (Fincham, Hokoda & Sanders, 1989; Nolen-Hoeksema, Girgus & Seligman, 1986), and metacognitive abilities (Carr & Kurtz-Coates, 1994), but not their attributional beliefs or self concepts (Carr & Kurtz-Coates, 1994). In a longitudinal study, mathematics teachers' expectations predicted changes in student achievement beyond effects accounted for by previous achievement and motivation (Jussim & Eccles, 1992), particularly for low achievers (Madon, Jussim & Eccles, 1997). When average achieving students were assigned to advanced mathematics classes, they received higher level mathematical content, more active teaching, and achieved at a higher than expected level (Mason, Schroeter, Combs & Washington, 1992). While neither meta-analyses of the experimental research (such as Raudenbush, 1984; Rosenthal & Rubin, 1978), nor naturalistic studies (see Brophy, 1983; Jussim & Eccles, 1995, for reviews) supported the self-fulfilling prophecy as a general concept (Rosenthal & Jacobsen, 1968), these studies in mathematics suggest that teachers' perceptions are important indicators of student achievement.

Achievement in mathematics has been related to anxiety (Hembree, 1990), confidence (Reyes, 1984), self-concept (Marsh, 1986), self-efficacy (Bandura, 1977) and attributions for success and failure (Kloosterman, 1988). While studies have investigated students' attributions for success and failure in mathematics (eg Gentile & Monaco, 1986), none has considered the manner in which learned helplessness in the classroom is perceived by mathematics teachers and the relationship between these perceptions and students' subsequent motivation and achievement. Learned helplessness is likely to occur in mathematics (Gentile & Monaco, 1986), partly because of the nature of the subject matter (Dweck & Licht, 1980) and partly because many students, at least in America, believe that learning mathematics is more a question of ability than effort (McLeod, 1992). As the characteristics of learned helplessness include passivity, loss of motivation and lack of effort, students' low participation in the activities and lessons provided by the teachers are likely to also interact with their achievement (Brookhart, 1994), creating a vicious circle.

As part of a longitudinal investigation into motivational variables likely to influence primary and lower secondary school students' achievement in mathematics, teachers rated the behavioural characteristics of students in the classroom as well as their achievement. These measures of academic helplessness and achievement, as perceived by teachers, were compared with student achievement data and self-reported motivational indices one year later.

Subjects

In November, 1994, 58 teachers in 31 schools in an Australian city rated 258 students from Grades 4 to 8. One year later, 243 of these students (see Table 1) in 26 primary and 24 lower secondary schools completed a test of mathematics achievement, and questionnaires of explanatory style, depression and attitudes towards mathematics.

Table 1 Numbers of students by grade level and gender in 1995

Gender	Grade 5	Grade 6	Grade 7	Grade 8	Grade 9	Total N
Male	8	28	21	28	24	109
Female	10	34	22	38	30	134
Combined	18	62	43	66	54	243

Instruments

1. The **Student Behavior Checklist** (Fincham *et al.* 1989) was composed of 12 learned helplessness items and 12 mastery orientation items. Teachers rated student

behaviour on a five point scale ranging from 1 (*not true*) to 5 (*very true*) and gave a single estimate of achievement in mathematics from 1 (*excellent*) to 5 (*poor*).

2. **Progressive Achievement Tests in Mathematics** (PATMaths) (ACER, 1984), consisted of three standardised multiple choice format tests (Tests 1, 2, and 3) at different grade levels and levels of difficulty. In each test, items covering general mathematics topics were arranged in content groups in order of increasing difficulty.

3 The **Children's Attributional Style Questionnaire** (CASQ), (Seligman, Peterson, Kaslow, Tanenbaum, Alloy, & Abramson, 1984), a dispositional measure of optimism and pessimism, was a forced choice pencil and paper instrument of 48 items of hypothetical good and bad events involving the child, followed by two possible explanations. A total score was formed from the positive and reversed negative items.

4 **Your Feelings in Mathematics: A Questionnaire** (Yates, Yates & Lippett, 1995), adapted from the *Motivation Orientation Scales* (Nicholls, Cobb, Wood, Yackel, & Patashnick, 1990), measured the task involvement and ego orientation dimensions of goal orientation beliefs in mathematics. Students rated their attitudes towards mathematics on a five point scale ranging from (5) a *strong yes* to (1) a *strong no.* Each item commenced with the stem "Do you really feel pleased in maths when ... " followed by a statement that related to student mathematics behaviour.

5 **The Children's Depression Inventory** (CDI) (Kovacs, 1992), a self-rating symptom orientated scale contained 26 items, as Item 9 concerning suicide ideation was omitted. For each item, students rated one of three sentences which described them best for the past two weeks.

Procedure

Teacher ratings: In Term 4, 1994, the *Student Behavior Checklist* and rating of student's achievement in mathematics was completed by 58 teachers in 31 schools.

Student data: In Term 4, 1995, 243 students were administered Form A of the *PATMaths*, using a timed standardised format. Students in Grades 5, 6 and 7 were administered Test 1 or 2, Grade 8 Test 2 or 3 and Grade 9 Test 3. The students then completed the CASQ, *Your Feelings in Mathematics: A Questionnaire* and the CDI.

Results

Calibration of the instruments

Each instrument and the student data were analysed with the Rasch scaling procedure, so that the students' estimated ability or attitude was independent of the items, while the difficulty level of the items was not dependent on the sample of students who took the items (Hambleton, 1989). The items and the persons were thus brought to common interval scales. The *PATMaths* had been Rasch scaled previously (ACER, 1984).

The item response model employs the notion of an inherent latent trait dimension (Weiss & Yoes, 1991). This criteria of unidimensionality had been met in the construction of the CASQ and the CDI. Unidimensionality of the *Student Behavior Checklist* was established with confirmatory factor analysis of a one factor, two factor, hierarchical and nested model (Yates and Afrassa, 1995). Acceptance of the one factor model indicated a scale of *Academic Behaviour,* as there was no evidence to support the two separate factors of *Learned Helplessness* and *Mastery Orientation.* From the factor analysis of *Your Feelings in Mathematics: A Questionnaire* two separate scales of *Task Involvement* and *Ego Orientation* were established.

The final Rasch scaled instruments were composed of only those items with infit mean square values in the predetermined range of 0.83 and 1.20. From the 24 items in the *Student Behavior Checklist,* six learned helplessness items and four mastery items measuring teachers' perceptions of effort, motivation, reaction to failure and persistence fitted the Rasch scale (Yates and Afrassa, 1995). All 48 items of the CASQ were retained (Yates and Afrassa, 1994; Yates, Keeves & Afrassa, 1997), while the final *Task Involvement* scale contained 12 items, the *Ego Orientation* scale five items (Yates & Yates, 1996; Yates, 1997) and the CDI 20 items.

Rasch scaled scores were then estimated for each student for the teacher ratings as well as the self-report measures. Students' scores for the *PATMaths* were placed on a single Rasch scale irrespective of whether they took Test 1, 2 or 3.

The Relationships of Teachers' Ratings to Student Achievement in Mathematics
In Table 2 significant correlations were evident between the teacher ratings of achievement and classroom behaviour and between both of these variables and student achievement one year later. Teacher ratings of achievement were also significantly correlated with subsequent student task involvement. However, with direct entry multiple regression, only the teachers' ratings of achievement in the previous year were found to be significantly predictive (see Table 3). Students' task involvement and explanatory style was also significantly related to their achievement in mathematics.

The Relationship between Teachers' Ratings and Students' Motivational Indices
There were significant correlations between the teachers' rating of both mathematics achievement and academic behaviour and the subsequent measures of students' depression, task involvement and explanatory style (see Table 4). When these variables were examined with multiple regression (see Table 5), the teachers' prior ratings of academic behaviour were predictive of subsequent student self-reported depression at the ten per cent level of significance. Students' self reported task involvement and explanatory style were also significantly related to depression.

Table 2 Correlations with students' achievement in mathematics

	Variables	2	3	4	5	6
1	1995 Mathematics achievement	**-0.40****	**0.33****	**0.13***	-0.08	-0.07
2	1994 Tch. rating of maths ach.	-	**-0.68****	**-0.14***	0.05	-0.11
3	1994 T. rating of classrm behav.		-	0.08	-0.04	0.07
4	1995 Student Task involvement			-	**0.26****	**0.34****
5	1995 Student Ego orientation				-	0.09
6	1995 Student Explanatory style					-

N = 243, ** $p < 0.001$, * $p < 0.05$

Table 3 Regression analysis: predicting mathematics achievement by teacher ratings, motivational orientation, and explanatory style.

Variable	r	Beta	t	Sig t
1994 Teacher rating of maths achievement	-0.40	-0.32	-3.97	**0.00**
1994 Teacher rating of academic behaviour	0.35	0.11	1.34	0.18
1995 Student Task involvement	0.13	0.16	2.43	**0.02**
1995 Student Ego orientation	-0.08	-0.08	-1.40	0.16
1995 Student explanatory style (CASQCT)	-0.07	-0.16	-2.57	**0.01**
N = 243 R = 0.45 $R^2 = 0.20$				

Table 4 Correlations with students' self reported depression

		2	3	4	5	6
1	1995 self reported depression	**0.16***	**-0.18****	**-0.28****	-0.01	**0.36****
2	1994 Tch. rating of maths ach.	-	**-0.68****	**-0.14***	0.05	-0.11
3	1994 Tch rating of academic behav.		-	0.08	-0.04	0.07
4	1995 Student Task involvement			-	**0.26****	**0.34****
5	1995 Student Ego orientation				-	0.09
6	1995 Explan. style (CASQCT)					-
N = 243, ** $p< 0.001$, * $p < 0.05$						

Table 5 Regression analysis: predicting student self reported depression in 1995 by teacher ratings, motivational orientation, and explanatory style.

Variable	r	Beta	t	Sig t
1994 Tcher rating of maths achievement	0.16	-0.00	-0.03	0.97
1994 Tcher rating of academic behaviour	-0.18	-0.15	-1.81	**0.07**
1995 Student Task involvement	-0.28	-0.19	-2.86	**0.01**
1995 Student Ego orientation	0.01	0.06	0.10	0.32
1995 Student explanatory style (CASQCT)	-0.36	-0.29	-4.67	**0.00**
N = 243 R = 0.43 $R^2 = 0.18$				

Discussion of the variables

Teachers were asked to rate their perceptions of students' motivation, effort, persistence, and reaction to failure in their mathematics classroom. One year later these ratings significantly correlated with students' perceptions of their own approach to mathematics as indexed by their task involvement, as well as their self reported optimism, pessimism and depression. In developing the *Student Behavior Checklist*, Fincham *et al.* (1989) highlighted the need to tap teacher perceptions of learned helplessness as a means of either supplementing or replacing student self report measures. While teacher ratings of academic behaviour were not predictive of subsequent achievement, they were important indicators of depression, which in turn was associated with students' explanatory style and task involvement in mathematics.

The correlation ($r = -0.40$, p , 0.001) between the teachers' subjective rating of achievement in mathematics with the objectively measured achievement on the

PATMaths one year later was surprisingly strongly predictive, suggesting that teachers' expectations, as indexed by this single rating, predicted achievement over time (Jussim & Eccles, 1997). While the correlation between teacher rating of achievement and subsequent student task involvement was weak ($r = -0.14$, $p < 0.05$), it nevertheless was important as students' belief that effort, persistence and engagement in the learning enterprise would lead to success in mathematics, was significantly related to their concurrent achievement in mathematics.

Implications for the Psychology of Mathematics Education

Perceptions of success and failure in the mathematics classroom affect both teachers and students alike. For many students, how they perceive the causes of their success and failure has a decisive effect on their attitude towards and engagement in mathematics (Kloosterman, 1988), but teachers' perceptions of students' attributions have yet to be investigated. In this study, teachers' rated students' overt behaviour in the classroom, but the question of whether the *Student Behavior Checklist* actually measured "helplessness" in a manner independent of actual student achievement was unanswered. Although the ten items possessed acceptable psychometric properties, there was no way of knowing if teachers' conceptualisations of "helplessness" reflected the internal motivational processes occurring in students some time later. Whether students were influenced by competitiveness in the classroom was also essentially unanswered as the ego orientation scale was inadequate. As this study used questionnaires and an objective test of achievement, interviews with both teachers and students and actual measurement of these variables in the classroom would be useful.

In their day to day interactions with students, teachers make judgments about both their classroom demeanour as well as their achievement. In this study, teachers' perceptions were predictive of subsequent student achievement and depression, with the results suggesting a subtle interplay between student achievement and classroom behaviour. It was not clear to what extent teachers' single subjective ratings of achievement were influenced by students' achievement, academic behaviour or both. How teachers form their judgments of achievement in relation to student behaviour in the classroom is clearly an important area of research, particularly for low achieving students who exhibit characteristic learned helplessness traits.

This paper has focussed on only a subset of a more complex design. Future reports will examine the impact of students' earlier achievement levels, measured two years prior to the present data set. Towards this work, path analyses are being carried out, along with the analysis of gender, grade level and school site.

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STUDENTS' UNDERSTANDINGS OF THE ROLE OF COUNTER-EXAMPLES

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The present study examines students' understanding of the notion of counter-example and ways in which students use counter-examples. In particular, the study reveals difficulties students encounter in using counter-examples. Two hundred and four students participated in the study. The students were top level 9th and 10th grades students from four different schools.

Counter-examples play a significant role in doing mathematics and have influenced some major developments of mathematics (Lakatos, 1976; Rissland, 1978, 1991). Yet, examining mathematics text books for secondary school and being familiar with the common practice, it seems that very little attention is given to facilitating students' use and understanding of counter-examples and their special role in disproving mathematics statements.

In the last decade, an increasing emphasis has been given to facilitating students' ability to reason mathematically (NCTM, 1989). Students are encouraged to make conjectures, to gather evidence, and to build arguments to support or refute their conjectures. In order to build such arguments, students need to appreciate the role of counter-examples in refuting false mathematical statements and to be able to use counter-examples in the process of establishing the truth of mathematical statements.

There is rather little empirical evidence regarding students' understanding of the role of counter-examples in the process of rejecting conjectures, and their ways of generating and using counter-examples in mathematics. These issues are mostly addressed in connection to understanding the notion of proof, its universal nature, and the role of examples in proving (Fischbein, 1982; Vinner, 1983; Chazan, 1993; Schenfeld, 1991). Many students are not convinced by a counter-example and view it as an exception that does not contradict the statement in question (Galbraith, 1981; Harel & Sowder, in press). Ballacheff (1991) identified six kinds of ways in which students treated counter-examples. Zaslavsky and Peled (1996, 1997) observed some difficulties encountered by mathematics teachers and pre-service teachers in generating counter-examples.

Objectives

The study was designed in order to answer the following research questions:

1. To what extent do students understand the special role of counter-examples in refuting false mathematical statements?
2. To what extent do students try out of their own initiative to suggest counter-examples to false mathematical statements with which they are presented?
3. To what extent do students succeed in generating correct counter-examples to false mathematical statements?
4. What are the difficulties students encounter when refuting false mathematical statements?

Method

One hundred and fifty students participated in the study. The students were top level 9^{th} and 10^{th} grades students from four different schools. Prior to the study, all the students who participated in the study had an opportunity to use counter-examples in order to refute false mathematical statements and to discuss their role.

Most of the data was collected through written questionnaires. In addition, a number of interviews were conducted in the process of developing the research instruments, as well as a number of classroom observations.

Research Instruments

The main research instrument consisted of two written questionnaires, each comprised of five parts. The two questionnaires had the same structure, and differed only in content. While one focused on algebra the other focused on geometry. Each student was given at random only one questionnaire. The five parts of the questionnaire were given to students separately -- one by one. Table 1 describes the structure of the questionnaire.

Table 1: The Structure of the Questionnaire

Part	Section	Given	Task	Goal
A		Six mathematical statements (four false and two true).	To identify the two false statements and to explain why they are false.	1. To check the frequency of cases in which students reason with and/or use counter-examples out of their own initiative; 2. To identify characteristics of students' attempts to generate counter-examples.

Table 1(continued): The Structure of the Questionnaire

Part	Section	Given	Task	Goal
B	preface	A false mathematical statement followed by descriptions of three students examining the truth of the statement		
	1	A response of a student who claims that the given statement is true, supporting his claim by specific example(s).	To determine whether the student is right, and to explain why.	To identify students who accept a special case as evidence (or proof) that a statement is true.
	2	A response of a student who claims that the given statement is false, supporting her claim by a counter-example.	To determine whether the student is right, and to explain why.	To identify students who accept the use of a counter-example for refuting a false mathematical statement.
	3	A response of a student who agrees that the given statement is false, but does not accept a counter-example as sufficient evidence, thus, suggests a different argument.	To determine whether the student is right, and to explain why.	To identify students who do not accept a counter-example as sufficient evidence for refuting a false mathematical statement.
C		A false mathematical statement followed by an assertion that it is false and by descriptions of four students trying to explain why the given statement is incorrect.	For each explanation, to determine whether it is valid, and to indicate the most convincing explanation among those that were considered valid.	To investigate the kinds of arguments that students find convincing, with reference to the use of counter-examples.
D	1-8	Statements expressing conceptions of and attitudes regarding the role of counter-examples.	To mark the extent of agreement with each statement.	To identify students (declarative) conceptions of and attitudes towards the use and role of counter-examples .
E	1-4	Four false mathematical statements.	For each statement, to generate various counter-examples and justify why, indeed, each one is a valid one.	To examine what kinds of counter-examples students produce when explicitly requested to.

Findings

In this paper we present in detail the findings for part A of the questionnaire and refer to findings for other parts of the questionnaire only in a general way.

In the analysis of students' responses, we use the term "incorrect counter example" for an example that was given as a counter example although it does not satisfy the conditions of a counter-example. For example, if a student gave a square as a counter-example to the statement "If the diagonals of a quadrilateral are equal in length, the quadrilateral is a rectangle", the response was classified as an attempt to give (or use) a counter example, or as a response including an incorrect counter example. In this context, a square is considered an incorrect counter-example because it is a special case of a rectangle, and the fact that in a square the two diagonals are equal, does not contradict the statement.

Even when only a hint was made to the fact that there exist counter-examples, the response was classified as making reference to a counter example. Thus, for the above statement, the following response was classified as making reference to a counter-example without actually using a counter-example: "The (above) statement is false because there are other quadrilaterals with two equal diagonals that are not rectangles".

Table 2 presents the distribution of types of responses to part A of the questionnaire. The findings of part A indicate a difference in responses depending on the specific statement. For example, in statement #2, 96% of the students who determined that it is false, justified their claim with reference to counter-examples, while in statement #4, only 17% of the students who determined it was false made reference to counter-examples. Interestingly, for both geometric statements much more reference was made to counter-examples than for the two algebraic statements. This finding supports Perkins and Salomon's view (1989) that the ability to produce counter-examples depends a lot on the context. Findings from other parts of the questionnaire suggest more kinds of differences between geometry tasks and algebraic tasks with respect to the role of counter-examples. Students were much more persistent in their view that a counter-example is sufficient for refuting a geometric statement than for an algebraic one.

Another interesting finding has to do with the fact that for all four false statements there were at least 33% of the students who were not able to determine that the statement was false. It seems that for these students, the reluctance to use counter-examples impeded their ability to correctly determine the validity of the given statements.

Table 2: Types of Responses to Part A of the Questionnaire

The False Statement:	#1	#2	#3	#4
	If in two quadrilaterals all sides are respectively congruent, the quadrilaterals are congruent	**If the diagonals of a quadrilateral are equal in length, the quadrilateral is a rectangle**	**The function $f(x)=x^2+2x-2$ is an odd function**	**The function $f(x)=-x^2-2x$ decreases in its domain**
Types of Responses (N=150)				
Did not assert that the statement is false	81 (54%)	50 (33%)	49 (33%)	68 (45%)
Asserted that the statement is false, with:	69 (46%)	100 (67%)	101 (67%)	82 (55%)
No justification	4	1	-	1
Correct justification without a C.E.	-	-	9	46
Partial justification without a C.E.	38	3	52	5
Incorrect justification without a C.E.	-	-	19	16
Justification with some reference to a C.E. (without giving a C.E.)	1	10	-	1
Incorrect reference to C.E.: treating an example that is not a C.E. as if it were	2	63	-	-
Correct use of C.E.	24	23	20	13

Figure 1 depicts the distribution of students' responses to all four false statements in part A. It is interesting to note that in the context of examining these four false mathematical statements, only 10% of the students correctly used counter-examples in more than one case. Two thirds of the students either did not find it appropriate to use counter-examples or were not able to correctly use counter-examples for any statement.

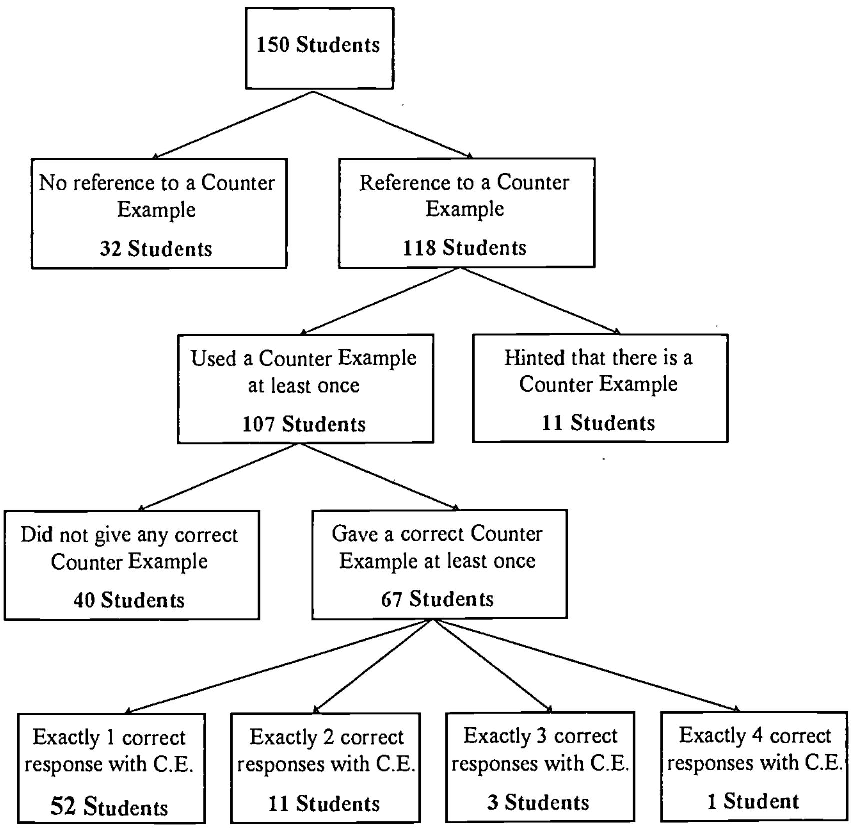

Figure 1: Students' responses to the four false statements in Part-A Task

Findings from other parts of the questionnaire suggest that many students who accepted a counter-example as sufficient evidence for refuting a mathematical statement were not able to distinguish between an example that satisfies the conditions of a counter-example and one that does not satisfy them. Thus, some accepted (or used) a counter-example to the converse statement as if it were a counter-example to the given statement. Others accepted (or used) an example that satisfies the statement as if it contradicts it. These findings resemble some of Zaslavsky and Peled's (1996) findings, in which mathematics teachers and student teachers generated incorrect counter-examples for a given statement.

Another phenomena that was found in the current study has to do with the way students generate counter-examples. In the process of generating counter-examples, students were trying to force many conditions upon an example. Consequently, they came up with "examples" that do not exist, and were not aware of it. For example, students suggested a triangle in which the sum of the lengths of two sides was not larger than the length of the third side. Similar findings were found by Peled and Zaslavsky (1997) for mathematics student teachers.

Discussion and Conclusion

The current study deals with many aspects of understanding the role of counter-examples. There is the logical aspect of understanding that has to do with accepting one counter-example as sufficient evidence for refuting a mathematical statement. This aspect is closely related to what Harel and Sowder (in press) call empirical proof schemes. Students' understanding of the role of counter-examples is influenced by their overall experiences with examples. The status of a counter-example is so powerful compared to the status of other examples. While one counter-example is enough to draw very definite conclusions, several supporting and verifying examples do not suffice. No wonder students often feel that a counter-example is an exception that does not really refute the statement in question.

Another aspect of understanding the role and use of counter-examples has to do with the conditions that a counter-example must satisfy. Many students who seem to understand the special role and status of counter-example, are not able to generate a correct counter-example. In an attempt to generate a counter-example they either give an example that does not satisfy the necessary conditions in order to qualify as a counter-example or an example that is impossible, i.e., does not exist.

The students who participated in the study were top level students. It is reasonable to assume that they have encountered as much experience with proving and refuting as the current secondary mathematics curriculum offers. The findings of this study, raise several concerns regarding the kinds of learning environment and experiences to which students are exposed. The differences in favor of geometry context, which is usually associated with more conjecturing and proving than other topics, may suggest that more emphasis on logical reasoning in other topics could improve students' understanding of the role of counter-examples in proving and disproving mathematical statements.

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SHORT ORAL COMMUNICATIONS

QUO VADIS? — THE "AC-ARROW" IN DATA-STRUCTURES LEARNING COMPARED TO MATHEMATICS LEARNING

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Abstract thinking is usually considered "superior" to concrete thinking. Researchers like Sfard (1991), support this approach also in mathematical thinking. We may consider then the metaphor of **an "AC-arrow"** lying between the Abstract and the Concrete, which in mathematics points towards the abstract from the concrete.

Data-Structures (DS) is an important topic in Computer-Science (CS). Now, "Quo Vadis DS?" What is the direction of the AC-arrow while learning DS? Surprisingly enough, despite the face-resemblance between DS and mathematical thinking, we argue that there are differences in the direction of the AC-arrows of the two. Note, that there is a subtle difference between the notion of abstraction in these domains (Leron, 1987): While in mathematics it usually means, "creating a higher-leveled mathematical element", in CS it usually means, "disregarding implementation".

The authors found different cases of relationships between DSs, which dictate different AC-arrows: If DS's implementation is not considered, they may be learned independently because the **abstract DS are isolated**. Otherwise, learning may perhaps progress most fruitfully following multiple AC-arrows resulting from "**DS abstraction-hierarchy through implementation**" — a global arrow and local ones. The first is pointed from the DSs which are more concrete (through implementation) to the more abstract ones, and for each DS a local arrow points from the DS's abstract face to its implementation. Locally, each abstract DS is first studied independently of other DS (unlike in mathematics). An example will be given in the presentation.

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THE RELATION BETWEEN ALGEBRA STUDENTS' MATHEMATICS ANXIETY AND THEIR COLLEGE ENTRANCE EXAMS

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Because of the high stakes of admissions testing, educators are concerned about equity issues surrounding college entrance exams. Aptitude measures and performance assessments do not account for all of the variance in achievement. Previous studies have found an inverse relation between anxiety and performance; students who score high on anxiety measures tend to perform relatively poorly on various ability tests. This paper presents the relations between mathematics anxiety and student performance on the Scholastic Aptitude Test-Mathematics Reasoning (SAT-M) among Algebra II high school students.

The study's sample consisted of 680 Algebra II General, Algebra II Regular, and Algebra II Honors students. The Mathematics Anxiety measure utilizes a five question, five-point Likert scale. It is designed to measure students' feelings of insecurity, concern, and enjoyment while doing mathematics. This self-reporting instrument is similar to ones used in the Second International Mathematics Study (Kifer & Robitaille, 1989).

This study found that students who reported high Mathematics Anxiety scored lower on the SAT-M. Mathematics Anxiety scores decreased as Class Type (General, Regular, and Honors) difficulty increased. The most compelling conclusions that can be drawn relate to the magnitude of effect sizes in this study. The effect size of Class Type was 2.18 and the effect size for Mathematics Anxiety was 0.66. Surprisingly, the effect size for Gender was only 0.16.

Results of this study will be shared and the findings summarized to help educators enhance mathematics performance on tests critical to students' careers.

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Graphic Calculators: An Interpretative Framework

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Much literature on the graphic calculator *implicitly* makes claims for the power of the technology in terms of its amplification or cognitive re-organisation effects (Pea, 1993). I submit that a Vygotskian approach to learning, with its emphasis on *mediated* activity within a particular sociocultural context, is useful for *explicitly* interpreting the relationship between the mathematical learner and the different sign systems of the graphic calculator. Within the Vygotskian framework, Pea's distinction between amplification and re-organisation can be usefully understood in terms of a (short-term) increase in the size of the zone of proximal development or in terms of a (long-term) restructuring of human activity respectively. Furthermore and in line with a Vygotskian approach, I maintain that it is essential to situate any interpretation of the effects of the technology within the sociocultural framework so that "ways in which mind reflects and constitutes a specific sociocultural setting " can be identified (Wertsch, 1990: 71).

This theoretical framework is used to interpret qualitative data from a study of first-year mathematics students in a South African university using the graphic calculator as an *add-on* tool. In this study, the effects of the graphic calculator were primarily amplification effects. I argue that the particular sociocultural context, in which the role of the graphic calculator was *not* integral to the course, resulted in less conscious reflection (Vygotsky) or mindfulness (Salomon, 1990); as a result the effects of the technology were short-term and there was little evidence of cognitive re-organisation in the students.

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ENCOURAGING EVERYDAY COGNITION IN THE CLASSROOM

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This research investigates what kinds of lesson structures encourage students to build from their everyday mathematics knowledge. A team of researchers and teachers designed a four week introduction to algebra for beginning secondary students that emphasized a meaningful thematic situation to encourage the use of everyday knowledge. Over the course of the unit students examined different characteristics of three pizza businesses in order to choose a pizza provider for their school cafeteria. Written posttests demonstrated that students who participated in the experimental curriculum were better able to solve algebraic word problems and to use multiple representations for understanding functional relationships when compared to students who received traditional pre-algebra instruction (Brenner et al. 1997). This presentation focuses upon the classroom processes that may have contributed to the students' learning outcomes.

Four teachers and their students were videotaped regularly as they did the lessons in the experimental unit. It was found that teachers used three different kinds of lesson structures. When a teacher used a traditional lesson structure, s/he modified lessons to make them more like a textbook by prescribing methods or goals that had been left open-ended in the original curriculum. In conforming lesson structures, teachers followed the lessons as presented in the written materials. In contextualized lessons, teachers added their own examples and created more opportunities for students to use everyday mathematics.

Students were quite sensitive to these differences in lesson structures. When traditional structures were used, the students focused upon finding numerical answers and rejected their peers' attempts to bring any everyday knowledge into the problem solving process. This effect was also seen when teachers used the conforming structure, although the materials were written with the intent of encouraging the inclusion of everyday knowledge. It was only when teachers contextualized their lessons that students actively engaged in everyday reasoning, and this in turn enhanced their ability to answer the problems presented in the curriculum. Interestingly, each teacher used a variety of lesson structures and, as a result, students engaged in widely varying amounts of everyday problem solving over the course of the unit.

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THE RATLOR EFFECT

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Curriculum 2005 brings to South Africa changes in assessment practices. Assessing diverse abilities with diverse methods is meant to provide a truer picture of student capabilities. But there is something in some assessment practices that undermines student confidence to such an extent that the student fails to achieve and even gives up trying to pass. We need to understand this process in order to guard against these more comprehensive assessment practices resulting in a more extensive lack of confidence in some pupils. Coining the term the Ratlor Effect to refer to the systematic undermining of pupil confidence by the teacher (conscious or unconscious), I draw from a case study in "extra-lesson" teaching to explore the importance of confidence in the classroom. Confidence is conceptualised as the belief that the student can "learn to do". Here I refer to the director systems developed by Skemp (1979). Using an extension of Vygotsky's Law of Cultural Development (1987) I argue that confidence develops in social relationships, and hence there is the possibility for social relationships to impede confidence. By emphasising the relationship between confidence and control, I argue that the teacher needs to understand the student's perspective on the assessment processes, and where assessment practices serve to hinder learning, to stop using them.

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WHERE WAS THE MATHEMATICS?

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In a study of U.S. high school mathematics teachers implementing a standards-based curriculum, Brombacher (1997) found that teachers believed that they had made profound changes in their teaching of mathematics as a result of implementing the new materials. Factors that contributed to the teachers' positive experience of the transition included extensive in-service training, the creation of time for reflection on the transition—usually one fewer class to teach in the first year of implementing a year of the course—and a range of support provided by both the materials and local program co-ordinators. Aspects of the transition that teachers found uncomfortable included dealing with a perceived shift from student mastery of content to a more general understanding of content by students, an increase in teacher workload resulting from a shift in assessment practices and the teachers' inability to grasp the "big-picture" of the curriculum without having taught a number of years of it.

One striking finding of the study was the difficulty teachers experienced in trying to describe the mathematics in both the earlier curricula they had taught and in the standards-based one. When pressed teachers described the mathematics of the earlier curricula as traditional, boring and irrelevant, features they considered to be in stark contrast with the standards-based curriculum. Asked to describe the mathematics in the standards-based curriculum the teachers said that it was more complex than the content of previous curricula, that some of the topics were previously only taught at post-secondary level and that it was "more difficult." These perceptions regarding the mathematical content seemed to satisfy the teachers that the standards-based curriculum was not only mathematically sound, but in fact superior to the previous curricula.

A possible explanation for the teacher's inability to describe the mathematics of the curricula is that the teachers in the study did not regard mathematics as being something that was under their control. Earlier curricula and the new curriculum, "prescribed" the mathematics to be taught leaving to teachers decisions regarding the presentation of the content and management of the classroom. One implication of this finding which could profitably be researched further is whether or not any reform of mathematics that is communicated to teachers through materials developed by "others" can ever achieve the visions of the reformers.

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DEVELOPING A MATHEMATICAL COMMUNITY THROUGH INVESTIGATIVE WORK

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In mathematical investigations students work like mathematicians. They explore and try to understand some phenomena, gather and organise information, formulate conjectures, test and eventually prove some of them. The students need to communicate and defend their ideas as well as listen and think about those proposed by their colleges. However, in order to do this, they need to develop habits and disposition, besides having some knowledge and competencies. In fact, Pólya pointed out that there are "moral qualities" required to do mathematics such as intellectual courage, intellectual honesty and wise restraint. In short, this means that we must be able to revise and change our beliefs, but only in result of a good reason. We also know from the work of Piaget and others that construction of knowledge takes place when one reflects on one's own ideas and actions, as well as on the ideas and actions of others. All this points to the social character of learning, calling our attention for the importance of the interactions between students and between teacher and students.

In this presentation, I analyse some of these aspects as they show up in a classroom episode. We will see an ordinary 9th grade class engaged in a collective discussion, where some of the students present the results of their work in a mathematical investigation. My analysis focus on the role of the teacher, the way she organises the work, the rules she sets, and also the kind of support she offers to students troughout the activity and during the communication of ideas and conclusions. Besides that, it is necessary to analyse the roles that students take, how they present their ideas and how they argue them, and also what kind of mathematical reasoning they engage. Also important is to discover some connections between the way the teacher acts and the way students behave towards the investigation. What kind of roles the teacher may play which will improve student's performance in the investigation? How should that role depend on the student's difficulties and knowledge of other sorts?

Some preliminary results from this research suggest that the way the teacher acts makes a strong influence on pupils' attitude towards the investigation. If they ought to feel the investigation as their own, the teacher has to balance his or her support between giving them enough help, so they are not lost, but avoiding to withdraw their responsibility. That is very clear when students move on to prove conjectures. Also the explicit mathematical reasoning that teacher engages while trying to understand students' ideas plays an important role. As a result, the students feel their work is valued. And, what is more important, they stand as an apprentice watching as someone more competent engage in mathematical activity, which is essential for their learning to mathematics.

HOW DO CHILDREN COMPARE THE SIZES OF PLANE SHAPES?

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A first exploratory study was made on the potential of some activities for the collection of information on strategies and justifications, used by children in the comparison of sizes of plane shapes. We then selected 11 activities, which were used in interviews with 15 children in 2 villages in Inhambane province. 13 of the children were primary school pupils from grades 1 to 6; 2 of the children did not attend school.

During the interviews we used drawings of plane shapes on the one hand and cut out shapes from cardboard on the other, in order to leave the children the option of using one form or the other. Some of the shapes were polygons, others had more irregular shapes; some were divided in different units of area.

Through these activities we wanted to conduct a survey of the ideas of area and size which are used by the children. E.g., when a child says that one shape is bigger than another, is it comparing areas or other magnitudes, like the length of sides or the perimeter?

During the interview, the children had at their disposal the following materials: ruler, string, razor blade, scissors, compasses, pencil, paper and a protractor.

Some results

We found that the children used the following strategies for the comparison of sizes: direct visual perception, superposition, counting of area units, measuring with the ruler and decomposition/composition of shapes.

The study shows that the length of the sides or the border are interfering negatively in the idea of area of a shape. In some cases children considered a shape as bigger, because it was longer and used the length in order to justify their answer; these children used a ruler or direct visual perception. At least one child explained that it had compared the length of the border, using a string: a greater length found on the string corresponded to a bigger shape. Some children changed their strategies: from comparing areas in one case to comparing the length of the border or of some sides in other cases. In some cases they even used one magnitude for comparing, and another for the justification.

In some cases it was not possible to identify the procedures which were used; possibly because of difficulties in expressing themselves, when asked to explain "How do you know that this shape is bigger?" Interviewing was done in Portuguese, i.e., the language used in schools but not the mothertongue of the majority of the children.

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FROM WOVEN PATTERNS TO NUMBER PATTERNS BY EXPLORING A WEAVING BOARD

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The paper reports on a recent study undertaken with pupils of grade 11 and 12, aiming to improve their skills in reasoning. The pupils final task was to formulate and to generalise conjectures related to the woven patterns vs. number patterns, either by common words or by mathematical symbols, working with weaving boards.

A weaving board is a simple kit to perform weaving. It consists of a frame of cardboard strips attached to a stiff cardboard, and of loose strips to be interwoven with those of the frame. The border of the stiff cardboard is numbered to make the description of the path easier.

By exploring a weaving board I mean the heuristic process in which pupils were involved to find by themselves (or somehow guided to find) any regularity in the woven patterns.

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LINKING HOME AND SCHOOL: IN PURSUIT OF A TWO-WAY MATHEMATICAL DIALOGUE[1]

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In this paper I describe a research project that has as one of its main goals the development of mathematics instructional innovations in classrooms composed of predominantly minority working-class students. What are the implications for the mathematical education of these children, if we take their experiences and backgrounds as resources for learning in the classroom? What may the learning environment look like if we develop a more participatory approach towards their learning of mathematics (similar to what these children experience in their out-of-school lives)? How can we develop a teaching innovation which successfully combines in-school and out-of-school mathematics?

Our research model is based on a sociocultural approach to education and has four inter-related components: 1) Household Visits: the teachers visit the homes of some of their students to <u>learn</u> about the funds of knowledge in these households. By seeing and learning about their students' experiences, they develop a firsthand understanding of such experiences, as opposed to being told generalities about the "minority culture." 2) Teacher-Researcher Study Groups: here is where the pedagogical transformation of the findings from the household visits takes place. These sessions have led us to a constant examination of what we mean when we say that a certain activity or practice has mathematics in it. They also include actual mathematical investigations. 3) Classroom Implementation: the project tries to adapt to the teachers' different interests and needs. For example, one of the teachers wants to explore whether "rigorous" mathematics can be developed from everyday mathematics. She has developed a garden theme grounded on students' ideas and experiences, that has led her to explore mathematics in ways she had never done before (e.g., measurement and graphing). 4) Parents as Learning Resources: through regular workshops with a core group of Spanish speaking, working-class mothers, we are developing a "two-way" dialogue to learn from each other. On one hand we learn about their everyday uses of mathematics as well as their beliefs about mathematics in particular as they relate to their children's education. On the other hand, we engage them in mathematical investigations similar to those that their children are likely to do at school.

Several bodies of literature inform our work including research on the social construction of mathematics, studies on in-school and out-of-school mathematics, and research in critical mathematics education. Key questions that apply to our work (both, with the students and with the parents) are: what kinds of mathematics can we extract from their experiences and practices? and how can we relate them in a truthful manner to the content and the ways of school mathematics? How can we develop learning situations that while capitalizing on their experiences outside school allow them to "advance" in their academic mathematics? Extracting the mathematics from the everyday practices can be particularly difficult -- in part due to our views about mathematics but also to our knowledge of the subject.

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UNDERSTANDING THE MATHEMATICS TEACHER ELEVEN YEARS ON.

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In 1987 Charles Desforges and I wrote *Understanding the Mathematics Teacher*. In it we discussed a range of constraints operating within the classroom environment which, in our view, limited the development of pupils' higher order thinking skills. Since then numerous student teachers at the University of East Anglia have been introduced to our research and some of the possible implications for classroom practice. Additionally, over the years in the U.K., there has been an increasing emphasis on the importance of process - rather than simply product - in mathematics. Such factors may, or may not, have major impacts on classroom practice depending on the extent that all - or any - of the following are applicable:

a) it is no accident that methods of instruction have gradually evolved making change difficult (Greeno, 1980)
b) new teachers generally reject their university training within a semester of qualifying (Wehlage, 1981)
c) schools tend to operate a production line model stressing products rather than processes (Marshall, 1988)

This research report will focus on a study of twelve of my former students. It will consider their thoughts and actions in mathematics sessions and discuss the influence of their training and the impact of recent Government initiatives on their classroom practice. Data from interviews with their pupils will also be presented as will details of session observations. More specifically,

1) the children's knowledge of the purpose of mathematics and their higher order thinking skills will be explored.
2) the balance between mathematical processes and products within each classroom setting will be examined.

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The Third International Mathematics and Science Study: The Role of Language for South African students

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The Third International Mathematics and Science Study (TIMSS) was conducted in 1995. Five grade levels were tested in about 50 countries. South Africa was the only country in Africa that took part in the testing and did so for grades 7, 8 and 12. South Africa was rated lowest of the 41 countries that participated at the grade 7 and grade 8 levels with average scores about 30% lower than the international average. Previous comparative studies that were done by the IEA (International Association for the Evaluation of Educational Achievement) had an impact on educational policies of many countries. It is of great importance for South Africa to analyse the results of TIMSS in detail. This paper focusses on mathematics performance of South African students at grade 8 level.

Item Response Theory enables the ordering of questions from most difficult to easiest, even though students might have been tested on different selections of questions out of a bank of questions. A graph representing the difficulty values of the items, shows that the average scores of South African students do not follow the same tendency as the average international scores. South African students scored lower on each and every item, but not in a consistent way.

The fact that more than 75% of the South African students wrote the tests in a language other than their first language was investigated by the researcher as a possible reason for the above mentioned deviations. South African students who wrote the tests in their first language significantly outperformed students who wrote the tests in a second or third language. Item analysis showed that for quite a few of the questions, using a minimum of language (words), the South African students scored higher than the South African average. South African students scored very low on some of the easier questions, possibly due to difficult language in the instructions. Frequencies on the different distractors furthermore indicate that guessing played a considerable role. Some serious misconceptions among South African students are also evident from the item analysis.

WHICH ARE TEACHERS' PROFESSIONAL KNOWLEDGE CONCERNING TEACHING THROUGH A MATHEMATICAL INVESTIGATION?

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The recent trends about school mathematics have insistently pointed the development of higher order skills, namely those associated with reasoning and problem solving, as a major goal for students. On the other hand, the research in the area of mathematics education has been suggesting that such goal require a deep change in the nature of the learning activities in the classroom. For this purpose, tasks involving mathematical exploration and investigation carried out by the student themselves can play a significant role.

However, to develop this kind of task is far from being an easy work for the teacher. It is necessary not only to have adequate materials disposable to the teacher, but also to understand possible ways to use them in the classroom in order to achieve the intended objectives. It is absolutely indispensable to understand the key factors that permit the development of an appropriate teachers' role, necessary not only for today but also for the teachers' training in the future. Questions as these ones have to have an answer: How the teacher' s professional knowledge, developed in others classrooms' contexts, is able to answer this new challenge? Are they components of the professional knowledge that can be transfer? Are they new ones that only can be create in these new situations? Which are the most significant difficulties that the teacher has to face? Is the number of years of practice a facilitator aspects or is not influencing the new role of the teacher?

This paper presents some preliminary results concerning the study of two different mathematics' teachers. Both of them worked with students with 15-16 years old. The preparation of the lessons where the students will work on a mathematical investigation had been made in collaboration with the researcher and audio recorded. The sequence of these lessons has been observed and video recorded and later transcribed some episodes. Finally, a new session of work has occurred, where teacher and researcher has developed a reflection about the students and teachers' work.

OVERCOMING MATHEMATICS ANXIETY – A CASE STUDY

MARCIA DANIELS

Mathematics anxiety relates strongly to students' tendency to avoid mathematics at high school or college levels (Hembree 1990; Wigfield and Meece 1988). There is thus a need to find ways of helping students overcome their anxiety to mathematics. This case study focuses on the strategies used by a group of first year students at the university of Auckland in New Zealand to overcome their anxiety to mathematics.

In order to determine which students had experienced mathematics anxiety and were able to overcome their anxiety to mathematics, a questionnaire was issued to all the students doing the mathematics paper 445.101. The questionnaire had four sections. 1) Demographic questions. 2) Questions relating to whether they had experienced any phobia or anxiety about mathematics, asking them to describe how they felt or to give an example of this phobia. 3) Whether they were able to overcome their phobia and how they did this. 4) Gave the students a list of items from which to select how they felt about mathematics generally. These were used as prompts during the interviews. Sixteen students were self-selected from their answers to sections two and three. Ten students gave their permission to be interviewed or turned up for the interviews. Unstructured interviews were used in which the questions were open, but still accompanied by an interview schedule. This ensured that the interviews were more focused around the question of how these interviewees had overcome their anxiety to mathematics.

The most dominant theme was that of an overarching motivation. Six of the ten interviewees fell into this category. This motivation was mainly extrinsic and varied greatly from student to student. Two of them required Statistics to complete their degrees and thought this paper was a good way of getting back to facing mathematics again. Two of them saw this as a good paper for scoring good marks to get into medical school or law school. One needed to upskill in his line of work as an industrial pattern maker, and the other wanted to become a teacher and work in his community. The other four interviewees all experienced anxiety to mathematics earlier in their schooling and were able to overcome their anxiety in various ways. An inspiring teacher; a good tutor; a maths book with many visuals and illustrations; a husband who was a tutor in *family maths* where practical examples were used.

One of the things lacking in a maths anxious student is a positive self-image. Strategies used to build up their self image was another theme that emerged during these interviews as these students learned to overcome their anxiety to mathematics.

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SPACE IN MATHEMATICS, ART AND GEOGRAPHY: GRAPHIC REPRESENTATION AND FIGURATIVE COMMUNICATION

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1. The conceptions of space pervade all fields of life, thought, and contemporary art. So they pervade all teaching school subjects, and notably mathematics, art and geography.

2. A Franco-Japanese pluridisciplinary cooperative research project, dealing with the theme "Mastery of space and the learning of geometry", has brought to light the fact that some aspects of the mastery of space are dependent to a certain extent on the cultural and school variables. Within the frame of this project, several didacticians in mathematics, art and geography were able to confront their approaches and interact while studying the students' graphic productions facing problem-solving situations concerning graphic representation, including figurative communication, in any one subject.

3. Representing space by means of drawings raises the question of their readibility and the quality of their production, both for teachers and students. The modeling and transference of geometric, graphic and artistic properties in specific contexts call for a kind of knowledge that is often lacking. Moreover, the specific language produced within each subject is not really understood. It seemed therefore suitable to make the hypothesis that a common work might help overcome this obstacle.

Following pluridisciplinary Franco-Japanese research, a study group was set up (IREM-Université Paris 7) with the aim of building a specific basis of knowledge on the representation of space, including an appropriate glossary.

4. The analysis of various activities concerning space and its representation offered, within in-service teacher training for the three subjects, showed an evolution in the communication established between the people concerned.

5. Does the fact of bridging the gap between the various registers of the three subjects lead the learner towards a synthesis of his or her knowledge? Or is it only a myth?

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THE ROLE OF GRAPHICS CALCULATORS: INITIAL TEACHER PERCEPTIONS

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The coincidence of 1998 as international year of science and technology and the introduction of Curriculum 2005 for South Africa is remarkable. While Curriculum 2005 is seen by most as an important innovation for South African education, it has brought with it challenges of teacher readiness for implementing it. For instance, what technology will be appropriate and viable for South African schools? Berger (1997) points to the ambivalence of evidence for the use of g.c as a learning tool. While this ambivalence cannot be ignored, South African studies (Laridon,1993; Julie, 1991,1993, in Myburgh, 1994) and beyond pointed to the benefits of using g.c in learning mathematics. What perceptions do teachers and educators have about the g.c. and its role in mathematics education? This paper reports on a pilot study that explored teachers' perceptions on the role of graphics calculators (g.c) in the teaching and learning of mathematics.

A two-day workshop was run for 25 teachers, 6 of whom had worked with the g.c in their Wits FDE course. For the rest it was new. The participants were given a questionnaire to complete which probed their perceptions on the g.c.

Teacher responses can be classified into three main categories:

- **G.c can improve pedagogical practice**

This was a significant feature in the responses, as teachers felt the graphics calculator will help them teach a number of concepts at a time, and have student attention and concentration substantially drawn.

- **Link and relate different sections**

While those teachers who first met the g.c at the workshop were more fascinated by the machine and its mechanical features, those who had contact with it in the FDE course, commented more on conceptual issues.

- **Effect on Participation and Cooperative learning**

The teachers see the g.c as a resource to enhance student participation and cooperative learning in the mathematics classroom.

Significant information can be extrapolated. *First,* teachers overwhelmingly see a graphics calculator as an important resource for their practice. *Second*, the three principles highlighted are integral to the critical outcomes of Curric. 2005, suggesting that a g.c can have a positive contribution to the latter. *Third,* useful information beyond just mechanical operations or fascination would probably come with more exposure to and interaction with the tool.

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PUBLIC ATTITUDES TOWARDS MATHEMATICS

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This paper reports findings of a modest investigation into public attitudes towards mathematics. It reports and discusses the range of attitudes as measured in terms of reported liking/dislike using a survey questionnaire given to 544 members of the public. The data is analysed in terms of gender, age band and occupational grouping. Major differences in reported liking of mathematics are revealed among the different subgroups. Some striking findings are summarised as follows:

1. More than half (53.9%) of the total sample reported liking while one third reported disliking of mathematics. These results challenge the widespread view that the majority of the public has negative attitudes towards mathematics.
2. Comparison by age groups and occupational groupings indicates that the youth group [age between 17-20 years old] and the non-mathematics students reported the lowest liking of mathematics. This result causes concern because this group represents the future human resources of the nation, and also represents those members of the public who have most recently experienced mathematics in school. Moreover, negative attitudes towards mathematics (and science) are often cited as contributing factors to the recent decline in recruitment into higher education courses in mathematics, science, technology and engineering noted in the UK and a number of other anglophone countries.

THE INFLUENCE OF LANGUAGE IN ADVANCED MATHEMATICAL PROBLEM SOLVING

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There is evidence that the use of mathematical formalism (representation of relationships, interpretation of expressions, translation from and into everyday language) is a major obstacle to mathematical learning even at university level. Students' problem solving strategies seem to be severely affected by the linguistic form of the statement of the problems. So it is interesting to observe the effect of slight variations of the data or of the statement of the problems. During the fall 1997 a group of 51 freshman computer science students were given a set of problems; each problem was presented in 2 or 3 versions, which were similar as to the structure and the contents involved but different in the statement. In some cases two versions were conceptually equivalent but different only as to the linguistic form (wording, layout, symbolism, É) and the third referred to a particular example of the objects described in the others. For example, one of the problems required to answer to some questions on an integer M, whose description was more formal in version A (e.g.: "there exists an integer p such that M=2p+1") and nearer to ordinary language in version B (e.g.: "M is odd"). Inversion C the (odd) number M= 3^6á5^8á7^5 has been explicitly given. The questions were the same for each version, and required to determine whether a given statement (e.g.: "M+7is a multiple of 21") was true, false or other.

From a standard mathematician's viewpoint, one could expect that the different presentations of the data should not affect students' behavior at all, because the properties involved are the same and students have been taught to translate verbal sentences into equations and vice versa. Nevertheless, people with version A showed fewer efforts to interpret the data and design some strategy, but often performed calculations or formal manipulations. Version B induced the subjects to apply everyday-life reasoning patterns. These different behaviors are more or less successful according to the question to be answered. So students with version A answered better to questions asking for counterexamples, whereas students with version B gave more satisfactory answers to questions asking for complex reasoning similar to everyday-life arguments. In spite of its 'concreteness', version C proved the most difficult one, as a number of students could not select the relevant data from the representation of M they have been given and represent them in some way (more or less formally).

Generally speaking, the influence of the statement or even of the structure of problems on students answers proved to be strengthened by other factors: the custom of working within one representation system only, neglecting the coordination of different representations; the weak use of mathematical knowledge (mainly if not explicitly mentioned in the statement); the non-cognitive attitudes and behaviors that induce students to try to solve a problem without understanding the problem situation or the basic ideas involved.

Title: 5th Grade Students Work on Sequences: Algebraic Thinking

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The purpose of this investigation is to better understand how students at 5th grade develop and deal with algebraic thinking.

This study took place in a 5th grade mathematics classroom in Rio de Janeiro, Brazil. The students were divided in groups of four children each, one group was selected for observation.

The framework was based on Lins (1993) Theoretical Model of Semantic Field presented in PME 95 and 96. Considering that knowledge is a pair (belief, justification) we elaborated a model to analyze the dialogue that raised within that group (Frant, Rabello and Oliveira 1997).

Model A: Different beliefs and different justifications
Model B: Equal beliefs and different justifications
Model C: Different beliefs and equal justifications
Model D: Beliefs become justifications and vice-versa.

In order to promote dialogue, teacher interventions were always questions instead of explanations. The sequence we will discuss in our presentation is 1,5,7,1,5,7,1,5,7...

Teacher asked the students to find the next term in this sequence, the sequence ratio, the 20th term and if it was possible to find a rule for describing any term in this sequence.

The activity was audiotaped and transcribed and students' enunciation were classified. We will discuss a dialogue and its analysis, here is a sample:

I: So what is the 20th term?
S_1: I think it is like that ... you have to add 157 plus 157 plus 157.
S_2: The 20th is 1.
I: Why?
S_2: If the ten will be 1, now I'll see if it works 1,2,3,4,5,6,7,8,9,10,11, 12, 13, 14, 15, 16, 17, 18, 19, 20, it isn't!
S_1: It will be the 5.

It is important to observe that it is not clear when they talk about cardinal or ordinal numbers. We found here a belief and a justification as follows:

B1: The 20th term is 1
J1: The 20th term will be the same as the 10th.

In our taxonomy we developed four types of thinking about sequences: Next term, Relation between Term and Position, General Term and Isomorphism .of Object Structure. We observed that S2 made a relation among the 20th term and its position with the 10th term and its position.

We conclude observing that listening to students beliefs and justifications can provide a richer environment for learning than listening only for answers.

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Diagnostic teaching with computers in the mathematics classroom

- the role of teachers and support material

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A few years ago I investigated the use of computers in mathematics classroom of students of age 11 - 14, with the aim of supporting a diagnostic teaching method (Fuglestad, 1996a; Fuglestad, 1996b). The students used computers in several topics during the year, but the topic of decimal numbers was the main focus of my study.

In order for the students to develop their concepts of decimal numbers it is necessary to reveal their misconceptions which often develop from their generalising too far previous knowledge. By challenging their incomplete concepts and provoking discussions, the computer software may provide good support for further development of the concepts (Bell, 1993).

Different kinds of software were used in the teaching experiment, but I found only few of the pieces of software gave opportunity for a diagnostic teaching approach. The spreadsheet turned out to be in particular helpful for this purpose. I think other software could also be utilised, but certain qualities and characteristics of the software is necessary, and so is carefully designed support material .

Looking closely at some episodes from the classroom, the teachers role appears to be a crucial point in implementing the diagnostic teaching approach. In order for the students to meet the challenges, the teachers invention was necessary. However, in my experience, even after been challenged to use a diagnostic teaching approach, the teachers were only in a limited extent conscious of their teaching method. Teachers often appear to follow their traditional teaching style, rather than conscious reflection of their methods of teaching (Bigge & Shermis, 1992).

In this presentation I will report on some experiences from my own research and from research of my students to illustrate the points given above.

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A COMPARATIVE STUDY ABOUT STUDENTS' BELIEFS ON AUTONOMY IN THEIR DOING MATHEMATICS

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The importance of beliefs is becoming more and more widely recognized in mathematics education; this particularly applies if one agrees with the constructivist understanding of teaching / learning. It is important to distinguish between beliefs which are strictly due to the domain of an individual's character and beliefs which are formed via interaction between the individual and the society. Of course, these two kinds of beliefs are interwoven. An individual's mathematical beliefs system may have many sources, e.g. his personal character, personal experiences in school and outside of school.

Due this double character - personal and interacting with society - the studies on beliefs may concern not only the single individual, but the whole context where he operates. This makes particularly interesting to include beliefs in the comparative studies - as the one presented in this communication - among different countries. We note that there is a quite large amount of comparative studies on students' performances in mathematics. Of course these studies are of paramount relevance to know the level and the kind of the mathematical instruction in the countries in question, nevertheless they give only partial information.

In the present study we give an example of a comparative study between two countries which is centered on the study of beliefs. Its interest may be both in the insights given on students' beliefs and in the possible model it offers for such a kind of study. We describe and discuss an investigation on the beliefs held by seventh-grade students, 260 Finns and 246 Italians. The data were collected by means of a questionnaire constituted of 32 closed questions dealing with different aspects of the mathematics teaching / learning and three open questions on students' good / poor experiences in learning mathematics and wishes for a good teaching. We have confined ourselves to the items which can be referred to the students' autonomy in doing mathematics. The analysis was carried out both considering the trends of the beliefs in each country and comparing the trends of the two countries.

The aim of the paper is twofold. From one hand we try to have insights on what students think about the possibility of autonomy in doing mathematics; from the other hand we try to outline a model for carrying out comparative studies.

We have identified a core of common trends in the two countries: The trend is the same in some items, but in other it is very different. For example, using the Pollak's metaphor the Finns are more oriented "to move around in mathematical territory in a flexible manner", while the Italians "like walking on a path that is carefully laid out through the woods".

DEVELOPING A STATISTICS ANXIETY RATING SCALE

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The concept of 'number anxiety' appeared in the research literature many years ago, when it was shown that there was a significant correlation between this newly conceived construct and final mathematics grades (Dreger & Aiken, 1957). Subsequently, there have been numerous investigations which have focused on 'mathematics anxiety', defined as "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems" (Richardson & Suinn, 1972, p. 551). Researchers have shown that, in general, mathematics anxiety correlates significantly with a number of mathematics-related performance outcomes, as well as with mathematics attitudes (Schwarzer, Seipp & Schwarzer, 1989).

In the field of statistics education, the concept of 'statistics anxiety' has emerged, frequently in connection with the many students in the human and social sciences who are required to study statistics. Such students are apprehensive when faced with a compulsory statistics course and often start with quite negative attitudes toward the subject (Glencross & Cherian, 1992). At the University of Transkei, students who register for Psychology as a major subject, are required to take a course in Statistics, and many are known to express feelings of anxiety about this course. It is important to know the extent to which students exhibit feelings of anxiety and the effect of this anxiety on their performance in Psychology.

This paper describes the design and development of a Statistics Anxiety Rating Scale and reports on the results of a pilot study, in which the scale was administered to 138 volunteer psychology students in September 1997. The initial analysis of the data shows that the scale has high internal consistency (Cronbach α=0.92), while principal components analysis indicates that the scale is essentially unidimensional. A majority of the students (86%) showed moderate to high levels of statistics anxiety. A follow up study, which will make use of semi-structured interviews, has been planned for 1998.

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Starting a research project with immigrant students: challenges and procedures

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In recent years, in the region of Spain known as Catalonia (capital Barcelona) there has been increasing immigration mainly from North Africa (Magreb), but also from other parts of Africa, North and South America, Asia and East European countries. A research project began in 1997, funded by the Catalan Ministry of Education, concerning finding more appropriate ways to teach mathematics to immigrant students in primary and secondary schools. This paper reports the first stage of the project and, in particular, will highlight the procedures used and the challenges of doing research in a highly politicized social-cultural context.

The project has several **aims**, including:

- understanding more about students' "out-of-school mathematical knowledge",
- uncovering the values the students associate with in-school and out-of-school mathematics, and how these can help or interfere with the learning of mathematics,
- making teachers aware of the different cultural backgrounds immigrant students have from the point of view of mathematics teaching, and
- developing practical examples of mathematical curriculum adaptation and classroom practices for possible use by other teachers.

The research **procedures** being followed are similar to those used by Abreu (1995) and Presmeg (1997). Among the **challenges** faced during the first year were: overcoming teacher-researcher conflicts, gaining the trust of the students, establishing the credibility of the research team within the immigrant communities, seeking support of teachers' groups within the schools, encouraging the involvement of school principals and inspectors, and overcoming language and cultural barriers for gathering information. More details will be given on how the research is developing, during the session.

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ARE MATHEMATICS HIGH SCHOOL TEACHERS READY FOR CURRICULUM 2005? Research on the impact of in service mathematics workshops and its implications for the new outcomes based curriculum 2005.

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The Mathematics High schools In service Project (MHIP) was formed in February 1996 at the RADMASTE Centre. It runs workshops with teachers in clusters of schools in and around Johannesburg. Workshops are highly interactive and deal with both content and methodological issues.

The workshops aim to help teachers move away from the so called traditional transmission approaches to mathematics teaching towards empowering teachers to use a wide range of process based methodological practices which are more in line with the new outcomes based curriculum. They emphasise the development of teaching skills which will facilitate students attaining the mathematical outcomes outlined by the new curriculum. Regular support visits and classroom observations are conducted to help achieve these aims.

The research work presented here was conducted in four different clusters of schools in traditionally neglected and under resourced areas around Johannesburg namely: Crocodile Valley, Eldorado Park, Soweto and Magaliesburg. 22 schools and 35 high school (and grade 7) teachers attended weekly workshops which ran in each cluster over approximately 4 - 6 months. The research began in Eldorado Park and the Crocodile Valley from August 1996 to June 1997 and in Soweto and Magaliesburg from May 1997 to February 1998. The research examines the current classroom practices and beliefs of mathematics teachers in these areas. It further examines the extent to which the workshops influenced such practices and beliefs.

Data was collected in the form of pre and post questionnaires, lesson observations and interviews and processed in three main areas namely: teachers' beliefs, attitudes and experiences of mathematics education, pupils' beliefs, attitudes and experiences of mathematics education, and the testing of pupil knowledge at grade 7 level. The research findings in each of these areas reveal difficulties and challenges for the in service preparation of teachers to meet the needs of the new curriculum currently being proposed.

This paper will take a brief look at those parts of the research which pertain particularly to <u>teacher beliefs and teacher practices</u> and the implications these have for the successful implementation of curriculum 2005.

THE USEFULNESS OF EVERYDAY MATHEMATICS IN THE SENIOR SECONDARY PHASE: A CONTROLLED EXPERIMENT

Claudine Grinker

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The focus of this discussion is on a controlled experiment I performed at a Johannesburg secondary school, which I will call Com-Tech High. The experiment was of a quasi-experimental nature. The aims of the experiment were to see if teaching through methods related to the use of materials based on everyday mathematics (realistic/ethnomathematics) improved pupils' academic performance in any significant way, and to observe if teaching in this way had any effect on pupils' attitudes to school mathematics.

The experimental material was created from an ethno-mathematical perspective bearing Dewey's experiential learning in mind. The pupils in the experimental group worked in a social constructivist manner.

Unexpected results showed that although everyday based activities improved pupils' attitudes immensely, unless they had a sound instrumental understanding of related concepts, ethnomathematical methods did not improve their mathematical performance. The results are explained in terms of Kant's epistemology of the *a priori synthetic*, Piaget's constructivism and Skemp's relational understanding.

The Object/Process Duality for Low Attaining Pupils in the Learning of Mathematics

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In this project, case studies were developed for 8 pupils. Data was collected from: individual interviews focusing mainly on numerical questions, paired work in Logo involving problem-solving tasks, individual projects, Logo-based work undertaken away from the computer and which involved the resolution of conflicts, individual interviews and written work based on Logo tasks. The analysis of the evidence from these sources enabled me to focus on how these pupils gave meaning to the mathematics embedded within the tasks on which they work. In particular I explored the nature of the object/process duality within their thinking as they worked on the tasks.

The meaning that the pupils were able to give to the work was directly related to the objects with which they worked and the processes they were able to perform on the objects. Thus in order to analyse the work of the pupils, Tall's (1993) ideas were developed to include the notion of primitive objects, operations on primitive objects which give rise to complex objects, and then operations on the complex objects. The study shows how pupils learnt to use the object/process duality of mathematical entities in order to develop their understanding of the mathematics within the tasks being undertaken.

Gray E & Tall D (1993) Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic, in Journal for Research in Mathematics Education. Vol. 25 No. 2 pp. 116-140.

USING PALMTOP COMPUTERS TO SUPPORT GRAPHING

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The **PIGMI** (Portable Information Technologies for supporting Graphical Mathematics Investigations) Project is investigating the potential uses of portable information technologies - particularly graphic calculators and palmtop computers - in supporting university entry and secondary school level mathematics learning. Our extensive review of the literature (Hennessy, 1997) indicates that these increasingly cheap and powerful tools present a compelling opportunity to help students develop understanding and skills in a traditionally difficult curriculum area. Using classroom observation, questionnaires and written worksheet records of graphing activities, we aim to elaborate upon the role which portable technologies can play in facilitating development of graphing concepts and skills. The main objectives are to investigate how these technologies mediate learning to handle graphical representations and to develop and test appropriate activities involving investigations of real world problems.

Our PME presentation will report upon a series of pilot studies with classes of students aged 12-14 using Acorn Pocket Book palmtop computers to display, manipulate and graph data concerning body measurements from different age groups. The first investigation concerned the relationship between shoe size and height. Almost all pupils produced and sketched a coherent graph showing a visible linear trend; they were able to identify the trends of height increasing with shoe size and of both variables increasing with age. They could detect anomalous data points and interpolate from their graphs. Despite certain physical limitations, the Pocket Book proved a useful tool for provoking thinking about graphs and relationships between quantitative variables in pupils of this age.

We are about to begin a follow-up study with one class of 13- to 14-year-olds who are now accustomed to the Pocket Book machine's spreadsheet and graphing capabilities. This group will undertake an open-ended investigation using their school satellite weather station. The students will collect temperature data from inside and outside the school as well as those from other local and world locations. Our presentation will report upon their learning from experiences with plotting graphs on the palmtops which compare the data from different locations.

Portable computers are a potential catalyst for curriculum change; they are already forcing educators to re-evaluate what and how they teach with respect to graphing. The outcomes of our studies will hopefully help to guide the introduction of future forms of portable technology into education.

S. Hennessy (1997). *Portable technologies and graphing investigations: Review of the literature.* Computer-Assisted Learning Research Group technical report no. 175, Open University.

(Also submitted for publication, January 1998)

UP-CLOSE AND PERSONAL: DEVELOPING PEDAGOGIC CONTENT KNOWLEDGE OF PROSPECTIVE MATHEMATICS TEACHERS

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Recent documents, both in South Africa and internationally, dealing with mathematics education have stressed the need to reconstitute mathematics and mathematics education programmes for intending mathematics teachers (The NCTM, 1991).These documents stress the need for teacher preparation to stay one step ahead of the introduction of the changes to the school curriculum. The need in teacher education in South Africa is that of preparing our teachers for the Curriculum 2000 due to be introduced into South African schools from 2005. In particular, notice is taken of the critical outcomes developed and adopted by the South African Qualifications Authority to encourage integrated, holistic approaches to teaching and learning (Brodie, 1997).

Swafford (1995, pg. 171) notes that after seventy-five years, programmes for the preparation of teachers still consists of the three-pronged approach: academic subject matter, professional education, and supervised practice. Although the substance of these components has changed over the years, the separation remains intact. By separating these issues and treating them distinctly, I believe we increase the risk of exaggerating the need to correct procedure, and of emphasizing the ritualistic, surface features of the mathematics at the expense of broader and deeper insights into the mathematical content. Separating the mathematical content from the pedagogical discourse I feel further prevents the students from meaningful reflection. As a result growth in knowledge and understanding of mathematics in a manner suited to its use in teaching, that is, in pedagogical content knowledge may be stunted.

In this paper I will briefly describe the Mathematics Insights Course (MI), designed to develop student teachers pedagogic knowledge through direct engagement with and reflection on the subject matter. The aim was to effect changes in the beliefs and knowledge of the students. Data collected during the course assessing both subject matter and pedagogic content will be analysed, focusing on changes in subject and pedagogic knowledge.

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Cooperative Mathematics Learning: Effects of Strategy Training

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This research project attempted to answer the following question: What are the effects of training in social or/and cognitive strategies on the learning outcomes of cooperative learning in secondary mathematics?

To answer this question two studies were developed to investigate the effects of strategy instruction on the learning outcomes of students. In the first study of this research project three programs for cooperative learning were compared: a cognitive program fostering strategic and reflective thinking and a social program instructing student to use effective strategies for group work, and one control program without any special instruction. During the second study two programs were compared: an experimental program in which the instruction of the social and cognitive programs were integrated, and a control program without any special instruction.

In the first study students in both experimental programs outperformed the student in the control program. Besides this program effect, the low achievers in the experimental program outperformed the low achievers in the control program, while the high achievers in all program had about the same learning gain.

The results of the second study, in which the cognitive and social strategies were combined in an experimental program, showed that students in the experimental and control program had the same mean score. However, in contrast to the first study, the low achiever in both program had about the same learning gain, while the high achieving in the experimental program outperformed the high achievers in the control program.

These results will be discussed during the short oral communication.

EPISTEMOLOGICAL ISSUES OF SCHOOL MATHEMATICS

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For pupils, mathematics acquire its meaning through its school content as well as the relationships and interactions developed among the social and cognitive partners of the classroom (teacher - pupils). In addition, the organisation of the elements of the content, their interrelations, their relationship with problems, situations and representations and the evaluation and interpretation of these elements within the classroom's and the individual's functioning inside or outside the classroom also play an important role in pupils' attempt to make sense of school mathematics (Sierpinska and Lerman (1996), Seeger (1991), Steinbring (1991)).

In this context, the study of the nature and of the organisation of the mathematical content, that is, of the epistemological features of school mathematics acquires a particular importance. This study requires an analysis of the ways in which the epistemological elements of mathematics appear and become the subject of negotiation in the classroom, especially in the processes of (linguistic) interaction.

The data collected for this study consist of 30 mathematics lessons from various classes of the last three grades of the elementary school (10 - 12 years old), observed over a week. The lessons were analysed along the following two dimensions:

- the organisation and the interrelationships of the various elements of the mathematical content (concepts, definitions of concepts, processes, mathematical properties)
- the organisation and the selection of the elements of the mathematical activities (solving processes, checking and evaluating solving processes and solutions)

The results showed that the teaching approach used did not allow pupils to distinguish among mathematical concepts, processes and representations. Furthermore, the value of the solving processes followed did not constitute a matter of negotiation in the mathematics classroom. As a consequence, children seemed unable to conceive the epistemological differences of mathematics from other subjects of the curriculum, often holding vague and even incorrect notions of mathematics.

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THE ELUSIVE PARAMETER

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What is a parameter? This question is of great significance since the use of letters as parameters and variables lies at the foundation of algebra and analysis. Dealing with variables and parameters establishes the basis for understanding algebra.

Although variables have been thoroughly insufficient effort was devoted to the parameter, the "elusive" concept, as it was dubbed by Furnghetti & Paola (1994). The notion of parameter stems from a variable, yet it has its own vital role in mathematics. If a concept is elusive, teaching it might be problematic, and it is thus important to understand its mathematical uses.

An open-closed questionnaire on variables and parameters was given to 106 participants: 43 pre-service teachers and 73 advanced mathematics students in their final school year. In addition to that, another class of 5 mathematics teachers, 6 pre-service and 5 students were interviewed and asked similar questions.

Relating to the concept of parameter as "elusive", seems to indicate that understanding it is not simple. Only 44% of the participants gave acceptable answers. The following explanations were used: *An individual case of a variable, but determined in advance (8%).

* Letter marking a constant number (36%).

Most participants brought up the linear or the parabolic function in order to demonstrate what parameters are, with 56% finding no distinction between the two concepts. Some 15% of the study population stated that a parameter is a type of variable, but did not know what distinguishes it as a parameter, except for the fact that in their opinion parameters are designated by letters a, b, c... while variables are designated by letters x, y, z... Characteristically they deduced from the interview that the difference between variables and parameters is the letter per se, and not its essence, or focused on the choice of letters.

The concept of parameter is thus unclear, and part of the participants confuses parameter with a constant number. This confusion apparently stems from the difficulty of understanding the stages of solving quadratic function and linear function problems, where numbers are substituted for parameters converting these into constants. Almost all examples submitted by the participants were from algebra, which is where students first encountered parameters and so could draw on that experience.

It is of great importance to thoroughly understand the concept of the variable, the parameter and the constant, as well their relationships. The findings of the study show that the knowledge of pre-service teachers and students regarding the concepts of variable and parameter is sadly lacking. The pre-service teachers will soon be teaching these subjects. This fact is of special concern in view of the suspicion that they lack the comprehensive and well organized knowledge of the mathematics which they are about to teach.

Furnghetti, F. & Paola, D. (1994). Parameters, Unknowns and Variables: a Little Difference? Process PME 18, University of Lisbon. Lisbon-Portugal, 2, 368-375.

SUBITIZING AND GESTURE COMPUTATION FOR THE LEARNING OF THE BASIC FACTS OF ADDITION AND SUBTRACTION

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One of the main objectives stated in Mozambique's primary mathematics syllabus is that at the end of Grade 1 pupils should be able to "carry out, rapidly and mentally, the basic exercises of addition and subtraction". Research shows that even in Grade 3 a considerable number of pupils has to resort to finger counting in order to find elementary sums and differences (Kilborn 1990). Educational traditions from different countries show considerable differences between the moment that the basic facts of addition and subtraction should have been memorized: e.g.: the end of Grade 3 (Sweden), the end of Grade 1 (Japan, Korea).

In our research we designed an experimental program, based on subitizing and gesture computation, in order to help Grade 2 pupils to overcome the need of finger counting. A group of ten pupils of below average ability were selected through two diagnostic tests. As an example of gesture computation that was encouraged we show the sum $8 + 6 = 14$:

8 is shown on the left hand as 5 (full hand) and 3 fingers; the full hand has to be remembered.

6 is shown on the right hand as 5 and 1 finger; the full hand has to be remembered.

Adding 8 and 6 means joining two full hands, which gives 10. The two hands have to be remembered

Finally add 3 and 1, 4; plus 10 equals 14.

Results: after four weeks of instruction, all children had abandoned finger and toe counting and used only gesture computation. In general they used the specific strategies that were encouraged by the researcher, but they also discovered alternative strategies, equally efficient. The proportion of basic facts for which pupils gave immediate answers had increased considerably.

The gestures that were practiced are close to local traditional practice (Gerdes 1993), and although verbalized in Portuguese, correspond to the base five/ base ten numerals used in the local Gitonga, Cicopi and Xitshwa languages (Gerdes 1993). This gesture computation could be a powerful resource in the first grades, as an alternative for the counting of pebbles or sticks, as is common nowadays.

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Analysis of gender attitudes and study of mathematics at University level: The case of Swaziland University

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The study conducted in 1997 involved student interest, confidence and performance in maths by gender, enrolment, dropout and performance in the subject.

The sample included B.Sc, B.Ed and Post-graduate students in education (PGCE) in the 1996/97 academic year. It also included university maths lecturers, some students and maths teachers from high schools in the country.

The enrolment, dropout and performance of students in maths for the period of eight (8) years i.e. 1989 to 1997 was considered.

Collected data was presented in percentages, histograms and frequency curves.

The findings of the study come up with recommendations on some of the considered variables and the emphasis is made on the maths curriculum to be gender sensitive at all school levels i.e. from primary to university levels.

INTERPRETATION AND IMPLEMENTATION OF A LEARNER-CENTRED APPROACH: THE CASE OF SOME MATHS TEACHERS IN MASERU

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The need for in-service education in maths teaching stems from the changes in the needs of the society and from the changes in maths itself (Reynolds, 1981). In-service education plays a valuable role in the introduction of these changes. However, Cornbleth (1990) points out that teachers may not be able to put the knowledge acquired from such courses into practice due to different contextual factors. In this paper, I will argue that optimal conditions are not the main determinant factor for implementation of new ideas such as learner-centred teaching.

This paper presents findings of the study that was set to investigate how maths teachers participating in one INSET course in Maseru interpret and implement learner-centred approach to teaching and learning of maths. Data were collected through the use of both classroom observation and semi-structured interviews.

Across the interviews with 9 teachers, there was a strong presence of learner-centred rhetoric. However, across observations in 6 of the 9 teachers' classrooms, there was no evidence of such practice. Teachers' justifications included the expected "no time, not enough resources, big classes". What was interesting is that 2 of the 3 teachers whose teaching did embrace learner-centred ideals, were working in materially difficult conditions. In this presentation, I will describe 2 cases to illustrate my argument. One teacher who embraced learner-centred ideals in his teaching, had 65 students in his class and most of them did not have textbooks and calculators, yet mathematical activity was present. On the other hand, a teacher whose practices were not learner-centred, had 21 students, all with textbooks and calculators.

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THE PROMOTION OF ALGEBRAIC PROBLEM SOLVING PERFORMANCE BY AFFECTIVE FACTORS

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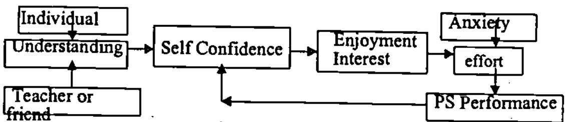
There has been much research on the importance of affective factors in problem solving. Our study is based on the hypothesis that a relationship exists between algebra problem solving ability and the affective domain that is described on the basis of statistical analysis of the groups and individual student interviews.

The data for the present study comprises 90 form 3 students (14 years old) from 7 secondary schools. Tests designed to measure affective factors (self-concept, self-perception interest, anxiety, and enjoyment.) and an algebra test were used. The methodology has been described in Kota & Thomas (1997). These tests were given twice, in the first half and at the end of the school year.

The range of possible scores on the algebra test was 5-25. The students were divided into lower and higher achievers (LA and HA), demarkation being the median score (15). A sufix defined whether it was first (I) or later (II) test. The four possible outcomes, LAI to either LA II or HAII and HA I to either LA II or HA II delineate four groups of students.

From the first to the second affective factor measurement, those students with dropping results in the second algebra test showed a significant decline in 3 of the factors, with another also falling (except enjoyment) and the other group whose algebra score remained below the median in both tests also showed more dramatic decline being significant (weakly for enjoyment) for all the five factors. From these two sets of results we hypothesised that the level of these five affective factors was linked to performance in the algebra test, with both a relatively low algebra score, or a decline in performance, accompanying a significant drop in the factor levels. For those students who were either improving their performance, or maintaining it in the higher achieving band, none of the five factors show a significant change but all are very stable in their values. In addition the student interview data leads us to propose a preliminary model of the relationship as seen in Figure.

Conclusions



This study provides evidence linking the maintenance of high levels of affective factors with higher standards of algebraic problem solving, but falling levels of these factors with a lowering of this performance. Although the process is cyclical with better work increasing confidence and the way into the cycle appears to be via understanding, affective factors drive the cycle, not the performance. (From the interviews, it is observed that self-confidence is an expression for self-concept and self-perception for these early adolescent students)

Kota, S. & Thomas, M. O. J. (1997). Gender Differences in Algebraic Problem Solving: The role of affective factors. *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*, Lahti, Finland, 3, 152-159.

Project Work in Botswana New Maths Curricula: A Pre-implementation Study.

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Project work (PW), now an integral part of secondary school mathematics curricula in Botswana (CD&E, 1996), evolved from the revised National Policy in Education (RNPE, 1994) which demanded continuous assessment to be fully recognised with some weighting for grading. Project work in maths is recognised as an effective means for providing learning opportunities appropriate to the world of work and for developing other work related skills such as communication, investigation and problem solving using real life situations and cultural experiences. It also aims to improve learners' understanding of mathematical concepts and achievement, ability to work co-operatively as well as to address issues of access and gender equity in maths. Thus, the incorporation of PW into Botswana school maths programs is consistent with international trends (Cockcroft, 1982, NCTM, 1989). However, its implementation in classroom setting poses several problems (e.g. lack of teaching materials and assessment criteria, etc.) which lead to the suspension of its assessment, thus practice.

This study investigated the level of beliefs about the importance and knowledge of PW possessed by mathematics teachers and inspectors at junior secondary schools, using a questionnaire, interviews and written description of the concept. Findings indicate inadequate teacher knowledge and low perceived importance of PW activities in maths. Very few teachers were able to give a concise written descriptive concept of project work in mathematics. Large class size, high teaching load and lack of appropriate teaching materials are identified by teachers as major factors that will hinder their smooth practice of PW activities in school maths. Most teachers indicated training needs in classroom based pedagogy including use of technology.

Department of Mathematics and Science Education (DMSE), University of Botswana, now has the responsibility of overseen the implementation of the new senior secondary school maths and science curricula. DMSE is to assist in developing exemplar teaching-learning tasks, expand its in-service training of Mathematics and Science teachers and initiate them into the successful practice of these new programmes using its "DO IT" approach. Consequently, Mathematics Education group (Chakalisa, et al; in progress) are developing teaching-learning materials and will be organising teacher in-service training to support implementation of Mathematics project work activities in schools. These research-integrated initiatives will address some of the findings.

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A REFLECTION ON GENDER RELATED DIFFERENCES ON MATHEMATICS IN MOZAMBIQUE

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Gender differences in mathematics learning is a topic continuing to attract mathematics educators attention. In various studies it has been found that girls are entering in mathematics at much lower rates than boys and are dropping out of mathematics majors in greater number than boys in the same grades.

In Mozambique the participation and mathematics achievement of young men are better than that of young women. Thus the purpose of this study is to find the factors that affect women's participation in mathematics.

From a reflection on this topic, mathematics classroom in Mozambique is affected by socio-cultural factors. Some classroom observations and semi structured interviews done and some cutural practices allows to say that stereotypes about mathematics are taught by our families or parents, teachers, peers, and books.

As an example the stereotype about mathematics being only for males contributes to some female feelings that mathematics is not appropriate for them, that they are unable to succeed in mathematics.

As family members and as mathematics educators we have to understand clearly what are the findings and shifts accepted, we must teach in ways that engage girls and provides girls' role models to emulate their confidence in their ability to succeed in mathematics and to not self select out from mathematics.

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USING DATABASE TO EXPLORE STUDENTS' CONCEPTIONS OF MEAN AND CARTESIAN AXIS

Sandra Magina and Maria Cristina S. de A. Maranhão – P.U.C.-S.P

There already exist many studies (Hancock et al, 1992; Healy & Hoyles, 1994; among others) which have investigated the impact of data collection and data analysis over students who had been worked with database. These authors generally agree with the idea that to collect data help pupils to analyze these data.

For this study we chose Tabletop software, which is a database that allows users to explore various representations such as Venn diagram and Cartesian axes. The purpose of this study was to observe how 30 secondary students would make sense of a database, looking specifically at their ideas of Cartesian axis and mean. Students were divided into two groups (group A and B); each group was divided into 5 groups of three pupils. Group A spent two 1 and half an hour sessions to elaborate the question they would like to investigate and also to collect and inset the data on computer; whilst Group B had two sessions exploring another database, using the same software. Afterward both groups were asked to answer the group A question by manipulating the Tabletop representations. The question was: *"Can we state that a student gets his/her best mark in his/her preferred discipline?"* From the data it was possible to conclude that although students got good marks in their preferred discipline, it was not true that all students best marks were in that discipline. However, students came up to different conclusions according the group they belong to.

The main result we obtained from both groups was that students considerably increased their understanding of means and Cartesian representation. We attribute this good result to the several manipulations pupils did exploring different representations. Specific results showed that Group A answered the question in a better way: only 1 out of the 5 groups answered YES for the above question; 1 group answered NO, and the remainders 3 groups said that it would DEPEND ON the discipline. These 3 groups went beyond the question, and concluded that the main concern here was the frequency of cases, i. e., the question was true for the most of the cases. On the other hand, in Group B 4 out of the 5 groups answered YES, justifying that once the discipline they like most is the one they get marks higher than 70, thus there was no reason to analyze another discipline. The remainder group said simply NO.

From the results we conclude that (a) data handling is a very good way to introduce the concept of mean and to give meaning for Cartesian axis representation, and (b) data collection phase was important for the analysis phase.

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TEACHING FUNCTION IN A COMPUTATIONAL ENVIRONMENT

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The authors are the actual team of Differential and Integral Calculus' teachers for the Science of Computation Course in PUC-SP and have decided to investigate if the topic "associated functions" could be explored in a computational environment to provide a better understanding of that discipline. Associated Functions, because the teaching practice of the group and recent results in Mathematics Education research say that function is a basic concept for the comprehension of a lot of others such as limit, derivative and integral. Computational Environment, because it has been used in the last two years in that discipline and it has the meaning of a change in the interactions of the triangle teacher-pupil-knowledge, with intriguing results.

In 1997, a sophomore students' class has been chosen for observation. In the first half of the scholar year, seven Computational Laboratory activities have been applied to these students, each one of them exploring a function f(x) and its associated f(x)+k, f(x+k), kf(x), f(kx), |f(x)| e f(|x|). The main goal was to verify the comprehension of some notions such as domain, range and graphics, as well as the handling of the algebraic expression of a function. The quantitative analysis of the data coming from the production of the pupils has answered some of the questions, but not all of them. Besides, new ones have emerged there. In the second half of the scholar year three new activities have been produced and applied to the students, all of them involving questions which could solve the doubts.

This research is the analysis of some of the many informations which could be extracted from the data. It has been done a qualitative treatment from a quantitative analysis of the data obtained in connection with some notions related to that of function. Some partial results can be described here. The observation group shows good results in the questions asking the algebraic expressions of functions. In these about domain and range the result is worst when the domain is a limited interval. A highly positive fact which can be noticed in these students is that they don't use to identify the graphic of any function to that of a linear or quadratic anymore.

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Measures of Length - Conventional and nonconventional measuring tools

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Researchers from different theoretical paths have presented studies involving measures of length. Douady et alli (1986) suggest the learning of the processes of construction of a measure, from the graduation of straight lines using nonconventional tools, for the solution of several measurement problems. Nunes et alli (1993) defend that the use of conventional tools, such as rulers, is possible and advisable, since the early years of the first grade.

This study involved two groups, A and B, each one with 8 pairs of Brazilian students aged 9-12 years. In the two groups we conducted clinical interviews, with a problem based on a straight line segment graduated in 3 points. One of the resulting segments of the graduation was of 1,2 cm long, corresponding to a course of 10 km. Students were asked to locate other points. The problem was posed in such a way that its solution, using rulers, was difficult due to the decimal numbers involved. However, it could be easily solved by geometrical procedures, in this case, using *only* folded or cut out paper strings. Both groups have used rulers for more than two years. Only group B had performed a sequence of activities, based on Douady' study, for the construction of a measure of lengths, which included the building of several measuring tools. We analyzed the use of those tools, the discussions and the answer sheets.

Despite knowing how to use rulers, as per Nunes' study, such knowledge was not enough for the correct solution of the problem by group A students. None of them used a procedure abolishing the ruler or numerical calculations. Besides, even after they affirmed that the measured segment had 1.2 cm (corresponding to 10km) they still considered it as being 1 cm long. On the other hand, all students in group B solved the problem correctly and accurately. After measuring the segment, they stated that they could avoid complicated calculations by folding paper strings. From this experiment we concluded that the students gained flexibility to choose the best procedure to solve the problem, because they knew how to build and use various measuring tools other than the conventional ones.

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CHANGES AS SEEN BY TEACHERS, PRINCIPALS AND OTHERS - AFTER A FIVE YEAR TEACHER DEVELOPMENT PROJECT

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A five year project for improving the teaching and learning of mathematics by fostering teacher development, was designed and conducted in elementary schools in an Israeli development town. The teachers of all eight schools in the town participated in the project.

The project focused on: a) guiding all the teachers in workshops and at the school sites - in the first three years b) developing a team of local leaders - in the third and fourth years c) gradually passing on the responsibility, to the team of local leaders - in the fourth and fifth years.

The research study focused on changes in teaching and learning mathematics as seen by the teachers themselves, by the principals, by the supervisor, by the junior high math coordinators and by representatives of the municipality and the Ministry of Education.

The analysis of interviews and open ended questionnaires indicates that each group of participants specified different dimensions of change. For example, the municipal representative said that one of the main changes was the fact that all schools worked together. He also mentioned the improvement of students' scores in exams given by the Ministry of Education. The principals and the supervisor mentioned that the teachers appeared to be much more sure of themselves when teaching mathematics. The junior high math coordinators noted that the students who enter seventh grade have much better understanding and are more able to think mathematically. Teachers said that they had changed their way of teaching mathematics and said that they are much more aware of students' different ways of thinking. Almost all participants in the research study said that it seems that students now learn mathematics with much more fun and only very few suffer from math anxiety.

All of the teachers, five out the six principals and the representatives of the municipality and the Ministry of Education expressed their opinion that in order to preserve the changes, the project should continue by means of limited continued communication with the team of local leaders.

ORGANIZING THE LEARNING IN A CABRI ENVIRONMENT FOR A JOURNEY INTO THE WORLD OF PROOFS

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The traditional vision of mathematical proofs, shared by most secondary school teachers nowadays, is that of verification of the correctness of mathematics statements by rigorous formal methods. However, recent research shows that proofs have various functions and roles both for mathematicians and students (De Villiers, 1990), and that there is a long journey, through several stages, for students to complete their understanding of the roles of proofs in mathematics and to improve their ability to perform proofs.

We are engaged in a research project aimed to introduce secondary school students into the world of mathematical proofs: We try to help the students to move from the stage of regarding a proof as an explanation of their conclusions or conjectures, based on the observation of examples (Van Hiele level 2), to the stage considering it as a justification of their validity, based on general arguments and deductions (Van Hiele level 3). Here we shall outline the parts of the research.

The main components of the theoretical framework are the types of proofs and the organization of classroom discussion introduced in Balacheff (1988), and the Van Hiele levels of reasoning, as a way to organize the teaching units and to assess students' evolution (Jaime, Gutiérrez, 1994).

Previous research has proved that the students' activities of collaborative discussion and production of proofs are facilitated if they take place in the context of a microworld (Mariotti et al., 1997). The software Cabri provides adequate environments to support such activity. We have elaborated a teaching unit with several sets of problems, to be solved with Cabri, related to different concepts of plane geometry. When we introduce the students into this microworld, the roles of elements such as figures and drawings, dragging, and group discussion are negotiated between teacher and students. Through their work on the problems followed by group discussion, the students go on along different meanings of mathematical proof.

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SOLVING PROBLEMS THAT ARISE IN REALISTIC CONTEXTS

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The research discussed here is part of an ongoing National Science Foundation-funded project entitled, *Connecting In-school and Out-of-school Mathematics Practice.*[1] During this project we are (a) investigating how middle school students use mathematics concepts and processes in a variety of out-of-school situations, and (b) working with a middle school teacher to develop and implement curriculum materials for building on students' out-of-school mathematics practice. The research framework guiding the study is Saxe's (1991) Emergent Goals Framework.

Through our analyses of data on six sixth-grade students' out-of-school mathematics practice collected during the first year of the project (1995-1996), we found a number of contexts (a) in which we saw our respondents using mathematical thinking, and (b) that seemed potentially rich mathematically for use in the classroom. Over several months we negotiated with our classroom teacher and settled on a context that was familiar to our six original respondents and with which we thought most students in our teacher's 1996-1997 sixth grade classes would have some experience. This context—miniature golf—was used to investigate geometry and measurement ideas (as well as other mathematical ideas, such as ratio, that arose). For four weeks in March and April 1997, the students in three sixth grade classes participated in a teaching experiment where they investigated mathematical ideas through the context of miniature golf.

The entire unit was centered around designing a miniature golf course. As the students worked on different phases of the design project, they encountered problems with which they had to deal. One such problem was how to measure and reproduce a curved side of a miniature golf hole. This problem arose when the students visited a miniature golf course and took measurements and made sketches with the eventual goal of making scale drawings of the holes. We found that students used a variety of strategies that could lead to fairly good approximations of the curve. In the presentation, we will discuss this problem in more detail, and show student responses to this problem which arose while working within a context.

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AN EXPLORATORY STUDY ON MULTIPLICATION WITH DECIMALS

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Pupils should leave grade 5 proficient in the use of decimal numbers because some of them will have no chance to study further (Draisma et al. 1986). The aims of this research were to collect the ideas of primary teachers and grades 5 and 6 pupils related to multiplication with decimal numbers (Fischbein et al. 1985) and attempt an approach of dialogue between pupils and teacher. In this paper I will present some examples of experimental lessons. Twenty experimental lessons took place with 45 pupils of grade 5 (9 – 13 year-olds). During the lessons many questions had no common answers. For example, in lesson 17, the pupils were asked to find out the result of 0,94 x 0,81. Answers:

0,94	0,94	0,94	0,94
x 0,81	x 0,81	x 0,81	x 0,81
94	0,94	94	0,94
+ 752	+ 75,2	+ 752	+ 75,3
0,7614 (1 pupil)	76,14 (4 pupils)	7,614 (10 pupils)	76,14 (pupil at the blackboard)

Other students had no answer at all to this question.

Teacher (T) – "Well, the head (pupil at the blackboard) has finished. Let's see. Is there anyone with a different idea?" Pupils – "..."

T – "Is there anyone with different idea, a different result?" Pupils – "..."

T – "I'm asking, is this correct?" A pupil – "It is not correct because there, on top, there are two numbers (0,94) and following there are two numbers as well (0,81)".

In lesson 9: T – (To a pupil) "How much is eight plus zero comma one hundred and sixty two (8 + 0,162)?" The pupil – "..."

T – "Eight plus zero, is how much?" The pupil – "..."

T – "Eight fingers (holds up 8 fingers) plus nothing, is how much?" The pupil – "..."

T – "One mango, you don't receive another mango, how much will you have?" ...

From the extracts we can notice difficulties with dialogue. Reasons: on part of the teacher: long interventions (some times monotonous and bewildering), repeated questions, not responding to the specific difficulties of individual students (Baroody 1987), wasting lesson time. The question: "Is there anyone with a different idea, a different result" is not relevant because it wasn't answered before either. On the part of the pupils: lack of experience of the learning process when asked to discuss with teacher . A child said: "We are not used to always explaining why". We can also see how the lack of basic knowledge related to the topic makes dialogue difficult.

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MEASURING PROJECT IMPACT IN THE CLASSROOM: AN ANALYSIS OF LEARNERS' SOLUTION STRATEGIES

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RUMEP's Farm School Project aims to sustain mathematics education through ongoing development of farm school teachers in the Eastern Cape. RUMEP is currently working with 97 teachers in thirty-nine schools scattered in the district. At the same time it is important to gather data that will determine the impact that the project is having on students' learning of mathematics.

The Farm School Project operates through various support programmes that include teachers' interactive workshops and also school-based support. Workshops are followed-up in classrooms by working alongside teachers in their own rural environments.

As a means of measuring impact on student learning two matched groups of schools were selected, three project schools and three control schools. Assessment tasks were administered to both groups of schools in grade 3 and grade 5 (9 and 11 year olds respectively, in farm schools). All the farm school classes were **multigrade**, with either one or two teachers per school. In both age groups the project schools scored significantly better than the control schools, except in one case ie. in addition and subtraction calculations.

Of particular interest was the analysis of the students solution strategies from both groups of schools. Incorrect responses were classified in the following manner. Learners used either inefficient methods or completely misunderstood the concept or made calculation errors, particularly when applying the standard algorithm. In situations where correct solutions were carried out, they sometimes used inefficient methods and a much smaller number of learners used efficient solution strategies.

An implication of the research is that it is useful for educators to think about their learners' mathematical thinking. The research provided insights into pupils' responses rather than focussing only on incorrect and correct answers made by the learners.

Although learners should be encouraged to create their own informal strategies, at the same time these informal strategies need to be **refined**. This refinement of strategies is what is still missing in the project schools. The need for a data-base of assessment tasks aligned to the goals of the new national curriculum is an aspect for consideration.

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BUILDING ON GEOMETRIC CONCEPTS

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This paper reports on an ongoing (preliminary) research study on concept development that is based on Klausmeier's (1974) model of concept formation and the Van Hiele model of geometric thinking. The study was conducted in a grade 8 class of Cape Town high school late in the school year of 1997.

Pupils in the class were grouped in heterogeneous small groups (mostly pairs) that according to the mathematics teacher included high and low proficiency English language speakers. Afrikaans is the spoken language in the area.

Data was collected using two different diagnostic tools, a group activity and the assessment activity. In the group activity, subjects were given shapes drawn on the dotty paper that they had to group in such a way that those shapes possessed similar features. In the assessment activity, subjects were to reflect on the previous lesson. They did this by means of writing a review exercise that explains the activities of the previous lesson.

The activities provide opportunities for pupils to talk and listen, read and understand, write, and reflect as they approach the exercise through problem-solving, using strategies and techniques which to a great extent are best known to themselves. This active learning (Meyer; Jones, 1993) assumes that (1) learning is by its very nature an active process and (2) that different people learn in different ways (Kolb, 1984; Briggs-Myers, 1980). Coupled with these assumptions are observations that pupils learn by doing and that teachers fail to reach a significant number of pupils in their teaching, thus leaving pupils and themselves dissatisfied.

The main focus of the study is on (1) local classification (the generation of categories and the relations **within** categories), and (2) general classification (the generation of a general category or categories the relations **between** categories associated with local classification).

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UNDERSTANDING THE DERIVATIVE USING GRAPHING UTILITIES

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This study investigates the conceptual change that occurs in the minds of students when they learn about the derivative with the aid of graphing utilities. The graphical representations produced by the dynamic graphing package used in this research, show how the derivative is the function generated at each x-value, by taking the value of the gradient (of the tangent to the graph of the original function at this x-value) as the corresponding y-value.

When students recreate the process of manipulating graphical notation in the same way that the computer does, they treat notation as a tool. A tool provides continual perceptual experience, which allows for the development of descriptions of the notation with regard to the objects and processes that it represents. It is through this manipulation that the mind develops the OPERATIVE IMAGE SCHEMATA (Dörfler, 1991) that affect the way in which students can interpret information about new mathematical objects. Notation behaves as a sign when it reflects to the mind both the structural and procedural aspects of the mathematical objects that it represents. The specific notational tool that is being transformed into a sign is called a carrier. Carriers consist of either actual or imagined notation that the mind interacts with.

One particular student, called Entle B, initially believed that the graph of the derivative would have the same shape as the graph of the original function. He did not use the fact that the graph he was given in the test was that of a cubic function with two turning points. He actually drew a cubic derivative for a cubic function. The zeroes on his derivative matched the turning points of the function, but most of his derivative should have been reflected about the x-axis.

The operative image schemata that developed in the course of the computer tutorial were sufficient to challenge his belief. Through using the computer package he started treating graphical notation as a carrier. In an interview I had with him a few days after the test, he had to sketch the derivative of a cubic similar to that in the test. This time he used his operative image schemata to sketch that part of the derivative which had a similar shape to the given cubic. He could not complete the rest of the parabola because of the cognitive dissonance he experienced based on the shape of the derivative emerging from the gradient analysis that he was doing on the graph of the original function (i.e. a parabola) and on the shape he expected for the derivative (i.e. a cubic).

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A CASE-STUDY ON DEVELOPING 3D-GEOMETRY AT SCHOOL*

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We present here some initial results on an ongoing case-study about valorization of geometry and space in French primary school. For at least thirty years, French primary school programs, concerning 7 to 11 years old pupils, have given a reduced place to geometry, whose teaching, centered on plane geometrical figures, is subordinated to numerical aspects. The aim of our project is to give a greater importance to geometry and space at school. We are developing a longitudinal case-study, in course of implementation in a primary school of Lille, concerning two groups of pupils during their school-attendance. The main assumptioms of the project are:

1) The first step to introduction of geometry at school concerns the space: for us, the beginning of geometry at school gives priority to the physical space, which should be a central point of geometrical learning; this entrance by the space should be done in articulation with geography, another subject-matter studying the physical space.

2) A didactical progression is clearly assumed and developed along the school-attendance: from physical space to plane; from plane to line; from line to point. Activities of problematisation are decisive for the transition from physical space to plane (A.L.Mesquita et al., 1998).

3) A special attention is given to the different registers of representation (in the sense of R. Duval (1995), i.e., the semiotic systems of presentation of knowledge) used with space and plane (at this level) and to their articulation.

With pupils of first year, our main preoccupation is centered on the passage from physical space to the plane (in both senses), which is an important step to children's learning of geometry, in general neglected by teaching. The presentation will focus on some interventions designed to implement our assumptions, and on an analysis on the effects of these interventions on the learning of first graders.

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The Discrepancy Between Policy and Practice in Mathematics Education in Black Colleges of Education.

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In line with the educational transformation as a consequence of the 1994 democratic elections, the first policy on teacher education (TE) in South Africa known as the Norms and Standards for Teacher Education was developed and implemented in 1995. This was a paradigm shift from a product input, content-based model to a process output, competence-based education (CBE) model in TE. The committee which generated the Norms and Standards promised to organise nationwide workshops to support teacher educators in implementing the new CBE.

The study on the impact of the policy on TE was conducted in 1997.The sample consisted of 20 black college mathematics teacher educators representing eight colleges and one province. The choice of colleges for the study was based on the information in the EduSource Survey (1997); that more than 70% of mathematics teachers are college graduates.

A questionnaire was developed and administered. This paper will discuss the following findings:

50% mathematics teacher educators were introduced to the policy by the colleges or departments of education at least six months after its publication.

10% have developed mathematics programmes in accordance with the new policy.

5% have implemented these developed programmes in mathematics classes.

0% cite positive changes in mathematics classes as a result of the new policy.

0% cite any change in their professionalism as a result of the policy.

15% understood what competence-based education implied in their practice.

It is alarming that the paradigm shift policy in TE had such little impact on one of the areas it was supposed to address, that is, the quality of the learning- teaching process at the crucial initial phase of the development of mathematics teachers.

The Norms and Standards have been reworked in line with the Curriculum 2005. These were published for discussion in November 30, 1997. To avoid a discrepancy between policy and practice as evidenced by the results of the study, it is recommended that support plans for implementation must be developed. Especially for colleges who were disadvantaged by the centrally developed curriculum which promoted a culture of dependency and lack of creativity.

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SOME GEOMETRICAL CONSTRUCTS AND PUPILS' CONSTRUCTION OF MINIATURE WIRE TOY CARS

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This study attempts to identify some of the geometrical constructs that may be embedded the construction of miniature wire cars by boys. The study deriving as it does from the perspective of ethnomathematics seeks to increase the socio-cultural relevance of mathematics and mathematics education. A group of five in-school boys with the age range of ten (10) to fourteen (14) in classes ranging from standard two (grade four) to standard five (grade seven) were observed and interviewed as they went about constructing toy wire cars. The analysis of subjects' protocols as well the observed artifacts was able to reveal certain geometrical concepts, ideas and principles embedded in the construction of the toy cars. The educational implications of these findings for making geometry more meaningful and accessible to more and more pupils are discussed.

Mathematics Classrooms in Hong Kong

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It is generally known that classrooms in East Asia conform to a stereotyped image of teacher-centered, with a tense atmosphere demanding mechanical learning and stress in rote learning (Stevenson and Lee, 1997). Yet this fails to capture many salient features. A curriculum reform, known as Target Oriented Curriculum (TOC), attempting to shift this prevailing nature to one which is characterized as pupil-centered, group-based learning using a variety of resources has been initiated by the Hong Kong Government since 1993. A key feature of TOC is the encouragement of developing the five fundamental ways of learning and using knowledge as a central element of the cross-curricular framework, viz., communication, inquiring, conceptualizing, reasoning and problem solving.

Classroom observations of more than 100 primary one and two mathematics lessons were conducted in an evaluation project in the year 1995-97. Preliminary analysis indicated that whole-class teaching was still the dominating style, and the prescriptive programme of study, textbook tasks and examination requirements were powerful influences on the pedagogy used. However, many teachers used a combination of pedagogies which involved whole-class teacher-student interactions, group work and individual student work designed to develop students' fundamental learning skills. These are not adequately explained by reference to various models of group processes developed in the west which are premised on the value of student interaction and collaboration (Good, Mulryan and McCaslin, 1992).

Selected lessons will be presented to give a scenario of the TOC primary mathematics lessons in Hong Kong, contributing to an understanding of a culture of mathematics classrooms and effective whole-class teaching.

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IDENTITY AND THE PERSONAL IN MATHEMATICAL WRITING

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There is a 'common sense' assumption in school and in academia that, just as knowledge is unbiased, factual, neutral, so the reporting of knowledge in the form of texts written by or for students and researchers is neutral and impersonal. The ideal of scientific objectivity is transformed into forms of language that suppress human agency and involvement in the subject matter (Halliday & Martin, 1993). In mathematics, in particular, the public image of the subject is impersonal: mathematical knowledge is seen as abstract and separate from people, social issues and the real world (although it may be 'applied' to these areas). The language in which mathematics is written is seen to be similarly abstract, with the extensive use of algebraic symbolism serving to distance mathematical texts and their readers from what we term 'natural' language. The absence of the personal and the author from the text is taken for granted. Thus a publication of the Mathematical Association of America can state: 'In most technical writing, "I" should be avoided, unless the author's persona is relevant.' (Knuth, Larrabee, & Roberts, 1989, p.2) with the unwritten assumption that, in general, the author's persona is <u>not</u> relevant.

Yet, moving away from such 'common sense', we know that, where learning and doing mathematics is involved, the feelings and personal identities of the participants are significant. This raises questions about the relationship between the individual's experience of doing mathematics and the ways in which he or she presents that experience and its results to the rest of the community. What is the picture of mathematics and mathematical activity constructed through the text? How do the author and reader appear to be positioned relative to one another and as members of the mathematical community?

In fact, when actual mathematical texts are examined using critical linguistic tools based on Halliday's functional grammar (Halliday, 1985; Morgan, 1996b) it is apparent that while some authors do consistently obscure their own agency and, by their absence from the text, conform to the conventional ideal of abstract impersonal neutrality, others produce texts which construct a personal identity for the author as an individual mathematician. Such constructed identities may well have consequences for the ways in which readers judge the authority and validity of the text and are hence of significance to student writers who are to be assessed by their teachers and to academic writers who are to be judged by reviewers, editors and their peers (Morgan, 1996a).

The short oral presentation will present some examples of the analysis of the construction of the author's identity in mathematical texts produced by secondary school students (including some responses to these constructions offered by teacher assessors reading the texts) and in academic mathematics research papers. The implications for students learning to write mathematically will be discussed.

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AN EXPLORATORY STUDY ON TWO TILING TASKS

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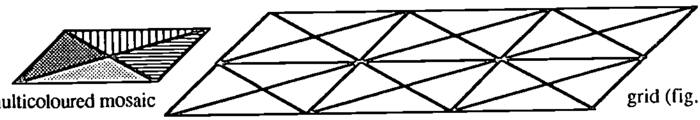
Dickson et al. (1984) compared the ability of 2 to 7 year olds to construct quadrilaterals using sticks and the ability to draw them in a paper after touching them. Freudenthal (1982), suggested tiling using mosaics to explore properties of quadrilaterals. This paper describes children's behavior doing two tiling tasks. The children filled grids using squares, rhombus, trapeziums and rectangles but this paper only focuses on parallelograms.

1. A group of children received a grid of parallelograms drawn on paper (see fig.1) and 16 equal, multicoloured mosaics as shown aside the grid. The mosaics were painted front and back with the same colours (at the back of blue was blue and so on). The children's task was to fill the parallelograms shown in the grids using the mosaics as beautifully as they could. They were encouraged to verbalize their movements with mosaics and their observations concerning diagonals.



2. Another group of children had received the same grid and one multicoloured mosaic was drawn aside the grid. The researcher said that he had forgotten the real mosaics at home and asked them to think on a set of concrete mosaics like the one drawn. He said also that the mosaics were painted front and back side with the same colours (at the back of blue was blue and so on). The children were encouraged to think about different ways to use the drawn mosaic. They should explain verbally how to use the mosaic and later should paint the parallelograms on the grid to show the position of the mosaics.

The children in group 1 filled the grid faster than those from group 2. The researcher had to stop the children from group 1 to make them think about their work. Children in group 2 looked concentrated on the task (reasoning?). Trial and error in group 1 showed children that there were mosaic positions that they could not fill the grid with (immediately they adjusted to the right position). Conditions for trial and error in group 2 were different. Sometimes children painting the parallelograms placed colours not in a proper place. (they had to think longer how to adjust the colours). The question "How did you fill it?" made children in group 1 to remove the mosaics in the grid and fill again often followed by a sentence *"I did so and so"* in front of the researcher's eyes avoiding long verbalization while in group 2 the verbalization was an important tool of learning. This paper intends to show how these two studied tasks helped to develop different skills.

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ERRORS MADE IN DIFFERENTIAL CALCULUS BY FIRST YEAR TECHNIKON STUDENTS

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The study sought to investigate the errors made in elementary differential calculus by students studying engineering at a technikon. A sample of 45 first year students from a technikon's engineering faculty were interviewed and questioned on their understanding of ideas considered to be important in elementary differentiation. Differentiation tasks were used to determine the kind of errors first year technikon students make in elementary differential calculus. Subsections of the tasks were regrouped to form twelve items, each item relating to one aspect of differentiation. These aspects were grouped into four sections: elementary algebra, rate of change, limits and infinity and differentiation. The errors in the four sections were analyzed according to a classification of errors. This classification of errors was linked to concepts in cognitive theory. Analysis of the data reveals that there were more structural errors than executive or arbitrary errors in the sections elementary algebra, rate of change and differentiation. There were more executive errors than structural errors in the section limits and infinity. The structural errors were due to the students not applying the correct group of principles to the tasks while the executive errors were due to the students either omitting or replacing one substage in a correct rule by an inappropriate or incorrect operation. It is recommended that the errors can be alleviated by the use of appropriate computer technology.

Change in Mathematics Education:

A case of beliefs and preservice training of mathematics teachers

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School mathematics classrooms need to be characterised by *teaching by negotiation*, rather than *teaching by imposition*. To this end mathematics teachers need to possess fundamental beliefs and conceptions about school mathematics and its teaching and learning, that can support an *open culture of growth and change* in their classrooms, rather than a *closed formalist* one. Mathematics teachers' preservice training has to contribute significantly in this regard. This report concerns an investigation in this respect, conducted at seven training institutions in the North West Province of South Africa.

The failure of many an attempt to 'marry' teaching and learning in classroom settings can be attributed to a lack of *grounded teaching theory*. Hence, a postpositivist theoretical model of effective school mathematics teaching had first been developed, upon which the field survey was then based. The model explains effective mathematics teaching in terms of six coherent features: *intention*, *teacher*, *learner*, *interaction*, *content* and *context*. Final year mathematics students, at the respective institutions (n=366), and their lecturers' (n=26) key beliefs about these features, as well as those of practitioners, including all mathematics subject advisors (n=16) and a sample of teachers (n=31), were then analysed and compared.

The results show that both primary and secondary school students hold *practically significant* less 'positive' beliefs than the lecturers involved in their preparation as mathematics teachers ($d \geq 0,8$; $p < 0,05$), especially with respect to the features *teacher*, *interaction* and *context*, which are of particular interest, viewed from a constructivist or interactionist perspective. The only exception is a group of primary school students that followed a four year long methods course largely based on constructivist principles, compared to the conventional exam-driven training elsewhere. Differences between the students and the practitioners are mostly *insignificant* ($d < 0,2$). Results indicate that the current conventional preservice mathematics and methods practices are reinforcing, rather than curbing the 'negative' impact of transmission school mathematics teaching on people's fundamental school mathematics related beliefs, especially in the case of students coming from historically 'disadvantaged' areas.

Changes in mathematics education - changes in assessment?

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A theoretical analysis was used to determine what effective assessment of mathematics in the senior primary phase comprises. After a theoretical framework for the effective assessment of mathematics in the senior primary school phase had been created, it was tested in senior primary classrooms (n=28) where the problem-centred approach (PCA) was supposed to be used. Questionnaires were used to evaluate the following: the way in which the teaching and learning of mathematics takes place; which aspects of pupils' knowledge, mathematical abilities and dispositions towards mathematics are assessed; and whether assessment forms an integral part of teaching-learning situations. Interviews were conducted to determine: which assessment procedures were used by PCA teachers involved; in what way they used the collected assessment information; and whether assessment formed an integral part of their teaching-learning programmes.

The results show a deficiency in the assessment programmes used in the PCA mathematics classes involved in the study: Although some teachers use different assessment procedures, namely observation, oral assessment (questions and interviews), and assessment of written work (reports, classwork and homework, journals, tests and examinations, and portfolios), they do not effectively keep record of this information. In addition, teachers do not use all of the collected assessment information, in particular "formative information", in a systematic way in teaching-learning situations. Furthermore, there seems to be a discrepancy between the time allocated for the different assessment procedures, and the ways in which the collected information are used to compile a pupil's final mark: The use of different assessment procedures during the year shows trends of "problem-centred" assessment, whereas a pupil's final mark for mathematics is compiled in a "traditional" way by mostly using summative test and examination results. This indicates a "traditional" test-driven tendency in the teaching-learning programmes employed by the teachers concerned, showing that assessment is still not utilised as an integral part of the teaching-learning events in these PCA classrooms. Thus, there does not seem to be an "automatic" connection between PCA teaching and learning practices and the use of alternative formative assessment measures.

In conclusion, there is a definite need to specifically equip inservice and preservice mathematics teachers with the necessary knowledge, skills and dispositions to develop and implement effective integral assessment programmes in their teaching-learning.

ANALYSIS OF STUDENTS' BEHAVIOUR REGARDING BADLY DEFINED PROBLEMS

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The solving of "well defined" or "well structured" problems forms part of the contents of all scientific curricula; this theme has been and is being widely researched. However, "badly defined" or "badly structured" problems have not received research attention in mathematics or other scientific disciplines which are concerned rather with the well defined type.

Our aim is to study solvers' behaviour when faced with badly defined find problem (with too few data and too many data) in various contexts: arithmetical, algebraic and geometrical. We take our characterization proposal as a competence model (mentioned in Noda, Hernández and Socas, 1997) that can be used as a reference for solvers' performance. The aim of this analysis is to discover whether or not there are regular forms of behaviour (invariants) in the transformations the solvers carry out at the "preparation" stage before they proceed to the production or judgement stages (Bourne et al., 1979). In this respect, we consider in our analysis three highly differentiated categories: (1) Identification of the problem in terms of well or badly defined. (2) Analysis of the actions carried out. (3) Analysis of actions performed on the conditions of the problem.

The experiment was carried out with 13 third-year students from the Infant Education Diploma course at La Laguna University's Centro Superior de Educación (Tenerife, Spain).

To sum up, we can point out that the local competence model constructed is of great use in characterizing the various badly defined find problems and in the analysis of solvers' actions; that badly defined problems cause some difficulty when students attempt to differentiate them as such, the influence of the context being noteworthy in this respect - the arithmetical context causing least difficulty and the algebraic most difficulty for this group of solvers; that actions determined by the problem objective predominate, while actions determined by the conditions are more apparent in too few data problems and mainly in those problems with an algebraic context, and finally, that the actions on the conditions of the problem are determined by the typology of the problem, adding or (adding and eliminating) data when dealing with problems with too few data, and eliminating or adding data when dealing with problems with too many data.

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Whither Relevance? Mathematics teachers' espoused meaning(s) of 'relevance' to students' everyday experiences

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The debates around relating school 'relevance' or relating school mathematics to everyday experiences of students have attracted much research in relation to meaningful mathematics learning (Carraher *et al*, 1985). Consideration of the relationship between school failure (particularly in mathematics) and social class structure has led to suggestions that in order for students to enjoy mathematics as mathematicians do, the teaching of mathematics should move from the interpretation of symbolic information to an emphasis on situating it in the realm of everyday experiences of people (Volmink, 1994). However, the specificity of school learning illuminated by Lave and Wenger's (1991) *legitimate peripheral participation* in a community of practice has led to some suggesting that the role of everyday contexts in school learning is dubious (Ensor, 1997). Others who do not dichotomise the relationship, recommend facilitating learners in crossing the bridge from 'everyday' discourse into 'educated discourse' (Mercer, 1995).

Set up in South Africa in a period of critical curricular change, this study focused on the teachers' awareness of possible problems of 'relevance'. A questionnaire and a focused group interview were used to investigate the ways in which teachers in this study talked about 'relevance'. Findings of this study indicate that teachers regard 'relevance' as inducing positive associations for students. Teachers' discussions of possibly problems were mainly limited to obvious shortcomings such as that the practice is time-consuming. However, implicit in discussions were some of the more intricate difficulties that arise when "reality" is brought into the classroom. In the presentation I will illustrate these findings and argue for the crossing of bridges rather than the dichotomy that is sometimes set between school learning and everyday experiences.

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ROLLE'S THEOREM DE-CONSTRUCTION: AN INVESTIGATION OF UNIVERSITY STUDENTS' MENTAL MODELS IN DEDUCTIVE REASONING.

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Deductive reasoning is a form of expert mathematical thinking. Nevertheless in traditional teaching, it is often considered easily assimilable through a set of axioms and formal inferential rules. Students elaborate a personal version of mathematical concepts, which may be different from the teacher's ideas.

A theorem can be seen as a "logic toy" which can be disassembled in order to investigate students' concepts. The first theorem on differentiable functions, well-known as Rolle's Theorem (Smirnov, 1964) fits well an articulate didactic analysis of the deductive reasoning, namely concepts and processes concerning theorems: Rolle's hypothesis consists of three distinct statements, whose validity can be accepted, refused or substituted in various arrangements; thus the related consequences concerning the thesis can be highlighted and discussed. Furthermore all assumptions and their negations are susceptible of an immediate graphical visualisation, very suitable to investigate how students correlate analytical statements with their geometrical representations and how they need a concrete visualisation to understand the meaning of abstractly enunciated features.

Students' conceptual obstacles in deductive reasoning are analysed in terms of the framework proposed by Johnson-Laird and Byrne (1991). According to mental model theory, novices reason by means of a three-step semantic process in which they: a) set up an initial and essential model based on the meaning of the premises and containing only true contingencies; b) derive a putative conclusion from the model; c) try to build further models in which the conclusion could be false. Difficulties in any of these three steps lead novices in logical fallacies.

We present some preliminary results of an exploratory research with seven university students attending a course in Higher Mathematics. Students were asked to express their beliefs about the fundamental concepts of deduction, and to solve a related set of tasks about Rolle's Theorem. The problems concerned: the implications of the negation of each hypothesis on theorem validity, the analysis of assigned functions, the generation of examples under given conditions. Students' errors were analysed in terms of mental models that violate expert deductive reasoning. The most common difficulties were:

1. Recognising examples in which the thesis is fulfilled although the hypothesis is not, and their conceptual meaning (hypothesis only sufficient for the thesis).
2. Dealing with hypotheses necessary only for the thesis, and discussing the consequences of their negation.

Novices tend to believe that if A implies B, then negation of A implies negation of B, e.g.: "If a function is differentiable then it is continuous, therefore if it is not differentiable, then it is not continuous". Educational implications are discussed.

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The Use of Children's Worksamples for Deciding Learning Outcomes and Level of Development in Spatial Thinking

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Noel Geoghegan, University of Oklahoma

Do Space strand classroom tasks used for deciding learning outcomes produce the same profile grade? Do grades relate to scores on a test of spatial thinking? The tasks were (a) making a map from home to school, (b) a tangram task, and (c) making a described simple shape in a barrier game. The responses to the tasks were graded so that there were two scores for each of the three levels recommended by the curriculum profiles (1994). Descriptors for the mapping-task grades are:

1. Knows meaning of place and movement words, draws or acts it
2. demonstrates recognition of between, opposite, and in another direction
3. places in order and between
4. shows right angles, parallel lines, left/right
5. makes a plan, interprets its use; uses language for sequence of movements
6. predicts sequence by visualising, has a good degree of accuracy of lengths

The test requires students to recognise shapes in different orientations, to disembed shapes in other figures, to decide on the number of tiles needed to cover a shape, to add parts to complete shapes, to fold a shape to make an open cube, and to recognise angles in different figures or orientations (shortened version of Owens, 1992). 103 students (school years 1 to 5), undertook the test but not all students completed these three tasks (teachers had a choice).

Spatial thinking scores ranged from 14 to 39 out of 40, with an average of 25 and standard deviation of 7. Correlations of task grades with spatial thinking scores were statistically significant at $p<.05$ level.

Correlation between Scores for Spatial Thinking and Grades on Tasks

	Map to school	Tangram shapes	Barrier game
Spatial thinking	= .287, p = .046, n = 49	r = .318, p = .023,n = 51	r = .342, p =.048, n = 34
Map to school		r = .347, p = .082,n = 26	r =-.098, p = .64, n = 25
Tangram shapes			r = .220, .p = 339, n = 21

The lack of correlation between tasks could have resulted from the small number of students or from the distinct outcomes. Only 17 out of 34 students who had completed two of the tasks were classified at the same national profile level for all tasks (that is, graded 1 and 2, or 3 and 4, or 5 and 6). These results bring into question the validity of using individual tasks for **grading** students' spatial development.

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INTUITIVE MODELS FOR MULTIPLICATION AND DIVISION WORD PROBLEMS

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Researchers today attribute much more mathematical knowledge and understanding to young children than they once did, giving them credit for their own informal methods which in the past would have been regarded as "inferior" or "not mathematical".

This paper reports on findings of a study investigating young children's intuitive strategies for multiplication and division word problems in a Problem-Centered approach. Nineteen pupils from the Junior Primary Phase participated in a 10 week study, in which, the instruction was generally compatible with the principles of Socio-Constructivism and the Problem-Centered Mathematics Approach. The results indicated that the children, who had not received any form of instruction on multiplication and division algorithms or associated concepts, used a range of intuitive strategies. From the analysis of these intuitive strategies seven models for multiplication and division word problems were identified, **three** models for multiplication and **four** models for division. This paper will discuss these models in detail.

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THE USE OF COMPUTERS IN THE MATHEMATICS TEACHING: FOCUSING TEACHERS' PROFESSIONAL DEVELOPMENT[1]

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Recent studies have focused on the participation of teachers as new technologies are brought into schools (Bottino & Furinghetti, 1996; Penteado Silva, 1997; Borba, 1996; Hoyles, 1992). In this paper, we will present partial results of an ongoing case study in which a teacher was first interested in bringing computers to the classroom as a means of overcoming her students' lack of interest in mathematics. She believed that computers could motivate them.

When the teacher sought help from our research group (GPIMEM), we showed her different software packages which were available in our laboratory. She chose software named *Fracionando* (Bordin et al, 1995) because it fit better the mathematical content she wanted to teach. As researchers, we considered to be a non-flexible software which tries to imitate a book and takes very little advantage of the computer as a medium. In any case, we respected her choice and decided not to discuss with her our ideas regarding the software. She then taught three out of four weekly mathematics classes in the computer laboratory. We observed her teaching a fifth grade class for two months and interviewed her several times. There were also meetings in which the researchers worked with the teacher on questions that arose about the software.

At the end of these two months, the teacher believed that her attempt to use the computer was successful since the students were engaged in solving the problems proposed in the software. She thought that this was quite a contrast with their attitudes in previous classes when paper and pencil were the main media being used.

We also believe that the computer could have been a major source of the change; however, there are also other possibilities which are currently being analyzed based on the data we collected. Regardless of this analysis, we believe that the step taken by the teacher was a very important one towards change even though her use of a new medium was still very traditional; she did not change other aspects of her teaching other than introducing a new tool in the classroom. However, from the perspective of professional development, it is relevant that she wanted to change and that she was able to feel comfortable with this new medium. Moreover, we foresee that other changes may come about as result of the interaction with our research group.

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MATHEMATICS TEACHERS' PROFESSIONAL KNOWLEDGE REGARDING STUDENTS' INVESTIGATIVE WORK

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Mathematical investigations in the classroom require that students themselves define working objectives, choose tools and strategies, lead many experiences, formulate and test conjectures. A mathematical investigation usually begins with the presentation of the task by the teacher. This is an open situation, formulated in a way that challenges students but leaves them still a significant work to do in refining it and making it specific. This may be followed by a period of work carried out in whole class, individually or in small groups. The class may end with a summary of the conclusions, jointly made by the students and the teacher — what questions were studied, what conjectures were drawn, how they resisted the tests made, for what conjectures was it possible to find a proof, what were the most significant conclusions of the work done. In this process the students may engage in many forms mathematical reasoning. Besides the investigative work, they have to find ways of representing data and relationships and to recall concepts previously studied. Listening (to their pears and to the teacher), arguing their conclusions, and, often, writing a report, are also an integral part of the activity.

On the other side, the teacher engages in what we may call didactic reasoning regarding the setting up of the task and the orientation of students' work. Previously to the class decisions need to be made regarding, for example, how to integrate this activity the curriculum. During the class, the form of presentation, the sort of support to provide regarding individual or group work, and the strategy to conduct classroom discussions are some of the questions that need to be faced. Previously to the activity, the teacher may have done quite a lot of mathematical reasoning regarding the task. However, it is quite likely that students' work will raise new questions and issues, not yet foreseen, requiring the teacher to engage again in mathematical activity during the activity.

This paper presents in detail a framework to analyze this kind of work in the mathematics classroom and illustrates it through the analysis of a video recorded and later transcribed episode concerning a mathematical investigation in a 8th grade class. The framework included the (mathematical) reasoning processes experimented by students and the roles that they take during the activity, and also the (didactic and mathematical) reasoning processes carried out by teacher and his/her role regarding the student. It is argued that there is a mutual relationship between both kinds of reasoning processes and the roles taken by each actor and a strong conditioning of roles and processes between teacher and student, regulated by the power relations between them.

Observations and Thoughts on The Ethnomathematics Research Project at The Centre for Research and Development in Mathematics, Science and Technology

Colin Purkey

The Ethnomathematics Research Project has the aim of investigating the place of ethnomathematics in the secondary school mathematics curriculum in South Africa.

Two observations made during current research are particularly highlighted in this paper. Both involve teachers who had agreed to use the context of rectangular Basotho huts in a lesson on construction of a right-angle and Pythagorean triples. It was interesting that both teachers, a standard 6 teacher and a standard 9 teacher, chose what appeared to be the most conventional activity from the choice of ethnomathematics materials.

The first teacher's lesson was appropriately contextualised in the introduction. During the lesson an interesting innovative technique with matchsticks was used to construct a right-angle. Nevertheless it was striking that the ethnomathematics was all in the introduction and the rest of the period was mathematics per se. This raises serious questions about the design of ethnomathematics material and lessons.

The second teacher did not bother to prepare an ethnomathematics lesson. The lesson given was completely conventional and textbook based. There were no references to Basotho huts. I was perplexed by this lesson and spent time reflecting on the number of things that may influence a teacher's performance in class; observation, new material, new approaches to teaching and perhaps political reasons.

Ethnomathematics is dealing with issues that can be highly sensitive. Perhaps the mathematics in ethnomathematics detracts from or even trivialises, the meaning of the cultural phenomenon in question?
There are many issues that are raised for pedagogy by ethnomathematics. As a consequence there is much scope for research. But, these issues also need to be taken into account when research projects are designed.

TEACHING OF GEOMETRY WITH REFERENCE TO THE ELEMENTS OF LEARNING PSYCHOLOGY

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Let us look at the process of learning and teaching within the general framework of communication [1]. Learning satisfies the human need for discovery of connections between the known and the unknown. This takes place through the mediated social practice of teaching. The aim of the teacher is to impart his or her professional knowledge in the best possible way to pupils at any level, with due regard to their goals, abilities and psychological peculiarities. The information requirement of a pupil is determined at the same time by his or her age and the socio-cultural environment with its norms, values and expectations. School pupils in the primary forms need not at all be motivated to study mathematics, whereas an interest born out of a conscious want of information is common to many pupils of senior forms, especially to tertiary students of natural and applied sciences. In any case, it is the primary task of the teacher to create a positive attitude, motivation to study, and interest, in respect of the material to be handled. In mathematics this can be done through acknowledgment of the possibilities of application. At the primary level geometry provides direct acquaintance with figures and their properties through folding, pasting, Cabri geometry, etc. Later, interest will arise and grow through the solutions of concrete problems of application, enveloped in the area of geometry by examples from art, architecture, and on a higher level also from economics and theoretical physics. This form the *context* for conveyance of knowledge, and a *cognitive basis* is prepared for its reception. There follows the *contact-preserving phase,* the one that needs to see the start of emotion-driven creativeness and pursuit. In the teaching of geometry there is an especially large device of ways and levels of solution of a given problem. One can only look at a problem on the level of *visualization* and comprehension of its contents. Visualization of the problem often leads to a suitable method of attack. It is possible to continue with an exact *analytical solution*, using the means of arithmetic and algebraic methods or the methods of differential and integral calculus. With an eye of applied objectives of the teaching of geometry, it is hardly possible to bypass the needs of engineering sciences for the *graphical methods* as a means of solving geometrical problems. It would be ideal to use all these possibilities in parallel at some stage of teaching process, since a variety of ways of presentations and points of view will help in overcoming *stickiness* and *barriers* in an individual manner of thinking, and take maximum advantage of peculiarities of a styles of cognition and learning [2,3]. As result it would enable us to arrive along the course of studying and teaching at *concrete knowledge,* which are based on certain mathematical facts and skills to operate with them by the comprehended mathematical models.

Profoundly understood and assimilated mathematical knowledge will help in moulding an individual who is rational, creative, and one who thinks in terms of alternatives independently from his or her cognitive and learning styles [2,3].

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MATHEMATICAL PRACTICE OUT OF SCHOOL: HOW DOES IT HELP TO UNDERSTAND SCHOOL MATHEMATICS LEARNING?

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Learning is becoming to be seen as a social practice (Lave, 1988) and used to look into school mathematics learning (Adler, 1996; Boaler, 1998; Santos & Matos, 1997). However, several questions emerge when we intend to think about school learning from this point of view. Lave's results come from studies of adults in situations with relevant differences from schooling. For instance, practices in which those adults were involved were deeply connected to a (chosen) process of *becoming*. Our main goal is to understand school mathematics learning and we believe that schooling is, for most of young people (12 to 15 years old) not explicitly associated to a process of *becoming* but it is a transitory life-space, that is, becoming is not the intentionality and purpose of their school practice.

Therefore we feel the need to clarify the meaning of learning as social practice, particularly in these aspects that we see as fundamental. In order to study this problem we are looking at youngsters practices which they don't see as something that takes them into a professional position but ones they are thrown to by their social and personal situation. For example, young people that need to earn money to help the family: selling goods in the street, washing cars and helping air passengers to carry luggage at the airport. As in school, these practices are socially and culturally organized.

In order to discuss learning as a social practice we are looking at three features that Lave (1996) called our attention to: *telos* (what is the direction of the movement of youngsters' learning?), *subject-world relation* (what is the general specification of relation between subject and social world?) and *learning mechanisms* (how does learning come about?). To address these features we are using concepts such as forms of life and language games (Wittgenstein, 1967). This analysis is supported with data collected in Cabo Verde among local youngsters' practice in the street as mentioned earlier.

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GRADE 7 PUPILS' PERFORMANCE ON CALCULATOR SKILLS
A comparison of a farm school, an informal settlement school and a township school.

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The ReMap (Relevant Mathematics) Project developed Grade 7 Mathematics materials to initiate schools into the newly proposed Curriculum 2005. All schools and teachers in the sample were assisted on a regular basis to implement the highly interactive materials. While working with these schools in historically disadvantaged communities, it was expected that pupils from similar socio-economic backgrounds, will have similar problems and that pupils' performance will not differ much across the different types of schools. The findings of this study revealed that each and every school situation had its own unique dynamics and those factors implicitly or explicitly affected the pupils' performance.

The purpose of this study is to highlight the relationship that exists between the mathematics performance of pupils and the socio-economic setting of the school. The predominant research approach was that of a case study. In the study a comparison is made between a farm school, a school in the informal settlement, and a township school. It is not the intention of the study to insinuate that pupils are passively formed by their cultural backgrounds, but what is acknowledged is that such contexts can impact to a large extent on the pupils' performance. For this study a chapter on calculator skills has been used for the comparison. None of the schools had ever taught calculator skills formally and very few pupils had been exposed to calculators informally. A calculator was made available to every pupil in the sample.

The findings show that, the pupils in the farm school, even though marginally very poor, performed better than the pupils from the township and the informal settlement schools. The informal settlement school took twice as long to complete the chapter on calculator skills and performed the poorest. This paper will attempt to shed some the factors that contributed to this status quo.

DEVELOPMENT AND EVALUATION OF A COMPUTER REPRESENTED SPATIAL ABILITY TEST

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Why represent spatial ability tests with a computer? - Computer represented spatial ability tests (CRSAT) have some advantages over corresponding paper and pencil tests: flexibility and economy in the organization of the test (e. g. the ordering and editing of the test items), economic and objective evaluation of the test scores, additional information on the tested person`s solving of the test-items (e. g. time of single item treatments, member of attempts to solve single items), the objective conducting of the test (for that reason a CRSAT must be self-explanatory) and specifically the use of animating and direct manipulating graphical objects (whereby the possibilities of computer graphics must be limited in order to provoke an adequate spatial thinking); CRSAT are a suitable test environment for computer supported training of spatial abilities (e. g. to avoide a change of media) and in addition a learning environment for improving spatial thinking.

The prototypical CRSAT "Geometric Modules Test" was designed by H. Schumann and programmed by T. Alavidze. The idea of the test is to find out, which two modules among four modules are components of a given solid. The test is of high face validity because the test persons need a lot of visualization and mental rotation capabilities to solve the items. The first testing took place in September/October 1995 with 350 students of German Mittelschule (grades 7, 8, 9). - A connected factor analysis enables one to prove the construct validity (one factor solution); an item analysis produces acceptable values for reliability and consistency of this test. An accompanying questionnaire provides interesting results on the relationship between the test performance and variables e. g. age, gender, attitude towards geometry, self evaluation of spatial ability, motivation etc. - The development of this CRSAT ist not yet been completed. Improvements for the item editing and the adaptive testing facilities (based on the first evaluation) are still to be implemented. - The psychometric orientated evaluation of the Geometric Modules Test is balanced by using case studies which investigate the individual processes of spatial thinking when solving the test items.

MATHEMATICAL THINKING IN A COLLEGE ALGEBRA COURSE

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A comprehensive view of a student's mathematical thinking is generally not provided by course assessment tasks and classroom communications. Schoenfeld (1992) suggested a structure for studying the complexity of mathematical thinking that consists of five categories: the knowledge base, problem-solving strategies, monitoring and control, beliefs and affect, and practices. My study focused on the whole of one student's mathematical thinking in a college algebra course and the intricate connections that existed between and among these catagories.

Allison, an 18-year freshman, was enrolled in a technology-rich college algebra course offered at a large university in the Southeastern United States. The instructor was a veteran teacher in the mathematics department. My role was that of a non-participant observer in the classroom. Data were collected in the form of informal interviews (at least once a week), daily classroom observations, student work, and an exit interview at the end of the course.

One of the cohesive elements in Allison's mathematical thinking was her *procedural disposition*, or a habit of thought that is primarily focused on the acquisition and use of procedures. Such a disposition seemed to be supported by her belief that classroom mathematics should be done the way the instructor did mathematics. Her problem-solving and metacognitive strategies were limited in this context because she frequently chose to attend to those strategies used by the course instructor. Allison's sense-making and interpretation of mathematics were strongly affected by this procedural disposition. Her attention to procedures seemed to diminish the importance of underpinning conceptual knowledge; hence, Allison often had to rely on intuition and insufficient mathematical connections for her understanding.

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CHANTING AND CHORUSING IN MATHEMATICS CLASSROOMS

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This paper focusses on some language practices in mathematics classrooms in South Africa. Using data collected in second language primary mathematics classrooms it examines the form and different ways in which chanting and chorusing are used.

According to current policy there are 11 official languages and flexibility for schools in determining their language of learning policy (Government Gazette, 9/05/97). It can however be predicted that most schools will not opt for mother tongue learning, since among speakers of African languages mother tongue policy has a bad image. It is associated with inferior education. It can be assumed, therefore, that English will remain the language of learning for many pupils who do not know the language before they go to school. It is also possible that practices such as chanting and chorusing will continue to be used by teachers to cope with the problem of teaching mathematics in English to pupils who cannot speak the language

In the data collected, both chanting and chorusing featured in all of the classrooms. This paper suggests that these practices are linked to the double task primary mathematics teachers have of teaching in English and teaching English at the same time (Adler, J; Lelliott, T; Slonimnsky, L. et. al., 1997). These practices are used both as linguistic and mathematical devices. The teachers used them to give pupils an opportunity to learn and practice the pronunciation of mathematics terms most of which the pupils were hearing for the first time (e.g vertex). Similarities in terms of what is chanted, when and how it is chanted were observed across the classrooms. Maths words and correct answers given were all chanted. Whatever is chanted is usually not just chanted once but it is repeated a few times.

What is absent in the data however is the creation of opportunities for learners to practice using the terms that they have learned to read and pronounce in a mathematical conversation. As Mercer (1995: 81) argued, learners can only develop familiarity and confidence in using new discourses by using them. They obviously do not need to be fluent first, because fluency comes with practice. So while chanting and chorusing are present in these mathematics classes, they are not being effectively used to scaffold the learners' entry into mathematical conversations. What Mercer calls educated discourse. In Vygotskian terms, use of language here is being limited to imitation and not being used for communicative and cognitive functions (1986).

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THE SENSE PROSPECTIVE TEACHERS MAKE OF THE PROPOSITION $2^0=1$

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There might be differences on what understanding mathematics implies. For the purpose of this paper, a definition offered by Hiebert et al (1997) is sufficient. They view understanding as knowing how things are related to each other and why they work the way they do. This definition is in line with Freudenthal's learning by insight (1983) and Schoenfeld's sense-making (1994). This paper explores the reasons that prospective teachers advance for justifying the validity of the proposition $2^0=1$. From their justifications, students' understanding of the concept and possible pedagogical implications can be inferred.

110 prospective teachers from three different colleges were required to justify, in writing, the validity of the proposition $2^0=1$. 35% of participants justified the proposition on the basis of an external source and 60% attempted to personally explain why the proposition held, although only 31% of these were 'correct'.

It is apparent that, for this item, most of the participants attempted to understand the underlying concepts behind the validity of the proposition. However, an attempt to give a meaningful account of a proposition does not necessarily result in a taken-to-be-shared explanation; therefore, some learners will still need to be empowered to acquire a correct understanding of the proposition. What is positive, though, is that the majority of participants do bother to make meaningful sense of the proposition.

Since teachers' perception plays a major role in informing the teaching methods, viewing mathematical concepts as meaningful will play a major role in discouraging the view held by some learners that mathematics is an external and inhuman body of knowledge.

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GENERALIZATION IN THE CONTEXT OF THE DIFFERENCE OF SQUARES IN ALGEBRA BY HIGH SCHOOL PUPILS: A CASE STUDY OF 4 PUPILS IN STD.7 (GRADE 9).

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Algebraic thinking is a project concurrently mounted by the mathematics education departments at the universities of Transkei and Zululand. The report is based on a case study of four standard 7 (grade 9) pupils who were interviewed and videotaped. Interviews are on subjects' responses to questions that direct the subject's attention to factoring a difference of squares. Problems posed demand that subjects express themselves verbally on the increase and decrease in factors and on the corresponding change in the product of factors.

Communicating enhances understanding (McCombs,1993). Through verbalising their observations subjects are expected to discern patterns in arithmetic situations, which can assist in the formulation of a generalisation using symbols. Subjects are also given an opportunity to perceive how the decrease/increase affects the product of factors. In this project subjects have to evolve even the actual writing of the statement of the difference of squares. The rationale here is that understanding develops from an examination of concrete situations.

From the interviews it is discovered that the majority of subjects could not ascend and descend Mason's spiral (Mason,1988). It was even harder to lead other subjects to see into the relationships that lead to generalisation. Those that easily articulate patterns are able to generalise and derive $(a + x) = a^2 - x^2$.

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ON THE GEOMETRY INVOLVED IN THE BUILDING OF TRADITIONAL HOUSES WITH RECTANGULAR BASE IN MOZAMBIQUE

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The aim of this research is to describe part of the building process of traditional houses and make a first survey of the mathematical aspects – in particular geometrical – involved in it, about which the builder may be conscious or not. This study is part of a larger program aimed at the incorporation of the geometry of traditional house building into mathematics education in Mozambique.

We will describe how some builders place pillars and laths of the basic structure of houses with a rectangular base and present a first overview of the mathematical knowledge used by the builders, without taking into account, for the moment, whether the builder uses them consciously or not.

Having marked the base rectangle of the house – another activity that we are studying, but not to be discussed in this paper – the builder indicates the places where to dig the holes where the pillars will be placed, making small holes with a bushknife or putting up small sticks. He then removes the arrangements made for the rectangular base and starts digging the holes. In order to put up vertically the first pillar, he needs the help of an apprentice or assistant, who has to carry out the orders of the master builder. They put the pillar in its hole and, whereas the apprentice holds the pillar, the master builder takes up a position at a certain distance. He verifies if the pillar is not inclined to the left or the right, looking up and down the pillar several times. If necessary, he gives orders to incline the pillar more to the left or more to the right. When he thinks that the pillar is standing up straight, he wedges the pillar to the right and to the left. He then takes up a new position, in a direction that is more or less perpendicular to the first direction, in relation to the pillar. As before, he verifies if the pillar is not inclined to to either side, giving instructions to eliminate the inclinations. He then goes back to the first position, in order to check whether the pillar is still in the vertical position. Finally gives instructions to wedge the pillar on all sides. This process is repeated for the other three pillars at the corners of the house.

In short, the builder looks at the pillar as if it were a straight line that must be perpendicular to the plane of the ground, i.e., it must be perpendicular to at least two straight lines in this plane. As he is working in a place that may not be level and with the naked eye, he takes up two nearly perpendicular positions that enable him to avoid errors. During the conference we intend to speak about the way in which the first horizontal laths are fixed.

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TEACHERS WHO TEACH TEACHERS: PROFESSIONAL DEVELOPMENT OF IN-SERVICE TEACHERS

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This paper reports on an attempt to provide in-service teachers with an opportunity to broaden and deepen their own mathematical thinking processes through a course which provides them with the ability and confidence to bring about improved mathematics teaching and learning.

For many years in South Africa minimal changes have been made in order to reform mathematics classroom practices. With the launch of a reformed curriculum (Curriculum 2005) educators are confronted with formidable challenges in the way they teach.

The Rhodes University Mathematics Education Project (RUMEP) recognizes the enormous task ahead of equipping teachers to meet the changes taking place, as advocated in the national curriculum.

RUMEP offers a Further Diploma in Education (FDE), an accredited course aimed at improving teachers' mathematical content knowledge and topics related to curriculum reform. This is achieved through a Key or (leader) teacher model whose task it is to help advance the goals of the new national Curriculum.

Evidence from research conducted by RUMEP on the FDE programme indicates that teachers drawn from rural areas (where there are most schools) have made more progress when there has been:

I) a programme of block release for teachers to engage intensively in mathematics content, curriculum reform, and mathematics change management, and have had the space to reflect on the issues involved,

ii) on-site support and guidance (which includes curriculum materials), to the key teachers in the field, when they return to their home areas to deliver school-based workshops for colleagues drawn from clusters of neighbouring schools,

iii) encouragement and official recognition of the work key teachers are doing in the field, from representatives of the Provincial Governments' Department of Education and Culture.

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THE ROLE OF TIME AND EFFICIENCY IN STUDENTS' ALGEBRAIC REASONING AND SYMBOL MANIPULATION.

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Current research on students' algebraic reasoning provides a useful explanation of the "cognitive gap" between learning algebra and arithmetic (Herscovics and Linchevski, 1994). In addition, research continues to identify pedagogical approaches to meaningful algebra learning (Kaput, 1995). The purpose of my research was to identify anchoring domains students bring to learning algebra.

To identify possible anchors I have investigated the metaphorical nature of students' mathematical reasoning as evidenced in their discourse (Lakoff and Nunez, 1997). The view that thought is structured metaphorically provided a useful theoretical framework for analyzing the data gleaned from didactic exchanges and interviews. The identified metaphors provided a useful window to students' algebraic reasoning and symbol manipulation.

Research data demonstrated students' attention to the efficacy of their strategies. Students verbalized their concern for the time required to complete a task or the efficiency of performing the appropriate operations in a task. Students' attention to time was evidenced by phrases such as "*quicker,*" "*short cut*" and "*the long way.*" Their attention to efficiency was evidenced by phrases such as "*easier*" and "*easy way.*" These phrases were used by students while simplifying algebraic expressions and generalizing pattern relationships. Some students were inclined to persist with more secure, if more time consuming strategies. Alternatively, some students, upon reflection, recognized the time consuming or inefficient nature of their strategy and proceeded to search for a more efficient approach.

At this time it is not clear if students' attention to time and efficiency enhances their desire to engage algebraic reasoning (rather than arithmetic strategies) in pattern generalization. In addition to the role of arithmetic as an anchor for learning algebra, students appear to introduce time or efficiency as a relevant anchor for their efforts to engage algebraic activities.

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Understanding Fractions Through Problem Solving

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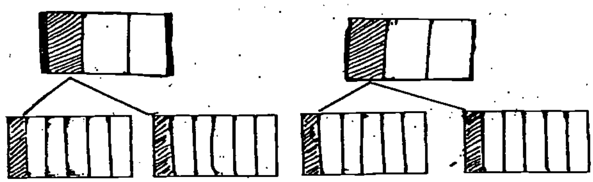
This work presents ideas that children develop in response to word problems that require working with fractions. It is based on observations of children in third through fifth grades whose mathematics classes were based on Cognitively Guided Instruction (CGI) (Carpenter & Fennema, 1992). In CGI, children solve problems and discuss their solutions, constructing their own strategies. A basic idea in CGI is that children possess a rich base of informal knowledge that can be effectively used to help them understand mathematics.

Research on fractions has shown that many children fail to obtain good conceptual understanding and rely on rote implementation of procedures, which often results in errors (Mack 1990). At the same time, studies have shown that children can construct meaningful knowledge of fractions when given the opportunity to solve problems and build on their informal knowledge, even in the first grade (Mack 1990, Baker, Carpenter, Fennema, & Franke, 1992, Baker 1994, Maher 1994).

This work extends our knowledge of how children develop conceptual understanding of fractions. Our third and fourth graders, who had studied in CGI classes for several years, devised a variety of strategies using drawings and manipulatives, constructing knowledge and providing a base for learning symbolic representations. The fifth graders had not studied previously in CGI classes and initially had difficulty solving problems. During the year they made significant progress and were able to present meaningful strategies for complicated fraction problems. The following example from the fifth grade illustrates the type of conceptual reasoning the children used:

Who gets more cake? A child at a table where 3 children are equally sharing 2 cakes? Or a child at a table where 6 children are sharing 4 cakes?



"Both children get the same amount of cake. Because at the first table they split 2 cakes into 3 pieces each. And at the second table 4 cakes into 6 pieces each. Splitting each piece from the first table into 2 pieces gives two pieces (sixths) at the second table, since two-sixths (2/6) equals one third (1/3)".

"A TRAPEZIUM IS A CONVEX QUADRILATERAL WITH TWO PARALLEL AND UNEQUAL SIDES" – IMPLICATIONS FOR LEARNING

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The lack of rigor in some definitions that appear in some schoolbooks and the lack of awareness of some teachers about the scope of the meaning of definitions or of justifications of certain geometrical representations contribute to an inconsistent learning.

As a lecturer for Didactics of Geometry in the LEMEP program (Master's Degree Program on Mathematics Education for Primary Schools), my students and I found several difficulties with the teaching and learning of quadrilaterals. We made a survey in order to find out whether it is a problem of learning and/or of teaching.

I discussed with my students the interviewing guides to be used with pupils in schools and guidelines for discussion with schoolteachers for the collection of information. Quite a bit of data were collected and discussed with the LEMEP students. The discussions on the inclusion relations of quadrilaterals consisted of:

- selection, resolution and analysis of activities
- elaboration of the guidelines for interviews and planning of discussions with teachers
- organization and analysis of the data that were collected through the application of the guides.

One of the striking results of the interviews with pupils and primary teachers is that they do not consider parallelograms as trapezia. And given a collection of different types of quadrilaterals and asked to identify all parallelograms, justifying their answer, pupils and teachers alike indicate only the parallelograms which are not lozenges, because "they have parallel sides two by two". Why does this happen?

Some contributing factors are:

- a limited relation between the definition and its geometrical interpretation. Frequently we obtained a correct definition for a given quadrilateral, but the geometrical interpretation of the definition is limited to one type of shape, favouring certain positions and dimensions and excluding others.
- problems of language: What is the scope of: "has parallel sides two by two"?
- the excessive trust of what is written in a book: "It is written in that book, so it must be true".

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FIRST YEAR ENGINEERING STUDENTS' LEARNING OF POINT OF TANGENCY, NUMERICAL CALCULATION OF GRADIENTS AND THE APPROXIMATE VALUE OF A FUNCTION AT A POINT

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As many students have difficulties in learning calculus concepts, there was a question of whether the use of computer in a 'realistic classroom settings' was helpful or whether it merely introduced another source of anxiety. It was therefore decided to investigate 147 first year engineering students' learning of calculus in two computer and two non-computer environments in 'realistic classroom settings' in England. This study was concerned with part of wider research (Ubuz, 1996). It was intended that the investigation would find answers to the following questions: what levels of understanding of the basic concepts and ideas concerned with calculus have the students acquired, what fundamental errors and misconceptions do the students hold in this area, and what implications does the investigation have for the learning and teaching of calculus?

In order to investigate students' learning of calculus, a diagnostic test including 10 essay-type of questions was developed and administered as pretest and post-test. Following the post-test, some students were interviewed from each group in order to substantiate inferences deduced from the data. In this study, the students' answers to a question including three tasks were analyzed. The question requires a student (a) to specify the point in which a line tangent to a curve (b) to calculate the gradient or slope of the tangent line and (c) to calculate or estimate the approximate value of a function at a given point.

The levels of understanding demonstrated by the students were assessed on a defined six-point scale, using specified criteria written for each task. In addition to that the different types of errors students made on each task were identified by analyzing their responses.

Results indicated no distinguishing features for computer and non-computer groups according to the classification of students' responses and errors. Results, however, incorporated three key principles to retain learning: concept-formation, strategy choice, and problem solving particularly understanding and looking back stage of Polya (1945).

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The Feasibility Of Curriculum 2005 With Reference To Mathematics Education In South African Schools

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This communication reports on current research undertaken by the presenter as part of his Ph.D. studies at the University of the Free State (S.A.). *Curriculum 2005* is an outcomes-based curriculum model that was introduced in all grade one classes in South Africa during 1998. It demands of both educators and learners a total paradigm shift from an *academic, content-centered* towards an *outcomes-based* curriculum model. Curriculum 2005 requires of educators and learners to focus their attention on two things (NDE 1997 : 9):

- the *results* expected at the end of each learning process;
- the *learning processes* that will take the learners to the results.

The above-mentioned foci have important implications for the teaching and learning of Mathematics and thus also for the Psychology of Mathematics education. Critics may argue that the new curriculum is based on two different philosophical and psychological paradigms: *traditional positivist-behaviourist* (because of the emphasis on the *products* of learning) on the one hand and *constructivist* (because of the emphasis on learning *processes* that are based on aspects like cooperative learning, critical thinking and problem-solving) on the other (cf. Schwartz & Cavener 1994 : 337).

The research undertaken will attempt to answer questions like these but will also involve research on the feasibility of the new approach in South African schools by analyzing, amongst others, the preparedness of learners, educators and schools to undertake this paradigm shift in the field of Mathematics education.

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EXPERIENCE: OBSTACLE TO REFORM OR AVENUE FOR LEADERSHIP?

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Many people believe that the cliché "Old dogs can't learn new tricks" applies to teachers, especially mathematics teachers close to retirement. To test this, we analyzed four veteran high school mathematics teachers who expressed an interest in improving the mathematics teaching in their schools by participating in a three-year teacher development program. At the beginning of the study, each of the four teachers, two male and two female, was at least 50 years old and had been teaching for more than 23 years.

The focus of the program, which resulted in a Mathematics Education Master's degree, was to prepare 30 secondary school mathematics teachers to become leaders and proponents of reform by exposing them to contemporary mathematics as well as current curriculum and pedagogical issues. The NCTM *Professional Standards for Teaching Mathematics* was used in conjunction with other reform literature to facilitate discussion and action.

In the current study, we investigated the role experience played in teachers' ability to navigate the rocky waters of reform. Questions that arose during the study include: the impact of administrative support, or lack thereof; the motivations of these teachers for wanting to change their practice at this point in their careers; the level of success they had in changing their practice; positive and negative factors in their quest for change; and the impact they were able to have on those around them.

Data used in this study was collected over a four-year period from a wide variety of sources including: an essay application to the program describing experience, professional development, and reasons for wishing to enter the program; reflective writing assignments related to classroom practice; annual extensive reflective final projects; impromptu journal entries; a pre-test and post-test project beliefs survey; and classroom observations followed by interviews. A thematic approach was used to analyze the data. That is, categories were allowed to emerge based on the data, instead of using the data to support pre-existing hypotheses.

The information gained from the study of these four veteran teachers provides insight into the impact of experience on a person's ability to engage in reform. In particular, it provides information for facilitating the use of veteran teachers' experiences as an avenue for leadership instead of an obstacle to reform.

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Changing the approach: Integrating Mathematics, Technology and Science in the Middle Years of Schooling

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The alienation of students in the middle years of schooling world-wide is well documented, and one of the recommended ways of addressing this alienation is to provide a curriculum that is relevant, negotiated and integrated (Cumming, 1996). Bean (1991) proposes a curriculum in which real life questions are posed through an integrated curriculum that motivates students and promotes wholeness and unity of content.

Sixteen Western Australian schools implementing integrated approaches to curriculum were studied to determine the ways that integration had been achieved. Data were collected by classroom observations, by an examination of teaching programs and by interviews with school staff members and students. The different forms of integration utilised included: *(a) Thematic Approaches:* where a theme was used to integrate the curriculum, and cooperating teachers developed a complementary program of work; *(b) Cross Curricular Approaches:* in which cross-curricular issues such as numeracy, literacy and computing skills served as a basis for integration between learning areas; *(c) Technology-Based Projects:* in which students were given a project which included technological, scientific and mathematical research components; *(d) Competitions:* wherein projects like the Science Talent Search facilitated integration; *(e) Topic Integration:* involving an integrated project amalgamating such topics as Statistics and World Environment; *(f) Integrated Assignments:* where teachers organised integrated mathematics and science investigative assignments; *(g) Synchronised Content and Processes:* wherein content was synchronised to allow similar processes to be taught at similar times; *(h) Local Community Projects:* in which local community activities integrating a variety of skills and content were introduced; *(i) Common Teaching Approaches:* which used similar frameworks for designing and implementing lessons in mathematics; science and other subjects *(j) Natural Approaches:* where integration happened naturally when teachers taught the majority of learning areas to the same students and integrated where opportunities arose.

These integration strategies, the issues they raised and their implications for teaching and learning will be discussed during the presentation.

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CONCEPTIONS AND GRAPHICAL INTERPRETATIONS ABOUT DERIVATIVE[1]

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In this paper we aim to present an analysis of an episode from a one-hour interview with two female students, aged 18 and 21 years old, while working with a mathematical question in a computational environment. These students were attending a first-year mathematics course for biology majors.

This interview was the forth of a set of four interviews conducted with these students. The interviews were videotaped and transcribed. After a first analysis of the data, this particular episode was selected for two reasons. First, it revealed some aspects of the students' thinking that showed a disconnection between mathematical statements and the graphical interpretations of them. Additionally, it brought to light some conceptions about derivative and its construction.

The episode also shows how strong is a conception of one of the students, who states that the growth of the derivative has to follow the growth of the function that generated it. In this way, a function and its derivative have similar shapes. Other authors (Nemirovsky, 1992; Scher, 1993) have indicated this trend, and we have also seen it in other episodes of our study.

The problem posed was to sketch the graph of the derivative of a function without knowing its algebraic expressions. Most of students' attempts were tied to previous relationships established between the growth of the function and the sign of its derivative and between the roots of the derivative and the critical points of the function. Although these relationships were correctly stated, some difficulties arose when graphical interpretations were required. The graph of the derivative constructed by the students was compared with the one the computer plotted. Finally, the students' conceptions were discussed.

The idea of knowledge as a web (Machado, 1995) helped us to analyse our episode. It appeared that the students had two "sites" that were connected with inappropriate "links". On one hand, they made mathematical claims that we considered correct; on the other hand, they worked graphically making interpretations which were not in agreement with such claims.

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[1]This research was developed within the activities of the Research Group *Technology and Mathematics Education* (*GPIMEM: http://www.igce.unesp.br/igce/pgem/gpimem.html*) at UNESP - Rio Claro.

KINDERGARTEN TEACHERS' KNOWLEDGE OF THEIR CHILDREN'S MATHEMATICAL THINKING: TWO CASE STUDIES

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Inherent in much of the recent literature in mathematics education is the belief that learning occurs as individuals add to and reorganize their knowledge structures and the view of teachers as facilitators of knowledge construction. It has been argued that, in order for teachers to facilitate the construction of knowledge by individuals, it is essential that they attend to the thinking of individual children (Carpenter, Fennema, Peterson, & Carey, 1988). Recent research indicates that teachers do attend to individual children's mathematical thinking (Weisbeck, 1992), but there is little evidence about factors that influence their ability to do so. This study provides such evidence.

Case studies were conducted of two kindergarten teachers using participant-observational fieldwork methods. The teachers had both participated in Cognitively Guided Instruction (CGI) workshops. The foci of the study were: the teachers' knowledge of the mathematical thinking of individual children in their classes; the ways in which they acquired that knowledge; and the ways in which they used that knowledge as the basis of instructional decisions. The teachers' beliefs about learning and teaching mathematics and about CGI, along with their knowledge of research-based information shared at the CGI workshops, were investigated in order to increase understanding of the foci.

The teachers' beliefs about learning and teaching mathematics were consistent with their interpretations of CGI and with the ways in which they taught mathematics, including the ways they learned about children's thinking and how they used what they learned. These, in turn, were related to what they knew about their children's mathematical thinking. The case studies confirm the importance of attending to teachers' beliefs about learning and teaching when attempting to understand their instruction. They also suggest particular beliefs that may contribute to or create barriers to teachers interpreting professional development projects in ways consistent with the reform movement in education.

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SEMIOTIC MODELS AND THE DEVELOPMENT OF SECONDARY SCHOOL SPATIAL KNOWLEDGE

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Senior secondary school pupils in spatial activities, e.g. descriptive geometry, in topology, in cartography, in machine production and in construction drawings are suppose to be on an advanced level of spatial thinking. Spatial thinking refers to the ability to successfully handle image types of full scale models, conventional graphic models and semiotic models. This implies that they have mastered the tactile, verbal, visual and mental skills in their mathematical education in order to execute the necessary transformations and indices of transformations. Those pupils will function not only on the third level of Van Hiele (abstract/ relational), but maybe also on his fourth (formal deduction) level of geometrical thinking. However this will be true for only a handful pupils in South Africa. The problems with second language teaching and not well developed teacher competencies are playing a major role locally.

In this discussion we will take a closer look at the role played by semiotic models in the process of developing spatial knowledge. The successful utilisation of semiotic models implies a well developed lingual and representational competence. Spatial knowledge in a child develops through the continuous interaction with full scale models, conventional graphic images (with respect to functional, assembly and structural diagrams) and semiotic models.

An important question to answer is How does the semiotic competence impact on the success of handling the different spatial demands when moving between positional transformations, structural transformations and combinations of structural and positional transformations which is reflected in Euclidean, topological and projective geometries?

An investigation was made into the following semiotic models, i.e. graphs, geographic maps, topographic maps, diagrams, chemical formulas and equations, mathematical symbols and other interpretative semiotic systems. Exemplary cases will be provided to illustrate the development from the real/concrete Euclidean transformations to complicated abstract transformational combinations.

Knowledge of the above-mentioned is of importance for researchers, the subject didacticians, lecturers at college/university level and teachers. This knowledge illuminates the composition and development of spatial thinking involved in the different aspects of human development and education. It is fundamental for our view on how children learn geometry. It has direct and important implications for mathematics teacher training, the designing of the curriculum and the teaching of certain sections of algebra, the whole field of spatial development, i.e. the spectrum of technical, science and IT teaching and learning. This should cover the vocational and professional training in many industrial and engineering fields of knowledge and production as well.

POSTER PRESENATIONS

K-12 CONNECTIONS IN UNDERSTANDING PROBABILITY & STATISTICS

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The long-term AAAS "Project 2061" promotes literacy in the closely connected areas of natural and social science, mathematics, and technology. In our books *Science for All Americans* (1989) and *Benchmarks for Science Literacy* (1993), we have attempted to define adult literacy in these areas and propose steps (at grades 2, 5, 8, and 12) by which that literacy might be acquired. A significant strategy in our work is mapping K-12 growth of student understanding — just what ideas have to be put together to get to each new step. The mapping work is based partly on the logic of the concepts and, insofar as possible, on the published research into how students understand and learn — both in general and with regard to specific concepts. This kind of map suggests what is required to arrive at each new idea and where it leads next, thus helping teachers to better understanding of the ideas themselves and of what to emphasize about them at different grade levels. The maps have been found very helpful in developing and evaluating learning goals, curriculum materials, instruction, and assessment tasks.

The draft growth-of-understanding map we have labeled "Uncertainty" portrays a coherent cluster of interconnected ideas in mathematical and scientific thinking, including mathematical modeling, proportional reasoning, interpretation of tables and graphs, summarizing data, sampling, correlation, logic, and habits of mind (such as skeptical critique of statistical claims in the mass media). Exhibiting the current draft of this map in a poster session is intended both to stimulate conference participants to think more about connections and to elicit their thoughts about how to expand and improve the map's representation of them.

PROBLEM SOLVING: TEACHING AND TEACHER EDUCATION

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Problem solving is one of the most important goals of mathematics teaching because it contributs to develop students' reasoning capacities and competencies (Charles, 1982; Fernandes, 1992). However, there are several complex and interdependent factors (cognitive, metacognitive, emotional, cultural, social) that affect the development of those capacities and competencies.

As mathematics educators, we have been realising that despite the optimism that our teachers students reveals about the teaching of mathematics and, particulary, about problem solving, their teaching practices seems to contrast with the idealistic visions they hold. There is a somewhat discouraging situation in portuguese middle and secondary schools (Grades 5-12). We all know that some particular school contexts do not facilitate the implementation of innovative ideas beginning teachers bring from preservice programs.

However, we do believe that preservice teachers can make a difference in the future development of mathematics education. We need to provide them adequate learning environments and we need to pay closer attention to them as persons (including their conceptions, beliefs, and knowledge) who are engaged in a life-long process of human and professional development.

Having these ideas in mind, we develop a research program including 1) a preservice program specially conceived and 2) the analisys of the effects of the program in the practices of some of the beginning teachers. The poster presents the research project, detailing teachers preservice program (philosophy, environments, materials and reflections about the experience that the participants lived) and the main findings about conceptions, knowledge, and practices of the teachers.

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* The research team includes also Domingos Fernandes and Isabel Cabrita (University of Aveiro), Pedro Palhares (University of Minho), Isabel Vale and Lina Fonseca (High School of Viana do Castelo), and Ana Leitão and Helena Fernandes (High School of Bragança)

An exploratory investigation about concept formation of quadrilaterals in second grade students

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According to NCTM Evaluation Standards (1989; p. 223) concepts are *the substance of mathematical knowledge and* assesment tasks *should focus on students' abilities to discriminate between the relevant and irrelevant attributes of a concept, in selecting examples and nonexamples, to represent concepts in various ways, and to recognize their various meanings.*

In previous studies Brito, Pirola e Lima (1997) studied some aspects of concept formation and the related levels. Other authors (Nakahara, 1995; Nasser, 1996; Mariotti, 1996; Gutiérrez, 1996) presented significant aspects of the formation of geometric concepts and the abilities related to them.

Based on the concept formation model (Klausmeier, 1977), according to which concepts is learned in four different levels (concrete, identity, classificatory and formal), the present investigation analised the process of concept formation of 4 quadrilaterals (square, rectangle, lozenge and parallelogram) by high school students and also analised the concept level presented by those student.

Subjects were 60 high school students (10^{th} and 12^{th} grades) from a public school located in a small town in South of Brazil. With the purpose of analising the irrelevant and relevant attributes used to define the different types of quadrilaterals the subjects answered questions, using a paper and pencil test, about the definition and properties of quadrilaterls, and choose among examples and nonexamples of some geometric figures. In adition, the students also described how their constructions were done and it permitted to analise the cognitive process used to include a particular type of quadrilaterals in a specific class and to recognise all of them as quadrilaterals.

The qualitative analysis of protocols and statistical analysis of the data revealed that most students were in the "identity level" for all kinds of quadrilaterals studied. The results showed also that, in general, students in the 10^{th} grade had a better performance compared to students in the 12^{th} grade.

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DEVELOPING TEACHERS PROFESSIONAL KNOWLEDGE IN A CONTEXT OF A CURRICULAR PROJECT

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In the context of a research project* aiming to study teachers' professional knowledge and development (Ponte, 1994), a one year in-service program for secondary teachers was created. This program promoted collaborative work and reflection (Schön, 1992) of the teachers for the conception, development and evaluation of a curricular project (Boutinet, 1990). The program included: 1) diagnosis of the problems teachers felt about the teaching of mathematics; 2) conception, implementation and evaluation of the curricular project.

The curricular project, evolving functions and graphs, was developed with 15-16 years old students and intended to help them to give sense of mathematics (the major dificulty teachers identified) through the proposal of tasks that relates mathematics with reality.

The curricular project was a powerfull context not only to develop teachers didactical knowledge about functions and graphs, namely the mathematical discussion of realistic examples of everyday life, but also to develop discussion and help teachers to reflect on their own practice.

The poster presents the research project, detailing in-service program (describes the work of a group of seven teachers including the characterization of their curricular project, some of the materials produced, and reflections about the experience they lived) and stressing the role of collaborative work and reflection in the context of a curricular project for teachers' professional development.

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* The research team includes presenting autors and also João Pedro Ponte, Henrique Guimarães, and Leonor Cunha Leal (University of Lisbon).

CLASSROOM CLIMATES THAT INHIBIT INVESTIGATIVE LEARNING FOR LIFE

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This paper relates the experiences of teaching Mathematics to B.Ed students, majoring or minoring in Mathematics at a tertiary institution, to the results of a research on interactions in mathematics classes at the primary school level. The research was directed to primary school level due to the fact that in Papua New Guinea, foundation for formal education is laid at the primary school level hence the teaching strategies used by the teachers at this level will have a lasting influence on pupils learning for life, which makes this level very crucial to the educational development of this nation.

From the complaints of the student-teachers, it is apparent that they expect the lecturers to explain mathematical concepts followed by solutions of problems related to these concepts which they should be allowed to copy and then provide work sheets with similar problems containing answers. Any teaching strategy that deviated from this was considered inappropriate. They demanded that teaching should confirm to what they are used to which does not include investigative approach. Most of the student-teachers in Mathematics, on peer teaching and practical teaching, demonstrated non-investigative approach in teaching. This made it necessary to obtain first-hand information about teaching at the primary school level, which initiated the research on interactions in mathematics classes at the primary school level.

Nineteen primary school teachers, teaching grades four to six, were involved in the study. Using a pre validated observation instrument, at least three mathematics lessons of each of the teachers were observed, coded and analysed.

Results indicated that teacher to whole class interactions far outweighed teacher to individual interactions (4:1), group to teacher responses were much higher than student to teacher responses (7:1), there was a lot of lecture while very little was done to relate the content to actual life situation(8:1), questions to challenge the students were never asked and students voluntary contributions were almost nil. In short, teachers dominated the lessons and students were passive participants in the class, which is in line with the expectations and the classroom practices of the student-teachers.

In order to improve the quality of teaching and learning of mathematics in the country, inservice training for primary school teachers and use of teaching strategies that urge learners to be investigative in their learning by teacher-educators is recommended.

INTERNATIONALISATION AND GLOBALISATION OF PROFESSIONAL DEVELOPMENT IN MATHEMATICS EDUCATION

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Mathematics education is one of the most internationalised areas of higher education. This is evidenced in similarity of the curriculum reforms happening around the world and the number of international conferences and research activities that have propagated during the past few decades. Yet, very few research studies have problematised the processes and outcomes of such internationalisation. This proposed study investigates issues related to internationalisation and globalisation of mathematics education with special emphasis on the professional development of educators with emphasis on four main regions: Australia, Asia, Latin America and South Africa.

Internationalisation of mathematics education is taken here to mean the integration of an international dimension in the curriculum, research and professional development including activities that promote inter-country collaborations. The term *globalisation of mathematics education* is taken here to refer to the phenomenon of knowledge, values, principles and curricula developed in some context gaining a global adherence.

Within the context of internationalisation, questions arise as to whether mathematics education, or parts of it, are becoming globalised. Usiskin (1992) has noted "the extent to which countries have become close in how they think about their problems and, as a consequence, what they are doing in mathematics education" (p. 19), but also hoped "that the new world order does not result in a common world-wide curriculum; our differences provide the best situation for curriculum development and implementation" (p. 20). Nebres (1995) has argued that "The more global and multicultural we seek to become, the deeper must be our local cultural roots" (p. 39). This proposed study is an investigation of how local context can be accounted for within international collaborative activities. It particular it will seek:

- to investigate issues and processes related to internationalisation and globalisation of mathematics education with reference to professional development
- to develop model(s) for cross-country collaborative activities in professional development of mathematics educators; and
- to develop and trial collaborative professional development activities based on the model(s)

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GRADE 1 STUDENTS' STRATEGIES FOR SOLVING SHARING PROBLEMS WITH REMAINDERS

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While the teaching and learning of fractions is problematic for primary school teachers and students, it has been documented that very young children can make sense of equal sharing problems with remainders (e.g. Empson, 1995). This poster reports on a base-line study in which Grade 1 students' solutions of such problems were investigated. A teaching approach in which fractions are introduced using such problems is currently being implemented to investigate whether limiting constructions associated with fractions can be prevented in this way, as has been suggested by, for example, Empson (1995) and Murray, Olivier and Human (1996).

61 Grade 1 students from 3 schools were interviewed by members of the MALATI fraction group in 1997. The students were presented verbally with problems that involved equal sharing of chocolate bars (3 between 2, 4 among 3 and 5 among 3). They were encouraged to draw their solutions and explain their reasoning. On the basis of these drawings and explanations, two project workers coded their strategies.

In this poster, students' work showing their strategies as well as their misconceptions will be presented. The frequency of each will also be indicated. The students used two main strategies, namely sharing the maximum number of whole chocolate bars and then dividing the left-over bar into a number of pieces, and dividing each chocolate bar into a number of pieces. Although some students created another friend (or mother!) for the left-over chocolate bar, most students were able to divide it into an appropriate number of parts. However, students did not always know what to call each friend's share, for example referring to $1\frac{1}{2}$ as 'two pieces' or $1\frac{1}{3}$ as $1\frac{1}{2}$. Some students were able to use only halves and quarters, although they were often aware that they had not shared all the chocolate equally.

Most of the Grade 1 students in this study were clearly able to make sense of the sharing problems with remainders, although they did not always have access to the correct terminology to express their solutions. We suggest that the teacher plays a crucial role in introducing the appropriate terminology when the need arises, and further facilitating the development of pupils' strategies so that they form a solid basis for the development of the concept of a fraction. Further research will indicate whether or not this has indeed been the case for these students.

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The Role of Visualization in Teaching Spatial Geometry

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This experience is part of a research that was accomplished at Colégio Pedro II, a federal school in Rio de Janeiro, Brazil, during three years, and that involved students aged 16 to 17.

Purposes: i) to identify the basic abilities to develop spatial thought; ii) to identify the factors that facilitate or hinder the development of spatial thought; iii) to verify how visualization and perception of geometric forms influence the construction of mental images.

Methodology: This research was divided in three parts: i) application of a diagnostic activity, called Parapris involving elementary concepts and inspired by the geometry teaching / learning processes developed by Hershkowitz; ii) researches accomplished with Mathematics teachers to identify their difficulties in teaching their courses of spatial geometry. Each teacher answered a questionnaire and participated of an interview which was tape registered; iii) application of activities that were elaborated starting from a transverse study through the four levels of development of knowledge based on Van Hiele theory (recognition, analysis, synthesis and formal deduction), and the necessary abilities for this development based on Alan Hoffer theory (visual, graph, linguistics and logic). In the next step the researchers built activities that were applied to the students and analyzed. The visual ability was the starting point of this analysis, the other abilities.were referred to this analysis.

Partial result: after the analysis of the third part activities, it was verified that the students could transfer a knowledge from one situation for another, did not need anymore the presence of the geometric illustration suggested by the proposals of activities, and that they could work with the representations of those illustrations to solve problems.The research showed those the four abilities were developed together, and for this reason, the visualization in the teaching of spatial geometry cannot be neglected.

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TEACHING AN INTRODUCTORY STATISTICS COURSE TO SOCIAL SCIENCE STUDENTS: A CASE STUDY APPROACH

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"[Statistical] techniques are often taught in isolation, with inadequate motivation and with no connection to the philosophy that connects them to real events" (Hogg, 1992, p. 36).

This presentation aims at exploring the use of a case study approach to address this rhetoric. By using problem-based learning illustrated with a real-life event, 'sadistics' might once again become statistics.

Concepts rather than complex formulas

Statistical notation can become dauntingly complex to a pre-graduate in the social sciences. The case study aims to introduce concepts and logical thinking in research rather than expecting students to master complex formulas.

Problem-solving rather than ideal solutions

The case study has built-in difficulties that could be encountered by inexperienced researchers. Errors made by statistics students are not often conveyed in traditional textbooks.

Interpretation and application rather than isolation

Students are expected to interpret results and apply them to the actual event outlined in the case study. This prevents students from seeing statistics as an end instead of as a means to an end. This approach aims to promote the usefulness of statistics in everyday social science research using a MSPowerpoint presentation better demonstrating the important issues than an oral presentation.

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The Contribution of Exploratory Talk to Mathematical Learning

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One of the key descriptors of classroom talk, as classified by Barnes (1976), is 'exploratory talk'. Barnes uses the term to describe the type of talk which contributes directly to learning taking place. Subsequently, Mercer (1995) elaborated types of classroom talk to include 'cumulative talk', which contributes to 'exploratory talk', but does not have such a direct effect on learning, and 'disputational talk' which contributes little to learning.

The use of pupil discourse in small group activities as a means of mathematics learning has been described in terms of theoretical models (Ernest 1992, Burton 1995 and 1996) and in small group cooperative work in classrooms (Webb 1995, Hart 1993). However there is little research linking the quality of mathematical thinking represented by the 'exploratory talk' and the use of small group problem-solving structures in secondary mathematics classrooms.

This study is designed to determine whether the incidence of 'exploratory talk' is affected by the degree to which teacher and pupils share common beliefs about the role of small group talk as a means to mathematical learning. Pupil-pupil talk talk within small group activity in a secondary mathematics classroom is analysed to identify the 'exploratory talk' category described by Mercer (1995). Interview data provides evidence of pupil perceptions and teacher perceptions about the role and function of small group activity in mathematics lessons. The results illustrate the relationship between the occurrence of 'exploratory talk' in secondary mathematics classrooms and the degree to which pupils and teacher share common goals relating to small group work.

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5th Grade Students Work on Assimilation Paradigms: Algebraic Thinking and Multicultural Environments

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This poster presents how students at 5th grade develop and deal with algebraic thinking specifically, how do teachers provide students with assimilation paradigms that will be useful for constructing knowledge. This is a component of a cross-cultural, longitudinal research that is being conducted by mathematics educators from Santa Úrsula University Brasil and Rutgers The State University of New Jersey.

This study took place in a 5th grade mathematics classroom in Rio de Janeiro, Brazil. One group of four students was selected for observation.

The framework was based on Robert Davis work. According to Robert Davis, students build new ideas from old ones.It is interesting to provide students with assimilation paradigms. Teachers should elaborate metaphors in order to favor the students to develop mathematical ideas and concept.

The methaphor in question should relate to negative numbers.

The first task was "Pebbles in the Bag". This activity was elaborated by Robert Davis for african students. It was part of their culture to play with pebbles so for these students it worked very finely and interestingly. When this game was played by brazilian students they were not quite interested in it. After three months the brazilian students were interviewed separately and they did not remember any part of the game but the "clap hands". So Brazilian team developed a "parking garage"activity. The students were involved all the time, came out with different ways to add and subtract negative numbers. We conclude that it is very important to pay strong attention to multicultural diversity when elaborating a lesson. Because it is not that the students were not able to understand one game or activity but it may be that such a game is not part of their culture. We suggest that culture including language, of course, influences the process of learning mathematics.

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CONTEXTUALISING DIDACTICAL KNOWLEDGE ABOUT STOCHASTICS IN MATHEMATIC TEACHER'S TRAINING

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In this presentation we emphasize the role that the didactical analysis of the meaning of mathematical objects and their adaptations to different educational institutions can play in the mathematics teacher's education. We also highlight the necessity of studying the knowledge produced by didactic research into teaching and learning processes. Finally, we present an example of this kind of analysis, based on a problem situation concerning the teaching of elementary probability, which contextualise the epistemological reflection on stochastics and the study of didactic knowledge about this theme.

Since courses directed to teachers' training should reflect the methodological principles suggested for pupils' mathematical instruction, and. taking into account the constructivist and social perspective of mathematics education, we consider that didactic knowledge should be contextualised in situations meaningful for teachers.

This view poses the problem of designing and developing problem situations for teachers' didactic training. In particular, these situations would allow the teacher to didactically reflect on mathematics, and to study didactic research results on errors and learning difficulties, as well as teaching methods and resources.

Training teachers to teach stochastics has particular interest, due to the emphasis on this content in recent curricular reforms and to the specific features of stochastic reasoning and knowledge with respect to other topics in the mathematics curriculum.

The situation described allow us to contextualise the reflection about the meaning of elementary stochastic notions and the related didactical knowledge. In our proposal, we emphasize the epistemological reflection on fundamental stochastic ideas, the analysis of pupils' difficulties and obstacles, and the identification of the corresponding didactical variables.

The situation is an example of what we call 'didactical analysis of mathematical content', which includes the study of the meaning of mathematical objects in mathematics and other institutions, the particular meaning that they adopt in teaching and learning situations, and the study of pupils and teachers' relationships with mathematical objects.

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A MATHEMATICS TEACHER'S CONCEPTUAL ORIENTATION (CO)

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Current reform efforts in school mathematics in the United States and in South Africa, call for a conception of mathematics which brings knowing and doing mathematics in school closer to the discipline. An illuminating way of talking about a teacher's classroom practice in relation to these reform efforts, is to talk about what I call a teacher's conceptual orientation, (CO), which consists of her conception of mathematics, her theory of how one teaches mathematics and her theory of how one learns mathematics.

How does the teacher think about mathematics ? The formalist view or conception of mathematics which is dominant, is *one* aspect of mathematics that portrays it as deductive, purely formal and symbolic; but mathematics in the making appears as an experimental, inductive science Pólya (1945/1988). This second aspect is new in one respect: mathematics "in statu nascendi," in the process of being invented has never before been presented in quite this manner to the student, or to the teacher, or to the public in general (p. vii).

How does the teacher think about teaching mathematics ? Good, Grouws and Ebmeier (1983) make generic recommendations on the teaching of mathematics that are virtually free of mathematics content. However Lampert (1990) articulates a position on the teaching of mathematics that deliberately alters roles and responsibilities of teacher and students. In her classroom students are encouraged to make conjectures, abstract mathematical properties, explain their reasoning, validate their assertions, and discuss and question their own thinking and the thinking of others (p. 33). This kind of teaching is analogous to mathematics "in the making."

How does the teacher think about learning mathematics? Skemp (1987) articulates three different theories of learning mathematics. In fact, he writes about three kinds of "understanding" which he connects to three goals of learning and these are: instrumental understanding, relational understanding and logical understanding (p. 166).

The purpose of the poster session will be to invite discussion about my construct in relation to reform efforts in school mathematics in South Africa.

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USING PERFORMANCE ITEMS TO ASSESS MATHEMATICAL REASONING AND COMMUNICATION

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In this paper we examine the use of new levels of mathematical assessment from many angles, seeking to describe more clearly what learning is, where and how it happens best, how it can improve, and how research can help. From our observational studies we are able to describe and compare the strategies of good and poor learners and the effects of discussion, disagreement, and teamwork among learners. For the past five years we have examined learning in math classrooms with students from grades 3, 6, 8, and 10. Results from these studies will be shared and the findings summarized to help shape more effective instruction.

This paper focuses on how the use of open-ended mathematics tasks and holistic scoring rubrics can facilitate teacher-student interaction and communication in classrooms. Open-ended tasks provide students with opportunities to display their mathematical thinking, reasoning, and problem solving. The use of holistic scoring rubrics allows for the criteria for evaluating students' responses to the open-ended tasks to be transparent to both students and teachers.

Teachers who use performance tasks and analyze student responses qualitatively can obtain important information about students' solution strategies and misconceptions, and can make instructional decisions based on that information. This paper utilizes a qualitative analytic framework and presents detailed information regarding the types of representations and strategies students use and the nature of student errors and misconceptions. A set of steps for integration of the assessment information into classroom instruction is provided.

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TEACHING STATISTICS IN PRIMARY SCHOOL IN SLOVENIA

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Encouraging mathematical literacy is an essential part of mathematics education. In the early years of schooling mathematical literacy is best described by using and applying mathematics. Children need to know how to use mathematics in their everyday life, how to make mathematics alive. Elementary statistics is a subject in mathematics that enables children and teachers to view mathematics, especially in the primary school, as a practical subject. In Slovenia some attempts were made to enrich mathematics in primary school by statistic. We believe that statistics is, beside practical skills children gain when doing statistics, an important subject in mathematics mainly because of the processes children do when solving problems in statistics: they collect, reason, sort, represent data, discuss and draw some conclusions about data. We strongly believe that these processes deserve greater emphasis in a mathematics curriculum and that statistics might contribute to encourage them.

When introducing ideas of elementary statistics the main consideration is given to representation of mathematical ideas, language children and teachers use and methods of teaching mathematics. In terms of methods the emphasis is given to problem solving.

The poster illustrates an introduction of statistics in primary school in Slovenia which was developed within a research project from 1993 to 1996. We chose ten primary schools to work with us on this project. About twenty teachers agreed to try out the materials we prepared for them and for children. We constantly monitored the teaching-learning process of statistics, and improved the materials according to experiences teachers and children had. As a result of this project two workbooks for children aged from 7 to 10 and teachers' notes on statistics were published.

"Math Is Next" – A Computerized Database For Developing the Mathematical Thinking of Young Children

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"Math Is Next", a computerized database for the kindergarten teacher and the (teacher training for teaching young children), contains a variety of activities for developing the mathematical thinking of young children. The project consists of:

1. Writing activities for the computerized database.
2. Accompanying study in kindergarten and lower grade levels for evaluating the activities and the support these provide to the kindergarten teachers.
3. Constructing and operating a teaching program that integrates the database in the training program of teacher training colleges.
4. Follow-up study on changes in the attitudes and knowledge of students who have been exposed to the database.

The computerized activity database, "Math Is Coming Soon", offers many opportunities to kindergarten teachers and teacher trainees. Rapid access and availability of abundant material organized a methodical manner, many information crosslinks presented both in graduated and continuous form, as well as both personal and group control and follow-up, permit relating to the personal needs of each and every pupil. The encounter with the various tasks in the database enables the child to sense the environment in an experiential, multisensory manner, to define, characterize and classify, and to observe relationships integrated with mathematical language. Searching for activities in the database is effected by means of a table that on one side lists the academic skills (mathematical concepts and development of thinking skills), while integrating on its other side the cognitive processes through which we transmit information (processing of visual, audio, and sensomotor information).

Such information processing capabilities fill an important function in everyday life and in acquiring the foundations of academic study, reading and arithmetic. Junctions of coordinates form "cells", and we inserted in each such cell the appropriate activities for the two coordinates comprising the cell. The activity card contains a description of the activity plus additional information required to activate the children (the educational objective, activity level, aids, number of participants, preliminary requirements and vocabulary). Attached to the activity card are work sheets and suggestions for similar aids to be prepared for the children.

Database development was accompanied by a study and qualitative evaluation carried out in twenty-six kindergartens. The study examined the degree of suitability of the computerized database to the kindergarten teacher's needs, from the standpoint of contents, search criteria and examination of how the software was employed. Examined as well was the effect of using the computerized database on work methods and contents in mathematical thinking in the kindergarten.

The database is integrated in part of the studies in the various courses forming part of the training of teachers for young children, and serves as a model for teaching mathematics in accordance with didactic principles.

Drawn, Telegraphic, and Verbal formats for Addition and Subtraction Word Problems

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The purpose of this study is to investigate problem solving performances of second grade students in three different problem formats: Drawn, Telegraphic and Verbal (Moyer, Sowder, Threadgill-Sowder, & Moyer, 1984). The investigators attended to semantic structures of addition and subtraction word problems (Verschaffel and De Corte, 1996) and used realistic contexts (van den Heuvel-Panhuizen, 1996) to develop a test of 21 word problems. Each problem was then embedded each in all three formats, resulting a total of 63 test items. A test on general mathematics performances and a test of reading comprehension were also developed. A total of 473 second grade elementary school children participated in the study and completed all tests. The investigators looked at children's problem solving peformances in each problem format and analyzed its relationships with mathematics ability, reading abiltiy, semantic structure, and error patterns. Results indicated that (1) high perfoming children (in mathematics, in reading) outperformed the low performing in solving problems, regardless of problem formats; (2) for the low mathematics performing group, children's performance were best in Drawn format; and (3) when solving CHANGE and COMBINE problems, children's performances were best in Telegraphic format but when solving COMPARE problems, students' performances were best in Drawn format; and, (4) error patterns were associated with problem formats. Interviews were conducted to a stratified sample of 40 children. Children preferred Drawn format and best remembered a problem in Drawn format. Also, problems in Telegraphic format were most efficient in assisting children to recognise a familiar problem. Finally, children performance in checking their solution was best in Verbal formats. Findings from tests and interviews yielded instructional implications. While verbal format were predominanty used for instruction and assessment, other problem formats (e.g. Telegraphic, Drawn) are powerful in achieving other instructional goals. A consideration on problem formats and a balance on using them will add meaning to instruction and reduce biased in assessments.

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UNSKILLED WORKERS IN ADULT VOCATIONAL TRAINING: IDENTIFYING MULTIPLE INFORMAL METHODS AND COMPARING PROFICIENCY ACROSS CONTEXTS

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In 1997 a tightly structured interview was prepared containing a)multiple choice questions on experienced workplace needs and b)eight problems to be solved. Six of the eight problems were constructed as three 'problems in pairs'. Each 'problems in pairs' consisted of a formal problem and a semi-authentic problem with apparently similar mathematical essence. The two other problems were also semi-authentic and should uncover different informal methods. To each semi-authentic problem were concrete materials collected from common everyday settings: a supermarket sales ticket, a shirt with a discount sign, fruit syrup bottles, tape measure and folding rule. We named the problems semi-authentic as interview materials were everyday authentic while interview setting and questions were arranged for the sake of the research. We carried out interviews with 160 workers at five adult vocational training centres. The poster will show questionnaire with drawings of the enclosed concrete materials and photographs of how the interviews were conducted. The poster will demonstrate results on

- Multiple informal methods among adults of doing additions, coping with proportionals and treating percentages. Although methods differ, most methods led to correct solutions.
- Differences between proficiency in handling formal problems and handling semi-authentic problems with apparently similar mathematical essence. Results will be presented from the 'problems in pairs' concerning calculation of area and results from the 'problems in pairs' concerning algebraic expressions. It appeared that the adults had better results in semi-authentic context than in formal, especially the lowest educated adults.
- Experienced needs for math-containing skills at the workplace, such as counting blanks or money, using shop drawings, filling out forms and controlling salary statement.

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USE OF ENVIRONMENTAL ISSUES AS MANIPULATIVES IN MATHEMATICS INSTRUCTION: CONNECTING KIDS TO MATHEMATICS THROUGH THEIR REAL WORLD CONCERNS.

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This poster presentation offers classroom examples of math problems rooted in environmental issues. Each of these examples can be generalized to real world social, political, economic concerns which children connect with in a heartfelt way. The purpose of this study was to see if students struggling to understand math concepts are able to grasp them quickly if the examples are connected to their real world concerns. This constructivist technique is couched in a triadic approach to life long learning where meta-cognitive approaches like COGNET and reflective dialogue are joined with individualized instruction pedagogy. This broad approach is consistent with the goals of Curriculum2005's overall criteria for classroom approaches. This triadic approach is strengthened again when we connect math to specific heartfelt student concerns. This presentation will also introduce constructivist classroom environments that promote learner agency in learning mathematics. This presentation provides a visual map of how metaphorical examples serve to provide contextual connections to the real world that reach across cultural and discipline boundaries. Interesting examples encourage self-efficacy as the mediator helps each learner keep an optimal level of challenge in what Vygotsky refers to the learners Zone of Proximal Development (ZPD).

Thirty-one easily grasped frameworks provide the nuts and bolts of this aspect of the triad. This approach to teaching mathematics combines theoretical work on metacognition by Piaget, Haywood, Vygotsky, Feuerstein, and Greenberg with theoretical work on Reflective Practice and Dialogue by Mesirow and Peters with Pre-intern Teacher Mediation programs for individualized instruction developed by Rowell and others. These three broad perspectives contain those meta-cognitive tools that teach task approach planning, effective scanning techniques, problem identification, and other sub skills that combine to enable one to think effectively and critically. Specific classroom examples of some math lessons will be demonstrated.

REPRESENTATIONS OF MATHEMATICS' TEACHERS IN SÃO PAULO PUBLIC SCHOOL - BRAZIL

Ana Lúcia Manrique, Saddo Ag Almouloud, Cileda Coutinho, Tânia Campos, Célia Pires

PROEM - PUC-SP*

This poster brings the results of a diagnostic study that has been done in PUC-SP into the Continued Education Program. It involves 903 Mathematics' teachers from five Teaching Units in the Metropolitan area of São Paulo. It has been given a questionnaire for the teachers at the beggining of the project and their answers have been quantitatively and qualitatively analysed.

The "use of didactic and technological resources" has been pointed by them as an item of big importance in the capacitation process and "evaluation" is one of the less important ones. This opinion is maybe due to the behaviour of the teachers' teachers, which don't discuss the various meanings of the evaluation, such as "formative evaluation" and "somative evaluation"[1]. The first one has a formative influence and the second one is just a part of the scholar system. This proves that is most important the developing of such a concept and its various meanings between the teachers in their formative development.

The content "Operations with Absolute Rational Numbers" has been pointed as the most difficult in both the apprenticeship and in the mathematical sense. Besides, the pupils do a lot of errors. "Interest Rates and Percentage" and "Equations" have been classified as interesting and easy-going between the students.

The analysis of the data has revealed the importance of the content Geometry compared with the others one. The contact with the teachers shows that their practice in a classroom is different from their conceptions about Geometry. It can be said that the process teaching-learning, in Geometry, must create conditions in which the student goes from the level of the experimental geometry to that of the geometry with constructions with rule and straightedge and proving.

The detailed results of this research can be seen in the poster.

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THE MOBILIZATION OF MATHEMATICAL CONCEPTS IN STRATEGY GAMES BY MICROS

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After the advent of the computer, we have observed a big quantity of softwares concerned to the most various matters, including Education. In this context, one emphasizes the arising of researches about the use of softwares in teaching, and its effects in learning. Between them, we see several types of magazines and CD ROM'S about games. Corresponding, inside the Mathematical Education, the strategy games have been a target for the interestin of many studious, because of its concerning with mathematical concepts. This work is the result of a research in which we investigated in graduation students the mobilization of mathematical concepts in strategy games including in CD ROM'S and played by micros.

To participate of the research, we invited students of both sex, on four teaching instituitions, who where in mathematics graduation. Seven students of both sex, 20 - 28 years old acepted the invitation, and they were selected for the research. From October/97 to January/98, we observed those students at three-four sessions of strategy game, about forty minutes by session. Each session was registered for later transcription and analysis.

The major part of the students estabilished mathematical relations about the description of winnings strategy; they also estabilished the connection between the descripted relations and the directions of the game, and they ennounced the mathematical concepts mobilized during the sessions. From the research, we concluded that it's viable to make a work with strategy games played by micros with students, for to introduce mathematical concepts through games. We also concluded by constatation of the needing to enlarge and to deepen the researches about mathematical concepts lying inside strategy games played in micros, aiming to increase elements to the researches that were already in mathematical education about the matter.

ISSUES RAISED IN THE TEACHING OF THE MULTIPLICATIVE STRUCTURES IN ELEMENTARY SCHOOLS

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In many countries, not to say all, the teaching of mathematics poses problems with the pupils and the teacher. Although of a different degree, according to each country, all this problems raise the interrogation on understanding. The researchers are focusing on the question and especially stressed the understanding of the pupil in mathematics. Guy Brousseau was among the first ones to work the teaching aspect of understanding. The theory of the didactic situations is the result of this work.

Which situations are proposed to the pupils? Which progression is offered to them? Which mode of training are privileged on the teacher's side as well as the pupil's? Is the teacher always aware of the other occulted aspects of the concept taught?

Multiplicative structures

1. Multiplication

As lived in the observed classes, the teaching of the multiplication uses primarily the decontextualized statements where the pupils work with the numbers and requires, from the beginning, memory and a suitable language. Eventually, the only worked aspect is a procedure of calculation necessary to the procedure of resolution of a problem of the proportionality type. The following aspects: « writing » of a type of addition or of a number, means of enumeration, function seem absent from this teaching. The notion of function would facilitate the articulation with proportionality.

2. Proportionality

The teacher starts from situations, generally everyday life ones. It is necessary to await the teaching and learning of proportionality to give meaning to the multiplication learned previously, in order to serve as a tool of calculation in the procedure of resolution of a proportionality problem.

Functioning of learning

The analysis of the transcriptions of the sequences on proportionality (Kinshasa and Bordeaux) makes it possible to distinguish two great moments in the teacher's functioning, moments of uncertainty and certainty on the pupils'knowledge. This identification led me to release roughly three kinds of functioning of the teacher.

Science and Mathematics Teachers' Perceptions of their role as "Key" Teachers in the Eastern Cape Province

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One of the strategies utilised in the in-service teacher training and development of large numbers of teachers in a relatively short period of time, is the "cascade" strategy. This essentially involves training and equipping a select group of teachers to become "key" teachers. These "key" teachers then have the responsibility of training and developing their fellow teachers. However, little research exists which documents "key" teachers' perceptions of their role as teacher trainers and developers.

A project which adopts a "cascade" approach to teacher training and development in the Eastern Cape Province, is the Open Society Science and Mathematics Initiative (OSSMI). This project was launched to train and empower a group of maths and science teachers (referred to as "key" Teachers) at the University of Port Elizabeth and Rhodes University. The aim of the project is to utilise the expertise of these "key" teachers in training programmes in their respective geographical areas to reach the masses of under-qualified and un-qualified science and mathematics teachers.

Twenty nine teachers from different rural schools in the Eastern Cape are studying towards obtaining a Further Diploma in Education at the University of Port Elizabeth. A further 35 are being trained at Rhodes University.

This paper will focus on the perceptions of five of these "key" teachers, who have concluded formal training at the Unversity of Port Elizabeth regarding their role as "key" teachers. Information regarding their perceptions was obtained through evaluation questionnaires and structured interactive interviews.

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DEVELOPING MATHEMATICAL UNDERSTANDING

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In the research project entitled *Teachers' constructions of their roles in building mathematical understanding*, four case studies of teachers' work were developed. Mathematics lessons were videotaped and teachers were interviewed. This poster focuses on the procedure for handling the videotaped data.

The tapes were digitised, then teacher actions that seemed to fit pre-determined and emergent categories were identified. Snippets of video were pasted in a spreadsheet, with codes summarising their origins, start and stop frame numbers, descriptors to aid future recall and electronic searching, items to follow up in later interviews, etc.

Code		Teaching action	Start	Stop	Details
131.01		Stimulating recall	36	60	"With our work on endangered species ..."
131.02		Eliciting children's explanations	60	150	(Various children, until Brad mentions "scale"). * PRE-DECIDED CONCEPT PRIVILEGES RESPONSE. * OTHER PURPOSES OF THIS TYPE OF INTRO?
131.03		Using b/board	100	170	Writes scale of model car, to illustrate Brad's example.
131.04		Emphasising language	90	201	Repeats, stresses terms: *Scale*, *Compare*
131.05		Using example	82	201	"I was just saying ... no comparison for whale size."
131.06		Using aid	165	201	"On that chart ..."

A multimedia program was constructed so that different types of data can be linked and recalled. For instance, one category of actions that develop mathematical understanding is "Explaining". Some of its sub-categories are:

4. EXPLAINING

Claims about explaining
- Importance of
- Constraints to

Explaining strategies
- Verbal explication
- Using gestures
- Using diagrams or drawings
- Using aids
- Using recall, drawing on previous experience
- Using examples
  - Giving examples
  - Accepting uninvited examples
  - Eliciting examples

Eliciting children's explanations
- Questioning
- Prompting/provoking
- Inviting

Clarifying
- Providing clarification
- Seeking clarification
- Eliciting further information

Explaining activity
- Giving initial directions
- Explaining directions further
- Re-explaining directions

Interactive links have been constructed between each of these sub-category lables and snippets of the videos, interview audiotapes, field notes and quotations from references. This poster shows sample data from the spreadsheets and the multimedia program.

THE MATHAKUZANA GAME AS A DIDACTICAL RESOURCE FOR THE DEVELOPMENT OF NUMBER SENSE AND ORAL ARITHMETIC

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Personal classroom observations, carried out during pedagogical practice as a student of the Pedagogical University confirm the conclusion of Carraher et al. 1988, that teachers make little use of the mathematical knowledge that children develop outside school. Encouraged by ethnomathematicians like D'Ambrósio (1985) and Gerdes (1991), who maintain that mathematical knowledge is rooted in popular experience, I organized two groups of children, with whom I could study the mathematics used in popular games. One of the games we practiced and studied was Mathakuzana, an ancient game, which is found in different forms in many countries (Junod 1974).

Our Mathakuzana is played by two teams, of 3 to 5 children each. For each pair of players a hole is made where 12 pebbles or seeds – called 'children' – are put. The two teams sit opposite each other with the row of holes in the middle. Each pair of players has a bigger pebble, called 'mugungu'. The game is played in 2 phases. First, the players of one team throw the 'mugungu' up, remove all 12 'children' from their hole and have to catch the 'mugungu' in the air, using the same hand. Then they have to put the 'children' in the hole, 1 by 1, using the same method: throw the 'mugungu' in the air, put one 'child' in the hole and catch the 'mugungu' in the air. They repeat the activity, but moving the 'children' 2 by 2, then 3 by 3, and so on, until completing 'house twelve'. When a player fails, by not catching the 'mugungu' in the air or by not moving the right number of 'children', the turn goes to the adversary. The first phase ends, when one of the players manages to complete the 12 'houses' without failing.

Then starts the 2nd phase, during which each player puts the 12 'children' on top of his hands with the palms showing down. The player has to throw the 'children' up in the air and then try to catch in the air as many as possible. Each 'child' caught means one point for the team. All players have to announce the number of points made. Finally, each team has to calculate the total number of points made by its members.

During the first phase, the players use counting numbers, numerosity numbers, split 12 into 6 and 6, 5 and 7, 4 and 8, etc., have to recognize instantaneously number patterns, from 2 to 6 or 7. During the second phase, the players have to add three to five numbers that may vary from zero to twelve. The points made during one round have to be remembered because they are added after each round.

The two phases of the game develop different arithmetical skills, from subitizing to verbal adding of several addends, in part linked to fastness in observation and reaction. Teachers should study these mathematical elements and use them in teaching.

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'WHAT IS MATHEMATICS AND WHAT AM I LIKE AS A MATHEMATICIAN?': RESPONSES FROM SOME MATHEMATICS STUDENTS IN INITIAL TEACHER EDUCATION

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In this poster I report on the results of a questionnaire completed by 45 students shortly after their entry onto our initial mathematics teacher education programmes. The students were all enrolled on courses which included not only professional studies but an element of mathematical studies as well, that is, four year Bachelor of Education degrees for both primary and secondary and two year secondary Postgraduate Certificate of Education.

The students were asked to provide information about their previous education and work experience (a significant number of our entrants are mature); to give their views about the nature of mathematics; to assess themselves on a range of mathematical skills; and to reflect on their personal attitudes and attributes with respect to the doing of mathematics. These areas of enquiry were chosen because of their relationship to two key elements in the aims we have in teaching our courses. First, we consider it important for student teachers to recognise in themselves and others that effective self assessment and the establishment of self worth are affected by students' positioning with respect to sites of disadvantage (Open University, 1986). Second, a high priority is inviting students to reassess their views of the nature of mathematical knowledge: what each of us believes mathematics to be is a central factor in shaping our pedagogy (Ernest, 1991; Burton, 1995; Povey, 1997).

Some patterns emerge. I offer some speculations about how these might influence our pedagogy in initial teacher education for mathematics.

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PARSUS – A PROJECT FOR DEVELOPING NEW TEACHING AND LEARNING MATERIAL

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PARSUS stands for Partnership between schools and the University of Stellenbosch and is a joint venture between the University of Stellenbosch in South Africa and the University of Leuven in Belgium. Two main issues drive the project:

(i) research results that prove the importance of developing mathematical concepts from real life situations (Freudenthal 1983) and

(ii) the new proposed curriculum model for South Africa known as Curriculum 2005. Curriculum 2005 is a learner-centred and outcomes based curriculum that stresses the importance of bridging the gap between classroom experiences and the real world.

Real-life situations not only motivate the learning process, but are also believed to have a further advantage in that they trigger mathematical thinking in a special way (Van den Heuwel-Panhuizen, 1993). In South Africa however, most secondary school textbooks are written from the perspective of first the mathematics and then the real world. There is a need for a different approach.

Our project therefore aims at developing materials that promote the principle of moving primarily from a familiar environment to mathematics. Materials are prepared for teachers as well as students at grade 11 and 12 level. Fifteen secondary schools and about 1500 students are involved.The project is guided by research questions like the following:

- What difficulties do students have when using the new material?
- How can the material be improved?
- Does the material enhance the outcomes envisaged by Curriculum 2005?

This poster session will explain the methodology of the project, present some of the new material and illustrate initial research findings.

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PSYCHOLOGICAL FACTORS INFLUENCING MATERIALS DEVELOPMENT IN MATHEMATICS

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Increasing attention is being paid to pupils' learning and appreciation of concepts at primary schools. This poster reports on one such attempt by the Education Support Services Trust (ESST) in collaboration with a group of primary teachers, subject advisors and university staff to design classroom materials which make use of imagistic and strategic approaches to subject content in mathematics.

The poster contains a brief theoretical overview from the rejection of imagery in the 1970s and 1980s by cognitive psychologists because of its 'fuzziness' to attempts in the 1990s to clarify and question a number of associated concepts.

The greater part of school-based learning is often addressed to the rational domain and formal reasoning, sometimes neglecting the affective, motivational and imagistic domains. The poster illustrates pages of a Grade 3 text which encourages pupils to form **"pictures in the head"**.

The poster illustrates how learning materials can be used to encourage pupils to question the purpose of what they have been studying on a regular basis and locate their learning in a real-world and wider learning area.

Having stated these two main psychological factors which guided the collaborative materials development process, the next phase is outlined for the continuous assessment of the materials against formative development criteria. The poster gives examples of some of the evaluation instruments used by ESST to obtain data for classroom-based research.

The process of analysing Outcomes Based Education Curriculum 2005 frameworks and relating them to teachers and learning materials is also indicated, in a way which gives a base for Foundation Phase materials development. The research attempts to balance the psychological factors without losing the mathematics content, nor the mathematical processes involved within the content.

The poster lay-out demonstrates the learning-centred style of ESST materials.

INTRODUCING BASIC CONCEPTS OF CALCULUS WITH THE ALGORITHM FOR DRAWING GRAPHS OF FUNCTIONS

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"The acquisition of a wide repertoire of advanced mathematical thinking is a challenge which now faces university mathematicians". Accepting this challenge proposed by David Tall (1995; p.74), we developed an experience of using the computer in classroom to provide richer concept images of the most important notions of calculus, which are one of the main basis for the transition to advanced mathematical thinking.

The usage of the computer wasn't conceived only as a tool for the visualization or simulation of the concepts involved. On the contrary, we tried to explore the way it represents things. Tall (1992; p.504) states that magnifying graphs is a good cognitive root for the limit concept and our experience uses a similar principle.

To present the fundamental concepts of calculus, we ask the students to draw graphs in the computer in three different ways:

1 - Construct an algorithm to draw the graph of a function f in an interval $[a,b]$, calculating the values of f in near points, connecting them by line segments. In the limit, this process leads to the hypothetical necessity of drawing tangents, despite it isn't possible in reality because of the computer limitations. We used the fact that the first calculus course and the first computer course are simultaneous in our university to encourage students to do a program to draw graphs of functions, and from this point, we introduce the formal definition of derivative.

2 - Draw the graph of a function f in an interval $[a,b]$, given its value in a and the values for the slopes of its tangents in $]a,b[$. Varying the height $f(a)$, we would obtain various copies of the former graph. The notion of integral, in the sense of anti-derivative, and the Fundamental Theorem of Calculus can be naturally introduced.

3 - Generalizing the problem above mentioned, draw a curve given its initial point and the values of the slopes of its tangent vectors in some points. In this case, the variation of the initial point would generate a family of different curves. This exercise is pertinent to the discussion of the basic problem of differential equations.

We think that with this experience we can make the learning of calculus more procedural, escaping from the traditional "definition-theorem-proof-illustration" way of teaching at college level, motivating students to think mathematically instead of just learning enough to pass in examinations.

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Examining the second grade mathematics classroom from a social-constructivist perspective: the interrelationship of teaching, learning, learning to teach and teaching to learn

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In this research, the classroom community is viewed from a social-constructivist perspective where the instructional approach affects both how the children learn mathematics and how the student teachers learn for teaching mathematics.The classroom teaching experiment is used to look closely at the nature of that community of learners (Carpenter,et al, 1989; Cobb,et al, 1991; Hiebert & Wearne, 1993).

The main goal of the research was to uncover the relationship between the instructional approach in the mathematics classroom, the mathematics learned by the children, and the knowledge for teaching mathematics learned by the student teachers. This was accomplished by:

- Observation and identification of the features which characterized the two instructional approaches
- Evaluation of the children's mathematical knowledge from multiple perspectives
- Uncovering the student teachers' knowledge for teaching mathematics as represented in their reflections on specific and general aspects of the field experience

The methodology of the research was qualitative-naturalistic. It is a case study of four second-grade classrooms: two classrooms adopted the conventional, textbook-driven instructional approach and two classrooms adopted an alternative approach with an emphasis on problem solving and building the formal mathematics on children's intuitive knowledge. The research followed one student teacher in each classroom and the mathematics lessons which she taught.

The results of the research focus on the web of connections between the features of the instructional approach in the mathematics lesson , the knowledge of the children, and the knowledge of the student teacher for teaching mathematics. The mathematical activity which defines the classroom culture is viewed as the arena for reflection, for the children who are "learning" and the student teachers who are "teaching."

The implications of the research are in the areas of elementary mathematics education, pre-service education for teaching mathematics, and research on knowledge growth for teaching mathematics.

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STUDENTS' UNDERSTANDING OF ALGEBRAIC EXPRESSIONS: CONSIDERING COMPOUND EXPRESSIONS AS SINGLE OBJECTS

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Purpose and Methodology

So far, various studies have pointed out that the process/object nature is crucial for mathematical understanding. This study is trying to elicit some evidence for the difficulty that students have with the process/object nature of algebraic expressions. A written survey was administrated to 349 lower secondary shcoool students, and an interview followed.

Tasks given

Task 1: The simultaneous equation shown below was given.

$$\begin{cases} 2x + y = 5 \\ y = 13 - 3x \end{cases}$$

This simultaneous equation seems to be solved more easily by using the substitute strategy rather than addition-subtraction strategy. However about 30% of 8th graders insistently used the addition-subtraction strategy. The plausible reason is that students were taught the addition-subtraction strategy first, then moved to the substitute strategy. However the result of the survey revealed that the strategy used by students does not depend on the sequence of teaching of strategy for solving the simultaneous equations.

Task 2: If $a + 3b + 5c = 25$ then $a + 3b + 5c - 10 =$?

The interview concerning Task 2 has revealed the tendency that students who used the substitute strategy in Task 1 seemed to be more comfortable handling Task 2. In the interview a student expressed excitement when she was able to consider the part of the algebraic expression: $a + 3b + 5c$ as a whole, instead of considering it as an adding process of three terms.

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Programme for Early Mathematics

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To change teachers' beliefs about 'how to teach' is a challenge which this programme has undertaken with 11 primary schools in the Western Cape Province. Teachers from these schools have a background of a very traditional, restricted and inflexible way of teaching. This programme is endeavouring to change teachers' beliefs so that they will change their teaching approach to an approach where students:

- learn by working on meaningful problems
- reflect on what they did
- evaluate their work and its results themselves
- are required to explain systematically how they did so.

To develop and establish the above, teachers need to create an appropriate classroom atmosphere. The role of the teacher involves:

- selecting appropriate mathematical activities so that powerful mathematical thinking takes place;
- planning of lessons and organisation of the classroom set-up in such a way that every student is involved in mathematics;
- providing relevant information and facilitation of student discussion, but not to the extent that students abandon their thinking.

This programme entails workshops, continuous support through school visits, discussions with teachers and the making of essential teaching material.

The Research design: Grade 2 and 3 students' performance in April 1997 will be compared to the performance of Grade 2 and 3 students in the same schools in April 1998. The performance of Grade 2 and Grade 3 students was also tested in October 1997. The results from this first evaluation already showed a significant improvement in the average number of problems successfully solved ($p<0,001$ in each case) and the average total score as an indication of the level of mathematical understanding and expression ($p< 0,001$ in each case). The change in teacher beliefs was not as significant, but it did reflect a change in classroom organisation towards more group work, acceptance of students' own strategies and the role of word problems.

Content of Poster: Information on the Programme and a series of photographs with descriptions of what has been happening in primary schools involved in this Programme. The sequence of these photographs will depict the extent of the development of this Programme.

Transforming Geometric Transformations

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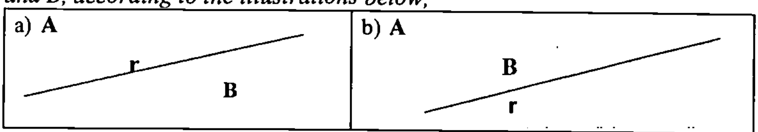
This paper is part of a research that is being performed by five teachers from MEM/USU in the project of teacher's Math training. The Subjects are Math Teachers who work in both private and public schools.

Goals: i) to develop visualization, perception and different representation of the same concept, so the teachers can work in real situations enabling them to work in their classroom teaching. ii)to make the relationship among the different representations performed by the teacher in the building of geometric ideas.
This research is based on the representation theories from Fischbein's and Vergnaud's works about intuition, perception and representation and also on Robert Davis's works in teaching formation.

Methodology: This research was performed using the plan geometric transformation to solve daily activities. Starting from simple activities enabling the teachers to gradually build the concept which was being studied, reaching the formalization either through Euclidean Geometry or Analitical Geometry tresspassing the epistemological obstacles. For example, starting from concrete material activities, they will later works with activities of this kind:
Find the point M in the line r, that minimizes the sum of the distances between the points A and B, according to the illustrations below;



In a later step the teachers were reguested to elaborate a real situation problem whose solution used the previous result. For example the problem was written by one of the teachers. *"A transmission eletrician on spot A needs to check the energization of the Aerial transmission line . So he can go to spot B where he should inform about this condition to the center of Operation of the System. To minimize the cost of this operation the itinerary that the helicopter has to follow should lie made using the smallest possible distance. Determine the ideal point to inspect the transmission line knowing that the points A and B are on the same side of the line".*

Partial results: The teachers: i)lived different approaches that led them to build and develop geometric concepts.ii) they started from the theory to solve these activities proposed and starting from these activities, they built geometric ideas.iii) the teachers thought of theses concepts under study from both the teachers' point of view and from the students' point of view as they were able to live during this investigation.

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CONNECTIONS BETWEEN PRESERVICE TEACHERS' BELIEFS ABOUT MATHEMATICS AND MATHEMATICS TEACHING AND THEIR INTERACTIONS IN CLASSROOMS USING A REFORM CURRICULUM

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Teachers generally look back on field experiences as the most valuable part of their formal teacher education program (Feiman-Nemser & Buchmann, 1985; Guyton & McIntyre, 1990; Lampert, 1988), yet research evidence suggests that field experiences may have a negative effect on the development of research-based and reflective teaching practices (Brown & Borko, 1992; Tabachnick & Zeichner, 1984). In particular, when teacher preparation programs respond more quickly to calls for reform than school classrooms, preservice teachers' field experiences are inconsistent with the expectations developed in their teacher education coursework.

The availability of teachers who have made a commitment to continually improve their teaching to meet the challenge of current calls for reform opens up new possibilities for beneficial field experiences. We are currently investigating what happens when preservice secondary school mathematics teachers are specifically placed in field experiences that support and amplify their NCTM *Standards*-based (1989, 1991) teacher education coursework.

As the first step in a four-year study of the effects of pre-intern and intern teaching in a reform environment, the authors assessed six preservice secondary school mathematics teachers' beliefs about mathematics and mathematics teaching. This poster session will look at these preservice teachers' beliefs and their initial interactions in classrooms that are using a reform curriculum.

The success of the current reform movement in mathematics rests in great part on classroom teachers. This, in turn, provides a tremendous challenge to teacher educators. As significant time, energy, and money is being spent to provide inservice teachers with the opportunities they need to prepare to teach in a radically different manner, we must make every effort to ensure that we are not increasing the task by preparing new teachers in old ways.

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