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ABSTRACT

Designed as an avenue of communication for mathematics educators concerned with the views, ideas, and experiences of two-year college students and teachers, this journal contains articles on mathematics exposition and education and regular features presenting book and software reviews, classroom activities, instructor experiences, and math problems. The first of two issues of volume 18 includes the following major articles: "When a Critical Point Is Not an Extremum" (Terry R. Tiballi); "A Variation of a Problem from Calculus of Two Intersecting Right Cylinders" (Leonard Casciotti and Donald Beken); "An Extension of Synthetic Division" (William Donnell); "The Sine Function CAN Have Values Larger Than One" (Paul Murrin); "Evaluation of a Gap Series" (Russell Euler); "Increasing or Decreasing?" (Jay M. Jahangiri and Herb Silverman); "A Journey with Self-Assessment as a Compass" (Agnes Azzolino); and "Spreadsheets in a Differential Equations Course?" (Richard F. Maruszewski). The second issue of the volume contains the following major articles: "Can You Recognize a Textbook that Supports the AMATYC Standards?" (Jack Rotman and Brian Smith); "A Classification System for Primitive Pythagorean Triples" (Neil Basescu); "Identifying Degenerate Conic Sections" (David E. Dobbs and John C. Peterson); "Summation of Arithmetic Sequences of Higher Degree" (Ayoub B. Ayoub); "An Unexpected Proof of an Unexpected Occurrence of e" (Sid Koplas and Steve Marsden); "Another Rotation of Axes" (Richard Quint); "An Alternate to Cantor's List" (Sandy Coleman); and "Responses to Teacher Feedback on Errors Differ by Age and Gender" (Sandra P. Clarkson and Wm. H. Williams). (TGI)

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Volume 18, Numbers 1-2
1996-97

Joseph Browne
Editor

THE AMATYC REVIEW

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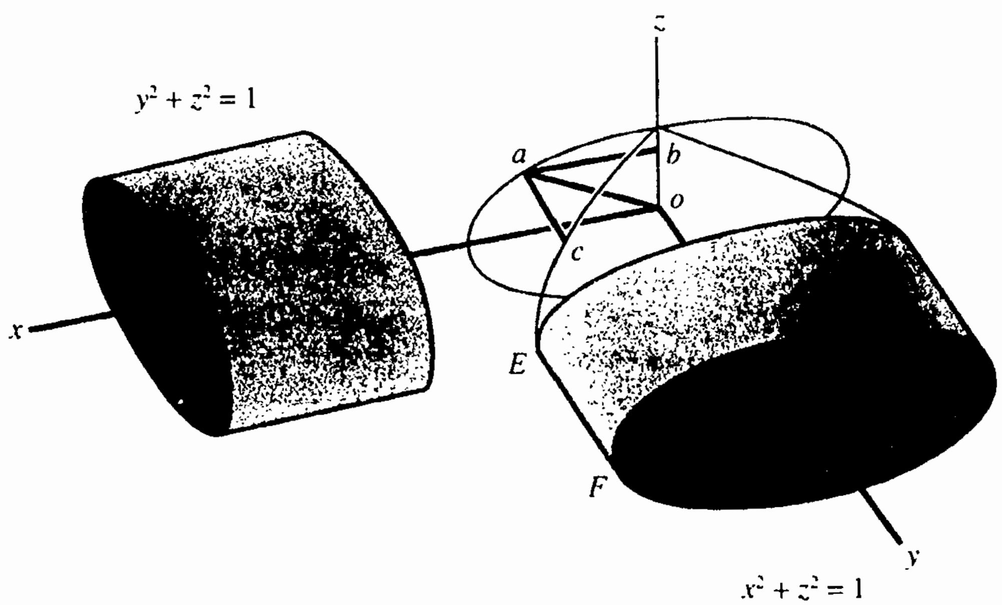
Intersecting Cylinders



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- Spreadsheets in Differential Equations Course
- An Extension of Synthetic Division
- Sin θ > 1?

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MISSION OF AMATYC: Recognizing the vital importance of the first two years of collegiate mathematical education to the future of our students and the welfare of our nations, AMATYC is committed to the following:

- to positively impact the preparation of scientifically and technologically literate citizens;
- to lead the development and implementation of curricular, pedagogical, assessment and professional standards for mathematics in the first two years of college;
- to assist in the preparation and continuing professional development of a quality mathematics faculty that is diverse with respect to ethnicity and gender;
- to provide a network for communication, policy determination, and action among faculty, other professional organizations, accrediting associations, governing agencies, industries, and the public sector.

The AMATYC Review provides an avenue of communication for all mathematics educators concerned with the views, ideas and experiences pertinent to two-year college teachers and students.

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About the Cover and Editor's Comments

If two cylinders intersect, what is the shape of the hole that one of them cuts through the other? A circle, of course, at least as viewed from the axis of the cutting cylinder. Leonard Casciotti and Donald Beken (see "A Variation of a Problem from Calculus of Two Intersecting Right Cylinders") take the subject a little further and ask this: Suppose you had to cut that hole in the material before it is rolled into a cylinder; What is the shape of the hole while the material is still flat? It's a tougher problem, but still solvable with the techniques studied in the calculus sequence.

There are a couple of staff changes to report. We welcome a new Book Review Editor in this issue, Sandra Coleman. We also are seeking a new Software Reviews Editor to replace Shao Mah who is retiring. If you would like to be considered for the position, please send me a letter (or e-mail) telling a little about your background and your ideas for the column.

Letter to the Editor:

I would like to respond to "Notes from the Mathematical Underground" edited/written by Alain Schremmer in the spring, 1996, issue of *The AMATYC Review*. As the editor at John Wiley & Sons, Inc. for CALCULUS by Hughes-Hallett, Gleason, et al., I would like to set the record straight especially in regard to two misleading statements about our widely-used and much admired textbook. First, this book is not being "dropped right and left" as was inaccurately reported in this article. Our sales results, which are based on real numbers and not on the wishful thinking of a competing publisher, demonstrate a solid pattern of growth and retention. Second, the book's focus is at all times on calculus—not data analysis as Schremmer implies. The highly regarded research mathematicians who participated as authors of this book as well as the respected mathematics departments who have adopted it offer ample testimony to the solidity of the mathematics. I also take issue with Schremmer's attitude toward textbook selection at community colleges. A significant number of community colleges have adopted Hughes-Hallett, Gleason, et al's CALCULUS, as well as other reform calculus books, giving thoughtful consideration to their students' educational needs and improving the quality of their programs.

Finally, I was disappointed to see *The AMATYC Review* choose to publish an article containing unsubstantiated (and incorrect) facts. Schremmer's admission that he failed to gain concrete evidence that the book was being "dropped right and left" should have been sufficient reason to excise his anecdote from the article. It should not be the practice of *The AMATYC Review* to print articles containing unsubstantiated and potentially damaging hearsay from an uninformed editor with an ax to grind.

Ruth Baruth,
Mathematics Editor
John Wiley & Sons, Inc.

Schremmer replies: In some ways, the Editor of John Wiley is more correct than she may think: I did not even try to gain concrete evidence since, as she acknowledges, this was just an anecdote. In other ways, she is not: This was not an article but a column and, further, I did not misrepresent the anecdote for anything else – but the anecdote itself is a fact. The conversation did take place. Besides, I also mentioned the "great successes that [my friends] were having with the 'Harvard Calculus'". So, why is mighty John Wiley & Sons so aroused by one paragraph in an obscure column? Am I not allowed to mention that "I was arguing that, by now, most so-called calculus books, a good example being the "Harvard Calculus" as it is now known, were really texts about what should more properly be called Data Analysis and that mathematics appeared to be going the way of Greek and Latin"?

As for her second objection, the books' focus, I was, of course, not talking about the "solidity of the mathematics": As I said in my column, "Yeah, but do [the students] learn some mathematics?" Of course, this depends on what you define as learning mathematics. I note in passing that there is no reply to the criticism of the Harvard Calculus I made in my Fall 1995 column. As for Ms. Baruth's taking "issue with [my] attitude toward textbook selection at community colleges", is it possible that she might be referring to an earlier column where I had written that "As any publisher will tell you, we just open the book at three or four places to see how the author handles certain pet items of ours and, if s/he does it our way, we like the book!"? Here, the possibilities for heavy sarcasm are indeed infinite, but I shall refrain. Still, I do not see that Wiley & Sons have made any point. I wish they had.

Lucky Larry #23

$$\frac{4}{8}+\frac{4}{8}-\frac{1}{2}=\frac{8}{16}-\frac{1}{2}$$

$$=\frac{7}{14}$$

$$=\frac{1}{2}$$

Submitted by Peter Collinge
Monroe Community College
Rochester NY 14623

MATHEMATICAL EXPOSITION

When A Critical Point Is Not An Extremum

by

Dr. Terry R. Tiballi
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Oswego NY 13126

After teaching for eleven years at North Harris College in suburban Houston, Terry R. Tiballi completed his Ph.D. in mathematics and is now an Assistant Professor at SUNY Oswego. He is currently completing a college algebra text that will be published in the fall.

At some point in the discussion of local extrema of functions in the first semester of calculus, students are shown that a number $x = c$ may be a critical point of a function f (hence $f'(c) = 0$ or $f'(c)$ does not exist) yet $f(c)$ may be neither a local maximum nor a local minimum value of f. Typically $f(x) = x^3$ is given as an example of a function which has a critical point at $x = 0$ but does not assume a relative maximum or minimum value there. However, (0,0) is a point of inflection for this function. Certainly a good student is capable of noticing this and wondering if a point of inflection must necessarily occur at a critical point where an extreme value is not taken. Is this true in general or is this example misleading?

If $x = c$ is critical because $f'(c)$ does not exist a counterexample can be found without too much trouble. Let

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ -\sqrt{x} & \text{if } x \geq 0 \end{cases}.$$

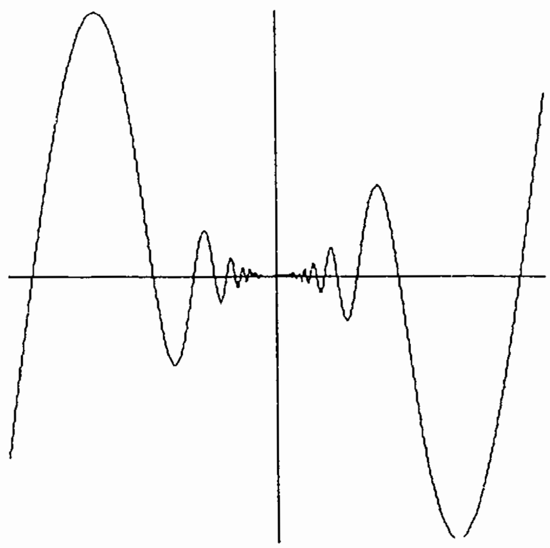
Though f is continuous at $x = 0$, it is not differentiable there. Thus $x = 0$ is a critical point. Since f is decreasing on $(-\infty,\infty)$, no maximum or minimum value occurs at $x = 0$. This function is concave upward on $(-\infty,0)$ and $(0,\infty)$, hence $(0,0)$ is not a point of inflection.

But what if f is differentiable at $x = c$? It seems hard to imagine a function that meets our hypotheses and does not have a point of inflection at $x = c$! Specifically let us assume that f is twice differentiable on an open interval containing c, except possibly at c, and that $f'(c) = 0$. If f has neither a relative maximum or minimum at c must $(c, f(c))$ be an inflection point?

As you might guess, an example to the contrary can be found. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Then f is twice differentiable except at $x = 0$, $f'(0) = 0$ and $f(0)$ is not an extreme value. Since there does not exist an open interval (a,b) containing 0 such that the graph of f is concave upward on $(a,0)$ and concave downward on $(0,b)$, or vice versa, $(0,0)$ is not a point of inflection.



$y = x^2 \sin\left(\frac{1}{x}\right)$

The function f described above has the peculiar property that f'' has infinitely many zeros on every open interval about 0. If we assume that there is an open interval about $x = c$ that contains a finite number of zeros of f'' then, no extremum at $x = c$ does imply that $(c,f(c))$ is a point of inflection. In our proof of this assertion we use the following theorem:

THEOREM: (Darboux Property) If a real function f is differentiable on an interval containing the numbers a and b and if μ is between $f'(a)$ and $f'(b)$, then there is a ξ between a and b such that $f'(\xi) = \mu$.

This theorem asserts that, like a continuous function, the derivative of a function that is differentiable on an interval assumes all values between any two of its values even though it may not be continuous. For a discussion of this theorem the interested reader may consult Malcolm Pownall's *Real Analysis: A First Course With Foundations*.

Suppose that f is twice differentiable on an open interval containing c, except possibly at c, and that $f'(c) = 0$. Then we have the following theorem.

THEOREM: If f'' has a finite number of zeros on some open interval containing c and if f attains neither a relative maximum or minimum value at $x = c$ then $(c,f(c))$ is necessarily a point of inflection.

PROOF: Since f'' has a finite number of zeros on an open interval containing c, there exist open intervals (a,c) and (c,b) on which $f'' \neq 0$. The sign of f'' must be constant on each of these intervals since otherwise the above theorem would imply the existence of a zero of f''. If $f'' > 0$ on (a, c) and $f'' < 0$ on (c,b), or vice versa, then $(c,f(c))$ is a point of inflection and we are done. We argue that the other two possibilities lead to a contradiction.

Suppose that $f'' > 0$ on $(a,c) \cup (c,b)$. Then f' is increasing on (a,c) and f' is increasing on (c,b). Since f' has the Darboux Property on $(a,c]$ and $f'(c) = 0$ it follows that $f' < 0$ on (a,c). (There is something to check here.) Because f' has the Darboux Property on $[c,b)$ and $f'(c) = 0$ it follows that $f' > 0$ on (c, b). By the First Derivative Test, $f(c)$ is a relative minimum value. This contradicts our assumption that no extremum is taken at $x = c$.

If $f'' < 0$ on $(a,c) \cup (c,b)$ a similar argument yields that $f(c)$ is a relative maximum value, which again is a contradiction. Hence, if no extreme value occurs at $x = c$, $(c,f(c))$ is necessarily a point of inflection.

This theorem shows that for most of the functions encountered in calculus (that is, for those functions whose second derivatives have only a finite number of zeros), a point of inflection must occur at a critical point where an extreme value is not taken.

Reference

Pownall, M.W. (1994). *Real analysis: A first course with foundations*. Dubuque, IA: Wm. C. Brown Publishers.

A good stack of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one.

Paul Halmos

❖ ❖ ❖

The value of a problem is not so much coming up with the answer as in the ideas and attempted ideas it forces on the would be solver.

I.N. Herstein

❖ ❖ ❖

We have to reinvent the wheel every once in a while, not because we need a lot of wheels; but because we need a lot of inventors.

Bruce Joyce

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A Variation of a Problem from Calculus of Two Intersecting Right Cylinders

by

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and
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Dr. Don Beken is an associate professor of mathematics at Pembroke State University. He received his Ph.D. in mathematics from the University of Mississippi in 1978. His research interests are in partially ordered algebras and semigroups.

One of the classical problems that students encounter in a calculus course is to calculate the volume of a region formed by the intersection of two right circular cylinders of the same radius. After the discussion of this problem, one of the authors was asked the question: How does one cut a cylinder in order that it will butt smoothly against another cylinder? In particular how would one construct a template that would wrap around a cylinder in order to mark a line representing the curve of intersection?

For the following discussion the radii are assumed to be one. Consider the equations of the cylinders:

$$y^2 + z^2 = 1$$

$$x^2 + z^2 = 1$$

in Figure 1.

We will now determine how the cylinder $x^2 + z^2 = 1$ must be cut in order to butt smoothly against the cylinder $y^2 + z^2 = 1$. Eliminating the z variable we have $y^2 = x^2$. That is $y = \pm x$. The circle formed by intersecting the cylinder $x^2 + z^2 = 1$ with the xz-plane (see Figure 1) can be parameterized in terms of θ the angle $\angle boa$ in xz-plane. The x-coordinate of point c will have the same value as the length of line segment ab. Now the circle has radius 1 yielding $x = \sin\theta$. Since $y = |x|$ on the curve of intersection of the two cylinders, we have the y-coordinate of point c equaling $|\sin\theta|$. The z-coordinate of point c has the same value as the length of segment ob which is $\cos\theta$. We can now write the curve of intersection in the parametric form

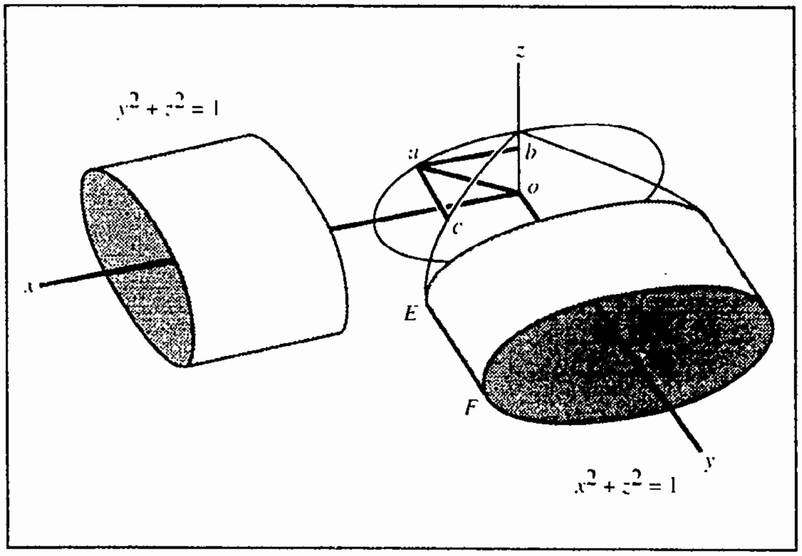

Figure 1. Intersecting cylinders.

$$(\sin\theta, |\sin\theta|, \cos\theta) \text{ with } 0 \le \theta \le 2\pi. \qquad (1)$$

If we cut along the segment EF lying on a ruling of the cylinder defined by $x^2 + z^2 = 1$ and roll it flat we would have a template (see Figure 2). Actually our

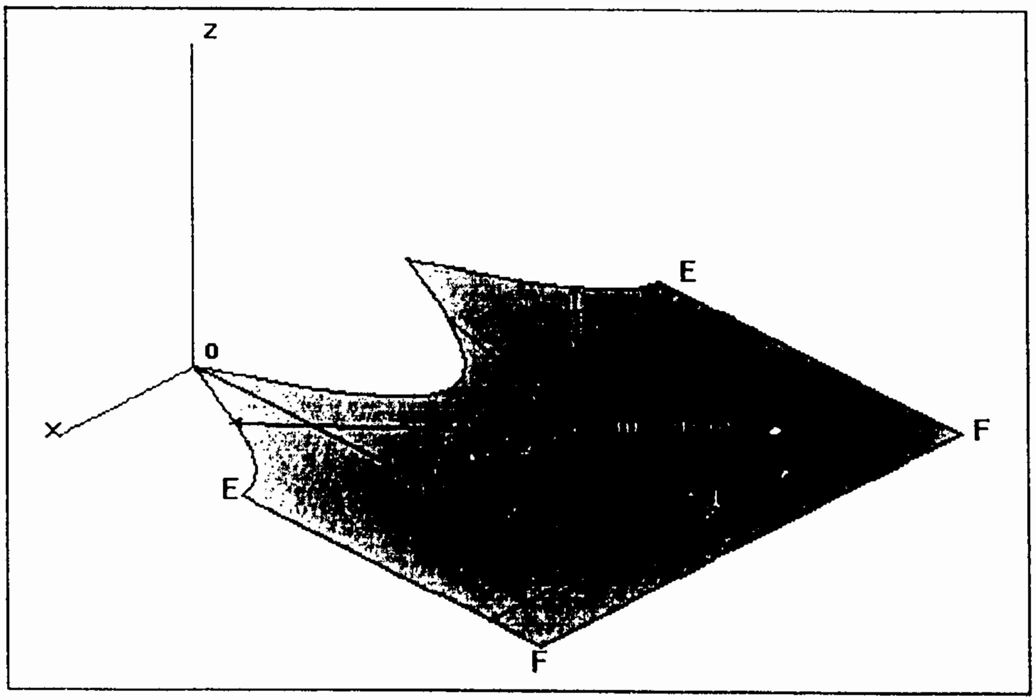

Figure 2. Cylinder rolled flat.

template would lie on the plane $z = 1$ or $z = -1$. In either case, if we project it onto the xy-plane we would have Figure 2. The parametric equation graphed in Figure 2 is

$$(\theta, |\sin\theta|, 0) \text{ where } -1.5\pi < \theta < .5\pi. \qquad (2)$$

A companion problem to that of butting a cylinder against another cylinder is that of developing the template needed to cut a hole in a cylinder so that another cylinder will fit smoothly into that hole. For instance, suppose you wish to weld two large diameter pipes together to carry a fluid. If the pipes are of a small diameter, then one would simply drill a hole of a diameter matching the pipe. For large diameter pipe one would need a template.

One approach would be to take another cylinder of the same diameter and use the above template to butt it against the cylinder to be cut. Mark the intersection and then cut the desired hole; however, that requires an additional cylinder and two cuttings. We decided to take a more direct approach.

In Figure 1, consider the perpendicular line segment from point c to the plane yz. That line segment lies on a ruling of the cylinder $y^2 + z^2 = 1$, and has *the* same length as the segment ab. The length of ab is $\sin\theta$ where $0 \leq \theta \leq 2\pi$. In Figure 3, the curve efg is traced out in the parametric form

$$(\sin\theta, 0, .5\pi - \theta) \text{ where } 0 \leq \theta \leq \pi. \qquad (3)$$

The curve ghe is the mirror image of the efg curve with respect to the z-axis in the xz-plane.

A generalization of the problem would be the case where the radius $\neq 1$. If the

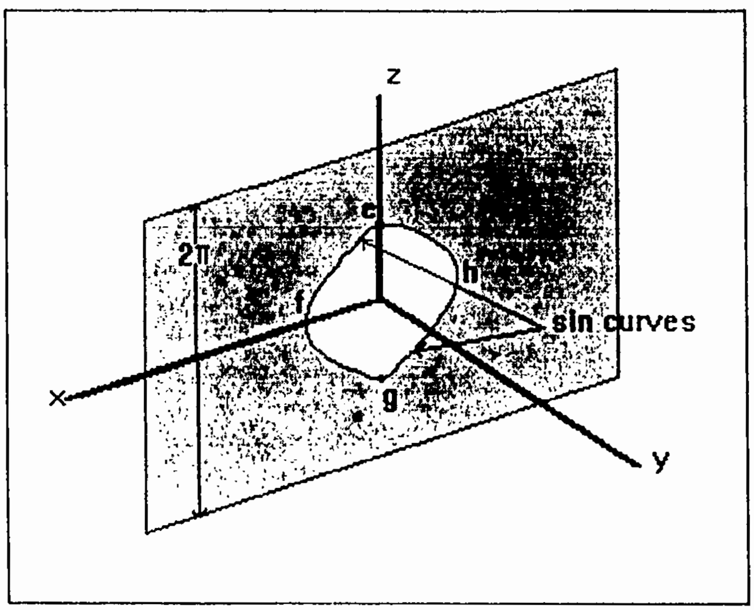

Figure 3. Cylinder rolled flat.

radius were given as R then the only changes that would need to be made are that the length of segment ob is $R\cos\theta$. From above, equation (1) is replaced with $(R\sin\theta,\ R|\sin\theta|,\ R\cos\theta)$ and equation (2) is replaced with $(R\theta,\ R|\sin\theta|,\ 0)$ and equation (3) is to be replaced with $(R\sin\theta,\ 0,\ R(.5\pi-\theta))$. The width of the template would then be $2\pi R$.

Observe that while the problem originated from a calculus problem, the actual mathematics needed is covered in most trigonometry classes. We suggest that before an analysis of the intersections is undertaken that the student be encouraged to experiment on paper cylinders. We found that the students seem to better appreciate the analysis after experimentation. The use of computer graphics would enhance the students' understanding, and frees the teacher from the arduous task of drawing representative figures.

> How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?
>
> Albert Einstein

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An Extension of Synthetic Division

by

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Synthetic division is a well known technique for dividing a polynomial $P(x)$ of degree m where $m \geq 1$, by a first degree polynomial of the form $x - k$. The number k is called the synthetic divisor. The division results in a polynomial $Q(x)$ of degree $m - 1$, called the quotient or depressed polynomial, and a constant polynomial $R(x)$ called the remainder. Students are able to learn this process rather quickly and seem to appreciate its simplicity and the ease with which they can carry it out. (Note: Synthetic division is sometimes called "Horner's Method," particularly in the context of numerical analysis.)

Some students ask if there is a similar process which works for divisors other than those of the form $x - k$. This paper gives a partial answer by showing an extension to include divisors of the form $x^n - k$, where $n \geq 2$. The technique is similar to the one the students have already learned and, therefore, not difficult to master. As with standard synthetic division, it is much faster and easier to carry out than long division of polynomials.

We begin with an example of ordinary synthetic division.

Example 1. Use synthetic division to divide $P(x) = 2x^5 + 3x^3 - x + 7$ by $x + 1$.

The coefficients of $P(x)$ are 2, 0, 3, −1, 7. The synthetic divisor is −1 since $x + 1 = x - (-1)$. The completed tableau is as follows:

	2	0	3	0	−1	7
−1	–	−2	2	−5	5	−4
	2	−2	5	−5	4	3

Note that a dash is placed inside the tableau in the space under the leading coefficient. The reason will become clear later. The coefficients of $Q(x)$ in decreasing order of powers are 2, −2, 5, −5, and 4. The remainder, 3, is boxed to separate it from $Q(x)$. In other words, $Q(x) = 2x^4 - 2x^3 + 5x^2 - 5x + 4$ and $R(x) = 3$.

An Extension of Synthetic Division

Suppose $P(x)$ is a polynomial of degree m which is to be divided by the binomial $x^n - k$ where $1 \leq n \leq m$. The division algorithm for polynomials enables one to write $P(x) = Q(x) \cdot (x^n - k) + R(x)$. This results in a quotient $Q(x)$ of degree $m - n$ and a remainder $R(x)$ of degree $n - 1$ or less. As with the usual synthetic division, arrange the $m + 1$ coefficients of $P(x)$ in order of decreasing powers in the top row of the tableau. Place the synthetic divisor k outside to the left. Next place n dashes directly beneath the first n coefficients of $P(x)$.

	a_m	a_{m-1}	$\cdots$	a_{m-n+1}	a_{m-n}	$\cdots$	a_1	a_0
k	–	–	$\cdots$	–				

Having set up the tableau, carry out the division as follows: Bring down the first n coefficients underneath the synthetic divisor symbol. Then multiply each of these n numbers by the synthetic divisor k, placing the result of each multiplication in the second row n positions to the right. Omit any products which would extend beyond the last coefficient a_0. Add the next n (or possibly less) coefficients to these products, placing the sums below the line (i.e. in the third row). Continue until the bottom row is filled with $m + 1$ numbers. Fortunately, doing this is considerably easier than trying to describe it. If you have trouble following this description, look at an example below and then return to the next paragraph.

Finally, put a box around the last n numbers in the bottom row; these are the coefficients of the remainder, $R(x)$. The $m - n + 1$ numbers before the box are the coefficients of $Q(x)$.

Example 2: Divide $P(x) = 2x^7 - 3x^5 + 4x^4 - x^3 + 2x - 1$ by $x^2 + 1$. First set up the tableau as follows:

	2	0	–3	4	–1	0	2	–1
–1	–	–						

Note that the –1 and two dashes in the second row imply division by $x^2 + 1$. Bring down the first two coefficients and complete the multiplication and addition process.

	2	0	–3	4	–1	0	2	–1
–1	–	–	–2	0	5	–4	–4	4
	2	0	–5	4	4	–4	–2	3

Thus $Q(x) = 2x^5 - 5x^3 + 4x^2 + 4x - 4$ and $R(x) = -2x + 3$.

Example 3: Divide $3x^6 + 5x^2 + 4$ by $x^4 - 2$.

	3	0	0	0	5	0	4
2	–	–	–	–	6	0	0
	3	0	0	[0]	[11]	[0]	[4]

Thus, $\dfrac{3x^6 + 5x^2 + 4}{x^4 - 2} = 3x^2 + \dfrac{11x^2 + 4}{x^4 - 2}$. Note that the number of dashes in row two is always the same as the number of boxed numbers in row three.

Example 4: All six complex sixth roots of negative 1 are roots of $P(x) = x^{12} + x^{10} + x^8 + 2x^6 + x^4 + x^2 + 1$. This can be shown by dividing $P(x)$ by $x^6 + 1$ as follows:

	1	0	1	0	1	0	2	0	1	0	1	0	1
–1	–	–	–	–	–	–	–1	0	–1	0	–1	0	–1
	1	0	1	0	1	0	1	[0]	[0]	[0]	[0]	[0]	[0]

Since $R(x) = 0$, the result is established.

A full proof of the method would be rather involved, if only for the notation, and we will not attempt it here. We can get an idea of why the method works by comparing the standard long division algorithm with the extended synthetic division algorithm for a specific case.

Example 5: Divide $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ by $x^2 - k$. Using the standard long division algorithm, we have

$$
\begin{array}{r|lllll}
 & ax^2 & +bx & +(c+ak) & & \\
\hline
x^2 - k & ax^4 & +bx^3 & +cx^2 & +dx & +e \\
 & ax^4 & & -akx^2 & & \\
\hline
 & & bx^3 & +(c+ak)x^2 & +dx & \\
 & & bx^3 & & -bkx & \\
\hline
 & & & (c+ak)x^2 & +(d+bk)x & +e \\
 & & & (c+ak)x^2 & & -k(c+ak) \\
\hline
 & & & & (d+bk)x & +(e+ck+ak^2)
\end{array}
$$

The extended synthetic division algorithm would look like this:

	a	b	c	d	e
k	–	–	ak	bk	$k(c+ak)$
	a	b	$c+ak$	[$d+bk$]	[$e+ck+ak^2$]

We see that in either case the result is $ax^2 + bx + (c + ak) + \dfrac{(d+bk)x + (e+ck+ak^2)}{x^2 - k}$.

This generalization of synthetic division can be mastered rather quickly since it is a natural extension of the algorithm already known. It is a useful and timesaving tool. A teacher could assign students the project of trying to discover a generalization of synthetic division. Another possibility is to assign students to read this paper, or have a student make a presentation on it to the class. This could be followed by an assignment which uses the technique.

Acknowledgment

The author expresses his appreciation to the editor for assistance in the preparation of this article.

SOFTWARE REVIEW EDITOR WANTED

Shao Mah, who has edited the Software Reviews column for many years, is retiring. We wish him well in his new endeavors. We seek a new editor who is acquainted with a wide range of mathematically related software and is not shy about requesting review copies of new products and asking colleagues to test them. If you think you might be the right person for the job, please contact the editor (address inside front cover).

Lucky Larry #24

Larry observed that if $\sin 2x = 2\sin x\cos x$, then clearly $\sin 5x = 5\sin x\cos x$. He used this to evalute the following trigonometric limit:

$$\lim_{x\to 0}\frac{\sin 5x}{2x} = \lim_{x\to 0}\frac{5\sin x\cos x}{2x}$$

$$= \lim_{x\to 0}\left[\frac{5}{2}\,\frac{\sin x}{x}\,\cos x\right]$$

$$= \frac{5}{2}\cdot 1\cdot 1$$

$$= \frac{5}{2}$$

Submitted by Steven Ira Lansing
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SHORT COMMUNICATIONS

The Sine Function CAN Have Values Larger Than One

by

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Some functions behave in unexpected ways with complex arguments. The thrust of this article is to remind us that the Sine function is not necessarily bounded between $[-1,1]$, provided z is complex. I mention this surprising observation to all of my trigonometry students at the end of each semester, because I spend all semester teaching them the Sine function is bounded.

If we let $z = x + iy$ and recall that $\sin(iy) = i\sinh y$ and also $\cos(iy) = \cosh y$, we have

$$\begin{aligned}\sin z &= \sin(x + iy)\\ &= \sin x \cos iy + \cos x \sin iy\\ &= \sin x \cosh y + i \cos x \sinh y\end{aligned}$$

Then

$$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$

and, since $a^2 + b^2 \geq 2|a||b|$, we have

$$|\sin z|^2 \geq 2|\sin x||\cos x||\sinh y||\cosh y| = \frac{1}{2}|\sin 2x||\sinh 2y|.$$

Since $\sinh 2y$ is unbounded as $y \to \pm\infty$, we see that $\sin z$ is also unbounded.

Solving the equation $\sin z = 2$ will demonstrate that the value of the Sine function can be ***Real*** and larger than 1. If we let

$$\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y = 2$$

then

$$\cos x \sinh y = 0 \quad \text{and} \quad \sin x \cosh y = 2.$$

If $\sinh y = 0$, then $y = 0$ and $\cosh 0 = 1$ would require $\sin x = 2$, which is not possible. If however, we let $\cos x = 0$, then $x = \frac{n\pi}{2}$, where n is an odd integer, and $\sin \frac{n\pi}{2} \cosh y = 2$. Since $\cosh y > 0$, we know $\sin x = \sin \frac{n\pi}{2} \neq -1$. Thus, $\sin x = 1$. Hence, $x = (4k+1)\frac{\pi}{2}$, where k is an integer, and $y = \cosh^{-1}2$. Putting this together gives us $z = x + iy = (4k+1)\frac{\pi}{2} + i\cosh^{-1}2$ (the principal value of which is $z = \frac{\pi}{2} + i\cosh^{-1}2 \approx 1.57 + 1.32i$) with $\sin z = 2$. We conclude that the Sine function can have ***Real*** values larger than 1.

> How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?
>
> Sir Arthur Conan Doyle

Evaluation of a Gap Series

by

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Russell Euler has been a member of the faculty at Northwest Missouri State University since 1982. He is a frequent contributor of papers and problems/solutions to this and other journals. Currently, Russell is the editor of the Pi Mu Epsilon Journal.

Let p be a positive integer. The purpose of this paper is to evaluate

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{pn}}{(pn)!}. \tag{1}$$

This power series is referred to as a **gap** series since it has many zero coefficients for $p \geq 2$. The techniques used to find a closed form for (1) consist of first obtaining a differential equation of order p which is satisfied by $f(x)$. This differential equation will be solved by the method of Laplace transforms to produce an alternative expression for $f(x)$. This provides a nonstandard application of Laplace transforms since we are reversing the usual situation where one starts with a differential equation and proceeds toward the solution.

Using the ratio test, it can be shown that the series in (1) converges for $|x| < \infty$. This follows from the fact that $\lim_{n \to \infty} \frac{(pn)!}{[p(n+1)]!} = 0$. The series converges uniformly on compact subsets of $(-\infty, \infty)$ and the series can be differentiated termwise. Differentiating (1) p times yields the initial-value problem

$$f^{(p)}(x) + f(x) = 1 \tag{2}$$

for all real numbers x with $f^{(j)}(0) = 0$ for $j = 0, 1, 2, \ldots, p-1$. Equation (2) can be solved using the Laplace transformation to give $(s^p + 1)\,\mathcal{L}[f(x)] = \frac{1}{s}$ and so

$$\mathcal{L}[f(x)] = \frac{1}{s(s^p + 1)}. \tag{3}$$

To find the partial fraction decomposition for the right-hand side of (3), first consider $s^p + 1 = 0$. Then $s^p = -1 = e^{(2k+1)\pi i}$ and so $s = e^{(2k+1)\pi i/p}$ for $k = 0, 1, \ldots, p-1$.

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Therefore, $\dfrac{1}{s(s^p + 1)} = \dfrac{1}{s\prod_{k=0}^{p-1}\left[s - e^{(2k+1)\frac{\pi i}{p}}\right]}$

$$= \frac{A_1}{s} + \sum_{r=1}^{p} \frac{A_{r+1}}{s - e^{(2r-1)\frac{\pi i}{p}}}$$

and so

$$1 = A_1(s^p + 1) + \sum_{r=1}^{p} \frac{A_{r+1}s(s^p + 1)}{s - e^{(2r-1)\frac{\pi i}{p}}}.$$

Using standard techniques, it can be shown that $A_1 = 1$ and $A_{r+1} = -\dfrac{1}{p}$ for $r = 1, 2, \ldots, p$. Then equation (3) becomes

$$\mathfrak{L}[(f(x)] = \frac{1}{s} - \frac{1}{p}\sum_{r=1}^{p} \frac{1}{s - e^{(2r-1)\frac{\pi i}{p}}}.$$

Taking the inverse Laplace transformation of this equation gives

$$f(x) = 1 - \frac{1}{p}\sum_{r=1}^{p} e^{xe^{(2r-1)\frac{\pi i}{p}}}$$

$$= 1 - \frac{1}{p}\sum_{r=1}^{p} e^{[\cos((2r-1)\frac{\pi}{p}) + i\sin((2r-1)\frac{\pi}{p})]x}. \qquad (4)$$

Table 1 on the following page is formed from equation (4) for some specific values of p.

If p is even, terms in the summation in (4) can be paired and simplified to give

$$f(x) = 1 - \frac{2}{p}\sum_{r=1}^{\frac{p}{2}} e^{x\cos[(2r-1)\frac{\pi}{p}]}\cos\left[x\sin\left(\frac{(2r-1)\pi}{p}\right)\right]. \qquad (5)$$

Similarly, if p is odd, then

$$f(x) = 1 - \frac{1}{p}e^{-x} - \frac{2}{p}\sum_{r=1}^{\frac{p-1}{2}} e^{x\cos[(2r-1)\frac{\pi}{p}]}\cos\left[x\sin\left(\frac{(2r-1)\pi}{p}\right)\right]. \qquad (6)$$

It is possible to express (1) in terms of a generalized hypergeometric function – specifically $a\,{}_0F_{p-1}$. This can be done as follows.

The factorial function is defined by $(a)_0 = 1$ for $a \neq 0$ and $(a)_n = a(a+1)\ldots(a+n-1)$ for $n \geq 1$. In particular, $n! = (1)_n$ and from an identity on page 22 of [1],

$$(pn)! = (1)_{pn} = p^{pn}\left(\frac{1}{p}\right)_n\left(\frac{2}{p}\right)_n\left(\frac{3}{p}\right)_n \cdots \left(\frac{p-1}{p}\right)_n n!.$$

So,

p	$f(x)$
1	$1 - e^{-x}$
2	$1 - \cos x$
3	$1 - \frac{1}{3} e^{-x} - \frac{2}{3} e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2} x\right)$
4	$1 - \cosh\left(\frac{x}{\sqrt{2}}\right) \cos\left(\frac{x}{\sqrt{2}}\right)$
5	$1 - \frac{1}{5} e^{-x} - \frac{2}{5} \left[e^{x \cos\frac{\pi}{5}} \cos\left(x \sin\frac{\pi}{5}\right) + e^{-x \cos\frac{2\pi}{5}} \cos\left(x \sin\frac{2\pi}{5}\right)\right]$
6	$1 - \frac{1}{3} \cos x - \frac{2}{3} \cosh\left(\frac{\sqrt{3}x}{2}\right) \cos\left(\frac{x}{2}\right)$
7	$1 - \frac{1}{7} e^{-x} - \frac{2}{7} \left[e^{x \cos\frac{\pi}{7}} \cos\left(x \sin\frac{\pi}{7}\right) + e^{x \cos\frac{3\pi}{7}} \cos\left(x \sin\frac{3\pi}{7}\right)\right.$ $\left. + e^{x \cos\frac{5\pi}{7}} \cos\left(x \sin\frac{5\pi}{7}\right)\right]$
8	$1 - \frac{1}{2} \left[\cosh\left(x \cos\frac{\pi}{8}\right) \cos\left(x \sin\frac{\pi}{8}\right) + \cosh\left(\cos\frac{3\pi}{8}\right) \cos\left(x \sin\frac{3\pi}{8}\right)\right]$

Table 1.

$$f(x) = 1 - \sum_{n=0}^{\infty} \frac{(-x)^n}{p^{pn} \left(\frac{1}{p}\right)_n \left(\frac{2}{p}\right)_n \cdots \left(\frac{p-1}{p}\right)_n n!}$$

$$= 1 - {}_0F_{p-1}\left(- - -; \frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \ldots, \frac{p-1}{p} \,\middle|\, \frac{-x}{p^p} \right).$$

One can now get a relationship between the ${}_0F_{p-1}$ and the expressions in (5) and (6).

Reference

Rainville, E.D. (1960) *Special functions*. New York: Chelsea Publishing Company.

We only think when confronted with a problem.

John Dewey

Increasing or Decreasing?

by

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and

Herb Silverman — College of Charleston, Charleston SC 29424

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Herb Silverman earned his B.A. from Temple University and his M.S. and Ph.D. from Syracuse University. Herb has been a mathematics professor at the College of Charleston since 1976. Herb has received the College of Charleston Outstanding Research Award and has been Speaker of the Faculty at the College of Charleston.

In this note, we discuss how a commonly used notation for graphing functions in algebra and calculus classes can confuse and mislead students and offer an alternative that seems to be working well with our students.

When we introduce the concept of increasing/decreasing curves (as graphs of increasing/decreasing functions), whether in an algebra or calculus class, the standard notation can be confusing to many students. Almost all our algebra or calculus students know the graph of $y = x^2$ is a parabol, and graph it as in Figure 1.

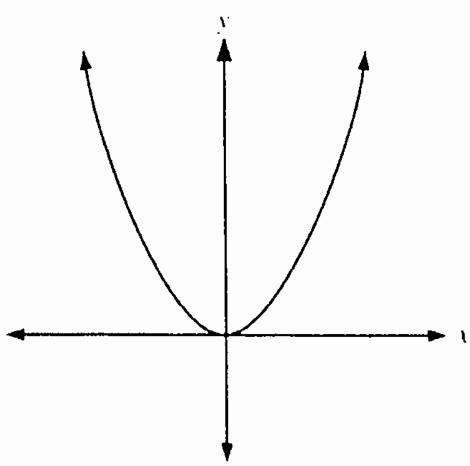

Figure 1

Why? Because they have seen this kind of graphing in their textbooks and even by some of their teachers. Students will often describe the graph as "increasing in both directions." It is difficult for some of the students to accept that the part of the graph that lies in the second quadrant, in Figure 1, is *decreasing* when they see the left tail pointing upward. In defining the Cartesian plane, we draw the x-axis and y-axis as in Figure 2.

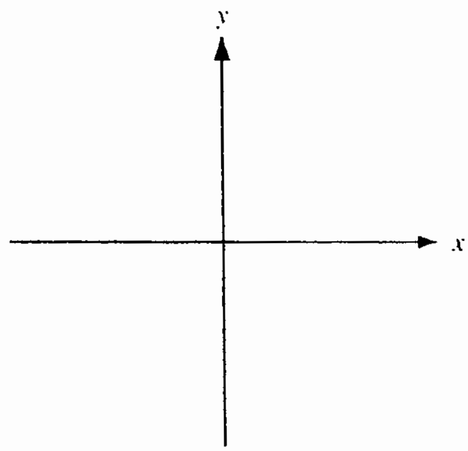

Figure 2

Many, students, however, even watching the board, will copy it down as in Figure 3.

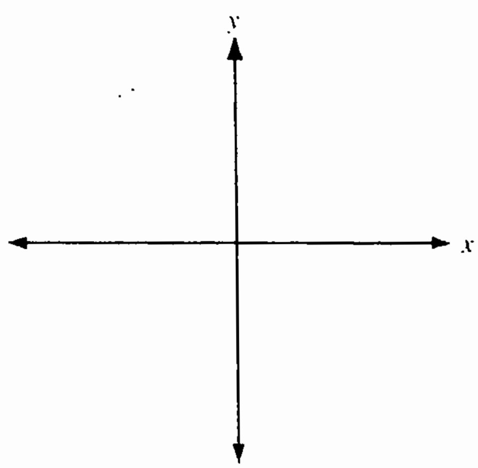

Figure 3

When we sketch the graph of $y = f(x) = x^2$ as in Figure 4 some of our students insist that an arrow is needed on the left tail of the graph. To satisfy these students,

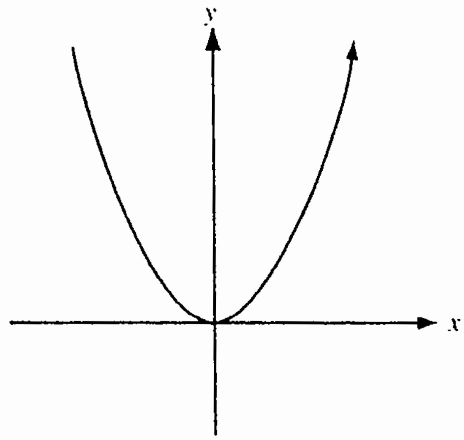

Figure 4

we have added a downward arrow (see Figure 5) to emphasize that the function is sliding down in the second quadrant and not climbing up. Replacing $\uparrow$ with $\downarrow$ on the left tail of the graph of $y = x^2$ does seem to successfully clear up the confusion on the part of most students.

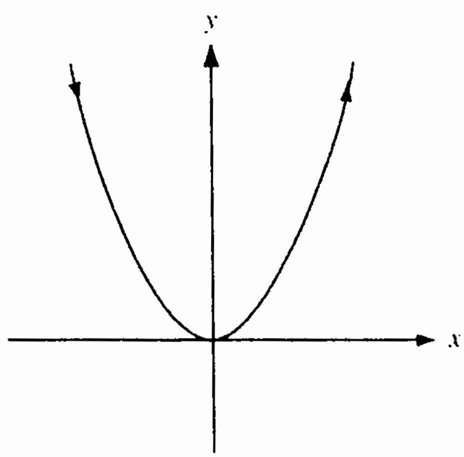

Figure 5

Many math teachers incorrectly assume that students have already accepted the convention so familiar to us, that the graph of a function moves from left to right on the Cartesian plane as the independent variable moves from left to right on the horizontal axis. The sliding arrows on the left tail of graphs helps fix in the minds of students that we are moving in our graph from left to right through the domain from $-\infty$ to $+\infty$. It is a simple matter to create for the students the correct perception with virtually no additional class time.

MATHEMATICS EDUCATION

A Journey with Self-Assessment as a Compass,© 1995

by

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Determine your position.

Decide your destination.

Move in that direction.

Continue to assess your position and destination so your direction may be modified.

The above strategy might be used to reach any number of destinations or attain any number of goals. In this case the goal is to provide activities, tests, exams, procedures, and assignments which maximize the outcomes of a course. The journey uses an instructor's continued self-evaluation of the self-written, graded, tests and assignments — not the parts students submitted for evaluation, but the parts in which the instructor has stated the task.

Determine Your Position

Before beginning the journey, determine your present position. Don't skip this first step. An inventory is provided. It is a vehicle for your self-analysis of the concrete examples of your communications with students, the tests and assignments you prepare.

Collect a sample of recent assignments or tests for each of the courses you teach and complete **Part One of the Math Assessment Inventory**.

Part One of the Math Assessment Inventory

Directions: Check (✔) the verbs you use ON GRADED ASSIGNMENTS OR TESTS. Unless the <u>entire assignment</u> is verbally stated, do not check verbs stated orally. Star (*) verbs you use but do not write on tests.

abstract	demonstrate	illustrate	rearrange
acquire	describe	infer	recall
add	design	integrate	recognize
analyze	detect	interpolate	relate
answer	determine	interpret	reorder
apply	develop	judge	report
appraise	devise	justify	represent
argue	diagram	lecture	restate
assess	differentiate	list	restructure
break down	discriminate	make	reword
categorize	discuss	match	rewrite
change	distinguish	modify	separate
choose	divide	multiply	show
classify	document	operate	simplify
combine	draw	organize	sketch
compare	employ	originate	specify
complete	estimate	outline	speculate
compose	evaluate	paraphrase	standardize
compute	explain	plan	state
conclude	extend	plot	subtract
consider	extrapolate	point plot	summarize
contrast	fill in	point out	support
convert	find	predict	synthesize
correct	formulate	prepare	tell
create	generalize	present	transfer
decide	give example	produce	transform
deduce	graph	propose	translate
defend	hypothesize	prove	use
define	identify	read	write

Our Tests Direct Our Development

Written works are concrete expressions of an author's thoughts. An author claims ownership of a written work saying, "This is what I think and feel. I stand by my work." This ownership in turn directs the author's future thoughts and development. A thought, weakly posed as a possibility yesterday, easily becomes the statement of today and the creed of tomorrow. As with a collection of an artist's work, a collection of an author's writings may well document its creator's developments and transition. The collection represents a sequence of snapshots of the thoughts of the author, a record of a progression through time. A teacher's tests represent no less a collection.

Tests and written assignments are concrete representations of what an instructor communicates to students as being of greatest importance. Students count "as worthy of their energy ... only those tasks or activities that were reviewed and then recorded in the grade book" (Wilson, 1993). Tests indicate to students what the instructor holds as most important to master.

Tests are prethought for the next semester. Instructors determine the future lessons by considering what was last tested. When grading a test an instructor might think, "Gee, they really did well on that question. I guess that lecture made sense." Or she might think, "They didn't get this at all. I guess I'll have to revise my presentation next semester." Experimentation with new strategies or presentations or ideas, is eventually reflected in what is evaluated. Increased use of a calculator or manipulatives should eventually be reflected in the tests students take on that material.

The goals we hold direct the tasks we set and the tests we write. Tests are the concrete evidence of the process, a record of present thoughts, and a workspace for future thoughts. The verbs of the test are the operative component of the goals, tasks, and future direction. By the analysis of the verbs, the status and direction of future progress may be evaluated. Through a change in verbs, a change in status and direction is made.

Part One of the Math Assessment Inventory pinpoints your present position. It records the verbs you use and through these reflects the goals you have for your students. Its completion began a self-assessment and possibly a course change toward specific goals. Through work on **Part Two of the Math Assessment Inventory** self-assessment continues. Through **Part Two** verbs are sorted according to Bloom's cognitive complexity categories: evaluation, synthesis, analysis, application, comprehension, and knowledge (Kibler, 1970). Yes, these are student behavioral objectives of the 1960s. These are also tasks the instructor considers important.

Use the Inventory to Decide Your Destination and Direction

Your inventory may be thought of in three ways: a picture of where you were before the inventory was compiled, a reminder of what possibilities exist, and a benchmark against which future assessments might be made. What message does your sorted inventory have for you? What are the categories you use the most? Do you have variety? Is the message to your students the message you wish to send? Is it the most appropriate one? What are its strengths and its weaknesses? How does

Part Two of the Math Assessment Inventory

©1992, Azzolino (Azzolino, 1992) (Kibler, 1970)

Directions: In the lists below, indicating the verb's level of cognitive complexity, mark only those verbs which were checked in **Part One of the Math Assessment Inventory**. At the bottom of each group write the number of verbs marked. Underlined words are considered critical thinking words.

Evaluate – judge

appraise	conclude	defend	hypothesize	predict
argue	consider	determine	interpret	speculate
assess	contrast	discuss	judge	standardize
compare	decide	evaluate	justify	support

Synthesize – produce

abstract	design	modify	propose	specify
categorize	develop	lecture	prove	summarize
classify	devise	originate	relate	synthesize
combine	document	plan	report	write
compose	formulate	present	rewrite	
create	give example	produce	show	

Analyze – identify errors or differentiate facts from assumptions

analyze	deduce	differentiate	organize	separate
break down	detect	discriminate	outline	
categorize	describe	discuss	point out	
contrast	develop	distinguish	recognize	
correct	diagram	infer	relate	

Apply – use in a new setting

add	detect	evaluate - expression	modify	simplify
apply	differentiate – compute dy/dx	generalize	multiply	solve
change	divide	graph	operate	subtract
classify	do	identify	produce	transfer
compute	employ	integrate - as $\int f(x)\,dx$	relate	use
demonstrate			restructure	
develop			sketch	

Comprehend – understand

abstract	discriminate	illustrate	plot	restate
change	draw	infer	prepare	reword
convert	estimate	interpret	predict	rewrite
defend	explain	interpolate	read	summarize
demonstrate	extend	make	rearrange	translate
differentiate	extrapolate	paraphrase	reorder	transform
distinguish	generalize	point plot	represent	

Know – remember, acquire

acquire	define	fill in	list	recognize
choose	determine	find	match	
complete	distinguish	identify	recall	

your inventory compare with your ideal inventory? With your vision of a poor inventory? With that of your colleagues?

Some readers might wish to know how they compare with other instructors. This additional information might be valuable before setting a new course. Samples of similar **Assessment Inventories** of other instructors follow. A variety of styles exist. Verbs are listed in columns, the higher the skill the further to the left the column is found. The six numbers represent the total number in each of the six categories: (1) evaluation, (2) synthesis, (3) analysis, (4) application, (5) comprehension, and (6) knowledge verbs.

Evaluate	Synthesize	Analyze	Apply	Comprehend	Know
Instructor 1 [2, 3, 1, 4, 2, 0]					
compare contrast	formulate specify write	contrast	graph sketch simplify solve	draw explain	
Instructor 2 [3, 2, 1, 3, 5, 1]					
compare contrast justify	tell write	diagram	graph identify solve	defend demonstrate estimate explain illustrate define	
Instructor 3 [0, 2, 0, 5, 0, 0]					
	combine write		compute graph sketch solve simplify		
Instructor 4 [0, 0, 0, 4, 3, 1]					
			graph operate simplify solve	convert draw estimate	identify

Of the group, Instructor 2 has the greatest variety of verbs, the highest number of higher level thinking verbs, and does not test the lowest level of cognitive skills. Instructor 4 may be the most typical test author, one who focuses on application and comprehension.

The following list indicates frequency of use for a group of 18 instructors.

13 graph	8 define	4 classify	2 summarize	1 demonstrate
12 solve	8 write	4 complete	2 predict	1 specify
11 simplify	8 explain	3 fill in	2 create	1 decide
11 defend	7 sketch	3 read	2 rewrite	1 point plot
9 draw	6 describe	3 justify	2 convert	1 use
9 compare	6 compute	3 interpret	2 fill in	1 change
9 estimate	6 plot	3 contrast	1 illustrate	1 apply
9 simplify	5 choose	3 rewrite	1 formulate	1 transform
	5 identify	2 classify	1 conclude	1 relate
	5 combine	2 translate	1 summarize	1 correct
	4 tell	2 diagram	1 list	1 organize
	4 identify	2 match	1 differentiate	1 operate

Their tests said to students, "In mathematics, we graph, solve, simplify, and defend. We also draw, compare, simplify, write, explain, sketch, describe, compute, and plot. Sometimes we do other things." What statement do your graded assignments make about what mathematics is?

Snapshots from One Instructor's Journey

There is potential for change. Consider these three samples written over a period of years by the same instructor. Each reviewed, debriefed, a lab on curve shifting or composition of functions. After graphs were produced with manipulatives, students summarized the lab using the exercises in the example.

Example 1: Sketch:

1. $y = x^2 - 2$
2. $y = -x^2$
3. $y = -x^2 + 3$
4. $y = (x + 2)^2$
5. $y = (x - 1)^2$
6. $y = -(x - 1)^2 + 3$
7. $y = |x + 2| - 1$
8. $y = \sqrt{x}$
9. $y = \sqrt{x} + 2$
10. $y = \sqrt{(x + 2)}$
11. $y = \frac{1}{x}$
12. $y = \frac{1}{x} + 3$
13. $y = -\frac{1}{x} + 3$
14. $y = -\sqrt{x} + 1$
15. $y = -|x + 2| + 3$
16. $y = \frac{1}{x - 2}$
17. $y = -(x + 3)^2 - 2$
18. $y = |x - 1| - 2$

Nouns provide the variety. In fact the focus is on the nouns rather than on the verbs. The instructor chose the verb "sketch," meaning "get to the message of the expression and depict it," rather than the verb "graph," meaning "show the exact relationship between input and output" and felt the students understood this meaning.

Example 2:

Questions 1–6: Sketch:

1. $y = |x|$
2. $y = |x| + 2$
3. $y = |x + 1| - 4$
4. $y = -|x| - 3$
5. $f(x) = 2|x|$
6. $f(x) = |x - 1|$
7. Compare the graph of $y = |x + 3|$ to the graph of $y = |x|$.
8. What does $+ k$ do in $y = f(x) + k$?
9. Discuss the graphs of $y = f(x)$, $y = y - f(x)$, and $y = f(-x)$.
10. Discuss the graphs of $y = f(x)$, $y = f(x - h)$, and $y = f(x) + k$.
11. Explain the term dilate as it relates to the graph of $y = a \cdot f(x)$. You may use examples to illustrate your written response.
12. Sketch $y = \frac{1}{x}$.
13. Sketch $y = \sqrt{x}$.
14. Sketch $y = \frac{1}{x - 2} + 3$.
15. List 5 important words or terms which were used today.
16. Explain 1 or 2 important ideas discussed in today's work.

In Example 2, more attention was paid to the possible questions an instructor might ask of a student. An awareness of the possibilities and a desire to have students think about mathematics, rather than just perform algorithms, prompted the shift. Here the instructor demanded that students step back to look at the bigger picture. Students were asked to discuss, to explain, and to compare, not just to sketch. Here the student was asked pointedly in question 7 to compare a translated graph to the parent function's graph rather than the sequence of question 8, 9, and 10 used in Example 1 which "covers the same idea." Here in question 8, the instructor pinpoints the same idea as Example 1's question pairs 2 and 3 or 8 and 9.

This very noticeable shift in focus caused the **Assessment Inventory** to be written. Example 2 improved the summary depicted by Example 1 and started a collection and organization of "math" verbs according to cognitive categories. A copy of the list next to the word processor still prompts experimentation and refinement of tests and assignments. It permits one to 1) fine tune the verb to fit the purpose, 2) increase and experiment with the variety of questions, and 3) include more higher-order thinking skills.

Example 3 highlights more recent summaries and questions.

Example 3:

1. Sketch the graphs of the identity function, the opposite function, the reciprocal, the squaring function, and the square root function.
2. Describe the graph of $y = .^2 + 3$.
3. Compare the graphs of $y = x^2 + 3$ and $y = (x + 3)^2$.
4. Predict the graph of $y = (x + 2) \cdot (\sin(x))$ based on the graph of $y = x \cdot \sin(x)$.
5. Verify your prediction using your calculator and explain why it was correct or how it was incorrect.
6. State the equation of the function [given a graph or a description of some sort or a table of values.]
7. Explain how the domain effects the graph of the function.
8. Suppose $y = f(x)$ represents the graph of a function. What does the graph of $y = f(x) + 3$ look like?
9. Create a quiz with answers to evaluate a student's understanding of ...
10. Collaborate with classmates and design a test for this unit.
11. Present a 10 minute lecture on the material of this section. Include written notes for this distribution at the time of your presentation.

Continue to Assess Your Position and Destination So Your Direction May Be Modified

Once a direction is determined, the instructor's choice of verbs directs student activity and sends a loud message to students about what skills are important. The challenge is to determine the most appropriate direction and choice of verbs.

If you desire flexibility in thought on the part of your students, increase the variety of ways in which you ask them to think, increase the variety of verbs. If you desire students to master higher order thinking skills, experiment with the use of higher order verbs. Take an old test and rewrite parts of it by replacing a lower order skill with a higher order skill. In lecture, pose thought provoking questions and demonstrate how one might go about answering them.

If you feel students should be communicating more, provide more opportunities for this to occur: ask students to pair up so they always have a designated individual with whom to discuss work; have them compare answers after each problem; have one student explain work to another student, to a small group, or to the entire class (either from the seat or in front of the room). If you feel students should write more: ask them to read and then summarize a page in the text; to write and solve a word problem; to write a cheat sheet; to write a set of lecture notes about a topic.

If you are undecided about which direction to take, move in the directions which are appropriate. Choose verbs which "head you in the 'right' direction." Keep the verbs visible when you write a test or assignment as a reminder of where you've been and where you're going. Continue to assess your position and destination so your direction may be modified.

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"I'd really like to ...

... get my students to apply mathematics."
... integrate technology."
... introduce functions earlier."
... use cooperative exercises."

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Spreadsheets in a Differential Equations Course?

by

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Introduction

Numerical methods have long been a part of any good introductory course in differential equations. When I started teaching this course, we would give our students a short course in FORTRAN or BASIC and then have them write programs to do such things as approximating the solutions to differential equations. When our friends in computer science began the push to structured programming, the language became something akin to PASCAL. I then began to notice that the assignments were less useful because the students were spending much more time worrying about the syntax of the language and much less time analyzing the algorithm under study. The pendulum then swung drastically in the other direction as more and more software packages became widely available on the market. Although these programs were expertly written and could be used to analyze problems that were inaccessible before, they were not written by the students and thus did not help in understanding algorithms. It is the purpose of this paper to suggest that the use of spreadsheets can be very helpful to augment the use of these software packages in a differential equations course. Not only are spreadsheets easy to use, but also their implementation helps the students to understand the classroom material and allows the class to discuss and review many of the important topics of calculus.

First Order Differential Equations

The software package that we use in our differential equations course is called MDEP (see reference). It is very easy to use. Basically all the students have to do is enter the differential equation and its boundary values and then select the type of approximation method such as Runge-Kutta or Euler. The package returns a graph of the solution and a table of values. Nice, but what about the algorithm?

To supplement this, we also approximate the answer using a Taylor polynomial. This method is very easy to implement on a spreadsheet and gives answers that

The Numerical Solution to $y' = g(x, y)$ using Taylor Series

The Problem: Given $y' = g(x, y)$ where $y(x_0) = y_0$, find $y = f(x)$, the particular solution to the differential equation and its boundary value.

Because $y = f(x)$ is the solution to the differential equation, it has derivatives and, therefore, a Taylor series:

$$y = f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 \ldots$$

where a is the center of the Taylor series and is chosen conveniently.

We can approximate $y = f(x)$ by truncating the Taylor series at the end of the three terms above, but recall that when we do this, the further that we are from the center, the worse that the approximation is. Because the x that we know the most about is x_0, it makes sense to begin with x_0 as the center of our Taylor series. We begin to present the approximate solution, $y = f(x)$, in table form:

x	y	y'
x_0	y_0	y'_0
x_1		

We find y' by evaluating our differential equation ($g(x,y)$ above) at (x_0, y_0), and we calculate x_1, by adding the increment, h, that we choose for our table to x_0. We now wish to find the value for y, denoted by y_1, which corresponds to x_1. Because x_1 is close to x_0, we may approximate the answer to y_1 by using a Taylor polynomial centered at x_0. We obtain:

$$y = f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \frac{f''(x_0)}{2}(x_1 - x_0)^2$$

$$= y_0 + y'_0(x_1 - x_0) + \frac{y''_0}{2}(x_1 - x_0)^2$$

But $x_1 = x_0 + h$, so $x_1 - x_0 = h$ and

$$y_1 = y_0 + y'_0 h + \frac{y''_0}{2} h^2.$$

We therefore can calculate y_1 provided that we know y''_0. We accomplish this by taking the derivative of both sides of our differential equation with respect to x. This gives us an expression for y'' which may be evaluated at (x_0, y_0, y'_0) to find y''. We now add another column to our table for y'' and complete the first two lines.

x	y	y'	y''
x_0	y_0	y'_0	y''_0
x_1	y_1	y'_1	y''_1

Figure 1: Handout

We may now proceed as above to add as many lines as we choose to our table. Each time that we complete a line, we move the center of our Taylor series to the x value for that line. Thus we are always near the center. (We do note, however, that we are building on approximations after the first line.) The equations used are:

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + y'_n h + \frac{y''_n}{2} h^2$$

$$y'_{n+1} = g(x_{n+1}, y_{n+1})$$

y''_{n+1} = the expression obtained for y'' evaluated at $(x_{n+1}, y_{n+1}, y'_{n+1})$

As an example, consider $y' = 2x - y$ where $y(0) = 1$. Taking the derivative of both sides of our differential equation with respect to x, we obtain $y'' = 2 - y'$. In this example, $x_0 = 0$, $y_0 = 1$, $y'_0 = 2(0) - 1 = -1$, and $y''_0 = 2 - (-1) = 3$. If we choose $h = 0.01$, $y_1 = 1 + (-1)(0.01) + \frac{3}{2}(0.01)^2 =$ 0.99015. And in general,

$$x_{n+1} = x_n + 0.01$$

$$y_{n+1} = y_n + y'_n(0.01) + \frac{y''_n}{2}(0.0001)$$

$$y'_{n+1} = 2x_{n+1} - y_{n+1}$$

$$y''_{n+1} = 2 - y'_{n+1}$$

Figure 1: Handout (continued)

compare very favorably to the package answers. Because the implementation is so easy, we have time to review Taylor series, including the choice of a center and the estimation of the size of the error caused by truncation. Also, since we are going to present the solution in table form, we can at the same time discuss the various ways that the functional answer can be presented and the effect of the choice of step size for the table on the table size and the accuracy of our answers. During these discussions, the light finally comes on for Taylor series for a significant number of students. The handout that we use for this is figure 1. Note that the Taylor polynomial is second degree and the differential equation is of order one. Thus, the students must differentiate both sides of the differential equation to obtain an expression for the second derivative, and we automatically find ourselves discussing implicit differentiation and the chain rule. There are many such nice little bonuses.

	A	B	C	D	E	F
1	x	y	y′	y″	dy/dx = 2x − y	
2						
3	0	1	−1	3		
4	.01	.9901500	−.970150	2.970150		
5	.02	.9805970	−.940597	2.940597		
6	.03	.9713381	−.911338	2.911338		
7	.04	.9623703	−.882370	2.882370		
8	.05	.9536907	−.853691	2.853691		
9	.06	.9452964	−.825296	2.825296		
10	.07	.9371847	−.797185	2.797185		
11	.08	.9293528	−.769353	2.769353		
12	.09	.9217977	−.741798	2.741798		
13	.10	.9145168	−.714517	2.714517		
14	.11	.9075074	−.687507	2.687507		
15	.12	.9007667	−.660767	2.660767		
16	.13	.8942920	−.634292	2.634292		
17	.14	.8880808	−.608081	2.608081		
18	.15	.8821304	−.582130	2.582130		
19	.16	.8764382	−.556438	2.556438		
20	.17	.8710017	−.531002	2.531002		
21	.18	.8658182	−.505818	2.505818		
22	.19	.8608853	−.480885	2.480885		
23	.20	.8562005	−.456201	2.456201		

Figure 2. Sample Spreadsheet Output for $y' = 2x - y, y(0) = 1$, $h = 0.01$

The construction of the table of functional values consists of entering a few constants (the boundary values) and then a formula for each column, perfect for the copy function of a spreadsheet. After we have completed our table, we compare our results to the MDEP answers, usually with great satisfaction. Figure 2 contains some sample spreadsheet output. We then discuss improving our accuracy. Should we add another term to our Taylor polynomial? All this means is that we have to calculate another derivative, add a new column to our spreadsheet, edit our Taylor polynomial formula, and copy. We could also choose a smaller h, another simple spreadsheet modification.

To get all this started, we spend about one day talking about spreadsheets, and I usually learn a lot from my students during this time. We have used both SuperCalc and Quattro Pro spreadsheets, but any good spreadsheet is fine. The point is that if we can reduce a problem to a little data entry, a few formulas, and lots of copying, spreadsheets are fast and easy.

Second and Higher Order Differential Equations

The usual method to approximate the solution to a second order differential equations is to rewrite the problem as a system and solve using some system technique. Instead, we simply generalize the ideas above. Assuming that our differential equation is in the form, $y'' = g(x, y, y')$, it is itself a formula for y''. Taking successive derivatives of both sides of the differential equation with respect to x yields formulas for any higher derivative needed. As for the first order problem, y can be approximated using a Taylor polynomial. That leaves us with y'. During this discussion, it always surprises me how few students think of y' as a function of x. Once we are by this hurdle, we realize that y' can be approximated by a Taylor polynomial in the same way as y. Therefore, the spreadsheet solution to the second order problem looks much the same as that for the first order with an added column for y''' which is used in the Taylor polynomial approximation for y'.

It is now easy to move to the general case, $y^{(n)} = g(x,y,y', \dots y^{(n-1)})$ (with the appropriate boundary values). The differential equation yields formulas for derivatives from "n" up. Lower derivatives and y values can be approximated using Taylor polynomials. We then proceed as above.

Systems

Most of the systems of differential equations that we encounter in an introductory course can be written in the form $y' = g_1(x,y,t)$ and $x' = g_2(x,y,t)$. (They may not originally be of this form but can be algebraically maneuvered into it.) The first equations yields formulas for all derivatives of y and the second for all derivatives of x. Therefore, x and y values can be approximated using Taylor polynomials and the initial values; so we can proceed as above. If either or both of the differential equations is higher order, we will use a Taylor polynomial to calculate the lower derivatives. In any case, we can produce our spreadsheet.

Conclusion

Of course, there are many other numerical methods that could have been used other than the technique above. Some of the reasons that we chose it are:

1. It leads to the discussion of many important (and not well understood) Calculus topic such as series.
2. This one method works for first order differential equations, higher order differential equations, and systems of differential equations.
3. It yields easy to understand formulas for the columns of our spreadsheet.
4. It's testable. You can ask the students to reproduce the first one or two lines of the spreadsheet on a test.

The use of spreadsheets in our differential equations course yielded some remarkable results. Students understood the numerical methods and the corresponding algorithms. They began to see their computer as an aid to their studies rather than a stumbling block. And last, but certainly not least, they understood the course material, as a whole, better.

Reference

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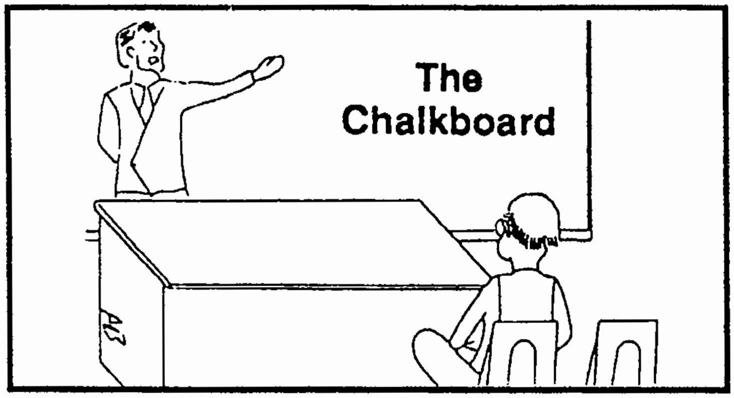
REGULAR FEATURES



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This column is intended as an idea exchange. We hope to facilitate an open exchange of ideas on classroom management, teaching techniques, tips for helping students get past the usual stumbling blocks, techniques for improving student participation, etc. We know there are lots of good ideas out there, and this is your chance to share them. Please send your contributions to Judy Cain. Our backlog is again nearly exhausted, and we would appreciate your participation! Items may be submitted by e-mail or regular mail; please include your e-mail address if available.

Cookies and Confidence Intervals

The use of candies, such as Skittles and M&Ms, in elementary statistics classes is a widespread practice. However, I use chocolate chip cookies to demonstrate confidence intervals. In the syllabus used at Suffolk, descriptive statistics and sampling are separated from estimation by a probability unit. The students often need a review of basic statistical concepts that this activity provides.

I begin the class with the question, "What is the average number of chips in a Chips Ahoy cookie?" After the students make some rather interesting guesses, I produce a bag or two of cookies. We review types of sampling, and we discuss the possible errors associated with the bag of cookies (such as the number of chips per cookie when the cookies are all from the same bag will vary less than if several different batches are sampled). Then I distribute two to three cookies per student and ask them to count the chips. This usually generates a discussion regarding definition of a chip and destructive tests. Again, we discuss other possible errors in the data.

After all the students have counted chips, the data are tabulated. Stem and leaf diagrams are reviewed, the shape of the distribution is discussed, and the students calculate the mean and standard deviation (usually difficult after a lapse of several weeks). At this point, the concept of point vs. interval estimate is introduced. Although the shape of the data is usually not normal (theoretically this is a Poisson process), the central limit theorem permits the use of the normal distribution, and a confidence interval is constructed.

I find that the students enjoy the activity, remember confidence intervals, and have excellent attendance for weeks afterward, wondering if cookies will be supplied again. Note: I also use two types of cookies (chewy and original) to motivate two-sample hypothesis testing.

Submitted by Jane-Marie Wright, Suffolk County Community College, 533 College Road, Selden NY 11784, wrightj@sunysuffolk.edu

> The arithmetic of life does not always have a logical answer.
>
> Inshirah Abdur-Rauf

Snapshots of Applications in Mathematics

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The purpose of this feature is to showcase applications of mathematics designed to demonstrate to students how the topics under study are used in the "real world," or are used to solve simply "charming" problems. Typically one to two pages in length, including exercises, these snapshots are "teasers" rather than complete expositions. In this way they differ from existing examples produced by UMAP and COMAP. The intent of these snapshots is to convince the student of the usefulness of the mathematics. It is hoped that the instructor can cover the applications quickly in class or assign them to students. Snapshots in this column may be adapted from interviews, journal articles, newspaper reports, textbooks, or personal experiences. Contributions from readers are welcome, and should be sent to Professor Callas.

Racing with Fire

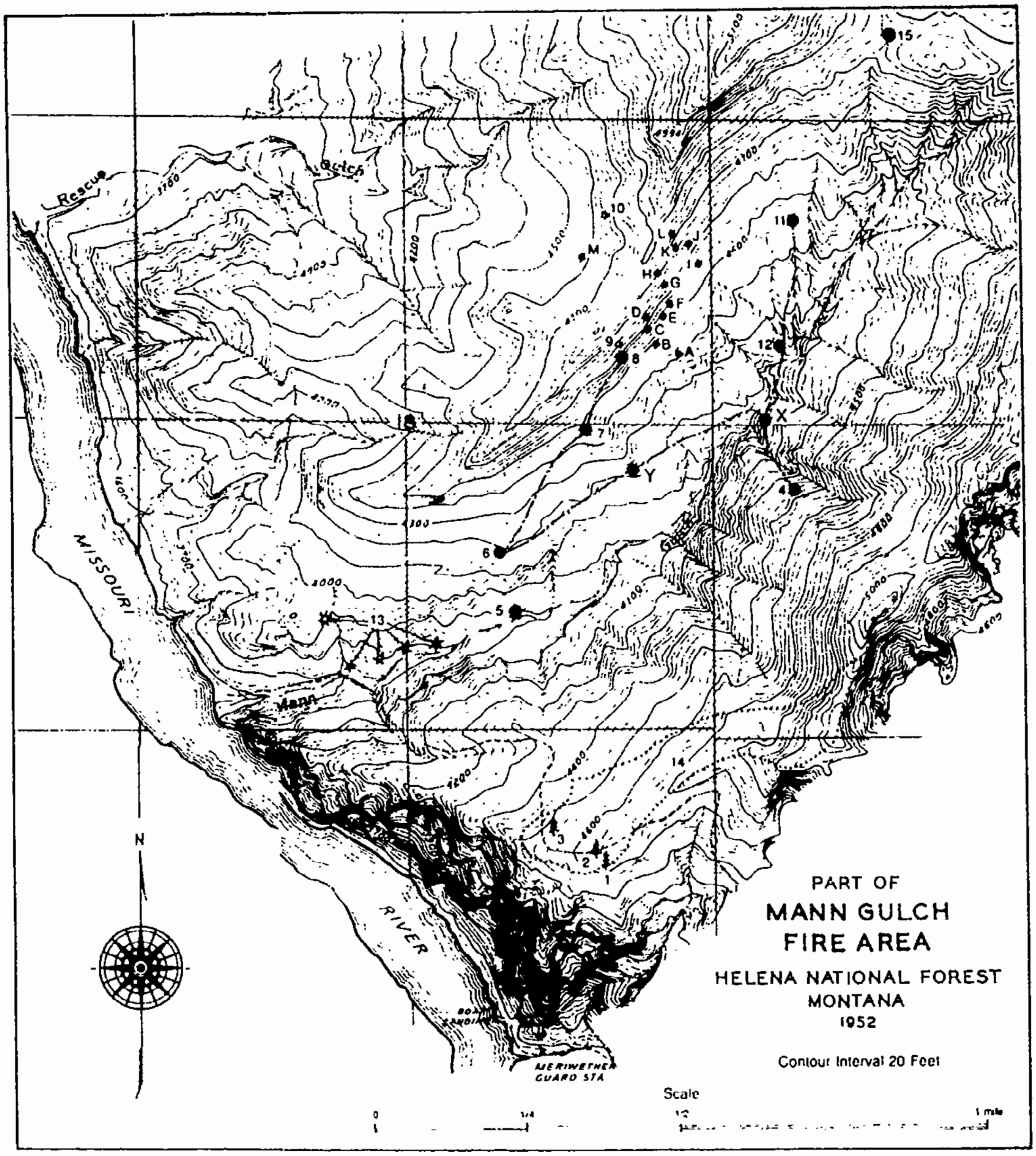
(to accompany an introduction to distance, slope and rates of change)

by James M. Sullivan, Massachusetts Bay Community College, Wellesley MA

The Mann Gulch forest fire is a tragedy that occurred on August 5, 1949 in a remote area of the Montana wilderness. A crew of 15 airborne firefighters, known as smokejumpers, landed in rough country with the purpose of getting a routine wildfire under control. What happened instead was a "blowup" or great explosion of flame and wind which caused a storm of fire to engage the young smokejumpers in a race that only 3 of them would survive.

This true story has become a case study for how western wildfires behave. The Intermountain Fire Sciences Laboratory (formerly called the Northern Forest Fire Laboratory) supports research projects that use mathematical models to predict the danger of wildfire and the rate of their spread. Some of the variables that have a significant effect on fire are fuel moistuie, fuel loading, wind velocity, relative humidity, slope and solar aspect. For instance, if a fire is burning on relatively flat terrain (between 1% and 5% slope), then its spread rate will double when it arrives at land with a 30% slope. The rate of spread will double again if the fire continues into steeper ground with a slope of 55%.

Referring to the contour map below, the smokejumper's path started down Mann Gulch toward the Missouri River. The perimeter of the fire was approximately in area 14, when the firefighters landed. Y indicates the crew's location at 5:40 and 13 the location of spot fires, caused by the lower end of the fire in area 14 jumping the gulch at 5:30. The crew initially moved toward these spot fires via point 6. Once they realized the danger they were in, they moved quickly away, moving toward G. G represents the approximate area where firefighters and fire literally collided. Points Y and 13 are approximately 400 yards from point 6.



LEGEND

1, 2, & 3 Lightning struck trees
4—Dodge met Harrison
X—Dodge ordered crew to north side of gulch*
Y—Dodge and Harrison rejoined crew; beginning of the crew's race*
5—Jansson turned back
6—Dodge and crew turned back
7—Dodge ordered heavy tools dropped
8—Dodge set escape fire
9—Dodge survived here
10—Rumsey and Sallee survived here
11—Jumping area (chutes assembled burned)
12—Cargo assembly spot (burned)
13—Spot fires
14—Approximate fire perimeter at time of jumping and cargo dropping (3:10 P.M.–4:10 P.M.)
15—Helicopter landing spot

BODIES FOUND

A—Stanley J. Reba
B—Silas R. Thompson
C—Joseph B. Sylvia
D—James O. Harrison
E—Robert J. Bennett
F—Newton R. Thompson
G—Leonard L. Piper
H—Eldon E. Diettert
I—Marvin L. Sherman
J—David R. Navon
K—Philip R. McVey
L—Henry J. Thol, Jr.
M—William J. Hellman

*Points X and Y added by the author

1952 contour map: "Part of Mann Gulch Fire Area."

Figure 1: Contour Map

The distance from Y to G, via point 6, is approximately 1320 yards or $^3/_4$ of a mile. However, there are 20 or 21 contour lines from Y to G, each representing an interval of 20 feet. Using 21 contour lines, the rise in elevation from Y to G is 140 yards (21×20 ft). Therefore, the crew did not travel 1320 straight yards. Rather, using cartographic terminology, they climbed a "slant distance" between Y and G, via point 6, of 1327.4 yards ($1320^2 + 140^2 = d^2$).

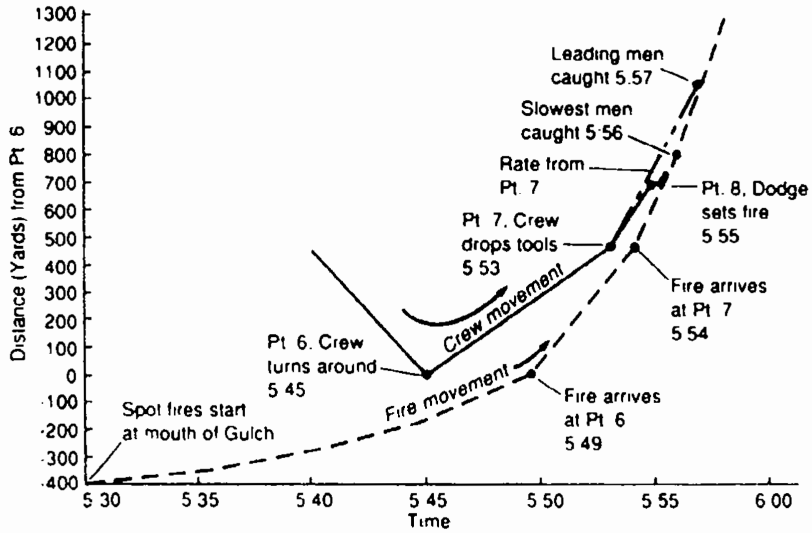
The crew was at point Y at approximately 5:40 and many met their death near point G at 5:56. Using the slant distance just calculated, we can determine their average speed. Given that

$$\text{Rate} = \frac{\text{Distance}}{\text{Time}}$$

the average speed of the firefighters $= \dfrac{1327.4 \text{ yards}}{16 \text{ minutes}} \approx 2.8$ mph.

The last stretch to point G and beyond was run on about a 75% slope of slippery grass and rock slides, on the hottest day on record. The percent slope implies the steepness of the climb could be represented by 3 feet of vertical change for every 4 feet of horizontal change.

Finally, Figure 2 shows the time and position from point 6 of the fire and crew, giving us more precise information of what happened during the crew's last 16 minutes of life. The intersection of the two curves graphically represents the conclusion to our story.



Time and position of fire and crew at Mann Gulch on August 5, 1949. Graph by Richard C. Rothermel.

Reprinted by permission

Figure 2

Exercises

1. After the Mann Gulch fire, crosses were placed to mark the site where each smokejumper's dead body was found. The representative cross, G, marking the end of the race was Leonard L. Pipers'. The cross closest to the top of the ridge belongs to Henry J. Thol, Jr. (point L) and there appears to be between four and five contour lines between them. What would you estimate as the difference in elevation between Piper's cross and Thol's cross if we use $4\frac{1}{2}$ contour lines as an approximation? Use this information to find the slant distance between the two crosses.
2. A typical jogging pace is about 6 mph. In the crew's race with fire they averaged only about half that speed. Were the smokejumpers out of shape? Justify your answer.
3. Explain the meaning and importance of the following statement: "The percentage increase in the spread rate (one factor of a fire) varies in proportion to the square of the percent slope (an environmental factor influencing the fire)." Why would this be considered a tragic statement for the smokejumpers of the Mann Gulch fire? Also, model the above statement with an equation in two variables x and y, with k as the constant of proportionality.
4. From 5:45 – 5:49 compare the fire movement with the crew movement. What does this tell us about the speed of each? Support your statement by comparing the fire movement slope and the crew movement slope in this time interval.
5. From 5:49 – 5:53 compare the fire movement with the crew movement visually and then by calculating the slope of each. How has the situation changed from the last time interval in problem 4?
6. After 5:53 the crew's average rate of change of distance with respect to time increased. Approximate this average rate and explain why you think this acceleration in speed occurred.
7. Although the crew had accelerated their speed at point 7, most of them were still in danger. Why?

Addendum

A 1994 wildfire in Canyon Creek, Colorado took the lives of 14 firefighters in a blaze that resembled the Mann Gulch Fire. After two days the fire covered 50 acres. Then the wind suddenly increased to gusts of 50 mph, a "blow up", and 2000 acres were engulfed in only two hours. There were 52 firefighters in the crew. While some of the crew was able to outrun the fire, others escaped into already burned areas and 9 crew members survived by stepping into large aluminum-coated fiberglass bags and letting the fire burn over them.

8. What was the average spread rate of the fire during the blow up?
9. If the front of the fire was a straight line, $\frac{1}{4}$ mile wide and the wind spread the fire in a constant direction, what was the speed of the fire?

10. Research how smokejumper foreman R. Wagner Dodge survived the Mann Gulch fire knowing there was no way he could outrun it. Remember, there were no aluminum-coated fiberglass bags back in 1949.

Related Readings

Maclean, N. (1992). *Young men and fire*. Chicago: The University of Chicago Press.

Rothermel, R. (1983). *How to predict the spread and intensity of forest and range fires*. Ogden, Utah. USDA Forest Service. Intermountain Forest and Range Experiment Station. Technical Report INT-143.

Rothermel, R. (1972). *A mathematical model for predicting fire spread in wildland fuels*. USDA Forest Service. Research Paper INT-115.

> The notion of infinity is our greatest friend; it is also the greatest enemy of our piece of mind.
>
> James Pierpont

Software Reviews

Edited by Shao Mah

New Software Editor Wanted

After many years of quality service to both Red Deer College and *The AMATYC Review,* Shao Mah is retiring. We wish him well in his retirement years.

This leaves us with a vacancy in the Software Editor position. We seek someone who is a regular computer user, is familiar with many of the software packages relevant to the teaching of mathematics, has contacts who would be potential reviewers, and will seek out both products and people to review them. If you would be interested in filling this position, send a letter to *The AMATYC Review* Editor describing your background and your thoughts about this column.

Title: *Mathcad 6.0 Student Edition*

Distributor: Mathsoft, Inc.
101 Main Street
Cambridge MA 02142

Computer: IBM PC or compatible with MS Windows 3.1 or later, at least 6MB RAM and 14MB free disk space.

Price: $69.95

Mathcad 6.0 Student Edition is useful software for learning and teaching mathematics for students and instructors at the high school and college levels. The software is Windows-based and does not require fancy computers. Due to its user-friendly interface, it is also convenient software for users of different levels.

Mathcad 6.0 Student Edition combines the live document interface of a spreadsheet with the WYSIWYG interface of a word processor. Users may typeset equations on the screen in the way they see them in a book. With *Mathcad 6.0 Student Edition*, users may read and transfer numerical data from a file and do mathematical chores ranging from adding up a column of numbers to evaluating integrals and derivatives and inverting matrices. Since the software is combined with an active spreadsheet, the numerical results and graphs can be updated or changed once a modification in the user's worksheet is made.

Mathcad 6.0 Student Edition can be a useful tool in the study of vectors and matrices. In a straightforward manner, the user may create vectors and matrices, define vector and matrix functions and use different vector and matrix operators supplied by the software.

As an example, if a square matrix having the following form is entered,

$$M = \begin{bmatrix} 19 & 71 & 10 & 10 & 4 \\ 7 & 14 & 9 & 70 & 19 \\ 87 & 19 & 2 & 6 & 1 \\ 3 & 22 & 38 & 19 & 5 \\ 61 & 11 & 4 & 6 & 19 \end{bmatrix}$$

the inverse, transpose, determinant and eigenvalues of the matrix can be obtained automatically by *Mathcad 6.0 Student Edition* as follows:

Inverse

$$M^{-1} = \begin{bmatrix} -0.003 & -6.94 \cdot 10^{-4} & 0.012 & 3.21 \cdot 10^{-4} & 6.67 \cdot 10^{-4} \\ 0.016 & -9.789 \cdot 10^{-4} & -0.002 & -0.004 & -0.001 \\ -0.008 & -0.007 & 6.989 \cdot 10^{-4} & 0.03 & 9.062 \cdot 10^{-4} \\ -0.003 & 0.016 & 0.01 & -0.002 & -0.015 \\ 0.004 & -6.47 \cdot 10^{-4} & -0.04 & -0.005 & 0.056 \end{bmatrix}$$

Transpose

$$M' = \begin{bmatrix} 19 & 7 & 87 & 3 & 61 \\ 71 & 14 & 19 & 22 & 11 \\ 10 & 9 & 2 & 38 & 4 \\ 10 & 70 & 6 & 19 & 6 \\ 4 & 19 & 1 & 5 & 19 \end{bmatrix}$$

Determinant

$$|M| = -2.349 \cdot 10^{8}$$

Eigenvalues

$$eigenvals(M) = \begin{Bmatrix} 107.346 \\ -0.305 - 51.789i \\ -0.305 + 51.789i \\ -50.04 \\ 16.305 \end{Bmatrix}$$

Notice that the symbol of multiplication is a dot instead of an "×".

All the information displayed above can be shown on the screen or printed out upon the user's wish. The entered data can be modified by the user at any time to obtain new results corresponding to the updated data.

Another advantage that the students and instructors may take from *Mathcad 6.0 Student Edition* is its manageability and straightforwardness in calculus computation. The form of derivative is

$$\frac{d}{dx} f(x) = 0.246$$

for a function value of $f(x) = x \cdot \sin(x) \cdot e^x$ at $x = 2$.

An integral of $f(x) = x \cdot \sin(x) \cdot e^x$ over the interval $[-5,5]$ is displayed on the screen as

$$\int_{-5}^{5} f(x)dx = -439.98.$$

In *Mathcad 6.0 Student Edition*, users may also solve a problem from the mathematics model to the final graphic analysis. Two-dimension Cartesian and polar graphs, contour plots, surface plots as well as some three-dimensional plots can be graphed by employing the software. An iteration model and the graphic result are analyzed here.

Mathcad 6.0 Student Edition uses a special variable called a *range variable* (an unfortunate name, since it actually refers to the domain of a function) to perform iteration. For a positive function $d(t) = 1600 - \frac{g}{2} t^2$, the user may compute results for a range of t values from 10 to 20. The results obtained are shown in Fig. 1(a) with the automatically plotted two-dimensional graph exhibited in Fig. 1(b). Users may gain additional flexibility by defining functions.

$t := 10, 11 .. 20$

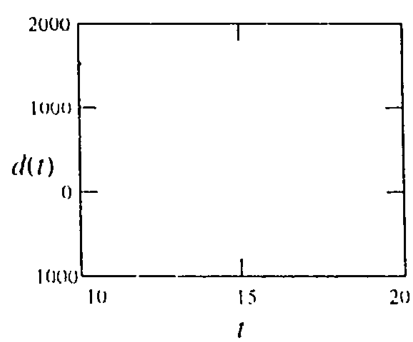
$g := 9.8$

$d(t) := 1600 - \frac{g}{2} \cdot t^2$

$d(t)$

$1.11 \cdot 10^3$
$1.007 \cdot 10^3$
894.4
771.9
639.6
497.5
345.6
183.9
12.4
−168.9
−360

(a)



(b)

Figure 1

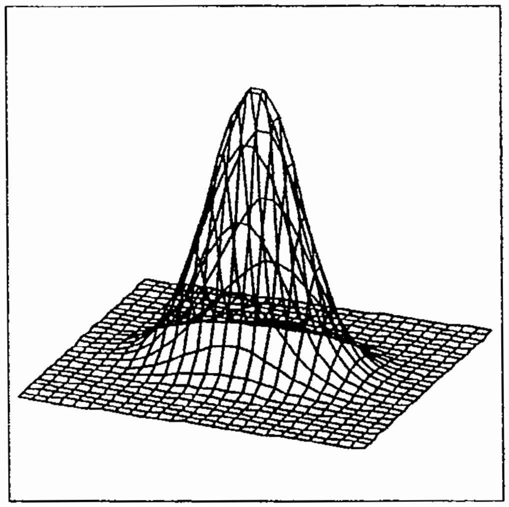
Some three dimensional graphs are shown in Fig. 2(a) and 2(b).

$f(x,y) := e^{[-0.6\cdot(x-0.5)^2 - 0.3\cdot(y-0.5)^2]}$

$i := 1..25 \qquad x_i := -10 + i \cdot \frac{1}{1.25}$

$j := 1..25 \qquad y_j := -10 + j \cdot \frac{1}{1.25}$

$M_{i,j} := f(x_i, y_j)$



M

Figure 2(a)

$f(x) := x \cdot \sin(x)^2 \qquad x := 0, = \frac{1}{\pi} .. 5$

$a := 0 \qquad b := \pi$

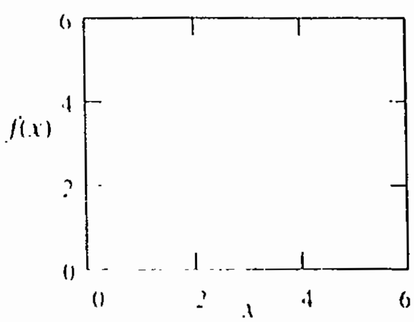
$xn := 35 \qquad rn := 25 \qquad i := 0 .. xn \qquad j := 0 .. rn$

$r_i := \frac{b-a}{xn} \cdot i + a \qquad s_j := \frac{2\cdot\pi\cdot j}{rn} - \pi$

$X_{i,j} := r_i \cdot \sin(s_j)$

$Y_{i,j} := r_i \cdot \cos(s_j)$

$Z_{i,j} := f(r_i)$



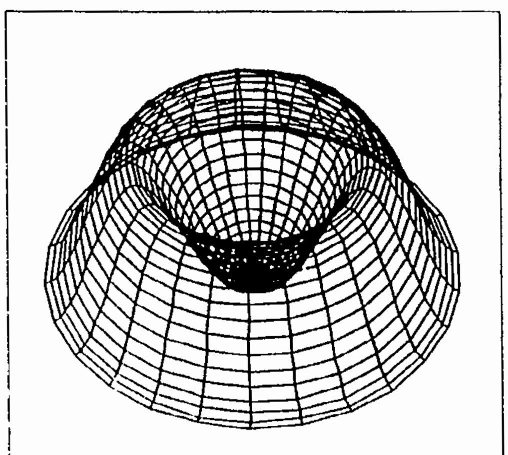


Figure 2(b)

There is a large variety of other fancy functions available in *Mathcad 6.0 Student Edition*. A short list of these additions includes animation, data analysis using statistical functions, electronic books for browsing, searching, copying information and making notes. Its ability to manage worksheets provides a new range of applications, such as opening worksheets on the Internet, including one worksheet inside another, and mailing a worksheet.

In summary, for users of Windows-based computers who write mathematics, *Mathcad 6.0 Student Edition* provides good output without too much effort. It offers an affordable, easy-to-use, powerful computer algebra/graphing system that can be brought very appropriately into mathematics learning and teaching. Generally speaking, most mathematical problem learning and teaching at the high school or college level can be solved and demonstrated by using *Mathcad 6.0 Student Edition*. However, the calculation and graph generation can take a longer time than one may expect if the RAM of a computer is not sufficiently large. Some of the mathematical symbols of the software are not conventional. Furthermore, change of fonts and format of page is not very easy in *Mathcad 6.0 Student Edition*.

Reviewed by: Liming Dai, Department of Mathematics, Physics and Engineering, Mount Royal College and Mingzheng Li, Department of Mechanical Engineering, the University of Calgary.

Book Reviews

Edited by Sandra DeLozier Coleman

A CONVERGENCE OF LIVES, Ann Hibner Koblitz, Rutgers University Press, 1993, xxxviii + 305 pages, ISBN 0-8135-1962-4 / 0-8135-1963-2 pbk.

A RUSSIAN CHILDHOOD, Sofia Vasilevna Kovalevskaia, translated by Beatrice Stillman, Springer-Verlag, 1978, xiii + 250 pages, ISBN 0-387-90348-8.

LITTLE SPARROW, Don H. Kennedy, Ohio University Press, 1983, ix + 341 pages, ISBN 0-8214-0692-2 / 0-8214-0703-1 pbk.

Readers should be aware that the translation of a Russian name into an English equivalent is not always consistent. Expect Kovaleskaia's name to appear in other forms, such as Kovalevskaya, Kovalevsky, or Kowalewskaia and to find her referred to variously as Sofie, Sophie, Sonya, and Sofa.

One afternoon last year I received a postcard from Rutgers University Press advertising a book by Ann Hibner Koblitz about the life of Sofie Kovalevskaia. The book is called *A Convergence of Lives*, because it reveals many different facets of the life of this remarkable woman. An entire book devoted to the life of one female mathematician seemed to me to be as rare a find even today as Kovalevskaia seemed to be in the university environment of the late 19th century. I was intrigued and ordered the book immediately. It arrived shortly and as I began to read I found the account of Kovalevskaia's success in establishing herself as a respected colleague among the many famous European mathematicians of her day to be both fascinating and inspiring.

She was born in Russia in 1850, at a time when the world of mathematics in her homeland was closed to all women regardless of their interests or talents. At the age of eighteen, Sofa Korvin-Krukovskaia's desire to find a place where she could further her education was so great, that she entered into a fictitious marriage with a young nihilist, Vladimir Kovalevskii, who agreed to help her travel to Germany and to leave her there to study mathematics. Though frowned upon by parents, such marriages of convenience were not unusual among the young revolutionaries of Russia who were convinced that furthering education, especially in the sciences, was the key to conquering all of the ills of society. *A Convergence of Lives* brings together images of Sofie as a revolutionary, a student, a teacher, a writer, a scientist, a poet, a mathematician, and an editor of one of the world's most respected mathematical journals. We also find here the very personal image of Sofie as a child, a sister, a friend, a wife, and a mother. We follow Kovalevskaia's footsteps as she boldly introduces herself to the famous German mathematician, Karl Weierstrass, and convinces him that she is capable of learning all that he has the time to teach her. We share her excitement as she earns the distinction of being the first woman outside of Renaissance Italy to be awarded a Ph.D. in Mathematics; we feel her frustration as she strives to obtain a satisfying teaching position in higher mathematics; and we enjoy her triumph as she goes on to win the Prix Bordin, an honor as great as winning the Nobel Prize today.

Koblitz's account makes us so keenly aware of the number and intensity of the obstacles Kovalevskaia faced, that men and women in the field of mathematics who read her story are sure to be inspired by the fact that she prevailed against such odds.

Although even the Koblitz biography contains more information about Kovalevskaia than is generally available about most famous mathematicians, I soon discovered, by exploring the suggested options for further reading, that it is possible to become so well acquainted with Sofie's life as to feel as though one has known her personally. Her own book, *A Russian Childhood*, which was translated by Beatrice Stillman and published by Springer-Verlag in 1978, gives a moving account of her early childhood and adolescence and provides much insight into the psychological and personal aspects of the life of this developing mathematician. I know of no other mathematician who has given the world such a detailed and personal account of childhood experiences. The story is enjoyable in its own right and is equally valuable for its documentation of the events in her life and its inside view of Russian society.

During her lifetime, Kovalevskaia received as much acclaim for her literary accomplishments as for her mathematical accomplishments. She began composing poetry at the age of five and wrote poetry all her life. This seems to be one of the things that Weierstrass admired about her, since he once wrote to her saying, "It is completely impossible to be a mathematician without having the soul of a poet." She also wrote literary reviews, novels and plays—some of which have been published in many languages. In *A Russian Childhood*, beginning with the day she first learned to say her name, Kovalevskaia unfolds the story of her life with such a passionate desire to have the world know who she was and what she was like that the reader soon begins to see her, even in the glory of her mathematical success, as just little "Sonya" all grown up! She holds nothing back—freely sharing her joys, her sorrows, her fears, and even her adolescent love for the great writer and family friend, Fyodor Dostoevsky. She ends her story with the end of that relationship and leaves the reader longing to know what happens next!

Fortunately, Don H. Kennedy in his account in the book entitled *Little Sparrow*, published by Ohio University Press in 1983, fills in all of the details and leaves the reader with an in-depth understanding of Sofie's experiences and personality. This book is dedicated to the author's wife because of her untiring efforts to locate and translate Russian books, periodicals, letters, and other original documents related to Kovalevskaia's life. In addition, the book contains recollections of stories about Sofie which have been handed down to the family members of those who knew her personally. It is in this book that the reader comes to truly understand her strengths as well as her weaknesses. We are given a more complete picture of what her life was like during the many years during which she and her husband maintained a fictional marriage. We learn why they decided to change that arrangement. The five years during which Kovalevskaia gave up mathematics almost entirely are explained in some detail. Her long-standing relationship with the famous German mathematician Karl Weierstrass is presented in such a way as to be an inspiration to both aspiring mathematicians and their mentors.

As one of the editors of the respected math journal *Acta Mathematica*, Kovalevskaia was familiar with all of the current developments in mathematics and corresponded with mathematicians throughout Europe. Her life is so intimately intertwined with the lives of such well-known scientists and mathematicians as Darwin, Bunsen, Mittag-Lefler, Kronecker, Cantor, Hermite, Poincaré, Picard, Chebyshev, and Sylvester, that reading her story sheds light on the whole picture of the interrelationships which played a role in the development of modern mathematics in the nineteenth century. As in the other two books, photographs of Sonya and those people who played prominent roles in her life give the story a completeness that I have not found in the accounts of the lives of other mathematicians. Details concerning both her literary and mathematical contributions are included in an appendix in the back. Among other mathematical accomplishments, she developed the final form of the Cauchy-Kovalevskaia Theorem for partial differentiation, solved a problem related to the revolution of a body about a fixed point which had baffled mathematicians for over one hundred years, and wrote three significant papers related to refraction of light in a crystalline medium.

In retrospect, of the three books, I would recommend *Little Sparrow* to the reader who would like to read only one of the three. Don H. Kennedy's determined efforts to locate and compile as much information from original documents as possible makes it an enlightening account of late 19th century European intellectual history, as well as a satisfying biography. Through his account, Sofie Kovalevskaia comes to life for the reader, but what is probably more significant, she is revealed as a person with whom the reader may identify. As this happens, her successes become inspiration. We see her face her weaknesses—some of which she overcomes, but some of which she matter-of-factly accepts. In either case, she continues to move forward to achieve her ultimate goal—to make a difference in the world—to make a lasting and significant contribution.

Reviewed by the Editor, Sandra DeLozier Coleman.

USING ALGEBRA, Ethan D. Bolker, 1990, xi + 298 pp., $19.95 paper. ISBN 1-55605-134-4, Wyndham Hall Press, Cloverdale Corporation, 52857 County Road 21, Bristol, IN, 46507-9460, (219) 848-4834.

Standard I-2 of the AMATYC "Standards for Introductory College Mathematics" (1995) states that students should "learn mathematics through modeling real-world situations." In my search for new teaching strategies to put more reality into my algebra lessons, I found Ethan Bolker's *Using Algebra*.

Real phenomena are the outstanding feature of *Using Algebra*. Bolker lets the story tell the algebra, restricting the introduction of algebraic skills to the level required by the application. Real data are interwoven with algebraic and graphical interpretations. Words rather than symbols, guess-and-check and estimation, attention to proper calculator use, and the choice of familiar and simple situations also support the student's comprehension of mathematical concepts.

The book is divided into four parts: linear relationships, exponential models, quadratic models, and trigonometry. The topics covered appear to be most consistent with college algebra or remedial elementary/intermediate algebra. However, using it as a textbook would entail a departure — in scope and treatment — from the traditional curriculum in either course.

Pedagogical strategies place emphasis on the concrete. Numerical data precede formulas and formulas are written with words before symbols are introduced ("total cost = overhead + dollar/panel * # of panels"). Graphing concepts like slope and intercept that typically are linked abstractly to the notation "$y = mx + b$" are here introduced as meaningful numbers representing, for example, overhead and cost per panel. Unnecessary abstraction is avoided when numerical calculations can be used. In the words of the author, "...the abstract truths that make mathematics beautiful and useful are best left implicit at this level. I have found that students tend to agree." (p.v) The value of abstraction is not altogether ignored by Bolker; all principles explored are summarized symbolically.

The demystification of '*e*' exemplifies this concrete approach. Bolker uses real data in the context of competing banks trying to top each other's interest rates. That, and the focus on comparing nominal with effective interest rates, provide the concrete handle needed to understand that 2.718... is simply the effective growth rate for a 100% nominal interest rate compounded continually! The comfort of that reality does not in any way compete with the more abstract representation of e as the maximum value of $\left(1 + \frac{1}{N}\right)^N$.

Connectivity among mathematical concepts is a highlight. Repeated themes from business and economics are threaded through the chapters. Because graphical representations are used so consistently to clarify each new model, they also add to the sense of connectedness. The idea of function is similarly repeated. (However, it is expressed always as the "dependence" of one quantity on another; the word "function" is not used.)

Unusual treatment of some topics strengthens the sense of connectivity. Breaking from the traditional equality-of-ratios viewpoint, Bolker defines proportion as "linear dependence when the intercept is 0" (p. 47). (Most textbooks fail to integrate the concepts of slope and proportionality, or proportion equations and linear graphing.) The use of a proportionality constant becomes a theme as it is reiterated in measurement conversions, trigonometric ratios, and quadratic models. An interesting approach to "mixtures" applications ties into proportionality. Problems about gasohol, nuts, diluting sulfuric acid, and planting wheat are all solved with a formula of the form $C = aP + b(1 - P)$, where P is the ratio of one of the two ingredients to the total.

Although linear, exponential, and quadratic models are presented in separate chapters, significant effort is made to compare and integrate their contrasts and similarities. The student's natural inclination to guess a linear model of inflation is acknowledged, then corrected by recalling that

> When you read about increasing prices, the units of increase are percentages... The proper question is: 'What do we *multiply* last year's

price by to get this year's price?' rather than 'What do we *add* to last year's price...?' (p. 110).

Both the conjectured linear model and the exponential model are graphed on the same axes to highlight their comparison.

Embedded in the development of the algebra are short reminders about arithmetic procedures, for example, rounding in the real world, order of operations, and adding fractions. Calculators are an essential part of every lesson, and instructions accompany every new technique. The calculator is not overused; its use is frequently preceded by instructions to begin with a mental estimate. Special attention is given to working with very large and very small numbers in problems involving the speed of light, the volume of the Earth, and electric power consumption.

The exercises ending each chapter expand the range of topics to areas as divergent as gas tank design, penicillin dosage, and proliferation of automobiles and magazines. These exercises are mostly not of the drill-and-practice genre. Some ask for missing information to be supplied; others ask "What question is implied here?", "What statement of proportionality can be made?", or "What do you think is the point of this exercise?" Some lead to the discovery of algebraic principles, as in the following:

(a) Calculate $\frac{1}{2.7} + \frac{1}{5.6}$

(b) Calculate $\frac{1}{2.7 + 5.6}$

(c) Your answers to (a) and (b) show that, in general, $\frac{1}{x} + \frac{1}{y}$ is not the same as $\frac{1}{x+y}$. Make up several more examples that demonstrate the same point. (ch. 2, problem 22)

Many questions may accompany the integration of these ideas into the current curriculum. Foremost may be the clash with the traditional emphasis on abstract manipulations. This textbook introduces, and asks the student to practice, only those skills needed to support the applications. There are no polynomial factoring exercises. Facility with exponents is supported by practice with scientific notation, rewriting simple exponential equations using e, and simplifying expressions such as a^b*a^c or $\frac{t^m}{t^n}$. Although some of the exposition assumes student familiarity with with properties of radicals, exercises involving radical expressions emphasize use of the calculator; simplifying by extracting perfect squares is omitted. Manipulations with rational expressions do not exceed the complexity level of adding $-\frac{b^2}{2a} + \frac{b^2}{4a}$ when deriving the quadratic formula (by investigating the "time to splash" for an object thrown upward from a bridge). Alternatively, attention is given to the manipulations needed to simplify units, for example, (ft/sec)/sec. Many in the educational community would place more emphasis on the practice of decontextualized algebraic skills. There may be some truth in both points of view, the challenge being to find a productive balance.

In conclusion, *Using Algebra* provides a plan for teaching algebra entirely through applications, or a resource for instructors who are interested in including more reality-based context in a standard course. Although many of the algebraic manipulations are treated in much less depth than in the traditional text, this approach brings far superior depth to connections among topics and to their applications in the real world. The book models many of the pedagogical and content standard selected for more emphasis by AMATYC and other advocates of reform in mathematics education.

Reference

American Mathematical Association of Two-Year Colleges. (1995). *Crossroads in mathematics: Standards for introductory college mathematics before calculus.* Memphis, TN: Author.

Reviewed by Sue White, Austin Community College, Austin TX 78736.

The AMATYC Review welcomes contributions of book reviews by its readers. We would like to continue to have reviews of books that would be of interest to a broad spectrum of persons associated with or interested in the world of mathematics. Reviews of individual books are welcome, although we would like to know about groups of books which complement each other in shedding light on a particular topic. Occasionally, reviews from several readers may be combined in order to present this type of selection.

Send reviews to: Sandra DeLozier Coleman, Mathematics Department, Okaloosa-Walton Community College, 4531 Parkview Lane, Niceville FL 32578-8734, SDColeman@AOL.COM

Many who have never had the occasion to discover more about mathematics consider it a dry and arid science. In reality, however, it is a science which demands the greatest imagination.

Sofia Kovalevskaya

❖ ❖ ❖

There is a fountain of youth: it is your mind, your talents, the creativity you bring to your life.

Sophia Loren

The Problem Section

Dr. Michael W. Ecker
Problem Section Editor
The AMATYC Review
909 Violet Terrace
Clarks Summit PA 18411

Dr. Robert E. Stong
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The AMATYC Review
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Greetings, and welcome to still another Problem Section!

The AMATYC Review Problem Section seeks lively and interesting problems and their solutions from all areas of mathematics. Particularly favored are teasers, explorations, and challenges of an elementary or intermediate level that have applicability to the lives of two-year college math faculty and their students. We welcome computer-related submissions, but bear in mind that programs should supplement, not supplant, the mathematical solutions and analyses.

Help! At this time I am virtually out of good problems. Contact me by one of these means:

E-mail: *mwe1@psu.edu* – via which I will acknowledge your problem, comment, suggestion, or whatever. In fact, I now prefer this method, although it is not required for participation. But, it allows faster receipt, faster replies, faster requests for clarification, etc. If you wish to include a solution with mathematical notation, I prefer Microsoft Word 6 for Windows and its equation editor. However, I am also able to handle Macintosh and other PC document formats if needed. Feel free to attach all such documents to your e-mail.

Regular mail: Send **two copies** of new **problem proposals** to the Problem Editor. Please submit separate items on separate pages each bearing your name (title optional; no pseudonyms), affiliation, and an address. If you want an acknowledgement or reply, please include a mailing label or self-addressed envelope or e-mail address.

In either case, if you have a solution to your proposal, please include same (two copies would be appreciated if sent by traditional mail) along with any relevant comments, history, generalizations, special cases, observations, and/or improvements.

One new point: Diagrams are welcome! In the past I have eschewed them, but I have a PaperPort Vx scanner attached to my Pentium with which I can now scan in graphics.

All solutions to others' proposals (except *Quickies*) should be sent directly to the Solutions Editor.

Dr. Michael W. Ecker (Pennsylvania State University, Wilkes-Barre Campus)

Dr. Robert E. Stong (University of Virginia)

Solved Quickies

Quickies are math teasers that typically take just a few minutes to an hour or two. Solutions usually follow the next issue, listed before the new teasers. All correspondence for this department should go to the Problem Editor.

Quickie #22: Proposed by Frank Flanigan, San Jose State University, CA.

The power series $1 + 2x - 3x^2 + x^3 + 2x^4 - 3x^5 + x^6 + 2x^7 - 3x^8 + x^9 + \ldots$ does not converge at $x = 1$, but it does represent on $(-1, 1)$ a function $f(x)$ that is analytic on $(-\infty, \infty)$. Calculate $f(1)$.

I received correct solutions – all virtually identical except that some included rigor – from Steve Plett (Fullerton College), John Rimar (Galveston College), M. Jerry Thornhill (Southwest Virginia Community College), Russell Euler (Northwest Missouri State University), Frank P. Battles (Massachusetts Maritime Academy), Carl Riggs (retired naval engineer in Largo, Florida), and Robert Stong (Solutions Editor). I hope I did not miss anybody in this list of solvers.

Solution Sketch: It's evidently a geometric series with first term = $1 + 2x - 3x^2$ and common ratio = x^3. So, $f(x) = \dfrac{1 + 2x - 3x^2}{1 - x^3} = \dfrac{1 + 3x}{1 + x + x^2}$, and $f(1) = \dfrac{4}{3}$, modulo any details.

Quickie #23: Proposed by Florentin Smarandache, Tuscon, AZ.

For each positive integer n, define the Smarandache function S by $S(n)$ = the least positive integer k such that n divides $k!$. Characterize $\{n \in Z^+ : S(n) = n\}$; i.e., when is $S(n) = n$?

Solution Sketch: Among the first few values yielding fixed points we have $S(1) = 1$ and $S(4) = 4$. Clearly $S(p) = p$ for all primes p. No other values are fixed. Thus $n = 1, 4$, or any prime.

Quickie #24: Proposed by the Problem Editor.

What is the average length of a chord in a circle of radius r?

Solution Sketch: As Bob Stong, Rick Armstrong, and others have pointed out, the question is ambiguous, much as are many probability problems in which the manner of selection of random objects and/or the measure is vague. As a result, there are at least three possible answers.

One interpretation that I consider "natural" is to use a semi-circle of radius r, viz.: $y = \sqrt{r^2 - x^2}$, get the average y-value of this function on $[-r, r]$, and finally double. The average is the area of the semi-disk divided by the interval length. After doubling, we get $\dfrac{\pi r}{2}$ for this interpretation.

Quickie #25: Proposed by the Problem Editor.

Given any modulus m, prove that every additive sequence of integers mod m is eventually periodic. (This result can be easily extended to other kinds of sequences. Note that an additive sequence f is one satisfying $f(n) = f(n - 1) + f(n - 2)$.)

Solution Sketch by Proposer: This is true by virtue of the pigeon-hole principle. Consider the $m^2 + 1$ pairs of numbers $(f(j) \bmod m, f(j+1) \bmod m)$, $j = 1, 2, \ldots, m^2 + 1$. There are only m^2 distinct elements of $Z_m \times Z_m$. (As usual, $Z_m = \{0, 1, 2, \ldots, m-1\}$ and $Z_m \times Z_m$ is the Cartesian product of sets.) Thus, there exist integers j, k in Z_m with

$$(f(j) \bmod m, f(j+1) \bmod m) = (f(k) \bmod m, f(k+1) \bmod m).$$

By additivity, $f(j+2) \bmod m = f(k+2) \bmod m$, as follows:

$$f(j+2) \bmod m = (f(j+1) + f(j)) \bmod m = f(j+1) \bmod m + f(j) \bmod m =$$

$$f(k+1) \bmod m + f(k) \bmod m = (f(k+1) + f(k)) \bmod m = f(k+2) \bmod m.$$

Similarly, $f(j+q) \bmod m = f(k+q) \bmod m$ for all positive integers q, by induction on q.

New Quickies

This time I am not using any separate *Quickies*. (If I were, I would remind you to send their solutions to me, not the Solutions Editor.) However, I am including some easier Problems in the regular section. It is understood that in such cases I seek elegance.

New Problems

***Set AE Problems** are due for ordinary consideration April 1, 1997.* However, regardless of deadline, no problem is ever closed permanently, and new insights to old problems – even *Quickies* – are always welcome. However, our Solutions Editor requests that you please not wait until the last minute if you wish to be listed or considered on a timely basis.

Problem AE-1. Proposed by the Problem Editor, but borrowed from elsewhere.

Find all rectangular solids whose sides have integral lengths and whose surface area is numerically equal to the volume (in that system of units).

Tip: I cheated and first wrote a program to count solutions. My source cited the same number. So, I am now asking for a mathematical argument (e.g., to prove there are no other solutions).

Problem AE-2. Proposed by the Problem Editor and Jim Preston (Minneapolis, MN).

Given that a proposed triangle is to have a given area, one given side, and the opposite angle given, find the other parts. If necessary, consider discussion of existence and uniqueness of such a triangle based on the given information.

Comment: Jim Preston is the man behind Rite Item, a small Minneapolis software company. He is also a subscriber to my *Recreational & Educational Computing*, to which he originally directed this. The immediate relevance is to his shareware program, AnyAngle (versions for DOS and Windows), which solves for any part of a triangle. A while back he decided to add to the program's capability, but was not sure how to do this one thing. I now have an elegant solution from Germany, so I challenge readers to find it – or another one as quick and as nice.

Problem AE-3. Proposed by the Problem Editor.

Suppose that you have an equiprobable choice of any one of the 180 angles of integral degree-measures 1 through 180. You randomly pick two such measures, one at a time, with replacement. In fact, after any one choice, the next choice is independent of the first.

What is the probability that the two chosen angles are complementary? ... supplementary?

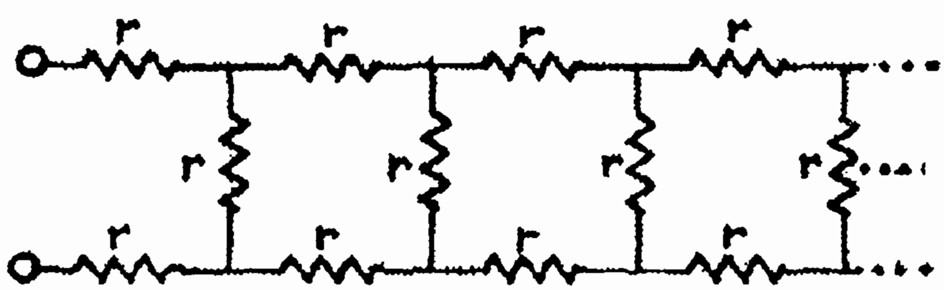
Problem AE-4. Proposed by the Problem Editor, but borrowed from elsewhere.

Consider any right triangle with legs a, b and hypotenuse c. Of course, $a^2 + b^2 = c^2$. Let $n > 2$. What can you say about $a^n + b^n$ vs. c^n? (Prove your answer.)

Comment: After I saw this, I had an interesting new insight. Consider $f(x) = a^x + b^x - c^x$ and its graph. It seems to have a unique zero. Now repeat for any triple (a, b, c). It appears that, for *any* one triple of positive integers (a, b, c) — not necessarily Pythagorean — there exists one and only one exponent x for which $a^x + b^x = c^x$. I have not yet proven this, but I believe it is true. Does this shed light on, or insight into, other results, notably Fermat's Last Theorem?

Problem AE-5. Submitted by Don L. Lewis, Bee County College (Beeville, TX).

Each individual resistor below has resistance r. Find the equivalent resistance of the infinite network. (Option: Write a terminating continued-fraction expression for a network of n loops).



Problem AE-6. Proposed by the Problem Editor.

Let $p(x)$ be the monic polynomial of degree n that fixes the first n positive integers. That is, consider the monic polynomial p with $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, ..., and $p(n) = n$. Find $p(n + 1)$.

Problem AE-7. Proposed by Frank P. Battles, Massachusetts Maritime Academy (Buzzards Bay, MA).

For which values of x is $\sqrt[3]{\sqrt{x^2+1}+x} - \sqrt[3]{\sqrt{x^2+1}-x}$ an integer? (This extends Problem AC-1, solved this issue, and to which the author invites you to compare this.)

Personal Notes: Due to lack of room, I am delaying publication of problems from Michael Andreoli and Ken Boback. In the absence of unforeseen new problems, these could and should appear next issue. However, I wish to thank all who have sent material to me over my first 15 years as Problem Editor, and I encourage others among you to do so now. Thanks!

Set AC Solutions

Root Route

Problem AC-1. Passed on by Harry J. Smith (Saratoga, CA) and the Problem Editor (Michael W. Ecker, Pennsylvania State University, Wilkes-Barre Campus).

There are numerous ways to evaluate the expression

$$\sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2}.$$

Evaluate and prove mathematically that your answer is correct.

Solutions by Jim Africh, College of DuPage, Glen Ellyn, IL; Charles Ashbacher, DecisionMark, Cedar Rapids, IA; Frank Battles, Massachusetts Maritime Academy, Buzzards Bay, MA; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Kenneth G. Boback, Penn State, Wilkes-Barre Campus, Lehman, PA; Mike Dellens, Austin Community College, Austin, TX; Celeste Elton, Clark College, Vancouver, WA; Matt Foss, North Hennepin Community College, Brooklyn Park, MN; Bernard G. Hoerbelt, Genesee Community College, Batavia, NY; William T. Long, Broward Community College, Coral Springs, FL; Gene Majors, Fullerton College, Fullerton, CA; Stephen Plett, Fullerton College, Fullerton, CA; Carl Riggs, retired naval engineer, Largo, FL; W. Grant Stallard, Manatee Community College, Bradenton, FL; Charles Stone, DeKalb College, Clarkston, GA; Wesley W. Tom, Chaffey College, Rancho Cucamonga, CA; Bruce Yoshiwara, Pierce College, Woodland Hills, CA; and the proposers.

Let $x = \sqrt[3]{\sqrt{5}+2}$ and $y = \sqrt[3]{\sqrt{5}-2}$. Then

Solution 1. One has $\left(\frac{\sqrt{5}+1}{2}\right)^3 = \sqrt{5}+2$ and $\left(\frac{\sqrt{5}-1}{2}\right)^3 = \sqrt{5}-2$, so $x - y = \frac{\sqrt{5}+1}{2} - \frac{\sqrt{5}-1}{2} = 1$.

Solution 2. One has $(x-y)^3 = x^3 - y^3 - 3xy(x-y)$ with $x^3 - y^3 = (\sqrt{5}+2) - (\sqrt{5}-2) = 4$ and $xy = \sqrt[3]{5-4} = 1$, so $x-y$ is a root of $z^3 + 3z - 4 = (z-1)(z^2+z+4) = 0$. This equation has roots 1 and $\frac{-1 \pm \sqrt{-15}}{2}$, and so $x - y = 1$ is the only real root.

Sum of Some Digits

Problem AC-2. Proposed by David Shukan, Los Angeles, CA (passed on by Problem Editor).

Consider any natural number n. Write n in bases 2, 3, 4, and 5. Add the digits in these representations and call the resulting natural number $f(n)$. Iterate to calculate $f^2(n) = f(f(n)), f^3(n) = f(f(f(n)))$, etc. Prove or disprove: For each n there exists a natural number k (which may depend on n) such that $f^m(n) = 10$ for all $m \geq k$.

Solutions by Charles Ashbacher, DecisionMark, Cedar Rapids, IA; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Matt Foss, North Hennepin Community College, Brooklyn Park, MN; Stephen Plett, Fullerton College, Fullerton, CA; and the proposer.

Since 10 is less than 2^4, 3^3, 4^2, and 5^2, a number n with k digits in base 10 requires at most $4k$ digits (each being at most 1) in base 2, at most $3k$ digits (each being at most 2) in base 3, at most $2k$ digits (each being at most 3) in base 4, and at most $2k$ digits (each being at most 4) in base 5. Thus, for n having k digits, $f(n)$ is at most $(4 \times 1 + 3 \times 2 + 2 \times 3 + 2 \times 4)k = 24k$. Since $24k$ is less than 10^{k-1} if $k \geq 3$, $f(n) < n$ if n has 3 or more digits, and $f(n) \leq 48$ if n is less than 100. It then suffices to check the result for $n \leq 48$, which is a routine calculation.

Function Count

Problem AC-3. Proposed by the Problem Editor.

Let $S = \{1, 2, \ldots, n\}$ for a natural number n, and suppose $f: S \to S$ hereafter. The number of such functions f is n^n, and of such permutations is $n!$ (n-factorial). The permutations all satisfy

$$\sum_{i=1}^{n} f(i) = \sum_{i=1}^{n} i.$$

However, there are surely other functions f that satisfy this condition. How many are there?

Solutions by Robert Bernstein, Mohawk Valley Community College, Utica, NY and Stephen Plett, Fullerton College, Fullerton, CA.

There is a one-to-one correspondence between the functions from S to itself and the monomials in the expansion of

$$A = (x + x^2 + \ldots + x^n)^n$$

assigning to each function f the monomial $x^{f(1)} x^{f(2)} \ldots x^{f(n)}$ using the exponent $f(i)$ from the ith factor. Thus the number of functions of the desired type is the coefficient of x^t in A, where $t = 1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$. This is also the coefficient of x^s in

$$(1 + x + \ldots + x^{n-1})^n = \left(\frac{1 - x^n}{1 - x}\right)^n$$
$$= \left\{\sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{kn}\right\} \left\{\sum_{r=0}^{\infty} \binom{n-1+r}{r} x^r\right\}$$

where $s = \frac{n(n-1)}{2}$. Thus, the number of functions of the desired type is

$$\sum_{k=0}^{\left[\frac{n-1}{2}\right]} (-1)^k \binom{n}{k} \binom{\frac{n^2+n-2}{2} - kn}{n-1}.$$

Going Out On A Lim

Problem AC-4. Proposed by Kenneth G. Boback, Pennsylvania State University, Wilkes-Barre Campus, Lehman, PA.

Let q be a prescribed positive constant. Find all values of the constant c that satisfy

$$\lim_{x \to \infty} \left(\frac{x+c}{x-c}\right)^x = q.$$

Solutions by Frank P. Battles, Massachusetts Maritime Academy, Buzzards Bay, MA; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Paul Casillas, Clark College, Vancouver, WA; Matt Foss, North Hennepin Community College, Brooklyn Park, MN; Steve Kahn, Anne Arundel Community College, Arnold, MD; Darrell P. Minor, Columbus State Community College, Columbus, OH; Stephen Plett, Fullerton College, Fullerton, CA; Carl Riggs, retired naval engineer, Largo, FL; W. Grant Stallard, Manatee Community College, Bradenton, FL; Charles Stone, DeKalb College, Clarkson, GA; Wesley W. Tom, Chaffey College, Rancho Cucamonga, CA; and the proposer.

One has $\lim_{x \to \infty} \left(\frac{x+c}{x-c}\right)^x = \lim_{x \to \infty} \frac{(1+\frac{c}{x})^x}{(1-\frac{c}{x})^x} = \frac{e^c}{e^{-c}} = e^{2c}$. Thus $e^{2c} = q$ or $c = \frac{\ln q}{2}$.

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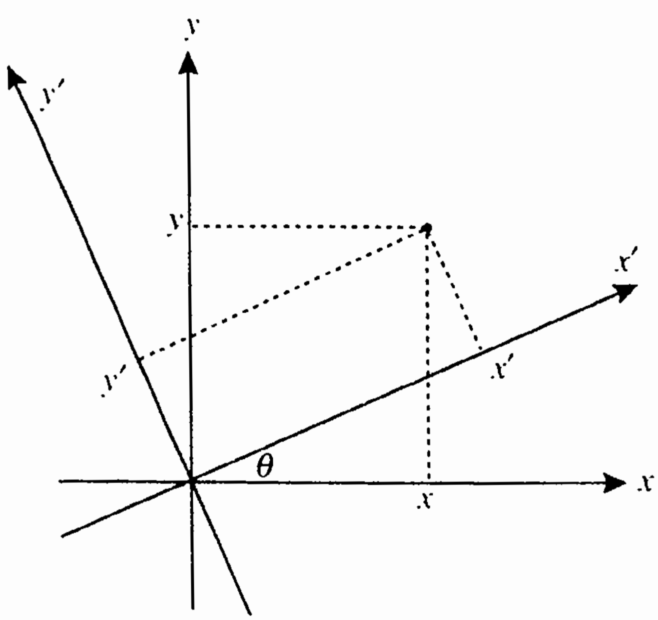
Rotation of Axes



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Also in this issue

- Identifying textbooks which support the AMATYC Standards
- Sums of powers of integers
- Student response to teacher feedback

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MISSION OF AMATYC: Recognizing the vital importance of the first two years of collegiate mathematical education to the future of our students and the welfare of our nations, AMATYC is committed to the following:

- to positively impact the preparation of scientifically and technologically literate citizens;
- to lead the development and implementation of curricular, pedagogical, assessment and professional standards for mathematics in the first two years of college;
- to assist in the preparation and continuing professional development of a quality mathematics faculty that is diverse with respect to ethnicity and gender;
- to provide a network for communication, policy determination, and action among faculty, other professional organizations, accrediting associations, governing agencies, industries, and the public sector.

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About the Cover

The rotation of axes is a topic which many colleges have either eliminated from their Calculus and Analytic Geometry course or are contemplating removing. One of the chief concerns in making such a decision is whether the student needs the material for what is to come later. In the case of this topic there is, perhaps, little chance that the student will be rotating axes in the foreseeable future. There is, however another facet to this topic which argues for keeping it in the syllabus. That is, the very nice way that developing the methods pulls together some rather widely distributed topics the student has studied before. In this issue Richard Quint shows a nice, and elementary, matrix approach to finding rotation equations.

New Editorial Team Sought

Even editors can't last forever, and the current team is approaching the end of its term. AMATYC is seeking candidates for the new editor, with duties to begin in 1998. If you would like to be considered, see the ad which appears in this issue.

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VIEWPOINT

Can You Recognize a Textbook that Supports the AMATYC Standards?

by

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Jack Rotman has been a faculty member at Lansing CC since 1973, and has been active in the Developmental Mathematics Committee of AMATYC since 1987. Besides the viewpoints generated by these perspectives, Jack also has reviewed many textbooks for publishers over the last twenty years.

Brian E. Smith is Professor of Mathematics at Dawson College. He received his M.A. in Mathematics from Trinity College Dublin, an M.Sc. in Computer Science from McGill University and his Ph.D. in Mathematics from Queen's University. He is Chair of the Technology in Mathematics Education Committee of AMATYC, and founder of the Two-Year College internet discussion group MATHDCC.

At the 1996 AMATYC Conference (Long Beach, California), we conducted a session on textbooks and the AMATYC "Standards" (**Crossroads**) (AMATYC, 1995).

The session was undertaken with the belief that people with experience and expertise in the learning of mathematics will have valuable insights into how textbooks could help support the implementation of the standards in the **Crossroads** document. Therefore, the basic objective of the session was to generate ideas on such textbooks and identify those which have consensus — at least among the group present at the session. These properties of a textbook might then serve to guide authors and publishers in the production of new materials.

In this article, we will:

A) Review the process used;

B) Present a brief summary of the reports made at the session; and

C) Give some interpretations.

The Process

The work of the session began with each person listing some properties of a textbook that supports the **AMATYC Standards** ("**Crossroads**"). Each of the eight small groups (representing 55 people) then sought consensus on these properties of a textbook. Then each group shared the results of their work with the large group. Originally, we had hoped to develop some consensus among the large group, based on the small group reports. However, the session proved to be too short for this purpose.

Summary of the Small Groups' Reports

Each of the small groups clustered their textbook properties into categories; as would be expected, there was not a uniform set of categories used. In this summary, we will impose one set of categories as a means to quickly communicate the results of the small groups.

Category 1: **Student-Centered**

Several small groups listed properties dealing with the interaction between the textbook and the student. Some properties in this category were:

Active Participation

Guided discovery

Concept development

Connect new concepts to old

Relate to students' world

Category 2: **Context for Learning, and Applications**

It was clear that many people felt there needed to be a context for the material in the student's mind — that the mathematics should be learned (at least partly) by examining applications and looking at skills within the contexts of these applications. Some properties in this category were:

Relevant to student experiences

Meaningful

Variety

Further application

Motivate concepts

Connect concepts to real world applications

Activities which emphasize usefulness of mathematics

Include real data applications

Integrated applications

Problems and skills revisited in expanded contexts

Include historical development where appropriate

Connections: Within mathematics; within other disciplines

Category 3: **Technology**

Since the content level of the textbook was not specified, the properties listed do not speak to specific technology; rather, they deal with broad issues.

Integrated

Appropriate

Student choice

Use technology to enhance student understanding

Encourage use of technology

To build understanding of math concepts

Text should motivate technology to enhance, explore, verify

Category 4: **Other Pedagogy**

A large number of properties dealt with the other factors in an ideal textbook's pedagogy. The properties in this category will be reported in 3 groups.

Group 4A: **Critical Thinking (including writing across the curriculum)**

This group of properties describes a textbook that goes beyond drill exercises and examples, to get at higher-level performance.

More than drill: writing and enhancement

Team/group debate and discourse

Incorporate open-ended problems that encourage exploration

Writing exercises

Patterning exercises

Activities for thought provoking explorations

More open-ended, non-standard exercises

Extending problem situations (revisiting in new contexts)

Anticipation of reasonable and unreasonable solutions

Group 4B: **Concept Development**

Along with the critical thinking properties, the ideal textbook would focus on concept development.

Utilize multiple representations ("Rule of *n*")

Various approaches: numeric, symbols, graphical

Approach concepts from more than one direction

Stress visualization

Communication: correct terminology

Group 4C: **Group Work and Collaboration**

The last group of properties deals with cooperative or collaborative learning.

Encourage collaborative learning

Group work

Projects for small groups: Long term and short

A few properties listed by a small group defy placement into a category, but speak to important issues; for example:

Weigh ≤ 5 pounds!

Varied ways for students to demonstrate what they can do

Affordable texts leaving money for technology

Interpretations of the Properties

We wish to make some comments about each group of properties, in order to explore their implications for mathematics textbooks.

Student Centered

The properties listed by the groups are consistent with the material in the **Crossroads** standards, and are also consistent with our understanding of current learning theory.

"Active Participation" and "Guided Discovery" imply that a textbook can not be simply a summary of knowledge with worked examples. Every section should include work (in the presentation) for the student, so that they can become involved. The properties imply that we need textbooks that are more sophisticated than the worktext/programmed learning type; the text should present the student with situations that are not "already covered" by prior material in the text.

"Concept Development" and "Connect New Concepts to Old" are cryptic phrases that may lack an agreed-upon meaning. However, we believe that these properties suggest a book that generates concept understanding by first exploring examples and non-examples, and that recycles concepts throughout the text. These properties give a clear contrast to the less-verbal textbooks and to the "modular" textbooks: words are needed to develop new concepts, and sections/chapters should present an integrated sequence.

The last property in this group presents a large challenge: With our diverse student population, how could any textbook relate to a sizable minority of the students — let alone most students?

Context and Applications

This is a complex group of properties, and involves an area of controversy: How much of mathematics education should be learned in a practical context? There is clearly a major difference between these types of texts:

a) a text that includes real-world applications throughout, and

b) a text that presents only topics within an application.

Our understanding of current learning theory is that it is not necessary to embed every topic in a practical context for learning, but that motivation to learn and beliefs about mathematics are enhanced by a significant presence of a context or real-world application.

None of these properties state the degree to which context and applications are needed in the textbook. We believe, however, that a textbook supporting the AMATYC standards would reach at least the midpoint of this continuum: realistic applications and contexts used consistently throughout the text, though not necessarily in every presentation.

We also believe that it is important to raise up the last two properties in this group, as they imply a little different perspective on "context". A historical perspective implies that we give a context for presenting current mathematics: How did we get here, in our understanding, our tools, and our symbolism? Emphasizing connections — especially to other disciplines — implies that we want a textbook to include some non-mathematical material. We believe that both of these properties support the general education function that is appropriate for many of our mathematics courses, as well as a more inclusive view of mathematics education reflected in our standards.

Technology

Some common themes relating to technology in the properties are: integration, appropriateness, building understanding, and exploration; these goals are clearly consistent with the AMATYC **Crossroads** standards document. While mathematics reform involves many aspects of pedagogy, nevertheless many see it as largely technology-driven, and there is a very significant challenge in merely keeping abreast of the rapidly changing developments in graphing calculators and computer algebra systems technology. As mathematics educators, many of us struggle to keep up with existing technologies, while new ones are constantly appearing on the scene. There is a belief that, in order to facilitate the use of technology by both faculty and students, it is essential that our teaching materials incorporate instruction in the use of specific technologies.

Authors must decide how much technology to include in a textbook. Should a text make only generic references to different types of technology, or should they refer to specific brand names and models of graphing calculators and computer packages? There is a big difference between a statement such as "Draw the graph of $\sin(2x)$ on your graphing calculator" and a specific set of instructions or keystrokes for a particular calculator. Clearly, while there is room for both types of material, hardware and software manufacturers have a significant stake in having authors refer to their products.

At one extreme is the calculator supplement or lab manual, which may be a stand-alone product or be linked to a particular textbook. By commissioning an author to write this sort of material in conjunction with a new edition of a text, the primary author(s) of a major text may be "off the hook" as far as incorporating technology into the text is concerned. While this approach has been adopted by publishers for some popular texts, we would argue that there is a significant benefit to having reference to technology, and explorations with technology, interwoven in the main text. This makes the use of technology an integral part of the course; technology should not be a side show, nor a barrier. Both instructor and student will be far more likely to think of the technology as an essential component of the course if it is integrated into the text.

We emphasize our belief that the use of technology is a tool to achieve other goals, such as active participation and realistic problems and applications. The emphasis with technology should be on the mathematics, not the hardware or software. There is a clear rationale for the use of technology, for a wide variety of reasons; textbooks should harness this power. We believe that a textbook needs to be written with a clear view of how technology will be used, with the presentations and content being consistent with this view. Technology usage is not an optional feature with a textbook; authors and publishers need to be willing to make a clear commitment. If the author(s) of a text accept "appropriate technology" that is "integrated", then there should be a significant change in topical coverage as well as avoidance of the contrived problems (i.e. — those with "nice" values for solutions).

Other Pedagogy

The properties in this large category cover many possibilities. We are faced with a decision that is not guided directly by the **Crossroads** standards: "How much of each pedagogy is appropriate — which is more important?" Furthermore, we believe there is a separate question: "Which pedagogies are appropriate for a textbook, and which are better delivered by the local college and/or instructor?" The first question is beyond the scope of this article, but we will address the second briefly.

The first group of properties (**Critical Thinking**) would be addressed in a textbook primarily in the student exercises. The additional variety of problems, in theory, reflects a realistic expectation of a textbook; it is our belief that a textbook supporting the standards would include significant quantities of these other types of exercises. This group includes one property that is not clearly a matter of exercises; we suspect that "anticipation of reasonable and unreasonable solutions" is a matter of the actual presentation and/or the supplemental materials (solutions manual or computer tutorial, for example).

The second group of properties (**Concept Development**) is closely related to the properties of "**Student Centered**" discussed previously. However, except for the communication and terminology property, this group of properties suggests that a text present concepts from multiple viewpoints, and that this multiple approach be used throughout the text. We believe that this is not a simple matter, and that a textbook must strike a balance between using multiple approaches and using a single approach; the rationale for this is mostly practical: using multiple approaches consumes quite a bit of space in the text. (We want to remind the reader of a property that defies a category: "Weighs no more than 5 pounds!")

The third group of properties (**Group Work and Collaboration**) seems, in our view, to be questionable as an expectation for a textbook supporting the standards. One rationale for this belief is that group work requires the professional monitoring of an instructor, which is difficult when the student has a set of directions in the textbook. A stronger rationale, however, deals with a basic principle of group work: The group work needs to be designed for the specific nature of the groups involved. The author(s) of a text will not have knowledge of the groups:

Are the same groups used throughout the course?

Are the groups heterogeneous or homogeneous?

Is the group work used for assessment?

Do the groups work in class time?

We believe that it is not appropriate to expect a textbook to provide group work, and that the instructor and/or college must have responsibility and control. However, we do recognize that there is a need for resources (especially given the large proportion of mathematics courses taught by part-time/adjunct faculty, who may lack curriculum development time.) Therefore, it is our belief that resources for group work, in a separate publication, should be provided by authors, and that these resources provide possible problems and projects to be used with a textbook. (Our statements here assume that the group work would deal with problems and/or projects of some length; clearly, some problems encouraging **Critical Thinking** are appropriate for group work.)

Summary

The AMATYC "standards" (**Crossroads in Mathematics**) lists 7 basic principles (pages 4 and 5); of these 7 principles, 4 are clearly related to textbooks:

The mathematics that students study should be meaningful and relevant.

Mathematics must be taught as a laboratory discipline.

The use of technology is an essential part of an up-to-date curriculum.

Students will acquire mathematics through a carefully balanced program.

A textbook having the properties outlined in this article will meet these standards. However, we have some concerns in closing.

Textbooks for all Purposes?

In the process for developing and marketing textbooks, there is a tendency to include *all* properties that might be desirable. A textbook that attempts to deal with all of these, along with a typical amount of mathematical content, tends to grow to mammoth proportions — or to pack features and material into every page. Either of these outcomes seems undesirable, in our view. The mammoth textbook might discourage student use because of its size, and a compressed format makes it difficult for most students to process the information. Therefore, it seems reasonable to raise the question:

Are there some properties that are significantly more important than others??

This is not a simple question in our real world. Some standards could be met by "properties of the instructor or classroom" instead of by the textbook — such as collaborative learning. However, the real world involves colleges who employ adjunct faculty who might lack preparation time and/or full-time faculty who might lack the commitment — as well as faculty in both groups who go beyond expectations in improving their classrooms. This situation results in one of the great pressures on textbooks — put it all in the book. In our view, for implementation of the standards, a different resolution needs to be sought.

Money, Social Classes, and Mathematics

As one group pointed out, and we paraphrase, students who must spend a large amount of money for a text are not likely to have much left for other supplies — such as technology. Due to the "textbooks for all purposes" pressure and other changes in the marketplace, textbooks have tended to become relatively more expensive — compared to the typical income of the two-year-college student. It is our concern that the cost of a textbook is such a burden that some groups of our students will be handicapped. The **Crossroads in Mathematics** includes a basic principle that we support: "Increased participation by all students in mathematics ..." (page 5). The question we raise here is:

> Is it possible, if faculty and publishers collaborate, to produce textbooks at a lower price that can be used to support the desired mathematics courses?

This issue is also complex. Recent years have seen a large number of the small and medium size publishers being bought or absorbed by diverse corporations, with both trivial and substantial consequences for our purposes. We would be pleased to see movement on the part of publishers to work with faculty on these issues.

References

American Mathematical Association of Two-Year Colleges. (1995). Crossroads in mathematics: Standards for introductory college mathematics before calculus. Memphis, TN: author.

> I believe that the ultimate caricature of good mathematical teaching is linear error-free programming. Under this scheme, instead of taking care to insure that every student is provided with the most stimulating challenges that he can react to successfully, people use their best efforts to create a situation where nobody is faced with any challenge at all.
>
> E.E. Moise

MATHEMATICAL EXPOSITION

A Classification System for Primitive Pythagorean Triples

by

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Introduction

Pythagoras' famous theorem states that "the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides." When all three sides of such a triangle have integer lengths they form a "Pythagorean triple," $[a,b,c]$. If the three lengths have no common factor, i.e., if the right triangle which they represent is not similar to a smaller integer-sided right triangle, then we may refer to $[a,b,c]$ as a "primitive" Pythagorean triple, or "pPt".

The field of Pythagorean triples is rather well plowed. All the more striking then, that the only structure to the infinite collection of such triples which has heretofore been identified is one which can be found upon cursory examination – namely, that they must either be of the form [odd, even, odd] or [even, odd, odd] when ordered from smallest to largest (Sierpinski, 1962). Until now, no further subclassification has been accomplished.

In this paper a system of classification of pPts will be derived, and a method for generating all such triples from a set of three restricted parameters will be presented. It will be shown that each of the two "Classes", (generated by a parameter f) can be further divided into an infinite number of "Types", (generated by a parameter r) with each type containing an infinite number of triples (generated by a parameter m).

A triple's "Type" will be defined as the difference between the largest and second largest of the three numbers, or equivalently, as the difference in length between the corresponding triangle's hypotenuse and the longer of its two legs. Perhaps surprisingly, while an infinite number of Type-1 and Type-2 pPts can be found, there are no Type-3 pPts. In fact the next pPts are Type-8. This will be proved, and all of the allowed Types will be derived.

Tables 1 and 2 show the first 10 members of the first 8 Types in Class 1 and Class 2, respectively. The meaning of the term "Class" and of the parameters f, r, and m will be described in the following section.

r	*f*	**Type**	*m*	*a*	*b*	*c*
1	1	**1**	3	3	4	5
			5	5	12	13
			7	7	24	25
			9	9	40	41
			11	11	60	61
			13	13	84	85
			15	15	112	113
			17	17	144	145
			19	19	180	181
			21	21	220	221
3	1	**9**	11	33	56	65
			13	39	80	89
			17	51	140	149
			19	57	176	185
			23	69	260	269
			25	75	308	317
			29	87	416	425
			31	93	476	485
			35	105	608	617
			37	111	680	689
5	1	**25**	13	65	72	97
			17	85	132	157
			19	95	168	193
			21	105	208	233
			23	115	252	277
			27	135	352	377
			29	145	408	433
			31	155	468	493
			33	165	532	557
			37	185	672	697
7	1	**49**	17	119	120	169
			19	133	156	205
			23	161	240	289
			25	175	288	337
			27	189	340	389
			29	203	396	445
			31	217	456	505
			33	231	520	569
			37	259	660	709
			39	273	736	785
9	1	**81**	23	207	224	305
			25	225	272	353
			29	261	380	461
			31	279	440	521
			35	315	572	653
			37	333	644	725
			41	369	800	881
			43	387	884	965
			47	423	1064	1145
			49	441	1160	1241
11	1	**121**	27	297	304	425
			29	319	360	481
			31	341	420	541
			35	385	552	673
			37	407	624	745
			39	429	700	821
			41	451	780	901
			43	473	864	985
			45	495	952	1073
			47	517	1044	1165
13	1	**169**	33	429	460	629
			35	455	528	697
			37	481	600	769
			41	533	756	925
			43	559	840	1009
			45	585	928	1097
			47	611	1020	1189
			49	637	1116	1285
			51	663	1216	1385
			53	689	1320	1489
15	1	**225**	37	555	572	797
			41	615	728	953
			43	645	812	1037
			47	705	992	1217
			49	735	1088	1313
			53	795	1292	1517
			59	885	1628	1853
			61	915	1748	1973
			67	1005	2132	2357
			71	1065	2408	2633

Table 1. The first 10 members in each of the first 8 Types of **Class 1** pPts

r	f	Type	m	a	b	c
1	2	**2**	8	8	15	17
			12	12	35	37
			16	16	63	65
			20	20	99	101
			24	24	143	145
			28	28	195	197
			32	32	255	257
			36	36	323	325
			40	40	399	401
			44	44	483	485
2	2	**8**	10	20	21	29
			14	28	45	53
			18	36	77	85
			22	44	117	125
			26	52	165	173
			30	60	221	229
			34	68	285	293
			38	76	357	365
			42	84	437	445
			46	92	525	533
3	2	**18**	16	48	55	73
			20	60	91	109
			28	84	187	205
			32	96	247	265
			40	120	391	409
			44	132	475	493
			52	156	667	685
			56	168	775	793
			64	192	1015	1033
			68	204	1147	1165
4	2	**32**	22	88	105	137
			26	104	153	185
			30	120	209	241
			34	136	273	305
			38	152	345	377
			42	158	425	457
			46	184	513	545
			50	200	609	641
			54	216	713	745
			58	232	825	857
5	2	**50**	28	140	171	221
			32	160	231	281
			36	180	299	349
			44	220	459	509
			48	240	551	601
			52	260	651	701
			56	280	759	809
			64	320	999	1049
			68	340	1131	1181
			72	360	1271	1321
6	2	**72**	34	204	253	325
			38	228	325	397
			46	276	493	565
			50	300	589	661
			58	348	805	877
			62	372	925	997
			70	420	1189	1261
			74	444	1333	1405
			82	492	1645	1717
			86	516	1813	1885
7	2	**98**	36	252	275	373
			40	280	351	449
			44	308	435	533
			48	336	527	625
			52	364	627	725
			60	420	851	949
			64	448	975	1073
			68	476	1107	1205
			72	504	1247	1345
			76	532	1395	1493
8	2	**128**	42	336	377	505
			46	368	465	593
			50	400	561	689
			54	432	665	793
			58	464	777	905
			62	496	897	1025
			66	528	1025	1153
			70	560	1161	1289
			74	592	1305	1433
			78	624	1457	1585

Table 2. The first 10 members in each of the first 8 Types of **Class 2** pPts

Parameterization and Classification

It has been known for over 2000 years (Neugebauer, 1945) that all pPts can be expressed in the form

$$a = 2xy$$
$$b = x^2 - y^2 \qquad (1)$$
$$c = x^2 + y^2$$

where the parameters x and y are positive integers with no common factors, opposite parity, and $x > y$. Since this discovery, numerous fascinating properties of pPts have been demonstrated (Koblitz, 1993). But until now, the above parameterization has not been improved upon, despite the fact that it provides little structure to the infinity of pPts.

The difference in length between the hypotenuse and the longer leg is a most useful parameter in the structure of pPts, and one which is subject to several important restrictions. It is this observation which has yielded the following parameterization and classification system. In the proposed scheme, a will always represent the smaller leg, b the larger leg, and, c will represent the hypotenuse. (It should be noted that this is not the case with the parameterization of equations (1). For example, in order to generate [3,4,5] one must set $(x,y) = \frac{3}{\sqrt{2}}, \frac{1}{\sqrt{2}}$. $(x,y) = (2,1)$ generates [4,3,5].)

The system proposed herein is based directly upon the value of $c - b$, which will define the pPts' "Type". The first pPts which most people discover are of the Type $c - b = 1$, henceforth "Type-1". Of these, the first is [3,4,5], the second is [5,12,13], and as will be proved in the next section, the n-th Type-1 pPt is

$$[a,b,c] = [2n + 1, 2n(n + 1), 2n(n + 1) + 1]. \qquad (2)$$

Naturally, each Type-1 pPt is similar to a non-primitive Pythagorean triple with $c - b = 2$: [6,8,10], [10,24,26], etc. But, between each of these doubled Type-1's there is a true (primitive) Type-2; [8,15,17], [12,35,37], etc. The n-th Type-2 pPt is

$$[a,b,c] = [4n + 4, 4n^2 + 8n + 3, 4n^2 + 8n + 5]. \qquad (3)$$

There are no Type-3 pPts. In fact, the next pPts are Type-8, and are of the form

$$[a,b,c] = [8n + 12, 4n^2 + 12n + 5, 4n^2 + 12n + 13]. \qquad (4)$$

Derivation and Proof

To prove equations (2), (3), and (4) and derive the more general parameterization from which they can be extracted, we first consider odd Types. Replacing c in the fundamental equation $a^2 + b^2 = c^2$ with $b + T$ and solving for a, one finds

$$a = T\sqrt{1 + \frac{2b}{T}}. \qquad (5)$$

From this expression it is clear that for a to be an integer, either 1) T must be a perfect square (so that the denominator can be moved out of the square root), or 2) b and T must have a common factor which, after reducing the second term in the square root, leaves a denominator which is a perfect square. But in case 2), the prefactor of T insures that a will also have that factor, yielding a non-primitive triple. Therefore, if T is odd it must be a perfect square.

Next consider even Types. Setting $c - b = T = 2T'$ and solving for a in terms of b and T' as before, one finds

$$a = 2T'\sqrt{1 + \frac{b}{T'}}. \tag{6}$$

By the same logic as above, T' must be a perfect square in order for the triple to be both primitive and Pythagorean.

Thus, the allowed Types of pPts fall into two "Classes": either the Type equals 2 times a perfect square (2,8,18,32...) or it equals the square of an odd number (1,9,25,49...);

$$\text{Type}(T) = c - b = fr^2 \tag{7}$$

where (f,r) = (1,odd) or (2,any), and have been chosen to stand for "factor" and "root". Note that the restrictions on the possible values of $c - b$ merely eliminate non-primitive Pythagorean triples.

Setting $c = b + fr^2$ in the Pythagorean theorem and solving for b as a function of a, f, and r yields $b = \dfrac{a^2 - f^2r^4}{2fr^2}$. From this expression it is evident that a must be divisible by r in order for b to be an integer. Defining $m = \dfrac{a}{r}$ yields the following expressions for a, b, and c in terms of the integer parameters m, r, and f:

$$\begin{aligned} a &= mr \\ b &= \frac{m^2 - (fr)^2}{2f} \\ c &= \frac{m^2 + (fr)^2}{2f} \end{aligned} \tag{8}$$

where the requirement that $a < b$ restricts the possible values of m, such that

$$m > rf\left(1 + \sqrt{2}\right). \tag{9}$$

Equations (2), (3), and (4) follow immediately from the above equations upon setting $f = 1$ and $r = 1$ for equation (2), $f = 2$ and $r = 1$ for equation (3), or $f = 2$ and $r = 2$ for equation (4).

Each of the infinite number of Types contains an infinite number of members, which can be generated by varying the parameter m, subject to the following further restrictions:

- Values of m which have a common factor with r/f are disallowed, since they yield non-primitive triples. For example, when generating Type-225 pPts ($f = 1$; $r = 15$), m's which are divisible by 3 or 5 are disallowed, since they yield similar triples to those in Type-25 or Type-9, respectively.
- If $f = 2$ ("Class 2"), odd r's require m's which are divisible by 4, and even r's require m's which are not divisible by 4 (6,10,14,...) in order to avoid non-primitive triples.

From this, there follows directly the fact that Class 1 primitive Pythagorean triples $[a,b,c]$ are [odd,even,odd], with m and r always odd, while Class 2 triples are [even,odd,odd], with m and r always even. This is consistent with the well known fact that all pPt hypotenuses are of the form $4n + 1$, where n is a positive integer such that $4n + 1$ is either prime or a product of such primes. For example 65 is allowed (and is the hypotenuse of the smallest Type-9 pPt) because 5 and 13 are both of the form $4n + 1$, but 115 is not allowed because one of its factors, 23, is not of the required form. The converse of this statement is one of Fermat's theorems, that every prime of the form $4n + 1$ is a primitive hypotenuse (Chrystal, 1964).

The author wishes to acknowledge the helpful suggestions of Dr. Richard MacKenzie.

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Lucky Larry #25

Larry ignored the need to combine only like terms and still got the right answer on this test problem.

$$6 = 2a + 4$$

$$6 = 6a$$

$$1 = a$$

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Identifying Degenerate Conic Sections

by

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Introduction

A standard result in plane analytic geometry asserts that if A, B, C, D, E, and F are constants such that not all of A, B, and C are 0, then the graph of the quadratic equation in two variables, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, is a conic section: see [Smith, 1956, article 167, page 226]. The "conic sections" referred to in this well-known statement include not only parabolas, ellipses, and hyperbolas, but a number of so-called "degenerate" cases. For example, the graph may consist of a point (the real graph of $x^2 + y^2 = 0$ consists of just the origin); a single line (the graph of $x^2 - 8xy + 16y^2 = 0$, or $(x - 4y)^2 = 0$, is the line $y = \frac{1}{4}x$); or a pair of lines (the graph of $x^2 - 9y^2 = 0$, or $(x - 3y)(x + 3y) = 0$, consists of the lines $y = \frac{1}{3}x$ and $y = -\frac{1}{3}x$).

Moreover, the real graphs of such equations may be empty. Examples such as $x^2 + y^2 + 4 = 0$ and $x^2 - 8xy + 16y^2 + 1 = 0$ show that empty real graphs may arise when the discriminant, $B^2 - 4AC$, is negative or 0. Changing the constant term F may produce nonempty real graphs, as in $x^2 + y^2 - 4 = 0$ and $x^2 - 8xy + 16y^2 - 1 = 0$, even though changing F does not change the discriminant, $B^2 - 4AC$. Such complications make it desirable to have a numerical or algebraic criterion for degeneracy which involves coefficients in addition to A, B, and C.

For graphs in which the coordinates of points may be complex numbers, such a degeneracy criterion has long been known. Indeed, [Smith, 1956, condition (iii) in article 37, page 43] states (using the above notation) that the graph in the complex plane of a quadratic equation of the above type is degenerate if and only if $\delta = 4ACF + BED - CD^2 - FB^2 - AE^2$ vanishes. In Smith's argument, the condition $\delta = 0$ is shown to follow algebraically if one assumes that $Ax^2 + Bxy + Cy^2 + Dx + Ey + F$ factors as $(p_1x + q_1y + r_1)(p_2x + q_2y + r_2)$. It was appropriate for Smith to analyze this type of factorization because, in the "complex" context, "degenerate" conics consist of either one or two lines. In the complex plane, graphs of the above type cannot be empty or reduce to a point. Nevertheless, by modifying the approach of [Smith, 1956, page 43], we have the following useful result.

Theorem (Criterion for Degeneracy)

Let A, B, C, D, E, and F be real numbers such that not all of A, B, and C are 0. Put $\delta = 4ACF + BED - CD^2 - FB^2 - AE^2$. Let Γ denote the real graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Assume that Γ is nonempty. Then Γ is degenerate (that is, is either a line, a pair of lines, or a point) if and only if $\delta = 0$.

How do you use the above "useful result"? The main purpose of this paper is to explain how to identify which type of degeneracy (point, line, pair of lines) is exhibited by a nonempty graph with equation whose $\delta = 0$. The seven examples given below show how to carry out the general process. In broad outline, our method is to solve for one variable in terms of the other and then use a graphing utility to graph these two equations as the two halves of the underlying degenerate conic section. Not only can this discussion be used as classroom enrichment material for the quadratic formula and complex numbers, but our methods also reinforce a popular way of using graphing calculators to graph the two halves of nondegenerate conic sections.

Examples

In each of the following examples, we determine whether the graph of the given equation is empty or degenerates to a point, a line, or a pair of lines. In the cases of degenerate graphs, we determine the point or an equation of each line. We begin with an example of two intersecting lines.

Example 1 $4x^2 - y^2 + 24x + 36 = 0$

Solution Let's apply the criterion in the above Theorem, with $A = 4$, $B = 0$, $C = -1$, $D = 24$, $E = 0$, and $F = 36$. The result is $\delta = 4ACF + BED - CD^2 - FB^2 - AE^2 = -576 + 0 - (-576) - 0 - 0 = 0$. Thus, the graph is either empty or degenerate (a point, a line, or a pair of lines).

Is it empty? No, it is not, because you can verify that $(-3, 0)$ lies on the graph. Thus, the graph is either a line or a pair of lines.

Of course, you cannot expect to always be lucky enough to guess a point on the graph. So, we proceed differently, by solving for y:

$$4x^2 - y^2 + 24x + 36 = 0$$
$$y^2 = 4x^2 + 24x + 36$$
$$y = \pm\sqrt{4x^2 + 24x + 36}.$$

How could we recognize these as equations of a line or a pair of lines?

The answer is to be found by factoring $4x^2 + 24x + 36$ as $4(x + 3)^2$. (In general, if r_1 and r_2 are the roots of the quadratic equation $ax^2 + bx + c = 0$, then we have the factorization $ax^2 + bx + c = a(x - r_1)(x - r_2)$.) Therefore,

$$y = \pm\sqrt{4x^2 + 24x + 36} = \pm\sqrt{4(x + 3)^2} = \pm 2(x + 3).$$

Thus, the graph consists of the lines $y = 2(x + 3)$ and $y = -2(x + 3)$. Notice that these two lines intersect at $(-3, 0)$, the point guessed earlier. ◀

You could also have found the above equations for the lines $y = 2(x + 3)$ and $y = -2(x + 3)$, by completing squares, rewriting the given equation as $4(x + 3)^2 - y^2 = 0$. This also leads to $y^2 = 4(x + 3)^2$ and $y = \pm\sqrt{4(x + 3)^2} = \pm 2(x + 3)$. However, the "completing squares" method is less effective in general, as will be seen in the next two examples which have nonzero coefficients of xy. The graphs in Examples 2 and 3 degenerate to two parallel lines and a single line, respectively.

Example 2 $4x^2 - 4xy + y^2 - 6x + 3y = 0$

Solution We find $\delta = 4ACF + BED - CD^2 - FB^2 - AE^2 = 0 + 72 - 36 - 0 - 36 = 0$, and so the graph is either empty, a point, a line, or a pair of lines. To see which possibility holds, rewrite the given equation as $y^2 + (-4x + 3)y + (4x^2 - 6x) = 0$, and use the quadratic formula to solve for y:

$$y = \frac{-(-4x + 3) \pm \sqrt{(-4x + 3)^2 - 4(1)(4x^2 - 6x)}}{2 \cdot 1}$$

$$= \frac{4x - 3 \pm \sqrt{9}}{2} = \frac{4x - 3 \pm 3}{2} = 2x,\ 2x - 3.$$

We see that the graph consists of the parallel lines given by $y = 2x$ and $y = 2x - 3$. ◀

Example 3 $x^2 - 4xy + 4y^2 + 2x - 4y + 1 = 0$

Solution Here, $\delta = 4ACF + BED - CD^2 - FB^2 - AE^2 = 16 + 32 - 16 - 16 - 16 = 0$, and so the graph is either empty, a point, a line, or a pair of lines. Rewriting the given equation as $4y^2 + (-4x - 4)y + (x^2 + 2x + 1) = 0$, we find via the quadratic formula that

$$y = \frac{-(-4x - 4) \pm \sqrt{(-4x - 4)^2 - 4(4)(x^2 + 2x + 1)}}{2 \cdot 4}$$

$$= \frac{4x + 4 \pm \sqrt{0}}{8} = \frac{4x + 4 \pm 0}{8} = \frac{x+1}{2}, \frac{x+1}{2}.$$

We see that the graph is the line $y = \frac{x+1}{2} = \frac{1}{2}x + \frac{1}{2}$. ◀

Next, we give an example of a graph which degenerates to a point.

Example 4 $x^2 + y^2 - 2x - 6y + 10 = 0$

Solution Here, $\delta = 4ACF + BED - CD^2 - FB^2 - AE^2 = 40 + 0 - 4 - 0 - 36 = 0$, and so the graph is either empty, a point, a line, or a pair of lines. In fact, it is a point! Rewriting the given equation as $y^2 + (-6)y + (x^2 - 2x + 10) = 0$, we find via the quadratic formula that

$$y = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(x^2 - 2x + 10)}}{2 \cdot 1}$$
$$= \frac{6 \pm \sqrt{-4x^2 + 8x - 4}}{2}$$
$$= \frac{6 \pm \sqrt{-4(x^2 - 2x + 1)}}{2}$$
$$= \frac{6 \pm \sqrt{-4(x-1)^2}}{2} = \frac{6 \pm 2(x-1)i}{2} = 3 \pm (x-1)i.$$

The only real values of x and y which satisfy this equation are $x = 1$ and $y = 3$. Thus, the graph consists of just the point (1, 3). Of course, you could also have discovered this fact by completing the squares and rewriting the given equation as $(x - 1)^2 + (y - 3)^2 = 0$. ◀

The next two examples present empty graphs. In Example 5, the discriminant $B^2 - 4AC$ is 0, while in Example 6, $B^2 - 4AC < 0$.

Example 5 $x^2 - 2xy + y^2 + 4 = 0$

Solution Here, $\delta = 4ACF + BED - CD^2 - FB^2 - AE^2 = 16 + 0 - 0 - 16 - 0 = 0$, and so the graph is either empty, a point, a line, or a pair of lines. In fact, it is empty! To see this, rewriting the given equation as $y^2 + (-2x)y + (x^2 + 4) = 0$, we find via the quadratic formula that

$$y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(x^2 + 4)}}{2 \cdot 1}$$
$$= \frac{2x \pm \sqrt{-16}}{2} = \frac{2x \pm 4i}{2} = x \pm 2i.$$

We see that the underlying equations for y involve the nonreal complex number $2i$. In particular, no real values of x and y satisfy the given equation, and so the (real) graph is empty. ◀

Example 6 $x^2 + y^2 - 2x - 6y + 12 = 0$

Solution Here, $\delta = 4ACF + BED - CD^2 - FB^2 - AE^2 = 48 + 0 - 4 - 0 - 36 = 8$, which is nonzero. Thus, the graph does *not* degenerate to a point, a line, or a pair of lines. The graph is then either a nondegenerate conic section (parabola, ellipse, or hyperbola) or empty. Which is it? As in the above examples, let's solve for y:

$$y = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(x^2 - 2x + 12)}}{2 \cdot 1}$$
$$= 3 \pm \sqrt{-x^2 + 2x - 3}.$$

Now, the real quadratic $-x^2 + 2x - 3$ is negative for all real x since *its* discriminant, $2^2 - 4(-1)(-3) = -8$, is negative and its parabolic graph opens downward. Thus, as in Example 5, since the equations for y involve a nonreal complex number, the real graph is empty. ◀

Another way to solve Example 6 is to complete the squares, since $x^2 + y^2 - 2x - 6y + 12 = (x - 1)^2 + (y - 3)^2 + 2 = 0$ is not satisfied by any real numbers x and y.

If $\delta \neq 0$, there are two possibilities: (1) the real graph Γ is empty; (2) Γ is nonempty and nondegenerate. Example 6 illustrates the first of these possibilities, while the next example illustrates the second.

Example 7 $x^2 + y^2 - 4 = 0$

Solution Here, $\delta = 4ACF + BED - CD^2 - FB^2 - AE^2 = -16 + 0 - 0 - 0 - 0 = -16$, which is nonzero. Thus, the graph does *not* degenerate to a point, a line, or a pair of lines. Hence, it is either empty or a nondegenerate conic section. The graph is not empty, as it contains the point (0, 2). In fact, the graph is the circle with center at the origin and radius 2.

Closing Remarks

One cannot ignore the hypothesis in the above Theorem that the real graph Γ is nonempty. For instance, consider the equation $x^2 + y^2 + 4 = 0$. Here, Γ is empty because $x^2 + y^2 + 4 \geq 4$ for any real numbers x and y. However, for this equation, $\delta = 4ACF + BED - CD^2 - FB^2 - AE^2 = 16 + 0 - 0 - 0 - 0 = 16$, which is nonzero. Thus, a nonzero value of δ does *not* ensure a nonempty real graph.

However, the question whether any given real graph of the above type is empty can be settled by using the quadratic formula. For instance, consider $x^2 + xy + y^2 + 5x - 5y - 5 = 0$. Proceeding as above, we solve for y, finding

$$y = \frac{-x + 5 \pm \sqrt{-3x^2 - 30x + 45}}{2} = \frac{-x + 5 \pm \sqrt{-3(x^2 + 10x - 15)}}{2}.$$

A graphing calculator quickly reveals that the real graph is nonempty (in fact, a parabola). To reach this conclusion algebraically, modify the reasoning given in Example 6, as follows. Notice that the real quadratic $-3x^2 - 30x + 45$ has two unequal real roots since *its* discriminant, $(-30)^2 - 4(-3)(45) = 1440$, is positive; hence, *its* parabolic graph is sometimes above the x-axis, for instance at $x = 1$. Now, $x = 1$ leads to $y = \frac{-1 + 5 \pm \sqrt{-3(1^2 + 10(1) - 15)}}{2} = \frac{4 \pm \sqrt{-3(-4)}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$ in the above equations, and so $(1, 2 + \sqrt{3})$ and $(1, 2 - \sqrt{3})$ each lie on the graph of $x^2 + xy + y^2 + 5x - 5y - 5 = 0$.

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The importance of group theory was emphasized very recently when some physicists using group theory predicted the existence of a particle that had never been observed before, and described the properties it should have. Later experiments proved that this particle really exists and has those properties.

Irving Adler

Lucky Larry #26

A common error and a little "one-sidedness" leads Larry to a correct answer.

$$
\begin{aligned}
14 - 3(p + 2) &< -5p \\
14 - 3p + 6 &< -5p \\
-6 \qquad -6 & \\
8 - 3p &< -5p \\
+3p \qquad &+3p \\
8 &< -2p \\
-4 &> p
\end{aligned}
$$

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Summation of Arithmetic Sequences of Higher Degree

by

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An arithmetic sequence of higher degree is a sequence whose mth term is a polynomial of degree greater than one. For example, $a_m = 8m^3 - 6m^2 + 1$ defines an arithmetic sequence of degree three. To find the sum of such a sequence, calculus books (see (Anton, 1995) for example) usually make use of the sum formulas of $\sum_{m=1}^{n} 1$, $\sum_{m=1}^{n} m$, $\sum_{m=1}^{n} m^2$, $\sum_{m=1}^{n} m^3$, ... However, if the degree is higher than three, the problem becomes challenging because the sums of $\sum_{m=1}^{n} m^4$, $\sum_{m=1}^{n} m^5$, ... are not given in most of these books.

In this note we will discuss two other approaches, which not only can handle these problems, but also may be used to derive the sum formula of any power of the natural numbers. In particular, we will use them to find the sum formula of $\sum_{m=1}^{n} m^4$.

The first approach uses the following set of sum formulas:

$$\sum_{m=1}^{n} 1 = n$$

$$\sum_{m=1}^{n} m = \frac{n(n+1)}{2}$$

$$\sum_{m=1}^{n} m(m+1) = \frac{n(n+1)(n+2)}{3}$$

$$\sum_{m=1}^{n} m(m+1)(m+2) = \frac{n(n+1)(n+2)(n+3)}{4}, \ldots .$$

While these formulas (Jolley, 1961) are not as well known as the previous ones, they have the advantage of being easier to remember because of the pattern involved. To apply them, we need to rewrite the mth term of the sequence as a linear combination of 1, m, $m(m+1)$, $m(m+1)(m+2)$, ... , using undetermined coefficients. In our example of $a_m = 8m^3 - 6m^2 + 1$, we write

$$8m^3 - 6m^2 + 1 = 8m(m+1)(m+2) + bm(m+1) + cm + d.$$

If $m = 0$, then $1 = d$.

If $m = -1$, then $-13 = -c + d \Rightarrow c = 14$.

If $m = -2$, then $-87 = 2b - 2c + d \Rightarrow b = -30$.

$$\text{Hence} \sum_{m=1}^{n} (8m^3 - 6m^2 + 1) = \sum_{m=1}^{n} [8m(m+1)(m+2) - 30m(m+1) + 14m + 1]$$

$$= 8 \cdot \frac{n(n+1)(n+2)(n+3)}{4} - 30 \cdot \frac{n(n+1)(n+2)}{3} +$$

$$14 \cdot \frac{n(n+1)}{2} + n$$

and $\sum_{m=1}^{n} (8m^3 - 6m^2 + 1) = n^2(2n^2 + 2n - 1)$.

Now we will use the same method to derive the sum formula for $\sum_{m=1}^{n} m^4$.

If $m^4 = m(m+1)(m+2)(m+3) + bm(m+1)(m+2) + cm(m+1) + dm$, then

$m = -1 \Rightarrow 1 = -d \Rightarrow d = -1$,

$m = -2 \Rightarrow 16 = 2c - 2d \Rightarrow c = 7$,

$m = -3 \Rightarrow 81 = -6b + 6c - 3d \Rightarrow b = -6$.

$$\text{Thus} \sum_{m=1}^{n} m^4 = \sum_{m=1} [m(m+1)(m+2)(m+3) - 6m(m+1)(m+2) + 7m(m+2) - m]$$

$$= \frac{n(n+1)(n+2)(n+3)(n+4)}{5} - 6 \cdot \frac{n(n+1)(n+2)(n+3)}{4} +$$

$$7 \cdot \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2}.$$

We then obtain

$$\sum_{m=1}^{n} m^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

The formulas we used here have the following general form:

$$\sum_{m=1}^{n} m(m+1)(m+2)\cdots(m+r-1) = \frac{n(n+1)\cdots(n+r-1)(n+r)}{r+1}. \qquad (*)$$

To prove our claim, let us denote the right hand side by S_n. Then

$$S_m - S_{m-1} = \frac{m(m+1)(m+2)\cdots(m+r-1)(m+r)}{r+1} - \frac{(m-1)m(m+1)\cdots(m+r-2)(m+r-1)}{r+1}.$$

After simplifying the expression above, we get

$$S_m - S_{m-1} = m(m+1)(m+2)\cdots(m+r-1).$$

If we sum each side from 1 to n, we get the equation denoted by (*).

The second method is based on the fact that the sum of an arithmetic sequence of degree k is a polynomial of degree $(k + 1)$.

For example, $\sum_{m=1}^{n} m^4 = an^5 + bn^4 + cn^3 + dn^2 + en$.

Since $n^4 = \sum_{m=1}^{n} m^4 - \sum_{m=1}^{n-1} m^4$,

$n^4 = an^5 + bn^4 + cn^3 + dn^2 + en - [a(n-1)^5 + b(n-1)^4 + c(n-1)^3 + d(n-1)^2 + e(n-1)]$.

Hence,

$n^4 = 5an^4 + (-10a + 4b)n^3 + (10a - 6b + 3c)n^2 + (-5a + 4b - 3c + 2d)n + (a - b + c - d + e)$.

If we compare the coefficients of the corresponding powers of n, we find that

$$1 = 5a \quad \Rightarrow \quad a = \frac{1}{5}$$

$$0 = -10a + 4b \quad \Rightarrow \quad b = \frac{1}{2}$$

$$0 = 10a - 6b + 3c \quad \Rightarrow \quad c = \frac{1}{3}$$

$$0 = -5a + 4b - 3c + 2d \quad \Rightarrow \quad d = 0$$

$$0 = a - b + c - d + e \quad \Rightarrow \quad e = -\frac{1}{30}.$$

Hence, $\sum_{m=1}^{n} m^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$.

After factoring, we get

$$\sum_{m=1}^{n} m^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

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> Sure, some [teachers] could give the standard limit definitions, but they [the students] clearly did not understand the definitions - and it would be a remarkable student who did, since it took mathematicians a couple of thousand years to sort out the notion of a limit, and I think most of us who call ourselves professional mathematicians really only understand it when we start to teach the stuff, either in graduate school or beyond.
>
> Keith Devlin

Lucky Larry #27

This problem was worth three points on a test, with one point deducted for each mistake. How do you explain to Larry that he received 0 points, but still got the correct answer?!?

$$\begin{aligned}
8-3(4x+2) &= 2(8-3x)+4\\
8-12x+6 &= 16-6x+4\\
14-12x &= 20-6x\\
+\ 6x \quad & \quad +6x\\
\hline
14-18x &= 20\\
-14 \quad & \quad -14\\
\hline
\frac{-18x}{-18} &= \frac{6}{-18}\\
x &= -3
\end{aligned}$$

Submitted by Jane Covillion
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An Unexpected Proof of An Unexpected Occurrence of e

by

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Glendale CA 91208

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Steve Marsden is Professor of Mathematics and currently Chair of the Math Division at Glendale College, where he has served on the faculty since 1975. He received both his A.B. and M.A.T. in Mathematics at UCLA.

"No, not Northward; upward;
out of Flatland altogether." (Abbott, 1884)

Introduction

In 1748, Leonhard Euler, in his *Introductio in Analysin Infinitorum*, introduced the symbol e as representing the infinite series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$, a special case of the Maclaurin series for e^x; he provided a decimal expansion of e to a remarkable accuracy of 23 decimal places (Dunham, 1994)! A most revered mathematical constant, e is found in applications such as growth and decay models, logistic and probability models, and continuous compound interest. To a mathematician, the result $\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = e$ is expected, but quite profound, while $\frac{d}{dx}(e^x) = e^x$ represents the ultimate in functional stability. And Euler's $e^{i\pi} + 1 = 0$ (the personalized license plate of one of the authors), one of the most famous of all mathematical formulas (Maor, 1994), combines e with the other four "holy" numbers of mathematics.

Such a wonderful irrational number, e occurs in a number of unexpected places:

- If numbers are randomly chosen from the interval $[0, 1]$ until a number is selected that is less than the one that preceded it, the expected number of selections is e.

- If a large group of people check their hats at a restaurant and later have them randomly returned, the probability that no one ends up with their own hat approaches $1 - \frac{1}{e}$.
- An employer must choose or reject an applicant for a job at the time of interview. To maximize the probability of choosing the best of all applicants, it can be shown that the employer should begin by initially interviewing approximately $\frac{1}{e}$ (as a percent) of the applicants, and then choose the **first** one thereafter who, based on some criteria, is superior to all of those initially interviewed; using this procedure, the maximum probability of choosing the best applicant is approximately $\frac{1}{e}$ (Leonard & Shultz, 1989).
- Using *e*'s eternal alter-ego, $\ln x$, one unexpectedly finds that the proportion of numbers less than x which are prime, when x is a very large natural number, is approximately $\frac{1}{\ln x}$ (Dunham, 1994).

This article investigates yet another unexpected occurrence of *e*: **if numbers are randomly chosen from the interval [0, 1], the expected number of selections necessary until the sum of the randomly chosen numbers first exceeds one (a trial) is *e*.** A computer simulation of 1000 trials suggests the average number of selections is about 2.7 (program available from the authors). This unexpected occurrence of *e* appeared in The 1961 William Lowell Putnam mathematical competition (Bush, 1961). Moreover, a proof of the convergence to *e* for the same criteria, but for numbers randomly selected (with replacement) from the set $\left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\right\}$, has also been established (Shultz, 1979). This article uses a different approach than the Putnam proof to establish convergence to *e*, with some interesting, unexpected results occurring along the way.

The Proof

The probability that the sum will first exceed 1 for 1 selection is clearly 0. The probability that the sum will first exceed 1 for 2 selections can be thought of geometrically in $\mathbf{R}^2$ (Flatland) as the area in the unit square above the line $y = 1 - x$ as shown in Figure 1. While this can be computed directly as the area of a triangle,

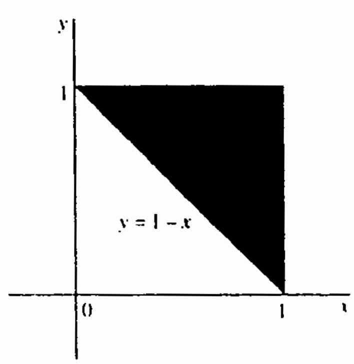

Figure 1

we use integration to establish a pattern:

$$p(x + y > 1) = \int_0^1 (1 - x)dx = \frac{1}{2!}.$$

We now leave Flatland altogether. The probability that the sum will **first** exceed 1 for 3 selections can be thought of geometrically in $\mathbf{R}^3$ as the volume **inside** the unit cube and **outside** the tetrahedron, but above the triangular area enclosed by the x-axis, the y-axis, and the line $y = 1 - x$ in the xy-plane (see Figure 2). While the volume can be computed directly, we again use integration to continue our pattern:

$$p(x + y + z > 1 \cap x + y \le 1) = \left(1 - \frac{1}{2}\right) - \int_0^1\int_0^{1-x}(1 - x - y)dydx = \frac{1}{2!} - \frac{1}{3!} = \frac{2}{3!}.$$

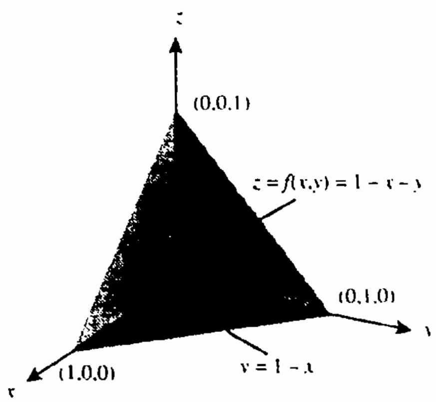

Figure 2

The probability that the sum will first exceed 1 for 4 selections can be thought of geometrically in $\mathbf{R}^4$ as the hyper-volume "inside" the hyper-unit cube, "outside" the hyper-tetrahedron, but above the tetrahedron:

$$p(x + y + z + w > 1 \cap x + y + z \le 1) = \left(\frac{1}{3!}\right) - \int_0^1\int_0^{1-x}\int_0^{1-x-y}(1 - x - y - z)dzdydx = \frac{1}{3!} - \frac{1}{4!} = \frac{3}{4!}.$$

Note that in the preceding three examples, the number of integrals used (n) is one less than the number of selections needed (s) for the sum to first exceed 1; that is, $s - 1 = n$ or $s = (n + 1)$.

We now generalize the process:

$$I = \int_0^1\int_0^{1-x_1}\int_0^{1-x_1-x_2}\cdots\int_0^{1-x_1-x_2-\cdots-x_{n-1}}(1 - x_1 - \dots - x_n)dx_n dx_{n-1} \dots dx_1 \text{ (n integrals).}$$

Examining the innermost integral: Let $u = 1 - x_1 - \dots - x_n$, and $du = -dx_n$. Substituting, and making the appropriate changes in the limits gives us

$$\int_0^{1-x_1-x_2-\dots-x_{n-1}} (1-x_1-\dots-x_n)dx_n = -\int_{1-x_1-x_2-\dots-x_{n-1}}^{0} u\,du = \frac{(1-x_1-\dots-x_{n-1})^2}{2}.$$ Using the substitution $u = 1 - x_1 - x_2 - \dots - x_{n-1}$ and making the appropriate changes in the limits for the second innermost integral yields

$$\int_0^{1-x_1-\dots-x_{n-2}} \frac{(1-x_1-\dots-x_{n-1})^2}{2} dx_{n-1} = \frac{(1-x_1-\dots-x_{n-2})^2}{2(3)} .$$

Repeating this process another $n-2$ times results in $I = \frac{1}{(n+1)!}$. Since $s = (n+1)$, $I = \frac{1}{s!}$. Moreover, $p(x_1 + \dots + x_n > 1 \cap x_1 + \dots + x_{n-1} < 1) = \frac{1}{(s-1)!} - \frac{1}{s!} = \frac{(s-1)}{s!}$ because the probability is restricted to the "hyper-volume" above the "hyper-tetrahedron" one dimension lower.

The results are summarized in the following table, along with the results from a computer simulation of 1000 trials (program available from the authors). (Probabilities do not sum to 1.0 due to rounding):

s_i = Number of selections required to exceed 1	p_i = Probability	Computer Simulation
1	$\frac{0}{1!}$	0
2	$\frac{1}{2!} = .5$	.52
3	$\frac{2}{3!} = .33\overline{3}$	.32
4	$\frac{3}{4!} = .125$	.14
5	$\frac{4}{5!} = .03\overline{3}$	.03
6	$\frac{5}{6!} = .007$	.01
.	.	
.	.	
.	.	
s	$\frac{(s-1)}{s!}$	

Table 1

The expected number of selections until the sum exceeds 1 is therefore:

$$E(s) = \sum_{i=1}^{\infty} s_i p_i = 1\left(\frac{0}{1!}\right) + 2\left(\frac{1}{2!}\right) + 3\left(\frac{2}{3!}\right) + 4\left(\frac{3}{4!}\right) + 5\left(\frac{4}{5!}\right) + 6\left(\frac{5}{6!}\right) + \ldots =$$

$$0 + 1 + \frac{2}{2!} + \frac{3}{3!} + \frac{4}{4!} + \frac{5}{5!} + \ldots =$$

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots = e,$$

as given by Euler in 1748. It is also given by the Maclaurin series for e^x,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

when $x = 1$.

Conclusion

The universal constant e, a single number in the infinitude of reals, is of singular importance. Not only is it a number deeply involved in the continuous processes of nature, but one that ties together many different areas of mathematics – from the law of continuous compound interest to the catenary, from the area under a hyperbola to Euler's famous formula $e^{i\pi} + 1 = 0$, from the inner structure of a nautilus shell to Bach's musical scales (Maor, 1994). When you least expect it, there it is – underlying seemingly random events, implying predictability in what appears to be unpredictable, showing up on license plates and within the laws of nature. This time, we found it by the least expected of techniques: continually going upward in dimension.

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> The purpose of education... is to create in a person the ability to look at the world for himself, to make his own decisions.
>
> James Baldwin

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SHORT COMMUNICATIONS

Another Rotation of Axes

by

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Richard Quint is a professor at Ventura College in Ventura, CA. He has a B.S. from Caltech and an M.A. from the University of Maryland.

Deriving the formula for the *Rotation of Axes* can provide a nice and straightforward application of many of the seemingly unrelated topics covered in a precalculus course – the definitions and identities of trigonometry, systems of linear equations, and matrix algebra. It is also a fine lesson in drawing a diagram, labeling its components, and writing down relations observed. Here is yet another way to derive the formula.

Suppose we draw the usual diagram of a point P whose coordinates are (x, y) in the original Cartesian plane and (x', y') with respect to axes rotated $0 < \theta < \frac{\pi}{2}$ in the in the counterclockwise direction. If P is in the first quadrant of both coordinate systems, we may drop perpendiculars from P to each of the four axes forming two pairs of congruent right triangles (all of which are similar), each of which has one side whose length is one of the coordinates and an angle which is θ. We can also label the other sides of the triangles to keep track of the various lengths. (The diagram and the labeling may change, depending on the size of θ and the location of P, but each possibility leads to an equivalent pair of equations.)

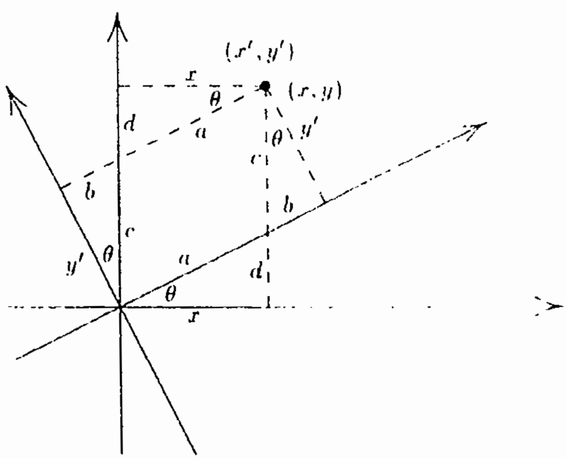

Figure 1.

Let us remember what we want: either x and y in terms of x', y', and θ or x' and y' in terms of x, y, and θ. It is easy to see $x' = a + b$ and $y = c + d$. The definitions of secant and tangent give us

$$\frac{a}{x} = \sec\theta \qquad \frac{b}{y'} = \tan\theta \qquad \frac{c}{y'} = \sec\theta \qquad \frac{d}{x} = \tan\theta.$$

Substituting into our equations for x' and y we get $x' = x\sec\theta + y'\tan\theta$ and $y = y'\sec\theta + x\tan\theta$. This looks complicated to untangle, but let us write them with x' and y' on the left and x and y on the right.

$$x' - y'\tan\theta = x\sec\theta$$
$$y'\sec\theta = -x\tan\theta + y.$$

Now let us see the effect of choosing the right notation. In matrix form these equations are

$$\begin{bmatrix} 1 & -\tan\theta \\ 0 & \sec\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \sec\theta & 0 \\ -\tan\theta & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We can quickly solve for either set of coordinates in terms of the other. Just find the inverse of one of the coefficient matrices and multiply it times the other coefficient matrix. If we find inverses using the identity, for a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

we may be reminded that the determinant of a triangular matrix is the product of the diagonal entries and we can use the identity $\sec^2\theta - \tan^2\theta = 1$ when we calculate x' and y' in terms of x and y.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{\sec\theta} \begin{bmatrix} \sec\theta & \tan\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sec\theta & 0 \\ -\tan\theta & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \frac{1}{\sec\theta} \begin{bmatrix} \sec^2\theta - \tan^2\theta & \tan\theta \\ -\tan\theta & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We can use our matrix equation to easily calculate the transformed coordinates of a point, *e.g.*, the new coordinates of $P = (2, 5)$ after the axes are rotated through an angle of $\frac{\pi}{6}$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{5+2\sqrt{3}}{2} \\ \frac{-2+5\sqrt{3}}{2} \end{bmatrix} \approx \begin{bmatrix} 4.23 \\ 3.33 \end{bmatrix}.$$

An Alternate to Cantor's List

The list begins with zero
And ends with reaching one.
As for the numbers in between –
I'll show how they're begun.
It's easy to imagine,
A binary array,
Which, though it can't complete them all,
Still lists them in a way.
Half of them start with zero –
The other half with one.
I'm sure you've seen a binary tree.
You know how branching's done.
Each digit grows two branches –
A zero and a one.
This way of growing branches
Goes on and on and on.
We may not make it to the end
Before we're out of time,
But though we finish none at all,
No "missing one" you'll find!
The problem, now, of counting them,
Though somewhat more involved,
May not be one which can't be done,
But merely one unsolved.
If we use aleph nought to mean
The first infinity,
I say the cardinal number
Of our set is plain to see.
Two-to-the-aleph-nought's the count
Of pathways we could find –
Each pathway corresponding to
A number well-defined.
That this is more than aleph nought
I grant may be the truth.
Still, I insist, to show that's true
Requires a different proof!

SANDY COLEMAN

Responses to Teacher Feedback on Errors Differ by Age and Gender

by

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and

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HUNTER

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Abstract

Many students enter Hunter College's developmental mathematics program committing errors (mis)learned years earlier. These errors typically persist into the adult years and it is important to correct them specifically; simply reteaching concepts is not sufficient. Furthermore, there is a strong correlation between completion rate and student perception of the instructor's concern. To address both factors, we developed and tested an instructional technique to see whether giving detailed feedback to students about their errors would facilitate progress through the course. We found that the use of the feedback method had a clear positive effect on women; while for men, complex age by sex interactions and a smaller male sample size made the results less clear.

Why Study Feedback

A first year Hunter College student faces many of the same hurdles faced by students in the first year of numerous community colleges. Approximately 67% are required to take developmental reading, writing or mathematics (Koutrelakos, 1979; Weisgal, 1996). The completion rate of the students enrolled in developmental mathematics courses is very important to the CUNY mission of providing a college education for all city residents who seek it. Many students enter Hunter College's developmental mathematics program committing errors (mis)learned many years earlier. Since these errors typically persist into the adult years, it is important to identify and correct them specifically (Ashlock, 1976); simply reteaching a concept is not sufficient. Furthermore, teacher evaluation forms used each semester in this developmental course indicate a strong correlation between the finish rate in a class and student perceptions of the instructor's care and concern. Motivated by these factors, we were interested in developing, and justifying, an instructional technique which

a) indicated to the students that their instructor was concerned about their individual progress in the course and was helping them individually, not simply dealing with them as a "class" of students;

b) gave students help specific to their own particular misconceptions and incorrect use of algorithms;

c) facilitated their movement through the course.

After consideration, we hypothesized that if teachers gave detailed feedback to students about the specific errors they had made, students would relearn the material correctly. Consequently, this study was planned to test whether giving detailed feedback to students about their errors would facilitate their progress through the course. The feedback method was effective for women; for men, complex age by sex interactions and a smaller sample size made the results less clear.

Review of the Literature

Many features of the developmental arithmetic course at Hunter were instituted because research indicated they would produce positive results. For example, the developmental arithmetic course has specified learning goals and strict deadlines for taking exams. Greenwood (1977) examined the effect of the imposition of external pacing on an individualized instructional program. When time goals and deadlines were provided, individualized instruction was more successful than traditional instruction in 25 of 27 studies; traditional was superior in two studies. When no pacing was imposed, traditional instruction was superior in one of four studies, individualized in none, and there was no difference in three studies.

Greenwood (1977) also found that the superiority of individualized over traditional instruction rests with the involvement of the instructor. If the teacher merely supervises the students' work, little is gained. Only when the teacher attends to individual problems and deficiencies does this system perform better than the traditional approach. Even without Greenwood's data, a teacher's intuition (skill from experience) would predict that this supportive role for instructors would produce superior results.

Research on error analysis is rather extensive. In particular, however, the types of errors that are common in adults are (a) wrong operation, (b) computational error, (c) defective algorithm, and (d) random response [see discussion in Clarkson (1981)]. Of these four types of errors, the latter is the only one that is not persistent. Someone answering randomly has, essentially, no clue about the problem and should be re-taught the material. In all the other error types, something about the problem has evoked an incorrect response; the error should be pointed out and, in most cases, must be specifically corrected to prevent its recurrence. An instructor who would take the time to write comments about individual errors would surely relay to the students that (s)he cared about that student's progress. And the value of such written feedback — feedback specific to student errors — has been demonstrated in several studies in traditional classrooms (Schoen and Kreye, 1974). And, since "teacher-learner-subject matter interactions...must be at the very core of every educational endeavor," (Wheeler, 1989), the teacher comments about student errors made up an important part of this, otherwise, individualized study program.

The Environment of the Study

Hunter College was founded in 1870 to educate young women. It became coeducational in 1964 and currently about seventy percent of the 19,500 students are women. The Dolciani Mathematics Learning Center (DMLC) is organized within the Department of Mathematics and Statistics. The DMLC has its own library, a laboratory, and a computer center and employs nearly 70 people, mostly as course instructors and tutors. The Learning Center offers several introductory mathematics courses. At the time of the study, Mathematics 001(MATH 001), an arithmetic/beginning algebra proficiency course, was the largest of these courses, with class sizes ranging from 75 to 90.

Upon entrance to Hunter, all undergraduates and nonmatriculated students must take two mathematics proficiency examinations, one mandated by CUNY and one by Hunter College. Depending on the results of the former exam, students may be required to take MATH 001 and, as a result, the students in MATH 001 are not there by choice. Successful completion of the placement examinations, or of MATH 001, is a prerequisite for any other math or science course at Hunter. MATH 001 is also the one course that can keep students from graduating. This, combined with the fact that most of the students have not had extensive training in mathematics, gives many of the MATH 001 students serious apprehensions about the course.

MATH 001 is a self-paced course (structured for completion within one semester, but with flexibility during the semester). Course attendance is mandatory but the student need not attend the instructor's lectures, held in a separate classroom. Instead, during class time students have the option of working in the DMLC lab, which is staffed with several tutors. When additional tutoring is needed, students may use the DMLC laboratory, library, or computer center during any open hours.

MATH 001 contains learning goals, called objectives—specific statements of what students are to learn. Students are required to pass individual mastery-based unit tests on eight areas of mathematics and a final exam, which covers topics on the placement test, as well as additional material needed for subsequent college level courses. The eight units covered in the course are arithmetic of whole numbers; fractions; decimals; ratio, proportion, and percent; signed numbers; and three units of beginning algebra. Both computational skills and applications are taught in this course. Passing the final exam means making a score that, had these students been entering students, would have exempted them from remediation. The final exam is a different version of the same exam they failed upon entry; passing this exam is required for exit from the course.

In the semester of the experiment, the eight unit tests contained a total of 60 objectives. Students were retested on each unit until they correctly answered two out of three questions per objective on all but one objective on the unit tests. Unit tests for all students were graded by the instructor on a pass/fail basis. No partial credit was given on these tests. Normally a student must pass all eight exams to be eligible to take the final examination. In the semester of the experiment, however, all students were allowed to take the final if they attended the course up until the end of the semester. Only students who had completed the required work and passed the exam were given a grade of Credit and allowed to progress to the next course.

Clearly, the completion rate of students enrolled in MATH 001 is very important to the overall success of the mission of the Dolciani Center. Not only are there obvious pedagogical reasons, but any reduction in the number of repeating students is very important financially.

Design of the Study

During the semester of the study, there were eleven sections of MATH 001 with about 75 students each. In each section, we focused on first semester freshmen who had never taken the course before. The freshmen were randomly placed in a control group and an experimental group, which we examined to ensure that they were balanced. These groups were balanced on all incoming measurements and test scores; statistically the two groups were as identical as possible. In all, 220 freshmen entered the study and 193 of them completed the final exam.

Students in the experiment were not singled out for any unique attention by the instructors. In fact, instructors did not know which students were in the experimental group and which were in the control group, although they did know which students were taking part in the study. Teacher training was routinely given to instructors to insure that the level of instruction they gave was consistent from section to section. In addition, a detailed course outline specified text sections, page numbers and problems that all students were expected to cover. The tests, taken by all students, were carefully constructed to be comparable from section to section. All students, including those taking part in the experiment, were given the opportunity to schedule their own exams, as long as they met deadlines. All these procedures were a part of the routine structure of the course and were not changed because of the experiment.

All students in the study — both control groups and experimental groups — had their exams graded by one person, the researcher. The only difference between the treatment of the two groups was the fact that the experimental group received error feedback on their exams.

Encouraging remarks were given to all students. For example, when a student passed all objectives on the exam, (s)he was given a statement like "Good work!" or "Keep up the good work!" or "Way to go!" or "Bravo!" or some such comment. When students were careless, they were cautioned "Be careful!" or "Check your work" or "Read the problem carefully." In addition, however, students in the experimental group (as opposed to those in the control group) were given specific information on their errors.

For example, a student whose errors were caused by incorrect multiplication by 6 was told what the error was; the suggestion was then made to review the multiplication facts for six. Students solving equations incorrectly were given details about their unique errors and a complete, correct solution was also given to them. Students who copied their answers incorrectly from their scrap paper to the test sheet were told to number the problems on their scrap paper and to carefully check that they had copied their answers correctly from their work to the test. Remarks to the control group included positive and cautionary remarks comparable to the experimental group. However, their errors were not specifically discussed. Periodically, an incorrect digit was circled, but no written explanation was given.

The Data

Many measurements were obtained on each student, including information relating to before, after and during the student's attendance in MATH 001. The variabıes relevant to this analysis are described below.

1) Sex
2) Age — Ages of the students' ranged from 18 to over 50. However, there were not many men above 25 nor women above 33.
3) CUNY—This is the student's score on the entering CUNY mathematics exam. This score is the basis for placing a student in MATH 001. Scores are the number correct out of 40; a student who scores 25 or over exempts the course.
4) Final Exam -- The final exam is simply a different version of the CUNY mathematics exam that the students took upon entrance to the college. As a result, the CUNY score and the final exam score are directly comparable.
5) Change Ratio (CR) — The ratio of the CUNY exit exam to the CUNY entrance exam. A higher ratio indicates greater improvement than a lower one and can be less than one.
6) Percent Possible Gain (PPG) — The actual gain, or loss, a student makes in the course as a percentage of the *possible* gain that the student can make,

$$\frac{\text{Final Exam} - \text{CUNY}}{40 - \text{CUNY}} \times 100$$

The Initial Results

We first analyzed the change ratio (CR) with the expectation that this would be higher for the experimental group than for the control group. However, this was not the case. The CR ratio for the control group was 1.597 and the experimental group 1.589, indicating that the control group showed slightly more improvement in the course than the experimental group (see Table 1). This difference between the experimental and control groups is statistically insignificant with a small t value of 0.134.

	Count	Mean	St. Dev.	St. Error
Control	99	1.597	0.487	0.049
Experimental	94	1.589	0.361	0.037

Table 1. Analysis of Change Ratios – Final Exam/CUNY

Based on these results, one might be tempted to conclude that providing feedback has no effect on Math 001 students, but as we shall show, this would be an incorrect conclusion. Table 2 extends Table 1 to include sex. The important feature of Table 2 is that women in the experimental group improved while the change ratio of the men dropped. While the main effect difference between the control group and the experimental group, and the main effect difference between males and females are not statistically significant without adjustment for the interaction, the *p* value of the interaction between treatment group and sex is 0.1351. While this is slightly above standard significance levels, why such an interaction would appear at all is unclear. Did the women in the experimental group prefer the extra help, while the men in the experimental group detested it?

	Females	Males	Combined
Control	1.586	1.694	1.597
Experimental	1.614	1.432	1.589
Combined	1.599	1.546	1.593

Table 2. Change Ratios by Group and Sex

At essentially the same time that the treatment group/sex interaction was observed, a second interaction was also observed. When the data were rotated in three dimensions, CR vs. age vs. incoming CUNY score, and the CR vs. age dimensions came into view, we could clearly see a dramatic difference in the behavior patterns of the men and women. In particular, the regression of CR on age for women drops slightly, but significantly, and the regression for men rises rapidly. However while the male regression is also statistically significant, the oldest male was 32 and the total number of males was under 30.

These clear indications of different behavior for men and women demanded that the data be studied more closely.

The Extended Analysis

The relationship between CR and the incoming CUNY scores exhibits a clear, statistically significant, downward slope with a correlation of -0.663. This implies that CR is a biased measurement of change because students with lower incoming CUNY scores have a greater potential for improvement. To correct this bias, we instead calculated each student's gain as percentage of that student's maximum possible gain, percent possible gain, PPG. The relationship between PPG and CUNY shows no evidence of a biasing relationship and indeed, the correlation between the new measure, PPG, and CUNY is negligible, (0.041). In addition to a negligible correlation with CUNY, the correlations of PPG with all of the other incoming student variables are also very small. This clear lack of systematic bias in PPG makes it a far superior measure of student gain. This uneven potential for gain has been observed before; for example, in 1994 Mary Margaret Shoaf-Grubbs used a graphical adjustment technique to correct the bias. We believe the use of PPG to be new.

Tables 3 and 4 exhibit analyses of PPG. Table 3 shows that the experimental treatment helped women but the men did noticeably **worse.** The interaction between treatment group and sex is very clear. The women improved from 45.89 to 49.21 while the men's scores declined significantly from 60.43 to 36.81. These results are not surprising. The interactions, which we have already noted, mask the differences between the sexes within the experimental groups. It is worth noting again that the men and women coming into the program were equally able, on average, in that the incoming CUNY scores for men and women in the two experimental groups showed no significant differences. In Table 4, we can see that the treatment group/sex interaction is significant with a p value less than 0.005 and that the feedback treatment is also significant.

	Females	Males	Combined
Control	45.89	60.43	53.16
Experimental	49.21	36.81	43.01
Combined	48.62	47.55	47.42

Table 3. Percentage Possible Gain (PPG)

Source	df	Sums of Squares	Mean Square	F-ratio	Prob.
Treatment (Tr)	1	2056.85	2056.85	4.7078	0.0313
Sex	1	22.9026	22.9026	0.05242	0.8192
Tr*Sex	1	3622.85	3622.85	8.2922	0.0044
Error	189	82573.6	436.898		
Total	192	88276.20			

Table 4. ANOVA of PPG

Finally, recall that the three dimensional graphs we described earlier suggested an interaction of improvement with the age of the student. The regression of PPG on age for women has an estimated slope of −0.34 and is significant (p = 0.021) and for men the slope is 4.04 with a p value of 0.0069. But we are currently studying age further because local smoothing of the data hints that, for young men, the regression first slopes downward and after that turns quickly upward.

Altogether, the results of this analysis permits the following summary,

i error feedback methodology does have a significant effect, ii women react positively to this particular procedure, men tend to react negatively.

iii feedback appears to have a lessening effect on older women and an improving effect on older men.

Discussion

The error feedback program had strong effects on the students, although the interactions were complex. A few explanations for the differential effects on males and females are plausible. Mathematics 001 is required for students who have not achieved an acceptable level of arithmetic and algebra skills. It is possible that the men, who typically expect themselves to be competent in mathematics, may have had a negative reaction to having their errors pointed out. Unlike their female counterparts, they did not appear to learn from their mistakes. It is likely, when told what they did wrong, the men "stored" that information, but did not act on it. Without practice, students do not retain the information from class to class.

Women, on the other hand, seemed to use the feedback information to help them learn the mathematics. There is a considerable body of research of elementary and middle school-age students that indicates that on algorithmic (step-by-step) computational problems, females outperform males (Zambo and Follman, 1994). Here, even in the "relearning" of such material, females do seem to excel.

Age had a significant effect. Although, women reacted positively to the detailed feedback and men negatively, there was some indication that this might reverse with age. Older men seem to react better than younger men: perhaps they understand better that they need help with the material and so may be less subject to peer pressure. In contrast, older women tend to do worse than younger women.

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...by a phenomenon that everybody who teaches mathematics has observed: the students always have to be taught what they should have learned in the preceding course. (We, the teachers, were of course exceptions; it is consequently hard for us to understand the deficiencies of our students.) The average student does not really learn to add fractions in an arithmetic class; but by the time he has survived a course in algebra he can add numerical fractions. He does not learn algebra in the algebra course; he learns it in calculus, when he is forced to use it. He does not learn calculus in a calculus class either; but if he goes on to differential equations he may have a pretty good grasp of elementary calculus when he gets through. And so on throughout the hierarchy of courses; the most advanced course, naturally, is learned only by teaching it. This is not just because each previous teacher did such a rotten job. It is because there is not time for enough practice on each new topic; and even if there were, it would be insufferably dull.

R.P. Boas

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The mind is not a vessel to be filled, it is a fire to be kindled.

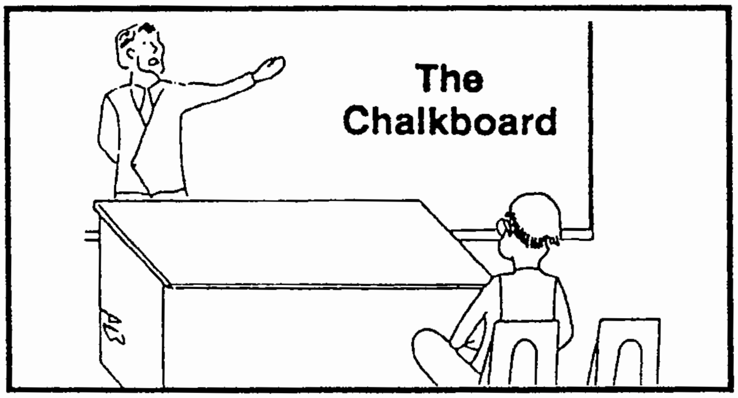
Plutarch

❖ ❖ ❖

The goal of teaching is learning, not teaching.

Hugo Rossi

REGULAR FEATURES



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This column is intended as an idea exchange. We hope to facilitate an open exchange of ideas on classroom management, teaching techniques, tips for helping students get past the usual stumbling blocks, techniques for improving student participation, etc. We know there are lots of good ideas out there, and this is your chance to share them. Please send your contributions to Judy Cain. Our backlog is again nearly exhausted, and we would appreciate your participation! Items may be submitted by e-mail or regular mail; please include your e-mail address if available.

A Random Walk Experiment

Here is a Markov Chain problem that can be investigated by a simple computer simulation. It can be used for any class in which probability is discussed. By changing the initial value of the stock, or the probability of the stock's increasing in value, the exercise can be used over and over again.

You own stock that is now worth $25.00 per share. The probability that its value increases is 0.6. The probability that its value decreases is 0.4. (Assume that each increase or decrease is five points, or $5, and the time interval is one day.) After an increase or decrease, the stock will again increase in value with probability 0.6 and decrease with probability 0.4. When the stock is worth $40.00 per share, you will sell it and go to graduate school. If it falls to $15.00, you will buy more, using the last of your disposable income. Buying and selling, in this case, are called absorbing states — once you reach an absorbing state, you remain there. Some questions of interest are:

1. What is the probability of selling the stock?
2. What is the probability of buying more stock?

3. What is the average length (time elapsed until an absorbing state is reached) of the walk?

Design an experiment using appropriate technology to answer the questions above. Explain clearly your design and conclusions.

Submitted by Rosemary Hirschfelder, University of Puget Sound, Tacoma WA 98416, hirsch@ups.edu or hirsch@halcyon.com

Exponential Functions

"Exponential functions get very large," I state to my class. The response I'd love to hear is, "How large do they get? How do they compare to the function $y = x^2$?" Instead, the best I might expect is a yawn. So last year I decided to wake them up a bit and let them visualize just how big $y = 2^x$ can get.

To do this, I started the unit on exponentials with an activity. It goes like this: Bring in a ream of paper so they can see how thick 500 sheets of paper are. Take out one sheet, fold it in half and ask how many sheets thick it is. Of course they recognize that there are two layers. Fold in half again and there are four layers. Continue folding in half a few more times so they understand the pattern that is developing: eight layers, 16 layers, etc. Then the clincher: Ask them to look at the ream of paper, which is about 1.5 inches high, and estimate how thick the paper would be if you could fold it 50 times. This is most effective if you just let them guess and don't allow them to use their paper or calculators to do any figuring.

At this point I close my eyes, pretending to meditate on their answers, and reply that I don't think any of them have guessed a large enough number, so they are to pick an even bigger number. A few protest that their numbers are already plenty large, but I encourage them to try again. Now the fun begins as you calculate the actual answer. After folding the paper 50 times, you would have 2^{50} layers. If you take a ream of paper to be 1.5 inches thick (which is a low estimate), you will find that the resulting height is 53,309,654.68 miles — more than half the distance from the earth to the sun!

There were no yawns in the room that day. We went on to look at the graph of the function $y = 2^x$ and other exponential functions. This activity helped the students to understand what was happening as the exponent gets larger. They could appreciate the enormity of the dependent variable as x increases, an appreciation my students in the past never had. (I want to give credit to Mitchell Bernstein for first bringing this paper-folding problem to my attention.)

Submitted by Jamie Thomas, UWC - Manitowoc Co, 705 Viebahn St., Manitowoc WI 54220, jthomas@uwcmail.uwc.edu or jthomas@moon.dataplusnet.com

Using Historic Coinage to Teach Base 2 Arithmetic

The use of Mexican coinage from centuries past offers hispanic prealgebra students in southern Arizona the opportunity to learn some interesting facts about their heritage and to gain dexterity with base 2 calculations.

Our decimal coinage system, and the dollar in particular, are so well established as a medium of exchange for settling debt, that we often forget that our modern base 10 monetary system is relatively new. In fact, the base 2 real (pronounced ra-AL) of Spain and the Americas was the standard for settling accounts from about 1500 to about the end of the 18th century. In the western United States, where American coinage was in short supply and Mexican reals were plentiful and contained a reliable metal content, this tradition continued throughout most of the 19th century as well. The real (R) was a coin about the size of a modern American dime, was made of 90 - 93% silver, and had a value of about 12.5 cents in 19th century U.S. currency. Silver reals were issued in denominations of 8R, 4R, 2R, 1R, $\frac{1}{2}$R, and $\frac{1}{4}$R. There was also a copper $\frac{1}{8}$R coin worth about 1.5625 cents. The 8R coin was about the size of an American silver dollar and was referred to as a "piece of eight." When change was scarce 8R coins were often cut up into eight pieces or "bits", each worth 1R. This was an interesting choice of words when one considers that each base 2 digit in a modern computer is also called a bit.

The real system enjoyed considerable and long-lived popularity because of its distinct advantages over decimal coinage systems in making change. In fact, it usually requires less coinage to make change in base 2 than in base 10.

Consider making 67 cents in change using a base 10 coinage system composed of quarters, dimes, nickels, and pennies. Two quarters, one dime, one nickel, and two pennies are required, a total of six coins. However, with reals we never require more than one coin of each denomination. If the reals are arranged into columns,

$$8 = 2^3\text{R} \quad 4 = 2^2\text{R} \quad 2 = 2^1\text{R} \quad 1 = 2^0\text{R} \quad \frac{1}{2} = 2^{-1}\text{R} \quad \frac{1}{4} = 2^{-2}\text{R} \quad \frac{1}{8} = 2^{-3}\text{R}$$

the heading on each column provides an instructor with the opportunity to discuss positive and negative powers of 2, and each entry below is either 0 or 1. No higher entry need be employed since two coins in any column would simply be an entry of 1 in the column corresponding to the next larger power of 2. Therefore, 67 cents would be represented as 0101011 in our table, which says that only one 4R coin, one 1R coin, one $\frac{1}{4}$R coin, and one $\frac{1}{8}$R coin (a total of four coins) would be required to make 67 cents. At this point it is very important to make the students realize that $\frac{1}{8}$R = 1.5625 cents (not 1 cent). If, however, it had been the case that $\frac{1}{8}$R = 1 cent, then the representation of 67 cents in our table would have been 1000011, which is the proper base 2 equivalent of 67 in base 10. With this background, it is now possible to discuss base 2 addition, since carrying operations involving 1 can be concretely interpreted as replacement of two coins of equal denomination by one of the next highest denomination.

Submitted by Richard Zito, Pima College, Tucson AZ 85709

> The title which I most covet is that of teacher. The writing of a research paper and the teaching of freshman calculus, and everything in between, falls under this rubric. Happy is the person who comes to understand something and then gets to explain it.
>
> Marshall Cohen

Snapshots of Applications in Mathematics

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The purpose of this feature is to showcase applications of mathematics designed to demonstrate to students how the topics under study are used in the "real world," or are used to solve simply "charming" problems. Typically one to two pages in length, including exercises, these snapshots are "teasers" rather than complete expositions. In this way they differ from existing examples produced by UMAP and COMAP. The intent of these snapshots is to convince the student of the usefulness of the mathematics. It is hoped that the instructor can cover the applications quickly in class or assign them to students. Snapshots in this column may be adapted from interviews, journal articles, newspaper reports, textbooks, or personal experiences. Contributions from readers are welcome, and should be sent to Professor Callas.

The Game of Life

(to accompany the study of matrix operations, computational theory, chaos)
by Dennis Higgins, SUNY College at Oneonta, Oneonta, NY

Who said life was a game? Fluid dynamics (including ocean waves and weather), thermonuclear reactions, and other phenomena are sometimes modeled using complex differential equations. Cellular automata, a much simpler mathematical tool, can be used to study many of the same mechanisms. The mathematician, John Von Neumann, developed a theory of cellular automata to create a universe where theoretical robots reproduced endlessly. Conway, Dewdney and other scientists continued Von Neumann's work. Cellular automata have been used to study water droplets, galactic spirals, and snowflakes. (See Figure 1.) They can even provide a technique to encrypt and decrypt messages.

Cellular automata can be implemented using a matrix (of one, two or more dimensions), initialized from a set of values (perhaps the integers zero and one) and updated according to a set of rules. If the rules governing our game are good ones, then the behavior of our "pattern" may be similar to that of the phenomenon it represents, and we may even learn something about real world entities from our experiment.

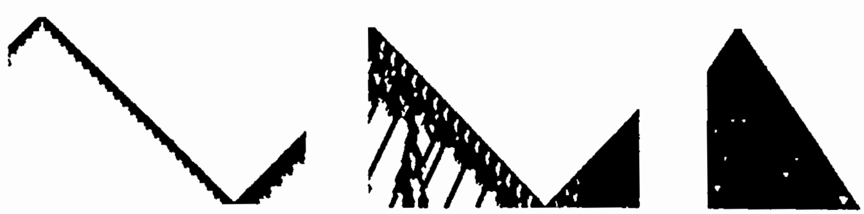

Figure 1. Three different cellular automata evolving geometric patterns.

Many cellular automata applications are complicated, but Conway's Game of Life easily demonstrates how cellular automata can be used to simulate real phenomena. Imagine a square board (a two dimensional matrix) filled with zeros and ones, where zero represents a dead zone and one represents a living entity. Conway played his game on a 10 by 10 board, but we will use 5 by 5 because it is easier to perform the computations by hand on a small board. Since life forms utilize resources to grow and reproduce, but also need other life forms for these processes, we can see that too many ones in a small area, or too few, might not be good for our little colony. Conway's Game of Life used the following rules:

1. Fill the board with some initial pattern of zeros and ones.
2. Compute the next "generation" on a separate, temporary board of the same size: If the 3x3 neighborhood containing cell (i,j) sums to 4, this cell will have a 1 in it in the next generation. If the neighborhood sums to anything else, put a zero in this cell for the next generation.
3. Copy the next generation board into the original board.
4. Go back to step (2) if you like.

Treat the cells on the edge as if they are surrounded by zeros. To do your computations, you might want to add a border of zeros around your 5x5 world. A sample computation is provided below.

1	0	1	1	0
0	1	1	0	0
0	0	0	1	1
1	0	0	0	0
1	0	1	0	1

Figure 2.
The original configuration

0	1	1	0	0
0	1	0	0	0
0	0	0	0	0
0	0	0	1	0
0	0	0	0	0

Figure 3.
The next generation

Although some patterns repeat endlessly, and often all life dies out after a few generations, no algorithm exists to determine a priori, for arbitrary rules and an arbitrary initial configuration, whether the pattern will die out, generate chaotic patterns, or evolve into a cyclic pattern. This is an important result in computational theory.

One dimensional automata are also interesting. Suppose our world consisted of one row with 10 cells in it. We can draw another row beneath the first representing the second generation. We can continue this process until our page is covered by 5, 10 or 50 generations. Using one dimensional automata, we let the second dimension represent time and thus we can see (appearing down the page or computer screen) what happens to our one-dimensional world over many generations. (See Figure 4.)

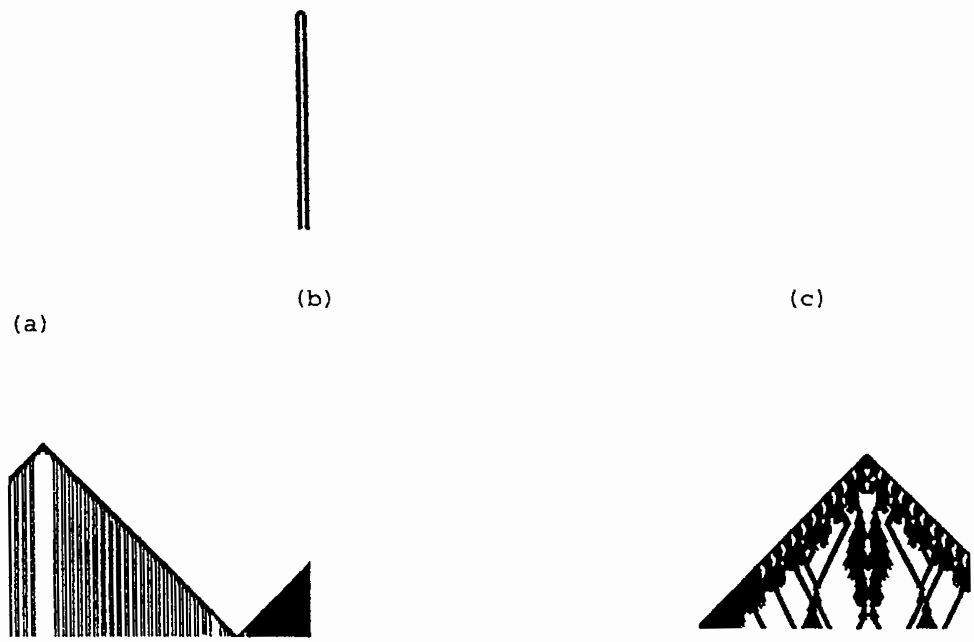

Figure 4. Illustration of three classes of one-dimensional automata as they evolve over time: They may form simple periodic or stable structures (a), or they may form chaotic patterns which seem predictable (b), or extremely complex (c).

Exercises

1. Make a 5x5 array and fill it with 0s and 1s. Draw a second array next to the first. Use the rules from the Game of Life to fill the second array with zeros and ones.
2. Can you guess what will happen to your organisms in the third generation? In the tenth or hundredth generation?
3. Repeat Exercise 1 but devise new rules for the game of life.
4. Draw a one-dimensional array with ten cells and fill it randomly with digits 0, 1, 2 or 3. Draw another array under the original one. By summing each cell and its two neighbors in the original array we might fill the second array according to the rule: If the sum is 0 1 2 3 4 5 6 7 8 9
   fill the spot with a 0 0 1 1 2 3 4 0 0 0
   Repeat this process for a few generations.
5. Now, let each digit represent a color, for example, 0 = white, 1 = blue, and so on. Color in the generations. Notice the pattern appears chaotic even though a rule governs our game.
6. If you can, write a computer program to play the Game of Life or to generate one-dimensional cellular automata.

This snapshot was based on a number of references, including those below, and was produced as a part of a project sponsored by the State University of New York and the National Science Foundation (Division of Undergraduate Education).

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Acknowledgements

My student, Paul Warren, provided the program and Steve Maniscalo provided graphics.

I must study politics and war that my sons may have liberty to study mathematics and philosophy. My sons ought to study mathematics and philosophy, geography, natural history, naval architecture, navigation, commerce, and agriculture, in order to give their children a right to study painting, poetry, music, architecture, statuary, tapestry, and porcelain.

John Adams

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Tell me and I'll forget. Show me and I may not remember. Involve me, and I'll understand.

Native American saying

Advertiser's Index

Notes from the Mathematical Underground

by

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The opinions expressed here are those of the author and should not be construed as representing the position of AMATYC, its officers, or anyone else.

The "Calculus Initiative" turned out to be one more sad instance of the collective inability of college professors to understand what it means to *learn* mathematics. Consider, for example, (Zucker, 1996): To summarize, high school is OK. We, college professors, are OK. Students are not OK but redeemable if they would just apply themselves:

> The fundamental problem is that most of our current high school graduates don't know how to *learn* or even what it means to learn (a fortiori to understand) something. In effect they graduate high school feeling that learning must come down to them from their teachers. That may be suitable for the goals of high school, but it is unacceptable at the university level. *That the students must also learn on their own, outside the classroom, is the main feature that distinguishes college from high school.* (emphasis in the original)

The answer is "*so obvious that it is embarrassing* [...]:"

> It is possible to get college freshman to learn calculus fairly well, without resorting to utopian tricks such as enforced group projects. All we have to do is get the student to accept that learning is something that will take place mostly outside of class; that is *just insist that they grasp the underlying premise of college education.*

Specifically,

> Students must be told *immediately* that they are about to face a big jump in level from high school. Most high school teaching is justifiably set to the needs of the least talented students in the class: the better students often become convinced by habit that this level is right for them too. They should be helped to recognize that the change is both appropriate and manageable.

My reason for taking issue with this particular article is because of where it appeared and because it quite embodies much of what is wrong with mathematical academia, including two-year colleges. The reasoning in the article is based on three, more or less explicitly stated, Uniqueness Theorems:

- Calculus is a corpus of knowledge one and indivisible, like the French Republic.
- There is one and only one way to learn it — the article does not disclose it — and one and only one way to "teach" it — by clearly presented, ex cathedra

lectures. In fact, *"the instructor's job is primarily to provide a framework, with* some *of the particulars, to guide* [the students] *in doing* [their] *learning of the concepts and methods that comprise the material of the course."* The student is directed *"to read the* [fat?] *textbook for comprehension. It gives the detailed account of the material of the course."* Etc.

- There is one and only one kind of student. *"Of course, the way in which mathematics education in any particular college or university can be improved depends on the composition of its student body. The overall theme should be the same though."*

This is overwhelming and impossible to rebut here. To illustrate what I consider to be the fundamental problem, and since it is universally acknowledged to be one of the tasks with which the students have the greatest difficulties, let me take graphing: In none of the texts I have consulted is there a mention that it is an *investigation*, driven by *logic*. Invariably, students are given a set of instructions to be followed, more or less blindly. For example, (Stewart, 1987), even though he invokes Polya in the preface, gives a three page "Checklist of Information for Sketching a Curve $y = f(x)$". Fortunately, it is reduced by (Anton, 1988), in the case of $f(x) = \dfrac{P(x)}{Q(x)}$, to the following "How to":

Step 1 : Find the x-intercept of $P(x)$. At these values we have $f(x) = 0$, so that the graph intersects the x-axis at these points.

Step 2: Find the x-intercept of $Q(x)$. At these values, $f(x)$ approaches $+\infty$ or $-\infty$, and the graph has a vertical asymptote.

Step 3: Compute $\lim_{x \to +\infty} f(x) = 0$ and $\lim_{x \to -\infty} f(x)$. If either limit has a finite value L, then the line $y = L$ is a horizontal asymptote.

Step 4: The only places where $f(x)$ can change sign are at the points where the graph intersects the x-axis or has a vertical asymptote. Calculate sample values of $f(x)$ in each of the open intervals determined by these points to see whether the graph is above or below the x-axis over the interval.

Step 5: From $f'(x)$ and $f''(x)$ determine the stationary points, inflection points, interval of increase, decrease, upward concavity, and downward concavity.

Step 6: If needed, plot a few well-chosen points and determine whether the graph crosses any of the horizontal asymptotes.

Why not just plot? But these authors are of the old school. So, let us turn to some "Revitalized Calculus". On page 81 of (Hughes-Hallett et al., 1994), we read

> Rational functions are those of the form $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials. Their graphs often have vertical asymptotes where the denominator is zero. If the denominator is nowhere zero, there are no vertical asymptotes. They may also have horizontal asymptotes which occur if $f(x)$ approaches a finite number as $x \to \infty$ or $x \to -\infty$.

This is followed by the example $y = \dfrac{x^2 - 5}{x - 1}$, in which the behavior at infinity is obtained by a table of powers of 10 but the behavior near 1 is not investigated. After that, an admittedly brief search does not turn up any graphing. Similarly, (Ostebee & Zorn, 1995) just gives a definition and the example $r(x) = \dfrac{x^2}{x^2 - 1}$.

Well, I suppose that the investigation of rational functions is already going the way of Greek and Latin. To show why I deplore this disappearance, let me give an example of what I ask from my students after they have gone through a "toolbox" in which they see how all the local properties of a function can be read off the appropriate terms of its polynomial approximation at that point. Investigating a function then consists in finding out the extent to which global features can be derived from local ones. We start with polynomial and rational functions, but I will take an example which occurs later on in the course.

Investigate the function $f(x) = \dfrac{\sqrt{x} + e^x}{x^2 - 1}$.

The students are supposed to go through two phases: The investigative phase in which they can cooperate and the writing of a report which they have to do alone. *I only get to see the reports.* Here is *my* report. I use, in lieu of little ohs, the symbol (...) — pronounced "a little bit", as in $\dfrac{1}{3} = 0.33 + (\ldots)$.

Since f involves a root, we must first determine the *allowable* inputs. Since $\sqrt{x}$ is not defined for negative inputs and since this is the only restriction, all inputs from 0 on are allowable.

At 0, we have $f(0) = \dfrac{0 + e^0}{0^2 - 1} = \dfrac{+1}{-1} = -1$. It is worthwhile to find the local graph near 0.

We have
$$f(h) = \frac{h^{\frac{1}{2}} + 1 + h + \frac{h^2}{2} + (\ldots)}{-1 + h^2}$$
$$= \frac{1 + h^{\frac{1}{2}} + h + \frac{h^2}{2} + (\ldots)}{-1 + h^2}$$
$$= \frac{+1 + (\ldots)}{-1 + (\ldots)} = -1 + (\ldots)$$

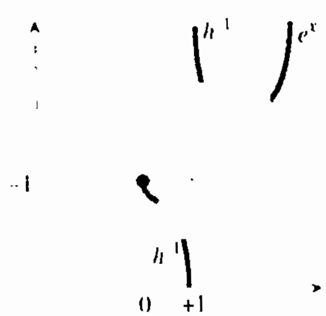
$$\begin{array}{r|l} & -1 - h^{\frac{1}{2}} \\ \hline -1 + h^2 & +1 + h^{\frac{1}{2}} + h + \frac{h^2}{2} \\ & +1 \qquad\qquad - h^2 \\ \hline & \quad + h^{\frac{1}{2}} + h + \frac{3h^2}{2} \end{array}$$

but, to get variation and concavity, we must divide *before* we approximate. We find $f(h) = -1 - h^{\frac{1}{2}} + (\ldots)$. So, right of 0, the graph starts straight down like the graph of $-\sqrt{x}$.

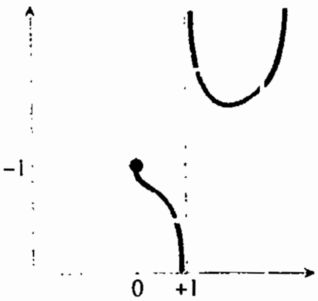
We continue by looking at the graph "outside the window".

For large *positive* inputs: $f(x)_{+\infty} = \dfrac{e^x + (\ldots)}{x^2 + (\ldots)} = e^x + (\ldots)$

We now must see if the graph could escape from the window through the top or the bottom. In other words,



we must see if f has poles. $x^2 - 1 = 0$ gives -1 and $+1$ as possible poles but -1 was ruled out by $\sqrt{x}$. We look at the graph near $+1$:



$$f(1+h) = \frac{\sqrt{1+h} + e^{1+h}}{(+1+h)^2 - 1} = \frac{1 + \frac{h}{2} + (...) + e[1 + h + (...)]}{1 + 2h + (...) - 1}$$

$$= \frac{e + 1 + \left(e + \frac{1}{2}\right)h + (...)}{2h + (...)} = \frac{e + 1 + (...)}{2h + (...)} = \frac{e+1}{2}\,h^{-1} + (...)$$

Altogether then, there must be an *even* number of (finite) zero points, an *odd* number of (finite) turning points and an *odd* number of (finite) inflection points. To find out if there might be fluctuations*, we should compute $f'(x)$ and solve $f'(x) = 0$ which may or may not be feasible.

*Pairs of opposite (finite) turning points.

Again, this is my report and I do not expect the students necessarily to write like that. Moreover, I agree with Zucker that, initially, the students have no idea of what I want. I keep telling them that they have to make a case as if they were lawyers in front of a jury and that I will act as the attorney on the other side, challenging what they assert. I demand that they explain (in writing) what they are going to do and why. In English. Otherwise I will not read them. I do end up getting readable reports which I then take apart. Of course, I have to read quite a number of them before they shape up. But eventually they do. And, that *is* doing mathematics.

Reporting on $f(x) = \dfrac{e^x}{x^2 - 1}$, a student once included a spreadsheet (Excel) plot that disagreed with her graph (Figure 1). I, too, was intrigued. First, looking at $[-25,+30]$, Excel just showed an exponential: $f(x)$ equals 0 until $x = 25$ and then moves rapidly to 20,000,000. The poles didn't show at all. But, much worse, Excel plots are quite unstable as they change qualitatively depending on very small incremental changes. Looking at $[-1.5,+1.8]$ by 0.05 increments, I, too, got the plot in Figure 2 which, since +1 is an odd pole (multiplicity 1), I found a bit ... odd. But then I tried 0.06 and, to my vast relief, got a plot I could believe in (Figure 3) even though it, too, gave no hint of $f(x)$ going to infinity as x goes to $+\infty$.

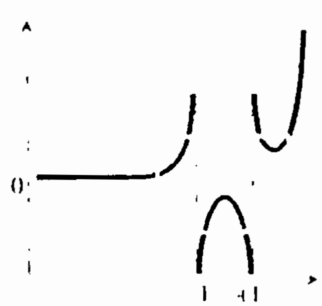


Figure 1

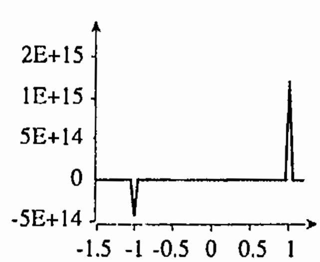


Figure 2

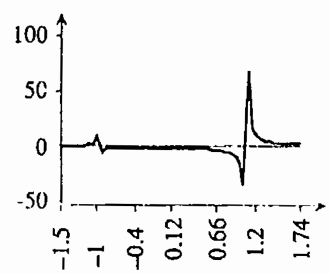


Figure 3

Lesson #1: No such thing as a *global quantitative* graph. Lesson #2: So much for Excel. Lesson #3: Graphing *qualitatively* with little ohs is nice.

While on the subject of high tech, and not to appear prejudiced, let me give an example of what I think certainly justifies its use. Back round the turn of the century, Poincaré showed that a lot of information could be obtained about the solution of a differential equation by looking at its phase portrait, a parametric family of curves. The trouble of course was that, apart from a few trivial equations, phase portraits were all but impossible to get in finite amounts of time. With the advent of specialized applications though, the situation has now changed to a point where the standard course in ODE, in which the goal is only to *obtain* solutions in closed form or series solutions, can be replaced by a course in Dynamical Systems in which one learns to *investigate* the behavior of the solution. See for instance (Artigue & Gautheron, 1983; Hubbard & West, 1991). Thus, here is a case where technology has allowed a complete rethinking of a course, changing it from empty formalism to useful meaning. Unfortunately, I don't know any other.

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Progress is impossible without change, and those who cannot change their minds cannot change anything.

George Bernard Shaw

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Education is what survives when what has been learned has been forgotten.

B.F. Skinner

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Software Reviews

Edited by Shao Mah

New Software Editor Wanted

After many years of quality service to both Red Deer College and *The AMATYC Review,* Shao Mah is retiring. We wish him well in his retirement years.

This leaves us with a vacancy in the Software Editor position. We seek someone who is a regular computer user, is familiar with many of the software packages relevant to the teaching of mathematics, has contacts who would be potential reviewers, and will seek out both products and people to review them. If you would be interested in filling this position, send a letter to *The AMATYC Review* Editor describing your background and your thoughts about this column.

Title: *EXP: The Scientific Word Processor Version 4.0 for Windows*

Author: Simon Smith

Distributor: Brooks/Cole Publishing Co.
511 Forest Lodge Road
Pacific Grove CA 93950
Phone: (408) 373-0728 or 800-214-2661

System Requirements: IBM PC with Windows 3.1 or later.
Minimum 4 MB RAM; 5 MB disk space.

Price: Simple copy $299.95; Site license $1500 + $30/CPU;
Student Edition $59.95.

There is an increasing number of mathematics and science teachers using word processors to write their own tests, examinations, reports and research projects. The advantages of these word processors are that they edit and modify the user's work easily and that they reduce clerical errors. However, many of the popular word processors on the market are either without mathematical symbols, or they are equipped with cumbersome mathematical equation editors. The EXP for Windows Version 4 is the solution to these problems.

The EXP for Windows Version 4 is a very easy to use scientific word processor, and it has many new features compared to its previous DOS versions. [The EXP Version 2.0 was reviewed by Professor J. Browne in Vol. 13, No. 2, Spring 1992 issue of this journal.] In fact, the EXP for Windows has almost all the features which can be found in other more expensive word processors. For example, a user can create the equation

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \pi$$

with EXP for Windows with ease:

When EXP is started up, the EXP window will appear. The user will see the various elements of EXP window which are similar to those found on other word processors' window; however, EXP offers on one of its toolbars the *Math Panel.* The *Math Panel* provides quick and easy access to common functions. In the example equation provided, the brackets and integral sign were done by clicking at the required symbols found under the *Insert Symbol* in the *Math Panel.* All that is left to do to complete the writing of the equation is to insert the *infinity* and *pi* which were inserted from the *Insert Symbol* in the Math pull-down menu. There are no worries about the sizing of delimiter symbols as they are self-sizing, which means that the delimiters automatically change in size to match the mathematical expression. This example clearly demonstrated that mathematical expressions can be freely mixed with text and/or graphs in EXP.

A displayed equation in EXP can, also, be created by using a *Display box.* The *Display box* changes the math style for the mathematical equation to display style, and typesets the equation to a new line of its own. Thus, the displayed equation can be either a single line or multiple lines called a *multiline displayed equation.* A *multiline displayed equation* can be entered by *Insert equation list* in the Math menu. In this case, the equation is treated as an equation box. A delete operation to be performed on an *equation list* will delete the entire *equation list* instead of the deletion of one line at a time. The following is an example of a *multiline displayed equation*:

$$\begin{aligned} \int_0^{\infty} \frac{1}{x^2+1}\, dx = \lim_{b \to \infty} &= \int_0^b \frac{1}{x^2+1}\, dx \\ &= \lim_{b \to \infty} [\arctan x]_0^b \\ &= \lim_{b \to \infty} \arctan b \\ &= \frac{\pi}{2} \end{aligned}$$

The chemical formula-like typesets is a delight to many scientists who use EXP to create special superscription and subscript. Example: the expression

$$\sum_{\alpha_i} \sum_{\beta_j} \left(W(i,j)_{\beta_j}^{\alpha_i} + \frac{F_{\alpha_i}(\alpha_i, \beta_j)}{\beta_j - k^2} \right)$$

where superscript and subscript can, also, have subscript, none of the popular word processors on the market have this feature.

Here are other important features of EXP. It has a table editor for creating tables and has the LaTeX templates for creating LaTeX compatible documents and for converting EXP documents to LaTeX styles. In EXP, there is a facility called *text libraries* which can be created by a user to store commonly used text and mathematical expressions to which he/she refers to frequently. Thus, the user can avoid entering the same text over and over again. Besides these features, a user can import graphics into an EXP document either by pasting via the Windows clipboard, or by inserting directly from a graphic file. After a graphic has been inserted into a document, it can be scaled and cropped with the use of the mouse to fit the user's needs.

EXP for Windows Version 4, also, has a student edition. The student edition has the full functionality of the professional version of EXP for Windows except that it has no LaTeX capabilities and no graphic conversions.

However, there are several problems the user should know about EXP for Windows Version 4. Firstly, the tables created from EXP are *box approach* which provides a single line of text in each cell. Therefore, a user who wants to create multiple-line entries in a cell, needs to use *Insert equation list* to overcome this problem. Secondly, a user should also be aware that the fonts used in *Math Input Mode* which is mainly for entering mathematical expressions is different from the fonts in *Text Input Mode*. Pressing the *Insert* key on the keyboard changes from one mode to the other mode. Finally, EXP for Windows does not indicate the line number and the column position of the cursor on the window; this is inconvenient for a user who needs to know the exact position of a word or an expression on a page.

As a first time user of EXP for Windows, the reviewer finds that EXP is quite simple to use. It produces a professional-like output for the document, and the mathematical expressions are automatically arranged and spaced properly. In short, EXP for Windows possesses many excellent features for mathematics and science users.

Reviewed by: Jerry Mah, Mathematics Education, University of Alberta, Alberta, Canada.

...it is the greatest achievement of a teacher to enable his students to surpass him.

John Kemeny

❖ ❖ ❖

The function of education is to teach one to think intensively and to think critically. Intelligence plus character – that is the goal of true education.

Martin Luther King, Jr.

Book Reviews

Edited by Sandra DeLozier Coleman

INFINITY AND THE MIND The Science and Philosophy of the Infinite, Rudy Rucker, Princeton University Press, 1995, xii + 342 pages, 107 figs., 3 tables, ISBN 0-691-00172-3 pbk.

IN SEARCH OF INFINITY, N. Ya. Vilenkin, translated by Abe Shenitzer with the editorial assistance of Hardy Grant and Stefan Mykytiuk, Birkhäuser/Boston, 1995, vii + 145 pages, ISBN 0-8176-3819-9/ISBN 3-7643-3819-9.

The infinite has always been a source of great fascination to the mathematical as well as the philosophical mind. Understanding even the most basic concept of infinity, that which we associate with the ellipses in the set {1, 2, 3, ...}, is no simple matter. To jump beyond and explore the multiplicity of forms presented in Cantor's "paradise" of transfinite numbers, John Conway's realm of the surreal numbers, Abraham Robinson's hyperreals, and the other mysterious highways and by-ways explored by such notable mathematicians as Hausdorff, Weierstrass, Sierpiñski, and Bolzano is not a task for one who is faint of heart or short of time. For those who, having glimpsed the key, cannot rest until the door is open wide, there is much to read to help us on our way. But beware! The path is treacherous and the wall is hard to climb!

Rudy Rucker in his book, *Infinity and the Mind*, attempts to be a travel mate and to guide us through the maze. I applaud his heroic efforts, but in all honesty, I must admit that he sometimes lost me—and often, just when I wanted most to follow his lead. Although he claims in the preface to have written the book with the average person in mind, he makes extensive use of notation developed in the writings and research of the pioneers of the infinite, but fails at times to make the meaning clear enough for the novice traveler to follow. He tumbles through his explanations in obvious ecstasy over his own hard-won measure of understanding, but his book might have benefitted from another year of revision and reflection before being thrust upon his hopeful audience, hungry for an equal measure of understanding. Rucker, himself, seems to be aware of the difficulty his audience might encounter in trying to understand all of the topics he attempts to discuss—from potential and actual infinities to the theological implications of the mathematical concepts of infinity. Indeed, he sums up his expectations by stating that he believes that "most of the main text should prove digestible, if chewed" [my emphasis].

One interesting feature of Rucker's book is the inclusion at the end of each chapter of a set of Puzzles and Paradoxes pertaining to the material just read. It might help to increase the reader's understanding to read these problems before embarking on the task of trying to understand each chapter. At least it gives the reading a focus and helps one to assess whether rereading is required. This approach has proven to be helpful in studying any math text—that is, the approach of reading through the homework problems before reading the textbook explanation and then working the problems afterward. Again, however, a word of caution is in order here. Through years of experience working textbook problems with answers provided in the back of the book, many of us have become accustomed, if not addicted, to the satisfaction that comes from working a problem and finding that our answer matches the answer given in the back— incentive enough for many to move ahead. Or, in lieu of the reward of a match, we may have found an answer which immediately made sense, so that at least we were able to put the problem out of our minds and go on to the next in line. No such satisfaction awaits us here, where one of the simplest problems reads:

> A recent science-fiction novel states that the following is a law of nature: "Every string which has one end also has another end." Is this necessarily true?

Of course, by the time the reader has reached page 92 of Rucker's book, he is not so gullible as to venture an answer based on his finite experience in the physical world! Of course, the answer in the back is, "No, consider a string of length ω." Now, why does that answer seem so right and yet so far from being satisfying? Ah! There's the rub! Were we *able* to do just that, i.e., visualize a string of such length, perhaps everything else we are asked to consider about the various numbers associated with a hierarchy of infinities would also seem meaningful, natural, logical, and correct. We might even be able to make a meaningful distinction between two strings — one of length ω and one of length Ω.

By the third chapter, where Rucker examines three well-known paradoxes, the reader begins to sense a sort of cavalier attitude toward the search for meaning in the concepts related to infinity. A few of the comments here were actually quite amusing. At least the author finally admits that many of the attempts to define concepts involving infinity are unsatisfying, so that we begin to rest easier and stop feeling inferior because of our inevitable confusion and inability to fully comprehend. The diversion into a light approach is short-lived, however, and we soon find ourselves deeply involved in detailed explanations of imaginary machines which theoretically generate sets of unimaginable size which supposedly prove the possibility of the existence of the incomprehensible. Readers who enjoyed that last sentence might actually have some hope of enjoying *Infinity and the Mind*, but I suspect that most of us would not find the rewards worth the effort.

Lest Dr. Rucker feel some concern that my words of caution would cause him to lose a few prospective readers, let me relate this review to one of my favorite stories in which two teachers are discussing their students' progress in the teacher's lounge. One asks the other how in the world she had been able to get her students to master the multiplication tables so quickly. To this the other teacher replies, "It was simple! I just told them that the multiplication tables were none of their business!" There may very well be many among our readers who will be interested in reading *Infinity and the Mind* just to see if it really is so difficult to follow. As for myself, I am caught up in Zeno's paradox, having understood about 1/2 of the book the first time I read it, about another 1/4 the second time, I am hoping to catch about another 1/8 in the next reading ...

In seeking another book which might prove a better choice for an introductory excursion into the realm of the infinite, I happened upon N. Ya. Vilenkin's

In Search of Infinity. Although both books deal with the same difficult topic, Vilenkin's book is much easier to understand and assumes no prior knowledge of the historical development or mathematical notation related to the concept of infinity.

The first of the four chapters in Vilenkin's book establishes the historical foundation for the idea as it is related to man's earliest attempts to understand the infinite nature of the universe. We read here of ancient Eastern parables and ancient Greek philosophical and mathematical writings which attempted to illustrate the concept of eternity or to use the polarity of finite and infinite as a tool for increasing man's knowledge of reality. Vilenkin makes us aware that from the beginning there has been disagreement among those who have made it their business to establish rules related to infinity. As early as the 5th century BC, the problem of paradoxes associated with assuming unlimited divisibility of space and time has complicated the problem of bridging the chasm between the discrete and the continuous, often leading to opposing schools of thought. We read about Aristotle's attempts to distinguish between potential and actual infinities and his view that all discussion of the infinite "entails walking on very shaky ground."

Later in chapter one, we learn about the role which the concept of infinity played in attempts to reconcile ancient philosophical ideas and the dogma of the Church in the 13th century and about the gradual demolition of ancient cosmology over a period of several hundred years. We learn about the contributions of Nicholas Copernicus, Giordano Bruno, and Galileo Galilei and the persecution which they endured for publicly expressing their ideas about the nature of the universe.

The connection between the evolving concept of infinitely small and infinitely large magnitudes and the development of the foundations of calculus independently by Newton and Leibniz in the 17th century is also discussed in chapter one. Here again, the problem of paradoxes led to objections by Gauss and other respected mathematicians to some of the methods of mathematical analysis. Gauss wrote to Schumacher in 1831, "I object to the use of an infinite magnitude as something completed; this is never admissible in mathematics."

Much of the remainder of chapter one is devoted to an exploration of the development of the current view of the nature of space and time, including discussions of the curvature of space, the expanding universe, theories related to the origin and future of the universe, and the concept of infinite divisibility of matter. We learn that even basic assumptions about the distribution of stars and other matter in infinite space lead to paradoxes and that many important questions remain unresolved.

Having given us reason for wanting to understand the concept of infinity, Vilenkin takes us by the hand in chapter two and walks us through the most basic concepts of set theory. He progresses slowly into the complicated world in which we discover not one, but many, levels of infinity. We learn about the critical role of establishing one-to-one correspondence between the elements of various sets in order to show that two infinite sets have the same number of elements. We read about Cantor's efforts to use the lack of such a correspondence to prove that there are infinite sets of unequal size—opening the door to a new view of the infinite as a hierarchy of numbers which follow rules of behavior quite different from these which we expect numbers to obey in ordinary arithmetic.

Vilenkin is sympathetic to the difficulty of grasping such complicated and illusive concepts. He tries, somewhat successfully, to make the ideas clearer by inventing stories in which infinite numbers of cosmic travelers seek accommodations in an extraordinary hotel which boasts an infinite number of rooms. He tries, somewhat less successfully, to validate Cantor's proof that the number of elements in the set of real numbers exceeds the first level of infinity, i.e. that which is associated with the natural numbers, the integers, the even integers, and all other sets for which a one-to-one correspondence with the natural numbers can be established.

In chapter three we discover that the "paradise" created by Cantor has not always been a peaceable kingdom. There have always been, and still are, many mathematicians who do not accept the notion of completed infinities. Strange new functions with infinitely many maxima and minima met with much controversy in the 19th century and were even referred to as mathematical monsters. Many of these "monsters" are explained in great detail and accompanied by helpful illustrations as we learn about the great mathematicians whose imaginations unleashed them on the innocent and orderly world of preCantorian mathematics.

Finally, in chapter four, we are led into the 20th century view of the infinite, where we find that controversy abounds and various schools of thought still exist. Cantor himself in 1895 wrote about the contradiction that arises in trying to conceive of a universal set greater than all sets. The axiom of choice and its consequences lead to argument over the very meaning of the expression "there exists" as it relates to mathematical objects such as infinite decimals for which no algorithm is defined for determining the course of their expansion.

In summary, I would say that Vilenkin's book is worth reading. He has done a good job of introducing the basic concepts of infinity and has provided enough detail and information about the relationship of infinity to mathematics, physics, and philosophy to convince us that the concept is worthy of careful thought and that its implications should not be over-simplified.

These are just two among many books related to infinity. For me, these two were enough to whet the appetite. The questions they bring to mind and the search for satisfying answers, like infinity itself, are endless.

Reviewed by the Editor, Sandra DeLozier Coleman.

The AMATYC Review welcomes contributions of book reviews by its readers. We would like to continue to have reviews of books that would be of interest to a broad spectrum of persons associated with or interested in the world of mathematics. Reviews of individual books are welcome, although we would like to know about groups of books which complement each other in shedding light on a particular topic. Occasionally, reviews from several readers may be combined in order to present this type of selection.

Send reviews to: Sandra DeLozier Coleman, Mathematics Department, Okaloosa-Walton Community College, 4531 Parkview Lane, Niceville, FL 32578-8734, SDColeman@AOL.COM

The Problem Section

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Charlottesville VA 22901

Greetings, and welcome to still another Problem Section!

The AMATYC Review Problem Section seeks lively and interesting problems and their solutions from all areas of mathematics. Particularly favored are teasers, explorations, and challenges of an elementary or intermediate level that have applicability to the lives of two-year college math faculty and their students. We welcome computer-related submissions, but bear in mind that programs should supplement, not supplant, the mathematical solutions and analyses.

At this time I am low in my supply of good problems. Contact me by one of these means:

E-mail: *MWE1@psu.edu* – via which I will acknowledge your problem, comment, suggestion, or whatever. If you wish to include a solution with mathematical notation, I prefer Microsoft Word for Windows and its equation editor. However, I am also able to handle Macintosh and other PC document formats if needed. Feel free to attach all such documents to your e-mail.

Regular mail: Send **two copies** of new **problem proposals** to the Problem Editor. Please submit separate items on separate pages each bearing your name (title optional; no pseudonyms), affiliation, and an address. If you want an acknowledgement or reply, please include a mailing label, self-addressed envelope, or e-mail address.

In either case, if you have a solution to your proposal, please include same (two copies would be appreciated if sent by traditional mail) along with any relevant comments, history, generalizations, special cases, diagrams, observations, and/or improvements.

All solutions to others' proposals should be sent by the deadline directly to the Solutions Editor.

Dr. Michael W. Ecker (Pennsylvania State University, Wilkes-Barre Campus)

Dr. Robert E. Stong (University of Virginia)

Changes

I've suspended the Quickies feature I have had for the past several years. However, I welcome elementary challenges of a non-routine nature, and you'll see several from now on. Please let me know how you like the new mix that includes more very elementary problems.

New Problems

***Set AF Problems** are due for ordinary consideration October 1, 1997.* However, regardless of deadline, no problem is ever closed permanently, and new insights to old problems are always welcome. However, our Solutions Editor requests that you please not wait until the last minute if you wish to be listed or considered on a timely basis.

Problem AF-1. Proposed by Jerry P. Kehn, Brookdale Community College, Lincroft, NJ.

Tim leaves Paris, driving at a constant speed. After a while he passes a mile-marker displaying a two-digit number. (Mile-markers start at 0 in Paris.) An hour later he passes a mile-marker displaying the same two digits, but in reversed order. In another hour he passes a third mile-marker with the same two digits (backward or forward) separated by a zero. What is the speed of Tim's car?

Problem AF-2. Proposed by Jerr P. Kehn, Brookdale Community College, Lincroft, NJ.

The French Ambassador gives a reception. Half his guests are foreigners, whose official language is not French. Each guest says "Bonjour" to the ambassador. To be polite, each guest says hello to every other guest in the official language of the person he is addressing. The French Ambassador answers "Soyez le bienvenu" to every guest. In all, 78 "bonjours" are said.

How many guests were at this reception?

Problem AF-3. Proposed by the Problem Editor, Pennsylvania State University, Wilkes-Barre Campus, Lehman, PA.

In mid-1996 the largest prime number known to humanity was the Mersenne prime $2^{1257787}-1$. Alas, I've forgotten how many decimal digits its base-10 representation has.

How many are there? (Without looking this up, calculate this number of digits.)

Problem Editor's Comment: Just a few months after this column was composed, a still larger Mersenne prime was found.

Problem AF-4. Proposed by the Problem Editor, based on a discovery of Gary Adamson, San Diego, CA.

Let sequence $<a_n>$ be defined by specifying $a_1 = 1$ and $a_{n+1} = na_n + 1$ thereafter. Show that $a_n = [[(n-1)!\, e]]$ for $n > 1$.

(The brackets indicate the greatest integer of the argument, and e is the familiar Euler's constant, 2.718.... This result was found by amateur mathematician Gary Adamson, who frequently re-discovers numerous seemingly obscure connections.)

Problem AF-5. Proposed by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO.

Let P be a fixed point interior to a circle with center O and radius r. Let A, B, C,

D be any four points on the circle such that AB and CD are perpendicular chords intersecting at P.

Find the maximum and minimum values of $AB + CD$ in terms of radius r and fixed length OP.

Problem AF-6. Proposed by Frank P. Battles, Massachusetts Maritime Academy, Buzzards Bay, MA.

While attending a mathematics convention, I stayed in a motel room that featured a large digital clock. I first noticed it when it was 2:12, which, ignoring the colon, is palindromic.

When viewed at random, the clock has what probability of displaying a palindromic time?

Problem AF-7. Proposed by Charles Ashbacher, Decisionmark, Cedar Rapids, IA.

For any positive integer n, the Smarandache function S is defined by $S(n)$ = the smallest nonnegative integer m such that n divides $m!$.

The first few values are $S(1) = 0$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, ...

Form the number $r = .023453...$ by concatenating the function values. Prove that r is irrational.

Problem Editor's Comment: I had thought that $S(1) = 1$, so I modified the wording explicitly to allow *nonnegative m* instead of the expected *positive m*. However, whichever was the original definition should have no real effect on the solution.

Problem AF-8. Proposed by the Problem Editor.

a) In the pool game variant called Chicago, you get points in a round by adding all the values (1, 2, ..., 15) of the balls that you successfully shoot into a pocket. Suppose you have a general number of balls, n, with values 1, 2, ..., n, to be sunk in one round. For which values of n is it possible to have a tie at the successful completion of one round if there are two players?

b) What if there are three players? For which values of n can there be a three-way tie?

Comment: I'm really asking for which values of n it is possible to split the sum $1 + 2 + 3 + ... + n$ into two equal sums (part a) or three equal sums (part b). For an easy example, $1 + 2$ cannot be split into two equal sums, but $1 + 2 + 3$ can be split into $1 + 2$ and 3. I have a solution for both parts, but I have not yet sharpened it up.

Problem AF-9. Proposed by Michael Andreoli, Miami-Dade Community College, Miami, FL.

A random sample of n people is taken from N married couples, with $2 < n < N$. Find the expected number of married couples in the sample.

Problem AF-10. Proposed by Kenneth Boback, Pennsylvania State University, Wilkes-Barre Campus, Lehman, PA.

Suppose the zeros of $ax^2 + bx + c$ are r and s. For the zeros of $x^2 + px + q$ to be r^3 and s^3, find p and q. (Problem Editor's Query: What about for the zeros to be r^2 and s^2? Is this too easy or too well-known?)

Set AD Solutions

Ace Avoidance

Problem AD-1. Proposed by Michael Andreoli, Miami-Dade Community College, Miami, FL.

A well-shuffled deck of 52 cards is to be turned over one card at a time. For each card that is not an ace, the player wins \$1. The player may elect to stop at any time and keep his accumulated winnings. However, if an ace is turned over, the player loses everything and the game ends. How many cards should the player turn over to maximize his expected winnings?

Solutions by Rick Armstrong, Florissant Valley Community College, St. Louis, MO; Charles Ashbacher, Decisionmark, Cedar Rapids, IA; Frank P. Battles, Massachusetts Maritime Academy, Buzzards Bay, MA; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Mike Dellens, Austin Community College, Austin, TX; Donald Fuller, Gainesville College, Gainesville, GA; Carl O. Riggs, Jr., Retired Naval Engineer, Largo, FL; and the proposer.

If k cards are turned over, the expected winnings are

$$E(k) = k \times \frac{48 \times 47 \times \ldots \times (49-k)}{52 \times 51 \times \ldots \times (53-k)}$$

so $E(k) > E(k+1)$ if and only if $k > (k+1)\dfrac{48-k}{52-k}$. This gives $5k > 48$ or $k > 9.6$.

Thus the maximum value of $E(k)$ occurs for $k = 10$ and is $E(10) = \$4.13$.

Fast Fall

Problem AD-2. Proposed by the Problem Editor.

The assumption of no air resistance or friction in short-distance, falling-body problems leads to the conclusion that heavier bodies fall no faster than lighter bodies. Suppose we more realistically include a force due to air resistance, one that is proportional to velocity at each moment. Prove (or disprove) that heavier bodies subject only to constant gravitation and such air resistance *do* fall faster when dropped than lighter ones.

Solutions by Frank P. Battles, Massachusetts Maritime Academy, Buzzards Bay, MA; Russell Euler, Northwest Missouri State University, Maryville, MO; Donald Fuller, Gainesville College, Gainesville, GA; Carl O. Riggs, Jr., Retired Naval Engineer, Largo, FL; and the proposer.

From $F = ma$ one has

$$m\frac{dv}{dt} = mg - kv$$

where m is mass, v is velocity, t is time, g is acceleration due to gravity, and k is the constant for air resistance. One has the initial condition $v(0) = 0$, since the body is dropped. Solving this equation gives

$$v = \frac{mg}{k}\left(1 - e^{\frac{-kt}{m}}\right).$$

Then

$$\frac{dv}{dm} = \frac{g}{k}\left[1 - (1 + x)e^{-x}\right] \text{ where } x = \frac{kt}{m}.$$

Since $(1 + x)e^{-x} < 1$ for $x > 0$, one has $\frac{dv}{dm} > 0$ for $t > 0$ and the velocity is an increasing function of the mass.

Divisor Dilemma

Problem AD-3. Proposed by the Problem Editor.

For each integer $n > 1$, an *aliquot* divisor is any divisor of n, including 1, other than n itself. Let $s(n)$ = the sum of the aliquot divisors of n, and similarly $p(n)$ = the product of these divisors.

a) Characterize the integers $n > 1$ according to whether $s(n) < p(n)$, $s(n) = p(n)$, or $s(n) > p(n)$.

b) Find all $n > 1$ for which $s(n) = p(n) = n$.

Solutions by Robert Bernstein, Mohawk Valley Community College, Utica, NY; Donald Fuller, Gainesville College, Gainesville, GA; Carl O. Riggs, Jr., Retired Naval Engineer, Largo, FL; and the proposer.

Given $1 < a < b$, we have $1 + a + b \leq b + b = 2b$ with equality only for $a + 1 = b$, and $ab \geq 2b$, with equality only for $a = 2$. Thus $1 + a + b \leq ab$ with equality only for $a = 2$ and $b = 3$. Thus, if n has three or more aliquot divisors, $p(n)$ is at least as large as $s(n)$ and one has equality only for $n = 6$. Clearly, n has one aliquot divisor only if n is prime and then $s(n) = p(n) = 1$, and n has two aliquot divisors when n is the square of a prime p, and then $s(n) = p + 1 > p = p(n)$. Thus $s(n) = p(n)$ if n is prime or $n = 6$, $s(n) > p(n)$ if n is the square of a prime, and in all other cases $s(n) < p(n)$. Finally $s(n) = p(n) = n$ only for $n = 6$.

Expected Urnings

Problem AD-4. Proposed by Michael Andreoli, Miami-Dade Community College, Miami, FL.

Balls numbered 1 through n are placed in an urn and drawn out randomly without replacement. Before each draw a player is allowed to guess the number of the ball that is about to be drawn. The player is told only whether his guess was right or wrong. The player decides to adopt the following strategy: Keep guessing 1 until correct. Then switch to 2 until correct, and so on. Find the expected number of correct guesses. What happens as n increases without bound?

Solutions by Rick Armstrong, Florissant Valley Community College, St. Louis, MO; Robert Bernstein, Mohawk Valley Community College, Utica, NY; and the proposer.

The expected number of correct guesses is the sum over k of the probability of getting correct k guesses. The probability of getting k guesses correct is the probability that the balls 1 to k are drawn in numerical order which is $\frac{1}{k!}$. Thus, the expected number of correct guesses is

$$\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}.$$

As n tends to infinity these values approach $e-1$.

Linear Iteration

Problem AD-5. Proposed by the Problem Editor.

Consider all linear functions $L(x) = ax + b$. For which pairs (a, b) is it true that there exists a unique real number h such that, for any real x, the sequence of function iterates $\langle L^n(x)\rangle$ converges to h? For such functions, find h.

Solutions by Robert Berstein, Mohawk Valley Community, College, Utica, NY; Joseph F. Conrad, Solano Community College, Suisun, CA; Russell Euler, Northwest Missouri State University, Maryville, MO; Donald Fuller, Gainesville College, Gainesville, GA; Tuan Nguyen, Orange County Community College, Middletown, NY; and the proposer.

One has

$$L^n(x) = a^n x + (a^{n-1} + a^{n-2} + \ldots + a + 1)b$$
$$= \begin{cases} a^n x + \dfrac{1-a^n b}{1-a} & \text{if } a \neq 1, \\ x + nb & \text{if } a = 1. \end{cases}$$

The sequence converges to h independently of x precisely when $|a| < 1$, and h is $\frac{b}{1-a}$.

> The merit of painting lies in the exactness of reproduction. Painting is a science and all sciences are based on mathematics. No human inquiry can be a science unless it pursues its path through mathematical exposition and demonstration.
>
> Leonardo da Vinci

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