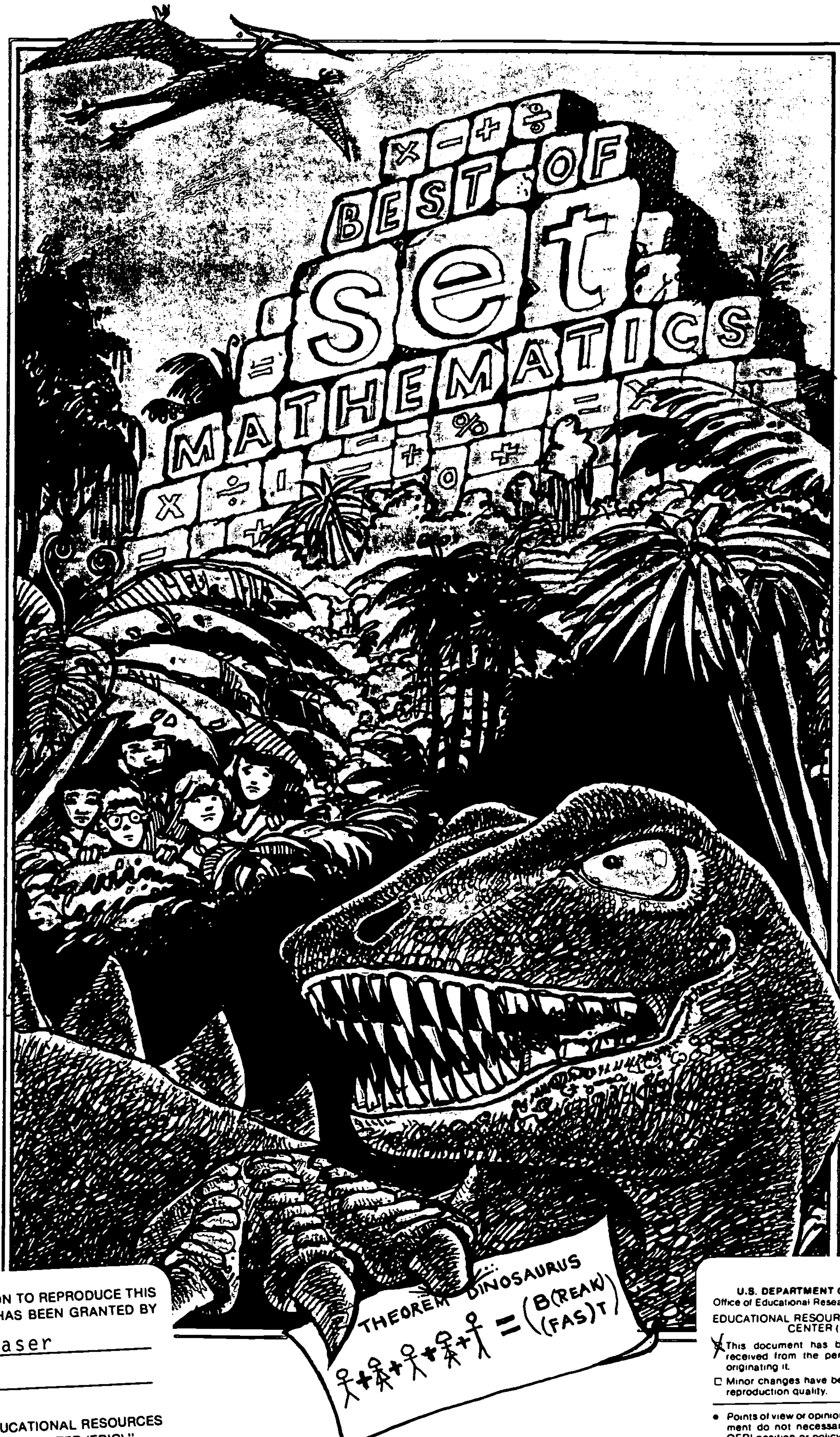
DOCUMENT RESUME

ED 405 176 SE 054 747

AUTHOR	Livingstone, Ian, Ed.; Izard, John, Ed.
TITLE	Best of SET Mathematics.
INSTITUTION	Australian Council for Educational Research, Hawthorn.; New Zealand Council for Educational Research, Wellington.
REPORT NO	ISSN-0110-6376; ISSN-0725-4873
PUB DATE	[93]
NOTE	73p.
AVAILABLE FROM	Australian Council for Educational Research, Private Bag 55, Camberwell, Victoria 3124, Australia ($19.95 Australian).
PUB TYPE	Collected Works - Serials (022)
JOURNAL CIT	SET: Research Information for Teachers; spec iss 1993
EDRS PRICE	MF01/PC03 Plus Postage.
DESCRIPTORS	Algebra; *Cooperative Learning; Elementary Secondary Education; Foreign Countries; Fractions; *Mathematicians; Mathematics Instruction; *Number Concepts; *Problem Solving; *Student Evaluation; *Writing Across the Curriculum
IDENTIFIERS	Australia; New Zealand; *Single Sex Classes

ABSTRACT

Set: Research Information for Teachers, is published twice a year by the New Zealand Council for Educational Research and the Australian Council for Educational Research. This document draws together 16 articles on mathematics from previous issues grouped into three categories: general, primary, and secondary. The titles are: (1) "Contents and Introduction," (2) "Twenty-nine Tough Questions," (3) "Unlocking the Great Secret," (4) "Evaluating Learning in Mathematics," (5) "Understanding Children's Mathematics: Some Assessment Tools," (6) "What Mathematicians Do," (7) "Number Skills in Junior Classrooms," (8) "Small Children Solve Big Problems," (9) "There are Numbers Behind the Piano," (10) "Being Both Right and Wrong," (11) "Beginning to Learn Fractions," (12) "Helping One Another Learn," (13) "A Child's Perspective of Algebra," (14) "What are the Benefits of Single-Sex Maths Classes?" (15) "Mathematical Needs of School Leavers," and (16) "How I Failed." (MKR)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Educational Consultant

Best of set: Mathematics

Contents and Introduction

From past issues of set we have extracted a number of articles of research on mathematics, and supplemented them by some new and more recent material. Ten of the items are reprints, some with minor modifications; the remaining six are new. They have been grouped into three categories: General, Primary and Secondary, for convenience, and we hope that they will be of interest to teachers, board members, and students, as the winds of change blow across the mathematics curriculum and assessment landscape in both New Zealand and Australia.

Contents

Secondary:

13. A Child's Perspective of Algebra
Nerida Ellerton

What is it that makes algebra a mystery to some children? A series of in-depth interviews, in which children talked through their solutions to 10 mathematics problems, emphasises the need for abstract mathematics to develop from children's own needs and experiences. From ***set*** No. 2, 1985 (Australian Edition)

14. What are the Benefits of Single-Sex Maths Classes?
Ken Rowe

A very carefully-run experiment in Ballarat High School shows definite benefits in single-sex classes. But would this be true in all secondary schools? The benefits claimed are detailed, along with critical comment. From ***set*** No. 1, 1990.

15. Mathematical Needs of School Leavers
Gordon Knight and colleagues

Adults were interviewed to discover the mathematics they actually used in everyday life and in their workplaces. The importance of problem-solving, simple estimating, optimising (often time or costs), 'finding the best strategy to follow' and the confident use of calculators is emphasised. From ***set*** No.1, 1994.

16. How I Failed
Anon

This short, witty piece, reprinted with permission from the *New Zealand Mathematics Magazine*, emphasises the danger of over-simplification of mathematical problems encountered in the real world.

Introduction

Mathematics Teaching Today

A Short Overview of what Research has to Offer

Ian Livingstone and John Izard

Recent developments in mathematics education in both Australia and New Zealand are not occurring in isolation, but are mirroring international trends. There is an increasing emphasis on applications of mathematics in context, rather than abstract or pure, mathematics. In the words of the recent *Mathematics in the New Zealand Curriculum* statement:

> Mathematics is a coherent, consistent, and growing body of concepts which makes use of specific language and skills to model, analyse, and interpret the world. Mathematics provides a means of communication which is powerful, concise, and unambiguous.

This may be a very incomplete statement to a pure mathematician, such as Derek Holton, whose item No.6 in this Best of *set* describes what professional mathematicians do. It is certainly not how philosophers of science would describe mathematics. However, it is how curriculum developers reflect the current emphasis, born of bad times, on the practical and pragmatic over the pure and esoteric - students want jobs when they leave school and there are precious few in pure mathematics. On the other hand industry and administration need, more and more, those who can apply mathematics.

With this increasing emphasis upon what mathematicians do (the mathematical processes) rather than on what mathematicians know (the mathematical content), mathematics is no longer seen as a pure, intact subject discipline in its own right, of value irrespective of its links with other areas of knowledge or with everyday life.

A Practical Emphasis

The teaching and learning of mathematics is now thought to be best carried out among the problems which are meaningful to students. There is good research backing for this approach, both from learning theory and because it will lead increased understanding of the way mathematics is applied in the world beyond school. Teachers are being asked to devote specific attention to such processes as reasoning, and problem-solving (including modelling and the techniques of investigation). Part of the problem-solving exercise will often involve the development of co-operative skills, something for which mathematics classrooms have not been noted in the past. Group work in mathematics is now to be encouraged. The ability to work on team projects, and to communicate findings and explanations, is now seen as an important outcome for mathematics education.

Probability and Statistics

Mathematics is used more and more in everyday life; as a consequence, statistical concepts, including probability, must have increased emphasis, at all levels in the curriculum.There are important recently-developed forms of exploratory data analysis, but classical statistical concepts will always have a place too. The advent of computers has meant an increased use of simple, graphical forms of data analysis, which have the benefit that they are not difficult for children to understand. They fit well with the new topics of problem-solving and investigation, and lend themselves very nicely to illustrating the results of simple experiments in science, social studies, and technology. The use of calculators has also meant that the computational difficulties (which come with real life problems) no longer slow down solutions nor get in the road of seeing problems as a whole. It often doesn't matter if the figures don't come out exactly to the 5th decimal place, but an increased understanding of approximation and estimation becomes crucial.

Electronics Revolution

The widespread use of electronic calculators, and the increasing availability of computers, are forces driving mathematics teaching in certain directions. The personal computer is an increasingly cheap and powerful tool. We can expect it to become more common than the personal car, and as liberating. Sensible use of computers demands high levels of simple numeracy, the ability to estimate the likely size of answers and to record them accurately, and most of all, an understanding of the correct processes to apply in any given situation. Heavy algorithmic computations (involved fractions, logarithms, trigonometric computations and so on) are out - the calculator or computer will see to them - an understanding of what to do, and the approximate answer to expect, are in.

Catering for All

This is a logical outgrowth of the general principle that education should cater for the needs of every individual. For example, there is widely-expressed concern that girls should not be disadvantaged by the content, teaching styles and assessment modes used in the mathematics classroom. This concern turns out to be only a mild worry - in New Zealand there is strong research evidence that girls do better than boys, in mathematics as in most other subjects, at least until the pre-university year. Research evidence in Australia is more difficult to interpret. Prior to the pre-university year girls tend to do better in mathematics than boys. At the final year of school the patterns are more complex due to the diversity of mathematical studies in the eight separate school systems within Australia.

There are other groups which need special attention. We must consider the prior experiences of all students. Many come from different cultures and subcultures. Not only do the actual problems we set need to be within their field of experience, but even the very language of mathematics itself needs to be considered. The development of mathematics vocabulary in Maori, for example, has taken enormous strides in recent years, the Ministry of Maori Development, the Maori Language Commission, and the NZ Council for Educational Research all being involved as well as practising teachers. A bulletin board and database of technical terms is maintained and kept up to date by NZCER. These developments are of great help in constructing mathematics curricula in the ever-growing number of bicultural schools in New Zealand. Similar issues face schools in Australia, particularly those with a diversity of first languages.

Curriculum Diversity

Many more students are staying longer at school. This is in part because there are so few jobs; in part it reflects a desire for better qualifications. The result has been a restructuring of senior secondary school programmes, a broadening of curricula available, and increased flexibility - allowing different combinations of subjects to be taken, often at different levels. Old specialist mathematics curricula just for the academically elite in upper forms are no longer appropriate, and new, more varied, less prescriptive curricula have been developed, suitable for the more diverse group now taking mathematics at that level. Because many have not tasted success in Mathematics at earlier levels, increased attention has had to be paid to diagnostic and remedial work, and rebuilding the foundations of concepts and processes which have not been securely laid in previous years.

Constructivism

The teaching of mathematics in schools in both New Zealand and Australia is being driven by two competing views of the ways in which students learn mathematics - do they learn from the teacher or by themselves?

According to one view, sometimes termed the behaviourist view, the syllabus drives the learning. The syllabus lists the content and skills; the teacher possesses them (or is supposed to), and the students will learn them from the teacher. Students are encouraged to work individually, and required to produce the information on demand. The function of the teacher is to transmit particular knowledge to the pupils, to find out who has learned what (by assessing many discrete pieces of information), and to ensure that this knowledge is retained until the examinations are over.

The alternative view, commonly called constructivism, sees [illegible]s deriving meaning in mathematics from the context in which the meanings are communicated, and places a premium on collaboration and communication, demanding

> a style of learning which relies upon learner responsibility, group communication and the negotiation of meaning between and with learners. It leads to a form of assessment which is reflective, that is self-questioning, rather than performance based.

The teacher provides work that the students acknowledge is worthwhile and relevant, and gets them to take responsibility for their own learning and improvement. The student is, for example, expected to participate (or negotiate) with the teacher in planning the next stage of work. Assessment is designed to empower the pupils to gauge their own progress (as well as to inform the teacher and others) and will usually involve gathering some evidence from more complex integrated activities.

If the learning environment is student-centred, it is important to appreciate what occurs in the interval between when students embark on a task and when they complete it. In some contexts, learning driven by the syllabus and the teacher's greater knowledge (the first perspective) might be appropriate. In other contexts, the second perspective offers more efficient learning.

> Acquisition, without commitment or deep consideration, of principles from teacher or other authority is *shallow learning*. The principles may be discarded as soon as the need for them - in a test or examination - has passed. Sometimes they do not survive that long. ... an individual's possession of conflicting views indicates that person's lack of awareness of what he or she knows, a lack of reflection on the meaning of knowledge.

Control over one's own learning involves

> ... constant use of positive strategies: getting the purpose clear, judging whether understanding is sufficient, searching for connections and conflicts with what is already known, creating images and thinking of relevant experiences.

The two views of the learner are thus contrasted - in one the student acquires more of what the teacher already knows; in the other the student shares some knowledge the teacher has, but also has some knowledge the teacher lacks. The former view implies that a detailed prescription of what everyone should be taught can be drawn up. The latter view implies that a single prescription cannot be drawn up; students start at different places, need different facts and skills. Each has to construct his or her own personal meaning, in other words, do his and her own learning.

Changes in Assessment

Assessment strategies must be appropriate to the curriculum goals. If the curriculum emphasises reasoning skills, the ability to choose and integrate information, to present it in oral and written form, and the development of a critical approach to knowledge, then the testing must test those skills, and not just recall of facts or the ability to calculate. Methods are needed to quantify, test, and report back on skill in reasoning and reflection, integration of relevant information and drawing conclusions, finding convincing arguments and conveying the consequences of an investigation, and recognising its limitations. Pencil-and-paper examinations distort what we are asking for, if they fail to allow for multiple (correct) solutions, and a diversity of problem-solving approaches. Nor are they appropriate where the skills looked for cannot be demonstrated easily at a desk.

Many traditional examinations assess skills indirectly. This is acceptable if the teaching has kept intact the relationships between the direct and indirect knowledge. However, there are pressures against this noble art. When students, teachers and administrators consider examination results are important, they can, mistakenly, infer that those tasks not examined are not important. To obtain 'better' marks they concentrate on teaching to the anticipated examination questions, rather than to the achievement itself. This intensive coaching on the test leads to an increased test score, but not to increased achievement in the full range of knowledge and skills you will find in the curriculum statement. The indirect measure (the exam) then ceases to be a valid measure of the true objectives.

'Direct' assessment is a good way to avoid teaching the strategies of test-wiseness, and teach the skills the curriculum intends instead. The use of projects and investigations to provide assessment evidence, in whatever form and at whatever level, can ensure that the tasks mirror the skills desired. This makes for valid assessment.

Such assessment is not a simple matter; assessment in authentic contexts involves assessment of complex behaviour. The much wider range of possible responses means that better approaches have to be devised. However, on the plus side, the procedures needed to deal with complexity also encourage better communication between the examinee and the examiner.

There should be more consistent and relevant acts of assessment, giving feed-back for further learning. Good examples of how the teacher can search for evidence of beliefs and explore the development of meaning can be found in other disciplines in the school. For instance, some excellent techniques are to be found in *Tapping Students' Science Beliefs,* by Doig and Adams.

In traditional assessment there is considerable emphasis on what students can do alone with just pencil-and-paper. This excludes the qualities associated with working co-operatively: sharing out tasks as a more efficient way of tackling problems, and being able to accomplish integrated and complex practical tasks. Yet living in a family and a society means being able to collaborate with others in doing such tasks. Working productively in groups, and recognising and solving local problems, are skills essential for survival and being part of the community. Some people involved in research studies into assessment have become more aware of this issue, but further work is necessary.

Project work is firmly student-centred; thus, when using projects, it is natural for the student to be involved in the assessment process. How are student views (on their own work and that of their colleagues) to be incorporated? What should be done about the problem of self-incrimination?

How should students be involved? The evidence being assessed will be complex – in many cases the evidence from a single project can be substantial and a considerable amount of time may be needed for students to gain assessment experience. If students are to make real contributions to their own assessments, then teaching staff must be sure that the factors considered have some integrity. Descriptors which reflect the student's own assessment of achievement will need careful development and testing in order to be accepted within any global scheme. Teachers will need to ensure that they are consistent when using such descriptors and that students understand what they mean. However, these problems must be faced if we are to begin assessing mathematics learning, rather than the ability to regurgitate facts and follow well-rehearsed procedures while the exam is on.

Notes

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Mr. Ian Livingstone is the immediate past Director of the New Zealand Council for Educational Research, Box 3237, Wellington, New Zealand.

The curriculum statement comes from

New Zealand Ministry of Education (1993) *Mathematics in the New Zealand Curriculum*, Wellington: NZ Ministry of Education.

A recent report on Mathematics in Maori is

Te Puni Kokiri; The Ministry of Maori Development (1993) *Pangarau – Maori Mathematics and Education*. Wellington: Te Puni Kokiri.

The database of Maori language mathematics vocabulary, (and indeed all new vocabulary for all subjects) can be accessed by modem. The initial application should be to

(FIDONET) 3:771/210

or

(INTERNET) bmaori@matai.vuw.ac.nz

and when you are accepted as a bona fide inquirer the database may be searched, downloaded, etc.

The quotation '...a style of learning which relies on learner responsibility...' is from

Burton, L. (1992). Who assesses whom and to what purpose? In M. Stephens & J. Izard. (Eds.) *Reshaping assessment practices: Assessment in the mathematical sciences under challenge*. (pp. 1-18).Hawthorn, Vic.: Australian Council for Educational Research.

The quotation 'Acquisition, without commitment or deep consideration...' has the author's emphasis, and is from

White, R.T. (1992). Implications of recent research on learning for curriculum and assessment. *Journal of Curriculum Studies,* 24, page 157.

The quotation '...constant use of positive strategies...' is also from White (1992), page 157.

The problem of keeping direct and indirect mathematical knowledge intact under exam prescription pressure is discussed in

Gasking, D.A.T. (1948). *Examinations and the aims of education*. Carlton, Vic.: Melbourne University Press.

and

Madaus, G.F. (1988). The influence of testing on the curriculum. In L.N.Tanner (Ed.). *Critical Issues in the Curriculum, 87th Yearbook of the National Society for the Study of Education, I,* (pp.83-121). Chicago: National Society for the Study of Education.

The full reference of the Doig and Adams package on beliefs and meaning in school science is

Doig, B. & Adams, R. (1993). *Tapping Students' Science Beliefs: A resource for teaching and learning.* Hawthorn, Vic.: Australian Council for Educational Research.

Involving students in assessing their own work, particularly that in projects, is explained in

Falchikov, N. & Boud, D. (1989). Student self-assessment in higher education: A meta-analysis. *Review of Educational Research,* 59, 395-430.

Twenty-nine Tough Questions

Andy Begg

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John Gillespie

In the best traditions of teaching, this special edition of *set* is accompanied by a test which is to be used twice: as a pre-test before the articles have been read, and as a post-test on completion.

To maximise the difference between these results (that is, the value added by this unit of work) candidates should be involved in study groups which use logic, problem-solving and communication, as well as technological aids.

In cases where the mark is to be used as part of an achievement-based assessment for the award of a learning unit, that is, as a real test, candidates may wish to not take the pre-test too seriously, thus increasing the difference between their two results.

set: Mathematics Test

Instructions

- This test has no time limit.
- Do as many questions as possible in the available time.
- Read all the questions carefully before beginning to answer them.

Question 1 (Do either option a or b)

(a) When did you last use Apollonius's Theorem?
(b) When did you last use a matrix to solve a practical transformation problem?

Question 2

Why do we teach mathematics? (Extra marks will be awarded for unique and honest reasons here rather than for quotations from well known sources.)

Question 3

Is mathematics education in a crisis or is it alive and well? Comment.

Question 4

Which of the following statements do you agree with?

(a) A teacher needs to know what prior knowledge and experience each student brings to the class.
(b) A teacher needs to know what students want to learn.
(c) The teacher knows best what students need to learn.
(d) Parents have an important contribution to make to each school's mathematics programme.

Question 5

List your responses to the following:

(a) As teachers, what do we bring to the classroom?
(b) As teachers what would we like to bring to the classroom?
(c) What do the experts think we should bring to the classroom?
(d) What extra skills do we need now in the classroom that were not required 10 years ago?

Question 6

Comment on the following two statements:

(a) Mathematics serves the needs of all students.
(b) Mathematics education should be concerned with the mathematical needs of each student rather than introduce the student into the world of the mathematician.

Question 7

Which of the following options do you prefer?
Statistics is taught by so many different subject teachers that a statistics coordinator should be appointed in every school to ensure that:

(a) all teachers learn from the mathematics staff how it should be taught.
(b) an agreed approach can be decided upon by the different subject teachers.

Question 8

Comment on the following:
Mathematics should be taught through its contribution to other subjects rather than as a separate subject. All schools need is a maths co-ordinator.

Question 9

Which of the following would you have preferred as a student? Why?

(a) Sailing and navigation or trigonometry.
(b) Banking and financial mathematics.
(c) School mathematics as at present.

Question 10

Which of the following would do most to improve the standard of mathematics teaching?

(a) Retraining programmes for surplus teachers from other subjects.
(b) Sabbatical leave for mathematics teachers.
(c) A 20% increase in pay for all mathematics teachers.
(d) A bigger quota of teacher trainees in mathematics courses.
(e) More teachers with majors in education rather than mathematics.
(f) More teachers with majors in mathematics rather than education.
(g) National assessment at ages 5, 8, 13, and 17.

Question 11

Comment on the following:

(a) Did the introduction of 'new mathematics' alter how people teach mathematics?
(b) What changes have you made in your teaching as a result of a recent document e.g.,
Australia: A *National Statement on Mathematics for Australian Schools;*
NZ: *Mathematics in the New Zealand Curriculum*?
(c) Do new text books have a greater effect on the author's bank balance or on the way that mathematics is taught?

Question 12

This mathematics test is boring as there are no conversations or illustrations.
(True or False?)

Question 13

Which pre-1980 network was formed first?

(a) Mathematics associations.
(b) Inspectors, curriculum officers and advisers.

Question 14

List the following networks in the order in which they were formed.

(a) Primary mathematics associations.
(b) Mathematics-gender affirmative action groups.
(c) Mathematics-gender equity groups.
(d) Aborigine bicultural and bilingual networks.
(e) Maori bicultural and bilingual networks.
(f) Pacific Island mathematics teachers' group.
(g) Gifted children mathematics association.
(h) Family maths networks.

Question 15

Draw a graph to show how much money was spent on the following last year:

(a) Mathematics research;
(b) Education research;
(c) Mathematics education research.

Question 16

(a) List all the institutions you know that are currently doing significant mathematics education research.
(b) Draw a graph to show the percentage of your country's GNP that is and has been spent on mathematics education research over the last 5 years.

Question 17

Rank the following questions in order of importance as topics for research grants:

(a) How do New Zealand and Australian students compare with their peers elsewhere in the world?
(b) What do students know when they come to school?
(c) How do students form mathematical concepts?
(d) What mathematics is useful in the real world?

Question 18

Write an essay on how to teach mathematics, or on any new maths textbook. Be sure to use the following terms – Problem-solving, logic, communication, calculation, computers, group work, relevance, assessment for better learning, and achievement-based assessment.

Question 19

Choose one of the following topics and list 10 points about it.

(a) Assessment in mathematics
(b) Diagnostic mathematics programmes using computers
(c) Assessment alternatives with particular reference to projects
(d) Why this review test is not a good assessment instrument.

Question 20

Comment on each of the following statements:

(a) The latest calculators can graph functions, solve equations, differentiate and integrate but will have no effect on what we teach or how we teach.

(b) Now that computer courses exist in most schools, computers should be banned from the mathematics classroom as they allow students to avoid the hard work of mathematics.

(c) Membership of the local chapter of the Luddites is a desirable attribute for all mathematics teachers.

Question 21

Now that computers can do so much, school mathematics programmes should be reduced to

(a) 4 hours per week.

(b) 3 hours per week.

(c) 2 hours per week.

Question 22

List the following topics from school mathematics programmes in order of importance for all students intending to study at a tertiary level.

(a) Calculus	(b) Statistics	(c) Geometry
(d) Computing	(e) Discrete Maths	(f) Calculators

Question 23

Name the most exciting mathematics education resource that you have seen this year. What were its deficiencies? When will resource producers get it right?

Question 24

List the resources that are needed to help teachers implement *Mathematics in the New Zealand Curriculum* or *A National Statement on Mathematics for Australian Schools*.

Question 26

Summarise your response to each of the following in 3 words:

(a) Who should take responsibility for teacher development and what needs to be done?

(b) My ideal maths education centre has...

(c) My own professional development is my own concern and I have plenty of time to devote to it.

Question 27

Who used to, who does, and who should control mathematics in the senior school?

NZ Questions

(a) Ministry of Education

(b) New Zealand Qualifications Authority

(c) Education Review Office

(d) Tertiary mathematics educators (University and Colleges of Education)

(e) Teachers

(f) Students

OR

Australian Questions

(a) The Federal Ministry

(b) State Ministries

(c) Tertiary mathematics educators (MERGA, MELA, ACMS, etc)

(d) Australian Association of Mathematics Teachers

(e) Teachers

(f) Students

Question 28

List 5 points for and 5 against the following proposition:
Australia, New Zealand, United States of America and the United Kingdom are all similar cultures and have similar problems in mathematics education.

Question 29

One does not get taller by being measured more often.
What effect does this assertion have on your participation in this test?

Question 30

Why is it that Maths educators can't count? Give an illustration from some test in your possession.

Instructions for Marking

* Each correct answer is worth four marks and the total can be regarded as a percentage because scaling marks is not currently fashionable.
* All students should be credited with question 25 as none gave an incorrect answer.

UNLOCKING THE GREAT SECRET

WRITING REVEALS THINKING

By **Graeme Withers**
ACER

Nicola Chadderton

HERE'S a common or garden situation found in every class. The children sit down to work through a set of exercises. It may be Maths, or it may be something else. The children may be 8-years-old, or 15. They do the work. It may be a test or a group activity.

The yield is always the same – a set of papers for the teacher to mark.

Say it was Maths, in Year 3 (Std. 2): ten examples of simple number work. Here we have a clever and resourceful teacher (with a lot of spare time). She analyses the papers, and discovers that Alison, who got five out of ten, was correct on a different five questions from Neil, who also scored five. With even more time she can discover exactly what it is that distinguished Alison's set from Neil's – what thought processes or reasoning abilities were called into play in each case; what was going wrong, what concepts are needed by each child, but not yet grasped.

Why not set up a situation where the students actually tell us answers to these larger, more fundamental, questions? Along the way we might make some surprising discoveries. Alison got five out of ten; so did another girl, Kylie, and she got the same five right. However they went about the problems in different ways. Is there any way of unlocking the secret of these individual differences, and building up a clearer picture of the abilities of the two girls? Yes: get them to write about how they thought.

Process writing across the curriculum

Over the past twelve months, I have been working with a group of Australian teachers in the mid-primary school, and with children of about the age of eight. The study which initiated this work is aimed at reviewing classroom practice in teaching and assessing literacy across the country. It involved getting the teachers to comment on their teaching philosophies, classroom strategies and assessment criteria, and a large selection of these comments will eventually be published for other teachers to share.

One strategy which just about all these teachers (there were over 50 of them) used in their rooms was the approach to students' writing which focussed on it as a process. A rough summary of their strategy might be the following:

1 **Thinking**
2 **Talking**
3 **First draft**
4 **Personal Edit**
5 **Conference**
6 **Final form**
7 **Publish**

Most of them made it clear that they used this strategy in other contexts than just 'Language Arts' or 'English' They also made it clear that it was a particularly useful strategy to support work in what they called, generally, 'problem-solving'. They had discovered a way of teaching not mentioned in the textbooks or in College courses. And it worked.

Process writing and problem-solving

Here is an adaptation, by one of the teachers, of the basic process. She was using it for problem-solving across the whole curriculum in her room. A wall-display showed to her 8-year-olds the points in **bold** in the following table. Her comments on what actually went on are added in ordinary type.

1 Listen to the problem.
The teacher or a student tells or reads the problem to the class. Teacher and class discuss the problem, underlining important words, and discussing other words or phrases that students don't understand.

2 Look at the problem.
In pairs, the children read the problem silently or aloud to one another.

3 Discuss the problem.
The pairs discuss: 'What are we being asked to do? How will we work with the problem?

4 Decide about the problem.
'Shall we draw a picture? Make a list? Make a table? Work backwards? Look for a pattern?'

5 Try the problem.
Students try, individually, one or more strategies to solve the problem.

6 Talk to your partner (or, if you're both stuck, to the teacher) about what you did.
This is the stage called 'Conferencing'.

7 Check your answer.

8 Publish and share your answer.

Donald Graves invented the techniques of 'process writing' after studying how 'real' authors write, and trying techniques out in New Hampshire classrooms. Even the teachers who hadn't read his books often gave pupils at the 6th stage (conferencing) a *conference card*. It has a top flap saying:
You will need:
- a pad;
- a pencil;
- a dictionary.

The main card reads:

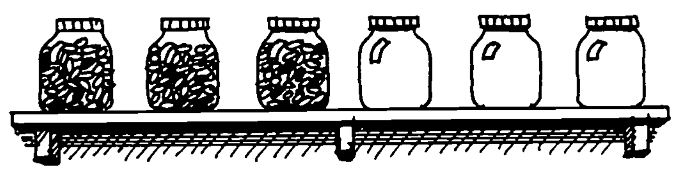
Publishing Conference
1 Author reads the work and others watch and *listen*.
2 Check:
- capitals;
- full-stops;
- commas;
- inverted commas;
- spelling;

3 Does it make sense?
4 How will the work be published?

This card was used during language work across the whole curriculum for Maths, Science, Social Studies, Health and so on, and not just when problem-solving was the main point of the activity. Children were very used to it, and cued in to the knowledge that special equipment (calculators and other instruments, for example) would also be needed during certain sessions, particularly Maths.

A simple example of process writing in Maths

In her Year 3 (Std. 2) class the teacher set up the following task: she issued each child with a Maths task sheet which depicted six jars on a shelf, three shaded (on the left) and three 'empty' (on the right). The printed stimulus for the task read:



Jelly Beans
Curly's boss had told him to set up the six jars as a display for the jelly bean promotion.
'How does it look?' asked Curly, about to leave for lunch.
'Well, I'd like it better if you alternated full and empty jars.'
Curly's in a hurry. What's the least number of jars he needs to move?

The children were asked to try to solve the problem of alternating the jars using some hands-on method (Unifix blocks were the most popular item resorted to, initially at least), and to record the results of this transformation of the printed problem to 'real-life'. A process writing procedure was undertaken, following the strategy set out on the wall-chart.

Here is a selection from the final versions of their solutions to the problem, produced by this class of 8-year-olds, after this process in Mathematics. They are ready to publish for the rest of the class and to be discussed by the class:

1. First I tried to use the unifix blocks but that didn't work out. Then I tried drawing a picture, but that didn't work out either. Then Catherine and I worked together using cups and marbles, the full ones had marbles the empty ones didn't. We took about 4 goes

to get it, and then we got it, well, Catherine did I should say. All we had to do was to move the marble from the second jar into the fifth jar. It was so easy, and the least number of cups we had to move was 1.

2. The answer was 1. At first I used unifix but I wasn't getting anywhere so Mrs Hockley said I could work with Jacki. Then Jacki and I got some mugs and very soon we found the answer. We poured the second into the fifth jar.
3. First I tried to work out the maths problem with unifix. But that didn't work because, I couldn't do what I wanted to do. Then I experimented on paper. But I still didn't get it. So on paper I drew six jars. I had 3 full jars at the start, and 3 empty ones at the end. I got the second full one and picked it up and tipped it into the second empty jar and there was my answer. It was 1 jar.
4. To fix a problem you need six cup and three marbles and you get the second morbel and put it in the fifth cup.
5. First I tried the unifix blocks. They did not work and then the paper did not work but then [I used] the caps I pot the second cup into the fifth cup.

These versions are what the children prepared for publication, and handed in for their teacher to review. Student number 3 also handed in his FIRST DRAFT, which showed her just how far he had come in organising and controlling his language during the conference process.

First Draft
first I treid to work it out with unfix but that didn't work because I couldnt do what I whanted to do. Then I experamented on paper but I still didn't cet it so on papper I drew some Jars and had 3 full jars at the start and 3 empety ones the I got the second full one and tiped it into the second empty and there was my arswer.

Published Version
[Above, Student No. 3]

Writing about thinking

Here is another example from Maths. You may skip this one if you don't teach Maths and go straight on to the last sections, *Across the curriculum*... and *Some implications*... But this example is a good illustration of the last implication of all – students can unlock for you the great secret, why they went wrong, for you to give them the teaching that will put them straight.

This time the material comes from a Year 6 class who were asked to do some writing about thinking. Once again the subject area is Mathematics, and the task a simple one, chosen by the teacher because she was curious about the differences she observed between students' abilities in the matter of number series. Here is the task:

Complete the following number series:
43 34 54 .... 65 76

And here is the solution:
43 34 54 **45** 65 **56** 76

The students used the process writing approach in preparing the descriptions of how they went about the task but they carried it out of their own, making notes as they went. Student number 1 derived the correct answer, and this is how she went about it:

What you have to do is to look at the first few numbers, and see what's going on. So I took 34 away from 43, and got 9. Then I took 34 away from 54 and got 20. So I sort of used that in the next bit. I took 9 from 54 and got 45, and then I added 20, and got 65 which was already there so I knew I was right. Then I took 9 away from 65, and got 56, and sort of checked it by adding 20, and there was 76, so I had to be right.

Student number 2, however, produced no answer, and offered the following piece of writing:

I couldn't do it – I don't know what you have to do.

These two represented the extremes of the work the teacher received. Between them emerged a number of other insights into student ability. Student number 3 gave the wrong answer, and, in describing how he went about it, provided the key to what the teacher had to do next to improve his understanding of number series and his ability to solve such problems.

Number 3's answer was 43, 34, 54, **63**, 65, **67**, 76. His writing told her that:

I looked for a number between 54 and 65, then I looked for a number between 65 and 76.

She took the issue further:

Teacher: Why did you choose 67?
Student: Because it's like 76, isn't it?

A sketchy understanding of how certain number series might work was mistakenly applied to this example – sometimes they do run in increasing order of size, but not in this case, as the appearance of 34 after 43 might have told the student. His selection of 67 'because it is like 76' was an interesting guess, in view of one other student's solution (discussed below), but no more than a guess.

Number 4 also derived the correct answer, but his description of how he got it happened to be faulty:

I had to find the formula. And the formula was +9 and take away 20. So I just went on adding +9 and −20 until all the spaces were full. And I got 45 and 56 for my answers.

Despite the fact that, in writing the 'formula' it was reversed from −9, +20, there is no doubt that the student could and did solve the number series in a fairly classic way.

Student number 5 took short-cuts in both solution and description: he wrote merely:

I took away 9.

There is obviously room for the teacher to point out subsequently that this approach to the problem will not always work, though it did clearly in this case. [65−9=56, 54−9=45]

However, for the teacher, the most interesting answer and description came from student number 6. She entered the correct numbers in the spaces, and also had 67 written on the end of the given series. Her comment read:

I really guessed because I can't do these things ever. Because 34 is 43 backwards I thought the first space might be 54 backwards, which is 45. Then the next space would be 65 backwards, and the last one 76 backwards, which is 67. I hope I'm right.

This completely visual approach to the problem was, in fact, one that had not occurred to the teacher as a possibility, and revealed that more ways of thinking than a computational one might be used to assist the solution. Again it pointed out a weakness in the particular student's understanding of number series ('I really guessed...'), but not one that would have emerged from a simple ticking or crossing of answers right or wrong. By that measure, student number 6 was as 'right' as numbers 1, 4 and 5, and various other students in the class.

When the resulting solutions were displayed ('published ') on the classroom wall, students were challenged to read the other explanations, and add to the list if they found another way of expressing a solution, which several did. The teacher capitalised on number 6's visual solution to offer other kinds of series in later exercises – everyone learnt, teacher included, from this whole sharing process. A rich process, indeed, when it can expose not only the accuracy, but also the diversity, of individual students' problem-solving processes.

Across the curriculum and across the grades

The process is generalisable, according to the teacher, beyond Year 3 and indeed beyond the primary school. Here, for example, is an expanded version of the process which might be used by teachers in secondary classrooms in setting up the strategy for their students. The words in brackets are interchangeable with the word in italics to indiciate how the basic scheme might fit other tasks in other classes.

1 **Thinking**
Consider the best methods of tackling the *problem* (essay topic/project/comprehension question). Get the criteria for assessment from your teacher in advance.

2 **Talking**
Share your ideas with a partner. Discuss a variety of different ways of meeting the criteria.

3 **First draft**
Set out your ideas on paper in point form. Experiment with different plans or ways of approach.
Try a rough draft. Don't worry about mistakes, but naturally you should try to be as accurate as you can.

4 **Personal edit**
Use a *calculator* (dictionary/instrument/work of reference) to check your draft. If you find an error, retrace your steps and find its source.

5 **Conference**
With a partner, check one another's *solution*(essay/project/answer). Incorporate the suggestions for improvement on your draft. Consult with the teacher if he/she is available.

6 **Final form**
Prepare a final version of your work. If your partner is available, use him/her to give it a final check.

7 **Publish**
Share your work with others in the class. And share theirs – they will have taken other paths which might be useful to you in the future.

Some implications underlying process writing

1 **Every teacher is a teacher of language**
You know the rules of *your* game, as far as language requirements are concerned – the English teachers know the rules of theirs. Only *you* can impart the rules of *your* game – but the English teaching profession can help with the rules (and the structures) of theirs.

2 **Language needs support**
It needs lots of dictionaries, thesauruses, specialised usage books. It needs dictionaries of different kinds, and at different levels (especially the ESL kids). And it needs spellers. And it needs them to be always available. Process writing in other disciplines sometimes needs specialist equipment to allow students to check their work. The word processor and spelling program is a huge boost to drafting, correcting, and publishing.

3 **Process writing does not mean more correction by you**
It does mean more correction, but by the students. It should mean less correction by you. If you find yourself doing more, then you're not doing it correctly. Initially, it might mean more in-class assessment, 'on the run' as it were, but you should find that even the need for this decreases as students become more familiar with, and more involved in, the process.

4 **Process writing does not mean more preparation by you**
By having the students participate in 'brainstorming' sessions about the possible outcomes for the work, which can be recorded and shared, you save yourself time – for thinking about the curriculum implications of the work rather than the details.

5 **Good writing partnerships are crucial**
Some will work best in pairs. Two arrangements are possible – students of equal ability, or one advanced and one less advanced. The former is probably preferable – otherwise advanced students miss out on getting the help that they need (and deserve), too. But you can judge best – you're the teacher. In some classes, students might work better in threes, or even fours.
Change the partnerships only when you see they need to be changed, when the pair or group are doing nothing. Students get used to each other's mistakes and are on the lookout for them.

6 **Self-reliance of the students is a key principle**
The more they participate in decision-making, the more they are committed to action (i.e., learning), and carrying out the whole task, rather than leaving it unfinished.

7 **Process writing takes class time**
'Will I get through the syllabus?' Yes; not everything has to be done using the process-writing approach. And you will save time if much of the students' drafting and personal editing is done at home.

8 **Language needs modelling by you**
You're the professional – you know the rules of the game, in the subject area being worked by the student. Sometimes these rules can be imparted by simple structures. But remember that occasionally you will have to show them how you would do it, quite directly.

9 **Students have to know the criteria for assessment before they begin**
A brainstorming session (five minutes) will collect as many criteria for the work in hand as you will need. Select from their suggestions, and remember – you don't have to assess everything all the time.

10 **The leading mode of assessment by you is diagnostic**
Students writing about how they went about a task will very often unlock, as the title of this article suggests 'the great secret' – just where they went wrong: what it was that they couldn't do: what concept they had failed to master.

Notes

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EVALUATING LEARNING IN MATHEMATICS

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'HOW are my students coping with the mathematical ideas they confront?' is a central question. As teachers we choose from a variety of different assessment techniques, formal and informal, to answer it. Does the choice of how to assess matter?

We are sure that it does. Our own observations and recent research suggests that no method of assessment is totally adequate. In this article we consider current practices in assessing mathematics learning in primary schools, we detail research that suggests some limitations in existing assessment practices, and we outline a constructivist approach to assessment.

There have been a variety of different approaches to the assessment of mathematics. Most are either
1) norm-based written tests,
2) mastery-based written tests, or
3) interview schedules.

A look around schools will find all these methods being used.

In several parts of New Zealand pre-testing and post-testing of students is carried out regularly. The tests may be made up by the teacher or be part of a commercial teaching package. The written tests are often multiple-choice or single-answer-required and come in several levels of difficulty. According to the pre-test results (or 'readiness' tests as they are sometimes called) the teacher may then group the class for instruction. At the conclusion of the unit a post-test (or 'mastery' test) is administered. In some areas this way of evaluating is very common.

Some schools have their own internal schemes for monitoring learning. Written tests may be administered by a syndicate, or group of classes. Schools may use some *PAT* norm-based tests recently revised to show trends and illustrate levels. The percentile rankings of individual children may be recorded on cumulative record cards, and quoted at parent interviews.

Interview guides are published in the New Zealand *School Mathematics* series and are included in the *Beginning School Mathematics* scheme under the name 'Checkpoints'. Typically these interviews provide the exact questions for teachers to ask, and also the model-answers that teachers should expect from children if they have successfully learned the idea being investigated.

The guidelines for assessing learning put out by the former New Zealand Department of Education (1989) suggest a descriptive approach with emphasis on what processes children use as they work. For example, the guide suggests that assessment should take many forms using a variety of methods best suited to the needs of the learner, and that assessment should be thought of as gathering information about the progress of learners. In Australian schools the equivalents of these techniques are also much to the fore.

Evidence of limitations

Are these ways of assessing mathematics adequate? We have found evidence that some frequently used ways are not.

Recently, after a group of student teachers had tried out several forms of assessment, they commented

> Administering the written tests raised many questions . . . By themselves, I found the tests didn't show a realistic indication of what the children knew.
>
> Children were very cautious of the (written) test and it took some encouraging to convince them that there was no pressure to pass or to do well.

The student teachers found more value in other assessment strategies.

By themselves I found that the tests didn't show a realistic indication of what the children knew. Only by backing up the written tests with oral questioning to each child, could I gain a true indication of what the children knew and how they thought.

In research interviews we have found that children use a large number of different strategies when solving mathematics problems (see Table 1).

Table 1

Proportions of a sample of 42 children (aged 8 and 9) giving ways of correctly solving two problems

Problem	Number of valid ways given 1	2	3	4	5 or more
5 + 7 = [ ]	29%	38%	13%	16%	4%
14 + [ ] = 26	42%	36%	16%	6%	

Note: The interviewer's question was 'How did you get your answer' followed by 'Can you tell me of any other ways of doing the problem'. The number of valid ways given by children is an *under-estimate* of the number of ways that they actually know.

For some problems almost every child knows of a number of different strategies. Children often switch strategies (sometimes in the course of solving a single problem). Many other researchers have noted the same. Also children approach school mathematics in different ways from the ways that they approach mathematics in the world and are often able to solve in real life mathematics problems that they can't do in the classroom. Researchers Denvir and Brown found that pencil and paper assessment procedures are not sufficient to indicate children's mathematical achievement, and others have found that teachers who rely a great deal on pencil and paper tests for mathematics assessment have lower learning gains in their class.

Problems of norm-based and mastery-based assessment

1. Mastery-based assessment methods often lack reliability or validity

The reliability and validity of mastery tests is commonly ignored. The assumption is that if items appear to measure the objectives of instruction that the test will be reliable and valid. It is equally important that reliability be established for mastery tests as for norm-based tests. Few mastery tests have their items scrutinised to ensure that the items

> . . . measure their respective objectives, are unbiased in relation to women and minority groups, and differentiate between groups of masters and nonmasters of the objectives (item discrimination). In addition they must be free of structural flaws that could cue or confuse students. (Berk, 1988, p. 367).

Both standardised tests and mastery tests do not take account of the strategy shifts that children may show. Thus, the reliability of these tests should be seen as reliability in a statistical sense only. For even though a child may get the same answers on a retest, the strategies used can be quite different.

2. Item scores in norm-based and mastery-based tests do not indicate the knowledge used in passing the item

In norm-based and in mastery tests there is a fundamental assumption that 'to pass is to know'. But we may ask 'to pass is to know what?' Children can answer the same questions using quite different strategies. Behind different solution strategies can lie marked differences in children's knowledge.

A further challenge to test validity is found in research that shows that children can solve real-world problems which (on the basis of pencil and paper tests) we would predict that they cannot do. This suggests that the validity provided by mastery and norm-based tests is not the kind of validity needed by teachers who want a pupil's learning to be relevant beyond school. Norm-based and mastery tests may show children's ability to solve 'school maths', but they are not able to indicate children's performance in a wider mathematics context.

3. Assessment can hijack the curriculum

If teachers take the results of test-based assessments too seriously, thinking that they do indicate real knowledge, then the curriculum can become orientated to the test and teachers will teach to increase pupils' scores on the tests.

Testing for standardisation can lead to teaching for tests so that the pupils can be seen to be successful.

Also almost all traditional mathematics assessment involves pencil and paper tasks and teachers and learners may come to see mathematics as pencil and paper exercises. If instead, assessment in mathematics takes children's ability to use mathematics in the world into account, then worksheets and textbooks will be seen to be insufficient.

A teacher may not be diverted by formal assessment procedures, but pupils will be. Erlwanger interviewed children engaged in self-paced maths programmes and found low levels of cognitive learning. The children viewed mathematics as a game in which they had to do no more than guess the answer in the answer key.

4. Assessment outcomes may limit children's learning experiences

It is often assumed that if a test shows that a child cannot do a topic then the child should not be introduced to other 'harder' topics. That is challenged by current research on two counts.

Firstly, so-called 'harder' topics are often easy for children. Curriculum developers can get 'easier' and 'harder' wrong. Young-Loveridge and Irwin have found that children often show facility with subtraction before addition.

Secondly, children's experiences affect the ease or difficulty they have in dealing with mathematical materials. Blades and Spencer found that young children can coordinate references in a grid if the grid is labelled with familiar symbols rather than numerals. We have observed that children from mathematically disadvantaged backgrounds find working number problems with money easiest because they have had experience with money. So children may be able to learn more advanced topics provided the topics link into familiar contexts.

5. Pupils are misclassified by tests

This is perhaps one of the most serious consequences of an over-reliance on standardised written test results. In well researched trials such as those of Denvir and Brown, it has been found that a significant number of students get misclassified (25% of students were misclassified, almost always to a lower level than they should have been). With unvalidated tests the level of misclassification is even higher. Thus relying on test results alone to control where pupils are in a mathematics programme limits children's progress.

6. It is difficult to get conditions for carrying out a valid assessment

Standardised tests require testing to be carried out in a silent non-distracting environment. Other pencil and paper tests, such as mastery tests, should also be carried out in such conditions. Invigilating is also required – we have observed that children will often exchange answers in order to hide their weakness. In the typical classroom it is difficult to get ideal test conditions.

Also, for any item to validly indicate mathematical understanding, the child has to understand the question or task. For that reason, the interview may be preferred. Unfortunately the interview requires an uninterrupted period of one-to-one interaction between teacher and child and if there is only one teacher per classroom such conditions rarely occur.

A constructivist alternative

The evidence is clear: teachers should not rely on written tests alone to assess learning in mathematics. There is need for a broader approach. A constructivist approach is one which emphasises the pupils' involvement in the assessment of their own learning; the teacher assists the learners in their own efforts to assess what they have learnt.

Examples are: having children describe the ways in which they have gone about solving problems, allowing children to discuss problems amongst themselves, getting the child's own description of mathematical concepts. Many teachers already engage in some assessment practices which we would call constructivist.

Constructivist assessments result in teachers gaining an idea of the mathematical ideas and strategies of the learner. The learner gains an appreciation of the knowledge that they have and where it is not sufficient to understand situations. Thus the goal is development of the processes and experiences by which learning occurs.

Practically, this approach may involve teachers and learners in exercises that seem more like learning and teaching than assessment. For example, the teacher may get children to describe to each other the way that they solve problems whilst the teacher listens or records. Or, children might write out a list of the problems that they get the wrong answers to, and then they reflect on whether there is anything about the problems that creates the difficulty.

Interviews are good when used constructively, but the realities of the classroom do not allow them to be used as the main method. However, if a teacher is in doubt about a child's readiness to start on more advanced topics then interviewing the child *as they do mathematics* will give a detailed picture of the child's knowledge. In the interview the teacher should pose similar problems in different contexts to find out if there is a solid foundation of out-of-school mathematical experiences that the child can draw upon. The interview can provide the teacher with information on how the child goes about solving problems, about sources of puzzlement, and about the questions they ask themselves – all information which is not available from written tests.

Conclusion

We need to reconsider the strengths and limitations of the assessment devices that we use in our classrooms. Norm-based tests are useful for ranking students. Mastery tests provide some indication of children's ability to do items from a particular domain of knowledge. But neither of these can give us precise guidance on how to improve the learning of those who have failed and those who have passed the tests. Only by observing and listening to learners as they solve mathematical problems and pursue investigations can we get the type of information that will help us plan for better learning.

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One of the key papers in evaluating primary school assessment of Mathematics is

Denvir, B. and Brown, M. (1987) The feasibility of class administered diagnostic assessment in primary mathematics, *Educational Research*, Vol. 29, No. 2, pp. 95-107.

The variety of approaches (norm, mastery and interview) is discussed by Denvir and Brown.

That children approach school and 'real-world' maths differently is detailed in

Saxe, G.B. (1988) Candy selling and math learning, *Educational Researcher*, Vol. 17, No. 4, pp. 14-21.

and

Carraher, T.N., Carraher, D.W. and Schiemann, A.D. (1987) Written and oral mathematics, *Journal for Research in Mathematics Education*, Vol. 18, No. 2, pp. 83-97.

That teachers do not do so well when they rely too heavily on pencil and paper tests is detailed in

Peterson, P.L., Fennema, E., Carpenter, T.P. and Loef, M. (1989) Teachers' pedagogical content beliefs in mathematics, *Cognition and Instruction*, Vol. 6, No. 1, pp. 1-40.

The quotation from Berk is in

Berk, R.A. (1988) Criterion-referenced tests. In J.P. Keeves (Ed.) *Educational research methodology and measurement: An international handbook*, Oxford, Sydney: Pergamon Press.

That children can answer questions using different strategies, and can change strategies can be found in

Siegler, R.S. (1989) Strategy diversity and cognitive assessment, *Educational Researcher*, Vol. 18, No. 9, pp. 15-20.

and

Siegler, R.S. (1988) Individual differences in strategy choices: Good students, not-so-good students, and perfectionists, *Child Development*, Vol. 59, pp. 833-851.

and

Carpenter, T.P. (1985) Learning to add and subtract: An exercise in problem-solving. In E.A. Silver (Ed.) *Teaching and learning mathematical problem-solving*, Hillsdale, New Jersey: LEA.

and

Loef, M.M., Carey, D.A., Carpenter, T.P. and Fennema, E. (1988) Integrating assessment and instruction, *Arithmetic Teacher*, November, pp. 53-55.

The quotation on testing for standardisation is from

Jaworski, B. (1988) 'Is' versus 'seeing as': constructivism and the mathematics classroom. In D. Pimm (Ed.) *Mathematics teachers and children*, London: Hodder and Stoughton.

Erlwanger's interviews are written up in

Erlwanger, S.H. (1975) The observation interview method and some case studies. *Proceedings of the Conference on the Future of Mathematical Education*, pp. 125-142. Tallahasse, Fl.: Florida State University.

Young-Loveridge's and Irwin's work showing some children master subtraction before addition is in

Young-Loveridge, J.M. (1987) The development of children's number concepts, *Research Report No. 87-1*, Christchurch: Education Department, University of Canterbury.

and

Irwin, K.C. (1989) Children's understanding of compensation, addition and subtraction. *Paper presented at the N.Z.A.R.E. Conference*, Wellington.

The alternative constructivist approach to assessment is best followed up in

Biddulph, F., Bell, B. and Carr, M. (1989) Constructivist assessment. *Submission to the assessment for better learning project*, Hamilton: Centre for Science and Mathematics Education Research, University of Waikato.

Understanding Children's Mathematics:
Some Assessment Tools

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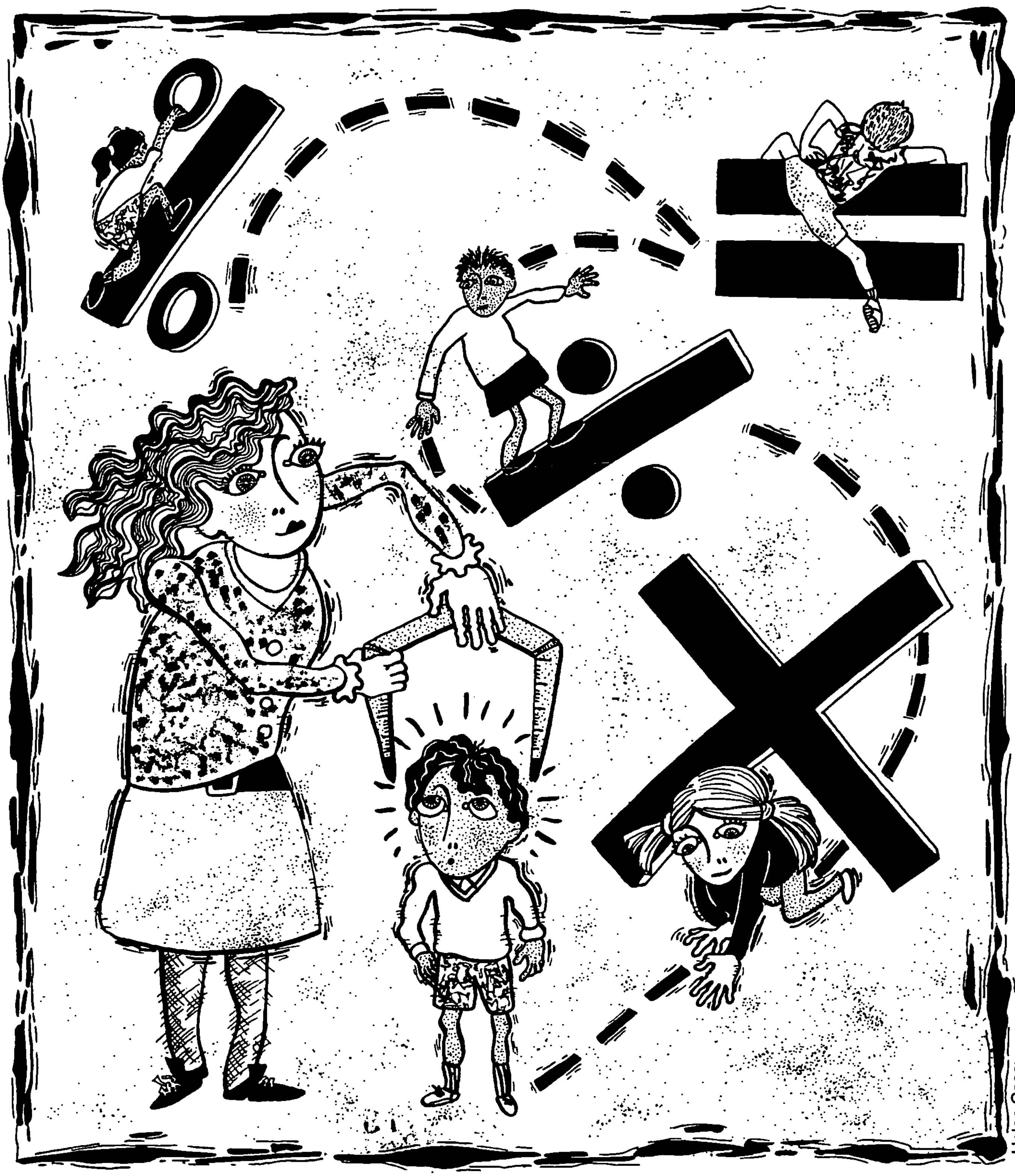
ACER



This item has been adapted by the authors from their chapter, 'Understanding Children's Mathematics: Some Assessment Tools', in *Assessment and Learning Mathematics*, edited by Gilah Leder, and published by ACER in 1992, and describes advances in formative assessment, and new helpful techniques.

Introduction

In our decisions about how and what to assess, the kinds of learning that we value (and consider worthy of recognition and reward) are made explicit. Our students, and frequently their parents, other teachers and the wider community, recognise this. In the final analysis, what we say we want to do (through our curriculum statements) is less important than what we emphasise through the assessment processes we use.

This principle has been long understood. It was recognised by Benjamin Bloom and his associates in the 1950s when they developed the *Taxonomy of Educational Objectives* as a means of encouraging curriculum planners and test developers to attend to a broader range of learning goals. It also underlies recent initiatives to broaden assessment procedures to include performance assessments, problem solving, independent research projects, group work, and the evaluation of portfolios of student work. Curriculum reform must be accompanied by assessment methods that support and reinforce reform. Most important of all, assessment methods must fit with how we learn - and provide feedback to assist learning.

Much mathematics assessment reflects a traditional view of mathematics teaching and learning in which teaching is the presentation, and learning is the mastery of specified bodies of facts and skills. Thus, in conventional tests and examinations, mathematics is treated as a relatively fixed body of knowledge, facts, algorithms and proofs. During instruction these facts and skills are organised and presented in logical sequences, and, in the jargon of more recent times, subsets of 'behavioural objectives' are specified for each 'instructional module'. In this traditional approach to mathematics teaching:

> Mathematics is assumed to be a static bounded discipline... Within each subject, ideas are selected, separated, and reformulated into a rational order. This is followed by subdividing each subject into topics, each topic into studies, each study into lessons, and each lesson into specific facts and skills.

Such an approach to mathematics teaching is a 'top-down' approach in the sense that its focus is on the delivery of knowledge. The order of presentation is logical and mathematically consistent, at least from an adult point of view. Programs of study and lesson modules are planned around sequenced instructional objectives, and the role of the teacher in the process is to present material in a manner that is motivating and understandable.

Learning, under a top-down approach, is largely receptive and passive. Students are presented with facts and algorithms to be committed to memory and recalled and applied when required. The desired outcomes are spelled out ahead of time as precise, observable bits of behaviour. Assessment is, therefore, relatively straightforward. The starting point is a list of the objectives of the course and a list of directly observable behaviour which can be reliably recorded as either present or absent. The objectives should be 'stated in terms which are operational, involving reliable observation and allowing no leeway in interpretation'. To achieve this degree of reliability, test constructors are encouraged to write items to assess the ability to perform unambiguous, observable tasks such as 'stating', 'listing', 'naming', 'selecting', 'recognising', 'matching' and 'calculating'.

Multiple-choice items have become especially popular in assessments of this kind because they can be scored quickly, unambiguously as right or wrong, and even by machine. There is usually no interest in a student's incorrect response beyond its value as evidence that he or she has failed ('to display the correct behaviour on that item').

Such tests are easily administered for *summative* purposes. In this case interest focuses on the proportion of mastered objectives. Have students mastered sufficient objectives to be awarded a 'pass' in the course? Have they mastered sufficient objectives to be awarded an 'A'?

However, even such conventional mathematics tests can, as well, be used for formative purposes i.e., for diagnosis and remedial teaching. Interest then focuses on the checklist of behavioural objectives and, in particular, on those objectives not yet mastered. Because learning is viewed as a process much like adding bricks to a wall, failed objectives are seen as 'gaps' in a student's learning. Diagnosis is the process of identifying missing knowledge and skill so that remedial teaching can be undertaken to fill those gaps, add those bricks.

A constructivist approach

School learning rarely occurs as a passive, receptive process of the kind implied by much of our past curriculum and assessment practice. Learning is an active, constructive process. Through it we develop our own interpretations, approaches and ways of viewing phenomena. We also, as we learn, relate new information to our existing knowledge and understanding.

Under this new, researched, view of learning, students, when addressing a problem, are rarely considered to have no understanding and no strategies. Even beginning learners are considered to be engaged in an active search for meaning, constructing and using naive representations or models of mathematics. Rather than being 'wrong', these representations frequently display partial understanding and are applied rationally and consistently by those who use them. In arithmetic, for example:

> ... it has been demonstrated repeatedly that novices who make mistakes do not make them at random, but rather operate in terms of meaning systems that they hold at any given time.

and

> Children are not passive learners who simply absorb knowledge. Children come to school with rich informal systems of mathematics. They actively structure incoming information and attempt to fit it into their established cognitive framework.

The recognition that much learning occurs as a constructive process has far-reaching implications for teaching and assessment. Most importantly, it shifts the focus of instruction from the delivery of static, 'correct' mathematical knowledge to the attempt to understand learning *from the perspective of the learner*. Teachers can assist learning if they first go to the trouble of investigating and understanding the naive and incomplete systems of mathematics that individuals invent and use. In other words, teachers must become not only deliverers of mathematics knowledge, but also researchers into their own students' learning. Through an appreciation of mathematics as experienced by learners, teachers are better able to assist students to modify or revise their personally constructed systems of mathematics.

A constructivist view of learning also has far-reaching implications for assessment. It becomes a process of collecting observations to build a picture of the learner's conceptions and systems of mathematics. This requires a new set of skills on the part of teachers and assessors, and new ways of thinking about the assessment process.

Formative assessment, for example, is less likely to begin with checklists of behavioural objectives and a search for 'gaps' (unmastered objectives) and is more likely to begin with an attempt to identify the conceptions students have developed (of particular mathematical processes) and to identify the rules with which they are operating. Often it may not be possible to extract information of this kind from written attempts at conventional mathematics tasks, and further investigation through verbal questioning may be required. Summative assessment is less likely to be concerned with establishing the percentage of test questions answered, and more likely to attempt to describe the levels and kinds of understanding achieved.

Developing assessment tools

A great deal of research has been done in recent years investigating the variety of conceptions, meaning systems, and 'mal-rules' that students invent for themselves in mathematics. Some consideration has been given to the implications for changing mathematics teaching, but very little to reforming assessment. In practice, most mathematics testing continues to reflect the view that mathematics learning is a process of memorising isolated facts and algorithms and recalling and applying these on demand.

New approaches consistent with our current understanding of how mathematics is learnt are required, and new assessment tools. These approaches and tools will have one overriding objective: to provide a better understanding of mathematics as it is experienced by learners. They will be research tools, available to teachers, to make the investigation of their own students' learning easy.

Three tools to help us understand mathematics learning follow. Each of these takes the form of a *map* for displaying and studying classroom mathematics learning. These maps are not abstract or theoretical. They are pictures of mathematics as experienced. They can be constructed only by making and recording observations of students attempting real mathematics tasks. The three maps we describe and illustrate in this chapter provide successively more detailed pictures of mathematics learning.

Tool 1: A curriculum map

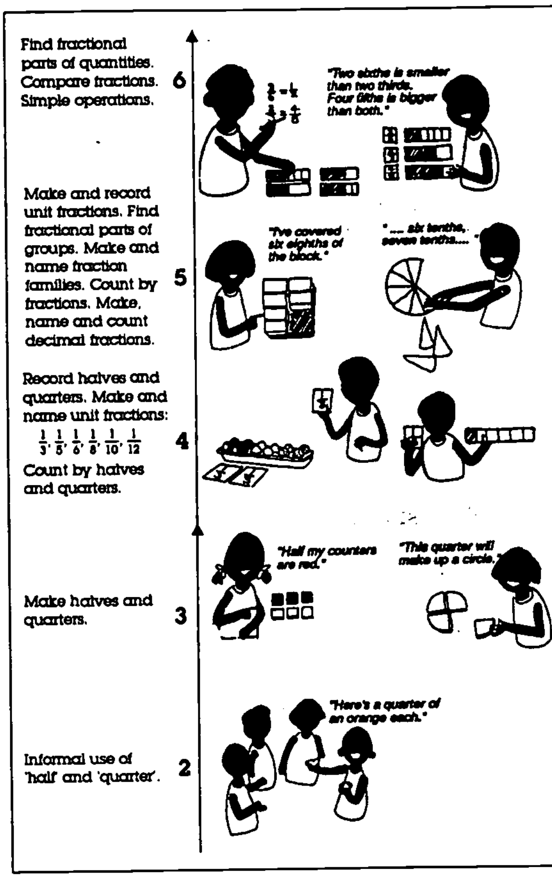
In traditional mathematics instruction, ideas are 'selected, separated, and reformulated into a rational order'. This reformulation provides instructional sequences which are logical and mathematically consistent from an adult point of view. Sequences of this kind can be found in most curriculum documents.

Chart 1, for example, shows a recommended sequence for teaching the naming of fractions to primary school children. This is one of a number of instructional strands described in *The Mathematics Framework P-10* of the Victorian Ministry of Education (1988). Under these recommendations, teachers begin (Level 2) with activities that introduce the terms 'half' and 'quarter' in informal conversations. Later (Level 3), children are given opportunities to *make* halves and quarters using objects such as counters and blocks. Once children have had some practice in making halves and quarters, they are introduced at Level 4 to conventions for *recording* halves and quarters in writing and are given opportunities to make and name other unit fractions (such as 1/3 and 1/6). Children carry out activities involving fraction families (6/10, 7/10...) at Level 5, and at Level 6, find fractional parts of quantities and compare quantities.

The sequence in Chart 1 is an apparently logical sequence of activities developed, no doubt, by experienced mathematics teachers. But to what extent does it reflect how *children* learn to name fractions? Is it based more on logical analysis than on observations of children? Could a more useful sequence be developed by paying closer attention to the ways in which most children learn to name fractions?

Chart 1

Five Levels of Progress in Naming Fractions



The central question we are asking is, 'How does this aspect of learning appear from the perspective of the learners?' This question is answered not by logical analysis but by empirical task analysis. A *curriculum map* is based on such an analysis of what children find easier and harder, in practice.

A small study we conducted several years ago provides a useful illustration of the construction of a curriculum map. We developed 12 fractions activities, each matched to one of the levels in Chart 1. The 12 activities (Chart 2) were based on familiar classroom objects such as fractions blocks. The activities were introduced (in sequence, starting with activity 1) to 28 Year 3 (Std. 2) children and 23 Year 5 (Std. 4) children in face-to-face interviews. Each child's attempt at each activity was recorded as either successful or unsuccessful.

Chart 3 shows a curriculum map constructed from the performances of these 51 children on the 12 fractions activities. This version of the map shows the difficulty level of each of our 12 activities and the average performance levels of the Year 3 and Year 5 children (8- and 10-year olds). (The maps in this chapter were constructed using a computer program we have developed for this purpose.)

A numerical scale runs up the middle of the map, but this can be ignored for the moment. More interesting are the relative difficulties of the 12 activities plotted on the right of the picture. These difficulties are based not on a logical analysis of what each activity entailed, but on an empirical analysis of children's success rates. The activity these children found most difficult was Activity 11 (naming and writing 3/8). The activity they found easiest was Activity 3 (naming a half). In fact, this one is shown off the scale at the bottom of the page because all 51 children could do it.

The average Year 3 (Std. 2) performance is near the middle of the activities on the scale, suggesting that many of these children are still learning to name fractions. The average Year 5 (Std. 4) performance is above the set of activities, suggesting that many of these children have a good grasp of naming fractions.

It is interesting to compare the empirical difficulty order of these activities as displayed in Chart 3 with their order by level in Chart 2. In practice, these children found Activity 3 and Activity 5 easier to carry out correctly than Activities 1 and 2. For these activities the 'logical' order of levels is not the order of experienced difficulty.

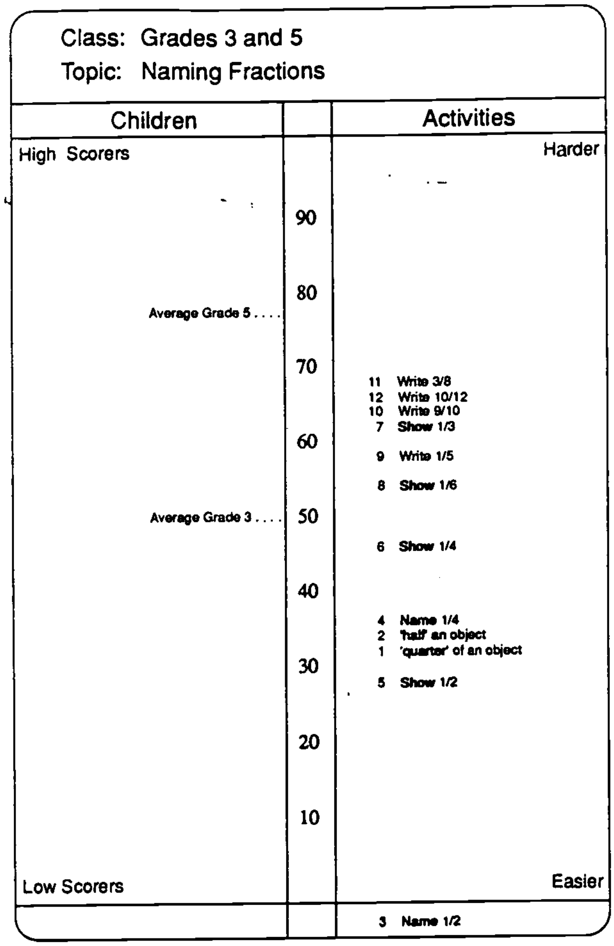
Chart 2

Twelve Fractions Activities

Level		Activity
5	12	Show student a unit and put 12 red twelfths on top. Take away 2 twelfths. 'Write down the fraction that is white.'
	11	Show student a unit and 8 green eighths. Place 3 eighths on the unit. 'Write down the fraction that is green.'
	10	Show student a unit and put 10 orange tenths on top. Take one away. 'Write down the fraction that is white.'
	9	Show student a unit and 5 orange fifths. Place one fifth on the square. 'Write down the fraction that is orange.'
4	8	Show student the cards with 1, 1/3 , 1/6 and 1/9 written on them. Give student a red sixth. 'Which card goes with this block?'
	7	Show student a unit, the thirds, sixths and ninths. Give student the card with 1/3 written on it. 'Show me a block that matches this card.'
	6	Show student the cards with 1, 1/2, 1/4 and 1/8 written on them. Give student a green quarter. Which card goes with this block?'
	5	Show student a unit, the halves, quarters and eighths. Show student the card with 1/2 written on it. 'Show me a block that matches this card.'
3	4	Put three green quarters on top of the white square. 'How much of the square is white?'
	3	Put down a white square. Place a green half on top. 'How much of the square is green?' or 'What fraction of the square is green?'
2	2	'Now give me half of an eraser.'
	1	Show student the three erasers, one whole, one cut in halves, the other in quarters. 'Give me a quarter of an eraser.'

Chart 3

Fractions Activities Calibrated on a Curriculum Map



An inspection of Activities 3 and 5 shows that they both involve the concept of a 'half', one requiring children to recognise and name half an object, and the other requiring the matching of the symbol 1/2 to half an object. While the levels described in the mathematics curriculum (Chart 1) suggest that halves and quarters be introduced together and that fractions activities be sequenced:

informal use of → making → recording

the children in our study found activities involving halves easier than activities involving quarters, no matter whether they involved **informal** use, **making** or **recording**.

We have used this observation to construct a second version of our fractions curriculum map (Chart 4). To construct this map we defined four activity types: tasks involving halves, tasks involving quarters, naming and recording other unit fractions, and naming fraction families. Chart 4 shows the difficulties of the 12 activities by type.

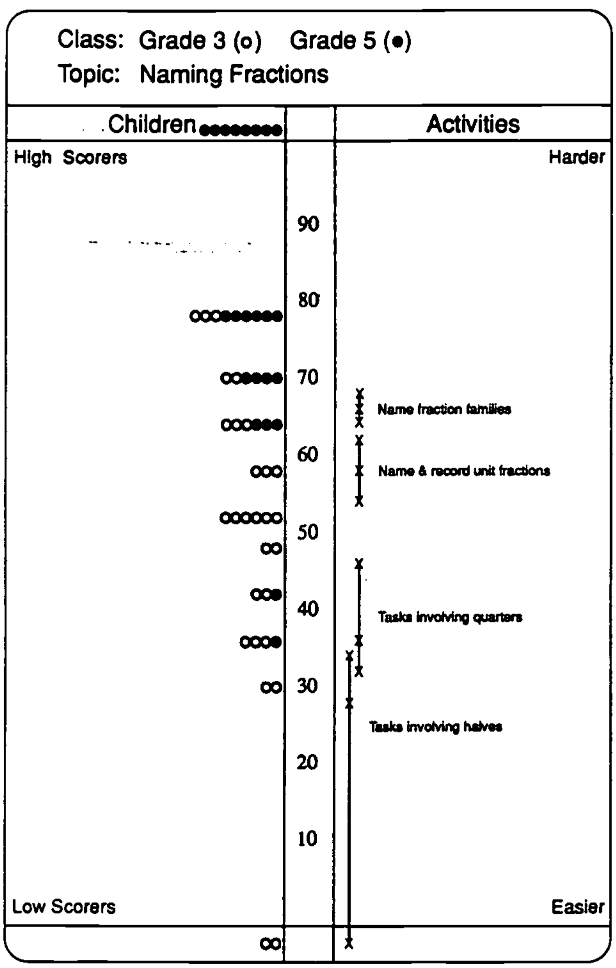
On the left of the map we have replaced the average Year 3 (Std. 2) and Year 5 (Std. 4) performance levels with the distribution of all students' performance levels. Notice that two Year 3 (Std. 2) students were unable to complete any of the activities, except Activity 3, and so appear off the scale at the bottom. Eight Year 5 (Std. 4) students completed all activities correctly and so are placed off the scale at the top.

Clustering the activities in this way assists in studying the kinds of activities students found easier and harder. It is tempting to conclude from these observations that, when teaching fractions, teachers should concentrate first on the concept of a 'half', giving children opportunities to use the term informally; make, identify and name 'halves'; and record halves using the symbol '1/2' before moving on to quarters. Our small study does not provide sufficient data to make such a recommendation with confidence. We use this example merely to show how a curriculum map can be useful in displaying the results of an empirical analysis to inform curriculum decisions.

There is another important feature of Chart 4. On a curriculum map, children's performance-levels and activities' difficulty-levels are recorded on a common scale. This enables statements to be made about what is typical of children at various levels of performance. The two Year 3 (Std. 2) children located at 30 on the scale are at a level where most children are coming to terms with the concept of a 'half'. The six Year 3 (Std. 2) children located at 52 are at a level where children (typically) have a good sense of halves and quarters but have not yet come to terms with other unit fractions (1/6, 1/7, and so on). The nine students at 78, although they did not complete all 12 activities correctly, have a good understanding of how to name, make and record fractions.

Chart 4

Fractions Activities Grouped by Type on a Curriculum Map



This approach to displaying children's performance can be contrasted with the traditional method of reporting test results. Traditional test results would show 9 correct out of 12, or a percentage (75%), with no attempt to describe what students with a result of 9 (or 75%) are able to do. The curriculum map provides an empirically-based description of what children at various levels of performance along the map are, on average, able to do.

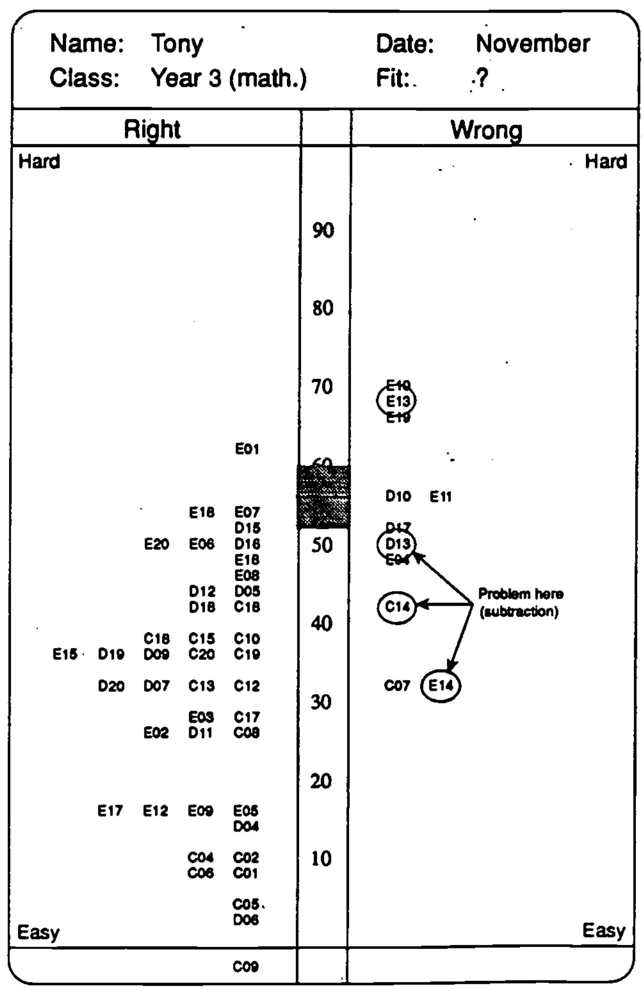
Tool 2: An individual map

A curriculum map is a useful picture of achievement for an entire group of learners. It shows which activities the group finds easier and which the group finds harder. This enables individuals' performances to be described and interpreted in the context of typical results. As such, it can be a valuable classroom tool. But a curriculum map does not show an individual's results on specific tasks. For that level of detail, we use an **individual map**.

Chart 5 shows an example of an individual map. This map displays the results of a single student, Tony, on a classroom mathematics test. Tony's level of test performance is marked by the shaded box centred on 56 on the vertical scale in the middle of the map. The code numbers of the questions Tony answered correctly are shown on the left of the page; the questions he answered incorrectly are shown on the right. As in Charts 3 and 4, items are located at their estimated difficulty levels on the scale. Students in this school found item E10 the hardest and item C09 the easiest to answer correctly.

Chart 5

Tony's Individual Map Showing Subtraction Errors



It can be seen from Chart 5 that, in general, the questions Tony answered incorrectly were the ones the entire class found more difficult. He had the 17 easiest questions (below 30 on the scale) right, and five of the six most difficult questions wrong. Of interest are questions towards the bottom right of the map: questions that most children found relatively easy but Tony had incorrect. Our computer analysis has drawn attention to the fact that Tony gave some surprising answers by printing a '?' against the word 'Fit' at the top of the page.

Four of the questions Tony answered incorrectly involved subtraction, suggesting he finds subtraction difficult. This was made clear in further classroom testing. While he appeared to have little difficulty with questions involving addition, multiplication and order of operations:

for example 2 x 2 + 3 x 2 = ?

he frequently gave incorrect answers to relatively easy questions involving subtraction:

for example 9 + 6 - 2 = ?

The value of an *individual map* of the kind shown in Chart 5 is that it displays results conveniently, making it easy for teachers to identify atypical/unusual patterns of success or failure. Once an area of difficulty is identified, teachers can undertake a more detailed diagnosis of the problem.

Tool 3: A response map

Assessment can be useful for identifying the kinds of understanding, the rules and the models that students have constructed and are using. An individual map is useful for drawing attention to areas of difficulty; however, more detailed investigation may be required. This may be possible through interviews and analyses of students' solution processes.

Research into mathematics learning is providing us with descriptions of students' common conceptions and solution processes. For example:

> The research on addition and subtraction has identified a progression of concepts and skills that is generally not reflected in instruction. Most instruction jumps directly from the characterization of addition and subtraction using simple physical models to the memorisation of number facts, not acknowledging that there is an extended period during which children **count on**, and **count back**, to solve addition and subtraction problems.

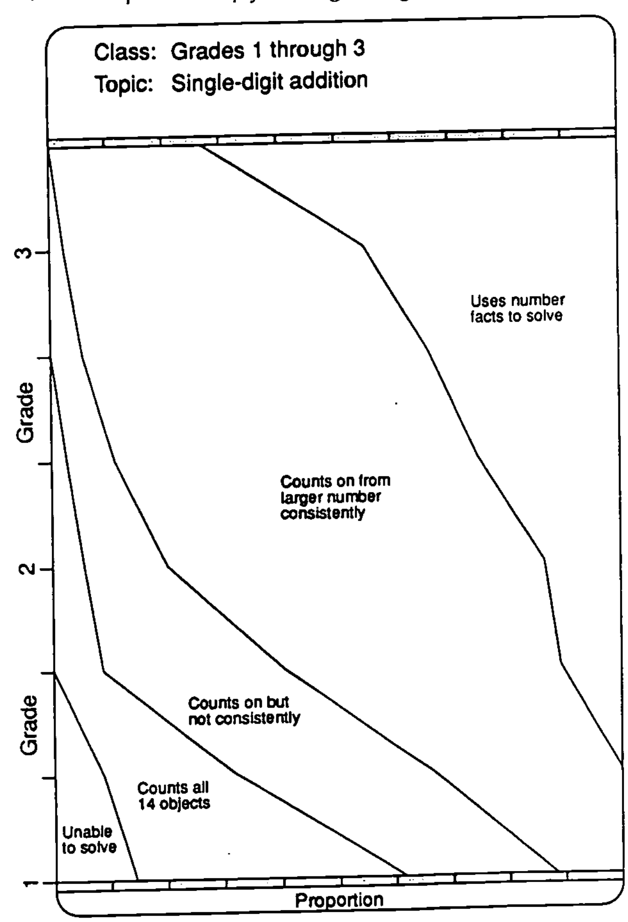
Similarly, Carpenter and Moser in 1984 identify several solution strategies to single-digit addition tasks which children commonly use. These are listed in Chart 6. They found that some 6-year-olds are unable to solve problems like 6 + 8 = ? even when given objects to count (Category 0). Others solve problems of this kind by counting out 6 objects and 8 objects and then counting all 14 (Category 1). Still others arrive at an answer by **counting on** either from 6 or from 8 (Categories 2 and 3), while older children use number facts (tables) to arrive at an answer (Category 4). This group of children were tracked from the beginning of Year 1 (J.3 and 4) to part way through Year 3 (Std. 2).

Chart 7 shows how the proportions of children using each of these addition strategies changed from the beginning of Year 1 (J.3 and 4) (bottom of Chart 7) to one-third of the way through Year 3 (Std. 2) (top of Chart 7). Notice that, at the beginning of Year 1 (J.2), 15 percent of these children were unable to solve 6 + 8 = ? This dropped to zero percent two-thirds of the way through Year 1. At the beginning of Year 1, nearly half the children in this study solved 6 + 8 = ? by counting out 6 objects and 8 objects and then counting all 14. But, by the time they were two-thirds of the way through Year 2 (Std.1), no student was using this strategy. The number of students using memorised number facts to solve 6 + 8 = ? increased steadily to about 70 percent of children one-third of the way through Year 3 (Std. 2).

Chart 6
Outcome Strategies for Single-Digit Addition of 6 and 8

Category	Description
4	Does not need to count objects, but uses number facts: 6 + 8 = 14
3	Always counts on from large number: 8; 9, 10, 11, 12, 13, 14
2	May count on, but not consistently from the larger number: 6; 7, 8, 9, 10, 11, 12, 13, 14
1	Counts out 6 objects and 8 objects and then counts them all: 1, 2, 3, 4, 5, 6; 7, 8, 9, 10, 11, 12, 13, 14
0	Unable to solve

Chart 7
Response Map for Single-Digit Addition



We refer to Chart 7 as a *response map* because it displays the different ways in which students respond to a task and shows how these responses change with increasing age or mathematics ability. This map draws attention to the range of strategies children use in their first few years of school. It also provides a framework for analysing and thinking about an individual child's progress:

> The ultimate goal of the research is not just to describe different strategies that students use. The objective is to clearly describe the **development** of addition and subtraction **concepts** and skills, and build **models** that specify the knowledge necessary for performance at each stage of development.

Chart 7 suggests little reason for concern if a child is not counting-on consistently from the larger number by the middle of Year 1 (J.2 and 3). But if the same child is still not using this strategy by the end of Grade 2 (Std. 1), that could be a cause for concern.

Conclusion

Students' pre-conceptions of number, the meaning systems they construct for themselves in school mathematics, and the variety of strategies they use to address mathematics problems, have been the subject of detailed investigation in recent years. This research has drawn attention to the many different ways in which students construct meaning for themselves and will have a growing impact on how mathematics is taught.

Parallel changes are required in the way we assess achievement. Much of our current assessment practice in mathematics is derived from a traditional view of learning as a passive, receptive process through which facts and algorithms are presented, absorbed and reproduced when required. A constructivist view aims not at adding up how many facts and skills students have mastered, but at giving a picture of each individual's conceptions and systems of mathematics.

Assessment of this kind requires a new set of skills by teachers and assessors, and new ways of thinking about the assessment process. There will also be a need for assessment methods and instruments that teachers can use as research tools.

The three maps described in this article have been developed as tools for understanding mathematics as it is experienced by learners. They have been found useful in large-scale testing with 60 000 students and in classroom studies with as few as 60.

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Notes:

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This is a shortened form of a chapter 'Understanding Children's Mathematics: Some Assessment Tools' in *Assessment and Learning Mathematics*, edited by Gilah Leder and published by ACER in 1992.

Bloom's Taxonomy of Educational Objectives can be found in

Bloom, B.S., Englehart, M.D., Furst, E.J., Hill, W.H. and Krathwohl, D.R. (1956) *Taxonomy of Educational Objectives, Handbook I: Cognitive Domain*, London: Longmans, Green and Company.

The traditional view of mathematics teaching can be found discussed in

Schoenfeld, A.H. (1988) 'When good teaching leads to bad results: The disasters of 'well-taught' mathematics classes', *Educational Psychologist*, Vol. 23, No. 2, pp. 145-66.

For a summary of the traditional approach to mathematics, and the quotation 'Mathematics is assumed...' see

Romberg, T.A. and Carpenter, T.P. (1986) 'Research on teaching and learning mathematics: Two disciplines of scientific inquiry'. In M.C. Wittrock (Ed.) *Handbook of Research on Teaching*, 3rd edition, New York: Macmillan, pp. 850-73.

That the starting point for assessment should be a list of course objectives that are directly observable, etc., see

Bloom, B.S., Hastings, J.T. and Madaus, G.F. (1971) *Handbook of Formative and Summative Evaluation of Student Learning*, New York: McGraw-Hill.

That these objectives should be 'stated in terms that are operational,' etc., see

Bloom, Hastings and Madaus, just mentioned.

That we, also, as we learn, relate new information to our existing knowledge and understanding, see

Masters, G.M. and Mislevy, R.J. (1991) 'New views of student learning: Implications for educational measurement'. In J. Fredriksen, R.J. Mislevy and I. Bejar, *Test Theory for a New Generation of Tests*, Hillsdale, New Jersey: Lawrence Erlbaum Associates.

The quotation about mistakes in arithmetic is from page 1117 in

Nesher, P. (1986) 'Learning mathematics: A cognitive perspective', *American Psychologist*, Vol. 41, No. 10.

The quotation 'Children are not passive learners...' is from

Romberg and Carpenter, mentioned above, page 858.

That constructivist assessment needs a new set of skills on the part of teachers, see

Kilpatrick, J. (1986) Editorial, *Journal for Research in Mathematics Education*, Vol. 17, No. 5, p. 322.

The full reference for The Mathematics Framework is

Victorian Ministry of Education (1988) *The Mathematics Framework P-10*, Melbourne: Victorian Ministry of Education

The quotation in the section on **Tool 3: A reponse map**, is also from

Romberg and Carpenter, mentioned above, page 856.

The work by Carpenter and Moser in 1984 on single-digit addition tasks is found in

Carpenter, T.P. and Moser, J.M. (1984) 'The acquisition of addition and subtraction concepts in grades one through three', *Journal for Research in Mathematics Education*, Vol. 15, pp. 179-202 and 309-10.

The quotation describing the ultimate goal of research is from

Romberg and Carpenter, referenced above, page 856.

In the **Conclusion** it is mentioned that there has been much research recently on children's pre-conceptions of number, etc. Such research can be found in

Nesher, referenced above

and

Marshall, S.P. (1988) 'Assessing problem solving: A short-term remedy and a long-term solution'. In I.C. Randall and E.A. Silver (Eds.) *Assessing of Mathematical Problem Solving*, Hillsdale, New Jersey: Lawrence Erlbaum Associates.

and

Silver, E.A. (1988) 'Teaching and assessing mathematical problem solving: Toward a research agenda'. In R.I. Charles and E.A. Silver (Eds.) *The Teaching and Assessing of Mathematical Problem Solving: Research Agenda in Mathematics Education*, No. 3, Hillsdale, New Jersey: Lawrence Erlbaum Associates.

and

Silver, E.A. and Kilpatrick, J. (1988) 'Testing mathematical problem solving'. In R.I. Charles and E.A. Silver (Eds.) *The Teaching and Assessing of Mathematical Problem Solving: Research Agenda in Mathematics Education*, No. 3, Hillsdale, New Jersey: Lawrence Erlbaum Associates.

The large-scale testing programme with 60 000 students can be found described in

Masters, G.N., Lokan, J., Doig, B., Khoo, S.K., Lindsey, J., Robinson, L. and Zammit, S. (1990) *Profiles of Learning: The Basic Skills Testing Program in New South Wales*, Melbourne: Australian Council for Educational Research.

Further discussion of the ideas contained in this article can be found in the book from which it was drawn

Leder, Gilah (Ed.) (1992) *Assessment and Learning of Mathematics*, Melbourne: Australian Council for Educational Research.

What Mathematicians Do and why it is important in the classroom

Derek Holton
University of Otago

THE ONLY NECESSARY PREREQUISITE for reading this article is that, at some time or other, you have failed to balance your cheque book.

1. What do mathematicians actually do?

I have wondered for years why more students don't enjoy mathematics. I spend, and enjoy, a great many hours each week doing something which is, apparently, anathema to almost everyone else. Why is this so?

Leaving aside obvious answers about my twisted personality, I think the problem is that schools and universities usually teach only a small part of what mathematicians do. I suspect that the most exciting part of mathematics is too often kept out of sight of children. So let's see what research mathematicians do.

1. *New Problems*

First, mathematicians look about for new, worthwhile, unsolved, problems; mathematics is not all sewn up, done, and finished. There are in fact, more open problems today than there have ever been before, but it is very difficult to explain most of them to a layman. These problems require too much preliminary mathematics; to even understand the question, you often need to have studied for many years.

But there are still some problems that can be easily explained. One of these is the so-called Fermat's Last Theorem. He wrote in the margin of a library book that he had a proof of the result 'but the margin is too small to contain it'. I hope he was banned from the library! He should certainly have been banned from the mathematicians' guild: his tantalising scribble was made about 1650 and there is no known proof of the result even today.

So what is the problem? Well, most people know Pythagoras' Theorem. The square on the hypotenuse (the long side) of a right angled triangle equals the sum of the squares on the other two sides. Algebraically we can write this as $x^2 + y^2 = z^2$, where z is the length of the hypotenuse and x and y are the lengths of the other two sides. There are whole numbers (integers) which we can use instead of x, y, and z and end up with a true statement. These are called solutions to $x^2 + y^2 = z^2$. The famous 3, 4, 5 triangle used by builders from ancient Egypt to the present day to get a right angle, is one solution since 9 + 16 = 25 ($3^2 + 4^2 = 5^2$). But a sort of 3-dimensional version of Pythagoras doesn't seem to work: ($3^3 + 4^3 \neq 5^3$); 27 + 64 does not equal 125. Does it ever work? What about $x^4 + y^4 = z^4$? Does that have integer (whole number) solutions for x,y,z?

As far as we know $x^n + y^n = z^n$ has *no* solutions in which x,y and z are whole numbers once n is greater than 2. But no-one has been able to prove this, despite Fermat's claim in the margin. So, there are still unsolved questions in mathematics. What else do mathematicians do, after they have found a problem?

2. *Experimentation*

When faced with a problem like that of Fermat, that can't be solved immediately, our mathematician plays around with it. In the case of Fermat's problem, the natural thing to do would be to try n = 3 and see if there are some integers x,y,z for which $x^3 + y^3 = z^3$.

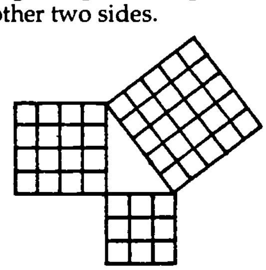
Having shown that there are no integer solutions in the case of n = 3, the next step is obvious. Test n = 4.

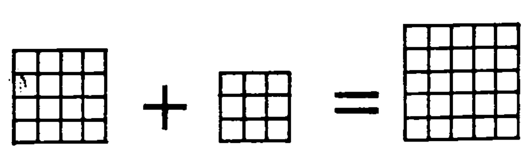
3. *Conjecture*

After some time working this way, the mathematician would be forced to conjecture.

C.1. For whole numbers n bigger than 2 you cannot get a whole number answer to $x^n + y^n = z^n$.

The conjecture has firmed up the problem (if it hadn't been firmed up from the start). Most frequently this is where the mathematician's experience and intuition come into play. Guessing the right conjecture is almost always the key step in finding new results.





4. *Proof*

The conjecture needs to be established as true, or condemned as false. We need a proof or a counter-example: just one case which doesn't fit the conjecture dooms it to failure.

And now the mathematician is on a see-saw. Try to find a proof. If you can't get all the way with a proof, find a counter-example. If you can't find a counter-example, try to find a proof following a different route. How this cerebral process works is anyone's guess. But generations of mathematicians have made progress this way. Even when a proof is found, the mathematician's work still isn't done.

5. *Generalise*

Suppose we have proved the conjecture C.1. How might we generalise or extend it? A generalisation might be

G.1. For whole numbers n bigger than 2 there are no **fractions** x,y,z such that $x^n + y^n = z^n$.

Since whole numbers are fractions ($5 = \frac{5}{1}$, etc.) G1 is a genuine generalisation of C1. That is, C1 is a special case of G1. Proving the generalisation G.1. would immediately settle the Fermat Conjecture. So far, it hasn't been done.

An *extension* is a result which somehow grows out of the first result but is not a true generalization. An extension of Fermat's Conjecture is

E.1. For whole numbers n greater than 2 you cannot get a whole number answer for $x^n + y^n + z^n = u^n$.

Now we are looking at *four* integers. So it is an extension rather than a generalisation. (By the way, even if you could get an answer, that would not completely prove the Fermat Conjecture, unless z can be zero.)

Mathematicians progress and develop a whole field of knowledge by this process of generalisation and extension. It is worth looking into any maths textbook to see how this process has worked there.

Although in the final published version, results are presented in large steps, they are usually developed in very small steps. Settling a host of little conjectures is usually the only way to achieve one reasonable-sized conjecture. It's often considered that knowing the right questions to ask, that is, making the right small conjectures, is really the heart of mathematics. A good mathematician is one who knows the right questions to ask.

Having developed a whole theory though, the mathematician is back at step one, looking for a problem. But if our mathematician is proud of the whole theory she or he will certainly want to spread it around.

6. *Publication*

Mathematicians don't accept *every* problem as worthwhile. Generally speaking, a result which doesn't have any generalisations or extensions which are of interest, is not thought worth spending research time on. Problems set for the International Mathematical Olympiad competition for secondary students are often in this category. They are not trivial questions but they don't lead anywhere; they are good for learning the mathematician's trade but they don't push mathematical frontiers forward.

What is considered a good problem may also be a matter of fashion. For instance, Gauss, one of the all-time-greats, worked on non-Euclidean geometry but did not publish his results because he thought other mathematicians would ridicule him.

Having solved what we think is a nice little problem, one which makes some contribution to an area of mathematics, we now have to write the problem up and hope that it will be published in some mathematical journal. For a Pure Mathematician this usually involves the statement of a theorem, its proof, finished off perhaps by a conjecture or a remark as to where the new result fits in to existing mathematical knowledge.

In this writing-up, just like a textbook, there is frequently no hint of all the trouble that went into the creation of the final work. Like a birth certificate it guarantees the existence of an individual but shows none of the effort (and fun) that its parents put into its creation, nor the attentions of the medical staff that helped bring it into the world.

Why do we publish such bare-bones information in Journals and textbooks when the really exciting part of mathematics is the creative part? Why do we show our students only the birth certificates? Is it embarrassment?

Once a paper has been submitted, the Journal editor passes the paper on to one or more referees. Is the paper correct? Have we indeed proved our result? And is the result sufficiently interesting to publish? If our paper passes the referees' tests, in due course it will appear in the journal and we will be provided with reprints that we can send to our friends to show them how clever we've been. We can put the paper in our curriculum vitae and, if we publish enough of sufficient quality, then we'll be promoted to Senior Lecturer or even Professor. It may even happen that our paper will spur someone else on to a generalisation or an extension and eventually solve a really major question.

Even when you think you've got it right though, the referee may find an error. Worse still, and this happened to me recently, even after the paper has been published, someone reading it may find a mistake. (Now that *is* embarrassing!)

Mathematics is not necessarily a game for hermits or recluses. Although many mathematicians still do work on their own, research papers are more and more often the product of two or more people. I personally find that joint research is more enjoyable and leads to fewer suicidal feelings when things are going badly. Usually too, results come more quickly because different people bring different experiences to bear on the problem in hand. And surprisingly 'talking' mathematics makes it clearer than mulling it around in your head. So mathematics does not have to be a game for one player.

I know very few mathematical research workers who don't enjoy what they're doing for its own sake – for the sake of treading where no-one previously trod. As a result, most of us spend far more than 40 hours a week on the job, for fun.

The Seven Steps of Mathematical Research

Step 1. Find a problem
Step 2. Experiment
Step 3. Conjecture
Step 4. Proof or counterexample
Step 5. Generalise or extend
Step 6. Publish
Step 7. Go to Step 1

2. What is mathematics?

I'm never totally sure that I know the answer to that question. However I will try to say not what mathematics is, in total, but rather what I think the most important ingredients are.

1. Objects

To me the most fundamental part of mathematics are its *objects*. These are the numbers, sets, algebraic quantities, graphs and so on, that are the basic items of study. Anything in the real or an *imagined* world is potentially an object for mathematics.

For instance, take points and lines. We think of them as real and use them to construct plans of houses, roads, aircraft, machines... which can be subjected to the rules of geometry as required. However, points and lines are imaginary objects in space; and that space is the one imagined by Euclid, which approximates to the real world on many occasions, but not all.

2. Algorithms

The second basic ingredient of mathematics are the rules of the particular game we are playing. These allow us to manipulate objects, kick them around, score goals if we're lucky. Called algorithms, they are well defined processes which do a particular job. A good algorithm will perform a particular function in a finite number of steps. Algorithms are almost always able to be converted into computer programs. All computer programs are algorithms.

The common algorithms that everyone has used in school are multiplication and division. The multiplication algorithm allows us to multiply two numbers. For instance, 24 x 17 is calculated as follows:

$$\begin{array}{r} 24 \\ \times\ \underline{17} \\ 168 \\ \underline{+\ 240} \\ 408 \end{array}$$

The division algorithm, in one form, works like this:

$$\begin{array}{r} \underline{24} \\ 17)\ 408 \\ \underline{-\ 34} \\ 68 \\ \underline{-\ 68} \\ 00 \end{array}$$

Although not often called an algorithm, we can think of the process of solving a linear equation as an algorithm. We illustrate the finite number of steps used in the example below.

$$\begin{aligned} \text{Solve}\quad 3x - 6 &= 0 \\ 3x &= 6 \\ x &= \tfrac{6}{3} \\ x &= 2. \end{aligned}$$

The algorithm implicit in these steps will solve any linear equation.

I regard mathematical formulae as algorithms because they give a well-defined method for solving certain problems. Hence $A = \pi r^2$ is an algorithm in the sense that it says, if you give me the radius of a circle, I will give you (in a finite number of steps) its area.

Similarly $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

is an algorithm for solving the quadratic equation $ax^2 + bx + c = 0$.

3. Theorems and Logic

At a slightly higher level in the mathematical hierarchy we have theorems and logic. Logic acts on theorems in much the same way as algorithms act on objects. But, so far, it is not as easy to programme logic to manipulate theorems as it is to programme algorithms to manipulate objects.

Given an object I can perform an algorithm on it. Given a known result, it can be proved by following a sequence of logical steps. There is nothing original here. We just follow the train lines from one station to the next. There is nothing unexpected; we know the algorithm will produce the right result; we know that logic will correctly prove the next theorem.

4. Creativity

How does creativity fit in? Creativity produces new discoveries. It extends our list of theorems, algorithms and logical tools. It gives us more control over the world we live in.

Creativity produces problems which lead to conjectures. It enables us to find counter-examples to conjectures, or proofs for new theorems. It is creativity that leads to generalisations and extensions and provides connections between previously unrelated sections of the subject. We know creativity when we see it, and we know what it does. *How* it does it is still locked inside the brain.

In the future it may be possible to program a machine to be creative. But this creativity will have to be more than applying logic to one theorem to produce new theorems. This is because most theorems produced in this way will not be useful, nor will they be interesting. Part of the creativity of a mathematician is to be able to see what is potentially useful and interesting and link that to the mathematics that is currently known.

So mathematics consists of certain planks, and tools – the objects, algorithms, theorems and logic. But mathematics also involves an imaginative side, the creative part that enables new tools to be invented and the wood to grow and to be of more use; that produces results that have not been thought of before; that produces processes to be developed, that can't be found in any textbook or mathematical paper; that is the most interesting and satisfying part of the whole subject.

3. What to teach?

1. Being Creative

We are extremely good at exploring the objects of mathematics and at teaching any number of basic algorithms. At university we show our students endless theorems and often expect them to learn the proofs.

I first got fun from mathematics because I could continually get the right answer. It was not till later that I realized that there was a completely different side to it all. It was possible to produce something entirely new. My entirely new results were rather like finding a dead tree in the Opaharua Swamp. I had to (like Newton) stand on the shoulders of giants to do it. But I was seeing things that no-one had seen before. It didn't matter that it was no more useful than a dead tree. It was *my* dead tree. And I kept hoping that, if I kept looking, there might be a swan behind the dead tree.

In recent years we've started to move towards more creativity in the mathematics classroom. This has been achieved to some extent by the introduction of problem solving. Students get a chance to take part in a search and sometimes get the feeling of discovery and creativity. Problem solving is the closest thing I know of at the school level, to real mathematical research. With good problems, students can go through all of the creative procedures: they can conjecture, find counter-examples, find proofs, generalise, extend, and even publish. In 1952, Ann Roe studied 64 eminent scientists. She found that the single most important factor in the final decision of these people to take up science, was the sheer joy of discovery. With problem solving we are giving children a taste of that joy.

There is a small bit of cheating involved here though: (i) the students generally do not dream up the problems for themselves, and (ii) the answer is already known to someone before the problem is posed. But it is a big step in the right direction.

2. Facts, and Using Them

The human race is storing up more and more facts each day. Probably they are being accumulated at an exponential rate. How can anyone even expect to learn them all? Luckily *knowing* facts is not as important as being able to *use* facts, and to use them in new and novel ways. It is necessary for our students to learn to think deductively and intuitively. We need to give them practice in creativity.

This is not to say that they should learn no facts. Of course, students need to know mathematical facts. They need to know about mathematical objects, algorithms, theorems, and logic. But they should also know how to generate *ideas*.

The reasons for this are because (i) creativity is an integral part of mathematics (and all other disciplines); (ii) we are trying to prepare our students for a world which will not be like today's world – possibly the best training we can give them is to be flexible, to think and to be creative.

There are clues from educationalists about how this may be done. Joseph Renzulli in his *The Enrichment Triad Model* has three types of enrichment which can lead students to genuinely creative experiences. I'll take up his themes in the next section, though they have already appeared implicitly. The only problem is likely to be a need for teachers to change their attitude – it is almost certain that some students will get away, will be off and running, so fast that teachers won't be able to keep pace with them. I often have this problem. Bright school students I've worked with and my graduate students often leave me behind. You just have to relax and enjoy it. Teachers should encourage their students to try out new ideas even if it means disrupting school timetables – perhaps for them to go to a library or visit an expert. And of course, not just for the study of Mathematics.

4. Renzulli was there first

Renzulli is an educator who developed his ideas through an interest in talented students in all areas, not just mathematics. At the heart of his programme are three kinds of enrichment activity – *Types I, II* and *III*. I have adapted his general ideas to the teaching of mathematics, and to the teaching of all pupils, whether obviously talented, or not.

Type I

Type I enrichment activities are general exploratory activities: the teacher provides experiences that put students in contact with topics or areas of study in which they may develop a sincere interest. Although the emphasis is on exploration, the students need to realise that it is purposeful exploration. Eventually students will be expected to conduct further study in one of their areas of interest.

Renzulli suggests 3 ways for students to explore: through interest centres, field trips and resource people. You may well be able to think of others. A mathematics **interest centre** could contain a range of books containing mathematics not met in the regular classroom, bibliographies of famous mathematicians, puzzle books, popular books on mathematically related topics, etc., appropriate to the level of the student. They could also contain a range of equipment. Apart from the obvious items, maybe you could find a theodolite, an astrolabe or some other exotic equipment. (At my last university I was custodian of three magnificent machines from the last century that variously found the arclength of a curve and constructed conics.) It's worth keeping your ear to the ground. It's amazing what equipment universities and companies discard. It's worth visiting antique shops too from time to time. Probably interest centres could also include these days, a computer and software. For example, there is some very nice fractal and chaos software available now.

It is fairly obvious what is meant by **field trips**. There are an increasing number of hands-on science museums springing up. These all have some mathematical items. My university encourages school classes to visit the university and I'm sure others do too. In Britain, I know that some airports are willing to take groups of students. That may be the case here as well. The Meteorological Office, a polling firm, an engineering design office, a standards testing lab, an actuary's department, a city or county council engineering branch, an economic forecasting firm etc., are all possibilities. The aim is to find local places where mathematics is actually being used in some form or other and see if they'll let you and your students in for a good look round.

Resource people could be friendly bearded professors, or people in government or industry. It's good for the students to see live mathematicians. Try to cultivate one or two that you know have a good story to tell or some good problems for the students to try.

In summary then, the *Type I* activities aim to broaden students' knowledge and experience by placing them in situations which are different to the regular classroom routine. It is hoped that this process may give students some particular interests that they will follow up to the *Type III* level.

Type II

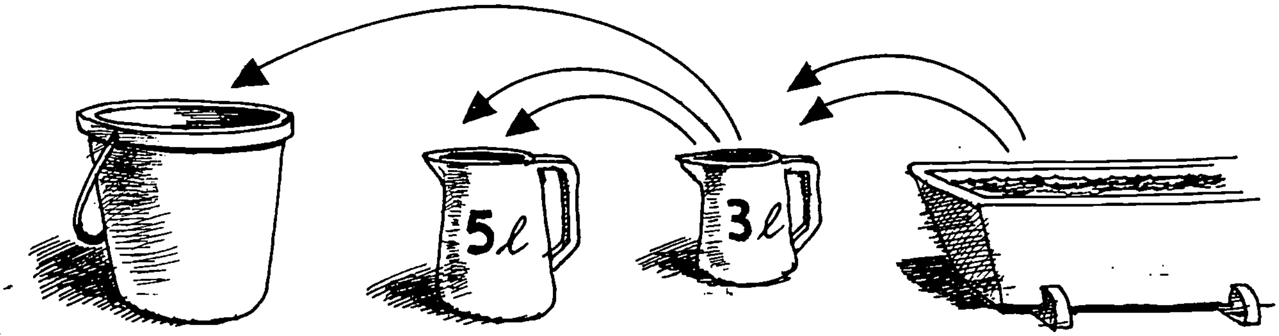
Type II activities are almost exclusively training exercises. But these are definitely *not* content-oriented. Hence *Type II* activities are meant to lead students to a mastery of the processes which enable them to deal more effectively with content. Learning how to use a formula or algorithm is **not** a *Type II* activity; but learning to select the right algorithm is. These activities include critical thinking, problem solving and brainstorming.

They should be planned so that they provide opportunities for developing thinking and feeling abilities to their highest potential. They should also give introductions to more advanced kinds of study and enquiry. It is hoped that they will provide skills and abilities to solve problems in a variety of areas and in new situations.

There are any number of mathematical problems and projects that students can undertake, on their own and in groups, that will facilitate *Type II* processes. These activities should highlight the processes at work, so that the pupils will be aware of their application in other situations. When is the Pigeon Hole Principle useful? When should counting techniques be used? Is now the time to generalise? What happens if we work from the other end? Is there just one counter-example? Could the computer do this for us? *Type II* activities should help students to learn the tricks of the mathematical trade.

Type III

In *Type III* enrichment activities the student becomes an actual **investigator** of a **real** problem using appropriate **methods of inquiry**. At this point students actually *become* mathematicians. They tackle a mathematical problem using the techniques that a mathematician would employ. Preferably (and this may be difficult to arrange) the problem they investigate is also a problem which they have taken an active part in devising.



It is very unlikely that students will be able to take complete responsibility for generating a problem which interests them. My experience is that (even at the Olympiad level) problem-producing does not come easily. However, the teacher needs to be ready to take an idea and help the student convert it into a viable problem.

This is not easy. After a while you develop an intuition for what might work, but you can never be sure. Sometimes the problems are too easy. Sometimes they are impossibly hard, in which case you have to learn to stop and go in another direction.

With original problems there is only one way to proceed: you have to jump in at the deep end. Therefore wise teachers stand by with a life jacket, just in case. If the student isn't able to swim it will be necessary to start again on a subset of the problem in order to produce something which will lead to a result.

Having produced the problem there may be no obvious technique to solve it. This is why *Type II* activities exist – to give some new techniques which might be useful. However, intuition and guesswork may be just as important.

Results

In a *Type III* activity it is important to communicate results. This should be more than putting the work on the classroom noticeboard. Perhaps the proof or extension could appear in a school magazine or newspaper. (After all, poems and stories appear there. Why not original maths?) A friendly academic may be able to recommend somewhere else, and exceptional work might be published in, for example, the *New Zealand Mathematics Magazine* or *Function*.

The attempt to publish or bring the result to the notice of an expert is important for three reasons. Firstly, it increases the chances that the problem has been truly solved, as experts will review it. Secondly, your solution may lead to new proofs and extensions by other people. Thirdly, and perhaps just as important, it gives recognition to the work; it says, 'Student, you have done well.'

So, following the three *Types* of activity Renzulli suggests, has led us to precisely what mathematicians do. Mathematicians search for a problem. Then they seek for ways to solve it. Finally they publish their result. Students can learn mathematics in this way too. In the process they will experience the joy of creating their own mathematics. Hopefully this will be more interesting and lead to better learning than the traditional algorithm learning approach to the subject.

5. Some Examples

Here are some examples of Type II activities developed into Type III activities. I hope you will see that, by taking almost any situation that you have been familiar with for a long time, can be developed into a Type III activity.

Example 1

(1.1) Given a 3 litre jug and a 5 litre jug and a water trough, can you measure out exactly 4 litres of water?

This is a fairly straightforward question that even young children can work at using trial and error. It can be developed into a project by going to the next question.

(1.2) Given a 3 litre jug and a 5 litre jug, what amounts of water can you measure?

This question is amenable to a rigorous proof by 11- and 12-year-olds. But first the students have to do some experimental work. (With pen and paper will do though it might be more fun with water.) The experimental work can lead to a conjecture and then the conjecture will need to be proved or a counter-example found.

The answer here is that any whole number of litres can be produced. The proof is to first get 1 litre. Once you have 1 litre you just keep adding 1 litre lots together and you eventually get any whole number of litres. It's important for students to

(a) get the idea of the proof, either through discussion in a group, or by their own individual experimenting,

(b) write up the proof.

At this stage brainstorming should come up with a whole host of problems related to (1.1) and (1.2). These will vary in sophistication depending on the maturity and ability of the students. With older students you might get them inventing the following problem (*Type III* activity).

(1.3) Given an x litre jug and a y litre jug, what amounts of water can you measure accurately?

A further extension : in (1.3) x and y will probably represent whole numbers.

(1.4) What if x and y are just *any* positive numbers?

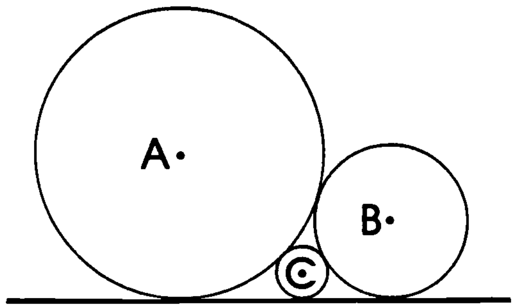
Students should work on whichever problem they come up with and produce a proof which is appropriate to their level of sophistication. For students who know the Euclidean Algorithm it is relatively straightforward to show that if x and y have no common factors, then all whole numbers of litres can be obtained.

(1.5) What if x and y have a greatest common factor of z though?

This activity can give students a chance to practice the creative side of mathematics. There is ample opportunity for experimentation, conjecture and proof, while all the time using known algorithms. (If an algorithm, such as the Euclidean Algorithm is needed but not known, then the students can be directed to a book to learn it for themselves, or it can be taught. This is potentially a *Type II* activity.) These activities can be used with small groups or a whole class. You could put the activities (1.1) to (1.5) on cards and let students work on one card at a time. But this leaves no obvious chance for *Type III* work.

How can it motivate a *Type III* activity? What possible problems come to mind as a result of this example? Of course, it's impossible to answer that question. It might be that the student goes for rods, x and y units long, and asks what lengths can be made with rods of these lengths. Who knows what they will come up with?

Example 2.



(2.1) In the diagram the circle centre A has radius 2cm and the circle centre B has radius 1cm. What is the radius of circle C?

This problem can be solved by an application or two of Pythagoras' Theorem.

While not a strong *Type II* activity it does require some ingenuity to solve the problem. A common error is to assume that there is a right angle at C. In fact no matter what the radii of the two larger circles it can be shown that the $\angle ACB$ is never 90°. One nice problem is to explore the values of $\angle ACB$.

Brainstorm. What questions arise from (2.1)?

(2.2) Suppose the largest possible circle centre D is placed (squeezed) between the circles A and C, and the horizontal line. What is the radius of D?

(2.3) Repeat (1.2). In other words, find the radius of the largest circle which lies between the circles with centres A and D, and the horizontal line. In this way there are a whole series of circles wedged in between the circle centre A, the horizontal line and the last circle. Find a formula for the radii of all these.

Students should now try to develop their own *Type III* activity based on (2.1). Two possibilities are suggested below but there are possibly infinitely many other questions that could be posed.

(2.4) Let P_1 be the first circle (centre A) and P_2 the second circle (centre B). Let P_i be the largest circle contained between the circles P_{i-1}, P_{i-2} and the horizontal line, where $i \geqslant 3$. Find a formula for the radius of circle P_i.

(2.5) Let $P_1 = Q_1$, $P_2 = Q_2$ and $P_3 = Q_3$, where P_1, P_2, P_3 are as in (2.4). Let Q_i for $i \geqslant 4$ be the largest circle contained between Q_1, Q_2 and Q_{i-1}. Find a formula for the radius of Q_i.

Example 3.

(3.1) Call a number n *good*, if the numbers 1, 2, 3, ..., n, can be split into two groups such that the numbers in one group have the same sum as the numbers in the other.

(For students who know about sets, let N = {1, 2, 3, ..., n}. We say that *n* is *good* if N can be divided into two sets A and B such that $A \cup B = N$, $A \cap B = \varnothing$ and the sum of the numbers in A is equal to the sum of the numbers in B.)

This is a nice *Type II* exploration that has a simple solution. The proof that I know does depend on knowing the sum of the numbers in N. However, a lot can be gained from this problem without knowing this. Primary school students should be able to guess the pattern even if they can't prove that their conjecture is true.

Taking this activity on to *Type III* is possible, perhaps with several children brainstorming. Perhaps they will break N into 3 sets to see what problems arise.

Example 4.

(4.1) Now 9 = 4+5 and 9 = 2+3+4. So 9 can be written as the sum of a *consecutive* string of positive integers in two ways. Are there any numbers that cannot be written as a consecutive string of positive integers?

Experiment. Start with 1,2,3,4,5, ... and so on. This is a classic chase. Do you see any patterns? Can you make a conjecture? Can you prove the conjecture? All the good old stuff.

This problem like many others can be dealt with at a series of levels.

(i) Primary school students can, if nothing else, get good practice in arithmetic by looking for examples of numbers which are *not* the sum of a consecutive string.

(ii) Many of these students can make a reasonable conjecture.

(iii) The proof is probably difficult to expect till secondary school. I have seen bright 10- and 11-year-olds get there with help.

(iv) What questions does this problem suggest? Brainstorming, may bring up questions that pupils (and teachers) can't solve. In that event, ring up a sympathetic academic.

Clearly the *Type III* suggestions I've made are not *Type III* in the strictest sense of the word because they are not the student's *own* problems. However, students will come up with questions that I haven't thought of. They do it all the time. They just need encouragement and help.

Notes

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He recommends a chapter by Alan Schoenfeld as reinforcing and extending the ideas in this *set* item. It is

Schoenfeld, Alan H. (1992) Learning to think Mathematically: Problem Solving, Meta Cognition, and Sense-Making in Mathematics, in, Grouws, Douglas (Ed) (1992) *Handbook of Research Teaching and Learning*, New York: Macmillan.

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Number Skills in Junior Classrooms

By Jenny Young-Loveridge
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What number concepts and skills do children have when they start school? How has their understanding changed after a year? How aware are teachers of what the children know? What number concepts and skills are taught in the first year? These questions are important. If we underestimate what children can do their achievement will be less than it might have been.

The results of this study were very cheering – teachers can, after only about 12 days teaching a child, make very accurate judgements about mathematical knowledge and skill. However, there are also some clear warning results about certain ways of using teaching programmes which do not bring about maximum progress.

The children

1. On Entry to School

The sample consisted of 47 girls and 34 boys who had been attending school for a month or less. The children came from 18 different primary schools in Christchurch, covering a wide geographic area and varying socio-economic status. The children were given an interview with 14 different kinds of number tasks, presented in the form of a game. (See Table 1.)

Table 1
Tasks Used in the Number Tasks Interview

	Extra items 1 year later
1. Rote Counting up to (10) (20) (30)	
2. Forwards Number Sequence – give no. just after (5) (16)	(29)
3. Backwards Number Sequence – give no. just before (5) (9)	(16)
4. Enumeration (5 objects) (9 objects)	
5. Understanding the Cardinality Rule (5 objects) (9 objects)	
6. Pattern Recognition for Small Numbers (3) (4) (5)	
7. Ordinal Numbers (first) (last) (second) (third)	(fifth)
8. Numerical Difference (of 1) (of 2)	
9. Recognition of Numerals (2) (5) (9)	(14)
10. Formation of Sets (2) (5) (9)	(14)
11. Addition with Concrete Objects (3 + 2) (6 + 3)	(9 + 5)
12. Subtraction with Concrete Objects (5 – 2) (9 – 3)	(13 – 6)
13. Addition with Imaginary Objects (2 + 1) (3 + 2) (4 + 3)	(7 + 4)
14. Subtraction with Imaginary Objects (2 – 1) (4 – 2) (5 – 2)	(9 – 4)

Many of the children performed very well on the number tasks. Even the least able children knew something about numbers. When their answers were scored to give a total out of 36, the average score was just over 20. However, children varied a great deal. At the top end of the range there was a child who could do all the tasks, including rote counting up to 100. At the other end was a child who could rote count up to 8, enumerate 5 blocks, and identify 'first'. Two other children who were slightly more successful overall, did not appear to be able to rote count at all. Table 2 is a summary of the tasks on which children were most and least successful.

Table 2
Tasks on which Children were Most and Least Successful on entry to School

	Number of children	%
Most Successful		
Identifying the ordinal position first	79	98
Formation of a set of 2	76	94
Identifying a numerical difference of 1	70	86
Enumeration of 5 objects	69	85
Application of the cardinality rule to a row of 5 objects	62	77
Least Successful		
Rote counting up to 30	8	10
Subtraction with imaginary objects of 2 from 5	18	22
Addition with imaginary objects of 4 and 3	21	26
Subtraction with imaginary objects of 2 from 4	24	30
Giving the number just after 16	25	31
Average across all tasks	45	56

2. After One Year at School

Approximately one year later, the children were interviewed again, using the same 14 different number tasks as before, plus a more difficult item for each of nine tasks.

The children's number concepts had advanced substantially. Several tasks were done correctly by all the children. The average score out of 45 was almost 36, the same as the full score at the beginning of the year. Individual children ranged from 15 to 45.

When the data was analysed it was found that the children who entered school with relatively little knowledge about numbers made greater learning gains than did their more knowledgeable peers. This pattern contrasts markedly with those of other studies, for example, that by Fogelman in 1983. These other studies show a 'snowballing' effect with more able students getting further and further ahead of their less able peers. Further analyses were done to find out why our children were different. The reason appeared to be that the school mathematics programme they were getting was well matched to the existing skills of the less knowledgeable, but was not well matched to the skills of the children who already knew a lot about numbers.

Once again, children's performance varied a great deal. Table 3 is a summary.

Table 3

Tasks on which Children were Most and Least Successful after a Year at School

	Number of children	Percent
Most Successful		
Enumeration of 5 objects	80	(100)
Recognition of the numeral 2	80	(100)
Formation of a set of 2	80	(100)
Rote counting up to 10	79	(99)
Pattern Recognition of 3	79	(99)
Identifying the ordinal position first	79	(99)
Identifying a numerical difference of 1	79	(99)
Least Successful		
Subtraction with imaginary objects of 4 from 9	18	(25)
Identification of the ordinal position fifth	24	(30)
Giving the number just after 29	32	(40)
Giving the number just before 16	33	(41)
Addition with imaginary objects of 7 and 4	33	(41)
Average across all tasks	63	(79)

The teachers

1. At the New-Entrant Level

The eighteen teachers were asked, with a questionnaire, which children could do each of the tasks. The questions were framed as if the teacher was interviewing the child, and the wording was the same as what was actually used in the interview with the children – that avoided any misunderstanding about terminology. Teachers were asked not to do the tasks with the children, but instead to think about how well the children could do on the tasks. Teachers' answers were compared with what the children could actually do, and then the numbers of overestimates and underestimates were calculated.

Teachers' judgements were remarkably accurate overall – that was even although the children had been at school only about 12 days, on average. These accuracy levels compare very favourably with those found in previous studies. Morine-Dershimer looked at teachers' expectations of pupil's success in reading and found that teachers tend to err on the side of being overly optimistic. However, my study found that teachers tended to err on the pessimistic side, with underestimates exceeding overestimates considerably. It may be that teachers are pessimistic about children's performance in mathematics because they themselves have a negative attitude towards mathematics. Comments made by teachers during my visits to schools did reflect a lack of confidence about mathematics.

Like other research, this study found that teachers tended to underestimate the mathematical knowledge of the high scorers more than that of low or middle scorers. The reason for this tendency may be that it is difficult to recognise tasks which are too easy when the children appear to be working cheerfully and industriously, even with tasks which hold no challenge for them. Table 4 is a summary table showing tasks on which children were judged most accurately and least accurately by their teachers.

Table 4

Tasks on which Children were Judged Most and Least Accurately by their Teachers

	Percentages of Children		
	Over-estimates	Under-estimates	Total mismatch
Most Accurate			
Formation of a set of 2	4	0	4
Identifying the ordinal position of first	3	4	7
Rote counting up to 30	12	3	15
Enumeration of 5 objects	7	9	16
Giving the number just after 16	1	16	17
Least Accurate			
Identifying a numerical difference of 2	8	50	58
Subtraction with imaginary objects of 1 from 2	12	46	58
Subtraction with concrete objects of 2 from 5	11	38	49
Addition with imaginary objects of 3 and 2	1	45	46
Addition with concrete objects of 3 and 2	16	28	44
Average across all tasks	11	19	30

2. At the End of the Junior One (J1) Level

Approximately one year later the teachers were asked whether or not they had taught the children any of the mathematical concepts and skills in the list. The questions were again framed as if the teacher was interviewing the child and the wording was the same as in the interview. Four different 'learning' outcomes were revealed: *already known, learned, forgotten,* and *not learned.* These were further subdivided according to whether or not the teacher said the skill had been taught to the child during the year.

A substantial number of children had been taught by their teachers skills they already had when they started school. For example, 85% of children were taught how to count 5 objects, how to identify 'first', and how to form a set of 2 objects, even though they already understood these concepts when they entered school. On the other hand, a considerable number already knew how to do simple operations, and were not taught (or re-taught) these skills during the first year of school. For example, over half of the sample (53%) could subtract 2 from 5 with concrete objects, and almost as many (48%) could subtract 1 from 2 mentally. Table 5 is a summary of these mismatches.

Table 5
Concepts Already Known by Most Children: Taught and Not Taught

	Already Known (percentages of children)
Taught	
Enumeration of 5 objects	85
Identification of the ordinal position first	85
Formation of a set of 2	85
Identifying a numerical difference of 1	81
Rote counting up to 10	74
Average across all tasks	41
Not Taught	
Subtraction with concrete objects of 2 from 3	53
Subtraction with imaginary objects of 1 from 2	48
Subtraction with concrete objects of 3 from 9	34
Addition with imaginary objects of 2 and 1	28
Addition with imaginary objects of 3 and 2	26
Average across all tasks	12

Forty percent of the children were taught, and learned, to recognise a pattern of 5 objects and to identify the numeral 2. Thirty-nine percent were taught, and learned what number comes just before 5 and how to count 9 objects. However, a similar number learned skills even although they had not been deliberately taught them. For example, 40% of children learned what number comes just after 16, the subtraction of 3 from 9 with concrete objects, as well as the subtraction of 2 from 5 with imaginary objects. Table 6 is a summary.

By putting Table 5 and Table 6 together I worked out that children were being given twice as many opportunities to practice existing skills as they were to learn new ones. In general, the more new skills taught the quicker the pace through the curriculum, but it is impossible to determine an ideal ratio of practice to new skills taught because the needs of individual children have to be taken into account. However, the children with higher abilities have their achievement levels most noticeably reduced by an unnecessarily slow pace through instructional materials.

Table 6
Concepts Learned by Most Children: Taught and Not Taught

	Learned (percentages of children)
Taught	
Pattern Recognition of 5	40
Numeral Recognition of 2	40
Giving the number just before 5	39
Enumeration of 9 objects	39
Formation of a set of 9	38
Average across all tasks	21
Not Taught	
Giving the number just after 16	40
Subtraction of 3 from 9 with concrete objects	40
Subtraction of 2 from 5 with imaginary objects	40
Subtraction of 2 from 4 with imaginary objects	39
Rote counting up to 30	34
Average across all tasks	12

Implications

The results of the present study have some important implications for Mathematics schemes and curricula in junior classes. For example, in New Zealand a new resource, the Beginning School Mathematics programme (BSM) is being introduced. Because BSM is a language-based activity programme which follows the theoretical principles of Jean Piaget, there is a lot of emphasis placed initially on sorting, matching, comparing, ordering, and classifying, and relatively little work in the area of number per se. A deliberate decision was made not to introduce the study of numbers until almost halfway through the programme (Cycle 4 out of the 8 cycles), to spread this over a year instead of a term, and not to go above the number nine.

Many of the concepts which a substantial proportion of the children in the present study knew on entry to school, (and even more knew after a year at school), are first referred to in BSM more than halfway through the programme. The most notable examples are enumeration to 9, and the joining and separating of sets. None of these are referred to in the programme before Cycle 6 of the 8 cycles to be covered in the first two years of school. At least half of Christchurch children were able to do these tasks before they started school.

The findings of the study have particularly serious implications if a lock-step approach to teaching mathematics is taken, with all children starting at the beginning of a programme (regardless of what they already know), and being taken through every activity, whether or not it is appropriate for them. If BSM is used this way the children would be most unlikely to reach the later cycles much before the end of the first year at school, if not the second year, and this has been clearly demonstrated in trial schools. As a consequence, the children would not be challenged from the word 'go' with activities which build on their existing knowledge: it is likely that they will lose interest in mathematics altogether. At the very least, their progress would be halted for a while, until more demanding activities are provided when the later units are finally reached.

Although it is intended that teachers use the BSM programme flexibly, starting individual children at points in the programme where there is room for learning new concepts and moving them thorugh it at a pace which is appropriate for their individual rates of learning, overseas research suggests that this may not actually happen in practice. For example, Bennett and his colleagues in 1984 found that teachers (who had been selected as 'good' teachers) stayed very close to the sequence of mathematics avtivities laid down for them. Particular activities were chosen because the teacher had reached the place in the programme where those activities are usually used, rather than because they were suitable for developing children's current skills a little further. Other researchers have observed that teachers tend to choose activities which maintain the flow of activity rather than those which are appropriate for moving students from their current level of skills towards objectives. If we are to use the BSM or other detailed step by step programmes flexibly (as is usually intended by the curriculum developers) we could do with the help of specific directions to this effect and concrete assistance.

Looking Back

The good news is that overall, teachers make quite accurate judgements about what children know, and can do so after a relatively brief period of acquaintance. They are well trained professionals: They clearly have the necessary skills for making decisions about which activities are appropriate to build on the existing skills of their pupils. (As a researcher I suspect that accuracy levels could have been even higher if some form of systematic assessment, e.g., checklists, had been used in addition to informal observations.)

The not so good news is that all of us may need to be reminded about the importance of using those skills of judging our students' current knowledge when we plan instruction. We need to choose learning activities which move our students from where they are now to where we want them to be. Otherwise many may waste precious time doing 'busy' work while they cover material with which they are already familiar. Current models of learning regard the amount of time spent actively engaged in classroom tasks as central to learning. Time on task seems to account for variation in school learning, not only among students and among classes, but also among nations. The long-term consequence of giving children number activities which are not challenging enough to provide opportunities for learning new skills, and instead, allowing them to spend a lot of time on skills over which they already have complete mastery, is that the total time available for mathematics *learning* over children's school years will be reduced. In the end, children's achievement will be less than it might have been.

We, as professionals, need to pay attention to both diagnosis and treatment.

Notes

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That understanding children's abilities leads to lower achievement is detailed in
Good, T.L. and Brophy, J.E. (1978) *Looking in Classrooms* (2nd ed.), New York: Harper and Row.
Good, T.L., Grouws, D.A. and Beckerman, T.M. (1978) Curriculum pacing: some empirical data in mathematics, *Journal of Curriculum Studies*, Vol. 10, pp. 75–81.

For details of the procedure when interviewing the children see
Young-Loveridge, J.M. (1987) *The development of children's number concepts*, Education Department, University of Canterbury, Research Report No. 87-1.

An example of a study which showed the 'snowballing' effect with able students getting further and further ahead is
Fogelman, K. (Ed.) (1983) *Growing Up in Great Britain: Papers from the National Child Development Study*, London: Macmillan.

Previous studies which looked at how accurate teachers were in judging children's progress are
Morine-Dershimer, G. (1978–1979) The anatomy of teacher prediction, *Educational Research Quarterly*, Vol. 3, pp. 59–65.
Wang, M.C. (1973) The accuracy of teachers' predictions on children's learning performance, *Journal of Educational Research*, Vol. 66, pp. 462–465.

Comments from teachers which showed a lack of confidence about Mathematics can be found in Young-Loveridge mentioned above.

Teachers tended to underestimate the mathematical knowledge of high scorers. Other studies which confirm this are
Bennett, N., Desforges, C., Cockburn, A. and Wilkinson, B. (1984) *The Quality of Pupil Learning Experiences*, Hillsdale, N.J.: Erlbaum.
Wunderlich, E. and Bradtmueller, M. (1971) Teacher estimates of reading levels compared with IRPR instructional level scores, *Journal of Reading*, Vol. 14, pp. 303–308, 336.

That it is difficult to recognise tasks which are too easy because the children are cheerful and industrious is mentioned by Bennett above.

I found that children were given twice as many opportunities to practice existing skills as to learn new ones. This ratio is slightly better (i.e., lower) than the corresponding ratio which Bennett and his colleagues (1984) found overall (i.e., 1 to 2.4), but not as low as they found for number tasks alone (where the ratio was 1 to 1.3).

Bennett and colleagues also mention that in general the more new skills 'learned' (and the fewer practised) the quicker the pace through the curriculum.

That the children with higher abilities have their achievement levels most noticeably reduced by a slow pace is discussed in
Barr, R. (1974) Instructional pace differences and their effect on reading acquisition, *Reading Research Quarterly*, Vol. 9, pp. 526–554.
Barr, R. (1975) How children are taught to read: grouping and pacing, *School Review*, Vol. 83, pp. 479–498.

The BSM programme is detailed in
Department of Education (1985) *Beginning School Mathematics*, Wellington: Government Printer.

The deliberate decision not to introduce numbers till almost halfway through the cycle is made clear in
Barriball, D. (no date) Major differences between programmes developed from the MIC resource, and the revised J.1 to Std 4 Syllabus and B.S.M. Unpublished paper.

Bennett and colleagues, above, found that teachers stayed very close to the sequence of mathematics topics laid down. Other researchers, mentioned as finding much the same include
Shavelson, R.J. and Stern, P. (1981) Research on teachers' pedagogical thoughts, judgements, decisions and behaviour, *Review of Educational Research*, Vol. 51, pp. 455–498.

Avoiding children wasting time on covering material they already know is discussed in
Bruner, J. (1960) *The Process of Education*, Cambridge, M.A.: Harvard University Press.

Current models of learning taking time-on-task into account include Bloom, B. (1980) The new direction in educational research: Alterable variables, *Phi Delta Kappan*, Vol. 61, pp. 382–385.

SMALL CHILDREN SOLVE BIG PROBLEMS

Lyn English
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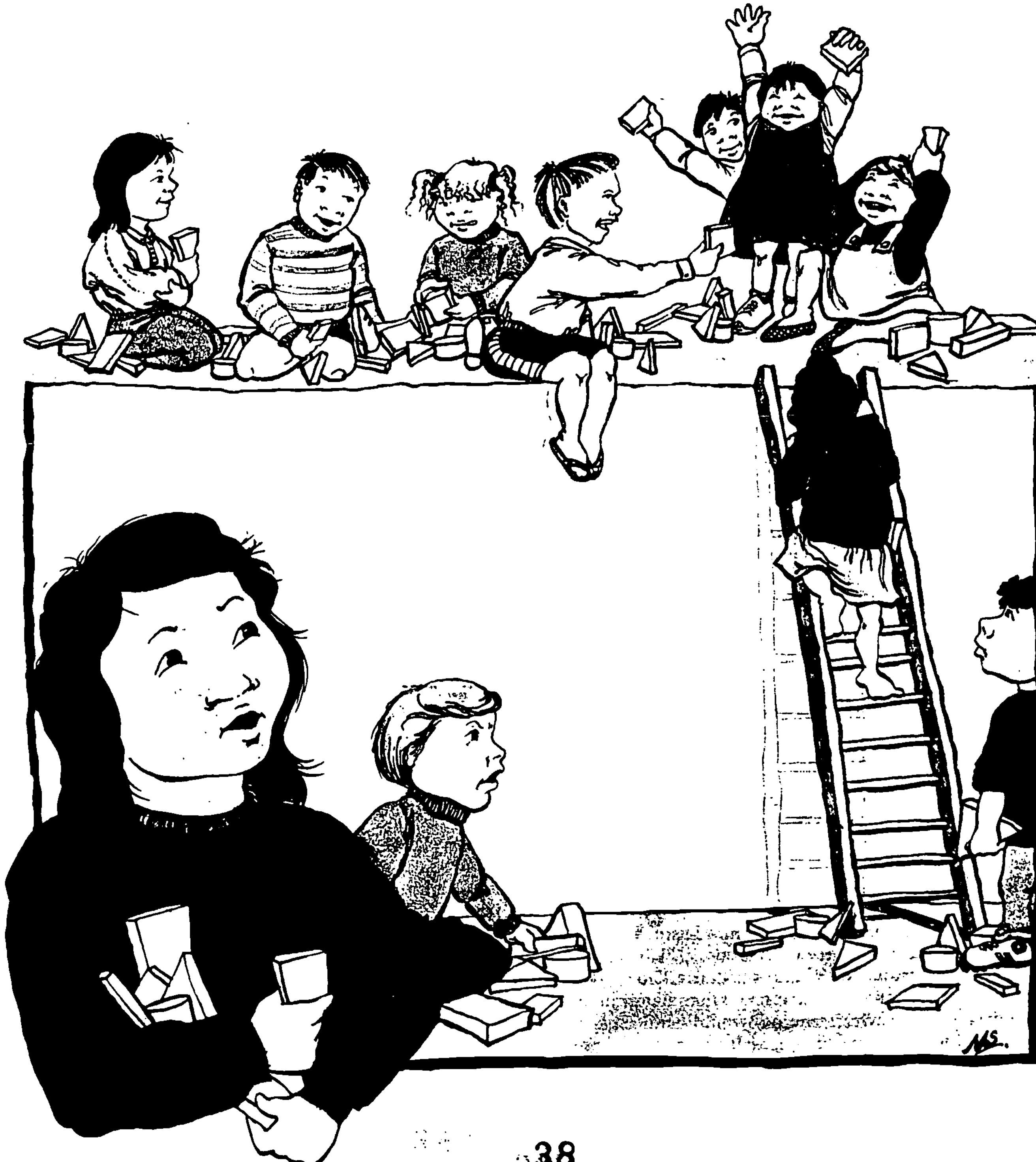


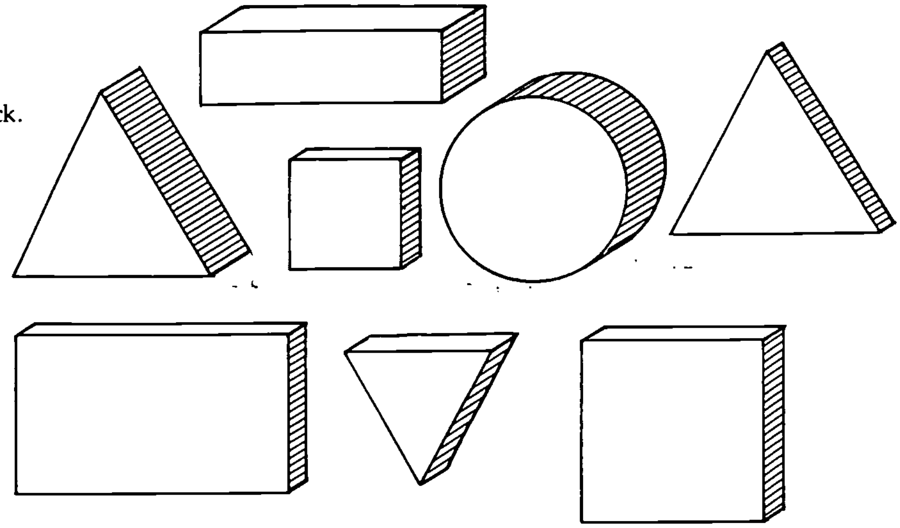
Nikki Slade

Figure 1.

Try your hand at this problem.

Use the clues to find the special block.

Clues:
The block is not thick.
It is not a triangle.
It is not large.



What method did you use to arrive at your answer?

THIS PROBLEM was given to 52 5-year-olds, in a quiet room of their school. The blocks were there and the clues spoken. Forty-six identified the right block and with very little delay; once the last clue had been given, the children pointed immediately to the correct block.

When we analyse the logical thinking required, we realise just how cognitively adept young children are when they commence their school lives. Yet research has tended to highlight what young children can *not* do. The result is that children's competence as thinkers and problem solvers has been underestimated. By highlighting the cognitive talents of young children we can see how to enrich children's problem-solving experiences.

Past Limitations

The theories of Jean Piaget have been the most influential on our thinking about children's thinking. However, they portray young children as limited in their ability to reason logically or inferentially or to think intelligently in general. If you set out to teach only the range of educational experiences which fit Piaget's stages of cognitive development you will restrict your teaching. Piaget's experiments were based on sophisticated scientific phenomena and were frequently accompanied by abstract instructions and unfamiliar apparatus. Consequently, children's poor performance on many Piagetian tasks could have been due largely to their failure to understand the instructions and to the meaningless nature of the task materials. In the 'colouring liquids' experiment, for example, children were presented with containers of different chemical substances which they were to mix in all possible ways. The chemical reactions of the substances produced varying colours which served to identify the combinations formed. Another experiment required the child to predict the conditions under which a rod would bend sufficiently to enable one end to touch the water in a basin. Weights and clamps were attached to the rods which varied in thickness, cross-sectional form, and substance. These variables had to be considered one at a time when determining the conditions under which the rod would bend. This meant that all the other variables had to be held constant. In other words, the child had to imagine all the possible combinations involving these variables. Given the sophisticated structure of these two tasks, it is not surprising that it was not till adolescence that children could do these tasks, that is, at Piaget's stage called 'formal operations'.

Young children can solve seriation problems

While acknowledging Piaget's outstanding contribution to child psychology, modern research is yielding a more positive and realistic account of the young child's cognitive abilities. For example, DeLoache, Sugarman, and Brown explored very young children's strategies in assembling a set of five nested cups. The cups were chosen because of their appeal to young children and because mistakes were immediately obvious to the children – a feedback feature. Children as young as 18 months arranged the cups with enthusiasm. They knew immediately when they had made a mistake and hence, were able to correct their errors. An analysis of the ways the children managed the corrections revealed increasing flexibility and broadening of thought and action with age, trends which otherwise might not have been detected.

Young children can reason deductively

It was once thought that only older children could manage to draw logical conclusions. However, six-year-olds have this ability. Within the context of a make-believe world, Dias and Harris presented young children with syllogisms whose premises ran counter to their practical knowledge, for example:

All cats bark.
Rex is a cat.
Does Rex bark? Why/why not?

It was found that the use of fantasy cut down the attention they gave to their practical world knowledge, enabling them to accept the premises as a basis for reasoning. Given a task with appropriate content, young children can reason deductively. I confirmed this in a similar project. I gave beginning school children syllogistic problems with fantasy creatures. One piece of reasoning was

Wobbles wear furry pyjamas to bed.
Animals that wear furry pyjamas love to eat grubs.
Morilla is a Wobble.
Does Morilla eat grubs? Why/why not?

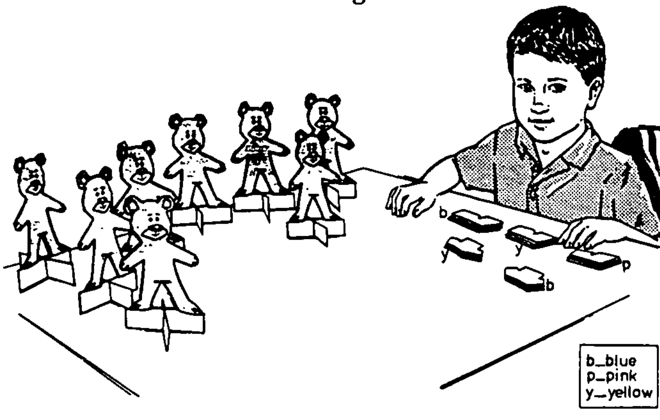
This was presented in two different ways. In one set of problems, children played with a stuffed animal called 'Morilla'. In the other set of problems, children were asked to pretend that *they* were the creature. For example, 'Let's pretend that you are a Wobble. Do you eat grubs?' In this instance, one would expect the children to say, 'No! I don't eat grubs'. However, many of the children not only drew logically correct conclusions, but also gave 'theoretical' reasons why: 'Yes, Morilla eats grubs because she wears furry pyjamas', or 'Yes, I eat grubs because I'm a Wobble and I wear furry pyjamas.' In other words, the children did not guess; they were able to ignore practical considerations and use deductive reasoning to draw conclusions. An interesting finding here was that children who adopted the creature role outperformed those for whom a toy was provided. But maybe the doll's pyjamas were not furry enough!

Young children can solve problems of combination

Reasoning in a logical manner is also evident when children systematically form all possible combinations of two items. The name for the mathematical study of the selection and arrangement of items in a finite set is *combinatorics*. An interesting research project looked into how well young children can handle such problems. The problems they were given did not have 'ready-made' solution procedures and hence, the children had to use their existing knowledge structures and thinking processes to solve the problems. The study was in three parts involving 115 children aged from 4.6 years to 9.10 years. Here is one of the investigations in which 50 children took part.

Each child was presented with a series of seven problem-solving tasks involving dressing toy bears in all possible combinations of tops and pants of various colours. Two of the tasks used items of the same colour but with varying numbers of buttons. The number of possible combinations ranged from five to nine.

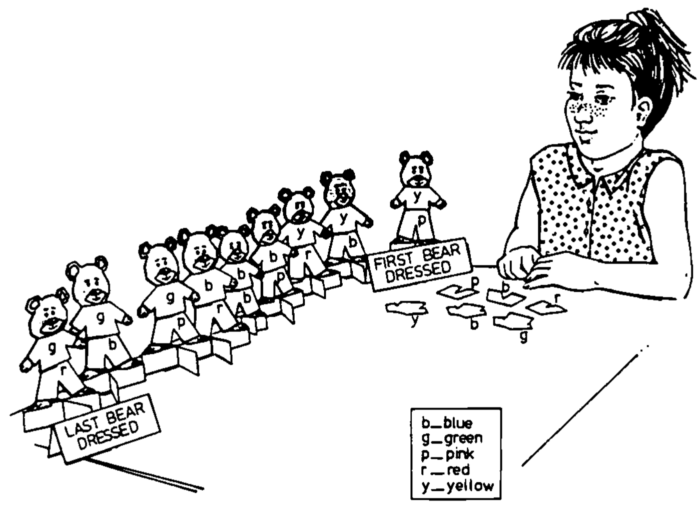
Figure 2



The children were videotaped and their problem-solving tactics later analysed. They all possessed a repertoire of problem-solving strategies which they almost all could apply effectively. All of the children commenced with trial-and-error; they did not employ any systematic method in selecting the clothing items but simply chose the tops and pants in a random fashion. This is a relatively inefficient way of solving such a problem yet 25 of the 50 children managed to get all the possible different outfits in the first task. The major reason for their success was their effective use of checking strategies. They employed a number of scanning actions to monitor their progress. Through careful scanning, the children were able to detect and correct any errors in the mix of items they had selected.

As the children got further through the set of similar tasks, 28 of the 50 children adopted more sophisticated solution strategies. They moved away from trial-and-error towards systematic solution procedures involving a pattern in item selection (e.g. *blue top, green top; blue top, green top...*). They refined their pattern until they had adopted an efficient 'odometer strategy', so named because of its resemblance to the functioning of a car's odometer (milage counter). This strategy involved taking one item of clothing (e.g. a yellow top) and systematically matching it with each of the items of the other type until all possible combinations with that item had been formed (e.g. *yellow top*/pink pants; *yellow top*/blue pants; *yellow top*/red pants). This procedure was repeated until all possible combinations of tops and pants had been formed, as shown in Figure 3.

Figure 3



While the problem (dressing dolls) was meaningful to young children, the mathematical domain underlying the task was new to them. Hence the children's initial attempts showed little knowledge of combinatorics, but considerable knowledge of informal problem-solving procedures. But once they had finished the tasks, many children displayed an implicit knowledge of combinatorial principles. Some could even make the principles explicit by explaining the 'best' way of solving the problem. Associated with this was a significant improvement in their ability to tackle non-routine problems outside the combinatorics field, with many children independently acquiring expert-like problem-solving strategies. In essence, these children were able to assume control of their own learning.

Conclusions

The findings of the studies have significant implications for the education of young school children. Traditionally, mathematical problem-solving in the early school years has focused on routine word problems involving arithmetic computations. Because problems of this type are usually 'predictable' in that they can be solved through the application of one of the four arithmetic operations, they rarely provide children with the opportunity to engage in diverse thinking processes. Problem-solving in the early school should encompass more than the application of previously learnt rules or procedures. Children need to engage in a wide range of problem-solving experiences (some of which have as many solutions as there are children in the class) and should be encouraged to *think* and *talk* about their means of solution. Schools should capitalise on young children's intellectual talents so that their thinking can be extended and enriched.

Notes

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The study first mentioned was undertaken by Lyn English during 1989. Further information may be obtained from her.

The underestimation of children's competence as thinkers and problem solvers is detailed in

Donaldson, M. (1985) The mismatch between school and children's minds. In N. Entwistle (Ed.) *New Directions in educational psychology 1: Learning and teaching*, London: The Falmer Press.

How Piaget's influential theories show young children as limited in logical reasoning is discussed in the same book.

A criticism of the sophisticated scientific phenomena in Piaget's experiments can be found in

Carey, S. (1985) Are children fundamentally different kinds of thinkers and learners than adults? In S.F. Chipman, J.W. Segal and R. Glaser (Eds.) *Thinking and learning skills, vol. 2: Current research and open questions*, Hillsdale, New Jersey: Lawrence Erlbaum.

The 'colouring liquids' experiment and bending rod experiment is written up in

Inhelder, B. and Piaget, J. (1958) *The growth of logical thinking from childhood to adolescence*, London: Routledge and Kegan Paul.

The nesting cups experiment is found in

DeLoache, J., Sugarman, S. and Brown, A. (1985) The development of error correction strategies in young children's manipulative play, *Cognitive Development*, Vol. 56, pp. 928-939.

The Dias and Harris experiment with syllogisms comes from

Dias, M. and Harris, P. (1988) The effect of make-believe play on deductive reasoning, *British Journal of Development Psychology*, Vol. 6, pp. 207-221.

Confirming research by the author on syllogistic reasoning is written up in

English, L.D. (1989) *Reasoning in a fantasy world: Deductive logic displayed by beginning first graders.* Paper presented at the Annual Conference of the Australian Association for Research in Education, Adelaide, November 28–December 2.

Research on reasoning to solve combinatorial problems was studied by the author and can be found more fully in

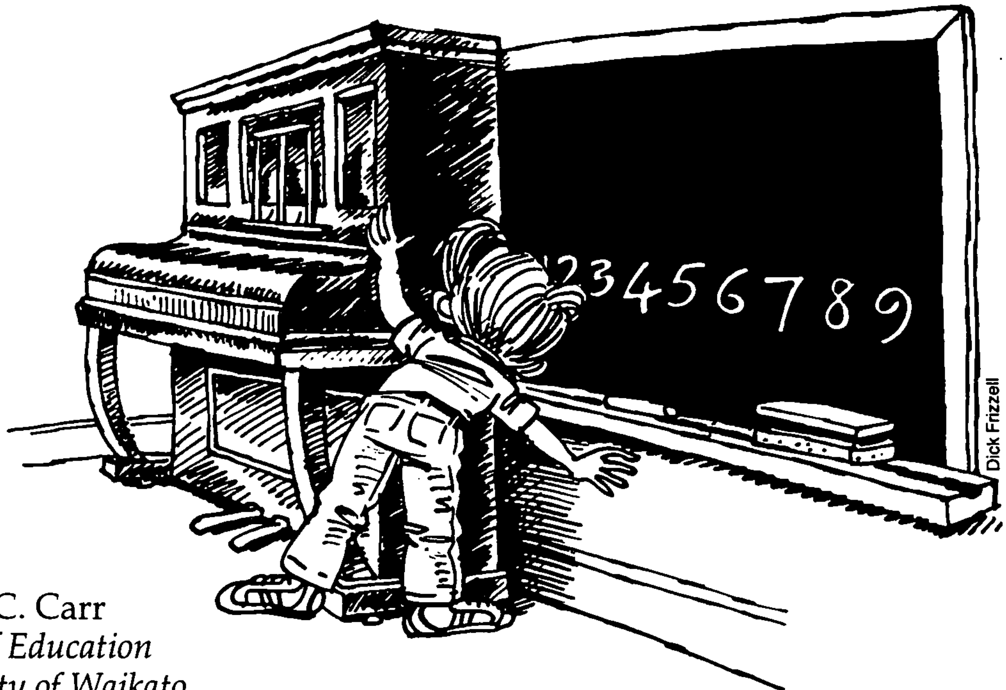
English, L.D. (1988) *Young children's competence in solving novel combinatorial problems*, Ph.D thesis, University of Queensland.

That schools should capitalise on children's intellectual talents is further developed in

Splitter, L. (1988) On teaching children to be better thinkers, *Unicorn*, Vol. 14, No. 1, pp. 40-47.

There Are Numbers Behind the Piano

Children's Construction of Meaning in Mathematics



By Ken C. Carr
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Introduction

The study of children's mathematical behaviour goes back many years. Piaget in the 1930s used clinical interviews to unearth children's ideas about number. Earlier than this, Brueckner had used tests that measured ability in computation and solving verbal problems, and tests that sampled various combinations of a particular skill (such as the addition of fractions).

In a short time the picture emerged that mathematics was an extremely difficult subject for many children to master. Some people, however, claimed that it was mathematics *education* that was failing.

To investigate the problems, large scale surveys, such as the National Assessment of Educational Progress (NAEP) in the United States of America, have been used. The latest NAEP assessments sampled 45,000 students. In Britain the Concepts in Secondary Mathematics and Science (CSMS) research programme assessed 10,000 students. Likewise, the International Association for the Evaluation of Educational Achievement (IEA) survey in mathematics used in New Zealand a sample of 5177 for the 'core test' as part of a study of secondary school mathematics in 23 countries.

At the other end of the continuum researchers have probed the mathematical ideas of individual children through interviews, in an attempt to uncover underlying cognitive processes. These studies usually focus on one specific topic within mathematics, and provide information on the learner's view of the processes involved.

Still other research adopts both techniques. Brown in 1981 used data from the CSMS survey and expanded upon this with interviews. Brown's study revealed that students in the 11 to 16 age group had considerable gaps in their knowledge in place value and decimal fractions. Brown and Van Lehn investigated the errors, or 'bugs', that students generated when confronted with multi-digit subtraction. Van Lehn listed 77 'bugs' (systematic errors) that students made!

Along with research into children's errors in mathematics, have come some studies that emphasise more positive aspects. Moser and Carpenter in 1982, and Gelman and Gallistel in 1983 noted how capable young children are in counting and solving verbal problems. Earlier, Donaldson and her co-workers found that children could conserve number younger than expected, and could understand the relationship between class and sub-class (in a set of objects). Other research revealed that, given alternative approaches, previously mathematically incompetent adults could make appreciable progress in computational skills and understanding.

The Evidence

In an analysis of particular items in the NAEP survey it was found that over half of the 13-year-olds could not calculate the area of a rectangle from its dimensions. Although most could identify common geometric shapes, fewer than 10% could use the knowledge that the sum of the angles of a triangle is 180 degrees to find the measure of the third angle. In other words relatively few students demonstrated knowledge of the basic properties of geometric shapes.

Problems were found to be just as frequent by the British CSMS study. In summarizing the results, Hart concluded that:

> The overwhelming impression obtained is that Mathematics is a very difficult subject for most children.
>
> And we have shown that understanding improves only slightly as the child gets older.
>
> And in the secondary school we tend to believe that the child has a fund of knowledge on which we can build the abstract structure of mathematics. The child may have an amount of knowledge but it is seldom as great as we expected.

Cockcroft in 1982 worked with 107 adults, as described in his report. He said,

> The extent to which the need to undertake even an apparently simple and straightforward piece of mathematics could induce feelings of anxiety, helplessness, fear and even guilt in some of those interviewed was, perhaps, the most striking feature of the study.

The recent IEA survey in New Zealand showed surprising weaknesses by third formers (13-year-olds). For example, only half of the children said 20% = 1/5, the most common error was 5% = 1/5. Fewer than half the 13-year-olds could successfully answer items on common fractions, decimal fractions, estimation of area, assigning points to a number line, and basic algebraic computation.

Some of the most revealing student misconceptions in mathematics have been unearthed by Stanley Erlwanger. One of the most frequently quoted examples comes from his conversation with Benny, an above-average 12-year-old. Benny's teacher, in fact, regarded him as one of her best pupils in mathematics. Benny's procedure for the addition of decimal fractions was as follows:– (E = Erlwanger; B = Benny)

> E: Like, what would you get if you add point 3 and point 4?
> B: That would be... oh seven... Point 07.
> E: How did you decide where to put the point?
> B: Because there's two points; at the front of the 4 and the front of the 3. So you have to have two numbers after the decimal, because... you know... two decimals. Now like if I had point 44, point 44 [i.e., .44 + .44], I have to have four numbers after the decimal [i.e., .0088].

In further exploring Benny's ideas and beliefs about mathematics, Erlwanger discovered that Benny considered that mathematics consisted of different rules for different types of problems. Benny's purpose in learning mathematics seemed to be to discover rules and use these to solve problems. There was only one rule for each type of problem, according to Benny.

During 1982 I studied the progress of eight 12-year-old students who were academically representative of their school class. The class was embarking upon four weeks of work on decimal numbers and decimal fractions and I interviewed the students before and after the work using a series of nine stimulus cards. These cards covered estimation, division with a divisor greater than the dividend, writing decimal numbers, problem solving involving decimal numbers, comparing numbers containing decimal fractions, and naming the place value columns in decimal numbers. These topics matched the instructional objectives for the teaching module. Individual interviews were audio-taped and the tapes transcribed.

Table 1 presents the results. The students are arranged in order of academic achievement – highest being 'Jo'.

Table 1
Response Movement on Stimulus Cards – Before and after Instruction.

Subject:	Correct-remaining-correct	Correct-to-incorrect (Change)	Incorrect-remaining-incorrect	Incorrect-to-correct (Change)
Tim	0	0	9	0
Mary	1	0	8	0
Bob	2	0	5	2
Sarah	4	0	4	1
Bevan	4	0	2	3
Sue	6	2	1	0
Oliver	4	2	1	2
Jo	7	1	0	1
Σ	28	5	30	7

Despite four weeks' work the 'Incorrect-remaining-incorrect' category is as common as the 'Correct-remaining-correct' class. In other words there was little change. However, worse still, there are 5 examples of Correct-to-incorrect. And these are among the more able students Sue, Oliver and Jo.

In attempting to explain the (apparent?) regression, Erlanger and Benny come to mind. Benny often constructed his own (unintended) meaning from the mathematics programme.

Because of the small sample in my study, it would be unwise to generalize from these results. As well, the teacher could be important, although from my observations the particular module of work was carefully planned and well taught by an able teacher. In spite of these limitations, the results do provide additional evidence that for many students slow progress in mathematics is the norm.

Lest we become too pessimistic, research evidence has also pointed out that children are surprisingly competent in mathematics. In particular, young children have more knowledge of the principles of number and counting operations than was previously supposed. For example, children as young as two and a half used the cardinal principle:

> For the child as young as two and a half years, enumeration already involves the realization that the last numeral in a set (at least in a small set) represents the cardinal number of the set.

Ninety percent of six-year-olds can solve addition verbal problems. About half of them use advanced counting-on procedures, that is, they enter the sequence at a place corresponding to one addend, then count forward as many words as indicated by the second addend in order to reach the answer.

The inventive powers of children in mathematics have been well documented, a good book being *Understanding Mathematics* by R.B. Davis, published in 1984. Teachers sometimes report to their colleagues the intuitive discoveries a child in their class makes. Alan Hall reported one 10-year-old's realization that negative integers could be recorded symbolically – this particular child realized there should be numerals on the number line 'behind the piano', the piano at the front of the classroom was obscuring that part of the number line to the left of zero.

Children attempt to construct some meaning from whatever they confront in mathematics. Sometimes this is the meaning that the teachers intend. At other times the children's active construction produces new errors, or stabilizes existing misconceptions.

A question researchers have asked is what meanings children construct from statements in mathematics that do not make sense? Do children attempt to answer bizarre questions? Children *do* attempt to answer bizarre questions about the world, for example, 'Is Red wider than Yellow?', but is this applied to questions in mathematics?

With the assistance of colleagues, I made up five bizarre questions, set out in Table 2.

Table 2
Bizarre Questions in Mathematics.

1. We are measuring using paces, or strides. It is 50 paces around a truck.
   How heavy would the load be on this truck?
2. It takes a person a day and a half to dig a hole and a half.
   How long will it take two people?
3. There are ten people. Each has five apples.
   Who ate them all first?
4. Some children sat a test. The top mark was 55.
   The bottom mark was 5.
   How many people sat the test?
5. It takes me ten minutes to bike five kilometers.
   How long will it take me to ride up a very steep hill?

The questions were written on cards. Eight academically representative eight-year-olds and eight ten-year-olds were interviewed. Seventy-eight out of a possible 80 answers were generated by the children – only one child claimed that answers were impossible, and for two questions only.

The responses given to the bizarre questions indicated that the children attempt to make sense of what is presented to them. Often they draw on their own experiences:

To question 1 (Truck's load?)

Julie (10):... about over a tonne... it seems that a truck could take that much 'cause it's built for that kind of thing.

Bill (10):... about 80 pounds... I just guessed it. A truck down River Road, it's a big Kenworth, and it worked for a meat company.

To question 5 (Ride up steep hill?)

John (8):... about 1/4 of an hour, or twenty minutes... well if it takes you ten minutes to ride five kilometres, then up a steep hill you'll be going slower... But *down* will be a lot faster.

Jason (10):... depends on what sort of bike... a 10-speed can zoom up a hill. Two minutes on a 10-speed. On a different kind of bike a bit longer. When I had my Cruiser it used to take me ages to climb hills.

With questions where there appeared to be the possibility of manipulating numbers, then children did so:

To question 2 (Time to dig hole and a half?)

Mike (8):... um... I wonder how big the hole is though?... 12 hours, half a day... if it takes one man to dig a hole, then two men 1/2 of 24... I reckon a day or 1/2 a day.

Rebecca (10):... 3 days... I added 1 1/2 and 1 1/2 together... I know 1/2 and 1/2 equals one, and one and one equals two. You add two and one.

To question 4 (How many sat test?)

Sue (8):... 11... fives into 55... five times 11 is 55... 'cause the top mark was 55, the bottom was 5.

Bruce (10):... About 15... well, usually most differences between people is about two or three percent, but usually it's greater than that, so about 15 sat the test.

In short, the children attempted to make sense of the situation in which they found themselves. This, of course, is exactly what they do each day in the mathematics lesson – for many students in our schooling system a considerable proportion of questions in mathematics must appear bizarre.

What, then, are the implications for teachers? How can we assist to construct appropriate meanings in mathematics?

Implications for Teachers

1. Children will not passively absorb what is presented to them. Children do not always learn that which the teacher intends. Keep this at the front of your mind while teaching. Explain concepts in different ways; build in regular maintenance; have realistic expectations for children; *listen* to children's explanations and questions.
2. We should assist children to take a greater responsibility for their learning. Children will attempt to make sense of even the most bizarre situations in mathematics when an *adult* is in charge. If children can be encouraged to become less dependent upon the teacher, then the knowledge becomes part of themselves. (This may be a difficult and different re-orientation for adults.)
3. Research shows that children can progress well in mathematics given the right environment. Such an environment includes: techniques that build upon existing knowledge, that involve the use of concrete materials (where needed), that promote discussion, that encourage children to ask questions, and interact with the teacher, and that reinforce important mathematical ideas (rather than polysyllabic labels).

4. We need to think critically about the materials we use. For example, the Form One (11-year-olds) textbook makes the following suggestion for slower children who need extra work on decimal fractions:

Remedial:

Those students who have difficulty with these pages may be asked to complete a pattern of multiplications in which the decimal point has different locations.

26 × 483	2.6 × 483	.26 × 483
26 × 48.3	2.6 × 48.3	.26 × 48.3
26 × 4.83	2.6 × 4.83	.26 × 4.83

Would the assigning of extra pencil-and-paper exercises such as these be of benefit to children struggling to cope with ideas behind operations on decimal fractions?

Likewise, care must be taken when using apparatus and visual displays (such as number line models). Children may view the particular teaching aid in quite a different way from the teacher.

5. It is unwise to rely too heavily upon the spiral curriculum approach; do not put too much faith in the notion that if children don't master ideas and processes one year, they will pick up that knowledge the next. For many children the spiral curriculum has become two-dimensional, never rising above one level. As teachers we should aim for mastery of concepts and processes, realizing that for most children progress is gradual and slow. If children miss some key mathematical idea one year they may never gain that knowledge; when the idea is next confronted it is usually at a more abstract level, and even less attainable!

Notes

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The early work of Piaget in maths can be found in

Piaget, Jean (1941) *The Child's Conception of Number.* London: Routledge and Kegan Paul Limited.

Brueckner's testing to measure ability in compilation and verbal problems are described in

Brueckner, L.J. (1938) *Techniques of Diagnosis.* Chapter Eight in the 34th Yearbook of the National Society for the Study of Education. Chicago: University of Chicago Press.

The claim that it is mathematics education that is failing students, not students that are failing maths, can be found in

Fey, J.T. and Sonnabend, T. (1982) Trends in School Mathematics Performance. Chapter Six in Austin, G.R. and Gardner, J. (eds.), *The Rise and Fall of National Test Scores.* New York: Academic Press.

and

Whitney, Haasler (1984) *Taking Responsibility for School Mathematics Education.* Paper presented at I.C.M.E.V. Adelaide, Australia.

The British CSMS survey can be studied in

Hart, K.M. (1981) *Children's Understanding of Mathematics: 11–16.* London: John Murray.

The New Zealand IEA study is written up in

New Zealand Department of Education (1982) *The Second I.E.A. Mathematics Study.* Wellington: Education Department.

Studies in which children's mathematical ideas are probed through interviews are numerous. Two good articles by Erlwanger are mentioned below. See also

Davis, R.B. (1975) A Second Interview with Henry – Including some Suggested Categories of Mathematical Behaviour. *Journal of Children's Mathematical Behaviour,* Vol.1 (3), pp.36–62.

Davis, R.B. and McKnight, C. (1980) The Influence of Semantic Content on Algorithmic Behaviour. *Journal of Mathematical Behaviour,* Vol.3 (1), pp.39–87.

Alderman, D.L., Swinton, S.S. and Braswell, J.S. (1979) Assessing Basic Arithmetic Skills and Understanding Across Curricula. *Journal of Children's Mathematical Behaviour,* Vol.2 (2), pp.3–28.

Carr, K. (1983) Student Beliefs about Place Value and Decimals: Any Relevance for Science Education? *Research in Science Education,* Vol.13, pp.105–109.

Knight, G.H. (1982) *A Clinical Study of the Mathematical Incompetence of Some University Students.* Ph.D Thesis, Massey University.

Wearne, D. and Hiebert, J. (1984) *The Development of Meaning for Decimal Symbols.* Paper presented at I.C.M.E.V., Adelaide, Australia.

Studies in which interviews and data from large-scale surveys are combined include

Brown, M.L. (1981) *Levels of Understanding of Number Operations, Place Value and Decimals Among Secondary School Children.* Ph.D Thesis, University of London.

and

Brown, J.S. and Van Lehn, K. (1982) Towards a Generative Theory of Bugs. Chapter 9 in Carpenter T.P. *et al., Additional Subtraction.* New Jersey: Lawrence Erlbaum.

Van Lehn's list of 77 'bugs' occurring in subtraction is in

Van Lehn, K. (1982) Bugs are not Enough: Empirical studies of Bugs, Impasses and Repairs in Procedural Skills. *Journal of Mathematical Behaviour,* Vol.3 (2), pp.3–72.

Studies displaying the capabilities of children:

Moser, J.M. and Carpenter, T.P. (1982) Young Children are Good Problem Solvers. *Arithmetic Teacher,* Vol.30 (3), pp.24–26.

Gelman, R. and Gallistel, C.R. (1983) *The Child's Understanding of Number.* Chapter 14 in Donaldson, M. (ed.), Early Childhood Development and Education. Oxford: Basil Blackwell.

Donaldson, M. (1978) *Children's Minds.* London: Fontana.

For the study on incompetent adults making good progress see

Duffin, J. (1978) Some Thought on numeracy. *Mathematics in School,* Vol.7 (5), pp.26–28.

That only 10% of 13-year-olds can use the 180° rule for triangles was revealed in

Carpenter, T.P., Matthews, W., Lindquist, M.M. and Silver, E.A. (1984) Achievement in Mathematics: Results from the National Assessment. *The Elementary School Journal,* Vol.84 (5), pp.485–496.

The quotation from Hart summarising the CSMS results is on p.209 of her book mentioned above.

Cockcroft's work with adults and the quotation can be found in

Cockcroft, W.H. (1982) *Mathematics Counts.* London: H.M.S.O.

Benny's conversations, and the other revealing misconceptions are in

Erlwanger, S.H. (1973) Benny's Conception of Rules and Answers in IPI Mathematics. *Journal of Children's Mathematical Behaviour,* Vol.1 (2), pp.7–26.

and

Erlwanger, S.H. (1975) Case Studies of Children's Conceptions of Mathematics – Part I. *Journal of Children's Mathematical Behaviour,* Vol.1 (3) pp.157–283.

The quotation on 2½-year-olds using the cardinal principle is from Gelman and Gallistell (p.198), mentioned above.

The work with 6-year-olds using advanced counting-on can be found in Moser and Carpenter mentioned above.

The full reference for Davis's book is

Davis, R.B. (1984) *Understanding Mathematics.* London: Croom Helm.

Is Red wider than Yellow? Such bizarre questions are asked in

Hughes, M. and Grieve, R. (1983) On Asking Children Bizarre Questions. Chapter 9 in Donaldson, M. (ed.), *Early Childhood Development and Education.* Oxford: Basil Blackwell.

The Form One text book quoted is

Duncan, E.R., Capps, L.R., Dolciani, M.P., Quast, W.G. and Zweng, M.J. (1980) *Modern School Mathematics: Structure and Use.* Boston: Houghton Mifflin Co.

That for some children the spiral curriculum in maths becomes two-dimensional is argued in Hart, mentioned above.

Being Both Right and Wrong

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Nicola Belsham

Mathematics is supposed to be one area of school knowledge that has right answers. Right?

Arithmetic is when the answer is right, and you look out of the window and the sky is blue.

And when the answer is wrong and you have to start all over again.

The mystery to **many** learners is knowing **when** the answer is right, and **why** that answer is right when they can **see** that a quite different answer should be right.

Competent adults look at the mathematics that primary school children are asked to do, and have no difficulty in telling a right answer from a wrong one, or even describing why it is right. Adults start from their understanding of mathematical foundations. It seems natural to teach mathematics by starting from simple concepts familiar to the learner and then building more complex ideas through a sequence of activities which grow in complexity. But David Tell from the University of Warwick points out that this ignores basic facts about the way in which children construct their own knowledge of mathematics. They do not start from mathematically basic concepts. They are operating in a complex environment and trying to understand, predict, and control the numerical concepts that they need in that environment.

It is generally accepted that children construct their own concepts in mathematics. The *Beginning School Mathematics* programme in New Zealand encourages a great deal of exploration of quantity and relationships. The introduction to another syllabus speaks of mathematics as a human endeavour which involves imagination, discovery, perceiving relationships, interpreting and communicating ideas and concepts. Many of the achievement aims and objectives in the syllabus start with the words *develop, find, devise.* Teaching is an essential part of this but the young people do their own construction of knowledge as a result of many factors which include exploring, listening to teachers, arguing with classmates and copying older siblings. And young people everywhere develop their own misconceptions or 'bugs' as they do so.

These 'bugs' are almost always right in one sense and wrong in another sense. For example, the child who puts down

$$\begin{array}{r} 27 \\ +\quad 36 \\ \hline 513 \end{array}$$

is accurately recording that 6 + 7 = 13 and that the two figures in the 10s column = 50, which we record as 5. What the child has not taken into account is that the answer as written is conventionally read as five hundred and thirteen rather than 5 and 13. So the child is both right and wrong.

This is just one example in which accurate concepts that a child has developed in one context produce a wrong answer in another - in this case in the eyes of an adult who does not see the child's thinking. A difficulty with school mathematics is that learners have to readjust their hard-learned concepts not once, but **frequently**. These points of readjustment have been called impasses, and for some children they become just that. An impasse is a point where a concept which is firmly believed (because it works in contexts which the learner has experienced) is found to be untrue in a different context.

Every time learners are required to realize that existing concepts are inadequate, or partly right and partly wrong, there is a possibility that they will find the new information or procedure too difficult to understand. They may give up trying to understand and just learn the rule for a new procedure. They may blame themselves as being inadequate or blame mathematics as being too hard. They may blame their teacher.

Three impasses will be discussed here. One usually occurs at age 6 or 7 and is about the order of written numerals and operations. The next concerns decimal fractions and is a common problem in the upper primary school years. And the last concerns operations with negative numbers, often met early in high school. They are by no means the only impasses, and for some children they are learned so easily that they are not impasses at all.

Directional Order in Written Numerals and Operations

Before children begin writing down their numerical experiences it makes no difference where the things that they are counting are. Beans, trucks, coins and counters can be counted in any order and still give the same sum. Toes and fingers can be counted from left to right or from right to left, without changing the number of toes or fingers a child has. Children going on a trip from an Early Childhood Centre can be counted in any order - as long as the same child isn't counted twice, and there should always be the same number of children. Similarly, when two sets are being joined, the set on the left may be moved to join the set on the right or vice versa without changing the quantity. When a small group is taken away from a larger group the direction in which they are taken doesn't matter.

The first operation that children write (or use number cards to record) is usually addition of single digits which total less than 10, and again order doesn't make too much difference. Children who are being encouraged to be flexible in their recording may specifically discover that the following statements all say the same thing:

$$3 + 5 = 8 \qquad 8 = 5 + 3$$
$$5 + 3 = 8 \qquad 8 = 3 + 5$$

A learner might decide that the only rule is that a sign has to come between two numbers, or that the big number has to be on one side of the equal sign and the two smaller numbers have to be on the other side. Children appear to learn very easily that the order of the addends is irrelevant, possibly because, in their non-writing world, spacial order is irrelevant. The very point of the exercise of writing this equation in different ways is that it can be written in both directions and still be true. If it can be written in both directions, it would be reasonable for the learner to think that it could also be read in both directions.

Then, suddenly, order does become important. It is of absolute importance in the way in which we write two-digit numerals. We write the numeral in the 10's column to the left of (and before) we write the number in the one's column. There is no reason for this other than convention. On a recent occasion I saw a J2 (Year 1) child who had written a page of sums that went:

10 + 1 = 11
11 + 1 = 21
12 + 1 = 31
13 + 1 = 41
and so on down to 18 + 1 = 91

Another child looked at her page and said 'Yes that's right, but that's not the way you **spell** them'. This seems a very good way of saying that the child was both right and wrong. And it did not diminish the first child's pride in her work.

A more difficult case is learning that the order of digits in subtraction does make a difference (even though the order in addition does not). Thus 7 - 2 = 5, but 2 - 7 does not equal 5. The child who treats order as irrelevant in subtraction is usually treating the operation as one of finding the absolute difference between the two numerals, but the equation (as written) does not mean that. In this case the child has to learn that one (7 - 2 =) is an operation that can be done with positive numbers and the other (2 - 7 =) is one that necessitates a negative answer if it is to be done. Subtraction is not commutative although addition is. This is more commonly an impasse in two-digit subtraction with renaming.

Decimal Fractions

There has been quite a bit of research which looks at the difficulties learners have when they begin to work in this area. Misconceptions are almost always instances in which learners have brought concepts that they have learned about whole numbers and applied them to decimal fractions. Two examples will be familiar to those who teach at this level:

11 is bigger that 2, therefore .11 is bigger that .2
13 + .62 = 75 because 13 and 62 are 75.

One particularly interesting study was carried out with students in Israel, France and the United States. It found that French and American students, who were taught about decimals after working with whole numbers, made mistakes of the kind shown above, in which whole number logic was applied, incorrectly to decimals. The Israeli children were different. They were introduced to decimals after working with fractions. They knew that ½ was larger than ¼ and many students therefore thought that .2 was larger than .4.

James Hiebert and Diana Wearne at the University of Delaware have studied children's concepts about decimals extensively (see also Carr in *set* No. 2, 1986, reprinted as item 9 in this folder). They have shown that those who learn procedures without a firm understanding of the underlying meaning of the decimal fraction are very likely to make errors like those in the examples above. This mis-learning is very hard to correct: children continue to make such mistakes even after instruction which carefully teaches the meaning of decimals through the use of Dienes blocks. They make similar incorrect generalizations from one procedure to another, such as believing that since it is important to line up the decimal points when adding decimals it is also important to do so when multiplying decimals.

The point here is not that the learners' incorrect procedures are wrong (in that they give the wrong answer), but that they are inappropriate applications of a procedure or concept (which works in one context) to a different context where it doesn't work. Should the learner who is told that .11 is not larger than .2 simply accept that information and reject previous understanding that 11 is larger than 2? That would require the learner to doubt all previous carefully constructed concepts, in this case ones which go back to early counting in conventional order as a means of knowing which is the larger number.

Operating With Negative Numbers

When you subtract, things get smaller. Therefore 4 – (⁻3) will be smaller than 4. Some learners think that since there are two negatives, it is going to be a lot smaller. This appears to be an area of mathematics in which most students 'learn the rule' (and often it is the wrong rule) and then apply it without thinking about what the rule means.

One competent child who knew numbers could go below zero but had not been taught to operate with them explained that ⁻5 – (⁻1) was ⁻4 because you had 5 negative things and took away 1 of them, leaving 4 negative things. However he then argued that 3 – (⁻3) was 3, because you didn't have any negative things to take away, so that was the same as 0, and 3 – 0 = 3. Mature educators have found that logic very appealing. Should that child be told that his answer is wrong, when so much of the logic is right?

The underlying meaning of double negatives is elusive and quickly forgotten. In an informal study of how educated adults understood this, it was discovered that unless the adults had some on-going experience with formulae, as do scientists, engineers, and mathematics teachers, the expression either meant nothing or brought up the response that they thought there was a rule for that but they couldn't remember what it was. When those who could operate with negative numbers were asked to explain why the rule worked none of those questioned gave the explanation that is found in most text books.

Two responses were common from people who did not remember the rule. One was to ignore one of the negative signs, treating 4 – (⁻3) as 4 – 3. The other was to believe that the operation could not be done. Their logic for this was that it was impossible to take away negative things, much like the untutored child mentioned above. Ignoring one of the negative signs simplifies an unfamiliar problem and relates it to something which the learner does know. The belief that you cannot take away a negative when you don't have one in the first place works from a concrete model for which the learner has a lifetime of experience.

What Should the Teacher Do?

The process that a learner must go through when faced with any of these impasses has been described as a paradox.

> ...when we push our ideas until we encounter a conceptual gap or obstacle, which signals that our ideas are in some way false, we can change those ideas only by remaining completely convinced of their truth. Only on looking back can we see how our knowledge has changed and how it has remained the same.

Thus it appears that the teacher's task is particularly delicate at these impasse points. There are techniques for teaching each of the concepts given as examples here. Directionality in writing numbers and equations can be related to both the reading process and the meaning of examples if read right-to-left or left-to-right. Decimals can be taught with concrete representation, such as dividing a chocolate bar, or cutting a piece of paper into ten parts, and then dividing each of those parts into ten further parts. If children have used Dienes blocks extensively for understanding whole numbers, they may become confused if the same material is now used to represent fractional parts. On the other hand, if they have not used them for representation of whole numbers they provide a useful way of representing decimal fractions. Adding and subtracting with negative numbers can be related to gaining and losing merit and demerit points, or positive and negative force; number lines, even when they stretch 'behind the piano', are useful too. It is probably impossible to identify all such impasses for individual learners. These can best be met by learning environments for mathematics which have three characteristics.

1. A learning environment which encourages as much flexibility in thinking as is mathematically sound, and relates concepts to concrete models whenever possible. This means the teacher must recognize mathematically sound alternative methods, (although not necessarily teaching what works for one learner to everyone).
2. A learning environment in which each learner's individual understanding is discussed and valued by the teacher and other students.
3. A learning environment in which the teacher knows when impasses are likely to appear, and gives explicit acknowledgment to the children in the cases when they are right. This also involves appreciation of the difficulty learners face in accepting a new concept, such as operating with negative numbers. For many learners it may be that full understanding will come only after considerable practice with a new procedure. This may mean learning a procedure for now, but with the understanding that both teacher and learner will keep working at understanding it.

The casualties of our mathematics education are not those who become confused at points of impasse but those who cease to believe that mathematics can be meaningful.

Notes

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The first quotation, 'Arithmetic is when the answer is right...' is from the American poet, Carl Sandburg.

The other syllabus referred to is:
Ministry of Education (1992) *Mathematics in the New Zealand Curriculum*, Wellington: Learning Media.

The work of David Tall of the University of Warwick can be found in
Tall, D. (1989) Concept images, generic organisations computers and curriculum change. *For the Learning of Mathematics*, Vol. 9, No. 3.

The work in Israel, France and the USA can be found in:
Resnick, L.B., Nesher, P., Leonard, F., Magone, M., Omanson, S. and Peled, I. (1989) Conceptual bases of arithmetic errors: The case of decimal fractions, *Journal for Research in Mathematics Education*, Vol. 20, pp. 8-27.

The work of Hiebert and Wearne at the University of Delaware on concepts of decimals can be found described in
Wearne, D. and Hiebert, J. (1988) Constructing and using meaning for mathematical symbols: The case for decimal fractions. In M. Behr and J. Hiebert (Eds.) *Number concepts and operations in the Middle Grades*, Hillsdale, NJ: Lawrence Erlbaum Associates.

The earlier printing of Ken Carr's work on decimals is
Carr, K.C. (1986) There are numbers behind the piano: children's construction of meaning in mathematics, *set* No. 2, 1986, Wellington: NZCER.

The quotation '...when we push our ideas...' is by Hermine Sinclair and can be found quoted in
Kilpatrick, J. (1987) Editorial in *Journal for Research in Mathematics Education*, Vol. 18, No. 5.

Two good articles for further study are
Resnick, L.B. and Omanson, S. (1987) Learning to understand arithmetic. In R. Glaser, (Ed.) *Advances in instructional psychology*, Vol. 3, pp. 41-95, Hillsdale, NJ: Lawrence Erlbaum Associates.
Resnick, L.B., Bill, V.L., Lesgold, S.B. and Leer, M.N. (1991) Thinking in arithmetic class. In B. Means, C. Chelemer, and M.S. Knapp (Eds.) *Teaching advanced skills to at-risk students: views from research and practice*, San Francisco: Jossey Bass.

Beginning to Learn Fractions

By **Robert Hunting**
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7/16ths, is it bigger or smaller than 8/17ths?

Fractions are notoriously difficult. Difficult to learn and difficult to teach. I have carried out, with help, some research which throws light on the questions 'When should we begin teaching fractions?' and 'What should our first lessons be like?'

A Typical Preschooler

Sarah is asked to cut a piece of macrame string into skipping ropes for two small dolls. She is told that the ropes will have to be the same or the dolls will be unhappy. Sarah cuts the string at about the mid-point. There is no sign that she made any preliminary estimation, no eye-movements nor physical folding of the string. She tells the interviewer that the dolls will be satisfied, but does not check the accuracy of her results. A follow-up problem is given: a longer piece of macrame string is to be shared evenly between *three* dolls. Sarah makes *three* cuts this time, resulting in three roughly equal pieces which are distributed to the dolls, and a smaller piece, which is ignored. Again there is no evidence of estimating nor of spontaneous checking after cutting. However she trims a small piece from the longest piece after being asked if the dolls would be satisfied.

Twelve cracker biscuits are to be shared evenly between three dolls. Sarah begins by giving out a single cracker to each doll, then stops. When the interviewer reminds her to give out all the crackers she continues giving out one cracker per doll in a systematic rotational cycle until all crackers are apportioned. Sarah moves the piles close to one another and compares their heights when asked if each doll has the same amount of crackers.

The interviewer rolls out a ball of playdough into a sausage shape and asks Sarah to cut it in half. Before giving Sarah a knife, he asks her to say how many cuts she will make (she says two) and how many pieces she will get (two). Sarah takes the knife, carefully places it near the mid-point of the sausage, and cuts. She then continues to subdivide each resultant 'half', until she has eight pieces.

A set of 12 picture swapcards is handed to Sarah, who is asked to help put half the cards in an envelope. Sarah places the cards down on the table, picks up eight of them, and places these in an envelope. She does not sort or count the cards.

Sharing, and the Dealing Procedure

The beginning of knowledge about fractions is in the action of subdividing, together with talk about the results. The social activity of sharing is very important. Through sharing, and the methods that children use to make equal shares, deeper meanings for fractions can be taught.

To teachers it is crucial that children learn to share into precisely equal amounts and that they learn to associate the special vocabulary of fractions with the number of equal parts created. However, my research with Chris Sharpley has shown that young children think about sharing in several different ways, depending on the social setting. For many children sharing does not necessarily mean that each recipient will be allocated a portion. Three dolls can share two skipping ropes if the dolls agree to pass the ropes around, just as humans do! Young children do not universally believe that a quantity to be shared is absolute. If the situation demands it, it is reasonable to expect more biscuits or milk will be provided from somewhere. Alternatively they may choose to ignore some of what has been provided. Amounts that are shared may depend on what the child considers appropriate, and what is appropriate may have, to them, nothing to do with using up the whole, nor with the creation of equivalent sub-units. For example, many children stopped sharing out after giving only one cracker to each doll. This was because of the size of the dolls; small dolls don't need lots of large crackers.

Bearing in mind that commonsense may lead to various ways of sharing, we found that 60% of pre-schoolers in one study of over 200 possessed a cognitive capacity we call the *dealing procedure* – a powerful systematic method for subdividing a collection of items into equal fractional portions or units. An analysis of the *dealing procedure* shows three nested components. The primary action is a matching of item to recipient; for example, one cracker to one doll. The secondary action is the completion of a cycle: the action is repeated until all dolls have received a cracker. If there are crackers left to be allocated, the cycle itself is repeated. Repetition of the cycle is the third component.

The dealing procedure has some interesting features. First, the method guarantees that each recipient will receive an equal number of items, even though the child very likely does not know how many items each recipient has received, particularly if each share has four or more items. Young children can successfully subdivide a collection into numerically equal subsets even though they themselves are *pre-numerical.* Many four and five year old children check whether the dolls have fair shares of crackers by comparing the heights of the piles, rather than by counting the crackers in each pile. Second, the dealing procedure works for any number of recipients; it is a general procedure which transfers to many different problems. Third, the dealing procedure will work for *any size* collection to be shared between *any number* of recipients. This feature is crucial for developing knowledge about fractions. The concept is relativistic. For example we can have one third of a collection of six (two), and one third of a collection of 39 (thirteen). The size of the fractional unit varies according to the size of the initial whole. Therefore the dealing procedure is a very attractive base, full of action and meaning, for the mathematical language and symbols used to represent fractional numbers. This is particularly true for unit fractions such as $1/2$, $1/3$, and $1/4$.

Sharing and Counting

A generally held belief that fractions should not be introduced until around 8-years-old or even later is under challenge from recent research. Conventional wisdom has it that since rational numbers are an extension of the whole numbers, children should become familiar with whole numbers first, in the early years of schooling. However the dealing procedure is widespread among young children. This means that many have effective means for creating equal fractional quantities. Appropriate vocabulary for identifying such quantities can be introduced naturally, as we shall see. Our observations were that many who can successfully share up a collection using the dealing strategy do not have well developed counting skills. In fact, the reverse of conventional wisdom often prevails; sharing, using discrete elements, is ideal for stimulating counting.

Furthermore, if teaching basic fraction concepts is left too late, children's knowledge of whole-numbers can dominate their interpretation of fractions. For example, $1/5$ is thought to be larger than $1/3$ because five is larger than three. Also, by the time children reach the second grade, counting is probably an integral part of sharing out a collection. Eight-year-olds who observed younger children dealing out items thought that it was necessary to count the resultant piles in order to make the piles equal. There are elements of both activities which are mutually beneficial. The one-to-one matching of item to recipient (in sharing) is intrinsic to successful counting, and counting can be used to quantify the size of a share, and later, to assist to predict share sizes. Fundamental mechanisms for learning whole numbers and fractions appear to develop side by side and interact, suggesting that initial instruction in whole numbers and fractions ought to be parallel and related.

The Fraction One Half

One half is a special fraction for children. Knowledge of this fraction appears to become established at an early age compared with other fractions. Children in the middle and upper primary years use it as a reference number when comparing other fractions. Children passing through the primary school must understand one half (as well as other fractions) at least at a *quantitative* level. The next sections explain how to get to this level, and what happens beyond it.

Categories of Meaning for One Half as a Qualitative Unit

This is a stage children go through before they understand one half as a mental object that can be represented precisely. Categories 1 to 4 are about continuous quantities such as lengths of string and licorice sticks. Categories 5 to 8 are about discontinuous quantities.

1. One half as a multiple sequence of subdivisions. Preschoolers often don't know when to stop. Some when asked to cut a length in half will subdivide by means of a *sequence* of cuts commencing at one end. Others will make repeated halving actions, making a subdivision at about the mid-point followed by further cuts at about the middle of each subsequent portion. There seems to be either a lack of awareness of 'one half' as implying just one cut, or no notion that a distribution process (to only two people) must follow. If you have a school child who, when asked for a half, subdivides in this way, then the best way to help is to find problems that provoke alternative means of responding – such as using symmetry for producing two near-equal quantities (e.g., cutting a paper chain of 2 dolls apart), or how to compose the multiple pieces they produced before into two lots.

2. One half as a single subdivision where there is gross inequality between each part. Children in this category may lack experience of the social imperative for equal shares. They may have shared with other people who are physically different in size, and so expected unequal shares. A simple activity is for one child to cut a food item into two pieces and for another child of the same age to have first choice (I cut, you choose).

3. One half as two subdivisions with remainder. In this category there is some attention to equality after the first cut – usually the child cuts a small part off the larger and ignores the remnant. Activities to promote the idea that the two halves must exhaust the material could include sharing a highly desirable object, such as a food item, or sharing a task where everything must be moved, such as shifting all of a load of wood or bricks.

4. One half as a single subdivision, all the material is used, and there is attention to equality. This is what we are aiming for. Estimation and checking are seen either before or after. There are eye movements darting back and forth between endpoints of the material as an estimate is made of the mid-point, back and forth movements of a finger or the cutter before doing the subdivision, and adjustment after the finger or cutter has been put in place, just before subdivision. A child who makes a first subdivision, checks the resulting portions by direct comparison, then proceeds to trim one or both pieces in order to equalize the quantities (not discarding the remainder) demonstrates an awareness of equality critical for a *quantitative* conception of one half.

5. One half as an unequal subdivision of a collection of items, no dealing. This is the case of the child with no systematic dealing procedure. This prevents a successful solution. Progress will be limited until a systematic dealing procedure matures. Social activities are needed. Try games and talking about the relative merits of systematic versus non-systematic dealing procedures.

6. One half as an unequal subdivision of a collection of items, with dealing. This is the case of a child who has a method for dealing in a systematic way, but has not associated this with the words 'one half'. Unless there is deliberate action by a teacher or parent enabling the child to bring actions and words together, progress will not be possible. Try playing with a collection of, say, toy farm animals asking for one half to go in one yard, the other half in the other yard.

7. One half as an equal subdivision of a collection of items, a result of visual check or estimate. As in Categories 3 and 4 the child may divide the items up and guess by the size of the stack or heap that they are equal. More sophisticated means such as counting or systematic sharing processes are not used, not seen to be appropriate. Has the child developed a dealing procedure? If not, social interactions – games and talk – about systematic and non-systematic dealing procedures would be beneficial. Try the notion of 'fairness'. If a dealing procedure is available, the farm game in Category 6 would be in order.

8. One half as an equal subdivision of a collection of items, using a dealing procedure. This is what we are aiming for. This child will confidently share a collection into two equal lots using a systematic dealing procedure and count to check that the outcome is right.

The next step in the child's progress will be to predict the size of the fractional unit using whole number facts and relationships. Social interactions should be planned so that the child will think about the function counting serves after the dealing is over. Counting is *not* necessary to determine whether the portions are equal; the dealing procedure guarantees that, but counting will confirm how many items are in each half if you didn't already know before you began. Children who can use whole number knowledge (and in particular doubles and halves), to anticipate the outcomes of problems involving finding one half of (small) collections understand one half as a *quantitative* unit.

A Structured Learning Environment for Introducing Fractions: The Farm

To bring this research to classroom practice here is a farmyard learning environment as a framework or template. The Farm is a robust framework because it lends itself to a range of levels of questioning and discussion – including situations that you can return to with the same group of children, or a different group, so that progressively more advanced ideas can be explored. The farm is suitable for children aged 5 to 8 years.

The farm has several advantages over subdividing a length of string or a sheet of paper. First, young children have effective ways of making equal shares using a systematic dealing strategy. Second, the created shares can be made precise, which is important for developing at the same time the concept of equality. Third, the teacher can evaluate a child's progress more accurately; unequal parts in the continuous case may as likely be the result of poor technique as immature conception of equality. Fourth, informal discussions of equivalence arise naturally, for example, one half, two quarters, and six twelfths of 12 sheep all number six sheep.

There are three phases to *The Farm*. In the first phase questions and activity center around developing confident dealing procedures (leading to equal subcollections). In the second phase the teacher assists the children to associate the conventional mathematical language of fractions with the results of subdivisions of collections of farm animals. In phase three children explore part/whole and whole/part contexts where the size of the 'wholes' varies. Assess how competent each child is, then begin them at the right phase.

Materials Needed

A rural mat or sandbox, play animals such as ducks, chickens, cattle, sheep, pigs, horses etc. Other useful materials include fences for making pens, yards, or paddocks, trees, buildings, toy trucks for transporting animals. Small groups of children can design and draw their own farm layout on large sheets of paper. One group might plan a three field farm; another group a four field farm, and so on.

Sample Discussion Starters

The following are indications of the sorts of questions and activity that might take place.

Phase 1: Consolidation of Sharing Processes
Here is a suggested introduction:

The farmer has just bought a farm, but he hasn't got any animals. So he goes to the saleyards and buys some pigs.

Let's load the pigs on the truck and take them home to the farm. The farmer has two yards and he wants each yard to have the same number of pigs. Can you put the pigs in the yards?

Allow a child to distribute the pigs, then ask:
Is there an equal number of pigs in each yard?
Why or why not? (Allow different children to respond).
Who thinks there is? Who thinks not? (These questions provoke attention to methods of checking).

Discuss and contrast systematic one-to-one cyclic methods, many-to-one cyclic methods, and trial and error methods as they arise. Encourage children to distinguish these different methods, and explain in their own words why a systematic method is superior.

Phase 2: Integration of Language and Action
One day the farmer bought some chickens (for example). *Here they are on the truck* (indicate). *He has two chicken coops.*
Can you help me put half the chickens in one coop, and half the chickens in the other coop?
Discuss what should be done. Allow a child to distribute the chickens.
Is that half the chickens? (Point to one coop).
Why or why not?
Encourage individuals to explain. Invite other children to evaluate the responses given:
(Child's name), do you think that's right?

Similar problems can be posed where animals are to be placed in three and four yards. Use the words one-third, and one quarter (one-fourth can be used interchangeably with one-quarter).

Phase 3: Extension to Reverse Problems, Variable Unit Sizes, and Notation
Place a small number (three, say) of animals in a paddock and say:
The farmer has lost some of his ducks because the fence was broken. These are the ducks left (indicate). *These are one-half of all the ducks that were in the yard. How many ducks did the farmer have at the beginning?*
The above type of problem can be repeated using a different unit size (for example, five).
After selling one-half of his pigs to his next door neighbour, these are the pigs the farmer has left (indicate). *How many pigs did he have to start with?*

If children find these reverse problems difficult, act out the situation using the total number at the outset. Such problems can be extended using ONE-THIRD, ONE QUARTER; also fractional units of varying quantities (for example, $\frac{1}{2}$ as three animals, four animals, five animals, etc). Able children can attempt similar problems solving non-unit fractions such as $\frac{2}{3}$, $\frac{3}{4}$, $\frac{2}{4}$. Fifths may even be introduced.

Other Environments

Other settings that are suitable for use with the three phase framework described above are introduced below. Space will only allow us to indicate how the first phase, Consolidation of Sharing Processes, begins.

The Birthday Party

Today is Jenny's birthday and she is six. She has brought along some food and some party things (biscuits, cake, balloons, hats, whistles etc).
There are four plates and each plate needs to have the same number of biscuits on it. Can you put the biscuits on the plates? (Allow a child to distribute the biscuits).
Is there an equal number of biscuits on each plate?

Mary's Garden

Each child in the group has their own set of materials: a shoe box containing soil, 12 red flowers, 12 flowers of mixed colours, 4 vases, and some popsicle sticks to make borders.

Mary Mary quite contrary wanted to have a beautiful garden with flowers in it (give children shoebox with soil).
This is what her garden looks like.
Early this morning Mary went to the nursery and bought all these flowers to plant in her new garden.
She has two special areas where she wants to plant all her red flowers (give children 12 flowers each).
Can you plant the red flowers so there's the same in each area?
Show me.
Is there an equal number of red flowers in each area?

Final Comment

In our research we asked questions about young children's ideas about fractions, identified an important intellectual tool called the dealing procedure, and advanced our understanding about the scope of children's knowledge of the fraction one half. Although we are not sure exactly how the dealing procedure develops, we believe teachers can use it to establish sound, durable fraction knowledge.

Traditionally books and maths equipment in primary schools have features-recognition activities (in contrast to constructive activities). For example, worksheets or textbook pages have emphasized memorizing links between symbols and pictures of fractions, such as shaded geometric shapes. Where manipulative materials were used these had subdivisions already on them. While all these materials have value, they are insufficient if children do not have the opportunity to develop and apply their own procedures for subdividing quantities. If a child is successful at recognition tasks this can lead to the false assumption that the child understands the concept.

More important than texts and equipment are the verbal interactions between teacher and child, and child and child, as different approaches to problems are discussed together. The adequacy of each child's ideas are, this way, tested against the ideas of others.

The distinction between continuous and discrete quantity settings and materials, cutting string and sharing crackers, is important. It is possible to subdivide a continuous whole such as a strip of paper into three approximately equal segments, though this is not an easy task. Alternatively you can subdivide a row of 15 buttons into three equal subgroups, and the outcomes seem to be essentially the same. However, from our work with both younger and older children we know that the mental processes used to make subdivisions are different in each case. In the discrete case children initially use dealing or partitioning strategy. Later, more powerful whole number multiplication and division relationships are substituted for the physical sequence of actions. In the continuous case, children use halving, where symmetry of the material is used, or informal measurement processes involving the estimation of a unit, its reproduction, followed by check and adjustment. These informal processes are difficult to develop into further mathematical knowledge.

Many teachers and parents comment that the sharing situations children experience in the home are predominantly of a continuous nature – for example, dishing out soup or pudding. Teachers rely heavily on continuous examples, such as cutting up an apple into halves and quarters. Yet our research indicates that where continuous and discrete materials are both available, children seem to prefer discrete materials. Most soup is dished out by the ladle-full, and potatoes are easier to dish out in their jackets than mashed! Certainly children achieve more accurate results using discrete material. We have to conclude that continuous experiences are important because of the child's prior knowledge base, but for fractions discrete materials should be emphasized because children have effective methods for making equal fractional units. Discrete materials allow the relativistic nature of fractions to be expressed in contextually varied ways to assist in the development of equivalence ideas.

Notes

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The research on which this *set* item is based can be found in detail in
Hunting, R.P. and Sharpley, C.F. (1988) Fraction knowledge in preschool children, *Journal for Research in Mathematics Education*, Vol. 19, No. 2, pp. 175-180.
and
Hunting, R.P. and Sharpley, C.F. (1988) Preschoolers' cognitions of fractional units, *British Journal of Educational Psychology*, Vol. 58, pp. 172-183.
The term *pre-numerical*, meaning the development of a child's counting competence in its early stages is discussed and defined in
Steffe, L.P., von Glasersfeld, E., Richards, J. and Cobb, P. (1983) *Children's counting types: Theory, philosophy, and application*, New York: Praegar Scientific.
and
Steffe, L.P. and Cobb, P., (1988) *Construction of arithmetical meanings and strategies*, New York: Springer Verlag.

That whole-number knowledge can come to dominate children's concept of fractions (one-fifth being regarded as larger than one-third) is discussed in
Hunting, R.P. (1986) Rachel's schemes for constructing fraction knowledge, *Educational Studies in Mathematics*, Vol. 17, pp. 49-66.

That counting is integral to sharing out by 7 years old is discussed in
Davis, G.E. and Pitkethly, A. (in press) Cognitive aspects of sharing, *Journal for Research in Mathematics Education*.

The relationship between counting and sharing is an important one and needs further investigation. See
Pepper, K. (1989) *The relationship between preschoolers' knowledge of counting and sharing in discrete quantity settings*, Masters' thesis in preparation, La Trobe University.

The three stages of knowledge of fractions – as a qualitative unit, quantitative unit, and abstract unit is discussed in detail in
Hunting, R.P. and Davis, G.E. (1989) *Dimensions of young children's knowledge of the fraction one-half*, Manuscript submitted for publication.
and in
Bigelow, J. Davis, G.E., & Hunting, R.P. (April 1989) *Some remarks on the homology and dynamics of rational number learning*. Paper presented at the Research Pre-session of the National Council of Teachers of Mathematics Annual Meeting, Orlando Florida.

The Farm. The application of the research to classroom practice was begun in
Hunting, R.P., Lovitt, C. and Clarke, D.M. (April 1987) *The foundations of number learning project: Where research, children, and teachers meet*. Paper presented in the symposium Early Mathematics Learning: Teacher-focused Curriculum Change, American Educational Research Association Annual Meeting, Washington D.C.

A discussion of language factors affecting early fraction learning can be found in
Hunting, R.P., Pitkethly, A., & Pepper, K. Language carriers and barriers in early fraction learning. In R.P. Hunting (Ed.), *Language issues in learning and teaching mathematics*. Forthcoming monograph.

That children prefer to deal with discrete (countable) material when they have the choice is demonstrated in
Hunting, R.P. and Korbosky, R.K. (1989) *Context and process in fraction learning*, Manuscript submitted for publication.

Helping One Another Learn

DISCUSSION IN JUNIOR MATHEMATICS

Gill Thomas

Dunedin College of Education

IN THE PRESENT-DAY classroom children talk to each other a lot. Independently Gill Thomas at Dunedin College of Education and Joanna Higgins at Wellington College of Education have been observing the conversations of very young children as they do mathematics. In this item and the next are two different, but complementary, analyses of the children's conversations.

A KEY FEATURE of New Zealand junior mathematics is the widespread use of *Beginning School Mathematics* (BSM). It provides a range of activities for children to use independently of the teacher. The most common organisational structure is to group the class into four groups. After some teaching of the whole class together the teacher works intensively with two of the groups in turn while the other groups work independently of the teacher on activities designed to supplement the concepts introduced by the teacher. It is called *independent activity time*. The result is that the children spend up to three-quarters of their mathematics time working in groups independent of the teacher. It is expected that the children will discuss the mathematics they are working on with their peers during this time.

What happens when children work in groups independent of the teacher? What discussion occurs within the groups? How do you encourage children to engage in 'meaning-making' discussion with their group-mates? My study (a pilot for a full-scale research project) sought to investigate these questions.

Method

The most advanced group from each of two classes was selected for this pilot study, along with the least advanced group from one class, the 24 children being between 6½ and 8½ years old.

In the first phase each group was videotaped during mathematics for a week. The videos were examined, transcribed, and what the children did was coded. In the second phase the two teachers and I developed a framework for group work. We hoped it would encourage the children to have meaning-making discourse. The framework had three parts:

1. The collective completion of tasks
   The activities chosen for *independent activity time* were modified to encourage collective work, rather than an individual or competitive approach.
2. The management of independent activities
   The children were encouraged to discuss aspects of management – for example the materials required and the rules – before the independent time activity was begun. The idea was to get these problems over first; then the subsequent discussion could be on the mathematics associated with the activity.
3. Helping one another learn
   The children were encouraged to question one another about the task as they worked on it. There were strategies introduced for giving and receiving help.

The teachers introduced the framework in whole-class sessions, over two weeks. I then worked with the groups we were studying, encouraging them to use the framework.

Findings

Phase 1

What was the nature of the discussion between child and child?

Most of the talk (83 per cent) was related to mathematics. However, most of that was related to task-management, with very little (7 per cent) being discussion which had an instructional element.

In order to provide substance to these findings, all the talk that had an instructional element was analysed for quality. Requests and responses turned out to be of differing complexity. So we made a distinction between higher order and lower order talk. Higher order requests were ones requiring an explanation of how to obtain a solution. For example, 'Will you show me how?' A lower order request simply required an answer without explanation. For example, 'What is 9 and 5?' Responses were similarly categorised.

Very few requests were made of other children (1 per cent). The requests that were made were equally of higher and lower order, although the majority received only a lower order answer.

Anna: *What does that equal* (points to equation)*?*
Lise: *8 take away 7 equals 1.*
Erica: *Is it 10 take away 5 or 5 take away 10?*
Lise: *You can't take 10 out of 5.*

There was only one instance of a child suggesting the use of teaching materials to help another child.

Lise: *8 take away 2 equals 6.*
Anna: *Does it? 8 take away 2.* (Holds up fingers, 4 on each hand, counts off 6 including thumbs which were turned down.)
It equals 4, silly.
Erica: *8 take away 2 does equal 6.*
Anna: *But look.* (Repeats with fingers.)
Lise: *Go and get some counters.*
Take 2 from 8. I reckon it equals 6.
Erica: *Yeah.*
Lise: *She says it equals 4 and I say it equals 6.*
(Anna returns with the counters.)
Lise: *Work it out with the counters. Get 8 counters.*
Anna: *No, Erica.*
Lise: *Leave it to Erica. She's doing it.*
Anna: *OK. 1 2 3 4 5 6 7 8.*
Lise: *OK. Now take 2 from it.* (Removes 2.)
Lise: *You see, it's 6. 1 2 3 4 5 6. It's not 4, it's 6.*

Eighty percent of the responses were unsolicited, that is, most of the children helped (or attempted to help) without being asked. Most of them were of lower order. Here are three.

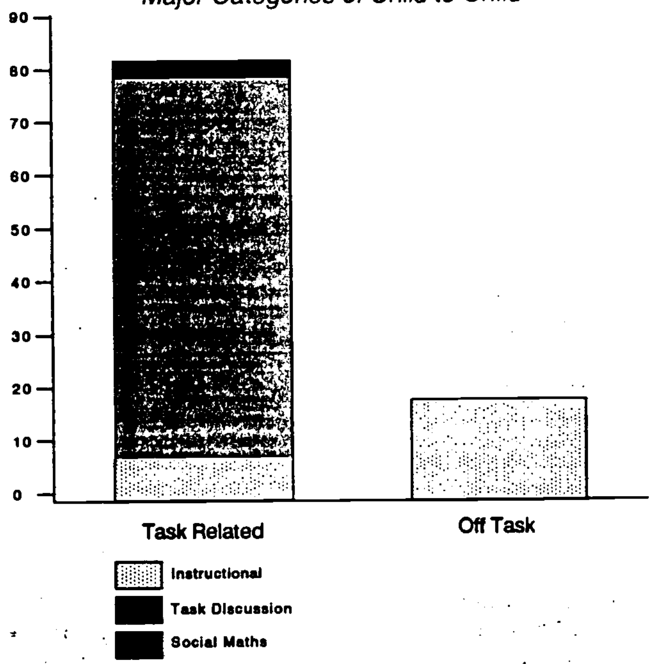
Sam: (To Steve.) *Sorry, Steve. 10 plus 2 is 12.*

Anna: (To Betty. The card has 9 – 6 on it.) *It's 3.*

Justin: (To Kay about a problem on the computer screen.) *I think you should go up; keep going up.*

Figure 1

Major Categories of Child to Child



Phase 2.

How do you encourage children to discuss mathematics in mathematics time, in a meaning-making way? When we began this phase we did not have the analysis completed so what follows is a summary of my reflections, and those of the teachers, on our attempts to encourage useful discussion.

1. Although there was an effort made to encourage the children to work together many of the activities the children chose to work at during the independent time were treated as competitive – for example *Snap* and *Memories*. Many of those activities had, of course, become popular before I arrived.
2. The hope that groups would clarify beforehand which materials and what rules they would use, was not fulfilled. The groups continued to make decisions about the rules as they worked on a task. It was felt by the teachers that the habit of looking at the materials and rules before starting would have to be introduced early in the year when work patterns for the classes were being established.
3. The children adopted the idea of checking one another as an activity was completed, but more often than not they failed to give an explanation if they identified an error. We wanted to use questioning as a means of encouraging the children to think about the mathematics they were working on. But this was fraught with difficulties. The children were willing to answer questions asked by the teacher but seemed unwilling (unable?) to ask questions of one another. It appeared to me and to both the teachers that many of the activities were not sufficiently challenging. If the children can complete the task with ease there is little motivation to help one another get it done.

Discussion

This study highlights a problem caused by a too ready acceptance of the idea that discussion between children is a tool for learning mathematics. It is necessary to explore the nature of young children's talk before assuming that in their talk they help one another learn. Although the quantitative analysis showed that the talk was usually task-related, this talk is extremely varied and often not useful to the task. Only a small amount of the talk (7 percent) was task-enhancing.

This finding poses the question of whether young children are capable of task-enhancing discussion with their peers. There were glimpses in every group. So I believe that they are capable of such discussion.

How then can teachers stimulate the type of discussion between children which will help them learn? The framework for encouraging discussion introduced into these classrooms did not appear to have any impact. The tasks and work patterns already established had perhaps set the children in their ways.

Another factor influencing the discussion is the social atmosphere in which it takes place. The working context within each group is important and the effect of dominance or conflict within each group needs to be considered.

The nature of the activity, and the way it is structured, is likely to influence the outcome also. There must be, for example, sufficient motivation to work together and to discuss the task, and the tasks must be capable of being done co-operatively. The dynamics of co-operative learning have been studied, for example by Johnson and Johnson, and can lead to better teaching.

Notes

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Beginning School Mathematics (BSM) was first published by the New Zealand Department of Education in 1985.

The coding of the transcribed video-recorded child-to-child talk was done following a system devised by Bennett and colleagues, with modifications. It can be found in

Bennett, N., Desforges, C., Cockburn, A., Wilkenson, B. (1984) *The Quality of Pupil Learning Experiences.* London: Lawrence Erlbaum.

Our finding that only a small amount of task-related talk was, however, task enhancing is consistent with Bennett *et al*'s findings.

All transcripts were coded by a single coder. A random sample of two transcripts from each group (115 utterances) was re-coded to establish test-retest reliability; the proportion of perfect agreement was 89%.

Research on co-operative learning has been carried out by Johnson and Johnson, with most encouraging results. They can be read in

Johnson, D.W. and Johnson, R.T. (1987) Co-operative Learning, Computer Assisted. *set* No.1, 1987, Item 13.

and

Johnson, D.W. and Johnson, R.T. (1988) Co-operative Learning Strategies for Mainstreaming/Integration. *set* No.1, 1988, Item 4.

and

Johnson, D.W. and Johnson, R.T. (1985) The Internal Dynamics of Cooperative Learning Groups. In Slavin et al (Eds.) *Learning to Cooperate, Cooperating to Learn.* New York: Plenum Press.

Item 13

A Child's Perspective of Algebra

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To a teacher, algebra may be just another topic to present; to a text-book author, just another chapter. But the children who must learn it, what do *they* understand of algebra? How do 11–13 year-olds cope with this new language? What is it that makes algebra a mystery to some children?

As part of a recent large-scale study on how children develop abstract reasoning in mathematics, one hundred and fifty children aged between 11 and 13 were interviewed while they worked through a set of mathematics questions.

The questions were:

simple numerical questions (e.g. $2 + 6 = 4 + \square$);
if... then questions
(e.g. A is a number. If $A + 3 = 10$, then $A + 5 = \square$);
more complex numerical questions
(e.g. $10 \div 2 = \square \div 10$);
algebra (e.g. Find the value of x which makes this true: $2x - 5x + 9x = 12$);
word problems;
completion of pattern problems.

For the interviews, children were selected at random from 6 schools – two primary schools which draw children from predominantly lower socio-economic areas, one primary school in an area of higher socio-economic status, one rural primary school, one rural secondary school and one surburban secondary school which draws from a wide cross-section of the community. Children from the final year of primary school and the first year of secondary school were chosen because many 11- to 13-year-olds are being exposed to abstract mathematical concepts while they have few means at their disposal to help them cope. It is at this age that problems in mathematics can become deeply rooted, sometimes influencing the children's future career options.

Why should algebra be a source of confusion? Does it derive from a conflict between what the teacher *says* and what the students *think*?

The Spoken Language of Mathematics

How often do children have the chance to *speak* the mathematical language they are learning? Opportunities include responses to a teacher's questions, initiating a question to the teacher, and discussion with peers. There is generally no time, though, to allow detailed interaction involving question/response/probing question/response in the normal classroom setting. By having a 1:1 interview it was hoped that children would respond to some of the questions they found difficult, and through faulty or inadequate responses, reveal some clues as to the reasons for their lack of understanding. It was anticipated, too, that some children who feel threatened by a teacher might feel less threatened by someone other than their teacher.

The interviewer initially explained to the child that we were interested in how she thought about mathematics and how she worked out problems. She would not be told whether the answer was right or wrong, we explained, as what we really wanted was to hear her 'think aloud'. All interviews were recorded, and lasted from 30 minutes to 1 hour, depending on the number of questions tackled and the detail given by the child. Sometimes the child needed prompting. 'How did you find that answer?' 'What are you trying out?' 'What are you going to try now?' Every attempt was made to use open-ended questions.

Six Children Talk About Ten Algebra Problems

Three girls and three boys of approximately equal (average) abilities have been chosen, at random, from those interviewed; three had had no previous instruction in algebra. They were asked to complete the following ten problems (which were interspersed with other mathematics questions not included here).

1. B is a number. What is B if $B + 4 = 27 - 4$?
2. A is a number. If $A + 3 = 10$, then $A + 5 = \square$
3. C is a number. If $C - 6 = 20$, then $C - 9 = \square$
4. $2b \times a = 2a \times \square$
5. $\frac{1}{3} = \frac{5x}{\square}$
6. If $x = 6$ and $y = 21$, what is the value of $\frac{2}{x} + \frac{y}{7}$?
7. Write as simply as you can: $5t + 3t - t - 7$
8. Find the value of x which makes this true: $2x - 5x + 9x = 12$
9. Find the value of x in $5 + 2x = 17$
10. Find the value of x in $11 - 3x = 2$

The Interviews

Sue, 13 yrs. Some prior algebra.

Problem 1:
C: Twenty seven minus four equals twenty three and twenty three minus four is B. So you minus four and you get nineteen.

Problem 2:
C: A must be seven because plus three equals ten. Then A which is seven plus five equals thirteen.

Problem 3:
C: Six from twenty is fourteen and fourteen plus nine is twenty three.

Problem 4:
I: What do you think it would be?
C: I'm not sure (pause). Does that equal $2b + a$?
I: What is written there?
C: $2b$ times a (long pause) Writes b in box.
I: Why did you write b?
C: The a has changed places to there (points to $2a$), so I've just put b in the box. I'm not sure if it's right or not though.

Problem 5:
C: One times five is five and three times five is fifteen (Writes $15x$).
I: Why did you write the x by the fifteen?
C: There was a letter there (points to $5x$) so I just put a letter there (points to $15x$).

Problem 6:
C: Two plus y which is twenty one equals twenty three and x which is six plus seven equals thirteen (writes the fraction $^{23}/_{13}$).

Problem 7:
C: What do you have to do for this one?

I: Try to write it more simply.
C: OK. (writes $7t$)
I: How did you get $7t$?
C: Five and three is eight (pause) t, minus one t is seven t. Then you have to minus seven from that (writes -7. Final written expression reads $7t - 7$).

Problem 8:
C: $5x$ and $2x$. You can't subtract two. Five from two gives negative three (writes $-3x$). You've got to put x there. Five plus nine. You've got to remember that that's negative (points to $-3x$), so it takes you back to six (writes $6x$). You have to double that (points to 6) to get twelve.
I: Does that tell you what x is?
C: No it doesn't.
I: You've got $6x$ equals twelve, then.
C: (Thoughtfully) $6x = 12$. (short pause) Does that mean $x = 2$?
(Subsequent conversation indicated that the verbalisation of $6x = 12$ had helped to establish the relationship between left-hand-side and right-hand-side which had not been obvious until then to the child).

Problem 9:
I: Why did you write down twelve first? (Child is busy rubbing out 12. Writes 6)
C: Yes, 17 minus 5 equals 12, and there's two x's which means you have to halve it which means x equals 6.

Problem 10:
C: I'm not sure. You have to find out what 2 into 3 is before you can get the answer. Because 11 minus 3 is 9 I don't know whether to put the x down or not. Then you've got the 9 and the 2.
I: Are you finding this one much harder than the last?
C: Yes.

Alex, 13 yrs. Some prior algebra.

Problem 1:
C: Twenty three.
I: How did you get that?
C: Well, B is 27 minus 4.

Problem 2:
C: A is seven. So A plus five is 12.
I: You found A first, then?
C: Yes, I found out what A was.

Problem 3:
C: C is 26, so the answer is 17.

Problem 4:
C: (writes $2b$ in box)
I: How did you get $2b$?
C: That's 2 (points to a) and $2b$ times $2a$ equals $2a$ times $2b$. See, a is 2. You've got a there (points to RHS) so you've got to put b there (points to box).
I: What made you choose a as 2?
C: Because it says here that a is 2 (points to $2a$ on RHS).

Problem 5:
C: How do you know what x is?
I: Do you need to know what x is?
C: I suppose it's just nothing (inferring that x has no meaning here) . . . (pause) one three is 3. Put 3 down here (writes 3 in box). Threes into five is once and two left over (writes $1\frac{2}{3}$).

Problem 6:
C: x is 6 so you put 6 there (crosses out x and writes 6) and y is 21 (crosses out y and writes 21). Two and 21 is 23; 6 and 7 is 13. Thirteen into 23 goes 1 and 10 left over so ten thirteenths. (Writes $23/13 = 1\,10/13$).

Problem 7:
C: You can only take away from the numbers that have t on them. Five t take firstly t is 5. Hang on. You can't tell what t is because you've got t there (points to $5t$) and t there (points to $3t$). It can be one of either of the numbers. So you just do 5 plus 3. You can't do this one (points to -7) because it hasn't got a letter on it. (writes 8 in box).
I: What about the other t? (points to $-t$)
C: You can't tell because there's a t there and a t there and you don't know what t is.

Problem 8:
C: $2x$ take away $5x$ is minus 3 plus $9x$ is 6. So x is 6.
I: Do you mean that x is 6 because you have $6x$?
C: Yes.

Problem 9:
C: x is 2 because you have $2x$.

Problem 10:
C: x is 3.
I: Let's think about a different problem. What if we had $11 - x = 2$. What would x be?
C: You wouldn't know what x is. You have nothing to tell you what x is.

Richard, 13 yrs. Some prior algebra.

Problem 1:
C: 23

Problem 2:
C: 15 (Although the interviewer did not probe this answer, the method the child has used for Problem 3 will give an answer of 15 when applied to problem 2.)

Problem 3:
C: If C take 6 equals 20, then C take 9 if C equals 20 equals 11.

Problem 4:
C: (reads problem aloud as tb times a equals ta times (short pause) b.
I: Why b?
C: Because that was tb (corrects himself) $2b$ times a equals $2a$ times by b.

Problem 5:
C: Ones into 5 goes 5. So times the one by 5 to get $5x$ so you have to times 3 by 5 to get 15 (short pause) x (writes $15x$ in box).

Problem 6:
C: (writes 6 near x and 21 near y) That must leave you with two sixths and twenty one sevenths which is twenty three thirteenths.

Problem 7:
C: $5t$ plus $3t$ equals $8t$. Take t equals $7t$ take 7 equals 0.

Problem 8:
C: $2x$ take $5x$ equals minus $3x$ plus $9x$ equals $6x$. Take 12 equals minus $6x$ so x would equal . . . (pause) . . .
(repeats whole statement) . . . $2x$ take $5x$ equals minus $3x$ plus $9x$ equals $6x$. Take 12 equals minus $6x$ so x would equal 1.

Problem 9:
C: Five take 17 equals 12. So it would be five plus 12 equals 17. So x would equal 12.

Problem 10:
C: x equals 3 because 11 take three threes are 9 equals 2. So x equals two (writes 2).

Anne, 12 yrs 6 mths. No prior algebra.

Problem 1:
C: Twenty seven minus four is twenty three.
I: So is that what B would be?
C: Yes.

Problem 2:
C: Seven plus three is 10. Then seven plus five is 12.

Problem 3:
C: Well, if C minus 6 is 20 then twenty minus 9 is 11.

Problem 4:
C: I made b two and a six. Then two times b is 4 and four sixes are 24. Then $2a$ is 12 and b is 2 so two times 12 is 24.
I: That's why you've put b as the answer?
C: Yes.

Problem 5:
C: (writes 15 in box) I made x five. Where's the rubber – I've made a mistake here!
I: Tell us what you did.
C: Well, I forgot you have to times it by 5 there (points to $5x$) I'm going to put x is 3. One three is 3 and three threes are 9 (writes 9 in box).
I: Is that the answer then? (child nods) What about the x? (points to the x)
C: x changes into . . . wait a sec. That would have to be 21 (points to 9). That's 15 (points to $5x$ and writes 15) and three 7's are 21 (writes $15/21$ next to question).

Problem 6:
C: y is 21. Twenty one over 7, I brought down into 3, and then two sixths equals one third. Then I added them together, three and one third.

Problem 7:
C: I have to work out what t is? (said as a question)
I: Do you need to work out what t is? Do you think you have enough information to work out what t is?
C: No, (reads problem out) there's not enough information there.
I: Can you write it a little more simply?
C: Well, I could add these together.
I: What would you get then?
C: I'd have to work out what t is.
I: Do you have to know what t is?
C: $8t$, and I could add those two together (points to $-t-7$). But I'd need to know what t is (writes $8t - t - 7$.)

Problem 8:
C: (long pause)
I: What are you trying?
C: 2
I: What's happening?
C: It didn't work out. Two 2's are four, five 2's are ten and nine 2's are eighteen. Four minus 10 is six, so it doesn't work out.
I: Had you tried anything else first? (child had written $45 - 15$ on paper first)
C: Yes, 5. That one didn't work either.

Problem 9:
C: (pause) six.
I: How did you get that?
C: Well, I took 5 from 17, that's 12, and two 6's are twelve.

Problem 10:
C: (short pause only) x is 3.
I: How did you get that?
C: I took two from 11 which is 9 and three 3's are 9.

James, 12 yrs 10 mths. No prior algebra.

Problem 1:
C: B is 27.
I: How did you get that?
C: 27 take 4 here. B plus 4 won't make any difference, so B is 27.

Problem 2:
C: A plus 5 is equal to 12.
I: What did you do there?
C: A plus 3 is 10, and A plus 5 is adding two more on so it must be equal to 12.

Problem 3:
C: Must equal 15. Since C take away 6 is 20, C take away 9 is three more so it will be 17 (wrote 15 first, crossed it out and wrote 17).

Problem 4:
C: I reckon that's 2 times 1.
I: What would make you choose that?
C: b's the second letter of the alphabet, and a is the first, so choose b as 2. Then $2a$ which is 1 gives (trails off, writes b after long pause).
I: In your mind, what did you do? Did you change a and b into numbers?
C: Yes, because a's the first and b's the second letter in the alphabet, you just use these numbers.
I: What did you get for $2b$ times a then?
C: Two, because two times one is still 2.

Problem 5:
C: I'm going to say x is 20. Then it's one over 3 times 520. It's really 3 times 520 (works out 520 × 3). That's 1560.
I: Would you leave this as the answer even though you put something for x?
C: Yes.

Problem 6:
C: (Reads) if x equals 6 and y equals 21, what is the value of $2x$ plus y seven? (Pause) Two times 7 which is 14, put the one up for the x and the 4 for the y so it's one x and y four (Writes $1/x + y/4$).
I: What about the fact that we've got $x = 6$ and $y = 21$?
C: I don't know. I don't understand that bit.

Problem 7:
C: Take away five plus three t take away 7. (writes $x - 5x + 3x - 7$)

Problem 8:
C: Two take 5 is 3 plus 9 is 12. Just don't worry about the x's. x is zero because you don't have to worry about the x's.

Problem 9:
C: The value of x is 10 because five plus two equals 7 and then x must be 10.
I: Before we go on, I'd like to explain what's meant by $2x$ and so on. $2x$ means two 'lots of' x. Does that change what you've just said, or will it stay the same?
C: It could be 6 because 2 times 6 would be 12 plus the 5 gives 17.

Problem 10:
C: x would be 9 because 11 take away 9 is 2.
I: The $3x$ would be 9?
C: Yes. Oh (short pause), the x would be 3 because you times 3 by 3 which is 9. Eleven take away 9 would be 2.

Jill, 12 yrs 5 mths. No prior algebra.

Problem 1:
C: (writes 23)
I: If B is 23 that would be 23 plus 4 equals 27 minus 4.
C: Oh, yes.
I: Would you be happy with that?
C: I suppose. I don't know these.

Problem 2:
I: Having letters A or B is just like having an empty box there.
C: Right-oh (pause)
I: Do I add A and 5? (writes 7) That's wrong. I've got to add them (writes 12).

Problem 3:
I: How did you get 23?
C: 6 and 14 is 20 and 14 and 9 is 23. I'm not sure how I got it.
I: Have another look.
C: Let's see. Twenty six. Twenty six minus . . . I'm always doing this. I don't look at it properly (writes 17 in box).

Problem 4:
C: (writes b)
I: Why do you think it's b?
C: I don't know. I just swapped it round; a went to there so I thought b would go there.

Problem 5:
C: What's x?
I: It takes the place of a number, any number at all.
C: Can it be zero?
I: Yes, or any other number.
C: I'll put 150 then (writes 150 in box).

Problem 6:
C: (Crosses out x and writes 6; crosses out y and writes 21. Writes $\frac{21+2}{42} = \frac{23}{42}$)
I: Can you explain how you did that one to me?
C: Two goes into forty two 21 times and 21 goes into 42 twice. Then I added them up.

Problem 7:
C: What does t stand for?
I: It can be any number at all. What do you think $5t$ plus $3t$ would be?
C: Zero. It could be anything. If I were to make it zero, it would be 50 plus 30 minus zero. Then minus 7 is seventy three (writes 73).

Problem 8:
C: Can I make it a double figure?
I: I'll help you a bit here. $2x$ actually means two times the number you put in for x. So if x is 3, $2x$ is two times 3 which is 6.
C: Right.
I: What would $5x$ be, if x is 3?
C: Anything. (pause) 6 (pause) 56. (She has evidently retained the 6 from the example of $2x = 6$ when $x = 3$, and has resorted to her earlier strategy of simply 'tagging on' the value of x to the coefficient).

Problem 9:
C: (points to $2x$) 12.
I: If $2x$ is 12, what would x be?
C: One. Is there one x or just straight out x? Is there just x?
I: Yes.
C: Then x is 6.

Problem 10:
(not attempted as time ran out)

What Do the Interviews Tell Us?

Their responses reveal that the children are almost totally preoccupied with trying to cope with the new language and ideas in the algebra problems. In doing so, they neglect to carry out simple arithmetic checks, or make careless mistakes such as Sue's 'seven plus five equals thirteen' for Problem 2.

Another example of the children's overriding concentration on the unfamiliar is their failure to remember how to add fractions in Problem 6. Most carried out the substitution correctly, but only one could carry the working through to a correct answer.

Although there is nothing unexpected in this preoccupation, it gives us important information about how children cope with an unfamiliar abstract situation.

All their resources become channelled into tackling the unknown, at the expense of the more familiar. They cannot apply a known and practised method *and* cope with the unfamiliar at the same time. And when the former breaks down, success in the latter is doomed.

Another aspect of children's mathematical understanding revealed is the speed and depth with which an idea is grasped and embedded. Unfortunately, this applied equally to correct and incorrect ideas, and perhaps accounts for some of the difficulties children encounter with algebra. An inappropriate idea may be grasped quickly and implanted long before the end of the first lesson on the topic, but discovering and correcting the misconception might take many months. Alex, for example, correctly answered Problems 2 and 3, but seemed to get the impression that $2b$ meant b = 2 in Problem 4. By the time Problem 9 was answered, his new strategy gave him the answer $x = 2$ very quickly. With Problem 10 , his answer $x = 3$ may well have been marked correct – but was his strategy correct? Despite his earlier solvings of Problems 2 and 3, he was unable to solve $11 - x = 2$. A new (and perhaps satisfyingly simple) strategy had overridden his earlier sound but possibly more laborious approach.

On a more optimistic note, Anne, who had already learned some algebra, found that by substituting values of the variable used, she could find a solution to the problems. Although her strategy failed her in Problem 8, it was firmly embedded and she applied it successfully in Problems 9 and 10.

The interviews reveal that the children's grasp of *speaking* mathematical language is *shaky*. A higher profile should be given to *speaking* mathematics in the classroom – confidence in *speaking* a language builds up confidence in *thinking* the language and in *writing* it.

What Do the Letters Mean?

Regardless of whether or not the children had previously learned algebra, a common difficulty was their search for the meaning of the letters. If t was used to represent a number, what was the number? As one wasn't given, a common reaction was to provide a value. But what value?

For questions like Problem 8, this compulsion to find a number was to *try out* possible numbers. Simplification of any kind was rarely thought of, thereby making what should have been a simple problem much more difficult.

The meaning of the letters was often misinterpreted. Jill, for example, took t in Problem 7 to represent place value. Early correction (Problem 8) failed to correct the misinterpretation, even though Jill had appeared to understand the explanation.

Ignorance – Is It Still Bliss?

A favourite strategy was to ignore whatever part of the problem could not be understood. If it wasn't within their personal realm of experiences, they felt it must have no real relevance to finding the solution to the problem. Alex for example, could not solve problem 5 because he felt that x had to have a value. He therefore concluded that x was unimportant and could be ignored. James showed a similar philosophy in Problems 1 and 8.

Abstract Ideas, Symbolic Notation and Errors

Problems noted during the interviews fell into six broad categories.

(a) Trying to keep track of abstract terminology and ideas and to manipulate simple numerals simultaneously seemed to present difficulties.

(b) Children tried to carry too much information in their heads. When some figures were written down, improved accuracy was usually noted. A difficult problem was rarely broken into two or more smaller parts that could be handled.

(c) Children commonly tried to give the letters numerical values at the earliest opportunity, whether this was appropriate or not. Presumably, this builds directly on their previous knowledge of manipulating numerals and on the current information that the letters denote numerals. With this philosophy ingrained into the children's background, it is clear that asking them to put aside these ideas is an immediate conflict.

(d) The meaning of place value had been drilled in Primary School mathematics. The use of $2t$ and $3x$ and so on is seen by some children as an extension of these ideas. (If t has a value of 0 then $2t$ can be written as 20). The notion that $2t$ means two multiplied by t is therefore a conflicting idea with what has gone before. Similarly, $2t$ was sometimes interpreted as $2 + t$ (James, Problem 9).

(e) Rapid establishment of strategies means that some children learn a correct approach quickly while others acquire an erroneous method just as quickly. In a classroom this can cause problems.

(f) Algebra was regarded by the children as a topic that 'we haven't done yet' or one which had been done a long time ago and forgotten. The use of pronumerals was seen as a new topic and certainly not as an extension of their previous mathematical knowledge.

Adult Ideas For a Child's World

With children such as James, a simple explanation of the meaning of $2t$ and $3x$ and so on clarified the problems. The interview was continued with James . . .

I: If you had $5 + y = 17$, would that be easier to do? (easier than $5 + 2x = 17$)
C: Yes, y would be 12.
I: What about $5 + \square = 17$. Would that be easier to do than $5 + y = 17$?
C: No, that would be harder.

This answer was unexpected. But why should we scrutinise children's ideas in the light of adult values and expectations? Isn't that precisely what we are doing when we teach algebra in the formal, abstract model that we ourselves were taught by? If we tried instead to look at algebra from a child's perspective, we might begin to develop algebra from the mathematical base with which the child is already familiar.

Most of the current textbooks introduce algebra by describing the use of pronumerals, for example:

> A letter which replaces a number or numeral is called a pronumeral . . . The number which stands before the pronumeral is called the coefficient and tells us how many pronumerals there are . . . E.g. in $5b$, 5 is the coefficient of pronumeral b. This means that there are $5b$'s .
> (Lynch *et al*, 1981)

Then follows a section on simplification of expressions, evaluation, recognition of like terms and so on. Equations and inequations follow (in the book quoted from) in a later chapter.

In the interviews, simplification and gathering of like terms was one of the more difficult tasks. The meaning of $5x$ or $5b$ in isolation was obscure. By contrast, the intuitive solving of If $A + 3 = 10$ then $A + 5 = \square$ was possible, as also were attempts at finding x when $5 + 2x = 17$, once the meaning of $2x$ was explained. Some may argue that the solution of the former problem when finding A was not included cannot be regarded as algebra. Surely, though, any problem that embodies some of the abstract ideas of algebra (as this problem does), is a useful bridge between fully numerical and fully algebraic expressions.

What Can a Teacher Do to Make Algebra More Meaningful?

1. Listen to the children. Their questions reveal what *they* think they don't understand, and may suggest what modifications need to be made to teaching approaches. If the text book you use goes

against the children's needs, don't follow it slavishly – select suitable exercises, add your own activities, and share ideas with other teachers.

2. Encourage the children to *write down* the steps they use in working out the problem. Show them how this means they don't have to remember the solution to the first part of a problem while they tackle the second part.

3. Turn the children's questions around. 'What do *you* think the next step is?' Frequently this approach will show up misconceptions or misinterpretations. The difficulty is, though, that no one feels comfortable revealing their lack of knowledge or understanding. It is up to the teacher to help the child to feel secure in discussing mathematics by demonstrating a supportive and not a destructive role.

4. Give individual encouragement. It is one thing to say to the class 'Have a try at these problems'. A child who is having difficulties with that topic will not necessarily respond, except negatively. A direct conversation with the child, though, helping him/her to find a starting point and offering encouragement can make a significant difference to the child's attitude; 'Perhaps I can do it if I try', rather than 'What's the point of trying?'

5. Above all, let algebra develop from the child's needs and experiences. Finding missing quantities, measuring perimeters (then letting children suggest an abbreviation for length and breadth, leading to simple formulae and substitution and checking by measurement), for example. *Abstractions need to be based on experiences if they are to be understood and generalised.*

Notes

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A 35-minute mathematics test was given to 10,500 children and adolescents aged between 11 and 18 years in the Adelaide metropolitan region in South Australia and the greater Wellington region in New Zealand. This 39-question test established a data base for a large scale study on the development of abstract reasoning.

For background reading in the use of interviews to diagnose errors and areas of difficulty in mathematics, the following references are useful: Ashlock, R.B. (1982) *Error Patterns in Computation*, C.E. Merrill Pub. Co., Columbus Ohio; Lankford, F.G. (1974) *What can a Teacher Learn about a Pupil's Thinking through Oral Interviews*, The Arithmetic Teacher, *21*, 26–32; Newman, A. (1983) *Language and Mathematics*, The Newman Language of Mathematics Kit, Harcourt Brace Jovanovich Group, Sydney.

The recent large scale study of how children develop abstract reasoning in mathematics can be found in Ellerton, N.F. and Johnston, L.C. (1984) *An Investigation into the Development of Abstract Reasoning*, presented at the 5th International Congress of Mathematical Education, Adelaide.

The quotations about pronumerals are all from Lynch, B.J., Picking, L.P., Anders, J.L., and Coffey, M.J. (1981), *Maths 7*, Sorret Pub. Co., Malvern 3144, Australia (quotations are on pages 170 and 172).

What Are the Benefits of Single-sex Maths Classes?

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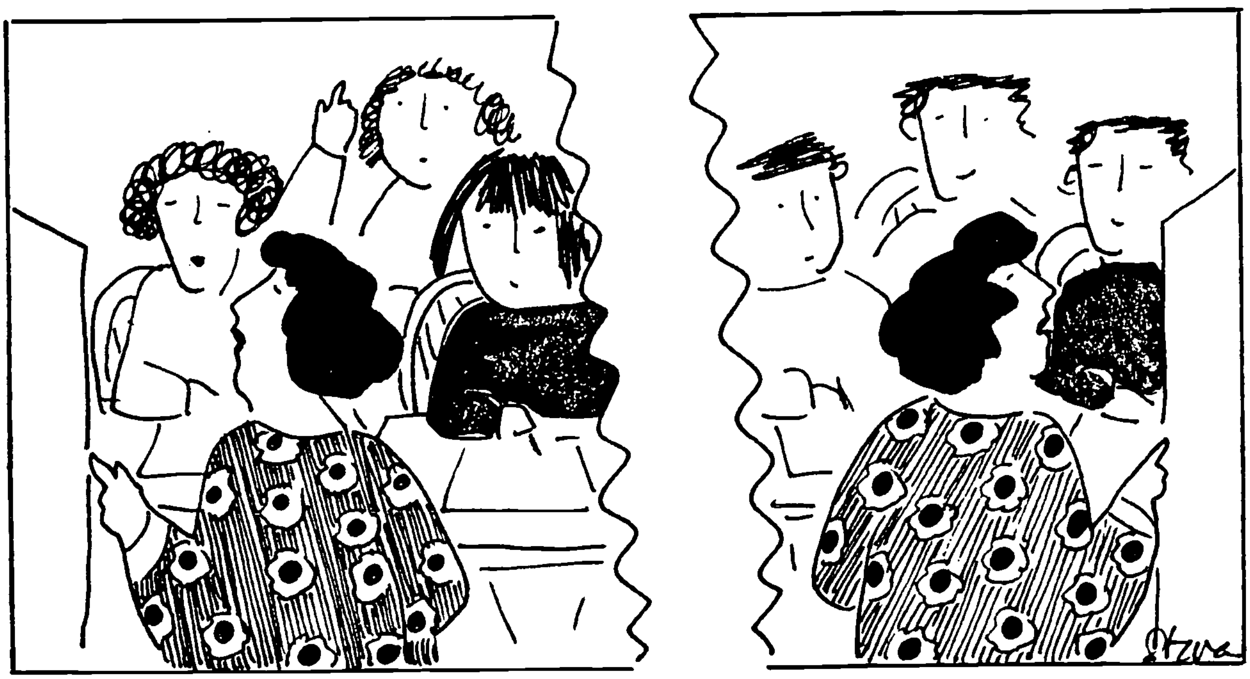

Shona Mclean

Introduction

THE MAJORITY OF GIRLS in Australian schools choose not to take mathematics and science subjects at higher levels. This choice is said to 'limit their career options to a narrow range of traditionally female vocations'.

One popular and highly visible method of doing something about this involves the establishment of single-sex classes within co-educational schools. This strategy follows classroom interaction research and literature indicating that, whether the teacher is male or female, boys consistently receive a greater proportion of a teacher's time and attention in mixed-sex classrooms. Spender in 1982 estimated that boys receive 'two thirds of a teacher's attention.... in mixed-sex classrooms' principally because teachers demanded more of the boys and the boys needed more discipline. This was confirmed in Australia: Leder in 1986 reported

> For the majority of contacts (those initiated by teachers) boys interacted more frequently than girls with their teachers, in both language and mathematics lessons. As well, boys consistently sought and received teacher attention more frequently than girls.

This is not just a finding in co-educational classes. In Victoria in 1987 Nix found that boys dominated mixed-sex classes but the teacher was more in control in single sex classes.

Such research, together with a growing literature contributing to the raising of consciousness of equal opportunity/sexist/feminist issues, places in question the claimed advantages of co-educational secondary schooling. For example, some writers argue that the 'normal' harassment of girls by boys, which typically occurs in co-educational schools, is particularly debilitating for girls and counterproductive for boys.

Gender research suggests that there are marked differences of *confidence* in learning mathematics. For example, Fennema and Sherman's study among senior elementary and secondary school students found that boys were consistently more confident in their ability to deal with mathematics than were girls. Confidence, if examined alone, might account for the major proportion of the variance in students' mathematics performance and enrolment patterns. Additional evidence exists to suggest that such gender differences in confidence may be created by parental, teacher and societal sex-role attributions. Do co-educational classes make things worse?

In response to such influences, single-sex classes in Australian co-educational postprimary schools, especially for mathematics and science curricula, are growing. However, there is some concern that many schools which establish single-sex or all-girls classes do not appear to be aware of alternative strategies (including organisational and curriculum options) designed to enhance the participation and achievement of girls in mathematics and science. A few valuable qualitative evaluations of all-girls classes have been conducted, but the movement to institute single-sex classes seems to be growing faster than the evidence to support it.

A 1987 review of current research on single-sex and mixed-sex schooling, commissioned by the Ministerial Advisory Committee on Women and Girls in Victoria, concluded 'that there was insufficient Australian research on which to base advice to the Minister'.

A previous study among 12-year-olds examined the joint effects of gender and class type on measures of students' mathematics achievement and attitudes towards mathematics. There were significant positive increases in both achievement and attitude scores by all students between two test occasions (nine months apart) and especially among the girls and boys in single-sex classes. There were moderate to strong correlations between students' achievement and confidence scale scores.

Following up the idea that confidence in mathematics may be the best predictor of student performance and participation, we began a longitudinal study to examine the effects of class type on student achievement and confidence in learning mathematics.

Context for the study

At a mathematics faculty meeting in 1984, the teaching staff of Ballarat High School (Victoria) expressed concern that, despite demonstrated ability in mathematics, girls were under-represented in senior mainstream mathematics classes at the school. It was agreed that girls were possibly being denied access to their share of teacher time and support in mixed-sex classes. A decision was made to institute single-sex classes as one approach but some means of determining its relative effectiveness was needed.

A request for assistance came to the Research and Development Section of the Curriculum Branch, Ministry of Education (Victoria). On the basis of discussions, it was decided to embark on a longitudinal study, beginning with the 1985 Year 7 and Year 8 classes (12- and 13-year-olds).

The school is located in a well-established residential area of a large Victorian provincial city. It has a long-standing reputation for encouraging excellence across the curriculum, including mathematics and science. Moreover, students attending the school are drawn from fairly homogeneous socioeconomic and sociocultural backgrounds, and generally have high aspirations.

Method

Students and Teachers

At the commencement of the 1985 school year, and within timetabling constraints, both the Year 7 [Form 2] and Year 8 [Form 3] student groups were randomly allocated to either single-sex classes ('treatment' group: three all-girls and three all-boys classes in each year level) or mixed-sex classes ('control' group: two in each year level), with approximately equal numbers of students in each class. For all other subjects of the curriculum, students were in mixed-classes. The teaching staff were also randomly allocated to the 16 classes involved. Timetabling constraints meant that the classes did not stay intact into the next year so the final numbers in the experiment are 180 treatment and 81 control. This is called the Intact Group. However, those children who shifted from single-sex maths classes to mixed-sex classes are also worth studying and are called the Shift Group.

Mathematics Achievement and Confidence Measures

On two occasions in 1985 (approximately nine months apart), the Year 7 and Year 8 students were administered selected items from four domain-referenced subtests of the *ACER Mathematics Profile Series: Operations, Space, Number; Measurement*. Towards the end of the 1986 school year (third test occasion), the same students (now Year 8 [Form 3] and Year 9 [Form 4] respectively) were again administered selected items from the four mathematics subtests.

On the three test occasions, students were also given selected attitude items adapted from the Fennema-Sherman Mathematics Attitude Scales. Of particular relevance to the present study, the Confidence scale contained eight items about confidence in learning and using mathematics.

Results

Class Type Effects on Achievement and Confidence – Intact Group

There were significant increases in mathematics achievement and confidence scores for all students, with the exception of those for boys in mixed-sex classes. Girls and boys in single-sex classes did better on the first test but not significantly so on the next two. The between-groups differences were not significant on the first and second test occasions. However, both girls and boys in single-sex classes indicated significantly higher levels of confidence as time went by. Note that the same things happened to both boys and girls, and the girls did not gain more than the boys.

In the intact group the only students whose achievement levels appear to have been adversely affected were boys in mixed-sex classes. Regardless of mathematics achievement, students in single-sex classes indicated consistently higher gains in confidence, over time, than those in mixed-sex classes, with confidence being a significant predictor of achievement.

Class Type Change Effects on Achievement and Confidence – Shift Group

The results for those children who had to shift to mixed classes or shift to single-sex classes are more complex. Those boys initially in single-sex classes, gained in both achievement and confidence. After relocation their achievement dropped a little and their confidence a lot. Girls initially in single-sex classes gained confidence but did not achieve better. On relocation they did improve their work but suffered a big drop in confidence.

In contrast, the achievement scores of students initially in mixed-sex classes notably improved following relocation to single-sex classes (significant for girls), and their confidence scores continued to improve significantly.

The structural relationship between confidence and achievement indicated that confidence was a significant predictor of achievement, for both class-type shift groups.

Class Type Effects on Participation

Did the children who had been in single-sex classes carry their confidence on to the next year? This was examined by looking at the numbers of Year 9 students who actually chose either Maths A (mainstream) or Business Maths for Year 10, and their class-type background.

The demand for Maths A in 1987 came predominantly (82% of the girls, 85% of the boys) from students who had been in single-sex classes for the previous year. The number choosing Maths A was up on previous years – a 30% increase!

Summary of Results and Discussion

The present study illustrates some of the practical difficulties associated with the conduct of applied longitudinal research in a school setting. However, despite some data-related limitations the results from the study yielded three major outcomes worthy of comment.

First, in contrast to the evidence available in existing reviews, it was found that gender differences in mathematics achievement and in confidence on the three test occasions were not significant. The results for the Intact Group indicated that although boys generally gained higher scores than girls on both the mathematics achievement and confidence measures (but not significantly higher), the most notable improvement across the three testing occasions occurred among girls in single-sex classes, followed by boys in single-sex classes. However, the fact that boys in the present study generally gained higher scores may have more to do with the nature of the measuring instruments used (i.e., multiple-choice/rating-scale-type tests and inventories).

Second, the results indicated a strong association between achievement and confidence, with confidence being a significant predictor of achievement, especially for students in single-sex classes. While the change in mathematics achievement over time, independent of confidence, was similar for all students regardless of class type, students in single-sex classes indicated significantly higher gains in confidence than those in mixed-sex classes.

Third, the major outcome of the study was that, for the Year 9 students, being placed in single-sex classes was associated with high levels of confidence which, in turn, significantly increased the likelihood of their subsequent participation in Year 10 mainstream Maths A classes. While it should be recognised that other factors not controlled for (e.g., teacher effects, Hawthorne effects and related factors) may have contributed to these outcomes, the evidence is sufficiently strong to suggest that single-sex classes were to the advantage of those students concerned.

By any criterion, the overall findings from the present study indicate that the institution of single-sex mathematics classes *at the school studied* has been a success. The findings also lend support for Chipman and Wilson's suggestion that 'confidence, if examined alone, might... appear quite important'. Nevertheless the outcomes of the present study possibly raise more questions than they answer. For example, explanations to account for the observed class-type effects on student confidence, independent of achievement, are not simple and require closer investigation. Part of the explanation for these outcomes, however, is to be found in additional qualitative data obtained from both students and teachers. While these data are too voluminous to report in detail here, a brief account is appropriate.

Observations of classroom interactions in mixed-sex classes supported the findings of related research, indicating a consistent tendency for boys in mixed-sex classes to demand higher levels of individual attention from their teachers than those demanded by girls. Teachers of single-sex classes (especially all-girls classes) reported 'improved working atmospheres' and that 'students identify closely with their group'. In single-sex classes of either gender, there was a notable reduction in the frequency and saliency of student attention-demanding behaviour as well as reduced student discipline and management problems (in most classes). Typical of the comments from girls was: 'It's easier to talk to the teacher with just girls, because boys sometimes laugh and make you feel stupid'. This resulted in perceived increase in both student and teacher 'time on task', impacted positively on teacher expectations of individual student and group performance, and encouraged teachers to match both curriculum content and teaching styles to specific gender- and group-related student interests.

Further explanations for the higher level of confidence among students in single-sex classes may be relevant to the outcomes of a recent study by Nelson-Le Gall and DeCooke in 1987, who found that both girls and boys typically seek help more frequently from classmates of the same sex than the opposite sex. The results of that study suggest that students' strong preferences for same-sex helpers may actually inhibit them from interacting with opposite-sex peers who are competent to provide needed assistance. Given that such availability is reduced in the context of mixed-sex classes, this factor may have been an important contributor to the observed outcomes of the present study.

In a recent well-controlled study of the effects of single-sex postprimary schools on student achievement and attitudes in the United States, Lee and Bryk in 1986 found that, for achievement, aspirations, locus of control, or attitudes and behaviour associated with school learning, single-sex schools deliver specific advantages to their students, especially to girls. Lee and Bryk conclude:

> What has been considered by some to be an anachronistic organisational feature of schools may actually facilitate adolescent academic development by providing an environment where social and academic concerns are separated.

On the basis of findings from the present study, it could be argued that similar sentiments may equally apply to single-sex class environments in co-educational postprimary schools.

In spite of the compelling nature of the data reported here (at least at the prima facie level) it should be stressed that, since they are derived from a study in only one school, the need for ongoing research of the kind reported here in additional schools is mandatory. The fulmination and emotion-laden rhetoric which frequently surround issues concerned with gender and education needs to be firmly placed in the context of appropriate longitudinal research.

In any event, the long-term effectiveness of single-sex class grouping as an appropriate intervention strategy to increase the participation of students in co-educational postprimary schools (for any area of the curriculum) has yet to be established. Moreover the implicated cognitive, affective, behavioural and psychosocial correlates operating in single-sex and mixed-sex classrooms require further investigation. If research findings continue to support the use of single-sex classes as an appropriate intervention strategy, the policy implications for government co-education systems are, needless to say, problematic. From the experience of the present author, school-based logistic problems associated with timetabling, school community and parental support, impose additional constraints which must be taken into account.

It may be that, as the current students are followed further into the more senior levels of postprimary schooling (remaining in single-sex classes), both girls and boys may be better represented in mathematics classes. Time and commitment will determine whether or not such an aim will be realised. Suffice to say at this stage, that additional data need to be gathered before an informed, responsible judgment advocating the universal efficacy of establishing single-sex classes in co-educational postprimary schools could be made.

Notes

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The assistance of Greg Tepper and the Mathematics staff at Ballarat High School is gratefully acknowledged. A full account of this research may be found in

Rowe, K.J. (1988) Single-sex and Mixed-sex classes: The Effects of Class Type on Student Achievement, Confidence and Participation in Mathematics, *Australian Journal of Education*, Vol. 32, No. 2, pp. 180–202.

That the majority of girls in Australian schools do not choose Maths and Science at higher levels is quickly ascertained from figures in, for example,

Commonwealth Schools Commission (1984) *Girls and Tomorrow: The Challenge for Schools*. Report of the Working Party on the Education of Girls, Canberra: AGPS.

That such choices limit their career options is discussed in

Bodna, K. and Scharwz, V. (1985) *Education for Girls*, Melbourne: Australian Education Council.

That boys receive most attention is detailed in many places, for example

Leder, G. (1987) Teacher student interaction: A case study, *Educational Studies in Mathematics*, Vol. 18, pp. 255–71.

Spender, D. (1982) *Invisible Women: The Schooling Scandal*, London: Writers and Readers.

The research by Nix can be found in

Nix, P. (1986) *Assessment and Reporting Practices for Girls in Mathematics*, Melbourne: Ministry of Education (Schools Division), Curriculum Branch.

A good example from amongst growing literature on equal opportunity/sexist/feminist issues is

Gray, A. (1986) *Are Girls the problem? Coeducation revisited*. Paper presented at the annual conference of the Australian Association for Research in Education, Melbourne.

That harassment of girls by boys is debilitating for girls and counterproductive for boys is argued, for example, in

Schwarz, V. (1987) Does Jill come tumbling after? Let's look at girls in education, *Unicorn*, Vol. 13, pp. 132-8.

Research on confidence, and large sex differences in maths classes, can be read in

Fennema, E. and Sherman, J. (1978) Sex-related differences in mathematics achievment and related factors: A further study, *Journal for Research in Mathematics Education*, Vol. 9, pp. 189–203.

The further evidence of where differences in confidence come from can be found, for example, in

Boswell, S.L. (1985) The influence of sex-role stereotyping on women's attitudes and achievement in mathematics. In S.F. Chipman, L.R. Brush and D.M. Wilson (Eds.), *Women and Mathematics: Balancing the Equation*, Hillsdale, NJ: Lawrence Erlbaum and Associates.

Options other than single sex classes are discussed in

Head, J. (1985) *The Personal Response to Science*, Cambridge: Cambridge University Press.

and

Hildebrand, G. (1985) The missing half: Women in science: Secondary school science: Successful strategies for improving the access and success of girls, *Lab Talk*, pp. 10–12.

and

Lantz, A. (1985) Strategies in increased mathematics enrolments. In S.F. Chipman, L.R. Brush and D.M. Wilson (Eds.), *Women and Mathematics: Balancing the Equation*, Hillsdale, NJ: Lawrence Erlbaum and Associates.

The few valuable qualitative evaluations of all-girl classes include

Dunn, J., Hammond, B. and Watson, N. (1984) *Evaluation of an All Girls Maths Class at Hawker College*, Canberra: ACT Schools Advisory Authority.

and

MacMillan, J., Hansford, B. and Thurgood, P. (1985) *Single-sex Classes in Co-educational Schools*, Melbourne: Education Department of Victoria, Equal Opportunity Unit.

and

Pummeroy, C. and Haynes, B. (1986) *An All-Girls Mathematics Class at Sale Technical School*, Melbourne: Ministry of Education, Participation and Equity Programme, Schools Resource Program.

For the 1987 review for the Victorian Ministerial Advisory Committee on Women and Girls see Schwarz, above.

The previous study among 12-year-olds is

Rowe, K.J., Nix, P.J. and Tepper, G. (1986) *Single-sex versus Mixed-sex Classes: The Joint Effects of Gender and Class Type on Student Performance in and Attitudes towards Mathematics*. Paper presented at the annual conference of the Australian Association for Research in Education, Melbourne.

The idea that confidence may be very important as a predictor of performance comes from

Chipman, S.F. and Wilson, D.M. (1985) Understanding mathematics course enrolment and mathematics achievement: A synthesis of the research. In S.F. Chipman, L.R. Brush and D.M. Wilson (Eds.), *Women and Mathematics: Balancing the Equation*, Hillsdale, NJ: Lawrence Erlbaum and Associates.

Confidence was measured using attitude items adapted from

Fennema, E. and Sherman, J. (1976) Fennema-Sherman mathematics attitude scales, *Catalogue of Selected Documents in Psychology*, Vol. 6, No. 1.

The statistical analyses of the results are described and tables given in the full report of the project in the *Australian Journal of Education*, Vol. 32, No. 2, 1988, mentioned above.

Summary of Results and Discussion. Some existing reviews, containing evidence opposite to that obtained in our research, are

Aiken, L.R. (1985) Mathematics, attitude towards. In T. Husen and T.N. Postlethwaite (Eds.), *The International Encyclopedia of Education*, Oxford: Pergamon Press, Vol. 6, pp. 3233–6.

and

Chipman, S.F., Brush, L.R. and Wilson, D.M. (Eds.) (1985) *Women and Mathematics: Balancing the Equation*, Hillsdale, NJ: Lawrence Erlbaum and Associates.

That boys do better in multiple choice tests, etc., is attested in

Harding, J. (1983) *Switched off: The Science Education of Girls*, Schools Council Programmes 3, Developing the Curriculum for a Changing World, London: Longman.

Observational research of classroom interactions in mixed-sex classrooms include Leder (1986) mentioned above, and

Croll, P.J. (1985) Teacher interaction with individual male and female pupils in junior-age classrooms, *Educational Research*, Vol. 27, pp. 220–3.

The research on children seeking help from similar sex partners can be found in

Nelson-Le Gall, S. and DeCooke, P.A. (1987) Same sex and cross sex help exchanges in the classroom, *Journal of Educational Psychology*, Vol. 79, pp. 67–71.

The U.S. study on single-sex post primary schooling is

Lee, V.E. and Bryk, A.S. (1986) Effects on single-sex secondary schools on student achievement and attitudes, *Journal of Educational Psychology*, Vol. 78, pp. 381–95.

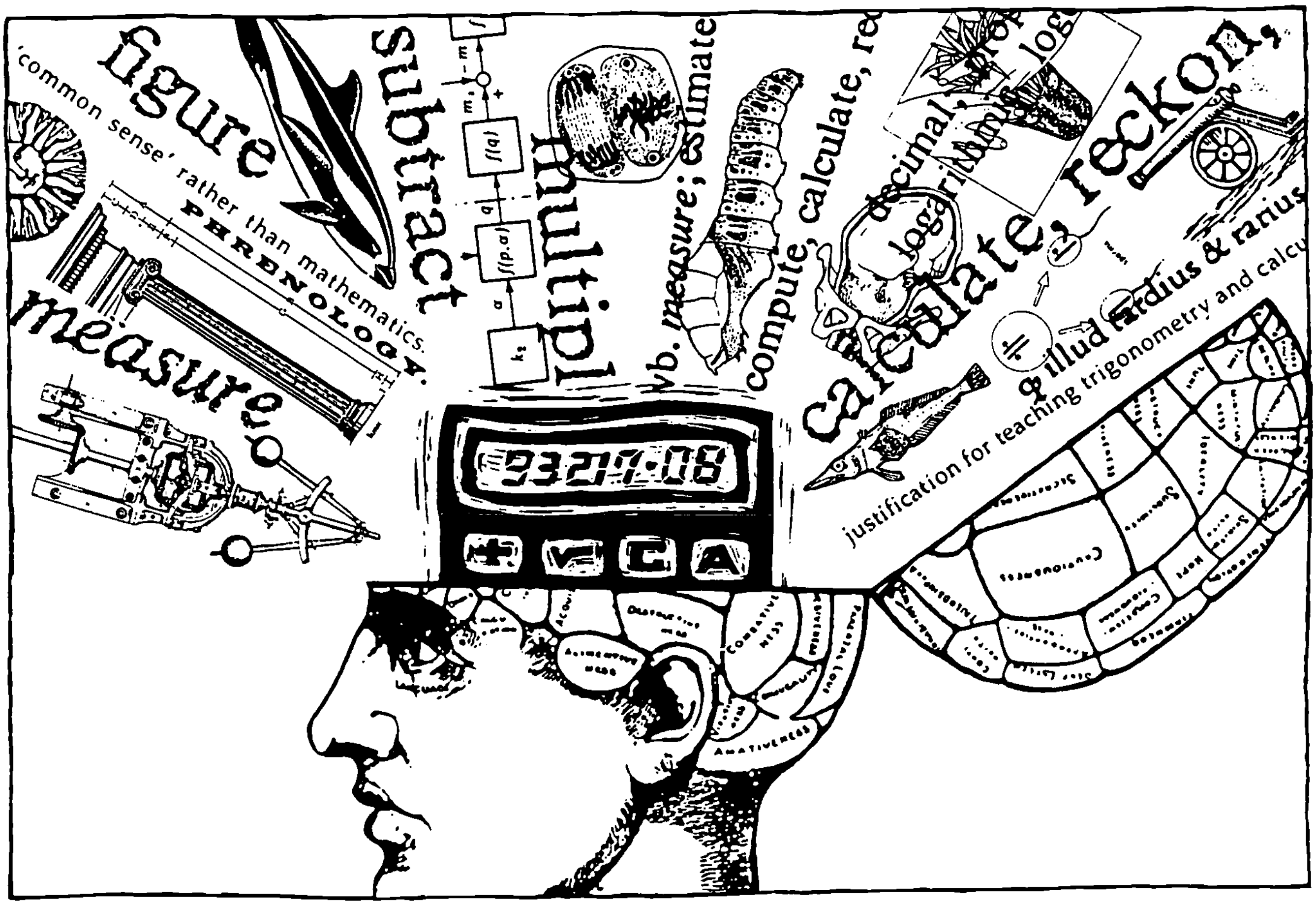
The age of the children may have a large bearing on whether they should work together or separately. Dr Martin Hughes of Exeter University found that 7-year-old girls did as well as the boys in LOGO Turtle-robot maths when paired with a boy, but poorly otherwise. See

'Single-sex teaching theory turned turtle'. *Times Educational Supplement*, 31.7.87, p. 8.

The Mathematical Needs of School Leavers

Gordon Knight, Greg Arnold, Michael Carter, Peter Kelly and Gillian Thornley

Massey University



Jennifer Lautusi

Introduction

What are

(a) the mathematical needs of everyday adult life;
(b) the mathematical needs of employment;
(c) the mathematical needs of those going on to further education?

We were asked to find out and to report any implications for current developments in curricula and national assessment.

Mathematics in Everyday Life

A random sample was chosen from the electoral rolls. These people were interviewed, with questions about a wide range of everyday tasks such as supermarket shopping, income tax returns, cooking, and hobbies. To avoid problems which had arisen in other similar research, every effort was made to avoid the impression that the survey was some form of mathematics test. While it would have been useful to know how *well* people are using mathematics in their everyday lives, it will come as no surprise to teachers that people are reluctant to take a mathematics test unless they have to! They are also unlikely to admit to using a mathematical technique if they know that they will be tested on it.

A further problem (apparent in other research too) is that if you ask people what mathematics they use in their lives, they interpret the word mathematics in a very narrow way, thinking of the pencil and paper methods of the classroom. They are most unlikely to consider the general quantitative and spatial judgements which we make every day in mathematics. These skills are more likely to be called 'common sense' rather than mathematics.

Consequently, the interviews were very much task-based and many of those interviewed were surprised how important mathematics was in their lives. One woman remarked:

You said you came to ask me about the mathematics I use. Now you know all about my life.

Table 1

Percentage of people who use mathematics in everyday tasks

Task	% using mathematics in the task Female	Male
Supermarket shopping	87	65
Gardening	46	74
Cooking	89	35
Knitting and sewing	78	9
Painting and wallpapering	22	84
Making curtains	39	9
Building, carpentry, etc.	7	67
Running a car	39	79
Holidays	70	74
Sport	35	58
Gambling	9	26
Hobbies	48	60
Cheque account	37	40
Tax return	41	63
Mortgage, investments, etc.	26	47

The mathematics used in these tasks was, not surprisingly, mostly elementary in character. Arithmetic, measurement and simple geometric concepts were predominant. The important feature was the ability to use these simple mathematical tools appropriately in a variety of situations. The ability to make sensible quantitative judgements based on mental arithmetic and estimation was at least as important as the ability to carry out pencil and paper calculations.

Calculators were widely used in those tasks, such as filling in an income tax form, in which an exact answer was required. 86 percent of people under 45 used a calculator. As might be expected, only 46 percent of those over 45 used one.

A significant feature of many of the tasks was that they involved solving problems. In supermarket shopping, for example, people have to decide whether or not they need to know the total cost of the groceries in their trolley before they reach the checkout, and if so how accurate this calculation needs to be. They will also make 'best buy' calculations, choosing between products and sizes. There was a great variety in the methods used. One woman, for example, rejected the usual comparisons based on weight or volume and price in favour of a 'meal size' price comparison. The best buy for her was the cheapest pack of mince which would suffice for one meal. The fact that a larger pack had a lower price per kilogram was unimportant if it would not all be eaten, or some would be thrown away, after one meal. Similarly a large, low unit-price, packet of biscuits might be uneconomic since, 'they will eat them all at one sitting anyway'.

Similar problem-solving situations were apparent in cooking, gardening, wallpapering, and making curtains. If schools are to prepare students for making these kinds of judgements, perhaps, instead of teaching specific procedures to solve problems. For example, to find the number of rolls of wallpaper needed to decorate a room we should be asking:

> How many ways can you think of to solve this problem? Discuss the advantages and disadvantages of each. Under what circumstances might one method be better than another?

There are obvious opportunities for group work in this approach.

Mathematics in the Workplace

For the workplace survey a variety of businesses were selected. The sample included small operations such as a hairdresser, a farmer and a motel operator; larger businesses such as a department store, a fibreglass product manufacturer and an insurance office; and major enterprises such as a coal mine, a car assembly plant, a government research establishment and a large financial corporation. The sample was checked against a Standard New Zealand Industrial Classification to make sure it was representative.

Within these workplaces individuals were chosen (for interview) to give a wide range of occupations. These included cleaner, truckdriver, manager, pilot, salesperson, mechanic, estate agent, matron, miner, graphic artist, reporter, market researcher, cook, labourer, etc.

Again we held interviews concentrating on *tasks* which might involve mathematics and then looking more specifically at the mathematics *skills* used in these tasks.

The task questions were of the type: "Have you ever in the course of your present job in this Company needed to...?"

As an example, in response to a question asking about data collection, 65 percent of the respondents said that they did collect numerical information in the course of their job. Examples included occupancy rates in a rest home, petrol pump readings, flight log-book information, financial records, spraying schedules, the results of flow testing of hydrants, container utilisation, and sales records.

Some of the results are presented in Table 2.

Table 2

Percentage of people who use mathematics tasks in the workplace

Task	% of workforce Female	Male
Count things?	98	95
Estimate sizes or quantities?	77	91
Measure anything?	59	81
Collect data?	63	66
Do any sampling?	13	39
Use a formula to calculate something?	49	66
Optimise anything?	69	91
Receive or pay out money?	74	43
Do any bookkeeping?	40	39
Use a calculator?	85	88
Use a computer?	61	54

In the skills section of the interviews we concentrated on the mathematical skills which were used in the tasks. For example, in order to gather information on the informal geometry which people might use, we asked them if they ever needed to position or arrange objects or shapes either to pack them into a small space, to make a functional layout, or to make an attractive pattern. 68 percent replied that they did.

Some responses to questions of the type: 'In the course of your job do you...?' are presented in Table 3.

Table 3
Percentage of people using specific mathematical skills in the workplace

Skill	% of workforce Female	Male
Summarize data	51	52
Organize data	67	67
Extract information from data	63	69
Make decisions on incomplete information	54	67
Try to determine a 'best strategy' to follow	74	85
Use symbols or letters to represent numbers	17	40
Use geometric concepts	56	76
Use trig functions	1	21
Use calculus	2	8
Use arithmetic	94	95

One of the more surprising findings is the large proportion (84 percent) of people who indicated that in the course of their job they needed to optimise something, that is, to make the best use of available resources. The techniques they used ranged from very simple ones based on experience to sophisticated computer modelling.

29 percent used informal methods based on experience to switch things around until they were satisfied with the performance. 12 percent were trying to make the best use of time, either their own or that of their staff. A further 35 percent were making decisions where costs and quantities were involved. Examples included optimising the use of two storage tanks, scheduling labour, organising routes for deliveries, minimising the cost of constructing doors, inventory control, and the application of fertilizer.

Another 6 percent used more sophisticated mathematical optimisation techniques, often using computers. For example, a government department used the critical path method in planning, and a large delivery firm used a computer simulation model to minimise the costs to customers, to optimise the use of manpower and vehicles, and to maximise profit.

Mathematics in Further Education

Questionnaires, not interviews were used. Teachers of courses in polytechnics, colleges of education and universities were asked what mathematical topics were needed by students entering their courses. The sample was not random: humanities courses were specifically excluded but otherwise an attempt was made to cover the range of programmes currently being offered in the tertiary sector.

The data collected were analysed both in relation to fields of study such as Teacher Training, Commerce, Science, etc., and according to the likely entry levels of students.

The analysis indicated that most students entering tertiary education courses (other than in the humanities) need:

- to be computer literate and to have good calculator skills;
- to have a thorough understanding of percentages, ratio and proportion;
- to be confident in making a variety of measurements and in calculating areas, and volumes;
- to be able to estimate quantities and to estimate the associated errors;
- to be able to use the algebra associated with the use of formulae, the changing of simple word statements into symbols and equations, and the solution of simple (linear) equations.

Some courses from all the fields of study (except service trades) also made use of:

- geometric skills
- trigonometry
- exponential functions
- vectors.

Mathematical Topics in the Curriculum

Table 4 gives a summary of the survey estimates of the use of mathematical topics in everyday life, in employment and in further education. The topics are ordered roughly from most used to least used.

Table 4
Percentage of people who use mathematics

Topic	% of population using the topic in everyday life	% of workforce using the topic in employment	% of surveyed further education courses using the topic
Arithmetic	96	100	98
Use of calculators	55	87	87
Measurement	84	82	45
Statistics	7	83	89
Geometry	71	68	37
Algebra-use of formulae	5	60	76
Use of computers	22	56	51
Algebra-solving equations etc.	1	20	59
Trigonometry-triangles etc.	1	11	27
Trigonometry-formulae	0	7	23
Calculus	1	6	18

The table indicates that some topics are likely to be used by more than half the students in their employment. These are arithmetic, calculators, measurement, statistics, geometry, algebraic formulae and computers. These, plus algebraic equations, (but not geometry) are needed in more than half the further education courses. Consequently, *utility* justifies teaching these topics to all secondary school students.

The justification for teaching trigonometry and calculus is less easy. It has to depend on a deeper consideration of the question of why we teach mathematics. Are there reasons other than pure utility? If so, what are these reasons, and are trigonometry and calculus the best topics for achieving these other objectives?

Problem Solving

In addition to the technical mathematical skills associated with the topics identified in table 4, the survey very clearly revealed the need for more generic problem-solving skills.

The role of problem-solving was particularly evident in the employment survey. It was estimated that of the workforce:

81% are involved in procedures which try to determine a 'best strategy' to follow;

84% have needed to optimize something in the course of their current job;

63% need to make decisions based on incomplete information;

67% extract information from data;

62% position or arrange objects or shapes to make a functional layout;

71% need some mathematics to read reports, articles, journals or research.

None of these activities is 'technique oriented'. For example, it would be possible to excel at the routine skills of adding fractions, solving quadratic equations, using the cosine rule, and differentiating using the product rule, but have very little skill in the process tasks above. This was also true in the everyday life survey where a number of situations called for judgement rather than the application of an established mathematical procedure.

Mathematics in the New Zealand Curriculum

This research was carried out in New Zealand and therefore in the full report the implications for the New Zealand curriculm and New Zealand national assessment are drawn out. The new mathematics curriculum from classes J1 to F7 which is being introduced in 1993 and 1994 includes some important changes in emphasis. These are:

(i) a strong emphasis on problem solving, with a specific 'strand' on mathematical processes;

(ii) an emphasis on statistics (which has its own "strand" from Level 1 to Level 8)

(iii) the recognition of the importance of calculators and computers.

The results of our survey give very strong support for these changes in emphasis.

Mathematics in Australian Curricula

Although this survey was carried out in New Zealand there are implications for Australian curricula, the mathematical needs of industry, adults, and children in both countries not being dissimilar.

The Status of Topics in the Curriculum

The survey produced clear evidence that the single most important mathematical ability required by people both in everyday life and in employment is the ability to make sensible quantitative judgements in problem-solving situations. These judgements are most likely to involve estimation and to be based on mental arithmetic.

It seems that such skills are undervalued in an educational system in which status is given, almost entirely, to topics which occur in examination prescriptions and papers. In New Zealand School Certificate and Bursary examinations dominate the secondary school curriculum. Their emphases are on algebraic, trigonometric, formal geometry and calculus skills. The survey shows these will be used by relatively few students.

Some schools provide courses which are not geared to these examinations, but they tend to be dismissed by students, parents and sometimes teachers, as 'vege maths'. Consideration needs to be given to ways in which these essential skills can be given status within the system. Students should have the opportunity to prove to themselves, to their parents, and to future employers that they have mastered the most important skills for functioning effectively at home and at work.

Gender Issues

There were very clear gender differences in the use of mathematics both in everyday life and in the workplace. Most of these differences seem to reflect the fact that (in New Zealand society at least) men and women tend to undertake different tasks in the home, to have different hobbies, and are employed in different sections of the workforce.

There was some suggestion, however, in the data, that men and women engaged in the same task involving mathematics, tend to approach it differently. This possibility will be the focus of further research.

Notes

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This research was carried out for the New Zealand Ministry of Education, in 1992. The full report is:

Knight, G., Arnold, G., Carter, M., Kelly, P. and Thornley, G. (1992) *The Mathematical Needs of New Zealand School Leavers: A Research Report*, Palmerston North: Massey University.

The sampling procedure for the workplace survey was designed to select a cluster of occupations selected from a random sample within strata defined by the NZ Standard Industrial Classification.

New Zealand curriculum information can be found in Ministry of Education (1992) *Mathematics in the New Zealand Curriculum*, Wellington: Ministry of Education.

How I Failed

by an ex-student of Mathematics

I suppose that some people get pleasure from the mere manipulation of algebraic equations and the integration of standard functions, but I don't think I am one of them: I can tell you I was delighted when I saw Question 8 in the Scholarship paper.

Question 8

An isolated island nation in the South Pacific decides to claim exclusive fishing rights in the zone within x miles of its coast. The island is triangular and has perimeter p miles.

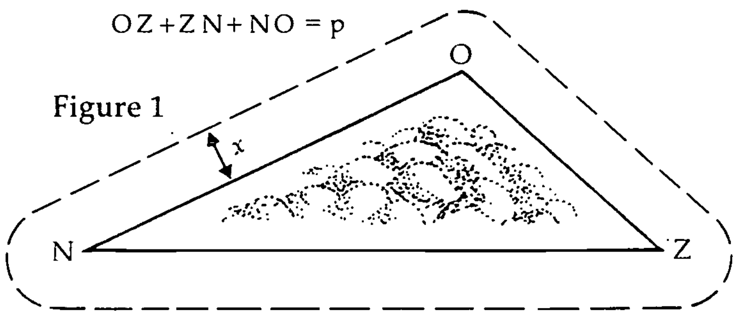


Figure 1

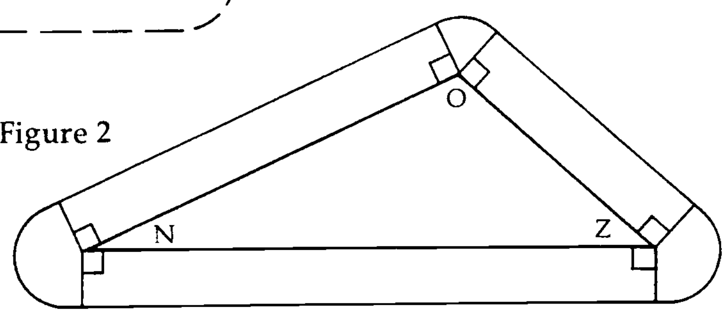


Figure 2

(a) Let A be the area of ocean with the fishing zone and let L be the outer perimeter of the new zone.

Show that $A = \pi x^2 + px$ and $L = p + 2\pi x$.

(b) The Ministry of Agriculture and Fisheries claims that the annual income from fishing the new zone will be \$13p/square mile. But the Ministry of Defence says that the annual cost of policing the zone will be \$8L/square mile.

For what range of values of x will the zone be profitable? What value of x maximizes the net annual profit? (Give your answers in terms of p).

I spent most of the fifteen minutes reading time studying this question and I moved right in on it once we had permission to begin.

It was clear how to begin: I drew the figure which is reproduced here as Figure 2. Being an island nation I supposed that, to within a very small error, the perimeter of the fishing ground was

$$p + (\pi - N) + (\pi - Z) + (\pi - O) = p + 3\pi - (N+Z+O)$$

I only had to calculate the sum N+Z+O, and that didn't seem too bad, given that the triangle was on the surface of a sphere.

Looking at the answer, I saw that I could anticipate that

$$N+Z+O = \pi$$

and that is where I first lost a bit of confidence. A sphere has positive Gaussian curvature and from what I could remember that would imply that all triangles on it would have angle sum greater than π! But it was only a momentary setback because my experience with scholarship questions had been that the difficulties tend to disappear in a flurry of cancellations at the last minute.

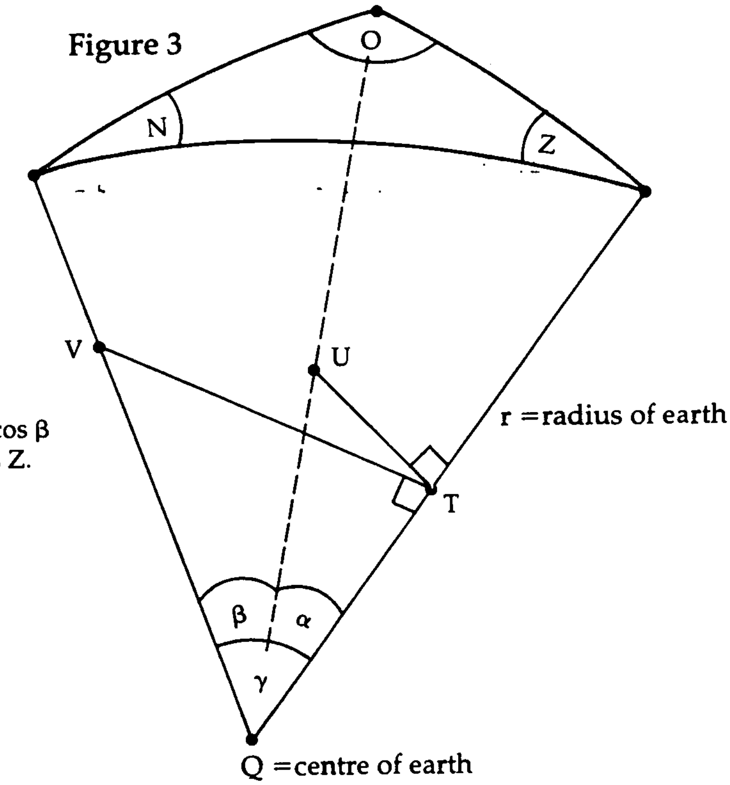
So, I set about analysing the triangle NZO on the surface of the sphere, keeping in mind that I had to calculate the sum N+Z+O. I drew what is here Figure 3.

In this drawing I first found

$$OZ + r\alpha$$
$$ON + r\beta$$
$$NZ + r\gamma$$

were r is the radius of the earth.

Figure 3



Hence

$$p = r(\alpha+\beta+\gamma).$$

Also

$$TU = QU \sin\alpha$$
$$TV = QV \sin\gamma$$
$$QU \cos\alpha = QV \cos\gamma$$
$$VU^2 = QU^2 + QV^2 - 2QU.\,QV \cos\beta$$
$$VU^2 = TU^2 + TV^2 - 2TU.\,TV \cos Z.$$

Putting all these together led to

$$\cos Z = \frac{\cos\beta - \cos\gamma\cos\alpha}{\sin\gamma\sin\alpha}$$

From symmetry it was clear then that

$$\cos O = \frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}$$

$$\cos N = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}$$

Keeping my mind on N+Z+O I thought the right thing to do might be to calculate cos (N+Z+O). I easily worked out

$$\cos(N+Z+O) = \cos N \cos Z \cos O$$
$$- \cos N \sin Z \sin O - \cos Z \sin O \sin N$$
$$- \cos O \sin N \sin Z$$

but when I began to substitute the values of these cosines and sines, my courage failed me. So I tried sin (N+Z+O), which turned out to be just as bad.

I looked at some other things, but with no success until I suddenly found that 2¾ hours had passed. With fear in my heart I spent the remaining fifteen minutes dashing off an answer to Question 1, but that was all I managed for the whole exam.

In the event, I ended up getting 25 percent. As this was more than full marks for the two questions I attempted, I was rather surprised, but maybe there was a lot of scaling.

Anyway, I am happy in the job that the Labour Department has found me, helping old people with their gardens, and I often find a bit of spare time to puzzle over this question. I still can't satisfy myself that a triangle on the surface of a sphere can have an angle sum of π, but I suppose there is something about it that I still haven't spotted.

Notes

This item first appeared as an article in The New Zealand Mathematics Magazine, Vol.18, No.1, 1981, in which it was called "How I Failed the Universities Entrance Scholarship Examination in Pure Mathematics in 1978." Permission to re-print was granted by the Editor, to whom our thanks.

The New Zealand University Scholarship exam is still held, though the University Entrance exam has been superceeded. Scholarship is sat by the students who schools regard as their top Form 7 (Year 12) scholars, and if passed with sufficient marks results in a small suppliment to any other grants.

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