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ABSTRACT

This paper explores a framework for research on the development of the angle concept based on theories of abstraction. The framework suggests that children initially acquire a body of disconnected angle knowledge situated in everyday experiences, group the situations to form angle contexts, and then form an abstract angle concept. The framework is explored using grade 4 students (N=36). The students appear to form distinct concepts of slopes, turns, hinges, bends, rebounds, and corners, but are often unaware of the common angle features of these contexts. The findings are discussed using a comparison with a previous study of grade 2 students and suggest that students' responses often show failure to use standard terminology rather than a lack of knowledge of the underlying physical relations. Contains 22 references. (DDR)

DEVELOPMENT OF THE ANGLE CONCEPT BY ABSTRACTION FROM SITUATED KNOWLEDGE

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ABSTRACT

A framework for research on the development of the angle concept, based on the theories of abstraction (Skemp, 1986) and situated knowledge (Brown et al., 1989) is proposed. It is suggested that children initially acquire a body of disconnected angle knowledge situated in a large number of everyday experiences; they then group situations to form angle contexts such as turns and corners; and finally they form an abstract angle concept by recognising similarities across several angle contexts. To investigate this framework, 36 Grade 4 students were interviewed to find how they classified and conceptualised six angle contexts (represented by models of six physical angle situations). The students appeared to have formed distinct concepts of turns, slopes, hinges, bends, rebounds, and corners, but were often not aware of the common angle features of these six contexts. Specific irrelevant features of each context appear to hinder abstraction of a general angle concept. In particular, it would seem that many students do not recognise turning in all contexts. Implications for curriculum and instruction are discussed.

DEVELOPMENT OF THE ANGLE CONCEPT BY ABSTRACTION FROM SITUATED KNOWLEDGE

Length and angle are arguably the two most basic concepts which we use to analyse our spatial environment, and they are also fundamental to the study of geometry. But whereas most students master length early in primary school, angle causes difficulties well into secondary school (Close, 1982; Mitchelmore, 1983). One difficulty for teachers is knowing what definition of angles to use—amount of turning, union of two rays with a common end point, intersection of two half-planes have all been suggested (Close, 1982; Krainer, 1989). Each definition has its advantages and disadvantages (Strehl, 1983).

We shall make the basic assumption that the angle concept, like all other elementary mathematical concepts, derives ultimately from our experience of the physical world through the process Piaget called reflective abstraction (Dubinsky, 1991). The curriculum equivalent assumption is that teaching should link abstract ideas to students' everyday experience (National Council of Teachers of Mathematics, 1989). It would therefore seem imperative to look at common physical experience as a possible explanation for the complexities of the angle concept and for students' difficulties in learning it.

There has, however, been very little research on how students conceptualize the angles in their everyday experience. (We exclude here research on perceptual and graphic skills as peripheral to the question of how the angle concept develops.) Studies of how students interpret the word "angle" give some insights. For example, Krainer (1989) obtained a vast number of responses when he asked 12-year-old students to name everyday situations involving angles. The majority of the examples involved some sort of turning motion, and most of the static angles included a horizontal or vertical arm. Other researchers have specifically asked students to say what an angle is. For example, Davey & Pegg (1991) surveyed students from Grades 1 to 10 and reported that their descriptions of an angle went through a sequence of four stages: (1) a corner which is pointy or sharp; (2) a place where two lines meet; (3) the distance or area between two lines; and (4) the difference between the slopes of the lines. Clements & Battista (1989) found that Grade 3 students most commonly described an angle as a sloping line, a place where two lines meet, the two lines themselves, or a turn (the last only by students who had studied Logo). Foxman and Ruddock (1984) reported that the description most frequently given by 15 year olds was the distance or area between two lines (29% of responses); and although only 4% spontaneously mentioned turning, 41% referred to amount of turn after being shown a movable angle model.

Such studies suggest that students do conceptualize their surroundings in terms of angles, and that the way they do this develops in sophistication as they get older. But there are many questions left unanswered. From what physical experiences do students derive their angle concepts? Do students conceptualize different physical situations in different ways? How does physical experience influence students' angle concepts? How do students eventually develop a general concept of angle out of their physical experiences? We have, in fact, found no study which investigates students' understanding of any of the physical situations which underlies the angle concept or of the relation between physical angles and their abstract representations.

As Davey and Pegg (1991) put it, "the key issue that has plagued work on angles is the lack of a theoretical framework from which to consider the results of various studies" (p. 1). We shall now propose such a framework, derived by integrating the two theories of situated knowledge and abstraction.

A New Framework for Research on the Angle Concept

The theory of situated cognition (Brown, Collins & Duguid, 1989; Greeno, 1989) is the first component on which our framework is based. According to this theory, "the activity in which knowledge is developed and deployed ... is not separable from or ancillary to learning and cognition ... it is an integral part of what is learned" (Brown et al., 1989, p. 32). Most of what humans do is situated in particular contexts, drawing on knowledge of that context which has been built up as a result of previous experience. Situated knowledge comes only from the particular situation—it is not the result of applying general, abstract principles to that situation. Recent ethnomathematical studies have, for example, revealed sophisticated, mathematics-like activity among illiterates who show no evidence of abstract mathematical knowledge (Nunes, 1992).

The proponents of situated cognition have argued that school primarily teach abstract principles which students cannot then apply to specific situations. "The primary concern of schools often seems to be the transfer of ... abstract, decontextualized concepts" (Brown et al., 1989, p. 32), based on the theory that "knowledge acquired in 'context-free' circumstances is supposed to be available for general application in all contexts" (Lave, 1988, p. 9). However, "much of what is taught turns out to be almost useless in practice " (Brown et al., 1989, p. 32), consonant with "the meager evidence for transfer garnered from a very substantial body of work [in cognitive psychology]" (Lave, 1988, p. 32). We have recently called abstract, decontextualized knowledge *abstract-apart* and given further clear examples of students' inability to use abstract-apart mathematical knowledge to solve application problems (White & Mitchelmore, in press).

On the other hand, abstract mathematics is very powerful simply because it can be applied in so many situations (Freudenthal, 1983). We call such widely applicable knowledge *abstract-general.* If schooling led to abstract-general rather than abstract-apart concepts, the knowledge would be much more useful in practice. But if knowledge is inextricably located in separate situations, how can abstract-general knowledge develop?

One answer is provided by the theory of abstraction in mathematics. This theory was developed by Piaget, Dienes, and Skemp earlier this century and recently extended by Dreyfus, Dubinsky, and Sfard (Mitchelmore & White, in press). The core of this theory is summarised by Skemp as follows:

> *Abstracting* is an activity by which we become aware of similarities ... among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class. ... To distinguish between abstracting as an activity and abstraction as its end-product, we shall ... call the latter a *concept*. (Skemp, 1986, p. 21, emphasis in original)

Skemp also distinguishes between *primary* concepts, those derived from our sensory and motor experiences of the outer world, and higher-order concepts, those derived from other concepts.

A mathematical concept is thus first formed when children recognise a communality between superficially very similar situations. For example, the concept of whole number is initially formed from the process of counting small sets of objects. After that, a concept develops in two different ways (Mitchelmore, 1994). Firstly, it gradually becomes more and more general as other situations are seen to be similar to the original situations. For example, the concept of a whole number is extended to large numbers which cannot be counted and to measurement situations. Secondly, several distinct concepts may be seen as similar to one another at a deeper level; a new, higher-level concept is then formed which embraces them all. For example, whole numbers, negative integers, and rational and irrational numbers become subsumed in the concept of real number. Both kinds of concept development result in a concept becoming more abstract—in the sense that it is general to an increasing variety of superficially different situations.

On the basis of the above theories and previous research on angle, we suggest that the angle concept develops in three broad stages.

1. Well before coming to school, children experience many physical situations which will later be seen to involve angles. Children make their first abstraction when they recognise angle-related similarities between repeated experiences of physically and socially very similar objects or events (for example, walking up and down steep paths). Each similarity defines a class of objects or events that we shall call a *physical angle situation.* Each class corresponds, in Skemp's terms, to a primary concept (in our example, a sloping path). The initial class may extend to very similar situations (such as riding up and down hills.) Children develop knowledge of many angle situations which is specific to each situation and makes no sense in similar situations. For example, knowledge of sloping paths (such as knowing that steeper hills are more difficult to ride up) has no applicability to roofs. We shall use the term *situated angle knowledge* to refer to specific knowledge of individual physical angle situations.
2. At some stage, children begin to recognise deeper similarities between physical angle situations. For example, they might see that paths and roofs both slope and then gradually extend the idea to other examples of sloping objects. We shall call such a class of angle situations a *physical angle context.* Each angle context corresponds to a concept such as slope, turn, bend, and corner. Children develop what we might call *contextual angle knowledge:* knowledge which applies to all the situations included in each context but not necessarily across contexts. For example, children may be able to compare two slopes, or compare two turns, without being able to compare a slope and a turn. The formation of angle contexts represents the second level of abstraction from children's physical angle experiences.
3. Students then become aware of even deeper similarities between physical angle contexts. They gradually see that each angle context involves two lines meeting at a point and some relation between them. As students see that more and more angular relations (previously described as inclination, turning, relative direction, and so on, according to context) are "really the same", an abstract angle concept emerges until it is general to all angle contexts. Concepts such as slope, turn, bend, and corner may therefore be regarded as angle

subconcepts. What we might call *abstract angle knowledge* is that which applies to all angle subconcepts.

The difference between a physical angle situation, physical angle context, and an abstract angle can be seen in the way they are modelled. (1) A physical angle situation is modelled by a realistic physical model. Many children's toys, such as Lego houses and toy cranes, model angle situations. (2) A physical angle context is modelled by representing that which is common to all situations in that context. For example, slopes may be modelled by a sloping pencil and corners in general by cut-out cardboard corners. We conjecture that children's abstract models of angle contexts would pick out the superficially obvious geometrical features of that context, so that different contexts would be modelled in different ways. (3) An abstract angle is modelled by that which is common to all physical angle contexts: two lines meeting at a point and some relation between them.. When students have formed an abstract concept of angle, they would be able to match the lines on such an abstract model to lines in each physical angle context in such a way as to demonstrate the appropriate angular relation between them.

In summary, we are proposing that the angle concept develops from situational → contextual → abstract knowledge.

The present study

The above theoretical framework has, we believe, the potential both to explain many research results and to guide the design of more effective teaching strategies. For example, the various definitions of angle used in curriculum materials and given spontaneously by students would seem to model different angle subconcepts. Possibly not enough is done in school to help students link these subconcepts into a single, general concept of angle. It is also possible that students do not link the abstract models of angle used in textbook diagrams to the various physical angle contexts with which they might be familiar.

Nevertheless, the framework is purely theoretical and needs empirical verification. For example, we do not know whether students categorize their physical angle experiences into situations and contexts as we have described them—or, if they do, what physical angle situations and contexts they recognise. We have extended Krainer's (1989) list of angle situations from our own experience and classified them into 14 putative contexts (Mitchelmore, in press). An exploratory study of Grade 2 students (Mitchelmore, 1995) suggested that there was some validity in this classification and provided valuable insights into the difficulties students encounter in abstracting a general angle concept. The present study was designed to explore the proposed framework further. It investigates the following questions among Grade 4 students.

- What situated knowledge of some typical physical angle situations do students have?
- Into what angle contexts do students classify these physical angle situations?
- How do students model the angles in the various physical angle contexts? In particular, do they tend to model the superficially obvious geometrical features of each context?
- What characteristics of various physical angle contexts hinder students from recognising (a) the abstract angles implicit in these contexts, and (b) the angle-related similarity between different contexts?
- How is students' modelling of physical angle contexts related to their formal knowledge of the angle concept?

METHOD

Design

Six common but varied contexts (unlimited turns about a fixed axis, linear slopes, hinges, bends, rebounds, and plane corners) were selected and one supposedly familiar situation was chosen from each context (body turns, hills, scissors, road bends, cricket, and tiles). These were the same contexts and situations as previously investigated (Mitchelmore, 1995). Each context was represented by a simple physical model of the situation chosen (see Figure 1).

<Insert Figure 1 about here.>

For the interviews, the six contexts were divided into three pairs: two contexts expected to be seen as similar (slopes and corners), two with some physical similarities (hinges and bends), and two with no obvious similarity (turns and rebounds). In the previous study, these pairs were recognised as similar by all, half and none of the students respectively.

A sample of 36 students, average age 9.6 years, was selected from two Grade 4 classes in two schools in western Sydney. Twelve students (3 boys and 3 girls from each school) were interviewed on each pair of contexts, with order counterbalanced.

All interviews were conducted by a trained research assistant. For each situation presented, she probed three aspects of students' understanding: (1) their situated angle knowledge; (2) the extent of the physical angle context in which they included that situation; and (3) their methods of modelling the angle implicit in that context. After students had responded to their two contexts, two further aspects were probed: (4) the similarities which students recognised between the two contexts; and (5) students' knowledge of the formal concept of angle. Details of each phase are given below.

The interviews followed a fixed protocol, the interviewer using neutral prompts such as "Good; anything else?" or "OK; show me how" whenever necessary to obtain a better understanding of students' responses. All interviews were audiotaped and transcribed the same day onto a response sheet by the interviewer; the authors then categorized non-quantitative responses and recoded them later.

Situated Angle Knowledge

The interviewer demonstrated or asked students to demonstrate various configurations of each physical model, including the most extreme configurations they could imagine. Angles of 30° and 45° were included in all contexts except turns, together with whichever of 0°, 90° and 180° were meaningful. The demonstrations were accompanied by questions intended to probe students' understanding of the consequences of varying the implicit angle. The following describes the general procedure for each context. The precise knowledge items assessed are given later in Table 1.

Turns. The interviewer turned the doll through various multiple, half and quarter turns, asking "How much did the doll turn?" The students were also asked to explain the effect of the various turns on the direction the doll faced.

Slopes. A model sports car was provided which could roll down the hill or could be pushed up it. The interviewer set the slope of the road at approximately 30°, 45° and 15°. Students were asked to predict whether the car would travel faster or slower (both uphill and downhill) as the slope was varied, and to explain how they reached their conclusions.

Hinges. Students were asked how they knew how much to open scissors when cutting different objects.

Bends. Students drove a toy ambulance as fast as they could from the "factory" on the left side to the "hospital" on the right, using the two possible routes. They were then asked to state which bends were the most difficult and which were the easiest to drive around. They were also asked to say how they could tell whether a bend would be hard or easy.

Rebounds. A ball was rolled along a groove to a "batter"—a block of wood, faced in plastic foam, which could be rotated to deflect the ball in various directions—and a second block of wood representing a fielder was used to provide a target. After the interviewer had demonstrated how to set the batter to deflect the ball to a fielder on the right, she asked the student to do the same for various positions on the right and to explain their method.

Corners. The interviewer first made a flooring pattern using nine 45° tiles. She then asked whether, if one of these tiles were to become damaged, it could be replaced by a 30° tile; and if not, why not. Students were then asked how the corners of the two tiles differed.

Context Definition

The interviewer next asked students to name other situations that they regarded as similar to the given situation in an angle-related sense. For example, for the doll model students were asked if they knew of "anything else which turns like this".

Abstract modelling

Seven abstract models were constructed to represent possible ways in which students might model the angles in various contexts (see Figure 2). The interviewer presented the abstract models in the order A to G and asked students if they could be used to show the given context. Students were also asked to demonstrate how each accepted abstract model would show the corresponding physical model. All models were set to show a 45° angle, except for the turns context where 360° was used. After all abstract models had been presented, students were asked to state what they regarded as the best model for showing the given context.

<Insert Figure 2 about here.>

Similarity Between Contexts

After students had responded to their two contexts, students were asked what similarities they saw between them. The interviewer first asked "Are these things the same, at all?" Students were then asked whether and how their best abstract model of each context could also show the other context; if they had chosen the same best model, this was simply pointed out to them. Finally, the interviewer asked students to look more closely at their best model(s) and to say again "what things are the same" in the two contexts.

Formal knowledge of angle

The interviewer first asked students if they had done "anything like this" in school. They were then asked to say what they understood by the word "angle" and if they knew what a "right angle" is. They were then asked if there were any angles and right angles in their two contexts, and to demonstrate them on the respective physical model.

According to the State mathematics syllabus (New South Wales Department of Education, 1989), which both class teachers said they adhered to, students should have met ideas of corner, sloping lines, angle as "the amount of turning between two lines about a common point" (p. 79), large and small angles, everyday examples of turning, methods of comparing angles, and square corners. Apart from the word "angle", no formal terminology is mentioned in the syllabus.

RESULTS

In interpreting the following results, recall that 12 students responded to each context.

Situated Angle Knowledge

Table 1 summarises student's responses to the interview questions probing their situated knowledge of the six angle contexts.

<Insert Table 1 about here.>

Items with 10 or more correct responses were interpreted as showing mastery modified by misunderstanding, lack of attention or ineffective prompting. There seemed to be three reasons why students did not reach mastery on all items.

Firstly, students' responses often showed failure to use standard terminology rather than lack of knowledge of the underlying physical relations. For example, all the students who called a forward-to-left turn "a little bit" recognised that a right-to-forward turn was the same size. Also, in explaining how to predict speed of ascent and descent, 10 students used the word "slope" at least once and the other 2 at least once referred to the "height" of the road. In explaining how corners fit, all students who did not explicitly use words like "sharp" or "pointy" correctly used terms like "skinny", "narrow" or "small".

A difficulty of a different kind occurred when students were asked to imagine extreme slopes and corners. Almost all responses which were not exact were approximately correct. A few students gave such explanations as "You couldn't drive up it if it was straight up" or "If it was any smaller, there wouldn't be any tile there." It seems likely that most approximately correct responses in the slopes and corners contexts were occasioned by such insights and not by an inability to imagine the extreme case. This was, however, not the case in the bends context. Only 7 students described the hardest possible bend within 10° and only 8 described the easiest bend as straight or almost straight.

Finally, the relatively poor performance on the cricket model can be put down to its unfamiliarity. In fact, all students did eventually set the batter correctly in all positions.

In summary, students seemed to have an excellent situated knowledge of the angle relations implicit in the six contexts. Only the extreme angles in the bends context were somewhat unfamiliar.

Context Definition

Table 2 summarises the responses students gave when they were asked to name situations which were similar to the given situations in an angle-related sense. In effect, the data show how students interpreted the phrases "turn like", "slope like", "open and shut like", "bend like", "bounce like" and "have corners like" the given models.

<Insert Table 2 about here.>

Angle-related similarities were predominant in all contexts except bends. Unlimited turning about a fixed axis was apparently very familiar and clearly distinguished from other types of turning. Sloping was also familiar and not limited to linear objects; it seemed that sloping 3-dimensional objects (such as a roof) were often represented by a cross-section, thus reducing them to the 2-dimensional case. Hinging was well-known but mostly limited to objects with two movable parts. Rebounds were mostly limited to ball games, but a few responses recognised the implicit V-shape. Corners were very familiar; all the abstract examples were 2-dimensional, but it was not clear whether students were thinking of the 2- or 3-dimensional corners of the concrete objects. By contrast, only a few responses to the bends context picked out the implicit V-shape of the abstract angle model.

Abstract modelling

Students' methods of representing the six angle contexts using the seven abstract models were easily classified into four categories as follows. (1) The abstract model was used in a standard manner, representing two lines on the physical model by two lines on the abstract model in a way that suggested that a concept of angle had been abstracted from that context. (2) The angle only partially modelled, indicating the abstraction of some angle-related features but not the complete angle concept. (3) Features not related to the angle concept implicit in that context were modelled. (4) The abstract model was rejected as a means of representing the given context. Table 3 shows the distribution of responses when students were asked to state whether and how each abstract model could represent a given context.

<Insert Table 3 about here.>

Non-standard but angle-related methods of modelling occurred most frequently for slopes, hinges and bends. For slopes, such responses used a single line to model the hill. For hinges, lines of the abstract models were laid on the arms of the scissors but did not meet at the pivot. For bends, lines on the abstract model were again laid somewhat casually on the road so that they were within the bend but not parallel to the edges of the two segments. All the angle-related methods of modeling turns represented a full turn by an angle of 180°.

Most non-angle related modelling occurred in the rebounds context, where 18% of the responses indicated that the ball travelled directly between the ends of the two arms of the angle model. One student started with the two arms together and then opened them quickly, suggesting that these responses were modelling the action of the batter in hitting the ball rather than the path of the ball. The rebounds context also showed by far the largest percentage of responses in which students could not explain any way of using the abstract models to represent the context.

In the introduction, we conjectured that students would represent physical angle contexts abstractly using the most obvious lines in each context. On the basis of this conjecture, we would expect an abstract angle model which clearly models the lines in a context to be selected more often than other models. The expected models for each context are given in Table 4, along with data on model selection. Only in the case of bends and rebounds was an expected model most often used in standard modelling and most often selected as the best model. In the slopes context, there was in fact no preferred abstract model—all models were used in a standard manner or selected as best with approximately equal frequency. The most frequently selected models overall were C, F and G.

<Insert Table 4 about here.>

Recognition of Similarities

Students' descriptions of the similarities they perceived between their two contexts were classified into four categories according to whether they indicated (1) common abstract features including two lines, such as angles; (2) common concrete features implying two relevant lines, such as corners; (3) common concrete features implying only one relevant line, such as a slope; and (4) non-angle related features, such as the rotation of a rolling ball, or no common features. In terms of the angle concept, these categories seemed to show four decreasing levels of similarity recognition. Table 5 shows the distribution of the highest level of recognition found in students' responses to the interviewer's two questions on similarity.

<Insert Table 5 about here.>

All students recognised some angle-related similarity between slopes and corners. However, nearly half of them could not get beyond explanations such as "If a tile is placed on edge, it slopes like a hill" which only matched one line in each context; and only one-third gave an abstract explanation such as "They both have two lines". The angular similarity between hinges and bends was also well recognised, almost all students picking out two lines. Most students gave initial explanations such as "Both have bends" or "The blades of the scissors can fit down both roads in a bend" and refined their explanations after looking at the abstract models. Two-thirds of the students eventually gave an abstract explanation. By contrast, only one-third of the students recognised any relevant similarity between turns and rebounds. Most students remarked that both the doll and the batter turned, despite the fact that the batter did not move during the rebound. In this last context pair, students did not change their responses greatly after looking at their abstract models.

Formal knowledge of angle

The majority of the students claimed not to have done "anything like this" in school, or gave non-angle related responses. Only 11% explicitly mentioned angles, a further 6% mentioning specific angle contexts (corners and tessellations). However, all offered a definition of an angle or demonstrated some examples. Many students gave several definitions which previous researchers have regarded as different; a total of 55 responses was noted. Of these, 45% stated essentially that an angle consisted of two lines meeting at a point, 22% the point where two lines meet, and 15% the area between two lines. Only 11% of the students mentioned turning at all. Correct demonstrations of right angles were given by 78% of the students—11% also showed some "left angles"!

Almost all students were able to show how each context involved angles. Most exceptions occurred with turns, where 33% of the students referred to the irrelevant corners of the base supporting the doll and 25% saw no angles. Also, 17% referred only to the sloping line in the slopes context and the same percentage referred only to the point in the corners context. In all contexts, about half the students were able to show a right angle.

DISCUSSION

As in the earlier study of Grade 2 students (Mitchelmore, 1995), the Grade 4 students in the present sample showed good situated angle knowledge in the contexts investigated. They seemed to be thoroughly familiar with the physical correlates of turns, slopes, hinges, bends, rebounds and corners, although several students showed a limited vocabulary for describing the relations involved. The only angle-related difficulty students had was in imagining the easiest and most difficult bends in a road.

Students' definitions of the six contexts were not always the predicted unlimited turns about a fixed axis, linear slopes, hinges, bends, rebounds, and plane corners. The contexts could be more accurately described as unlimited turns about a fixed axis, all slopes, 2-part hinges, all bends, ball rebounds, and all corners. In particular, the bends context appeared to include both bends which can be modelled by angles and bends which can better be modelled by curves. Nevertheless, all the contexts were well-defined and distinct from one another. The data confirm that, by age 9, most students have formed angle subconcepts of turns, slopes, hinges, rebounds and corners based on similarities between different physical angle situations. The concept of bends has in many students not yet taken on the status of an angle subconcept. Again, the results are not radically different from those obtained from Grade 2 students in the earlier study.

Contexts varied greatly in the way students modelled them using abstract angle models. The percentage of responses using standard angle modelling varied from 40% on rebounds to 89% on corners. Our conjecture that, if students did conceptualise a context in terms of angles, they would primarily represent the physically present lines, was only confirmed for bends and rebounds. Students who used standard modelling seemed to prefer to represent angles in all contexts by two half-lines with a common end point.

In five of the six contexts, there was a marked increase in the proportions of standard modelling from Grade 2 to Grade 4. However, comparisons are limited because of changes in the schools sampled and the models used. Hinges were probably made easier in Grade 2 (71% standard as compared to 60% in Grade 4) because the scissors in the Grade 2 study were represented by two thin strips of wood, thus making the geometrical structure clearer; and bends were probably made more difficult (15% compared to 65%) because the curved parts of the bends in the Grade 2 model were almost as long as the lines on the abstract models. The other four models were not altered significantly, and the change from Grade 2 to Grade 4 seems to represent genuine growth. The frequency of standard modelling of turns increased from 45% to 76%; slopes 27% to 62%; rebounds 14% to 40%; and corners 47% to 89%. In particular, a tendency among Grade 2 students to represent turns by a single line (17% of the responses) was not found at all in Grade 4. It seems at least possible that most of the gains were due to the intervening instruction in angle which, as noted, had a significant emphasis on locating angles in everyday situations.

Students who used non-standard methods of modelling angles showed through their methods and their incidental comments that each angle context has specific features which hinder the abstraction of the standard angle model. These features may be summarised as follows.

- *Turns*. The main problem seems to be that turning, although it is very familiar, leaves no physical trace. The context is best modelled by a rotating half-line which is often not physically present in a turning situation. If, however it is modelled by a line which extends on both sides of the centre of rotation, whole turns and half turns may be confused. Also, even if the half-line is present in the given situation, it cannot simultaneously be in both the initial and final positions as is required in the standard angle model.
- *Slopes*. Clearly many students see slope as a property of a single line. When they do represent a second line, it is probably the horizontal base or run-off road on the physical model rather than a general horizontal. The vagueness of the horizontal is confirmed by the fact that there was no strong preference for a particular abstract angle model. We conjecture that there would have been far fewer standard uses of the abstract angle models if the hill had been held in the air.
- *Hinges*. Although the two arms of the angle appear to be obvious in this context, the significance of the pivot as the vertex of the angle seems not to be so well known.
- *Bends*. As for hinges, the arms of the angle (assuming the bend is modelled by an angle, which is often not the case) are physically present but broad. Because of rounding, there is not even a point which the vertex of the angle could represent. Perhaps because roads cannot overlap (as the arms of scissors can), it is difficult to conceive of a 180° bend.
- *Rebounds*. In this context, both lines of the standard angle model have to be imagined. The turning movement of the batter (not present in other rebound situations) also has to be

ignored. It may also be difficult to conceptualise the path of the ball as a pair of straight lines.

- *Corners.* Perhaps because both lines and the vertex of the standard angle are physically present in corners, this context is most frequently modelled in a standard manner. Nevertheless, many students identify a corner with the point at or near the vertex. It is worth remarking on the fact that it is only in the corners context that students show any tendency to model the implicit angle by the point of intersection of two lines.

A further difference between the six contexts lies in the relation between the visual angle and its physical correlates. The size of a turn, the opening of a hinge, and a the size of a corner are all represented by the visual angle itself, as shown in Figure 3(a).. In these contexts, there is a direct relation between the size of the visual angle and its physical consequence. A "sharp" angle represents a small turn, opening or corner. However, the amount of bending and the deflection in a rebound are not represented by the visual angle, but by the angle shown in Figure 3(b). In these two contexts, there is an inverse relation between the size of the visual angle and the related physical quantity. A sharp bend or deflection makes a small visual angle. In the slopes context, the relation is direct if you represent slope by the acute angle between the sloping line and the horizontal and inverse if, as some students did, you represent slope by the obtuse angle (see Figure 3(c)).

<Insert Figure 3 about here.>

Our results show clearly that the specific features of each context strongly affect the probability of students recognising angle-related similarities. Hinges and bends were almost always recognised as similar, apparently on the basis that they both have broad arms with a rather indeterminate intersection. Slopes and corners both have one obvious slanting line and were always recognised at similar—but often only on this basis. Turns and rebounds have nothing obvious in common, so most students grasped at irrelevant similarities.

Many of the similarities students reported were a sort of lowest common denominator of the two contexts and therefore often omitted essential attributes of the angle context. If students adopted this approach consistently, the only angle concept they could abstract would be the attribute which is common to all angle contexts—a slanting line. (Clements and Battista (1989) report that many students do indeed conceive of angle in these terms.) In most contexts, lines have to be imagined—or at least made more definite—in order to construct a standard angle. What is it that drives this construction, that leads students to note that "a slope is like a corner if you imagine a horizontal line" rather than "one edge of a corner slopes if you put the other edge on a table"? It is surely this drive to find the *greatest* common denominator that eventually leads to a general angle concept.

Perhaps the answer lies in formal tuition. Although most of the students in the present sample did not spontaneously recognise the six contexts as related to the angles they had studied in school, they were able to identify relevant angles (which they usually defined as two lines meeting at a point). This was one of the objectives of their previous instruction, and it may have been activities related to this objective which led to the observed increase in standard modelling between Grades 2 and 4. A closer look at several contexts to identify exactly what lines meet at which point—under a teacher's guidance—could lead to an even bigger increase in standard angle modelling. Students would gain in their understanding of each context (for example, by clarifying the concept of horizontal) as well as in their ability to recognise the similarity between different angle contexts.

Although students had been taught that an angle was an amount of turning, very few gave this definition and more than half could not see angles in the turns context. The definition of angle as an amount of turning has been widely promoted in the curriculum literature; see, for example, Wilson and Adams (1992). There would seem to be three reasons why it has not been effective. (1) Although turning about a point is familiar, students have difficulties representing a turn as an angle. (2) Those contexts which do involve turning are very different, not least in the relation of the amount of turning to the size of the visual angle. (3) Some contexts (such as slopes and corners) do not appear to involve any sort of turning. It might be more productive to work with a definition of an angle as two lines meeting at a point, and to relate the size of the visual angle to its physical correlates in several different contexts separately.

In fact, it might be better not to work with any definition at all. We suspect that the reason students give so many different responses to the question "What is an angle?" is that the concept is so many-sided. In some contexts, the angle is two lines, in others the point is crucial, in others the angle is the turn between two lines—there are many other possibilities. Younger students probably define an angle as a point because they only know of angles in the corners context; definitions broaden as students try to include more and more contexts which they have recognised to involve angles. "Angle" is that which is common to all such contexts but, apart from the two lines meeting at a point, it is very difficult to say exactly what is common. So how can we expect a "correct" definition from students?

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Table 1

Frequency of correct responses to situated knowledge questions, by context

Context	Knowledge item	N
Turns	State amount of turning in demonstrated full turn	12
	State amount of turning in demonstrated two turns	11
	Indicate doll can turn large/unlimited number of times	10
	Indicate doll would face forward after 20 turns	12
	State amount of turning in demonstrated half turn	12
	Indicate turn from facing backward to facing forward is a half turn	12
	State amount of turning in demonstrated forward-to-left turn	4
	Make same turn as forward-to-left turn, starting from left	11
	Indicate right-to-forward is same turn as forward-to-left	12
Slopes	Predict relation of demonstrated slope to speed of ascent/descent	12
	Explain speed of ascent/descent in terms of slope	8
	Demonstrate hardest possible hill as vertical	9
	Demonstrate easiest possible hill as horizontal	8
Hinges	Show typical scissors opening	12
	Explain choice of opening	12
	Show maximum opening	12
	Show minimum opening	12
Bends	Identify easiest/hardest bend on two routes	12
	Explain difficulty of bend in terms of internal or external angle	12
	Demonstrate hardest possible bend as coincident half-lines	6
	Demonstrate easiest possible bend as a straight line	5
Rebounds	Make 45° hit at first or second attempt	7
	Make 30° hit at first or second attempt	7
	Make 90° hit at first or second attempt	9
	Explain need to set batter facing between bowler and fielder	9
	Make 0° hit at first or second attempt	11
	Explain need to set batter facing bowler for 0° hit	12
Corners	Explain non-fitting in terms of sharpness of corner	0
	Explain 30° corner sharper than 45° corner	7
	Indicate skinniest possible tile has 0° corner	3
	Indicate normal tile has 90° corner	10

Note. N is the frequency of correct responses to each knowledge item. The maximum N is 12 in each case.

Table 2

Classification of situations regarded as similar to given situations, by context.

Context	Category of situation named as similar	N
Turns	Unlimited rotations about fixed axis	30
	Limited rotations about fixed axis	4
	Other turns	2
	No turning	1
Slopes	Sloping linear objects	13
	Sloping plane or curved surfaces	7
	No slope	1
Hinges	Hinged objects with two arms	15
	Hinged objects with one arm	5
	Rigid objects	1
Bends	V-shaped objects	5
	Tracks and paths	6
	Closed regions	4
	Junctions	3
Rebounds	Rebounding objects	11
	V-shaped objects	4
Corners	Corners of objects in room	16
	Corners of abstract figures	14

Note. N is the number of responses falling into each category. N is occasionally greater than 12 because of multiple responses.

Table 3

Percentage frequency of each method of abstract modelling a typical configuration, by angle context.

	Method of abstract modelling			
Context	Standard	Angle-related	Non-angle	Rejected
Turns	76	13	0	8
Slopes	62	21	1	15
Hinges	60	29	4	8
Bends	65	25	0	10
Rebounds	40	4	19	37
Corners	89	1	0	10

Note. N = 84 responses in each row (12 students responding to 7 models).

Table 4

Expected models for each angle context, and the models most frequently selected and used in standard modelling.

Context	Expected abstract model(s)	Models most often used in standard modelling (percentage usage)	Model most often chosen as "best" (percentage use in standard modelling)
Turns	C, F	C (100%)	G (33%)
Slopes	B, E	F (65%)	F (25%)
Hinges	A, D	C, G (each 75%)	A (67%)
Bends	C, F	F (92%)	C (25%)
Rebounds	C, F	C, F (each 75%)	C (33%)
Corners	G	G (100%)	C (42%)

Note. All percentages are out of 12 responses.

Table 5

Percentage frequency of four levels of similarity recognition, by context pair.

Context pair	2-line abstract	2-line concrete	1-line concrete	Non-angle responses
Slopes and corners	33	25	42	0
Hinges and bends	67	25	0	8
Turns and rebounds	17	0	17	67

Note. N = 12 students in each row.

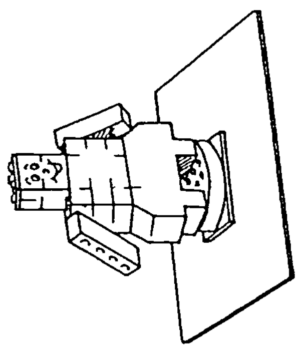
FIGURE CAPTIONS

Figure 1. Physical models of situations representing six angle contexts. Sizes: doll 10 cm high, ramp 32 cm long, scissors 18 cm long, roads plan 76 cm x 55 cm, cricket field 47 cm square, tiles side 5 cm.

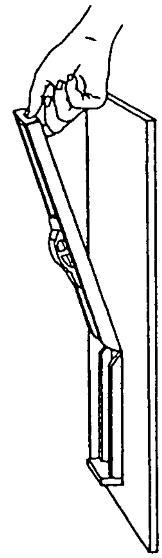
Figure 2. Abstract angle models. Models A-C were each made from two concentric plastic circles slotted together so that they could rotate relative to each other; one line was marked on each circle. Models D-F were made of straws joined either with thread (D) or a pipe cleaner (E, F). Model G was made from two plastic circles, each with one semicircle shaded black, fastened at and rotatable about their common centre. Radii: D-F 10 cm, others 8 cm.

Figure 3. Angles representing the physical correlates in various angle contexts: (a) turns, hinges and corners, (b) bends and rebounds, (c) slopes.

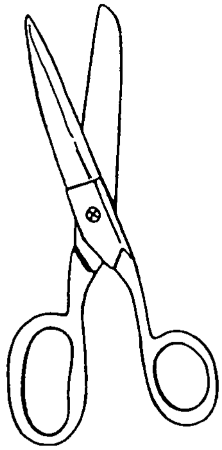
<u>Figure 1</u>



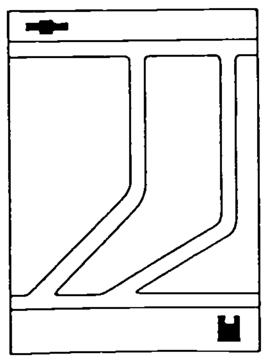
Doll (turns)



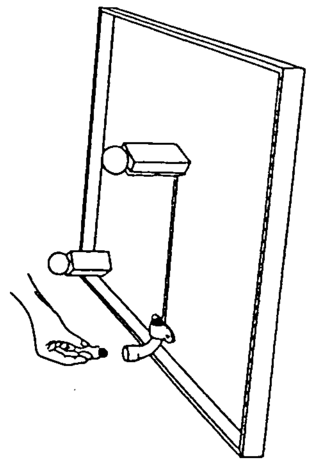
Hill (slopes)



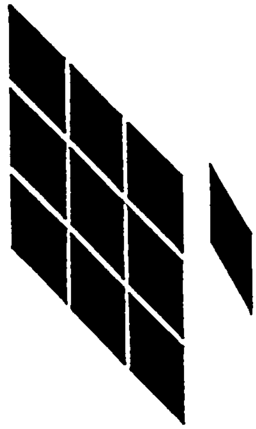
Scissors (hinges)



Roads (bends)



Cricket (rebounds)



Tiles (corners)

Figure 2

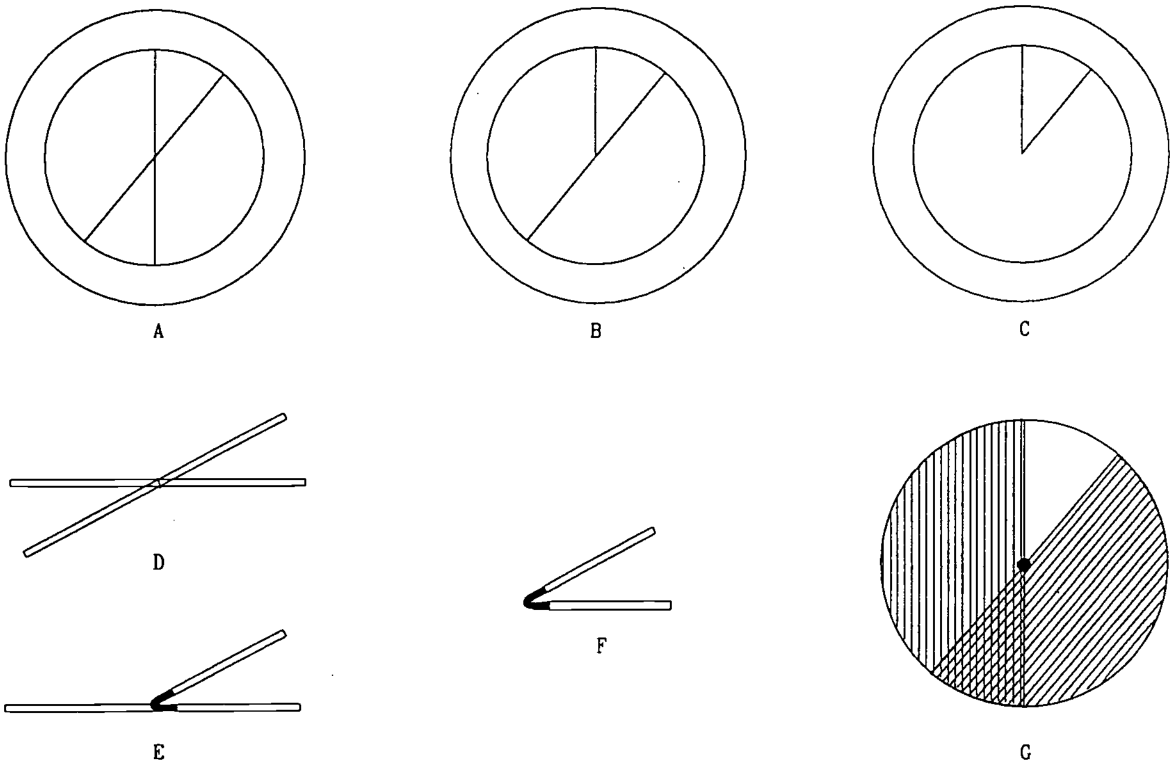
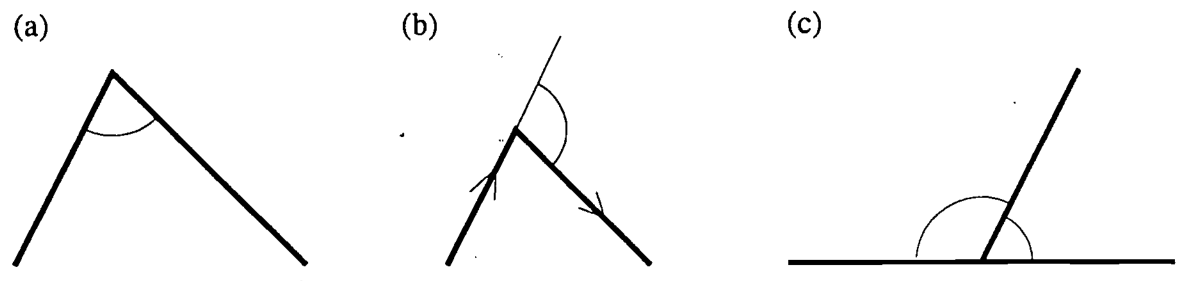


Figure 3



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