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ABSTRACT

This paper posits that, as educators' attention turns to elementary mathematics as preparation for algebra, it is important to begin by examining the kind of elementary mathematics content teaching that is aligned with reform. The paper calls for a close examination of students' development of operation sense. The role of children's engagement with the four basic operations in their preparation for algebra is also discussed. One method of data collection used in this study is episode writing. Six of the scenarios presented in this paper are taken from episodes written by teachers in a National Science Foundation teacher-enhancement project known as Teaching To the Big Ideas. Although the question of how elementary education can prepare students for algebra is addressed here, teachers at other levels still need to be prepared to help their students begin to make meaning of the language of algebra. Implications of this work include the idea that other notional systems such as diagrams, graphs, and tables in written and electronic forms are important for students to understand. Contains 62 references. (DDR)

Developing Operation Sense as a Foundation for Algebra

by

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Developing Operation Sense as a Foundation for Algebra[1]

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In recent years, concerns have been growing about the teaching of algebra. These concern are threefold: 1) Middle- and high-school algebra courses have been identified as filters for a variety of future career opportunities. That is, children who do not take algebra are shut out of mathematics-related careers in science, including medicine, in engineering, business, etc. (Chazan, in press; Moses, et al., 1989; Steen, 1993) 2) Students entering middle school are ill-prepared for algebra and those who study algebra (and may even earn passing grades) learn it only as a repertoire of manipulations applied to meaningless strings of symbols (Brown et al., 1988; Kieren, 1992). 3) Visions of a reformed mathematics pedagogy, as well as rapid advances in technology, have raised doubts that algebra as currently taught is adequate to the needs of today's (much less, tomorrow's) students (Coxford & Shulte, 1988; Kaput, in press-a, in press-b; NCTM, 1995; Wagner & Kieren, 1989).

Among those responding to these concerns are some who advocate (or discuss the complications of) "algebra for all," requiring all students at a given grade level to take an algebra course (Lacampagne et al., 1995; Silver, 1997); others pursue research into students' cognitive difficulties in learning algebra (Booth, 1988; Kieren, 1988, 1992; Sfard, 1991); and still others experiment with alternative curricula designed to engage students with the content of high-school algebra (Chazan, in press; Chazan & Bethel, 1996; Cuoco, 1992; Yerushalmy & Gilead, 1997).

It is also widely recognized that, to address the problems of algebra teaching faced at middle- and high-school levels, elementary mathematics education must be reconceived. Thus, there is currently much discussion of "early algebraic thinking" and the kinds of activities that might be introduced at the elementary level to prepare children for the algebra content they will be expected to "master" later on (Kaput in press-a, in press-b; National Council of Teachers of Mathematics, 1995, 1997).

These discussions are taking place in the midst of a larger effort to reform the teaching and learning of mathematics (National Council of Teachers of Mathematics, 1989, 1991; National Research Council, 1989). In response to research findings of the last two decades on children's

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mathematical thinking and the construction of conceptual understanding, the reforms propose a pedagogy that puts student thinking at the center of instruction. Teachers' roles no longer primarily involve demonstrating to students what they should learn. Instead, classrooms are organized as communities of inquiry in which students work on mathematical questions that are truly problematic for them, learn to pose their own questions, formulate conjectures, and assess the validity of various solutions (Ball, 1993; Heibert, et al., 1996; Lampert, 1988; Mokros, et al., 1995; Nelson, 1995; Schifter & Fosnot, 1993).

In this paper, I argue that, as the field turns its attention to elementary mathematics as preparation for algebra, it is important to begin with an examination of the kind of teaching of K6 content that is aligned with the reforms and is already in place (Bastable & Schifter, in press). Specifically, I argue for a close look at students' developing *operation* sense: How far will children's engagement with the four basic operations take them in terms of preparation for their studies of algebra? And what would be the next appropriate steps?

Why Should Practice be Reformed?

Research has demonstrated that students are capable of powerful mathematical thinking (Carpenter, 1985; Carpenter, et al., in press; Confrey, 1995; Fuson, 1992; Ginsburg, 1977, 1986; Kamii, 1985; Lesh & Landau, 1983; Resnick, 1987; Steffe, 1991), but that conventional instruction, rather than building on children's natural ways of thinking about mathematics, centers, instead, on memorization of facts and computational procedures (Cohen, et al., 1990; Goodlad, 1984). In the process of their schooling, many students lose contact with their own mathematical ideas and, unable to keep hold of the many procedures needing to be remembered, come to rely on faulty "rules of thumb" (Brown et al., 1988, 1989; Carpenter et al., 1981 1983; Kouba et al., 1988).

For example, young children learning the conventional U.S. algorithms for multi-digit addition and subtraction recall that, in order to perform the calculation, one must operate on the digits of each column separately. They also learn that "you can't subtract a larger number from a smaller." Thus, a common error in solving problems like 53 - 17 is to subtract the smaller digit from the larger in each column, producing the answer, 44.

An example from older grades: students asked to calculate 12/13 + 7/8 are likely to arrive at 19/21 by adding the numerators and adding the denominators. In fact, in the NAEP of 1980, when asked to approximate this sum and given the choices 1, 2, 19, and 21, students were more likely to choose 19 or 21 (the sums of the numerators or the denominators, respectively) than either 1 or 2 (Carpenter, et al., 1981).

These cases reveal students who lack a sense of the numbers they are operating on and of the action of the operation in question. In the first, they have lost touch with the idea that 17 from 53 must result in a number less than 43 (10 taken from 53), or, in the second, that both 12/13 and 7/8 are numbers slightly less than 1, and so their sum must be between 1 and 2.

Numerous studies have also shown that, when working on word problems, many students do not analyze the situation modeled in the problem when determining the operation to apply, but rather select that operation by guessing, by trying all operations and choosing the one that gives what seems to be the most reasonable answer, or by studying such properties as the size of the numbers involved (Greer, 1992; Graeber & Tanenhaus, 1993).

The weight of these and similar findings provides evidence of the need to transform mathematics pedagogy, creating classrooms in which making sense of mathematics is both the means and the goal of instruction.

Preparing K-6 Students for Algebra

As groups have come together to discuss what might be done to improve algebra instruction, it has become clear that there is little agreement on what algebra is about (Lacampagne, et al., 1995; Stanley, 1994). Is *function* the primary organizing concept, and, therefore, the necessary focus of the algebra curriculum (Chazan, in press, 1996; Confrey, 1992; Yerushalmy, M. & Gilead, S., 1997)? Or is it *algebraic structure* (Cuoco, 1996, in press; Ruopp, et al., 1997)? Or *model* (Kaput, in press; Nemirovsky, 1993)?

This paper need take no position on this controversy. What is assumed, however, is that on any definition, competence in the language of algebra is a *sine qua non* for doing algebra—students must be able to understand and use conventional notation, as well as other "symbolizing media" (including diagrams, graphs, spreadsheets, etc.), as descriptors of properties of operations and number systems, of relationships among quantities, and as models of situations with unknown quantities.

If in earlier grades, students have lost their ability to make sense of mathematics, are no longer able to represent situations with appropriate mathematical models or attach appropriate meanings to arithmetic expressions, upon what can their algebra be built?

For example, what can a student make of the statement "$a/b + c/d = (ad + cb)/bd$" if he/she thinks that $12/13 + 7/8$ is closer to 19 than to 2? Or, considering the somewhat simpler identity—$a(b + c) = ac + ab$—what might it mean to a student who has never thought about the distributive property in the context of arithmetic?

Or, how can a student write an algebraic equation to solve a word problem when he/she has not developed an understanding of the kinds of situations modeled by the operations?

And when a student learns techniques for solving equations—e.g., $7 + 5x = 17 \longrightarrow 5x = 17 - 7$—how can these steps have any meaning if he/she has never considered how the operations of addition and subtraction are related?

Now, in the context of the current effort to reform the teaching of mathematics, it is both possible and necessary to pose another, more fundamental, set of questions: *What happens if pedagogy is specifically designed to keep students in touch with their ways of making sense of mathematics? When instruction places student thinking at the center, gives students opportunities to articulate their own thinking, and encourages them to build on their ideas, how does their work in arithmetic prepare them for algebra? What experiences do students have with such notions as equivalent equations or the commutative, associative, and distributive properties of the operations they use?*

This paper scans a set of elementary school classrooms in which teachers are working to transform their pedagogy, and students, to articulate their own mathematical ideas. The particular lens employed focuses examples that illustrate the principles of pedagogical reform, offer information about students' developing operation sense, and provide clues about young children's preparation for algebra. The children from these classrooms have not been followed into their first year of algebra instruction. However, their work allows us to formulate hypotheses that should be pursued in other settings.

Where Do These Examples Come From?

Over the last decade, many programs have been designed to help in-service teachers reconstruct their practice around student thinking (Friel & Bright, in press; Fenemma & Nelson, 1997; McLaughlin, 1996). This paper draws its examples from Teaching to the Big Ideas (TBI), a 4-year teacher-enhancement project sponsored by the National Science Foundation, and jointly conducted by EDC, TERC, and SummerMath for Teachers (Schifter, et al., in press). In its first two years, 6 staff members and 36 elementary teachers (coming from urban, suburban, and rural communities) came together in two-week summer institutes, biweekly after-school seminars, and one-on-one biweekly classroom visits in order to examine the mathematics of the elementary classroom when teaching builds on children's thinking (Russell et al., 1995; Schifter, in press). In its final two years, the project has been oriented toward developing teacher leadership: first, through producing written materials in which teachers communicate what they have learned (Teaching to the Big Ideas, 1997) and, second, through implementing the staff development initiatives which TBI teachers are now conducting in their schools (Russell, 1997). The project was begun in the summer of 1993 and is currently in its fourth year.

As a major emphasis of the first three years, teachers and staff closely monitored student discussion, recording dialogue in order to identify central mathematical ideas as they arose naturally in classroom contexts. Those ideas were then analyzed to see how they shift, change, and grow as they are embedded in various mathematics topics at different grade levels. As teachers developed expertise, they became research collaborators with staff.

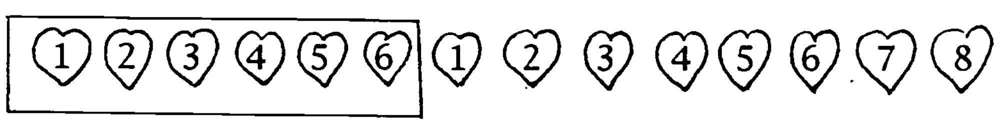
One mechanism developed for such investigations is "episode writing." Twice in the first year and as a regular monthly assignment in the second and third, teachers wrote scenarios—episodes—from their own teaching. The assignment was to write a 2- to 5-page narrative that captured some aspect of the mathematical thinking of one or more students, using transcriptions of classroom dialogue or samples of students' written work.

Of the seven scenarios presented in this paper, six are taken from episodes written by TBI teachers[2]. The seventh is based upon a teacher's episode, together with field notes and audio tape recordings made when the author visited that teacher's classroom.

A Glimpse into a Grade 1-2 Classroom: Illustrating the Principles of Reform

In the following episode, Jody Sorensen, who teaches a combined first- and second-grade class, describes her students' work on an arithmetic problem.

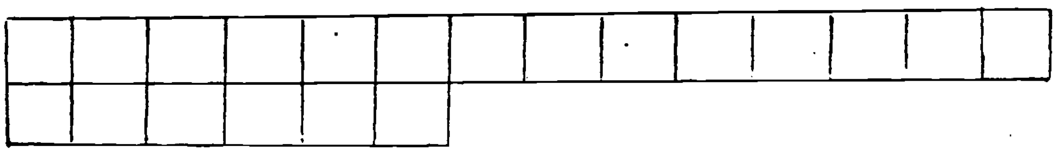
> When the children arrive in the morning each day, right away they work on the problem of the day, which I have written on a chart or on the chalkboard. One day last week, I had this word problem written on a chart:
>
> > *Sabrina and Yvonne have 14 stickers when they put their stickers together. Yvonne has 6 stickers. How many stickers does Sabrina have?*
>
> Solving the problem of the day has become a routine in my class in that, after they settle in, children just go and get any materials they need (cubes, links, counters) to solve the problem. They know that they have to keep a record of their strategies for solving the problem by using models, pictures, words, or number sentences so that they would be able to explain their thinking process to someone else.
>
> Liza is a first grader and this is how she solved the problem. First she drew 14 hearts on her paper. Then she put numbers 1 to 6 inside the heart shapes for the first 6 hearts. Then she started again from 1 to 8 on the remaining hearts. She marked off the first 6 hearts. Below her picture of hearts, she wrote, "6 + 8 = 14."



$$6 + 8 = 14$$

> Most adults would think of this as a subtraction problem, but Liza represented it with an addition sentence. When I saw what she was doing, I wanted to make sure that she was clear

[2]The first five episodes appear in the professional development curriculum, Developing Mathematical Ideas (Teaching to the Big Ideas, 1997).

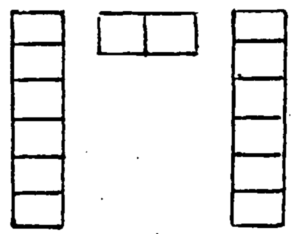
> on her process and that she understood the problem. I asked Liza what the 14 was. She said that this was the number of stickers Sabrina and Yvonne had together. When I asked her how many stickers Sabrina had she quickly pointed to the hearts labeled 1 to 8 that she had not put a [box] around and said, "Sabrina has 8 stickers." Her responses assured me that she understood what the problem was, and that her strategy was clear to her. . . .
>
> Maya is [also] a first grader. She took 14 cubes and made a tower. Then she took 6 cubes and made another tower. Then she put her 14-cube tower and her 6-cube tower next to each other like this:
>
> 
>
> Then she counted how many cubes there were beyond her 6 cube tower and found that there were 8 cubes.
>
> When I saw what Maya had done, I was struck by the fact that her model consisted of more than 14 cubes. I asked her what the 8 cubes represent and she said it was Sabrina's stickers. She said that the 6-cube tower below the 14-cube tower was only a way for her to remember how many stickers Yvonne had. She was using the extra 6 cubes as a marker so that she could easily see how many stickers Sabrina had. I then asked her what the 14 cubes represented, and she said that these were the stickers that Yvonne and Sabrina had together. Maya could explain her process clearly, using objects, but when I asked her if she could tell me a math sentence that shows what she had just done, she was not able to.

Children enter school with informal mathematical knowledge derived from experience. Before receiving formal instruction about addition or subtraction, children already can join, compare, or take away quantities by counting. Thus, when teachers give word problems, children can solve them by acting out the events described (Carpenter, 1985; Fennema, et al., 1993). Building on these perceptions, teachers who are working to transform their practice are learning to give their students opportunities to solve mathematics problems using the knowledge they bring to the classroom.

In Sorensen's lesson, Liza drew 14 hearts to represent the 14 "valentines" shared between Sabrina and Yvonne; Maya counted out cubes. The representations chosen by both children allowed them to count out all components of the problem. By keeping track of which valentines belong to which girl, they were able to conclude, in answer to the word problem, that Sabrina has 8. While Liza could model the situation with a math sentence, "6 + 8 = 14," Maya could not.

Their teacher asked questions of each of the children, giving them opportunities to explain their solution processes, at times probing to explore the extent of their understanding. Although it was important that Maya learn to represent word problems with arithmetic sentences, Sorensen did not insist that she use symbols that did not yet have meaning for her.

When her students were given many opportunities to use methods that made sense to them, were encouraged to solve problems using different strategies, and shared their solution methods with one another, they developed a repertoire of strategies that became increasingly sophisticated. As Sorensen monitored the children's work on the valentine problem, she found evidence of this. Some of her students used number strips and counted on from 6; others used cubes, also counting on from 6; and still others, able to keep track of the numbers without counting, reasoned numerically to a solution.

> Cecile, a second grader, did not use pictures or objects. . . . She explained her way like this:
>
>> I know that 7 + 7 =14, so I took 1 from one of the 7s and put it on the other 7 so now it is 6 + 8 and it's 14.
>
> I asked Cecile what she was thinking when she started with 7 + 7 = 14. She said, "7 + 7 is easier for me to think about and that makes 14, so if I move 1 from one of the sevens to the other, I have 6 + 8 and that is 14."
>
> Joe is a second grader and had this drawn on his paper:
>
> 
>
> When I looked at Joe's model I was confused at first so I asked him to explain his process to me. This is what he said:
>
>> I took two 6s and added them. That is 12. But this is not the correct number so I added 2 to the 12 and it is 14. So now it is 6 + 6 + 2 = 14. 6 + 8 = 14
>
> Joe's way is similar to Cecile's. They both relied on their knowledge of doubles to get to the right number. . . .

Indeed, these second graders, Cecile and Joe, did exhibit more sophisticated methods for attacking the valentine problem. In trying to find the missing addend, they worked with addition facts they knew (7 + 7 = 14; 6 + 6 = 12) and used them to solve it.

When Jessie, another second grader, worked on the problem, she offered two arithmetic sentences that provided the solution:

> If Yvonne has 6 stickers and they have 14 altogether I figured it out by minusing 6 from 14.
>
> 14 - 6 = 8
>
> I also figured it out like this:
>
> 6 + __ = 14

As Sorensen reflected on what she saw of her students' thinking in this lesson, she felt satisfied for where they now were.

> All the children had appropriate ways of solving the problem. They used methods that were familiar to them; some used number combinations that were easy for them to think about. They understood the problem and were able to explain their strategies and represent the problem in different ways. My goal for all my students is for them to feel comfortable in communicating their thinking process as well as to expand their repertoire of strategies for problem solving. I try to encourage them to solve a problem in more than one way and to share their strategies with someone else. I also would like my students to explore the properties of addition and subtraction. Jessie, who used the operations, knew that the problem could be solved by either addition or subtraction.

While student thinking is at the center of Sorensen's teaching, her role is not a passive one. In addition to selecting the tasks her students will work on each day, she is actively listening to her students and analyzing their thinking. Although not explicit in this episode, Sorensen's interactions with individuals and groups of children are designed to move their thinking forward.

> I feel that I need to be . . . asking some questions to elicit their ideas and helping them to begin to ask their own questions that will move them from their confusion.

Word Problems as a Context for Learning About Operations

A common practice in teaching mathematics is, first, to present rules for calculations, which students (sometimes) diligently memorize, and then to give word problems as an exercise in "application." In general, it is quite easy for students to find the numbers in the problem; the challenge is to identify which operation to use. The "correct" operation is that which, when applied to the numbers, produces the answer to the word problem.

To help students with this challenge, teachers frequently assign for memorization lists of "key words," words or phrases associated with one of the four operations. So, for example, if students see the word "left" in a problem, they will subtract; if they see the words "times more than," they will multiply. Students also frequently develop such strategies as selecting the operation according to the size of the numbers involved or testing out operations until they produce the answer that seems most reasonable. When working on word problems, the goal for both teachers and students is to select the operation that will provide the answer (Graeber & Tanenhaus, 1993).

Given this notion of the purpose of assigning word problems, Jody Sorensen's lesson would have to be assessed a failure. Indeed, of those students mentioned in the episode, only one, Jessie, used the number sentence deemed to be "correct": 14 - 6 = 8. However, Sorensen's purpose is quite different and, at the end of the lesson, she feels satisfied that "all the children had appropriate ways of solving the problem."

In fact, Sorensen did not teach her students rules for adding and subtracting before giving them the valentine problem. On the contrary, she set that problem in order to provide them with a context for constructing for themselves understandings of those operations. Similarly, many of the children worked at the problem without knowing the relevant number fact beforehand. Instead, that problem posited a situation which they could act out and, in so doing, make discoveries about the quantities involved.

As children work with different word problems over time, and as teaching is structured to encourage children to share their strategies and experiment with new ones, they experience the various types of problems that can be modeled by each operation. They also come to see how a single situation can be modeled by different operations.

In Sorensen's second grade classroom, Jessie observed that both 14 - 6 = 8 and 6 + __ = 14 represent the valentine problem. The episode does not indicate whether Jessie regularly explored the different operations that can model a particular situation. Nor do we know whether, or how, this idea developed in class in later lessons. However, when we look across classrooms, and across grades, we see that observations like Jessie's frequently arise.

Having learned mathematics in the traditional way, many TBI participants were initially unsure of how to proceed when their students' solution methods relied on operations other than the ones teachers intended. For example, Georgia Wilson was working on division with her combined third and fourth grade class.

> My initial thinking was, I wanted to expose kids to division problems. . . . But as I've begun my research, I'm learning much more about how kids think about division, what they call division, and how they define division. . . . Each week, I wrote story problems that I considered to be division problems, problems I would solve using division. Here are some examples of the problems and their responses to them.
>
> *Problem 1: Jesse has 24 shirts. If he puts eight of them in each drawer, how many drawers does he use?*
>
> Vanessa wrote, "24 - 8 = 16, 16 - 8 = 8, 8 - 8 = 0," and then circled "3" for the answer.
>
> *Problem 2: If Jeremy needs to buy 36 cans of seltzer water for his family and they come in packs of six, how many packs should he buy?*
>
> This time Vanessa added: 6 + 6 = 12, 12 + 12 = 24, 24 + 6 = 30, 30 + 6 = 36. . . .
>
> Other students use these same methods. Is it significant that sometimes they add and sometimes they subtract? What are their choices based on? I thought problems 1 and 2 were the same kind of problem, and yet they were treated differently. . . .
>
> Matthew worked on the following problem:
>
> *You go into a pet store that sells mice. There are 48 mouse legs. How many mice are there?*
>
> Matthew organized his work beautifully. He wrote a key and put his numbers in columns.

1 m	4 ℓ
2 m	8 ℓ
3 m	12 ℓ
4 m	16 ℓ
5 m	20 ℓ
6 m	24 ℓ

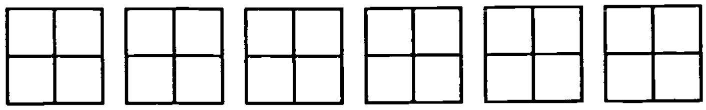
7 m 28 ℓ
8 m 32 ℓ
9 m 36 ℓ
10 m 40 ℓ
11 m 44 ℓ
12 m 48ℓ

> And then in a neat box he wrote 12m x 4ℓ = 48ℓ. And above the box is the number 12. What does this say about Matthew's understanding of division? He knows that 12 is the answer but he feels satisfied with a multiplication number sentence where the answer is not the answer to the problem; rather it is part of the problem. He knows how to find the answer; he knows how division and multiplication relate to one another, but instead of the answer I had expected, 48 ÷ 4 = 12, he wrote a multiplication number sentence.
>
> During a conversation with three kids about a similar problem, Matthew said, "This is another division problem. It's 63 divided by 9. What number times 9 is 63? 7." When I asked him to explain what about the problem made it a division problem, he said, "I don't know, but it is—but my thinking is multiplication."
>
> What does this say about kids' understanding of division, if they use all the other operations except division?

Wilson's observations about her students' different solution methods are significant. Vanessa had internalized the operations of addition and subtraction and used them to model the (imagined) actions of Wilson's "division" problems: putting shirts into drawers or loading up a grocery cart with 6-packs of seltzer. Matthew, it seems, could recognize a problem as division, but thought in terms of multiplication, finding a missing factor, to solve it. These children's understandings of division will build from here, deeply connected and in relation to other operations.

Sarita Worsley made a similar discovery when she presented a word problem to start a unit on division of fractions with her sixth graders.

> The students in my sixth-grade class have been solving fraction word problems for several weeks. During this class they were spread around the classroom and working in groups of twos and threes. The directions were to draw a picture to solve the problem, write a number sentence (equation) that explains the problem, and show any computation work.
>
> This was the problem they were given to solve:
>
> *You are giving a party for your birthday. From Ben and Jerry's Ice Cream Factory, you order 6 pints of each variety of ice cream that they make. If you serve 3/4 of a pint of ice cream to each guest, how many guests can be served from each variety?*
>
> . . . For the most part, once the students got started they didn't seem to find this a difficult problem. They drew pictures to represent the pints of ice cream and separated each pint into four equal sections. They explained that every three sections was what one person would receive and that was equal to 3/4. There was enough to do that eight times, so each variety would serve eight people.



However when they expressed this problem in a number sentence or equation, the range of answers was interesting. I use the word interesting because I am not sure whether the students are incorrect or that they just see things differently than I do. Although I could understand the thinking behind their equations, I didn't think that they were "correct." I had been taught that equations have to match what is going on in the problem. In my mind the students' equations seemed to be, while not necessarily wrong, just not correct!

I was trying to get the students to see that this was a division problem: $6 \div 3/4 = 8$. Instead they came up with the following equations and had very impressive reasoning to justify their thinking:

EQUATIONS	JUSTIFICATION
$24 \div 3 = 8$	There are 24 pieces, 3 pieces to a serving, 8 people can be served.
$8 \times 3/4 = 6$	8 servings of 3/4 of a pint each gives you 6 whole pints.
$3/4+3/4+3/4+3/4+3/4+3/4+3/4+3/4=6$	3/4 each gives you 6 whole pints.
$6-3/4-3/4-3/4-3/4-3/4-3/4-3/4-3/4=0$	Take 3/4 pint for each serving. You do this 8 times.

When Worsley wrote this episode, she was taken aback by her students' work—she had expected her students to recognize this as a division of fractions problem, "$6 \div 3/4 = 8$," and hadn't been sure how to respond. She has since come to appreciate the validity and the power of their observations and has realized that, in fact, these four equations *did* match what is going on in the problem. Once her students solved her birthday-ice-cream problem by multiplying, adding, or subtracting fractions, as well as by dividing whole numbers, they could explore how all of these are related to each other *and* to division of fractions.

As one examines the work of the children in these episodes, one recognizes, *implicit in their work*, observations and strategies which could be called algebraic. When children are challenged to find different solution methods for word problems, they see that the same problem can be modeled by different operations. Although the children do not use algebraic notation, the following equivalencies are implied in their work:

$$14 - 6 = x \longleftrightarrow 6 + x = 14$$

$$63 \div 9 = x \longleftrightarrow 9 \times x = 63$$

$$x \times 3/4 = 6 \longleftrightarrow 6 \div 3/4 = x \longleftrightarrow 24 \div 3 = x$$

Calculation as a Context for Learning about Operations

In conventional elementary mathematics teaching, a major—perhaps *the* major—goal, is to have children remember their basic math facts and the conventional computational algorithms. Children are commonly given sets of computation problems and asked to produce pagesful of correct answers as quickly as possible (Cohen et al, 1990, 1993; Goodlad, 1984).

Implicit in the conventional computational algorithms is use of the base-10 structure of the number system and application of the commutative, associative, and distributive properties. However, as calculation is conventionally taught, the emphasis is on remembering the sequence of steps to produce an answer as quickly as possible. Typically, lessons on calculation are not seen as opportunities to develop deeper understandings of place value or the properties of operations.

Yet, as the previous episodes demonstrate, if operations have meaning for children, they can devise a variety of appropriate procedures for calculating. When Cecile and Joe didn't know what number added to 6 yields 14, they recalled 7 + 7 = 14 or 6 + 6 = 12 and figured out how those facts could be used to answer the question. Eventually, Cecile and Joe—and, one hopes, all children who move through second grade—will have number facts like 6 + 8 = 14 firmly in place. In the meantime, as these children worked to solve the problem, they already displayed an important mathematical "habit of mind": *When faced with a problem that you cannot solve immediately, begin with a problem you do know how to solve and see if there's a way to transform it into the more difficult one.*

In the case of these second graders, 7 + 7 = 14 and 6 + 6 = 12 were facts they already knew. The transformation Cecile applied was to decompose a 7 to 6 + 1 and then group the 1 with the other 7 to produce 6 + 8 = 14. Joe saw that, from 12, he needed 2 to get to 14, so he added 2 to one of his addends, 6 + (6 + 2) = 12 + 2 = 14.

Soon the fact that 6 + 8 = 14 will be remembered. And, as these children are challenged to compute with larger numbers, they will exercise the same habit of mind, employing a variety of solution strategies—as did, for example, the second graders in Lynn Norman's class as they explored 2-digit subtraction.

> Fiona worked on a variation of the word problem that involved regrouping (of 37 pigeons, 19 flew away). She dropped the 7 from the 37 for the time being. She then subtracted 10 from 30. Then she subtracted 9 more. She puzzled for a while about what to do with the 7, now that she had to put it back somewhere. Should she subtract it or add it? I asked her one question: Did those seven pigeons leave or stay? She said they stayed, and added the 7.

37 - 19

30 - 10 = 20

20 - 9 = 11

$$11 + 7 = 18$$

It was interesting to me that Fiona was able to use that one question to clear up her confusion, and I think for the most part she subtracts this way and keeps it straight. As Fiona goes through the steps in her algorithm she is able to keep track of when to add and when to subtract. The 7 is being subtracted (from 37) and then added again (at the end, to 11). The 9 from the 19 is in a way added to the 10 in 19, but it gets subtracted, because Fiona needs to subtract all of the 19. The 7 is part of what is being subtracted from. The 9 is part of what is being subtracted. It is a complicated process and it is amazing to me that a second grader can make sense of it for herself. . . .

Paul also takes numbers apart to subtract. To solve 39 - 17, he takes the 17 apart:

$$39 - 10 = 29$$

$$29 - 4 = 25$$

$$25 - 3 = 22$$

Paul is keeping track of the 17 and breaking it into familiar chunks. Many children wondered where he got the 10, 4, and 3 from. How did he know what to subtract? How did he know when he was done?

Interestingly, Paul had questions for Nathan about how Nathan knew which numbers to put together for his answer. Here is Nathan's process for 39 - 17:

$$17 + 3 = 20$$

$$20 + 10 = 30$$

$$30 + 9 = 39$$

$$3 + 10 + 9 = 22$$

Nathan just about always adds, even for what seems like a straightforward separating situation like birds flying away. After Nathan told how he solved this problem, Paul said, "But how does he know what numbers to add up at the end?"

In Norman's three examples, the children began with a 2-digit subtraction problem and decomposed it into parts they could more readily manipulate. Their strategies aimed toward working with tens, which made handling large numbers easier, but they had to understand how the operations work in order to put the numbers back together correctly.

Working on 37 - 19, Fiona decomposed each number into tens and ones, and knew that each part needeed to be either added or subtracted to produce the result. The problem context provided a touchstone for figuring out what to do with which parts.

Paul and Nathan, who both solved 39 - 17, appeared to be employing their methods without reference to the problem context. Paul broke the 17 into three pieces and subtracted one piece at a time. Nathan started at 17 and added up to 39, keeping track of how much he added at each step.

Continuing her episode, Norman reflected on how the thinking of these students compared with that of those of her students who used the conventional algorithm.

I thought a little bit about children using the conventional algorithm. A few do sometimes, ever since I gave word problems for homework. So much for asking parents not to help them. If a child memorizes the procedure, there is no real "keeping track." They must learn the steps, but they do not need to keep track of what the 3 in 37 means or how much of the 19 they have subtracted so far. All they do is use the recipe. If they get confused or forget a step or go out of order, children using this procedure don't tend to go back and make sense of the numbers or the problem, or try to keep track of what is going on.

Norman's concern was not about the conventional algorithm *per se*, but how her students tended to use it. She found that when they apply a memorized algorithm whose steps are not meaningful to them, they stop thinking about the size of the numbers and the nature of the operation. For that reason, and in contrast to past practice, she now commits more class time to giving her students opportunities to devise their own procedures.

Finally, an almost unrelated observation: This year, for the first time, I have never seen a single child "subtract up" in the ones column if the bottom number is greater than the top one. I have always had many children do this other years.

$$\begin{array}{r} 37 \\ -\underline{19} \\ 22 \end{array} \quad \text{because } 3 - 1 = 2 \text{ and } 9 - 7 = 2$$

I am not sure what to make of this, but I hope it is because the children this year carry more of the meaning of the problem with them, because they are allowed to construct their own ways of solving it.

Susannah Farmer wrote about her third graders' first attempts to think through multiplication of multi-digit numbers. She had given them the following problem: "There were 64 teams at the beginning of the NCAA basketball tournament. With 5 players starting on each team, how many starting players were in the tournament?"

"Wow, that's hard," proclaimed Jenny loudly, and a chorus of protesters joined her. Undaunted, Julia presented her thinking:

> That would be 64 x 5. I use one 10, because I know 5 x 10 = 50. Then you do that six times. (She counted by 5s, not using her fingers, but moving her lips and nodding her head for each group of 5.) That's 30, I mean 300. Then you add 4 five times, which is 25, no 20. I added it all together and got 320.

$$\begin{array}{r} 5 \times 10 = 50 \\ 5 \times 10 = 50 \\ 5 \times 10 = 50 \\ 5 \times 10 = 50 \\ 5 \times 10 = 50 \\ 5 \times 10 = \underline{50} \\ 300 \end{array}$$

$$4 + 4 + 4 + 4 = 20$$

$$300 + 20 = 320$$

Chris, usually reticent and lacking in confidence, volunteered his thinking in a quiet, unassuming voice:

> 64 means 60 + 4. . . . So I did 60 five times, for 300. Then 4 x 5 is 20, so the answer is 320.
>
> 60 x 5 = 300
>
> 4 x 5 = 20
>
> 320

Chris returned to his seat in a way I can only describe as cocky. I was certainly impressed.

Jack, our resident goof-off, but intuitive math thinker, explained his strategy next:

> I split the 64 into four parts—20, 20, and 20. . . . I did each one separately.
>
> 20 x 5 = 100
> 20 x 5 = 100
> 20 x 5 = 100
>
> Then the last part, 4 x 5, is 20. All together, 320.

These were the ideas and strategies I'd tried so hard to explain and instill in my students last year—breaking up numbers in useful parts, recognizing which numbers are being multiplied and by how much, finding a way to multiply that makes sense. Posing the right questions and relying on the children to use what they'd been practicing all year proved to be the solution to teaching multiplication. Of course, not everyone had moved beyond adding, or understood what their colleagues were doing. But they were listening and would begin to develop new strategies as we continued to multiply.

. . . I was just about to hand out colored tiles for building arrays, when another multiplication opportunity presented itself.

> Teacher: We have 18 kids here today and each needs 12 tiles for the next activity. How can we figure out the number of tiles to give out?

It took me a second to realize this problem was a leap from the ones we'd just done. But it was "real life," so I let the question stand.

I was surprised that no one suggested using a calculator, their usual response to big numbers. But who needs a calculator when you have Josh!

> That would be 18 x 12, and I know 10 x 10 is 100 and 8 x 2 is 16, so if you add them together it would be 100 + 16 = 116.

Everyone seemed satisfied with the answer, whether out of agreement or lack of interest, I wasn't sure. After all, the process mimicked what they'd just been doing. I was thinking what to say that would help them see the error of their ways, when David's voice broke the quiet: "That's wrong."

"What do you mean, David?" I asked.

> I did 18 x 10 and got 180, but I thought at first I was wrong, so I double checked. I noticed that Josh didn't do 8 x 10, so my answer was right. (David is very knowledgeable about the workings of our number system, but leaves gaps in his verbal explanations. His mind races, and neither his mouth nor our brains can keep up.) I didn't do the 2 yet, so I do 18 x 2. Then you add it up—180 + 36.

"Wow," I thought, amazed at his understanding, but realizing that the rest of the class looked dazed. Luckily, there will be more chances for [these children] to show what they know about multiplication.

When challenged to solve computational problems with large numbers, Farmer's students, like Norman's, broke the given problem apart to work with smaller, more manageable bits. In this case, they had to hold onto what multiplication *does* in order to keep track of what to do with the parts. To solve the problem, 64 x 5, Farmer's third graders suggested various ways of decomposing 64, multiplying each part by 5, and totalling the partial products.

A problem with two 2-digit factors, 18 x 12, produced some confusion. Josh applied a strategy used effectively with addition—decompose the two numbers into tens and ones; combine the tens and combine the ones; add the results together—which does not work for multiplication. His classmate, David, recognized the error—"I noticed that he didn't do 8 x 10"—and offered a strategy that works: Decompose the 12 into 10 + 2 and multiply 18 by each of those parts; add the results.

As the children depicted in these episodes worked on computations with numbers beyond familiar facts, they found equivalent expressions containing numbers they did know how to work with. Implicit in their strategies is application of commutative, associative, and distributive properties of the operations.[3]

$$7 + 7 = (6 + 1) + 7 = 6 + (1 + 7) = 6 + 8 = 14$$

$$37 - 19 = (30 + 7) - (10 + 9) = 30 - 10 - 9 + 7$$

$$64 \times 5 = (60 + 4) \times 5 = (60 \times 5) + (4 \times 5)$$

$$18 \times 12 = (10 + 8) \times (10 + 2), \text{ which does } \textit{not} \text{ equal } (10 \times 10) + (8 \times 2)$$

Given opportunities to articulate their own ways of solving problems and challenged to make sense of those of their classmates, we see these children learning to navigate the number system with considerable fluency.

Algebraic Notions, Implicit and Explicit

A major goal of traditional elementary mathematics teaching is to have students learn to compute. To that end, children are expected to remember basic arithmetic facts and learn algorithms for adding, subtracting, multiplying, and dividing multi-digit whole numbers and rationals. Word problems are seen as opportunities to practice application of computational procedures (Kieren, 1992).

The classroom episodes presented above demonstrate that, for a reformed mathematics instruction, the goal of teaching computation is not replaced, but surpassed. When teaching is

[3]The claim is that the properties of commutativity, associativity, and disributivity are implicit in the children's work. This is not the same as the children using the notation presented in the following lines. In particular, there is no evidence in the episodes that the children have a sense of "=" as it is used in these equations.

designed to build on children's ways of solving mathematical problems, they begin by enacting those problems, counting out all of the quantities involved. Given opportunity and challenge, they develop more sophisticated strategies, strategies they employ with understanding. Indeed, we have seen how word problems and computational exercises can provide contexts for constructing meanings for the operations and the structure of the number system.

Algebraic methods are clearly implicit in the children's work described above. As they apply different operations to solve a single word problem, they evidence a sense of how the operations are related. For example, as the children come to see that any missing addend problem can be solved by subtracting, or that any division problem can be solved by finding the missing factor, they are acquiring experience of the inverse relationships of addition/subtraction and multiplication/division and, so, of equivalent equations. And as they develop fluency in a variety of computational strategies, they implicitly apply the laws of commutativity, associativity, and distributivity.

When students have learned arithmetic as a set of memorized procedures and have lost contact with their own abilities to make sense of calculations and operations, it is no wonder they have to rely on remembered rules and procedures to pass an algebra course. For example, what if Josh, the student in Farmer's third-grade class, never had confronted his error in solving 18 x 12 by calculating 10 x 10 + 8 x 2? What if there were no occasion in his elementary education to think through what multiplication does so that he comes to understand why 18 x 12 *can* be solved by calculating, instead, 18 x 10 + 18 x 2? If he someday enters algebra class without that understanding, how is he to learn how to multiply binomials? What else can he do to learn that $(a + b)(c + d)$ does not equal $ac + bd$, but memorize the correct identity?

In the classroom episodes presented above, we have seen children developing a sense of arithmetic that extends beyond finding answers to given calculations or word problems. Where instruction is designed to help children construct meaning for operations and devise procedures for navigating the number system, there is greater emphasis on examining the variety of problem-solving strategies the children employ than on the answers those strategies produce. Their approaches frequently involve solving problems with different, equivalent equations; or performing calculations by working with equivalent arithmetic expressions. Thus, when Josh and his classmates eventually confront "$a(b + c) = ab + ac$," they will bring to it their discussions of why 18 x 12 is equal to 18 x 10 + 18 x 2, rather than 10 x 10 + 8 x 2.

However, in algebra these notions are no longer implicit. Students must come to explicit understanding of distributivity and of the inverse relationship of addition and subtraction. When asked to factor $ab + ac$, they cannot take cues from the numbers because "a," "b," and "c" represent any number. Instead, they must rely on their understanding of multiplication and addition, of what those operations do. While children in the episodes freely employ what adults recognize as distributivity, are they able to abstract multiplication from its application to particular situations or particular numbers to make a claim about a general property of the operation? When children have rich and meaningful experiences in arithmetic, what is involved in learning how to recognize, articulate, and justify their observations of the properties of the operations, themselves?

This is the question that half[4] the TBI teachers and staff posed for themselves in the spring of 1996. In their monthly episodes, those teachers investigated their students' generalizations in the context of arithmetic. To what extent does children's natural curiosity draw them to observe and examine patterns in the number system? How do they articulate their observations and what is the extent of their generalization? And how do the children justify their claims of generality?

When Katherine Kline began to think about arithmetic generalizations in her second grade classroom, she wrote,

> I thought of an idea that quite a few children in my class have been quite vocal about and seemingly quite certain of: something they call "turn-arounds." Turn-arounds came up first when we were generating ways to "make" ten early in the year. The children made a list like 5+5, 4+6, 3+7—and then would suggest 6+4, 7+3 etc. and referred to them as "turn-arounds." Soon everyone was calling 4+6 and 6+4 "turn arounds" and it became almost a vocabulary term without ever really discussing its implications. So [since our TBI group has been thinking about "generalizations"] I decided to ask them to think about turn-arounds and see if they might define it or describe it or illuminate something about it for me.

It seems that Kline's second graders had themselves identified the commutative property of addition, which they captured with the expression, "turn-arounds." Sticking with the term they had coined, she decided to set up a lesson in which her students would explore the idea of turn-arounds, while she, in turn, would explore their thinking.

The children in Kline's class were ready to take on an investigation of the commutative property of addition. Although for months they had been talking about "turn-arounds" as a fact, they now engaged in this exploration with energy and curiosity. Among Kline's discoveries was that, though her students' earlier observations about turn-arounds seemed to imply recognition of a generalized property, not all of her students were convinced that it always works. Natalie, though, was fairly sure—"Turn arounds always work. I just know they do."

Teacher: How do you know?

Natalie: Well, look, 27 + 4 = 31 and 4 + 27 = 31.

Teacher: But does this always work, for any number, no matter how big it gets?

Natalie: Well, let me try it.

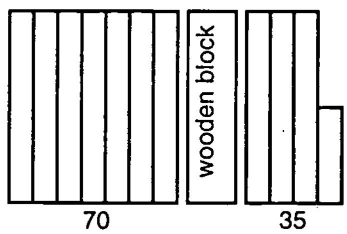
[4]The other half of the project investigated student thinking in the context of geometry.

> So Natalie tried numbers in the hundreds and added them together both ways and felt convinced that it always worked. Her reasoning seemed to be based on her having done many of them and having had them always work out to be the same answer.

Other children, less confident than Natalie about their claim, also checked it by testing out numbers larger than those they frequently encountered. For example, Ingrid, using a calculator, recorded, "22 + 100 = 122, 100 + 22 = 122; 8 + 99 = 107, 99 + 8 = 107; 100 + 3 = 103, 3 + 100 = 103," and expressed deep satisfaction with her work..

Emily was among those who were not at all sure. In order to explore the conjecture, she turned to a representation of addition frequently used in Kline's classroom: joining two sets of cubes. Kline's colleague, Lynn Norman, who was visiting Kline's classroom that day, described how Emily thought about 70 + 35.

> At the end of class, when the children shared what they worked on, Emily showed how she worked with adding 70 and 35. She had 70 cubes in stacks of 10, and this was separated by a wooden block from 35 cubes in stacks of ten and one five.



> She said she added 35 to 70 by counting on: 80, 90, 100, 105. Then she moved the two groups of cubes so that the 35 was to the left of the block and the 70 was to the right. She counted on again: 45, 55, 65, 75, 85, 95, 105. This pair worked because they added up to 105 in either order. Kline asked if Emily thought it would still work for different numbers. Emily said she didn't know because she only did these ones.

In contrast to Natalie and Ingrid, Emily decided to work with different representations of quantity and of addition to think about the question. Initially, she felt that her method demonstrated that 70 + 35 yields the same answer as 35 + 70, but she didn't know if it would also work for other numbers. However, as the class discussion moved on to other topics, Emily continued to think through just what she could learn from her representation.

> A few more children shared and then Emily raised her hand again. She said that she could use the same cubes but divide them up differently and it would still work. When asked to demonstrate, she moved the block to another spot and said that the two new parts would also add up to 105, no matter what order she added them. She said it would work no matter how she divided the cubes because there would always be 105.

Relying on her representation, Emily was convinced that, regardless of how she decomposed 105 cubes into two parts, the sum of the parts would always be 105. Thus (one might think of her conclusion as a corollary), any two parts can be added in either order. Norman wrote:

> It seems Emily has reached a general rule, at least for the number 105. For any [partitions] of 105, she seems to be sure that you can add them in any order and still get 105. Having the 105 cubes in front of her seems to help her be sure. As her classmate, Nathan, said, she will always get the same answer because she is always starting with the same number of cubes. Her apparently firm grasp of conservation of number is providing a base for her to make more generalizations about how numbers behave, and her work with 105 will probably help her to generalize further. When will she be sure that "it works for all numbers"?

Initially, Emily's conclusions had been limited to the specific configurations of her blocks. However, as she continued to reflect, and as her classmates, like Nathan, commented on her representation, she extended her claim to a generalization about any paired addends of 105. On the other hand, Nathan was prepared to go further: "She will always get the same answer because she is always starting with the same number of cubes." That is, Nathan's generalization does not rest on the specific number, 105, but is valid for collections of any number of cubes.

Students like Natalie and Ingrid tested commutativity by checking particular number facts, choosing numbers outside the range of familiarity. They treated addition as "something you do to two numbers to get a result." Emily chose a different tack, relying on a representation of addition—joining two quantities of blocks—which allowed her to think about how addends and their sum are related. Initially, she didn't see how her representation extended beyond particular numbers she had chosen: 70 + 35 and 35 + 70. However, after some thought, she saw that when 105 is broken into any two quantities, that sum is conserved no matter the order in which you bring the two quantities together. For Emily, the next step would have been to realize, as Nathan already had, that this relationship extended beyond those 105 cubes she had laid out, to any number whatsoever.

While many of Kline's students explored the generality of additive commutativity for numbers larger than those turn-arounds they first noticed, other students wondered whether the property generalizes across operations. Kline wrote:

> [Some] children felt that the turn arounds for subtraction worked if the answer to the problem was zero, like on Daniel and Kareem's page where they say 100 - 100 = 0 and 100 - 100 = 0. They said that 9 - 8 = 1 and 8 - 9 = 0 without any ideas about negative number possibilities. Several others supported this idea with examples like, "Well if you have nine fingers and you take eight fingers away you'll have one. But if you have eight fingers and you try to take away nine fingers you'll have none left."

Daniel, Kareem, and others of their classmates worked with a domain that does not include numbers less than 0. Thus limited to positive integers, their conclusion is valid—turn-arounds for subtraction don't work.

However, the children in the class who did have ideas about negative numbers began to see a pattern emerge:

> 6 - 5 = 1 and 5 - 6 = -1! When you turn around a subtraction problem you get the same answer but as a negative number. They were quite intrigued with this and pursued it for the math period.
>
> When I asked if this would always work, Adam and Tom simultaneously said yes/no. Then they looked at each other and laughed. They began showing all their examples that did work, but when I asked again, "So will this always work with turn-arounds in subtraction?" they didn't think so. They thought it was "luck" that they had found the numbers that they did.
>
> But when I watched them work they talked through several examples by picking two numbers, subtracting them and getting a positive answer and then immediately assuming that the turn-around of it would be the negative. So 25 - 12 = 13 meant that they immediately would say that 12 - 25 = -13. Tom determined that answer by counting backwards 12 to zero and then saying, "13 more to subtract the rest of the 25." But he did not seem inclined to prove it with the counting. . . . It would seem to me that they both had generalized this rule or pattern in relation to subtraction. But when asked if it always worked, they were indecisive and said it was "luck" that made the ones they had found work.

When Adam and Tom explored turn-arounds for subtraction, it *appeared* that they understood the relationships among the quantities involved. Working on 12 - 25, they counted back 12, from 12 to 0, and then subtracted 12 from 25 to figure out that they needed to count back another 13 steps, to -13. While they performed this action repeatedly, confident that it would give them the correct answer, they appeared unable to analyze their actions to explain how the numbers—12, 13, and 25—are used in 25 - 12 and 12 - 25. Unlike Emily, they did not employ a representation of the operation to consider how the quantities are related and could not articulate why their strategy for calculating a smaller number minus a larger will always produce the pattern they had identified Thus, the boys were indecisive when asked if turn-arounds for subtraction (you get the same answer but as a negative number) always work.

In this class, the justifications these second graders used are at different levels of abstraction. Some children relied on the view that addition is a procedure applied to two numbers to produce an answer. Their test of commutativity rested on repeatedly coming up with the same answer for pairs of number facts. Others, like Emily and Nathan, began to explain commutativity in terms of how quantities are related under addition.

Jean Boyer was working with her fourth graders on factors of multiples of 100, using a set of activities taken from *Investigations in Number, Data, and Space*, "Landmarks in the Thousands" (TERC, 1995). Boyer describes the activities:

> For two days, students worked on finding all of the factors of various hundreds numbers [i.e., multiples of 100]. Some worked alone, some in pairs, and there were two small groups of students. Although they were all doing the same assignment, strategies and approaches differed widely. Some students picked a number and skip counted on the calculator to see if they landed on their hundreds number. Some students divided their hundreds number by the factor they were testing, to see if the answer was a whole number. Some added with paper and pencil. Some tested out factors in their heads. Some picked numbers at random to try. Some tried every number in order. Some could predict fairly accurately which numbers would be factors and tried those.

The second day, we posted the number of factors that had been found for each hundreds number. We didn't say what the factors were, just how many had been found. Students developed ways to see if they had all the factors. They would check their lists with each other, challenging each other's listings and double-checking their own.

We began the third day by assembling an organized chart on the board from their investigations of the first four hundreds. [Under each hundreds number, 100 to 400, we listed its factors.]. . .

[Then I asked the following focus questions:] What patterns do you see in looking at the factors of different hundreds numbers? How might the patterns help you in figuring out the factors of a hundreds number that is not yet on our chart?

Students were ready to offer a variety of observations:

All of the hundreds numbers have 1 as a factor, and they have, most of them have 2, and some, all of them have 10.

All the second to last numbers in the columns go up by 50. For 100 the second to last factor is 50. For 200, it's 100. For 300, it's 150. For 400, it's 200.

Every one has it's own number.

All the numbers in the hundreds column, if you plus that number two times, you'll get something in the 200's column. Like 1, if you do it two times you'll get 2. There's a 2 in the second column. If you do 2 + 2 you get 4. If you do 4 + 4 you get 8. All the numbers in the 100 column, if you double them they're in the 200 column.

By the end of the class period, the students had come up with 25 observations, which Boyer wrote up on a single sheet of paper. (See Figure 1.) The following day, Boyer handed out copies of the observations and then asked students to pick whichever observations they wanted, determine if it is always or sometimes true, and offer a proof.

Always or Sometimes?

1. 1, 2, 4, 5, 10, 20, 25, 50, and 100 are factors of all other hundreds numbers.
2. The next to the highest factors go up by 50.
3. The highest factor of a hundreds number is itself.
4. If you double the numbers in the 100 column, you get numbers in the 200 column.
5. Every two- or three-digit factor of hundreds numbers ends in 0 or 5, except 12 and 16.
6. There are a lot of 5s in the factors of hundreds numbers (5, 25, 50).
7. The factors of a hundreds number are in its double.
8. Factors of 100 and 500 go odd, even, even, odd, even, even.
9. If you take the zeros off a hundreds number, you get one of its factors.
10. 300 is the only hundreds number with 3 as a factor.
11. If you double a factor of a hundreds number, you get another factor of that number.
12. Each hundreds number has all the factors of 100.
13. The first two digits of a hundreds number are a factor of the number. 100 has 10 as a factor; 200 has 20 as a factor.
14. All other hundreds numbers will have more factors than 100.
15. Hundreds numbers have a lot more even factors than odd ones.
16. 7 is not a factor of a hundreds number.
17. The highest factor of a hundreds number is that number.
18. Many factors are multiples of 10.
19. The smallest factor x the largest factor = the hundreds number.
20. The next to the smallest factor x the next to the largest factor = the hundreds number.
21. 300 has factors that the other hundreds numbers don't. (3, 6, 12, 15, 30, 60, 75, 150, 300)
22. The factors of a factor of a hundreds number will also be factors of the hundreds number.
23. The number of factors of a hundreds number increases or decreases by multiples of 6 as you work with each larger hundred.
24. Every number has 1 and itself as factors.
25. 300 has the most factors.

Figure 1. Class list of observations

In the opening discussion, Boyer talked to her students about what constitutes a proof. Is it enough to test out some numbers? How many numbers do you have to test in order to claim that something is always true? What else can you do to say that something is always true?

As they got to work, different children selected observations at various levels of generality. Their proofs relied on their representations of multiplication or division, highlighting different interpretations of what these operations do.

The first observation on the list was, "1, 2, 4, 5, 10, 20, 25, 50 and 100 are always or sometimes factors of all the other hundreds numbers." Some children chose to prove the claim for each factor separately. For example, Betsy suggested, "All hundreds numbers are even and all even numbers can be divided by 2, so 2 is a factor of all hundreds numbers."

Shavon went to the hundreds chart to show that 5 must be a factor of all hundreds numbers. First, she pointed to the 5's column and then, pointing to the 10's column, said that those numbers are multiples of 5. Since the hundreds numbers all appear in the 10's column, the hundreds numbers must have 5 as a factor.

When, considering 25 as a factor, Shavon said that she thought of dollars and quarters, Boyer asked the class to consider this idea. How many quarters in two dollars? three dollars? four dollars? five dollars?

Boyer: So what does that tell you about 25?

Joey: That 25 goes into 100 four times and, so, however many dollars, there are, umm, there will be four quarters for each one—or four 25s.

Boyer: So what does that prove?

Joey: That there's 25 in any hundred.

Still other children, also working from observation #1, interpreted it as "All factors of 100 are factors of other hundreds numbers." Thus, they offered a single proof for all 9 factors.

Chang, Ivan, and Khalid, working as a small group, thought about this claim in terms of skip counting. Although they *spoke* in terms of specific numbers, they explained that they were talking about *all* factors of 100. Khalid used 25 as his first example—you go up 4 for each additional hundred.

Chang interrupted, excited by his own insights:

> Because 100 has those numbers as factors. Say I'm skipping by 2. I need, umm, 50 skips to get to 100. Umm. Just add 50 more, then you got a factor of 200. Just keep adding by 50s, you get to higher numbers, the other hundred numbers.

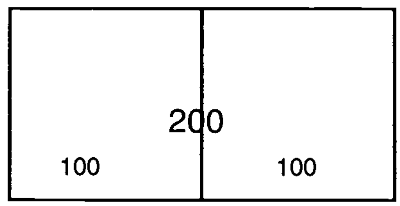
When asked if there are any hundred numbers that you can't get to that way, Ivan spoke up. "700 has 7 hundreds in it. So all those numbers are factors. You just have to take more jumps."

Virginia also wrote in terms of skip counting to prove that "the highest factor of a hundreds number is itself": "Yes, because you only have to make one jump."

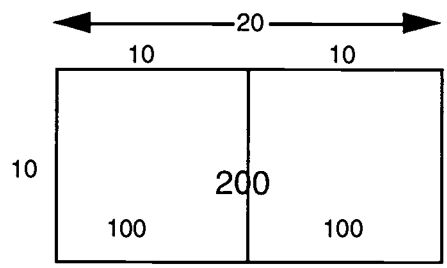
Jon, Williamson, and James worked on observation #4, "If you double the numbers in the 100 column you get numbers in the 200 column." [If you double a factor of 100, you get a factor of 200.] First they wrote:

> I think this works because 100 is half of 200 and therefore the factors of 100 are half of the factors of 200.[5]

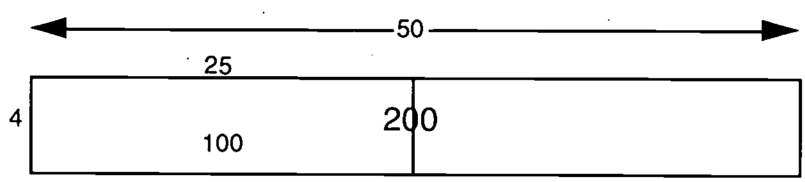
When Boyer suggested that they might explore this generalization using arrays, the boys excitedly got back to work. First, they drew the following picture:



When asked if there was any way to see factors in the picture, Jon filled in some more numbers. "See, 20 x 10. And inside here there is 10 x 10."



Jon's picture illustrates a rectangle whose area is 200, with vertical sides of 10 and horizontal sides of 20. Half the rectangle, whose area is 100, has vertical sides of the same length (10), and horizontal sides half the length (10). However, Jon did not stop to explain this. Instead, he went ahead and drew a picture to demonstrate the same notion with another pair of factors. Starting with 50 x 4 as factors of 200, he cut his rectangle in half to show that half of 50, 25, is the corresponding factor of 100.



These boys also made a general claim, but, like Chang's group, relied on specific examples to justify it. Yet, their justifications were not of the same type as those of Kline's students who claimed that "turn-arounds always work" for addition and tested the claim by checking particular facts. In this case, the children were using representations of multiplication to illustrate how the

[5]The boys did not consider any odd factor of 200.

quantities involved are related. Like Emily, they employed a representation with particular numbers. However, unlike Emily in her initial hesitation, they seemed to be saying that their representations demonstrate the general claim.

Tyrone, who wrote about observation #25, "300 has the most factors," ended up working at a higher level of generality.

> I don't think 300 has the most factors. Here's my proof. Because 600 is a multiple of 300 it has all the factors of 300 plus itself and 8.

An adult observer who came by as Tyrone was writing out his explanation asked about the claim that 600 has all the factors of 300. Here is Tyrone's written response:

> The reason for this is that any factor of a lower number will be a factor of any given multiple of that number. This is because you simply multiply the number the factor is being multiplied by as many times as the lower number goes into the multiple.

In this response, Tyrone was able to keep track of which number is "the lower number," which is "the multiple," which is "the factor," and which is "the number the factor is being multiplied by." Indeed, Tyrone's explanation provides a good example of the power of algebraic notation where the different quantities might be represented by *a, b, c, n*, and *m*. Thus, we can represent Tyrone's claims as:

> If a is a factor of b and c is a multiple of b, then a is a factor of c.
> This is because if $m \times a = b$ and $n \times b = c$, then $(n \times m) \times a = c$.

Conclusion

On the first day of my seventh-grade French class, the teacher held up a black plastic box and said, "Boîte." She walked over to the door, held it open, closed it, and said, "Porte." Then she crossed the room, held her hand against the window: "Fenêtre." She walked back to her desk, patted it, and said, "Pupitre."

My classmates and I were learning new sounds, words, but the objects those words represented were known to us. In the months ahead, we would learn to communicate in a new language: to introduce friends, to talk about going to the library, to complain of a headache. We would learn to use French to converse about things that were already familiar in our lives.

When students begin their study of algebra, they learn a new language, an efficient way of representing properties of, and relationships among, operations. To students who are already familiar with those properties and relationships, the challenge is to learn the conventional symbol system. But what about those who have never had the opportunity to develop operation sense? who do not have ideas about *what* this new symbol system is supposed to communicate?

When students first see an equation like "$a + b = b + a$," their attention is drawn to the unfamiliar symbols, "a" and "b." Do they realize that exactly the same idea is expressed by "$x + y = y + x$"? Do they realize, in fact, that it is not "a," "b," "x," or "y, which represent any number, that plays the central role in that equation, but "+"? That is, "$a + b = b + a$" is a statement about the operation of addition.

By the time Katherine Kline's students enter middle school, they will likely have dropped the term "turn-arounds." However, their second-grade explorations of turn-arounds, and the experiences that build on them, will provide grounding for the conventional term, "commutativity." They will know that addition is commutative, and they will learn that this idea can be expressed as "$a + b = b + a$." Furthermore, they will know that subtraction is not commutative, but that there is regularity in reversing the terms, which can be expressed as "$a - b = -(b - a)$."

Concern about the extent and quality of algebra learning in middle- and high-school courses has brought researchers, curriculum developers, and policy makers to begin thinking about the kinds of elementary school experiences that can prepare students for algebra. In this paper, I have attempted to make the case for an emphasis on the development of *operation sense* as critical to this preparation. Furthermore, I hypothesize that once the teaching of elementary arithmetic is aligned with reform principles, when classrooms are organized to build on students' mathematical ideas and to keep them connected to their own sense-making abilities, then children so taught, upon entering middle school, will be prepared for algebra. They will have had experience with commutativity, distributivity, equivalent equations, etc. and, so, will bring meaning to algebraic formulations of such notions.

In this paper, I have placed emphasis on middle school students learning and using formal algebraic notation. However, the same argument applies to other notational systems, as well—diagrams, graphs, tables in written and electronic forms, etc.

While this paper has taken on the question of how elementary education can prepare students for algebra, teachers of algebra cannot wait until students with such preparation arrive. When a middle-school, high-school, or even college teacher confronts a roomful of students who have learned to calculate but have not developed operation sense, they need to consider the kinds of experiences that can help their students begin to make meaning for the language of algebra.

It is especially critical to reconceive those courses taken by pre- and in-service elementary teachers. If the argument of this paper is valid, conventional algebra lessons are not a first priority for their professional development. Above all, these teachers must, themselves, have opportunities to connect with their own abilities to make sense of mathematics and to develop their own understandings of the basic operations. In addition, they must learn how to listen to children's mathematical ideas, to assess the validity of children's thinking, and to pose questions that are likely to challenge and extend children's conceptual understandings. This is no small task.

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