DOCUMENT RESUME

ED 412 262 TM 027 563

AUTHOR	Cai, Jinfa; Moyer, John C.; Grochowski, Nancy J.
TITLE	Making the Mean Meaningful: Two Instructional Studies.
SPONS AGENCY	Ford Foundation, New York, NY.
PUB DATE	1997-03-00
NOTE	40p.; Paper presented at the Annual Meeting of the American Educational Research Association (Chicago, IL, March 24-28, 1997).
PUB TYPE	Reports - Research (143) -- Speeches/Meeting Papers (150)
EDRS PRICE	MF01/PC02 Plus Postage.
DESCRIPTORS	Algorithms; College School Cooperation; Comprehension; *Concept Formation; *Elementary School Students; Elementary School Teachers; Grade 6; *Instructional Effectiveness; Intermediate Grades; Knowledge Base for Teaching; *Mathematical Concepts; Pretests Posttests; *Problem Solving
IDENTIFIERS	*Averaging (Mathematics)

ABSTRACT

Two studies are reported that explore the effectiveness of instruction in promoting students' conceptual understanding of the average concept using a leveling model and an open-ended problem-solving approach. One teacher and her 46 sixth-grade students participated in the first study. One year later, the second study was conducted with the same teacher, but with 42 different sixth graders. For both studies, pretests and posttests were used to examine the instructional impact on students' understanding of the concept of averaging. Tests in the second study were more comprehensive than those in the first study. The second study also documented lessons on teaching averaging. The teacher was provided with comprehensive inservice training on the mathematical content and pedagogical knowledge of arithmetic average, on the use of manipulative activities to introduce the averaging concept informally, and on how to make a transition to the use of a formal averaging algorithm to solve problems. Results of the studies show that from pretest to posttest, the students became more capable of applying the averaging algorithm to solve problems, as evidenced by increases in the number of correct answers and in the number of appropriate strategies, as well as by increases in the number and quality of the explanations of their solution processes. In addition, the results of the second study show that students were able to use their knowledge of averaging to solve novel problems. An analysis of the teacher's instruction revealed that the use of the leveling model in conjunction with an open-ended problem-solving approach mediated students' mathematical and statistical understanding of the averaging concept. Findings suggest that teachers' success in teaching with understanding is dependent on the encouragement and support they receive as they begin to change their approach to teaching. The collaboration of university professors and school teachers over extended periods of time is a possible way to help change occur. (Contains 2 figures, 5 tables, and 16 references.) (Author/SLD)

Making the Mean Meaningful: Two Instructional Studies

Jinfa Cai

University of Delaware

John C. Moyer

Marquette University

Nancy J. Grochowski

Edison Middle School

Running head: MAKING THE MEAN MEANINGFUL

Paper presented at the annual meeting of the American Educational Research Association; Chicago, IL; March 24-28, 1997.

Preparation of this paper was supported, in part, by a grant from the Ford Foundation. Any opinions expressed herein are those of the authors and do not necessarily represent the views of the Ford Foundation.

Making the Mean Meaningful: Two Instructional Studies

Abstract

Two studies are reported which explored the effectiveness of instruction in promoting students' conceptual understanding of the average concept using a leveling model and an open-ended problem-solving approach. One teacher and her 46 sixth-grade students participated in the first study. One year later, the second study was conducted with the same teacher, but with 42 different sixth-grade students. For both studies, pre- and post-tests were used to examine the instructional impact on students' understanding of the average concept. Pre- and post-tests in the second study were more comprehensive than those in the first study. The second study also documented the lessons on teaching average. The teacher was provided with comprehensive inservice training on the mathematical content and pedagogical knowledge of arithmetic average, on the use of manipulative activities to introduce the average concept informally, and on how to make a transition to the use of formal averaging algorithm to solve problems. Results of the studies showed that from pre-test to post-test, the students were more capable of applying the averaging algorithm to solve problems, as evidenced by increases in the number of correct answers and in the number of appropriate strategies, as well as by increases in the number and quality of the explanations of their solution processes. In addition, the results of the second study showed that the students were able to use their knowledge of averaging to solve novel problems. An analysis of the teacher's instruction revealed that the use of the leveling model in conjunction of an open-ended problem solving approach mediated students' mathematical and statistical understanding of the average concept. Findings of these studies suggest teachers' success in teaching with understanding is dependent upon the encouragement and support they receive as they begin to change their approach to teaching. The collaboration of university professors and school teachers over extended periods of time is one possible way to help change occur.

In the age of information and technology, society has an ever-increasing need for citizens who can judiciously use data in prediction and decision-making. The National Council of Teachers of Mathematics (NCTM) suggests that "it is important that students develop an understanding of the concepts and processes used in analyzing data," (p. 105, 1989). Arithmetic average or mean is one of the important and basic concepts in data analysis and decision-making. Statistical analyses and inferences are conducted almost exclusively through the determination of measures of central tendency such as the mean or arithmetic average and measures of dispersions such as the standard deviation. Data reported and used in daily life, scientific journals, and public media frequently include the average. Thus, average is not only an important concept in statistics, but also an important concept for informed citizens.

The arithmetic average is a statistic that is used to describe and make sense of a data set. It is also a tool, used in conjunction with the standard deviation, for summarizing a data set and comparing data sets (e.g., Gal, Rothschild, & Wagner, 1990; Mokros & Russell, 1995). Computationally, the arithmetic average is defined by adding the values to be averaged and dividing the sum by the number of values that were summed. Although the average concept seems to be as simple as the averaging algorithm suggests, previous research indicates that precollege and college students did not understand the properties related to the averaging algorithm and students experienced many difficulties using the algorithm to solve problems (e.g., Cai, 1995; Cai & Moyer, 1995; Cai, Magone, Wang, & Lane, 1996; Mevarech, 1983; Pollatsek, Lima, & Well, 1981; Strauss & Bichler, 1988). For example, Pollatsek et al. (1981) found that only about 40% of the college psychology majors were able to solve the following problem: "A student attended college A for two semesters and earned a 3.2 grade-point average (GAP). The same student attended college B for three semesters and earned a 3.8 GPA. What is the student's GPA for all his college work?" The most common incorrect answer was 3.5, which apparently resulted from the direct averaging of the GPA's for college A (3.2) and for college B (3.8).

Cai (1995) analyzed 250 sixth-grade students' responses to the following average-related problem: "Angela is selling hats for the Mathematics Club. She sold 9 hats in Week one, 3 hats in week 2, and 6 hats in week 3. How many hats must Angela sell in week 4 so that the average number of hats sold is 7?" He found that 50% of his sample had one of the following four errors in solving the problem.

Error 1: The student added the numbers of hats sold in week 1 (9), week 2 (3), week 3 (6), and the average (7), then divided the sum by 4. The student then gave the whole number quotient (6) as the answer.

Error 2: The student added the numbers of hats sold in week 1 (9), week 2 (3), and week 3 (6), then divided the sum by 3. The student then gave the quotient (6) as the answer.

Error 3: The student added the numbers of hats sold in week 1 (9), week 2 (3), and week 3 (6), then divided the sum by 3, and got 6. However, the average was 7. Therefore the student added 3 to the sum of the numbers of hats sold in the first three weeks, then divided it by 3, and got 7, and then gave the answer 3.

Error 4: The student added the numbers of hats sold in week 1 (9), week 2 (3), and week 3 (6), then divided the sum by 3, and got 6, 6 + 1 = 7. So the student gave the answer 1.

In these studies, the misconceptions were not due to students' lack of procedural knowledge for calculating the average. Rather, they were due to students' lack of conceptual understanding of the concept, which affected their ability to utilize the algorithm. Shaughnessy (1992) has argued that most students' understanding of the average is the "add-them-all-up-and-divide" algorithm, because the computational procedure is all they are ever taught. It is intriguing to hypothesize that, as implied by Shaughnessy, a change in teaching emphasis is needed to enrich students' understandings of the average beyond the computational algorithm.

Researchers have explored different approaches (e.g., a fair share or a balance model) to teach pre-college or college students' understanding of average (Hardiman, Well, & Pollatsek, 1984; Pollatsek et al., 1981; Strauss, 1987). The fair share and balance models provide ways of making sense of the arithmetic average, but each has limitations. A fair share model is not applicable in some situations. For example, many students are puzzled with a household having an average of 3.4 persons (Mokros &

Russell, 1995). The balance model is helpful for those who have a good understanding of balancing (Hardiman et al., 1984). However, many middle school students and even college students do not have a good understanding of mathematical balancing. Thus, there is a need to explore alternative approaches for teaching average with understanding. In addition, implementating change in mathematics instruction has been found to be complex and challenging for teachers (Cohen & Ball, 1990). Teachers need adequate professional support to make instructional change a reality in their classrooms.

This paper reports two instructional studies in which a teacher attempted to teach the arithmetic average with understanding to her students after receiving professional support from university mathematics education professors. The purpose of the two instructional studies reported here was to explore the effectiveness of a teacher's instruction using a "leveling" model and an open-ended problem solving approach to build a foundation for her students' conceptual understanding of the arithmetic average. The arithmetic average is both a central concept in statistics and a computational algorithm. Therefore, a conceptual understanding of average includes both an understanding of the computational algorithm and the statistical aspects of the concept. With a statistical understanding of the concept, students should be able to use the concept to summarize and make sense of a data set or to compare data sets. With a conceptual understanding of the averaging algorithm, students should be able to correctly apply the averaging algorithm to solve problems. That is, students should not only know the averaging algorithm, but also know when and how to correctly apply the algorithm to problem situations.

The leveling model emphasizes the average concept as a process of "averaging," rather than as a formal procedure (Bennett, Maier, & Nelson, 1993). Figure 1 shows the leveling process of finding the average of 5, 6, and 13. The open-ended problem-solving approach stresses students' knowing when and how to use average concept to solve contexually-based problems. Students were encouraged to solve average related problems using multiple solution strategies and representations (Cai & Moyer, 1995).

The emphases of instruction using the leveling model in conjunction with an open-ended problem solving approach are on both statistical and mathematical understanding: i.e., on the average as a tool for describing and making sense of a data set and as a mathematical algorithm for solving problems. Using manipulatives and concrete models, students in our studies were introduced to the properties of the average concept (Strauss & Bichler 1988). After students had established informal knowledge, they were introduced to the formal computational algorithm for calculating an average and for solving real life problems.

Insert Figure 1 about here

In this paper, we also explore the design of teacher learning and subsequent changes in instruction. Duffy and Roehler (1986) have theorized that four factors affect adaptation of a teaching innovation by a particular teacher: the teacher's understanding of the mathematics content, the teacher's concept of instruction, the environmental constraints that affect the teacher's classroom, and the teacher's desire for a smooth flowing school day. These four factors relate to the two underlying premises which guided the instructional component of the studies. The first two factors relate to the first premise: in order to develop students' conceptual understanding of the average concept, teachers themselves must have two types of well-established content knowledge: mathematical and pedagogical. That is, teachers must have not only a deep mathematical concept of average, but they also must have significant pedagogical content knowledge about how to teach the concept. This includes the ability to "...choose and develop tasks that are likely to promote the development of students' understandings of concepts and procedures in a way that also fosters their ability to solve problems and to reason and communicate mathematically" (NCTM, 1991, p. 25). The third and fourth factors identified by Duffy and Roehler relate to the second premise: teachers must be supported and encouraged to actually use their content knowledge to engage their students in thinking and sense-making of the average concept. According to Duffy and Roehler, this must be done within the constraints of their own particular classroom in such a way as to maintain a smoothly flowing school day.

STUDY I

PURPOSE

The first study was exploratory in nature. The purpose was to explore the effectiveness of teaching average with understanding by providing a teacher with resources and training in the use of the leveling model and an open-ended problem solving approach.

METHOD

Subjects.

Subjects consisted of a mathematics teacher, Ms. G, and her two classes of 46 sixth-grade students from a public middle school in an urban school district. Students were ethnically and culturally diverse, with 75% of them qualifying for free or reduced lunch. Students had been briefly exposed to the average concept in previous years. At the time these studies were conducted, Ms. G was involved in a mathematics education reform project. She has 15 years of teaching experience, with a master's degree in education. Ms. G and her students volunteered to participate in this study.

Teacher Training and Instruction.

As indicated above, two premises guided the organization and design of inservice work with Ms. G.: (1) Teachers must have not only a deep mathematical concept of average, they must also have significant pedagogical content knowledge about how to teach the concept; (2) Teachers must be supported and encouraged to actually use their content knowledge to engage their students in thinking about and sense-making of the average concept. Before the academic year started, Ms. G participated in a three-day workshop led by two university mathematics education professors. One of the three days was dedicated to a statistics unit designed to improve the teachers' content knowledge of statistical measures, including mean, median and mode.

During the school year, Ms. G, along with other middle school teachers, met once a week with two university professors. A number of the weekly meetings focused on the improvement of their mathematical and pedagogical knowledge of the concept of average. Over the course of those meetings Ms. G was introduced to a variety of instructional materials and techniques for fostering the development of the concept of arithmetic average. The activities were directly adopted or modified from Bennett, Maier, and Nelson (1993), Meyer, Browning, and Channell (1995), and Strauss and Bichler (1988). In addition, at the weekly meetings the teachers were provided with an overview of students' difficulties in learning the average concept, based on previous research. The university professors used discussions of the teachers' own performances on the averaging activities to develop their insights into students' difficulties understanding various properties of average, as identified by Strauss and Bichler (1988).

Although the teachers learned how to use various manipulative activities to teach the average concept, it was necessary to help them develop the confidence to use these activities with their own students. During the workshop and at regular meetings, Ms. G discussed with the university professors and other math teachers, her classroom management techniques related to the use of manipulatives and cooperative learning. She also developed a repertoire of questions she could ask her students to challenge them to learn the average concept meaningfully.

Through this process Ms. G came to understand the importance of building upon students' existing knowledge (Hiebert & Carpenter, 1992). Equally important, she developed a strategy she believed could be used to successfully help her students learn the average concept. The strategy was to use the leveling model to emphasize the average concept as an evening-off process. Lessons involved the evening-off of columns of tiles (Bennett, et al., 1993; Meyer, et al., 1995), the recognition that the height of the evened-off columns is the AVERAGE of the original columns, and the discussion of relationships between the heights of the original columns and the evened-off columns. In addition, Ms. G decided to de-emphasize the computational algorithm at the beginning, and to require those students who did use the algorithm to explain why it worked.

In her instruction, Ms. G helped ensure understanding of the average concept by using a contextually-based, open-ended problem-solving approach. She tried to stress that averaging is an effective technique for making sense of a set of data and that it is not simply a computational process. To help do this, she developed additional instructional activities, which were based upon the need to make sense of data that naturally occurred within classrooms. As part of the learning process, Ms. G guided the students to use verbal, pictorial, and symbolic representations to communicate multiple solutions to these open-ended problems. The instruction of the average unit was completed in five days.

Pre- and Post-Tests.

Problems 1 and 2, shown in Figure 2, are the two problems used as pre- and post-tests. Problem 1 requires students to figure out a simple mean of four numbers, and Problem 2 requires students to find a missing number when the first four numbers and the average of the five numbers (including the missing number) are presented graphically. Since open-ended problems seem to be better in eliciting students' thinking and reasoning, both tasks used in the study were open-ended. Students were allowed to use paper and pencils as well as manipulative materials, and were given about 15 minutes to complete the two problems. The post-test, which consisted of the same two problems as the pre-test, was given a week after the last lesson.

Insert Figure 2 about here

Data coding and analysis were completed using a classification scheme adapted from Cai (1995). In particular, each student's response was analyzed from four distinct perspectives: (1) numerical answer, (2) mathematical error, (3) solution strategy, and (4) representation. To ensure inter-rater reliability, two raters randomly selected 20% of the student responses and coded them independently. The inter-rater agreement ranged from 87% to 99%.

RESULTS

Numerical Answer and Mathematical Error

The numerical answer was what the student provided on the answer space for each task, and was judged correct or incorrect. With respect to the correctness of numerical answers, students improved significantly from the pre-test to the post-test. On the pre-test, only 35% and 13% of the students, respectively, answered Problems 1 and 2 correctly. On the post-test, however, 93% and 78% of the students respectively gave the correct answers for Problems 1 and 2. The percentage of students who gave correct answers for both problems increased significantly from 13% on the pre-test to 65% on the post-test ($z = 5.12$, $p < .001$). On the pre-test, 81% of the students who were able to solve Problem 1 failed to correctly solve Problem 2. On the post-test, 22% of the students who were able to solve Problem 1 were still unable to correctly solve Problem 2, but the percentage is statistically smaller than on the pre-test ($z = 5.24$, $p < .001$). This implies that after instruction students had a better understanding of the average concept and were more capable of correctly applying the averaging algorithm to solve problems.

Mathematical errors were similar to those found in studies by Cai (1995) and Cai et al. (1996). A common error that students made in solving Problem 2 was to incorrectly apply the computational algorithm. For example, some students added the numbers of John's blocks (9), Jeff's (3), Joyce's (7), Jane's (5), and the average (8), got a sum of 32, then divided the sum by 5. The students typically gave the whole number part of the quotient (6) as the answer. These students seemed to know the computational procedure for calculating an average (i.e., "add-them-all-up-and-divide"), but did not appear to know what should be added, what should be divided, or what they should divide by. Fewer students made errors on the post-test than on the pre-test. However, error analysis shows that students who did not correctly solve the problems tended to make similar types of errors on both tests. Thus, although student performance in solving the average problems improved significantly from pre-test to post-test, a small proportion of the students still showed a lack of conceptual understanding of the arithmetic average after instruction.

Solution Strategy

Three solution strategies were identified as described below.

Strategy 1 (Average Formula)
The student used the average formula to solve the problem arithmetically (e.g., $8 \times 5 - (9 + 3 + 7 + 5) = 16$ or algebraically (e.g., $(9 + 3 + 7 + 5 + x) = 8 \times 5$, then solve for x).

Strategy 2 (Leveling)
The student used visualization to solve these problems. For example, in solving problem 2, generally students viewed the average (8) as a leveling basis to "line up" the numbers of John's blocks, Jeff's, Joyce's, and Jane's. Since John had 9 blocks, he had one extra. Since Jeff had 3 blocks, 5 additional blocks were needed in order to line up the average. Since Joyce had 7 blocks, she needed 1 additional block to line up the average. Since Jane had 5 blocks, 3 additional blocks were needed. In order to line up with the average number of blocks, Bob should have 16 blocks.

Strategy 3 (Guess-and-Check)
The student first chose a number for Bob, then checked to see if the average of the numbers of blocks for five people was 8. If the average was not 8, then he/she chose another number for Bob and checked again, until the average was 8.

Table 1 shows the percentage of students who used each of the strategies for each of the problems in pre- and post-tests. On the pre-test, 20% of the students used Average Formula and 4% of the students used Leveling Strategy to solve Problem 1. On the post-test, these percentages significantly increased to 28% using Average Formula and 61% using Leveling Strategy to solve Problem 1. Similarly, for solving Problem 2, 7% of the students used Average Formula and 2% of the students used Leveling Strategy on the pre-test. These percentages increased to 13% using Average Formula and 54% using Leveling Strategy on the Post-test. Therefore, the percentages of students who had a clear indication of using appropriate strategies increased from the pre-test to post-test. In particular, 24% of the students who gave a clear indication of using appropriate strategies in solving Problem 1 on the pre-test significantly increased to 89% on the post-test ($z = 3.43$, $p < .01$). In solving Problem 2, 9% of the students gave a clear indication that he/she used an appropriate strategy on the pre-test. The percentage increased to 69% on the post-test (($z = 5.43$, $p < .001$).

Insert Table 1 about here

On the pre-test, students most frequently used the average formula to solve the problems. On the post-test, the number of students who used the Average Formula increased, but the increase was not as dramatic as that for the Leveling Strategy. In fact, only a few students used the leveling strategy on the pre-test, but over half of the students used the leveling strategy on the post-test. Only a few students used guess-and-check strategy in the post-test.

Nearly 67% of the students gave clear indications that they used solution strategies in solving both problems on the post-test. For the 67% of the students who gave clear indications of using solution strategies to solve both problems on the post-test, the majority of them (71%) used the same solution strategy for both problems. For example, if a student used the Leveling Strategy to solve Problem 1, he/she would most likely use the same strategy to solve Problem 2. About 20% of the students who gave clear indications of using solution strategies in solving both problems shifted their strategies from using Average Formula to Leveling. Less than 10% of the students shifted their strategies from Leveling to using Average Formula.

<u>Representation</u>

The responses were examined according to the way students represented their solutions. Three categories were used to classify the representations: verbal, pictorial, and symbolic. A representation was coded as verbal, pictorial, or symbolic if a student mainly used written words, a picture or drawing, or mathematical expressions, respectively, to explain how he/she found the answer. If a student response contained more than one representation, a dominant one was identified and coding was based on the dominant representation.

Table 2 shows the percentages of students who used various representations. From pre-test to post-test, the number of students who did not provide explanations or justifications of their solutions decreased. In particular, on the pre-test 14% and 19% of the students, respectively, did not provide an explanation in solving Problems 1 and 2; while on the post-test, only 2% and 6% of the students respectively did not provide explanations for Problems 1 and 2. Not only did more students provide explanations on the post-test than on the pre-test, but also the quality of student explanations improved from pre-test to post-test. For example, more students' explanations on the post-test than on the pre-test contained clear solution processes.

The type of representations also changed. In the pre-test, about 50% of the students used verbal representation in solving either Problem 1 or 2. These percentages decreased to about 25% on the post-test. On the other hand, more students on the post-test than the pre-test used pictorial representations in solving Problem 1 (from 11% to 54%, $z = 2.43$, $p < .01$) and Problem 2 (from 20% to 46%, $z = 1.58$, $p > .05$). This might be due to the fact that the representations students used appeared to be related to the strategies they employed. For example, when students used the Average Formula to solve the problems, they tended to use symbolic-related representations in their explanations. On the other hand, when students used Leveling Strategies, they tended to use pictorial representations in their explanations. However, not all students who employed the Leveling Strategy used pictorial representations. It was also true that not all students who used the Average Formula displayed symbolic representations.

Insert Table 2 about here

DISCUSSION

The results of Study I suggest that for the pre-test a majority of the students only knew the "add-them-all-up-and-divide" algorithm of calculating average. On the post-test, however, the number of students with conceptual understanding increased dramatically. The findings of this study provide evidence of positive instructional impact on students' understanding of the average concept. This evidence includes: (1) the number of students with correct answers increased from the pre-test to the post-test; (2) more students on the post-test than on the pre-test gave a clear indication of using appropriate strategies; (3) not only did more students provide explanations on the post-test than on the pre-test, more students provided higher quality explanations for their solution processes.

Thus, results of Study I show that, if the emphasis of instruction on average is changed appropriately, students can learn the concept of average in a meaningful way. The change in the teachers' classroom performances was fostered by a teacher-enhancement component which focused on improving their own understanding of the average concept as well as ways to teach it effectively. In particular, the results indicate that teachers may need to learn how to use activities and techniques, which encourage students to construct a meaningful concept of average and to build connections between the concept and the algorithm. This probably requires that teachers become comfortable using strategies like those implicit in the leveling activities found in Bennet, et al. (1993).

The "full" extent of the students' understanding of the average concept, however, needs further exploration. One may argue that teaching a leveling strategy influences more students to use a leveling strategy it is not very surprising. One may also argue that the process of leveling can itself be an algorithm devoid of meaning. The design of Study I does not allow us to determine if the leveling model will lead to students' understanding of the average as a tool for describing and making-sense of a data set (statistical understanding) as well as a mathematical algorithm for solving problems (mathematical understanding).

The focus of Study I was to examine the impact of changes of the instructional approach and materials on students' understanding of the average concept. Therefore, the teaching of the average concept in the classroom was not documented in this study. Further study is needed to document students' learning of the average concept through analysis of classroom instruction and to determine the robustness of the students' concept of average that results from this type of instruction.

STUDY II

PURPOSE

The purpose of Study II was to examine if the changes of instruction previously mentioned will lead to students' understanding of the average as a tool for describing and making sense of a data set (statistical understanding) as well as a mathematical algorithm for solving problems (mathematical understanding). In particular, Study II is intended to examine how the improvement of students' performances on the concept of average are related to teaching through documenting and analyzing classroom instructions.

METHOD

Subjects.

The same teacher and her two classes of 42 sixth-grade students from the same school participated in Study II. The characteristics of the students were similar to those in Study I. However, the students were totally different. Study II was conducted one year later than Study I.

Instruction.

Ms. G met twice with the same university professors prior to the instruction. In these meetings, the results from Study I were discussed. Various instructional activities were also discussed and the teacher reflected on her teaching of the average concept with understanding in Study I. Through discussion and reflection, a consensus was reached that the use of leveling to find an average itself was not the final goal of instruction. The goal of using the leveling model was to *mediate* students' understanding of the average as a tool for describing and making sense of a data set and as a mathematical algorithm for solving problems. Ms. G should use the leveling model in conjuntion of an open-ended problem solving approach to make the mean meanful to her students. Students should not only be provided opportunities to experience everyday use of the average concept and concrete processes of finding an average, but also be provided opportunities to solve real life problems using multiple solutions and representations.

The actual instruction of the unit was completed in seven days. Each lesson from one of her classes was video-taped and transcribed. The following narrative provides a description of Ms. G's instruction on the concept of average. Such narrative of classroom instruction served as a means to understanding the mechanisms by which classroom instruction helped students make sense of the statistical and mathematical aspects of the average concept.

Lesson #1-Introduction to Average Using Leveling. In this lesson, Ms. G first led a discussion that helped the students realize the importance of the concept of average in everyday life. One of her objectives was to help the students have an understanding of average, not just a technique for finding it. In particular, Ms. G, began her lesson by engaging her students in a discussion of everyday uses of the word average. Students gave more than 20 examples of daily use of average such as, average points per game, average weight, grade point average, average blood pressure, average cost of something, average hospital recovery rate, average cost of groceries, average number of votes, average 6th-grade attendance, average allowance, and average taxes.

This discussion gave Ms. G a preliminary idea of how well her class understood the concept of average, and helped her students realize the persuasiveness of the concept. Although the students easily came up with many examples, Ms. G realized that many of them had incomplete, fuzzy, or incorrect notions of average, because quite a few students gave examples that were not appropriate uses of average. For example, one of the students gave the example of "number of deaths," which probably was due to confusing average with total. Ms. G corrected students' inappropriate examples of average. Then, she asked her students: "Well, average is everywhere, you know. But, what does average mean?" "For example, what does average 6th-grade attendance mean? How can we find the average 6th-grade attendance?"

With these questions, Ms. G engaged her class in a discussion of sharing candy among three friends. Students discussed different ways to figure out fair shares: doling out a piece at a time, counting the total and dividing by three, making three equal piles. Finally, Ms. G led them to the leveling strategy using the overhead projector. She drew three columns labeled "you," "first," and "second." Then she placed 12 tiles in the "you" column. At first, there were no tiles in the "first or "second" columns. She said, "Okay, now instead of putting it in a pile, I am putting it in a column, okay? So I am going to start moving these down and giving them their own little towers of candy, okay? Take from the 'haves' and give to the 'have nots'." On the overhead, Ms. G took tiles from the "you" column and evenly distributed them to the "first' and "second" columns until each column had four tiles. "Now they are all level, all at the same height. So we call that leveling off. Bringing everybody's amount to the same level since they share equally."

Ms. G then turned the students to bowling situations. She told the students that when the class goes bowling they will keep track of the number of strikes each student makes in three games. Ms. G then made up some hypothetical outcomes for various pairs of students, and asked the class to find the average number of strikes made by each pair. In Ms. G's examples, each pair averaged seven strikes: 9 and 5, 12 and 2, 13 and 1. She demonstrated finding each average by laying out two columns of chips and then leveling them off. She was trying to help the students predict that any combination of strikes with a pair sum 14 would average to 7. In the end, the students came close to the prediction. Then Ms. G summarized that, "any combination as long as the total is 14 and we've only got two people, the average is going to always be seven, right, okay. So it hinges on the total, which you guys were talking about putting together."

After showing a few examples of using leveling to find an average, Ms. G passed out tiles to the students, and asked them to work in groups, and to use the leveling strategy to find the average miniature golf score of two students who scored 23 and 13. Afterwards, she led a class discussion in which the students were encouraged to analyze the leveling they had been doing. One of the students noticed that the number of tiles moved from the larger to the smaller column was half the difference between the two columns. At the end of class, Ms. G congratulated everyone for coming up with the correct answer, and told them they were ready for the next day's lesson.

Lesson #2-Comparing Five Averaging Methods. Ms. G's objective in Lesson #2 was for her students to review and compare methods for finding the average of two numbers. As lesson 2 began, Ms. G asked the students to recall the methods of finding average which they discussed in the previous class. The following five methods were: 1) Leveling off; (2) Find the midpoint; (3) Get the total, divide by two; (4) Find the difference and then subtract half the difference from the larger number; (5) Find the difference and then add half the difference to the smaller number. She then asked the students to explain how each method works for finding an average, while she wrote the five methods on the blackboard.

Next she challenged her students: "I have 5 hogs. Billy has 13. We are going to share, okay. Go through all the different ways you could find out how many we're going to have when we get done sharing...from getting the average, OK?" Ms. G walked around the room helping the students use all five averaging methods. In her discussion afterwards, she emphasized that, regardless of the method used, the number "4" [half the difference] played an important role. She stressed that by "leveling" half the difference between the two numbers, inequities were removed and the average was achieved.

However, Ms. G found that the midpoint method for finding average (Method 2) was a source of confusion for some students, who apparently confused it with the method for finding the median. The discussion of midpoint and middle reminded them of instruction they had received in elementary school on the concept of median. Unable to distinguish between methods for finding average and median, they thought that the average was the same as median. One of these students demonstrated finding the median by ordering five numbers (99, 75, 46, 4, 2) from largest to smallest and circling 46, the middle number. These students were convinced that this was the same as the "midpoint" method the class had been using to find the average. Pressed for time at the end of the class period, Ms. G simply told the class, "Okay, but that is not the average; it is the middle number of that group. What you are doing is related to another concept in statistics, but it is not finding an average."

Lesson #3-Using Leveling to Find the Average of Three or More Numbers. Interestingly, Ms. G did not come back to the confusion some students had about finding mean and median in lesson 3. However, she did come back to this point in lesson 7. This decision was based on her desire to focus on the leveling model. Therefore, she directly went to new content in lesson 3. Prior to this lesson, students had used the leveling strategy to find the average of pairs of numbers. In this lesson, the students were asked to use the leveling strategy to find the average of three or more numbers. The objective of this lesson was to extend and deepen their understanding of various properties of the average.

The students were first asked to make columns of numbers 4, 7, 2, 9, and 3, and then to visualize how many tiles would be in each column if they were leveled off. She told the class to predict the average of the columns by visualizing moving the tiles until all the columns were the same height (level). The reason Ms. G asked the students to predict the average was that she wanted to make explicit an important property of average, namely that the average is neither larger nor smaller than the largest or smallest numbers. After receiving a number of predictions, she said "Nobody is saying more than 9, right? And no one is saying less than 2. Okay, why couldn't it be more than 9?" One of the students said "Because the highest number is 9." Another student added, "'Cause you're going to have to take stuff off from 9." She then summarized their answers and told her students to "go ahead" and move tiles until all the columns were level. She emphasized again that the average must be within the range of the four numbers.

Some new problems were designed to illustrate new aspects of the average concept. In particular, the new problems were chosen to highlight different features related to finding an average: when the average is not a whole number, when one of the numbers being averaged is zero, and when one of the numbers is unknown, but the average is known. The following problem involved sets of numbers with an average that was not a whole number. There were six columns and 32 tiles.

Ms. G:	And after you got everyone to five you got some left over, right?
Student A:	Two left over.
Ms. G:	So those left over would have to be shared among how many columns?"
Student B:	"Six!"
Ms. G:	So one will be shared by these three, and the other one would have to be shared among the other three.
Student C:	So it would be one-third.
Ms. G:	That is a neat way to do it. So it would be five and one-third 'cause the extra ones have to be cut into pieces, right?

Next, Ms. G asked the students to figure out the number of tiles in a fourth column given the numbers of tiles in each of the first three and the average. "All right, one column is 9. Another column is 6. The next column is 8, and the last one you don't know. The average for these four columns is 6. What you need to do is figure out how many would be in that tower or column, okay?" Some students immediately said, "It must be less than 6 because if the average is 6 and you average 9 and 8, it can't go down." After a short time, one student had worked out the answer. He explained that he removed all the tiles in excess of 6 from the first three columns (9, 8, 6) and placed them in the fourth column. Now he had piles of 6, 6, 6, 5. He said, "It wouldn't be 6 [in the fourth column], so I added one there, so it made 1."

Lesson #4-Averaging in the Computer Lab. For their fourth consecutive lesson on averaging, Ms. G took her class to the computer lab. The objective of this lesson was to help students experience the value of technology and to examine properties of average. About half the time was spent on mechanical issues related to helping the students enter home run data into a spread sheet. After the data had been entered the students used the "=max"and "=min" functions to find the maximum and minimum of the data. The students also used the "=mean" function, and learned that mean is another name for the average. Students realized that the mean is always greater than the minimum and less than the maximum. The students were also instructed to make a bar graph of the data. Ms. G related the bar graph to their work on leveling the previous three days. Some students indicated that the bar graph looked like what they were doing when they had the tiles in towers or columns. Ms. G noted, "You can see how we could level off here, can't you?"

Lesson #5-Average Temperature. The objective of lesson 5 was to use average to describe a data set and compare two data sets. The instruction consisted of a class discussion of how meteorologists determine the monthly average temperature. Ms. G had temperature data from January and February for the class to compare. She started the lesson by asking her students to write the answers to three questions: (1) What does the meteorologist mean when he/she makes the statement: The average temperature for the month of January is 27 degrees? (2) What steps do you think the meteorologist took to get the number (27 degrees); (3) So far this month, Milwaukee's temperatures have been quite high (45 degrees on Saturday, 41 degrees tomorrow). How will our high January temperatures affect the average?

About 15 minutes later, Ms. G said "Okay, I'd like to talk about the answers you wrote and see if we can agree on what these meteorologists mean when they make the statement, 'The average temperature for the month of January is 27 degrees.' " One of the students answered, "Okay, he means the overall temperature is such and such degrees. He added up all the temperatures and then divided by how many days are in January, and then he got the average."

Ms. G asked, "Where do these numbers come from? When are we taking these numbers." After some discussion, another student said, "Okay, they take it in one day. The highest temperature and the lowest of that day." Ms. G said "Okay, so now we have the high and low. What good are those going to do us for that day? What else could we find?" Some students answered together, "Average. Adding them up and dividing them by 2." "Or you can level them off," some other students added. Ms. G took this opportunity to guide students to examine and compare two ways to find an average: leveling and using the average formula. Ms. G said "Okay, all right. So now how many Januarys do you think the weather man is talking about when he says the average temperature is 27 degrees?" One student said "He means all the Januarys to 1996." To which Ms. G replied, "Actually, I think our weather records go as far back as 1849, so we've got a lot of information that we could use, right?"

Thus, the students had collectively come up with all the pieces of a complicated method: (1) find the average of the high and low temperatures for each day of the month, (2) find the average of all the average daily temperatures in the month, (3) find the average of all the average January temperatures. No one, however, had described how to fit the pieces together to get the average January temperature. Ms. G realized time was getting short, so she showed them how the pieces could be put together rather than spend more time on this question.

As the period came to a close, Ms. G led the students to discuss the meaning of "The temperature for the month of January is usually five degrees lower than February." This discussion proceeded more smoothly than the discussion of the steps needed to find the average temperature; probably because the details were not as complicated. The important outcome was that the students realized the average can be used to summarize a data set or compare two data sets. In Ms. G's example, the average temperature for January (however it is found) can be used to represent all the January temperatures and can be compared to the average temperature for February, which represents all the February temperatures.

Lesson #6-Collecting and Representing Raisin Data. Based on the class discussion on the previous day, Ms. G decided to do more hands-on activity with averaging. Ms. G began lesson 6 by passing out a box of raisins to each student and asking the students to separate the raisins into two piles: raisins with stems and raisins without stems. She told the students to write down how many raisins had stems and how many did not. The students were to save those numbers for a future class on ratio and proportion.

Students spent about half of the class sorting their raisins. Ms. G then passed out graph paper to each student. "We're going to use graph paper to help us represent all the raisins you counted - just like we used tiles to help us with leveling off and we moved the tiles around. We're using another representation: a graph." Ms. G needed to help individuals decide what to put on each axis, what the scale ought to be, and how to label the graphs. The class put the names of the students in the class along the horizontal axis and the total number of raisins each student had on the vertical axis. There were 21 students in the class. The number of raisins ranged from 19 to 43. The students worked on the graphs for the remainder of the period. Ms. G told them they would need the graphs on the following day.

Lesson #7-Averaging Using Raisin Data. For this lesson, the students worked in groups to find the average number of raisins in a box. Ms. G began lesson 7 by asking the students "Now that we have been working on averaging, what is the average number of raisins we have?" Some students suggested they use the leveling strategy. Ms. G asked, "What is the first step in leveling off?" The students said they needed to find a leveling off point, which would be the least number, 19. Ms. G used this as an opportunity to discuss the range of the data, which was 43-19= 24. To help clarify the possible misconception that the midpoint of the maximum and minimum points is the average, Ms. G asked the students to figure out the midpoint. After some discussion, they agreed the midpoint was 31. She asked," Is that the average?" Ms. G also took this opportunity to help students distinguish between the mean and the median (which was the misconception raised during Lesson #2). She then discussed how the students could draw a line across their graphs at 19 and represent with tiles the raisin data in excess of 19. Then they were to use leveling to find the average and see if it was the same as the midpoint number, 31.

Most students started to represent raisins with tiles, which they laid out in columns similar to their bar graphs from the previous day's class. In reality, however, the total number of raisins was so great the students only had enough tiles to represent the raisins in excess of 19, the least number of raisins found in any box. Once all 21 boxes were represented, the students leveled the tiles to get the average. The average was not a whole number, so most students simply reported the average was a whole number part, which was 29, plus a remainder. At this time, disadvantages of using tiles to level off were discussed. In fact, most students ended up using combinations of tiles, graphs, calculators, drawings, and paper and pencil calculations to find the average. The students decided that using the average formula to solve this problem seemed to be favorable and more efficient.

Pre- and Post-Tests.

The pre-and post-tests were designed to capture students' mathematical and statistical understanding. In Study I, problems on the pre- and post-tests contained graph-like diagrams. These "graphs" may have invited students to use a leveling strategy. Therefore, pre- and post-tests in Study II contained problems with both graph-like pictorial representations and word-like verbal representations. In particular, Problems 1, 2, 3, and 4 were used as the pre-test. Problems 1 and 2 were the same as those used in Study I, with graph-like pictorial representations. Problems 3 and 4 had similar structures as problems 2 and 1, respectively, but they were presented in words with no pictorial representations. Problems 1, 2, 3, and 4, as well as problems 5, 6 and 7 were included in the post-test in Study II. See Figure 1. The post-test was given one week after the last lesson. Students' responses to problems 1, 2, 3, and 4 were analyzed using a scheme similar to Study I, with a focus on (1) numerical answer, (2) mathematical error, (3) solution strategy, and (4) representation. Analyses of students' responses to Problems 5, 6, and 7 were focused on correctness of answers and the nature of their explanations. Only those students who completed both the pre- and post-tests were included in the analysis, a total of 37 students.

RESULTS

The results are reported in two separate sections. The first section reports the results of students' performances on Problems 1, 2, 3, and 4 in both pre- and post-tests. The second section reports results of students' performances on Problems 5, 6, and 7.

Results from Problems 1, 2, 3, and 4

Table 3 shows the percentages of students on the pre- and post-tests who correctly answered each of the Problems 1, 2, 3, and 4 and used appropriate solution strategies. On each problem, percentages of students having the correct answer or appropriate strategies increased significantly from the pre-test to the post-test. These results suggest the positive impact of instruction on students' understanding of the average concept.

Insert Table 3 about here

Using the Average Formula and Leveling are the two strategies used by students to solve these problems. Table 4 shows the percentages of students using each of these solution strategies for each problem in their pre- and post-tests. On the pre-test, students mainly used the Average Formula to solve these problems. A small proportion of the students used the Leveling Strategy. On the post-test, the average formula and leveling strategy were evenly favorable to solve each of the problems. From pre-test to post-test, the percentage of students who correctly used the Average Formula and Leveling to solve each of these problems increased.

Insert Table 4 about here

Task representations appear to have an impact on the students' selection of solution strategies. Recall that Problems 1 and 2 were presented graphically, while Problems 3 and 4 were presented in words. However, Problems 2 had the same structure as Problem 3 and Problem 1 had the same structure as Problem 4. On the pre-test, proportions of students using the Average Formula and Leveling Strategy in solving Problem 1 are similar to those in Problem 4, but a larger percentage of students used the Average Formula in solving Problem 3 (22%) than in solving Problem 2 (11%). On the post-test, the proportion of students using the Average Formula or Leveling Strategy in solving Problems 1 and 2 are evenly distributed. However, in Problems 3 and 4, a larger proportion of students used the Average Formula than the Leveling Strategy.

Examination of solution representations also reveals changes from pre-test to post-test. Table 5 shows the percentages of students who used various representations or did not provide any explanation. Similar to the findings of Study I, the percentages of students who did not provide any explanations decreased from pre-test to post-test. In fact, on the post-test, each student provided an explanation for his/her solution process. From pre-test to post-test, the proportion of students who used written words to explain their solution processes decreased, but proportions of those who used symbolic and pictorial representations increased. It should be noted that, not all students who employed the leveling strategy used pictorial representations. It is also true that not all students who used the Average Formula displayed symbolic representations.

Insert Table 5 about here

In the pre-test, a larger proportion of students used symbolic representations in solving Problems 3 and 4 than in solving Problems 1 and 2, but only a slightly larger proportion of students used pictorial representations in solving Problems 1 and 2 than in solving Problems 3 and 4. On the post-test, a larger percentage of students used symbolic representations in solving Problems 3 and 4 than in solving Problems 1 and 2, but a larger proportion of students used pictorial representations in solving Problems 1 and 2 than in solving Problems 3 and 4. These results also suggest the impact of task presentation on students' selection of representations.

<u>Results from Problems 5, 6, and 7</u>

Problems 5, 6, and 7 were only administered as part of the post-test. These tasks were designed to assess if students could transfer their knowledge about average to solve more complex problems after instruction. About 32% (12 of 37), 35% (13 of 37), and 30% (11 of 37) of the students, respectively, had correct answers for problems 5, 6, and 7. About two-thirds of the students had a correct answer for at least one of these three problems. One-third of the students could not solve any of these problems.

Commonly, students flexibly used the average formula to solve Problem 5. In their explanations of solution processes, they either used mathematical expressions or written words. For example, one student explained in words that: "First I multiplied 87 $\times$ 10 and got 870. Then I subtracted 55 from 870 and got 815. Then I subtracted 95 and got 720. Then I divided 720 by 8 and got 90." Another student explained symbolically that: "$87 \times 10 - (55 + 95) = 720$. $720 \div 8 = 90$."

One of the students used a unique strategy to solve Problem 5. The student first used one of the properties of average and determined that the average of the remaining eight scores must be between 55 and 95. The student then drew 10 circles and put 95 on the first and 55 on the last, leaving eight empty circles. Using these numbers, the average would be 15 [$(95 + 55) \div 10 = 15$]. The student said that each of the eight blank spaces should get 15. But 15 is 72 less than 87 (the average for 10 scores). He then multiplied 72 by 10 and got 720. Since $720 \div 8 = 90$, 90 became the average of the remaining eight scores after the top and bottom scores were thrown away. In this solution, the student viewed throwing away the top and bottom scores as taking 15 away from each circle. By inventing this approach, which works for any problem like Problem 5, this student demonstrated an incredible understanding of averaging.

A few students used properties of average to estimate the average of the remaining eight scores. Their reasoning was revealed in the following explanation: "I think that the average for his remaining scores is between 55 and 95. But 87 is closer to 95 than 55. So the average must be bigger than 87. It may be 88 or 89." These students used estimations but no one obtained a correct answer. However, their explanations show a certain degree of understanding of the average. A few students tried to use the Leveling Strategy to find the average of the remaining set of eight scores, but they failed. Of those who got correct answers for Problems 6 and 7, a majority used the average formula as explanations to verify their possibilities. A small proportion of them used leveling to verify their possibilities.

DISCUSSION

The results of Study II were similar to the results found in Study I. The students' performance improved from pre-test to post-test. Not only did the number of students with correct answers increase significantly from the pre-test to the post-test, but more students on the post-test than on the pre-test gave a clear indication of using appropriate strategies. Moreover, one-third of the students were able to successfully transfer their knowledge to solve novel problems. The differences in the students' performances from pre-test to post-test suggest that students improved in their understanding of the average concept. Thus, the findings of Study II not only confirmed the findings from Study I, but also extended Study I and provided stronger evidence for a positive instructional impact on students' understanding of the average concept.

Examination of the seven lessons suggests that Ms. G focused her instruction on (1) both a concrete and abstract way of finding an average, (2) both a direct application of the averaging algorithm and a reverse application of the averaging algorithm, (3) both average as a computational algorithm and as a tool for describing and interpreting data, (4) both the properties of the average concept and the usefulness of average concept in real life, and (5) both finding the answers for average-related problems and finding these answers using multiple strategies and representations.

In particular, her instruction emphasized each of the following. (1) Ms. G. used the concrete leveling model for finding an average, which enabled students to establish a mental image of averaging. (2) Ms. G. used the concrete leveling model for finding an average, which served as a means to help students build knowledge about the seven properties of average identified by Strauss and Bichler (1988). For example, after a few columns of tiles were evened off, the height of the evened-off column is always higher than the lowest column and lower than the highest column (i.e., The average is located between the extreme values). The number of tiles which were moved away from the higher columns is always equal to the number of tiles which were received by the lower columns (i.e., The sum of the deviations from the average is zero). If the height of one of the columns was changed, the height of the evened-off columns was changed accordingly (i.e, The average is influenced by values other than the average). The average value is a representative of the values that were averaged. (3) Ms. G introduced the formal computational algorithm for calculating an average only after students had established a mental model of calculating the average using leveling of tiles. However, students were consistently encouraged to use two approaches to find an average. Ms. G consistently guided students to comparing the two approaches and discussing their advantages. For example, when asked to find an average for a set of larger numbers, using the average formula or algorithm may be more efficient than using the actual leveling approach. (4) Ms. G. used specifically designed problems or activities to show students both mathematical and statistical aspects of the average concept. For example, students were provided opportunities to average monthly temperatures and to compare the temperatures in different months. Students were also provided opportunities to flexibly use the average algorithm to solve problems.

Improvement of students' performances from pre-test to post-test and analysis of classroom instruction suggest that uses of the leveling model in conjunction of the open-ended problem solving approach for instruction lead to students' enhanced understanding of the average as an algorithm and as a tool for describing data. In particular, the results of this study showed that a significantly larger number of students were able to flexibly apply averaging ideas to solve complex problems. Students had better understanding of the properties of the average concept. Some students even explicitly used properties of the average concept to solve some of the problems in the post-test.

However, leveling itself can still be procedural. Analysis of classroom instruction suggests that using the leveling strategy alone to find an average could lead students to memorize leveling procedures without conceptual understanding of why they are doing so. Thus, caution should be taken when a teacher guides students to use a leveling strategy to find an average. Leveling itself is not the final objective of instruction. Leveling is an instructional means to mediate students' conceptual understanding. As long as they establish an initial model of averaging, students should be encouraged to use numerical approaches to find an average and statistical ideas to make sense of data.

From pre-test to post-test, the number of students who used the Average Formula to solve problems increased, but the increase was not as dramatic for the number of students who used the Leveling Strategy. In fact, in the pre-test, only about 3 to 5% of students used the Leveling Strategy, but the percentage increased from 15% to 30% in the post-test. These results may suggest that tasks with the same structure, but different presentation may elicit slightly different responses. Tasks with pictorial presentations more likely elicit the Leveling Strategy and visual representations in students' solutions, while tasks with presentations of written words more likely elicit strategies of using the averaging algorithm and symbolic representation. These results may also suggest that instruction with a leveling model influences students' selection of solution strategies. In fact, for those tasks with pictorial representation, 5% of the students used the Leveling Strategy in the pre-test, but it was increased to over 30% in the post-test. The change is not as big as that for tasks presented in written words. For these tasks, 3% of the students used the Leveling Strategy in the pre-test, but increased to about 15% in the post-test.

CONCLUSION

This paper reported two studies which explored the effectiveness of instruction in promoting students' understandings of the average concept. The leveling model illustrates that the average can be viewed as a "even-off" process. This conception, in turn, mediates the understanding of average as a tool for describing and making sense of a data set (statistical understanding) and as a mathematical algorithm for solving problems (mathematical understanding). These studies suggest a way to teach average for understanding. Most importantly, findings of these studies suggest that in order to teach mathematics with understanding, teachers themselves must have well-established mathematical content and pedagogical knowledge. Further, they must be able to actually use their content knowledge to provide students with opportunities and assistance to engage in thinking and sense-making of mathematics. Studies reported here also indicate that teachers' success in teaching for understanding may be related to the encouragement and support they receive as they begin to change their approach to teaching. This change does not happen easily and it does not happen in a short period of time. The collaboration of university professors and school teachers may be an effective way to help change occur.

REFERENCES

Bennett, A., Maier, E., & Nelson, L. T. (1993). Visualizing number concepts. Mathematics Learning Center.

Cai, J. (1995). Beyond the computational algorithm: Students' understanding of the arithmetic average concept. In L. Meira & D. Carraher (Eds.), Proceedings of the Nineteenth Annual Meeting of the International Group for the Psychology of Mathematics Education (Vol. 3, pp. 144-151). Recife, Brazil: Universidade Federal de Pernambuco.

Cai, J. & Moyer, J. C. (1995). Middle school students' understanding of average: A problem solving approach. In D. T. Owens, M. K. Reed, and G. M. Millsaps (Eds.), Proceedings of the Seventeenth Annual Meeting of the North American Chapter of the International Group of the Psychology of Mathematics Education (Vol. 1, pp. 359-364). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Cai, J., Magone, M., Wang, N., & Lane, S. (1996). A cognitive analysis of QUASAR's performance assessments and their sensitivity to measuring changes in middle school students' thinking. Research in Middle Level Education Quarterly, 19(3), 63-94.

Cohen, D. K., & Ball, D. (1990). Relations between policy and practice: A commentary. Educational Evaluation and Policy Analysis, 12(3), 327-345.

Duffy, G., & Roehler, L. (1986). Constraints on teacher change. Journal of Teacher Education, 37(1), 55-58.

Gal, I. Rothschild, K., Wagner, D. (1990, April). Statistical concepts and statistical reasoning in elementary school children: Convergence or divergence? Paper presented at the American Educational Research Association, Boston, MA.

Hardiman, P. T., Well, A. D., & Pollatsek, A. (1984). Usefulness of a balance model in understanding the mean. Journal of Educational Psychology, 76(5), 792-801.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 65-100). New York: Macmillan Publishing Company.

Meyer, R. A., Browning, C., & Channell, D. (1995). Expanding students' conceptions of the arithmetic mean. School Science and Mathematics, 95, 114-117.

Mevarech, Z. (1983). A deep structure model of students' statistical misconceptions. Educational Studies in Mathematics, 14.,415-429.

Mokros, J. & Russell, S. J. (1995). Children's concepts of average and representativeness. Journal for Research in Mathematics Education, 26, 20-39.

NCTM (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: NCTM.

Pollatsek, A., Lima, S., & Well, A. D. (1981). Concept or computation: Students' understanding of the mean. Educational Studies in Mathematics, 12, 191-204.

Shaughnessy, J. M. (1992). Research in probability and statistics: Reflections and directions. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 465-494). New York: Macmillan Publishing Company.

Strauss, S., & Bichler, E. (1988). The development of children's concepts of the arithmetic average. Journal for Research in Mathematics Education, 18(1), 64-80.

Figure 1

Leveling Process of Finding Average of 5, 6, and 13

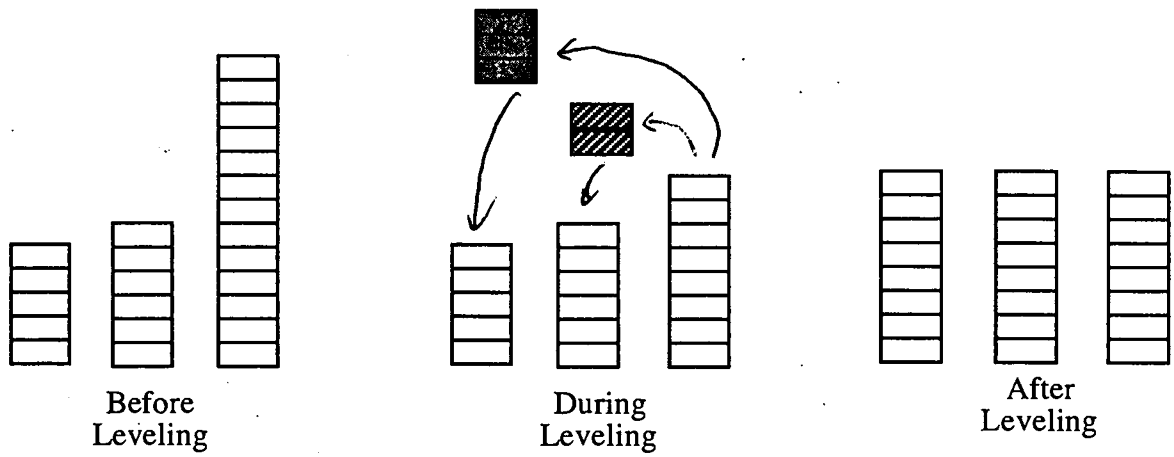
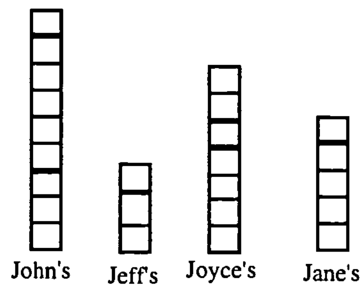


Figure 2. Pre- and Post-Tests for Studies I and II

Problem 1

John, Jeff, Joyce, and Jane each has a stack of blocks, which are shown below.



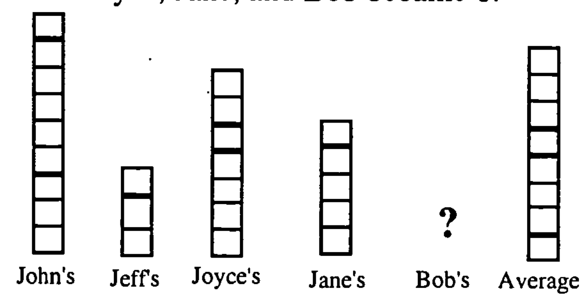
What is the average number of blocks for those four people?

Answer:__________________

Explain how you found your answer.

Problem 2

Later Bob joined them. When Bob came in, the average number of blocks for John, Jeff, Joyce, Jane, and Bob became 8.



How many blocks did Bob have so that the average for the five people was 8?

Answer:__________________

Explain how you found your answer.

Problem 3

For their club's food drive Jason has 12 cans of food, Deon has 5 cans, Tonya has 4 cans, and Essie has 3 cans. When Deandre came in with his cans, the average number of cans for Jason, Deon, Tonya, Essie and Deandre became 7.

How many cans did Deandre have so that the average for the five people was 7?

Answer:______________________________

Explain how you found your answer.

Problem 4

For a different club's food drive, Tasha, David, Jeffrey, and Dwayne each has some cans. Tasha has 11 cans, David has 6 cans, Jeffrey has 5 cans, and Dwayne has 2 cans.

What is the average number of cans for those four people?

Answer:______________________________

Explain how you found your answer.

Problem 5

The average of Ed's ten test scores is 87. The teacher throws out the top and bottom scores, which are 55 and 95.

What is the average of the remaining set of scores?

Answer:______________________________

Explain how you found your answer.

Problem 6

The average of five numbers is 14. One of the numbers is 8.

Name two possibilities for the other four numbers.

Possibility 1:____

Possibility 2:____

Show how you found your answer.

Problem 7

We took a survey of the family size of ten different families. The average family size for these ten families was 4.

What would the family sizes of these ten families be?

Answer:______________________________

Explain how you found your answer.

Table 1

Percentages of Students Using Each of Solution Strategies in Study I

	Percent of Students			
	Pre-test		Post-test	
	P1	P2	P1	P2
Strategy 1 (Using average formula)	20	7	28	13
Strategy 2 (Leveling)	4	2	61	54
Strategy 3 (Guess-and-check)	0	0	0	2
Strategy cannot be identified	76	91	11	31

Table 2

Percentage of Students Using Various Representations in Pre- and Post-test In Study I

	Percent of Students			
	Pre-test		Post-test	
	P1	P2	P1	P2
Verbal	55	44	24	26
Pictorial	11	20	54	46
Symbolic	20	17	20	28
Without Explanation	14	19	2	6

Table 3

Percentages of Students Having Correct Answer and Appropriate Strategies for Problems 1, 2, 3, and 4 in Study II

	Percentage of Students					
	Correct Answer			Appropriate Strategy		
	Pre-test	Post-test	P-value	Pre-test	Post-test	P-Value
Problem 1	57	89	$P < .01$	46	86	$P < .01$
Problem 2	19	70	$P < .01$	14	70	$P < .01$
Problem 3	24	68	$P < .01$	24	61	$P < .01$
Problem 4	51	81	$P < .05$	43	73	$P < .01$

Table 4

Percentages of Students Using Solution Strategy for Problems 1, 2, 3, and 4 in Study II

	Percentage of Students			
	Using Average Formula		Leveling Strategy	
	Pre-test	Post-test	Pre-test	Post-test
Problem 1	41	43	5	43
Problem 2	11	35	3	35
Problem 3	22	44	3	17
Problem 4	41	54	3	19

Table 5

Percentages of Students Using Representations for Problems 1, 2, 3, and 4 in Study II

	Percentage of Students							
	Verbal		Symbolic		Pictorial		No Explanation	
	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test
Problem 1	68	46	14	22	5	32	14	0
Problem 2	68	50	14	19	5	31	14	0
Problem 3	65	56	27	30	3	14	5	0
Problem 4	65	49	24	32	3	19	8	0