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ABSTRACT

The New Jersey Mathematics Curriculum Framework is based on the "Mathematics Standards" adopted by the New Jersey State Board of Education on May 1, 1996, and is intended to provide guidance to teachers, administrators, and districts that will help them translate a vision of exemplary mathematics education into reality. It includes information and resources for teachers at all grade levels and for school and district administrative personnel. It illustrates how each of the standards can be addressed at all grade levels, and provides information and guidance on the major issues that need to be addressed, on the process of systemic change, and on the interrelated areas of content, instruction, and assessment. The first chapter presents the vision for mathematics education in New Jersey and the standards that articulate that vision. The next chapter, "The First Four Standards," discusses processes of problem solving, reasoning, communicating mathematics, and mathematical connections that should underlie all student learning. Each subsequent standard has its own chapter that begins with a K-12 overview of the standard and continues with grade-level discussions that include grade-level overviews of the content standard followed by sample activities. There are two planning chapters which provide guidance on strengthening a district's technology component and focuses on the process of bringing about change. (JRH)

New Jersey Mathematics Curriculum Framework

Curriculum

A Collaborative Effort of the New Jersey Mathematics Coalition and the New Jersey Department of Education

Framework

New Jersey

Mathematics

Curriculum Framework

Joseph G. Rosenstein, Janet H. Caldwell, Warren D. Crown

(with contributions from many other New Jersey educators)

December 1996

A Collaborative Effort of the

New Jersey Mathematics Coalition

and the

New Jersey Department of Education

PREFACE

The *New Jersey Mathematics Curriculum Framework* is based on the *Mathematics Standards* adopted by the New Jersey State Board of Education on May 1, 1996. The *Mathematics Standards* are part of the *Core Curriculum Content Standards,* developed by the New Jersey State Department of Education, including standards in seven content areas and cross-content workplace readiness standards. Taken together, the *Core Curriculum Content Standards* describe what every New Jersey student needs to understand and be able to do at the completion of the 4th, 8th, and 12th grade.

PURPOSE: New Jersey's *Mathematics Standards* describe a vision of mathematics teaching and learning which involves high expectations for *all* students, and insists that *all* students can achieve these expectations.

The *New Jersey Mathematics Curriculum Framework* provides information and guidance to teachers and districts on how to help make that vision a reality.

Achieving this vision is an ambitious, long-term undertaking; there is no simple path to the goal. Achieving this vision will take time, effort, and a commitment to change. The recommendations of the *Mathematics Standards* cannot be implemented overnight, and results will not appear overnight. Changes will be needed in all areas — in curriculum, instruction, assessment — and will involve rethinking school practices and extensive professional development. The changes will require the commitment of teachers, administrators, school boards, parents, and policy makers, and the effort of the entire New Jersey community.

The *New Jersey Mathematics Curriculum Framework* is intended to be a resource, providing practical guidance to implement the *Mathematics Standards*. It includes information and resources for teachers at all grade levels and for school and district administrative personnel. Each chapter contains much information, and can serve as a basis for extended discussions involving teachers and administrators.

The *New Jersey Mathematics Curriculum Framework* is not intended to be read straight through. It is intended to be user-friendly; but to achieve that purpose, the user also has to be friendly, warming up to its contents a little at a time, and not shying away from it because of its bulk.

New Jersey's *Mathematics Standards* and the *New Jersey Mathematics Curriculum Framework* are directed toward one crucial goal:

GOAL: To enable all of New Jersey's children to move into the twenty-first century with the mathematical skills, understandings, and attitudes that they will need to be successful in their careers and daily lives.

The *Mathematics Standards* are based on the twin premises that all students can learn mathematics and that all students need to learn mathematics. They set high achievable expectations for all students, and call for teachers and parents to help all students strive toward and achieve those standards.

New Jersey's *Mathematics Standards* and the *New Jersey Mathematics Curriculum Framework* call for major changes, both in terms of what mathematics will be taught, and in how it will be taught. The recommendations provided here are very specific. Yet, it is not intended that they be implemented dogmatically; different situations call for different responses and different strategies. In education, as in other areas, there is a tendency to swing from one extreme to another. We hope that educators will utilize their common sense, judgment, and experience in finding appropriate ways of using the recommendations in this *Framework* to inform their decision-making. We expect that this *Framework* will be a major resource to teachers seeking to implement the *Mathematics Standards* in the classroom; we also expect it to be valuable to districts which are seeking to introduce mathematics curricula based on the *Mathematics Standards* and to provide professional development to their teachers based on the *Mathematics Standards*.

The publication of this document is the culmination of the New Jersey Mathematics Curriculum Framework Project, a collaborative effort of the New Jersey Mathematics Coalition and the New Jersey Department of Education, which was funded by an Eisenhower grant from the United States Department of Education. This effort is also a component of New Jersey's Statewide Systemic Initiative to Improve Mathematics, Science, and Technology Education. The *Framework* and the *Mathematics Standards* build on the Standards published by the National Council of Teachers of Mathematics in 1989 and 1991.

A *Preliminary Version* of the *New Jersey Mathematics Curriculum Framework* was published in January 1995. That *Preliminary Version* reflected the efforts of hundreds of New Jersey mathematics educators who worked together during 1993 and 1994 to develop materials that would be appropriate for a world-class mathematics curriculum framework. During the last two years, the *Preliminary Version* has been reviewed and used by many teachers, schools, and districts throughout the state. This new version of the *New Jersey Mathematics Curriculum Framework* reflects their comments and suggestions, and follows the organization of the *Mathematics Standards* adopted by the New Jersey State Board of Education.

Though published, the *New Jersey Mathematics Curriculum Framework* is not completed. We anticipate that it will continue as a living document on the Web site of the New Jersey Mathematics Coalition, where it is available at http://dimacs.rutgers.edu/nj_math_coalition/framework.html/. We hope to post additional resources relating to the *Mathematics Standards*, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

The efforts of all those who have contributed to the development of the *New Jersey Mathematics Curriculum Framework* are acknowledged below. This has been truly a state-wide effort of which we can all be proud. Let us all continue to work together to make the vision of New Jersey's *Mathematics Standards* a reality in the coming years!

For further information, please call the New Jersey Mathematics Coalition at 908/445-2894 or contact the New Jersey State Department of Education, Office of Standards and Assessment, CN 500, Trenton, NJ 08625-0500. We welcome your comments on the *Framework* and your suggestions about its future; please send them to joer@dimacs.rutgers.edu or the New Jersey Mathematics Coalition, P.O. Box 10867, New Brunswick, NJ 08906.

Joseph G. Rosenstein
Editor, New Jersey Mathematics Curriculum Framework
Co-Director, New Jersey Mathematics Curriculum Framework Project
Professor of Mathematics, Rutgers University
Director, New Jersey Mathematics Coalition
December 9, 1996

ACKNOWLEDGEMENTS

The development of the *Preliminary Version* of the *New Jersey Mathematics Curriculum Framework* was a broad-based effort. As evidence, I submit the following section (pages v-xi) which appeared in the *Preliminary Version* acknowledging all those who had roles in the development of that version of the *Framework*. They should be pleased to see that all their efforts have now come to fruition. Many thanks to all of those who played a role in developing the *Framework*!

By contrast, this version was the result of an intense effort by a small number of people. Warren D. Crown, Professor of Mathematics Education at Rutgers, Janet H. Caldwell, Professor of Mathematics at Rowan College of New Jersey, and Joseph G. Rosenstein, Professor of Mathematics at Rutgers University wrote this entire document, based on the materials in the *Preliminary Version* and responding to the comments and suggestions offered by all those who used and reviewed that document.

Assisting in the writing process was Karin Rupp who collected and organized all of the comments and suggestions, and developed additional information that grew out of those comments and suggestions. Also assisting in the writing process were those who carefully read various chapters of the *Preliminary Version*, and recommended many changes. This includes Robert Davis, Frank Gardella, Evan Maletsky, and Maureen Quirk. Especially important was the contribution of those who reviewed the chapter dealing with Discrete Mathematics, a topic which has never before been subjected to a K-12 grade-level analysis; this includes Valerie DeBellis, Emily Dann, Bobbie Goldman, Janice Kowalczyk, Evan Maletsky, Claire Passantino, and Michael Saks, as well as the many teachers in the Leadership Program in Discrete Mathematics who shared their classroom experiences with these topics.

Before this version of the *New Jersey Mathematics Curriculum Framework* could be written, the *Mathematics Standards* had to be adopted by the New Jersey State Board of Education. So acknowledgements are appropriate here to those who served on the Governor's Review Panel for the Mathematics Curriculum Standards — Janet Amenhauser, Joyce Baynes, Janet Caldwell, Warren Crown, Barbara Graham, Patricia Klag, Paul Lawrence, Evan Maletsky, Paula Norwood (Panel co-Chair), Jean Paige, Robert Riehs (Department of Education), Joseph G. Rosenstein, William Smith (Panel co-Chair), Dorothy Varygiannes (Department of Education), and Allen Wesley.

The editing of the *New Jersey Mathematics Curriculum Framework* was the work of Joseph G. Rosenstein, with the dedicated assistance of Karin Rupp. Both have read each word of this document ... over and over. Meeting the deadline imposed by the Eisenhower grant period — this document had to be printed by December 31, 1996 — involved, simply put, many long days and nights.

The document was prepared by the staff at the Center for Mathematics, Science, and Technology Education at Rutgers University, including the staff of the New Jersey Mathematics Coalition. Most of the document was put into its final form by Stephanie Micale and Debby Toti who have cheerfully put up with the *Framework*, and its Editor, full-time for the last three months. Chris Magarelli did all the computer graphics, and others provided important assistance when it was needed — Janet DeBellis, Valerie DeBellis, Lisa Estler, Bonnie Katz, Stephanie Lichtman, and Peter Sobel. There were times when six people were working on the document simultaneously! Thank you for all your help.

Thanks also to Dolores Keezer of the Department of Education, who has served as co-Chair of this project and has helped ensure the dissemination of this document, and to Robert Riehs of the Department of Education who has helped ensure that the contents of this document reflected the initiatives of the Department. Thanks also to Stephen Bouikidis for providing expert advice on the format of this document and to him and Keith Kershner for ensuring that its cover would do justice to its contents. In addition to their assistance, the Mid-Atlantic Eisenhower Consortium for Mathematics and Science Education at Research for Better Schools also provided a graphic artist for the cover and contributed to the distribution of the *Framework*. Thanks also to Rutgers University and the Center for Mathematics, Science, and Computer Education for serving as a home for this project, and to its Director, Gerald A. Goldin, for his consistent support.

Finally, thanks to my family for their patience during the last few months while I was intensely wrapped up in the writing and editing of the *New Jersey Mathematics Curriculum Framework*.

Joseph G. Rosenstein
Editor, New Jersey Mathematics Curriculum Framework
Co-Director, New Jersey Mathematics Curriculum Framework Project
December 9, 1996

ACKNOWLEDGEMENTS

(from *Preliminary Version* — 1995)

The development of the *Preliminary Version* of the *New Jersey Mathematics Curriculum Framework* was made possible by a grant from the United States Department of Education to the New Jersey Department of Education and the New Jersey Mathematics Coalition. Joseph G. Rosenstein has served as co-Director of the New Jersey Mathematics Curriculum Framework Project for the New Jersey Mathematics Coalition; serving as co-Directors for the New Jersey Department of Education have been Charles Mitchel, Karen Sanderson, and Dolores Keezer.

In addition to a collaboration between the New Jersey Mathematics Coalition and the New Jersey Department of Education, this document represents a collaboration involving many of the most knowledgeable mathematics educators in New Jersey and many other members of the community. Some served on the New Jersey Curriculum Standards Panel that developed the draft version of the *New Jersey Mathematics Standards*. Others were active members of the Curriculum Framework Project Advisory Committee of the New Jersey Mathematics Coalition. Still others served as members of Task Forces which developed recommendations and drafted materials for the framework. Lists of members of these groups are provided on the following pages[1]; please bring corrections or omissions to our attention so that modifications can be made in subsequent versions.

We also acknowledge the 294 individuals who submitted comments on the draft version of the *New Jersey Mathematics Standards* and the 29 District Leadership Teams (DLTs) who reviewed an earlier version of the framework as part of their participation in the pilot implementation program of the New Jersey Mathematics Curriculum Framework Project. Serving as Project Coordinator of the pilot implementation program have been Irwin Ozer and Karin Rupp.

Special mention must be made of the following individuals who have served the Leadership Team in various capacities, attending long and arduous planning meetings, chairing Task Forces, and writing, reviewing, and editing endless drafts of sections of this document.

Janet Caldwell, Professor of Mathematics, Rowan College of New Jersey
Warren Crown, Associate Professor of Mathematics Education, Rutgers University
Karl-Heinz Haas, Mathematics and Science Supervisor, Saddle Brook Public Schools
Dolores Keezer, Director, Academy - Central, New Jersey Department of Education
Phyllis Klosowski, Project Coordinator, New Jersey Mathematics Coalition; Rutgers
Paul Lawrence, Mathematics Supervisor, Livingston Public Schools
Rebecca Lubetkin, Director, Consortium for Educational Equity, Rutgers University
David Marain, Mathematics Staff Developer, Tenafly Public Schools
Elizabeth Marquez, Teacher, North Brunswick Township Schools
Bruce Normandia, Superintendent, Brick Township Public Schools
Anthony Piccolino, Associate Professor of Mathematics, Montclair State University

[1] Members of the New Jersey Mathematics Curriculum Standards Panel are listed on page vii, of the Curriculum Framework Project Advisory Committee on page v, and of the various Task Forces on pages viii-ix. A list of the 60 districts comprising the District Leadership Teams (DLTs) and the DLT Coordinators appears on pages x-xi.

Joseph G. Rosenstein, Professor of Mathematics, Rutgers; Director, NJ Mathematics Coalition
Maureen Quirk, Center for Communications Research
Karen Sanderson, Assistant Principal, Hamilton (Atlantic County) Public Schools
Roberta Schorr, Mathematics Specialist, New Jersey Statewide Systemic Initiative (NJ SSI)
Sharon Sherman, Senior Project Leader, Science Educ. Program, Princeton Plasma Physics Labs
William Smith, Mathematics Supervisor, Haddonfield Public Schools
Dorothy Varygiannes, New Jersey Department of Education
Linnea Weiland, New Jersey Department of Education
Larry Wiley, Mathematics Consultant, (formerly) New Jersey Department of Education

Special thanks to Phyllis Klosowski who has served as Project Coordinator of this project since its inception, and has been a source of information and assistance to the Leadership Team and to Mary Ann Reilly, who developed the initial draft of the framework upon which this *Preliminary Version* is based. Thanks also to the Mid-Atlantic Eisenhower Consortium for Mathematics and Science Education located at Research for Better Schools for providing refreshments for those reviewing framework drafts.

That this document now exists is a tribute to a team of five people each of whom has spent hundreds of hours converting an idea into a reality:

Karen Sanderson, currently Assistant Principal, Hamilton (Atlantic County) Public Schools, who served for nearly two years as Project co-Director for the New Jersey Department of Education;

William Smith, Mathematics Supervisor, Haddonfield Public Schools, who served as Facilitator of the *New Jersey Mathematics Standards* Panel;

Janet Caldwell, Professor of Mathematics, Rowan College of New Jersey, and
Warren Crown, Associate Professor of Mathematics Education, Rutgers University, and Associate Director, New Jersey Mathematics Coalition

who were principal authors of the *Preliminary Version* of the *New Jersey Mathematics Curriculum Framework* with Joseph G. Rosenstein, writing, rewriting, and weaving together the contributions of many others; and

Joseph G. Rosenstein, Professor of Mathematics, Rutgers University, and Director, New Jersey Mathematics Coalition

who served as Project co-Director, and headed the team and managed the effort to create this document.

Congratulations and thanks to all who have participated in developing the *Preliminary Version* of the *New Jersey Mathematics Curriculum Framework*.

New Jersey Mathematics Standards Panel

Ray Allen	Atlantic City, Teacher
Janet Y. Amenhauser	East Windsor, Teacher
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Irene Benn	Toms River, Special Education
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Donald Dearborn	Ocean City, Superintendent
Ann Faubl	Sterling Regional, Teacher
Marjorie D. Ford	Hoffmann-LaRoche, Manager of Community Affairs
Toni Fultz-Fauntleroy	Galloway Township, Teacher
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Elizabeth Marquez	North Brunswick Township, Teacher
Bruce Normandia	Brick Township, Assistant Superintendent
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Laurence Wiley	New Jersey Department of Education, Mathematics Supervisor

(affiliations as of completion of the work of the Panel in June 1993)

Curriculum Framework Project Advisory Committee

New Jersey Mathematics Coalition

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Darnell Davis	Teacher, University High School, Newark Public Schools
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Susan Simon	Mathematics Teacher, Morristown Beard School
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Task Force Members

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Task Force Members (continued)

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Arlene Riegel	Teacher, Scotch Plains-Fanwood High School
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Gloria Sanok	Teacher, Wayne Schools
Roberta Schorr	New Jersey Statewide Systemic Initiative
William Smith	Mathematics Supervisor, Haddonfield Memorial High School
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Patricia Wang-Iverson	Research for Better Schools
Linnea Weiland	New Jersey Department of Education
Larry Wiley	New Jersey Department of Education
Joseph Zahn	New Jersey Department of Education

District Leadership Teams (DLTs) and DLT Coordinators

New Jersey Mathematics Curriculum Framework Project

Regional Leadership Team I

Atlantic City DLT	Ray Allen
Mainland Regional DLT (including Linwood, Northfield, Somers Point)	Daniel Kortvelesy
Ocean City DLT (including Sea Isle, Upper Township)	Sherry Bosch
Stone Harbor DLT (including Middle Township, Dennis Township, Avalon)	Lynn Mathis, Ronald Bonner, Tom Cushane

Regional Leadership Team II

Haddonfield DLT	William Smith
Bridgeton DLT	Rita Sherman
Pitman DLT	Barbara Michalsky
Salem City DLT (including Quinton)	Esther Lee
Pittsgrove Township DLT (including Elmer)	Carol Taylor Winkie, Jacqueline Murphy

Regional Leadership Team III

Florence Township DLT	Dana Paykos
Northern Burlington County Regional DLT (including Chesterfield, Mansfield, North Hanover, Springfield)	Ellen Mulligan
Trenton DLT	Mary Mitchell
Metuchen DLT	Joseph Bulman
South Brunswick DLT	Kaye L. Crown

(continued)

District Leadership Teams (DLTs) and DLT Coordinators (continued)

New Jersey Mathematics Curriculum Framework Project

Regional Leadership Team IV

Pinelands Regional DLT (including Eagleswood Township, Little Egg Harbor Township, Tuckerton Elementary, Bass River)	Harry Bublin
Southern Regional DLT (including Stafford Township, Long Beach Island Township)	Deborah Mayeux
Eatontown DLT (including Monmouth Regional)	Diane Bloom
Ocean Township DLT	Leona Worth
Rumson-Fairhaven Regional DLT (inlcuding Rumson, Fairhaven)	Louis W. Mitchell

Regional Leadership Team V

Midland Park High School DLT (including Ho-Ho-Kus)	Patricia A. Dunn, Victoria Kniewel
East Orange DLT	Howard Walker, James Leutz
Dover DLT (including Mine Hill)	Gary Meyer
Jefferson Township DLT	Mary Ann Tierney
Sparta Township DLT	Pat Spagnoletti, Karl Mundi

Regional Leadership Team VI

Union City DLT	Silvia Abbato
Washington Township DLT (including West Morris)	Dolly Cinquino
Franklin Township DLT	Jane Taglietta
Bernardsville DLT	Louis Rodriguez
Linden DLT	Walter Tylicki

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INTRODUCTION TO THE *FRAMEWORK*

Overview

The publication of the *New Jersey Mathematics Curriculum Framework* is the culmination of the New Jersey Mathematics Curriculum Framework Project, a four-year collaborative effort of the New Jersey Mathematics Coalition and the New Jersey Department of Education, which was funded by an Eisenhower grant from the United States Department of Education. This effort is also a component of New Jersey's Statewide Systemic Initiative for Improving Mathematics, Science, and Technology Education.

The purpose of the *New Jersey Mathematics Curriculum Framework* is to provide a guide to individual New Jersey teachers, administrators, and districts that will help them translate a vision of exemplary mathematics education into reality. It is anticipated also that the *New Jersey Mathematics Curriculum Framework* will serve as a model for other states.

The *New Jersey Mathematics Curriculum Framework* intends to help educators base their district's mathematics curriculum on the recommendations of New Jersey's *Mathematics Standards*. It illustrates how each of the standards can be addressed at all grade levels, and provides information and guidance on the major issues that need to be addressed, on the process of systemic change, and on the inter-related areas of content, instruction, and assessment.

Defining *Standards* and *Framework*

We may think of *standards* as expressing our common goals — first in terms of a vision, and then in terms of clear statements (called *standards*) of what we want to accomplish. A useful metaphor is that of a road map, where the goal is simply a common destination. You *do* have to know where you're going before you can figure out how you're going to get there. Moreover, since we are all starting at different places, we will take very different routes to arrive at our common goal.

We may think of a *framework* as an instrument to help us determine which route to use, how to structure our efforts, in order to achieve our goal. A useful metaphor is that of a skeletal structure. The framework is not a completed building. It is, however, the scaffolding that provides initial support, definition, and direction to our efforts to achieve our goal.

New Jersey's *Mathematics Standards* are intended to describe our goals; the *New Jersey Mathematics Curriculum Framework* is intended to help us achieve those goals. It is intended to provide policy-makers, instructional leaders, teachers, and community members with the support, definition, and direction necessary to re-envision and reconstruct mathematics education here in New Jersey and across the United States. The *New Jersey Mathematics Curriculum Framework* is not a finished product — it is not a curriculum; it does however provide the support necessary for educators who wish to generate and implement a new vision of how mathematics can be taught and learned in their schools.

... All Students

The vision that is presented in New Jersey's *Mathematics Standards* and the *New Jersey Mathematics*

Curriculum Framework is articulated in high standards which are indeed achievable by all New Jersey students. All students *need* to achieve these standards if they are to be productive in the 21st century; all students *can* achieve these standards if we create environments in which learning is both possible and expected. There may be exceptions, but these must be exceptional.

At the same time, our attention to "all students" must not diminish our dedication to providing full encouragement and opportunity to explore mathematics in greater breadth and depth to those students who have interest or talent in pursuing careers which require additional mathematical achievement.

Standards and Frameworks in a National Context

This document builds on the *Curriculum and Evaluation Standards for School Mathematics* (1989) and the *Professional Standards for Teachers of Mathematics* (1991), published by the National Council of Teachers of Mathematics.

The 1993 report from the National Governors Association to the National Education Goals Panel entitled *Promises to Keep: Creating High Standards for American Students* recommended and announced the development of national standards documents in seven other content areas. A basic theme of both the Goals 2000 legislation (March 1994) and the Improving America's Schools Act (October 1994) is the importance of developing high standards of learning for all students. A national consensus has been building around the importance of agreed-upon standards in improving the education of our country's students.

Standards and Frameworks in a New Jersey Context

A draft version of the *New Jersey Mathematics Standards* was developed by a panel of thirty-one individuals who met extensively during the 1992-1993 school year. Crafted by a broad range of New Jersey elementary, middle school, and secondary teachers, supervisors, administrators, college mathematics educators, mathematicians, and representatives of business and industry, the draft *New Jersey Mathematics Standards* was intended to provide a clear vision of exemplary mathematics learning and to define and then articulate the standards necessary for achieving quality mathematics education.

After the completion of the draft *New Jersey Mathematics Standards*, over 7000 copies of the document were distributed for review across the state. At the same time, efforts began to extend the draft *New Jersey Mathematics Standards* into a mathematics framework. The *Preliminary Version* of the *New Jersey Mathematics Curriculum Framework*, published in January 1995, contained a revised version of the standards which addressed many of the comments of both the reviewers of the draft standards and the drafters of the framework materials. As a result of this process, the standards in the *Preliminary Version* represented a statewide consensus of what mathematics educators believe are high achievable goals for all students.

During 1995, a new working group — the Governors's Review Panel for the Mathematics Curriculum Standards — built upon these draft standards and, together with similar working groups in other content areas, engaged the public in an extensive review process that resulted in modest modifications of the draft standards in mathematics. This process culminated in the adoption on May 1, 1996 by the New Jersey State Board of Education of the *Core Curriculum Content Standards*, which includes the *Mathematics Standards*, standards in six other content areas, and cross-content workplace readiness standards.

The Organization of the *New Jersey Mathematics Curriculum Framework*

The *New Jersey Mathematics Curriculum Framework* begins with a chapter entitled *New Jersey's Mathematics Standards*. This chapter presents the vision for mathematics education in New Jersey on which the *Mathematics Standards* are based. It presents the standards that articulate that vision, and it enumerates *cumulative progress indicators* that futher define and elaborate on those standards. It describes them in terms of student experiences, providing a number of vignettes that both illustrate the vision and clarify the recommendations of the standards. The *Mathematics Standards* include the sixteen *content standards* that were adopted by the New Jersey State Board of Education, and two *learning environment standards* that were developed and approved by the task forces that prepared the *Mathematics Standards* and appear in the New Jersey State Department of Education's *Core Curriculum Content Standards*; however, since they were not considered content standards, they were not presented to the New Jersey State Board of Education for adoption.

The next chapter *The First Four Standards* discusses the processes of problem solving, reasoning, communicating mathematics, and mathematical connections that should underlie all student learning; among the "connections" discussed are the connections between mathematics and science.

Each subsequent standard has its own chapter. Each of these chapters begins with a K-12 overview of the standard, and continues with grade-level discussions for each of the K-2, 3-4, 5-6, 7-8, and 9-12 grade levels; these grade-level discussions include grade-level overviews of the content standard followed by sample activities — about 1500 altogether — for achieving the expectations enumerated in New Jersey's *Mathematics Standards* at those grade levels.

The *New Jersey Mathematics Curriculum Framework* is designed so that a teacher can easily extract information about all content areas for a particular grade level, and so that a teacher or an administrator can easily extract information about a particular content area for all grade levels.

There are two additional "planning" chapters, *Implementing a Technology Plan*, which provides guidance on strengthening a district's technology component, and *Planning for Change*, which focuses on the process of bringing about change. All of the recommendations in the *Mathematics Standards* involve significant changes in how mathematics will be taught and learned. System-wide changes involve decisions and actions at all levels: at the district level, at the school level, at the department level, and in the classroom. This chapter discusses how change takes place, both in general and in the specific contexts of professional development for school personnel and aligning school and district policies with mathematics education reform. *Planning for Change* will help you understand the change process and function as a "change agent."

Using the *New Jersey Mathematics Curriculum Framework* ...

A long document like this is not written with the expectation that it will be read from cover to cover. However, it is expected that every reader will begin by reviewing this *Introduction* and the chapter *New Jersey's Mathematics Standards* which follows.

Each chapter of the *New Jersey Mathematics Curriculum Framework* can serve as a basis for extended discussions involving teachers and administrators, and readers are encouraged to form groups in their schools and districts for this purpose.

The *New Jersey Mathematics Curriculum Framework* addresses two audiences. First, it speaks to school and district personnel who intend to implement the standards comprehensively and systemically, by bringing about change in all of their classrooms. Second, it addresses teachers who are interested in implementing the standards in their own classrooms. How each of these groups might use this document is discussed in the next two sections.

... for Systemic Change

For school and district leaders, the first and last chapters of this *Framework*, those dealing with the *Mathematics Standards* and *Planning for Change*, are critical. Chapter 20 provides a model for understanding systemic change, and describes specific processes to follow in order to successfully bring about change. Key to the success of efforts designed to bring about systemic change is enlisting the involvement and support of all those affected by the change.

But what changes should be made? From the outset it must be acknowledged that "implementing the standards" cannot happen overnight, that there is no "magic bullet," that there is no one action which will transform all of our classrooms, all of our teachers, all of administrators, and all of our students, so that they all manifest the vision. Bringing about change involves a long process, with many inter-related components. Each district must choose specific areas with which to begin its efforts.

We suggest that, in addition to this chapter, you also peruse the various other chapters in the *Framework*, together with your colleagues. Try to reach a tentative agreement on which specific areas in these sections should be the focus of your attention. Those chapters can then be the subject of intensive review and discussion, and subsequently the focus of efforts to improve the mathematics curriculum.

It should be noted, however, that the success of such efforts will depend on whether sufficient attention is devoted to the issues raised in Chapter 20, *Planning for Change*. Thus, for example, decisions about where to focus a school's attention should involve all those within the school who will be involved in implementing those decisions.

... for Change in the Classroom

Teachers should begin by reviewing the chapter on the *Mathematics Standards*, and should then review the grade-level sections in each of the subsequent chapters. The information in each of these chapters is organized by grade-level. Thus, for example, a 5th grade teacher can easily review the grade-level 5-6 material for all of the content standards; this will include overviews of each of the content standards for this grade level, cumulative progress indicators regarding student performance at this grade level, as well as activities intended to help achieve the expectations for each of the standards. The 5th grade teacher should, however, also review the grade-level 3-4 material — to find out what the student is expected to bring to the 5th grade — and the grade-level 7-8 material — to find out what the student will be expected to do at the next grade level.

Summary

The *New Jersey Mathematics Curriculum Framework* presents a vision and a working guide to help educators create the changes necessary to achieve world-class mathematics programs in all New Jersey classrooms. This *Framework* is intended to serve as a vehicle for change, to generate commitment, and to encourage and facilitate the leadership necessary to transform mathematics education in the state. As the authors of the report *Everybody Counts* concluded: "The challenges are clear. The choices are before us. It is time to act." So too, we must accept the challenges, recognize the choices, and take action now.

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On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

New Jersey's *Mathematics Standards*

Two Vignettes

*Somewhere in a New Jersey **elementary** school:*

The students in Mrs. Chaplain's fifth grade class eagerly return from recess, excited by the prospect of working on another of her famous Chaplain's Challenges. Mrs. Chaplain regularly uses a Challenge with her math class, and today she has promised the children that the problem would be a great one. She believes that all of her students will be up to the Challenge and expects that it will engage them in an exploration and discussion of the relationship between area and perimeter.

> *Suppose you had 64 meters of fencing with which you were going to build a pen for your large pet dog. What are some of the different pens you could build if you used all of the fencing? Which pen would have the most play space? Which would give the most running space? What would be the best pen?* *

As the students file into the classroom, they stand, scattered around the room, reading the Challenge from the board even before finding their seats. Then they begin to ask questions about it. *What is fencing? Wouldn't the pen with the most play space be the same as the one with the most running space? What shapes are allowed?* Mrs. Chaplain answers some of these questions directly (she has brought a sample of fencing to class so she could show the students what it looks like), but, for the most part, she tells the students that they can discuss their questions in their regular working groups.

Continued in the left column on the following pages

*Somewhere in a New Jersey **high school:***

Ms. Diego's algebra class and Mr. Browning's physical science class are jointly investigating radioactive decay. The two teachers, with the support of the school administrators, have worked out a schedule that enables their classes to meet together this month to explore some of the mathematical aspects of the physical sciences. Both teachers regularly incorporate some content from the other's discipline in class activities, but this month was specially planned to be a kind of celebration of the relationship between the two areas. By the end of the month, they expect that the students will really appreciate the role that mathematics plays in the sciences, and the problems that are presented by the sciences that call for innovative mathematical solutions.

The classes are average. Nearly every student in the high school takes these two classes at some point during their stay and, over the past few years, because of exciting real-world problems like the one on which they are working this week, the classes have become two of the most popular in the school.

Monday's class begins with a presentation by Mr. Browning about the process of carbon dating. He describes the problem that archaeologists faced in the 1940s with respect to determining the age of a fossil. They knew that all living things contained a predictable amount of radioactive carbon that began to diminish as soon as the organism died. If they could measure the amount that remained in some discovered fossil and if they knew the rate

Continued in the right column on the following pages

*This problem was adapted from one that appears in the *Professional Standards for Teaching Mathematics*, National Council of Teachers of Mathematics, 1991.

In the elementary school ...

The groups begin their exploration by discussing how to organize their efforts. One of the first questions to arise is what kinds of tools would help solve the problem. The students have used a variety of materials to deal with Chaplain's Challenges, and they have often found that the groups that fashioned the best models of the problem situations were the ones that found it easiest to find solutions. Today, one group decides to use the 10-by-10 geoboards, figuring that they can quickly make a lot of different "pens" out of rubber bands if they let the space between nails equal four meters. Another group decides to get some graph paper on which to draw their pens, because that gives them a lot of flexibility. Still another group, reluctant to be limited to rectangular shapes and work spaces, thinks that the geometry construction software loaded on the computers in the back of the room would let them draw a variety of shapes and even help them measure various characteristics of the shapes. One last group, striving for realism, decides to use a loop of string sixty-four inches long. As the session progresses, the groups of students make many sample pens with whatever materials they have chosen to use. Some groups switch materials as they perceive other materials to be less restrictive than the ones they are using. Keeping the perimeter a constant 64 meters, they measure the areas of the pens using some of the strategies they developed the week before. Mrs. Chaplain circulates around the room, paying careful attention to the contributions of individual students, making notes to herself about two particular children, one who seems to be having difficulty with the concept of area, and another who is doing a nice job of leading her group to a solution.

Gradually, the work becomes more symbolic and verbal and less concrete. The students begin to make tables to record the dimensions and descriptions of their pens and to look for some kind of pattern, because they have learned from experience that this frequently leads to insights.

In the high school ...

at which the carbon "decayed," they could figure out the age of the object. An American chemist named Willard Libby developed a technique that allowed them to do so. Ms. Diego explains that the classes will spend the next few days exploring the concept of radioactive decay and, toward the end of the week, they will be able to solve some of the same kinds of problems solved by those archaeologists.

On Tuesday, working at stations created by the teachers, the students begin to explore both the mathematical and scientific aspects of radioactive decay. Working in groups, the students use sets of 50 dice to simulate collections of radioactive nuclei. Each roll of the collection of dice represents the passage of one day. Any time a die lands with a "1" showing, it "spontaneously decays" and is taken out of the collection. The students plot the number of radioactive nuclei left versus the number of days passed in an effort to determine the half-life of the element — the amount of time it takes for half of the element to decay. Because the experiment is relatively well controlled, each group working on the task produces a graph that effectively illustrates the decay, but, because the process is also a truly random process, each group's results are slightly different from those of other groups.

On Wednesday, in a very different kind of activity, students use graphing calculators in a guided activity to discover properties of exponential functions, and the effects on the graphs of various changes to the parameters in the functions. Working from a worksheet prepared by the teachers, they start with the general form of an exponential function, $y = ab^x$. Using the values $a = 1$ and $b = 2$, they input the equation into the calculators and study the resulting graph. Then, they systematically change the values of a and b to discover what each change does to the graph. They are directed by the worksheet to pay particular attention to the effect of changing b to a value between zero and one, because graphs of

In the elementary school ...

One group follows the teacher's suggestion and enters their table of values for rectangular pens into a computer, generating a broken-line graph of the length of the pen versus its area.

Toward the end of the class, the students become comfortable with their discoveries. Mrs. Chaplain reflects again on how glad she is that the faculty decided to organize the school schedule in such a way as to allow for these extended class sessions. When she sees how involved and active the students are, how they try to persuade each other to follow one path or another, how their verbalizations either cement their own understandings or provide opportunities for others to point out flaws in their thinking, she realizes that only with this kind of time and this kind of effort can she do an adequate job of teaching mathematics.

The summary discussion at the end of the session allows the students an opportunity to see what their classmates have done and to evaluate their own group's results. Mrs. Chaplain learns that everyone in the class understands that if you hold the perimeter constant, you can create figures with a whole range of areas. Moreover, she feels that a majority of the class also has come to the generalization that the more compact a figure is, the greater its area, and the more stretched out it is, the smaller its area.

But the students still have very different answers to the question, *What would be the best pen?* That fits her plans perfectly. For homework, Mrs. Chaplain asks each student to design the pen that he or she thinks is best, draw a diagram of it, label its dimensions and its area, and write a paragraph about why that particular pen would be best for the dog. Mrs. Chaplain plans to move on from this activity to others where the students concentrate on more efficient strategies for finding the areas of some of the non-rectangular shapes they explored in this Challenge.

In the high school ...

that type will be especially important for their work with radioactive decay. The culminating problem on the worksheet is a challenge to try to find the values of a and b that produce a graph that looks like the ones that resulted from the experiment with the dice. The students enjoy the problem and use their calculators to quickly check and refine their solutions, zeroing in on the critical numbers. There is a lot of discussion about why those numbers might be the correct ones.

On Thursday, the students discuss a reading that was assigned for homework the night before, focusing on carbon dating and addressing some of the mathematical processes used to determine the age of fossils. This discussion is led by the two teachers, who have brought in some fossilized samples to better acquaint the students with the kind of materials they read about. Ms. Diego then leads a session to develop the computational procedures for solving the carbon dating problems using exponential functions. The students will be given some homework problems of this type and will spend tomorrow's class discussing those problems and wrapping up the unit.

The teachers are very pleased with what the classes have accomplished. The active involvement with a hands-on experiment simulating decay, the symbolic manipulations and graph explorations made possible by the graphing calculator, and the study of a particular scientific application of the mathematics have been very productive. By working together as a team, the teachers have been able to relate the different aspects of the phenomenon to each other. The students have learned a great deal of both mathematics and science and have seen how strongly they are linked.

The Vision

The vision of the *Mathematics Standards* of the New Jersey State Department of Education's *Core Curriculum Content Standards* revolves around what takes place in classrooms like those described in the previous pages. It is focused on achieving one crucial goal:

GOAL: **To enable ALL of New Jersey's children to move into the twenty-first century with the mathematical skills, understandings, and attitudes that they will need to be successful in their careers and daily lives.**

As more and more New Jersey teachers incorporate the recommendations of the *Mathematics Standards* into their teaching, we should be able to see the following results (as described in *Mathematics to Prepare Our Children for the 21st Century: A Guide for New Jersey Parents*, published by the New Jersey Mathematics Coalition in September 1994).

Students who are excited by and interested in their activities. A principal goal is for children to learn to enjoy mathematics. Students who are excited by what they are doing are more likely to truly understand the material, to stay involved over a longer period of time, and to take more advanced courses voluntarily. When math is taught with a problem-solving spirit, and when children are allowed to make their own hands-on mathematical discoveries, math can be engaging for all students.

Students who are learning important mathematical concepts rather than simply memorizing and practicing procedures. Student learning should be focused on understanding when and how mathematics is used and how to apply mathematical concepts. With the availability of technology, students need no longer spend the same amount of study time practicing lengthy computational processes. More effort should be devoted to the development of number sense, spatial sense, and estimation skills.

Students who are posing and solving meaningful problems. When students are challenged to use mathematics in meaningful ways, they develop their reasoning and problem-solving skills and come to realize the potential usefulness of mathematics in their lives.

Students who are working together to learn mathematics. Children learn mathematics well in cooperative settings, where they can share ideas and approaches with their classmates.

Students who write and talk about math topics every day. Putting thoughts into words helps to clarify and solidify thinking. By sharing their mathematical understandings in written and oral form with their classmates, teachers, and parents, students develop confidence in themselves as mathematical learners; this practice also enables teachers to better monitor student progress.

Calculators and computers being used as important tools of learning. Technology can be used to aid teaching and learning, as new concepts are presented through explorations with calculators or computers. But technology can also be used to assist students in solving problems, as it is used by adults in our society. Students should have access to these tools, both in school and after school, whenever they can use technology to do more powerful mathematics than they would otherwise be able to do.

Teachers who have high expectations for ALL of their students. This vision includes a set of achievable, high-level expectations for the mathematical understanding and performance of all students. Although more ambitious than current expectations for most students, these standards are absolutely essential if we are to reach our goal. Those students who can achieve more than this set of expectations must be afforded the opportunity and encouraged to do so.

A variety of assessment strategies rather than sole reliance on traditional short-answer tests. Strategies including open-ended problems, teacher interviews, portfolios of best work, and projects, in combination with traditional methods, will provide a more complete picture of students' performance and progress.

Learning environments like this **should and can become the reality in virtually all New Jersey classrooms before the turn of the century**. Making this vision a reality is both necessary and achievable.

The Necessity of the Vision

Perhaps the most compelling reason for this vision of mathematics education is that our children will be better served by higher expectations, by curricula which go far beyond basic skills and include a variety of mathematical models, and by programs which devote a greater percentage of instructional time to problem-solving and active learning. Many students respond to the current curriculum with boredom and discouragement, develop the perception that success in mathematics depends on some innate ability which they simply do not have, and feel that, in any case, mathematics will never be useful in their lives. Learning environments like the one described in the vision will help students to enjoy and appreciate the value of mathematics, to develop the tools they need for varied educational and career options, and to function effectively as citizens and consumers.

Preparing our students for careers in the twenty-first century also requires that we make this vision a reality. Our curricula are often preoccupied with what national reports call "shopkeeper mathematics," competency in the basic operations that were needed to run a small store several generations ago; yet very few of our students will have careers as shopkeepers. To compete in today's global, information-based economy, students must also be able to solve real problems, reason effectively, and make logical connections. Jobs requiring mathematical knowledge and skills in areas such as data analysis, problem-solving, pattern recognition, statistics, and probability are growing at nearly twice the rate of growth of overall employment. To prepare students for such careers, the mathematics curriculum must change.

We must take seriously the goal of preparing *all* students for twenty-first century careers. In order to do this, we must overcome the all too common perception among students that they simply lack mathematical ability. *Everybody Counts*, a 1989 report prepared by the Mathematical Sciences Education Board of the National Academy of Sciences, notes the following:

> Only in the United States do people believe that learning mathematics depends on special ability. In other countries, students, parents, and teachers all expect that most students can master mathematics if only they work hard enough. The record of accomplishment in these countries — and in some intervention programs in the United States — shows that most students can learn much more mathematics than is commonly assumed in this country (MSEB, 1989, 10).

Curricula that assume student failure are bound to fail; we need to develop curricula that assume student success.

Not only will our students need to find employment in the twenty-first century, but our state and country will need to find employees. American schools have done well in the past at producing a relatively small mathematical elite that adequately served the needs of an industrial/mechanical economy. But that level of "production" is no longer good enough. The global economy in which graduates of our schools will seek employment is more competitive than ever and is rapidly changing in response to advances in technology. Products and factories are being designed by mathematical models and computer simulations, computers are controlling production processes and plants, and robots are replacing workers on assembly lines. Our state and our country need people with the skills to develop and manage these new technologies. In the past, industry moved in search of cheap labor; today, industry frequently moves in search of skilled labor. Our unemployment problem is not only one of too few jobs, but also one of too few skilled workers for existing jobs. We must not only strive to provide our graduates with the skills for 21st century jobs, but also to ensure that the number of graduates with those skills is sufficient for the needs of our state and our nation.

Toward Achieving the Vision

To achieve the vision, the first step is to translate it into specific goals. That is the purpose of the *Mathematics Standards*. The term *standards* as used here encompasses both *goals* and *expectations*, but it also is meant to convey the older meaning of *standards*, a *banner*, or a *rallying point*. These mathematics standards are intended to be a definition of excellent practice, and a description of what can be achieved if all New Jersey communities rally behind the standards, so that this excellent practice becomes common practice.

This vision of excellent mathematical education is based on the twin premises that *all* students *can* learn mathematics and that all students *need* to learn mathematics. Therefore, for all of the reasons mentioned previously, it is essential that we offer students the very highest quality of mathematics education possible. The *Mathematics Standards* were not designed as minimum standards, but rather as world-class standards which will enable all of our students to compete in the global marketplace of the 21st century.

Sixteen mathematics core curriculum content standards, describing what students should know and be able to do, have been adopted by the New Jersey State Board of Education. Two additional standards explicitly address how the learning environment in classrooms can foster success in mathematics for all students and can link assessment to learning and instruction.

These eighteen standards define the critical goals of mathematical education today. In addition to more familiar content, there are many topics which have not been part of the traditional curriculum. Included also are new emphases on the whys and hows of mathematics learning: posing and solving real world problems, effectively communicating mathematical ideas, making connections within mathematics and between mathematics and other areas, active student involvement, the uses of technology, and the relationship between assessment and instruction.

In the future these standards may undergo revision. The standards must be dynamic, and we must be prepared to revise them with changes in mathematics and its use.

Overview of New Jersey's *Mathematics Standards*

Background

A draft version of the *New Jersey Mathematics Standards* was developed by a panel of thirty-one individuals who met extensively during the 1992-1993 school year. Crafted by a broad range of New Jersey elementary, middle school, and secondary teachers, supervisors, administrators, college mathematics educators, mathematicians, and representatives of business and industry, the draft *New Jersey Mathematics Standards* was intended to provide a clear vision of exemplary mathematics learning and to define and then articulate the standards necessary for achieving quality mathematics education.

After the completion of the draft *New Jersey Mathematics Standards*, over 7000 copies of the document were distributed for review across the state. At the same time, efforts began to extend the draft *New Jersey Mathematics Standards* into a mathematics framework. The *Preliminary Version* of the *New Jersey Mathematics Curriculum Framework*, published in March 1995, contained a revised version of the standards which addressed many of the comments of both the reviewers of the draft standards and the drafters of the framework materials. As a result of this process, the standards in the *Preliminary Version* represented a statewide consensus of what mathematics educators believe are high achievable goals for all students.

During 1995, a new working group built upon these draft standards and, together with similar working groups in other content areas, engaged the public in an extensive review process that resulted in modest modifications of the draft standards in mathematics. This process culminated in the adoption on May 1, 1996 by the New Jersey State Board of Education of the *Core Curriculum Content Standards*, which includes the *Mathematics Standards*, standards in six other content areas, and cross-content workplace readiness standards.

Building on the National Standards

Although philosophically aligned with the *Curriculum and Evaluation Standards for School Mathematics* of the National Council of Teachers of Mathematics (NCTM, 1989), New Jersey's *Mathematics Standards* are designed to reflect conditions specific to New Jersey, as well as national changes in mathematics education since the NCTM document was written.

New Jersey is a state on the forefront of industrial and academic uses of technology and the national leader in numerous scientific industries. Our work force, it could be argued, has a greater need for mathematical and scientific fluency than that of any other state in the country. At the same time, the state is highly urbanized, has a tremendously diverse population, and presently delivers its educational programs through a network of more than 600 independent school districts. These educational and demographic characteristics present a truly unique setting in which to establish standards.

The standards rest on the notion that an appropriate mathematics curriculum results from a series of critical decisions about three inseparably linked components: content, instruction, and assessment. The standards will only promote substantial and systemic improvement in mathematics education if the *what* of the content being learned, the *how* of the problem-solving orientation, and the *where* of the active, equitable, involving learning environment are synergistically woven together in every classroom.

The mathematical environment of every child must be rich and complex and all students must be afforded the opportunity to develop an understanding and a command of mathematics in an environment that provides for both affective and intellectual growth. Particular to New Jersey's *Mathematics Standards* is the definition of an appropriate mathematical learning environment.

New Jersey's *Mathematics Standards* also contain a strong focus on the use of technology as a regular, integral part of school mathematics curricula at every grade level. The state mandate for the use of calculators on statewide assessment is but one indication of the strong movement that has already begun in this direction. Teachers and students who adopt these standards will understand, and develop the abilities to use, powerful, up-to-date mathematics and technology.

Although ours is a geographically small state, it has a widely diverse population. Children enter our schools from a tremendous variety of backgrounds and cultures. One of the roles of New Jersey's *Mathematics Standards*, therefore, is to specify a set of achievable high-level expectations for the mathematical understanding and performance of *all* students. The expectations included in the standards are substantially more ambitious than current expectations for most students, but we believe that they are attainable by all students in the state. Those New Jersey students who can achieve more than this set of expectations must be afforded the opportunity and encouraged to do so.

A Core Curriculum for Grades K-12

Implicit in the vision and standards is the notion that there should be a core curriculum for grades K-12. What does a "core curriculum" mean? It means that every student will be involved in experiences addressing all of the expectations of each of the sixteen content standards. It also means that all courses of study should have a common goal of completing this core curriculum, no matter how students are grouped or separated by needs and/or interests.

A core curriculum does not mean that all students will be enrolled in the same courses. Students have different aptitudes, interests, educational and professional plans, learning habits, and learning styles. Different groups of students should address the core curriculum at different levels of depth, and should complete the core curriculum according to different timetables. Nevertheless, all students should complete all elements of the core curriculum recommended in the *Mathematics Standards*.

All students should be challenged to reach their maximum potential. For many students, the core curriculum described here will indeed be challenging. But if we do not provide this challenge, we will be doing our students a great disservice — leaving them unprepared for the technological and information age of the 21st century.

For other students, this core curriculum itself will not be a challenge. We have to make sure that we provide these students with appropriate mathematical challenges. We have to make sure that the raised expectations for all students do not result in lowered expectations for our high achieving students. A core curriculum does not exclude a program which challenges students beyond the expectations set in the *Mathematics Standards*. Indeed, the *Mathematics Standards* call for all schools to provide opportunities to their students to learn more mathematics than is contained in the core curriculum.

The issue of a core curriculum, and its implications, is discussed at greater length in the chapter on Standard 16.

New Jersey's *Mathematics Standards*

The *Mathematics Standards* consist of eighteen statements which describe what is essential to excellent mathematics education and presents a view of mathematics teaching and learning that integrates the processes of mathematical activity, the content of the mathematics, and the learning environment in the classroom. The following sixteen standards were adopted by the New Jersey State Board of Education.

1. All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.
2. All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.
3. All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas and the roles that mathematics and mathematical modeling play in other disciplines and in life.
4. All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers.
5. All students will regularly and routinely use calculators, computers, manipulatives, and other mathematical tools to enhance mathematical thinking, understanding, and power.
6. All students will develop number sense and an ability to represent numbers in a variety of forms and use numbers in diverse situations.
7. All students will develop spatial sense and an ability to use geometric properties and relationships to solve problems in mathematics and in everyday life.
8. All students will understand, select, and apply various methods of performing numerical operations.
9. All students will develop an understanding of and will use measurement to describe and analyze phenomena.
10. All students will use a variety of estimation strategies and recognize situations in which estimation is appropriate.
11. All students will develop an understanding of patterns, relationships, and functions and will use them to represent and explain real-world phenomena.
12. All students will develop an understanding of statistics and probability and will use them to describe sets of data, model situations, and support appropriate inferences and arguments.
13. All students will develop an understanding of algebraic concepts and processes and will use them to represent and analyze relationships among variable quantities and to solve problems.
14. All students will apply the concepts and methods of discrete mathematics to model and explore a variety of practical situations.
15. All students will develop an understanding of the conceptual building blocks of calculus and will use them to model and analyze natural phenomena.
16. All students will demonstrate high levels of mathematical thought through experiences which extend beyond traditional computation, algebra, and geometry.

In addition, the New Jersey State Department of Education's *Core Curriculum Content Standards* includes, in its Introduction to the *Mathematics Standards*, the following two "learning environment standards."

17. All students' mathematical learning will embody the concepts that engagement in mathematics is essential, and that decision-making, risk-taking, cooperative work, perseverance, self-assessment, and self-confidence are frequently keys to success.
18. All students will be evaluated by using a diversity of assessment tools and strategies, to provide multiple indicators of the quality of every student's mathematical learning and of overall program effectiveness.

Descriptive Statements and Cumulative Progress Indicators

Accompanying each of the *Mathematics Standards* is a "descriptive statement" and "cumulative progress indicators," which together provide a brief elaboration of the standard. The descriptive statement expands the simple statement of the standard into a paragraph which outlines the meaning and significance of the standard. The cumulative progress indicators describe what students who are working to achieve the standard should understand and be able to do at each of grades 4, 8, and 12. These materials begin on page 15.

Vignettes

Following the descriptive statements and cumulative progress indicators are a series of short vignettes which suggest how the standards can be effectively integrated in classroom settings. The vignettes are intended to make the standards user-friendly; they serve as examples, as illustrations, of how individual educators can incorporate the standards into their classroom instruction.

Implementing New Jersey's *Mathematics Standards...*

This set of standards is not an end in itself. It represents, instead, a beginning — the beginning of a process intended to mobilize all segments of the education community and the state at large to truly reshape our approach to mathematics education, to achieve the vision.

... Through Statewide Efforts

Statewide efforts to implement the New Jersey's *Mathematics Standards* are proceeding in a number of areas, including curriculum framework, assessment, professional development, and public information.

New Jersey Mathematics Curriculum Framework

An important step in implementing the *Mathematics Standards* is the development of this document, the *New Jersey Mathematics Curriculum Framework*, with the support of a three-year grant from the United States Department of Education. In accepting the grant, the New Jersey Department of Education and the New Jersey Mathematics Coalition also accepted the challenge to develop and implement a world-class curriculum framework which will serve as a model for other states.

The *New Jersey Mathematics Curriculum Framework* contains chapters dealing with each of the standards.

Each chapter provides overviews of what the standard means at each of five grade level clusters (K-2, 3-4, 5-6, 7-8, 9-12), and activities which illustrate how the cumulative progress indicators can be achieved at each of those grade levels. The *Framework* will provide assistance and guidance to districts and teachers in how to implement these standards, in translating the vision into reality.

Statewide Assessments

In order for these standards to be implemented, our statewide assessment program must be based on the *Mathematical Standards*; and if these standards truly represent what we value in the learning of mathematics, then that must be reflected in what we assess and how we assess it. The New Jersey Department of Education will continue to develop a statewide assessment program which reflects the *Mathematics Standards*. A fourth-grade statewide mathematics assessment aligned with these standards is now being developed, called the Elementary School Proficiency Assessment (ESPA), and should replace the kinds of standardized tests currently in use which tend to reinforce a traditional low-level, drill-based curriculum. The Eleventh-Grade High School Proficiency Test (HSPT) and the Eighth-Grade Early Warning Test (EWT) will continue to evolve to reflect the *Mathematical Standards*.

Professional Development

In order for the *Mathematics Standards* to be implemented, there must also be a concerted effort to provide professional development activities to enable teachers to achieve in their classrooms the vision described in this document. Teachers at all grade levels will need to understand and utilize new content material, new orientations toward problem-solving and reasoning, and new strategies for helping all students achieve success. To do this, they will need extensive assistance, through expanded opportunities for professional development throughout the state, and commitment and encouragement from their schools and districts to take advantage of those opportunities.

Public Information

At the level of public information, there will be a concerted effort to inform parents and the public about New Jersey's *Mathematics Standards*, and to enlist their cooperation and advocacy in the implementation of the standards. The New Jersey Mathematics Coalition developed and distributed in 1995 a booklet, *Mathematics To Prepare Our Children for The 21st Century: A Guide for New Jersey Parents*. This booklet conveyed the vision and the substance of the standards to the parents of the state, and encouraged them to support the direction and efforts represented by the national and state standards. A revised guide for parents will be published and disseminated widely. Moreover, the Coalition will continue to develop other vehicles for conveying the message of the standards to New Jersey parents and the public, such as activities for parents during Math, Science and Technology Month (MSTM) each April and presentations to parent organizations, and will continue to provide parents with mathematical experiences that reflect the vision of New Jersey's *Mathematics Standards*.

... Through Local Efforts

The *New Jersey Mathematics Curriculum Framework* is designed to provide assistance and guidance to schools and districts in their efforts to implement the *Mathematics Standards*. However, the vision is not something that can be achieved overnight; there is no "magic wand" which will suddenly transform a classroom or a curriculum into one which implements these standards. A decision by a school or a district to work toward achieving the vision involves an ongoing commitment to a process of change. That process should begin now.

Chapter 20 of this *Framework* provides a model for understanding systemic change, and describes specific processes to follow in order to successfully bring about change. Key to the success of efforts designed to bring about systemic change is enlisting the involvement and support of all those affected by the change.

An important first step in each school is to encourage teachers to review and explore together the *Mathematics Standards* and the *New Jersey Mathematics Curriculum Framework*. Each chapter of the *Framework* can serve as a basis for extended discussions involving teachers and administrators, and school personnel are encouraged to form discussion groups for this purpose. They might begin by:

- reviewing together the *Mathematics Standards*, discussing what each standard means, the extent to which they are already addressing each standard, and what next steps they can take;
- selecting individual content chapters from the *Framework* for more extensive review, discussion, and implementation;
- reviewing together the vignettes at the end of this chapter and discussing how their own classroom practices can reflect the diversity of strategies described there;
- using the chapters on the two learning environment standards as a basis for review of their instructional and assessment techniques; and
- developing their own recommendations concerning how the school or district can begin its efforts to achieve the vision.

However, teachers cannot carry out these suggestions without support. To facilitate this process, we encourage schools and districts to:

- create opportunities for teachers to meet together regularly, possibly through the scheduling of common planning times;
- actively encourage teachers in their explorations, providing resources to support such activities;
- take steps to provide more flexible scheduling, permitting extended periods for exploration and contiguous periods for collaboration among mathematics and science teachers; and
- make meaningful professional development activities for teachers an important priority.

All of these activities will be valuable. However, to realize the vision throughout the state, virtually all elements in our educational system must be rethought. Some of the areas of concern and questions which arise are these:

Mathematical Disposition: How can a community of mathematics learners best be created and then fostered in a school setting? How can the positive affective characteristics that we hope for in students be extended to and reinforced by their parents and community?

Equity: What teaching and administrative strategies result in the inclusion of all students in mathematical activities? How can we plan for and achieve true equity in our mathematics classrooms?

Instruction and Assessment: What are the most effective strategies for assuring the adequate integration of curriculum, instruction, and assessment? How can we best assure that appropriate connections are made among mathematical topics and between mathematics and other disciplines, that technology truly becomes a tool for mathematics learning, that mathematics learning is active, involving, and exciting? What materials and resources are necessary to assure our success? What means of evaluation will best allow us to measure our progress toward these goals?

Professional Development: What are the most effective strategies for preparing teachers at both the pre-service and in-service levels to teach in a manner consistent with these standards? What efforts are necessary to develop and nurture the cadre of mathematics teacher leaders that will be needed to move the vision beyond this document? What is the role of the state's college and university faculties in this process?

School Organization: What type of school culture and ethos must be in place before these recommended changes in orientation can begin to take hold and then grow and flourish? What changes are needed in school scheduling and time allocation practices to promote the kind of teaching and learning envisioned here? What implications for staffing and teacher assignment are inherent in the standards?

Educational Policy: What changes in the state administrative code or the body of state mandates would further encourage the reform suggested here? What local school district policies either inhibit or promote these efforts? What are the roles of local school administrators and school board members in support of the standards? How can this document be used as a vehicle for change? What mechanisms are in place to assure that the vision it embodies changes and grows with time?

These questions are raised in the *New Jersey Mathematics Curriculum Framework*, and should be part of the ongoing discussion at the local level, as well as at the state level, of how the *Mathematics Standards* can best be implemented in New Jersey schools.

Summary

The *Mathematics Standards* presented in this chapter offer a powerful challenge to all teachers, all schools, and all districts in New Jersey — to enable all of our students to step forward into the next century with the mathematical skills, understandings, and attitudes that they will need to be successful in their careers and daily lives.

The *Mathematics Standards* also offer a powerful tool to help us meet that challenge, providing a vision and standards which both inform us and rally us in our efforts.

It will not be easy to meet this challenge, nor can it happen overnight. But it can only happen if all of us together decide to make it happen. There are many obstacles, but we must not let our awareness of the obstacles become yet another obstacle. Let us work together to make the vision of New Jersey's *Mathematics Standards* a reality by the end of this century!

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Mathematical Sciences Education Board. *Everybody Counts*. Washington, DC: National Academy Press, 1989.

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New Jersey State Department of Education. *Core Curriculum Content Standards*. Trenton, NJ, 1996.

New Jersey Mathematics Coalition. *Mathematics to Prepare Our Children for the 21st Century: A Guide for New Jersey Parents*, 1994.

New Jersey Mathematics Coalition. *Preliminary Version* of the *New Jersey Mathematics Curriculum Framework*, 1995.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

MATHEMATICS STANDARDS*

Descriptive Statements and Cumulative Progress Indicators

STANDARD 1*	All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.

Descriptive Statement

Problem posing and problem solving involve examining situations that arise in mathematics and other disciplines and in common experiences, describing these situations mathematically, formulating appropriate mathematical questions, and using a variety of strategies to find solutions. By developing their problem-solving skills, students will come to realize the potential usefulness of mathematics in their lives.

Cumulative Progress Indicators

By the end of **Grade 4,** students:

1. Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand mathematical content appropriate to early elementary grades.
2. Recognize, formulate, and solve problems arising from mathematical situations and everyday experiences.
3. Construct and use concrete, pictorial, symbolic, and graphical models to represent problem situations.
4. Pose, explore, and solve a variety of problems, including non-routine problems and open-ended problems with several solutions and/or solution strategies.
5. Construct, explain, justify, and apply a variety of problem-solving strategies in both cooperative and independent learning environments.
6. Verify the correctness and reasonableness of results and interpret them in the context of the problems being solved.
7. Know when to select and how to use grade-appropriate mathematical tools and methods (including manipulatives, calculators and computers, as well as mental math and paper-and-pencil techniques) as a natural and routine part of the problem-solving process.

*Note that in the *Core Curriculum Content Standards* of the New Jersey State Department of Education, the *Mathematics Standards* are numbered 4.1, 4.2, 4.3, etc., since they are preceded by standards in three other content areas.

8. Determine, collect, organize, and analyze data needed to solve problems.
9. Recognize that there may be multiple ways to solve a problem.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 4, 5, 6, 7, and 8 above, by the end of **Grade 8**, students:

10. Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand mathematical content appropriate to the middle grades.
11. Recognize, formulate, and solve problems arising from mathematical situations, everyday experiences, and applications to other disciplines.
12. Construct and use concrete, pictorial, symbolic, and graphical models to represent problem situations and effectively apply processes of mathematical modeling in mathematics and other areas.
13. Recognize that there may be multiple ways to solve a problem, weigh their relative merits, and select and use appropriate problem-solving strategies.
14. Persevere in developing alternative problem-solving strategies if initially selected approaches do not work.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 4, 5, 6, 7, 8, 12, and 14 above, by the end of **Grade 12**, students:

15. Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand the mathematical content appropriate to the high school grades.
16. Recognize, formulate, and solve problems arising from mathematical situations, everyday experiences, applications to other disciplines, and career applications.
17. Monitor their own progress toward problem solutions.
18. Explore the validity and efficiency of various problem-posing and problem-solving strategies, and develop alternative strategies and generalizations as needed.

STANDARD 2 All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.

Descriptive Statement

Communication of mathematical ideas will help students clarify and solidify their understanding of mathematics. By sharing their mathematical understandings in written and oral form with their classmates, teachers, and parents, students develop confidence in themselves as mathematics learners and enable teachers to better monitor their progress.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Discuss, listen, represent, read, and write as vital activities in their learning and use of mathematics.
2. Identify and explain key mathematical concepts, and model situations using oral, written, concrete, pictorial, and graphical methods.
3. Represent and communicate mathematical ideas through the use of learning tools such as calculators, computers, and manipulatives.
4. Engage in mathematical brainstorming and discussions by asking questions, making conjectures, and suggesting strategies for solving problems.
5. Explain their own mathematical work to others, and justify their reasoning and conclusions.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 1, 2, 3, 4, and 5 above, by the end of **Grade 8,** students:

6. Identify and explain key mathematical concepts and model situations using geometric and algebraic methods.
7. Use mathematical language and symbols to represent problem situations, and recognize the economy and power of mathematical symbolism and its role in the development of mathematics.
8. Analyze, evaluate, and explain mathematical arguments and conclusions presented by others.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 1, 2, 3, 4, 5, 6, 7, and 8 above, by the end of **Grade 12,** students:

9. Formulate questions, conjectures, and generalizations about data, information, and problem situations.
10. Reflect on and clarify their thinking so as to present convincing arguments for their conclusions.

STANDARD 3	All students will connect mathematics to other learning by understanding the interrelationships of mathematical idea and the roles that mathematics and mathematical modeling play in other disciplines and in life.

Descriptive Statement

Making connections enables students to see relationships between different topics, and to draw on those relationships in future study. This applies within mathematics, so that students can translate readily between fractions and decimals, or between algebra and geometry; to other content areas, so that students understand how mathematics is used in the sciences, the social sciences, and the arts; and to the everyday world, so that students can connect school mathematics to daily life.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. View mathematics as an integrated whole rather than as a series of disconnected topics and rules.
2. Relate mathematical procedures to their underlying concepts.
3. Use models, calculators, and other mathematical tools to demonstrate the connections among various equivalent graphical, concrete, and verbal representations of mathematical concepts.
4. Explore problems and describe and confirm results using various representations.
5. Use one mathematical idea to extend understanding of another.
6. Recognize the connections between mathematics and other disciplines, and apply mathematical thinking and problem solving in those areas.
7. Recognize the role of mathematics in their daily lives and in society.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 1, 2, 3, and 4 above, by the end of **Grade 8,** students:

8. Recognize and apply unifying concepts and processes which are woven throughout mathematics.
9. Use the process of mathematical modeling in mathematics and other disciplines, and demonstrate understanding of its methodology, strengths, and limitations.
10. Apply mathematics in their daily lives and in career-based contexts.
11. Recognize situations in other disciplines in which mathematical models may be applicable, and apply appropriate models, mathematical reasoning, and problem solving to those situations.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 1, 2, 3, 8, 9, 10 and 11 above, by the end of **Grade 12,** students:

12. Recognize how mathematics responds to the changing needs of society, through the study of the history of mathematics.

STANDARD 4	All student will develop reasoning ability and will become self-reliant, independent mathematical thinkers.

Descriptive Statement

Mathematical reasoning is the critical skill that enables a student to make use of all other mathematical skills. With the development of mathematical reasoning, students recognize that mathematics makes sense and can be understood. They learn how to evaluate situations, select problem-solving strategies, draw logical conclusions, develop and describe solutions, and recognize how those solutions can be applied. Mathematical reasoners are able to reflect on solutions to problems and determine whether or not they make sense. They appreciate the pervasive use and power of reasoning as a part of mathematics.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Make educated guesses and test them for correctness.
2. Draw logical conclusions and make generalizations.
3. Use models, known facts, properties, and relationships to explain their thinking.
4. Justify answers and solution processes in a variety of problems.
5. Analyze mathematical situations by recognizing and using patterns and relationships.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 2, 3, and 5 above, by the end of **Grade 8,** students:

6. Make conjectures based on observation and information, and test mathematical conjectures and arguments.
7. Justify, in clear and organized form, answers and solution processes in a variety of problems.
8. Follow and construct logical arguments, and judge their validity.
9. Recognize and use deductive and inductive reasoning in all areas of mathematics.
10. Utilize mathematical reasoning skills in other disciplines and in their lives.
11. Use reasoning rather than relying on an answer-key to check the correctness of solutions to problems.

Building upon knowledge and skills gained in the preceding grades, and especially demonstrating continued progress in Indicators 2, 5, 8, 9, 10, and 11 above, by the end of **Grade 12**, students:

12. Make conjectures based on observation and information, and test mathematical conjectures, arguments, and proofs.
13. Formulate counter-examples to disprove an argument.

STANDARD 5	All students will regularly and routinely use calculators, computers, manipulatives, and other mathematical tools to enhance mathematical thinking, understanding, and power.

Descriptive Statement

Calculators, computers, manipulatives, and other mathematical tools need to be used by students in both instructional and assessment activities. These tools should be used, not to replace mental math and paper-and-pencil computational skills, but to enhance understanding of mathematics and the power to use mathematics. Historically, people have developed and used manipulatives (such as fingers, base ten blocks, geoboards, and algebra tiles) and mathematical devices (such as protractors, coordinate systems, and calculators) to help them understand and develop mathematics. Students should explore both new and familiar concepts with calculators and computers, but should also become proficient in using technology as it is used by adults, that is, for assistance in solving real-world problems.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Select and use calculators, software, manipulatives, and other tools based on their utility and limitations and on the problem situation.
2. Use physical objects and manipulatives to model problem situations, and to develop and explain mathematical concepts involving number, space, and data.
3. Use a variety of technologies to discover number patterns, demonstrate number sense, and visualize geometric objects and concepts.
4. Use a variety of tools to measure mathematical and physical objects in the world around them.
5. Use technology to gather, analyze, and display mathematical data and information.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 1, 2, 3, 4, and 5 above, by the end of **Grade 8,** students:

6. Use a variety of technologies to evaluate and validate problem solutions, and to investigate the properties of functions and their graphs.

7. Use computer spreadsheets and graphing programs to organize and display quantitative information and to investigate properties of functions.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 1, 2, 3, 5, and 7 above, by the end of **Grade 12,** students:

8. Use calculators and computers effectively and efficiently in applying mathematical concepts and principles to various types of problems.

STANDARD 6	All students will develop number sense and an ability to represent numbers in a variety of forms and use numbers in diverse situations.

Descriptive Statement

Number sense is defined as an intuitive feel for numbers and a common sense approach to using them. It is a comfort with what numbers represent, coming from investigating their characteristics and using them in diverse situations. It involves an understanding of how different types of numbers, such as fractions and decimals, are related to each other, and how they can best be used to describe a particular situation. Number sense is an attribute of all successful users of mathematics.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Use real-life experiences, physical materials, and technology to construct meanings for whole numbers, commonly used fractions, and decimals.
2. Develop an understanding of place value concepts and numeration in relationship to counting and grouping.
3. See patterns in number sequences, and use pattern-based thinking to understand extensions of the number system.
4. Develop a sense of the magnitudes of whole numbers, commonly used fractions, and decimals.
5. Understand the various uses of numbers including counting, measuring, labeling, and indicating location.
6. Count and perform simple computations with money.
7. Use models to relate whole numbers, commonly used fractions, and decimals to each other, and to represent equivalent forms of the same number.
8. Compare and order whole numbers, commonly used fractions, and decimals.
9. Explore real-life settings which give rise to negative numbers.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8,** students:

10. Understand money notations, count and compute money, and recognize the decimal nature of United States currency.
11. Extend their understanding of the number system by constructing meanings for integers, rational numbers, percents, exponents, roots, absolute values, and numbers represented in scientific notation.
12. Develop number sense necessary for estimation.
13. Expand the sense of magnitudes of different number types to include integers, rational numbers, and roots.
14. Understand and apply ratios, proportions, and percents in a variety of situations.
15. Develop and use order relations for integers and rational numbers.
16. Recognize and describe patterns in both finite and infinite number sequences involving whole numbers, rational numbers, and integers.
17. Develop and apply number theory concepts, such as primes, factors, and multiples, in real-world and mathematical problem situations.
18. Investigate the relationships among fractions, decimals, and percents, and use all of them appropriately.
19. Identify, derive, and compare properties of numbers.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 12,** students:

20. Extend their understanding of the number system to include real numbers and an awareness of other number systems.
21. Develop conjectures and informal proofs of properties of number systems and sets of numbers.
22. Extend their intuitive grasp of number relationships, uses, and interpretations, and develop an ability to work with rational and irrational numbers.
23. Explore a variety of infinite sequences and informally evaluate their limits.

STANDARD 7	All students will develop spatial sense and an ability to use geometric properties and relationships to solve problems in mathematics and in everyday life.

Descriptive Statement

Spatial sense is an intuitive feel for shape and space. It involves the concepts of traditional geometry, including an ability to recognize, visualize, represent, and transform geometric shapes. It also involves other, less formal ways of looking at two- and three-dimensional space, such as paper-folding, transformations, tessellations, and projections. Geometry is all around us in art, nature, and the things we make. Students of geometry can apply their spatial sense and knowledge of the properties of shapes and space to the real world.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Explore spatial relationships such as the direction, orientation, and perspectives of objects in space, their relative shapes and sizes, and the relations between objects and their shadows or projections.
2. Explore relationships among shapes, such as congruence, symmetry, similarity, and self-similarity.
3. Explore properties of three- and two-dimensional shapes using concrete objects, drawings, and computer graphics.
4. Use properties of three- and two-dimensional shapes to identify, classify, and describe shapes.
5. Investigate and predict the results of combining, subdividing, and changing shapes.
6. Use tessellations to explore properties of geometric shapes and their relationships to the concepts of area and perimeter.
7. Explore geometric transformations such as rotations (turns), reflections (flips), and translations (slides).
8. Develop the concepts of coordinates and paths, using maps, tables, and grids.
9. Understand the variety of ways in which geometric shapes and objects can be measured.
10. Investigate the occurrence of geometry in nature, art, and other areas.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8,** students:

11. Relate two-dimensional and three-dimensional geometry using shadows, perspectives, projections and maps.
12. Understand and apply the concepts of symmetry, similarity and congruence.

13. Identify, describe, compare, and classify plane and solid geometric figures.
14. Understand the properties of lines and planes, including parallel and perpendicular lines and planes, and intersecting lines and planes and their angles of incidence.
15. Explore the relationships among geometric transformations (translations, reflections, rotations, and dilations), tessellations (tilings), and congruence and similarity.
16. Develop, understand, and apply a variety of strategies for determining perimeter, area, surface area, angle measure, and volume.
17. Understand and apply the Pythagorean Theorem.
18. Explore patterns produced by processes of geometric change, relating iteration, approximation, and fractals.
19. Investigate, explore, and describe geometry in nature and real-world applications, using models, manipulatives, and appropriate technology.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 16 and 19 above, by the end of **Grade 12**, students:

20. Understand and apply properties involving angles, parallel lines, and perpendicular lines.
21. Analyze properties of three-dimensional shapes by constructing models and by drawing and interpreting two-dimensional representations of them.
22. Use transformations, coordinates, and vectors to solve problems in Euclidean geometry.
23. Use basic trigonometric ratios to solve problems involving indirect measurement.
24. Solve real-world and mathematical problems using geometric models.
25. Use inductive and deductive reasoning to solve problems and to present reasonable explanations of and justifications for the solutions.
26. Analyze patterns produced by processes of geometric change, and express them in terms of iteration, approximation, limits, self-similarity, and fractals.
27. Explore applications of other geometries in real-world contexts.

STANDARD 8 — All students will understand, select, and apply various methods of performing numerical operations.

Descriptive Statement

Numerical operations are an essential part of the mathematics curriculum. Students must be able to select and apply various computational methods, including mental math, estimation, paper-and-pencil techniques, and the use of calculators. Students must understand how to add, subtract, multiply, and divide whole numbers, fractions, and other kinds of numbers. With calculators that perform these operations quickly and accurately, however, the instructional emphasis now should be on understanding the meanings and uses of the operations, and on estimation and mental skills, rather than solely on developing paper-and-pencil skills.

Cumulative Progress Indicators

By the end of **Grade 4,** students:

1. Develop meaning for the four basic arithmetic operations by modeling and discussing a variety of problems.
2. Develop proficiency with and memorize basic number facts using a variety of fact strategies (such as "counting on" and "doubles").
3. Construct, use, and explain procedures for performing whole number calculations in the various methods of computation.
4. Use models to explore operations with fractions and decimals.
5. Use a variety of mental computation and estimation techniques.
6. Select and use appropriate computational methods from mental math, estimation, paper-and-pencil, and calculator methods, and check the reasonableness of results.
7. Understand and use relationships among operations and properties of operations.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicator 6 above, by the end of **Grade 8,** students:

8. Extend their understanding and use of arithmetic operations to fractions, decimals, integers, and rational numbers.
9. Extend their understanding of basic arithmetic operations on whole numbers to include powers and roots.
10. Develop, apply, and explain procedures for computation and estimation with whole numbers, fractions, decimals, integers, and rational numbers.
11. Develop, apply, and explain methods for solving problems involving proportions and percents.

12. Understand and apply the standard algebraic order of operations.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicator 6 above, by the end of **Grade 12**, students:

13. Extend their understanding and use of operations to real numbers and algebraic procedures.
14. Develop, apply, and explain methods for solving problems involving factorials, exponents, and matrices.

STANDARD 9	All students will develop an understanding of and will use measurement to describe and analyze phenomena.

Descriptive Statement

Measurement helps describe our world using numbers. We use numbers to describe simple things like length, weight, and temperature, but also complex things such as pressure, speed, and brightness. An understanding of how we attach numbers to those phenomena, familiarity with common measurement units like inches, liters, and miles per hour, and a practical knowledge of measurement tools and techniques are critical for students' understanding of the world around them.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Use and describe measures of length, distance, capacity, weight, area, volume, time, and temperature.
2. Compare and order objects according to some measurable attribute.
3. Recognize the need for a uniform unit of measure.
4. Develop and use personal referents for standard units of measure (such as the width of a finger to approximate a centimeter).
5. Select and use appropriate standard and non-standard units of measurement to solve real-life problems.
6. Understand and incorporate estimation and repeated measures in measurement activities.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8,** students:

7. Use estimated and actual measurements to describe and compare phenomena.
8. Read and interpret various scales, including those based on number lines and maps.
9. Determine the degree of accuracy needed in a given situation and choose units accordingly.
10. Understand that all measurements of continuous quantities are approximate.

11. Develop formulas and procedures for solving problems related to measurement.
12. Explore situations involving quantities which cannot be measured directly or conveniently.
13. Convert measurement units from one form to another, and carry out calculations that involve various units of measurement.
14. Understand and apply measurement in their own lives and in other subject areas.
15. Understand and explain the impact of the change of an object's linear dimensions on its perimeter, area, or volume.
16. Apply their knowledge of measurement to the construction of a variety of two- and three-dimensional figures.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 12,** students:

17. Use techniques of algebra, geometry, and trigonometry to measure quantities indirectly.
18. Use measurement appropriately in other subject areas and career-based contexts.
19. Choose appropriate techniques and tools to measure quantities in order to achieve specified degrees of precision, accuracy, and error (or tolerance) of measurements.

STANDARD 10	All students will use a variety of estimation strategies and recognize situations in which estimation is appropriate.

Descriptive Statement

Estimation is a process that is used constantly by mathematically capable adults, and that can be mastered easily by children. It involves an educated guess about a quantity or a measure, or an intelligent prediction of the outcome of a computation. The growing use of calculators makes it more important than ever that students know when a computed answer is reasonable; the best way to make that decision is through estimation. Equally important is an awareness of the many situations in which an approximate answer is as good as, or even preferable to, an exact answer.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Judge without counting whether a set of objects has less than, more than, or the same number of objects as a reference set.
2. Use personal referents, such as the width of a finger as one centimeter, for estimations with measurement.
3. Visually estimate length, area, volume, or angle measure.

4. Explore, construct, and use a variety of estimation strategies.
5. Recognize when estimation is appropriate, and understand the usefulness of an estimate as distinct from an exact answer.
6. Determine the reasonableness of an answer by estimating the result of operations.
7. Apply estimation in working with quantities, measurement, time, computation, and problem solving.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 5 and 6 above, by the end of **Grade 8,** students:

8. Develop, apply, and explain a variety of different estimation strategies in problem situations involving quantities and measurement.
9. Use equivalent representations of numbers such as fractions, decimals, and percents to facilitate estimation.
10. Determine whether a given estimate is an overestimate or an underestimate.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicator 6 above, by the end of **Grade 12**, students:

11. Estimate probabilities and predict outcomes from real-world data.
12. Recognize the limitations of estimation, assess the amount of error resulting from estimation, and determine whether the error is within acceptable tolerance limits.

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STANDARD 11	All students will develop an understanding of patterns, relationships, and functions and will use them to represent and explain real-world phenomena.

Descriptive Statement

Patterns, relationships, and functions constitute a unifying theme of mathematics. From the earliest age, students should be encouraged to investigate the patterns that they find in numbers, shapes, and expressions, and, by doing so, to make mathematical discoveries. They should have opportunities to analyze, extend, and create a variety of patterns and to use pattern-based thinking to understand and represent mathematical and other real-world phenomena. These explorations present unlimited opportunities for problem-solving, making and verifying generalizations, and building mathematical understanding and confidence.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Reproduce, extend, create, and describe patterns and sequences using a variety of materials.
2. Use tables, rules, variables, open sentences, and graphs to describe patterns and other relationships.
3. Use concrete and pictorial models to explore the basic concept of a function.
4. Observe and explain how a change in one physical quantity can produce a corresponding change in another.
5. Observe and recognize examples of patterns, relationships, and functions in other disciplines and contexts.
6. Form and verify generalizations based on observations of patterns and relationships.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8,** students:

7. Represent and describe mathematical relationships with tables, rules, simple equations, and graphs.
8. Understand and describe the relationships among various representations of patterns and functions.
9. Use patterns, relationships, and functions to model situations and to solve problems in mathematics and in other subject areas.
10. Analyze functional relationships to explain how a change in one quantity results in a change in another.
11. Understand and describe the general behavior of functions.

12. Use patterns, relationships, and linear functions to model situations in mathematics and in other areas.

13. Develop, analyze, and explain arithmetic sequences.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 12,** students:

14. Analyze and describe how a change in an independent variable can produce a change in a dependent variable.

15. Use polynomial, rational, trigonometric, and exponential functions to model real-world phenomena.

16. Recognize that a variety of phenomena can be modeled by the same type of function.

17. Analyze and explain the general properties and behavior of functions, and use appropriate graphing technologies to represent them.

18. Analyze the effects of changes in parameters on the graphs of functions.

19. Understand the role of functions as a unifying concept in mathematics.

STANDARD 12	All students will develop an understanding of statistics and probability and will use them to describe sets of data, model situations, and support appropriate inferences and arguments.

Descriptive Statement

Probability and statistics are the mathematics used to understand chance and to collect, organize, describe, and analyze numerical data. From weather reports to sophisticated studies of genetics, from election results to product preference surveys, probability and statistical language and concepts are increasingly present in the media and in everyday conversations. Students need this mathematics to help them judge the correctness of an argument supported by seemingly persuasive data.

Cumulative Progress Indicators

By the end of **Grade 4,** students:

1. Formulate and solve problems that involve collecting, organizing, and analyzing data.
2. Generate and analyze data obtained using chance devices such as spinners and dice.
3. Make inferences and formulate hypotheses based on data.
4. Understand and informally use the concepts of range, mean, mode, and median.
5. Construct, read, and interpret displays of data such as pictographs, bar graphs, circle graphs, tables, and lists.

6. Determine the probability of a simple event, assuming equally likely outcomes.
7. Make predictions that are based on intuitive, experimental, and theoretical probabilities.
8. Use concepts of certainty, fairness, and chance to discuss the probability of actual events.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8**, students:

9. Generate, collect, organize, and analyze data and represent this data in tables, charts, and graphs.
10. Select and use appropriate graphical representations and measures of central tendency (mean, mode and median) for sets of data.
11. Make inferences and formulate and evaluate arguments based on data analysis and data displays.
12. Use lines of best fit to interpolate and predict from data.
13. Determine the probability of a compound event.
14. Model situations involving probability, such as genetics, using both simulations and theoretical models.
15. Use models of probability to predict events based on actual data.
16. Interpret probabilities as ratios and percents.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 12**, students:

17. Estimate probabilities and predict outcomes from actual data.
18. Understand sampling and recognize its role in statistical claims.
19. Evaluate bias, accuracy, and reasonableness of data in real-world contexts.
20. Understand and apply measures of dispersion and correlation.
21. Design a statistical experiment to study a problem, conduct the experiment, and interpret and communicate the outcomes.
22. Make predictions using curve fitting and numerical procedures to interpolate and extrapolate from known data.
23. Use relative frequency and probability, as appropriate, to represent and solve problems involving uncertainty.
24. Use simulations to estimate probabilities.
25. Create and interpret discrete and continuous probability distributions, and understand their application to real-world situations.
26. Describe the normal curve in general terms, and use its properties to answer questions about sets of data that are assumed to be normally distributed.
27. Understand and use the law of large numbers (that experimental results tend to approach theoretical probabilities after a large number of trials).

STANDARD 13	All students will develop an understanding of algebraic concepts and processes and will use them to represent and analyze relationships among variable quantities and to solve problems.

Descriptive Statement

Algebra is a language used to express mathematical relationships. Students need to understand how quantities are related to one another, and how algebra can be used to concisely express and analyze those relationships. Modern technology provides tools for supplementing the traditional focus on algebraic techniques, such as solving equations, with a more visual perspective, with graphs of equations displayed on a screen. Students can then focus on understanding the relationship between the equation and the graph, and on what the graph represents in a real-life situation.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Understand and represent numerical situations using variables, expressions, and number sentences.
2. Represent situations and number patterns with concrete materials, tables, graphs, verbal rules, and number sentences, and translate from one to another.
3. Understand and use properties of operations and numbers.
4. Construct and solve open sentences (example: 3 + ___ = 7) that describe real-life situations.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8**, students:

5. Understand and use variables, expressions, equations, and inequalities.
6. Represent situations and number patterns with concrete materials, tables, graphs, verbal rules, and standard algebraic notation.
7. Use graphing techniques on a number line to model both absolute value and arithmetic operations.
8. Analyze tables and graphs to identify properties and relationships.
9. Understand and use the rectangular coordinate system.
10. Solve simple linear equations using concrete, informal, and graphical methods, as well as appropriate paper-and-pencil techniques.
11. Explore linear equations through the use of calculators, computers, and other technology.
12. Investigate inequalities and nonlinear equations informally.
13. Draw freehand sketches of, and interpret, graphs which model real phenomena.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 12**, students:

14. Model and solve problems that involve varying quantities using variables, expressions, equations, inequalities, absolute values, vectors, and matrices.
15. Use tables and graphs as tools to interpret expressions, equations, and inequalities.
16. Develop, explain, use, and analyze procedures for operating on algebraic expressions and matrices.
17. Solve equations and inequalities of varying degrees using graphing calculators and computers as well as appropriate paper-and-pencil techniques.
18. Understand the logic and purposes of algebraic procedures.
19. Interpret algebraic equations and inequalities geometrically, and describe geometric objects algebraically.

STANDARD 14	All students will apply the concepts and methods of discrete mathematics to model and explore a variety of practical situations.

Descriptive Statement

Discrete mathematics is the branch of mathematics that deals with arrangements of distinct objects. It includes a wide variety of topics and techniques that arise in everyday life, such as how to find the best route from one city to another, where the objects are cities arranged on a map. It also includes how to count the number of different combinations of toppings for pizzas, how best to schedule a list of tasks to be done, and how computers store and retrieve arrangements of information on a screen. Discrete mathematics is the mathematics used by decision-makers in our society, from workers in government to those in health care, transportation, and telecommunications. Its various applications help students see the relevance of mathematics in the real world.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Explore a variety of puzzles, games, and counting problems.
2. Use networks and tree diagrams to represent everyday situations.
3. Identify and investigate sequences and patterns found in nature, art, and music.
4. Investigate ways to represent and classify data according to attributes, such as shape or color, and relationships, and discuss the purpose and usefulness of such classification.
5. Follow, devise, and describe practical lists of instructions.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8**, students:

6. Use systematic listing, counting, and reasoning in a variety of different contexts.
7. Recognize common discrete mathematical models, explore their properties, and design them for specific situations.
8. Experiment with iterative and recursive processes, with the aid of calculators and computers.
9. Explore methods for storing, processing, and communicating information.
10. Devise, describe, and test algorithms for solving optimization and search problems.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 12**, students:

11. Understand the basic principles of iteration, recursion, and mathematical induction.
12. Use basic principles to solve combinatorial and algorithmic problems.
13. Use discrete models to represent and solve problems.
14. Analyze iterative processes with the aid of calculators and computers.
15. Apply discrete methods to storing, processing, and communicating information.
16. Apply discrete methods to problems of voting, apportionment, and allocations, and use fundamental strategies of optimization to solve problems.

STANDARD 15	All students will develop an understanding of the conceptual building blocks of calculus and will use them to model and analyze natural phenomena.

Descriptive Statement

The conceptual building blocks of calculus are important for everyone to understand. How quantities such as world population change, how fast they change, and what will happen if they keep changing at the same rate are questions that can be discussed by elementary school students. Another important topic for all mathematics students is the concept of infinity — what happens as numbers get larger and larger and what happens as patterns are continued indefinitely. Early explorations in these areas can broaden students' interest in and understanding of an important area of applied mathematics.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Investigate and describe patterns that continue indefinitely.
2. Investigate and describe how certain quantities change over time.

3. Experiment with approximating length, area, and volume, using informal measurement instruments.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8,** students:

4. Recognize and express the difference between linear and exponential growth.
5. Develop an understanding of infinite sequences that arise in natural situations.
6. Investigate, represent, and use non-terminating decimals.
7. Represent, analyze, and predict relations between quantities, especially quantities changing over time.
8. Approximate quantities with increasing degrees of accuracy.
9. Understand and use the concept of significant digits.
10. Develop informal ways of approximating the surface area and volume of familiar objects, and discuss whether the approximations make sense.
11. Express mathematically and explain the impact of the change of an object's linear dimensions on its surface area and volume.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 12**, students:

12. Develop and use models based on sequences and series.
13. Develop and apply procedures for finding the sum of finite arithmetic series and of finite and infinite geometric series.
14. Develop an informal notion of limit.
15. Use linear, quadratic, trigonometric, and exponential models to explain growth and change in the natural world.
16. Recognize fundamental mathematical models (such as polynomial, exponential, and trigonometric functions) and apply basic translations, reflections, and dilations to their graphs.
17. Develop and explain the concept of the slope of a curve and use that concept to discuss the information contained in graphs.
18. Develop an understanding of the concept of continuity of a function.
19. Understand and apply approximation techniques to situations involving initial portions of infinite decimals and measurement.

STANDARD 16 All students will demonstrate high levels of mathematical thought through experiences which extend beyond traditional computation, algebra, and geometry.

Descriptive Statement

High expectations for all students form a critical part of the learning environment. The belief of teachers, administrators, and parents that a student can and will succeed in mathematics often makes it possible for that student to succeed. Beyond that, this standard calls for a commitment that all students will be continuously challenged and enabled to go as far mathematically as they can.

Cumulative Progress Indicators

By the end of **Grade 12**, students:

1. Study a core curriculum containing challenging ideas and tasks, rather than one limited to repetitive, low-level cognitive activities.
2. Work at rich, open-ended problems which require them to use mathematics in meaningful ways, and which provide them with exciting and interesting mathematical experiences.
3. Recognize mathematics as integral to the development of all cultures and civilizations, and in particular to that of our own society.
4. Understand the important role that mathematics plays in their own success, regardless of career.
5. Interact frequently with parents and other members of their communities, including men and women from a variety of cultural backgrounds, who use mathematics in their daily lives and occupations.
6. Receive services that help them understand the mathematical skills and concepts necessary to assure success in the core curriculum.
7. Receive equitable treatment without regard to gender, ethnicity, or predetermined expectations for success.
8. Learn mathematics in classes which reflect the diversity of the school's total student population.
9. Be provided with opportunities at all grade levels for further study of mathematics, especially including topics beyond traditional computation, algebra, and geometry.
10. Be challenged to maximize their mathematical achievements at all grade levels.
11. Experience a full program of meaningful mathematics so that they can pursue post-secondary education.

STANDARD 17	All students' mathematical learning will embody the concept that engagement in mathematics is essential, and that decision-making, risk-taking, cooperative work, perseverance, self-assessment, and self-confidence are frequently keys to success.

(This "learning environment standard" was developed and approved by the task force that prepared the *Mathematics Standards* and appears in the Introduction to the *Mathematics Standards* chapter of the New Jersey State Department of Education's *Core Curriculum Content Standards*; however, since it was not considered a "content standard," it was not presented to the New Jersey State Board of Education for adoption.)

Descriptive Statement

Engagement in mathematics should be expected of all students, and the learning environment should be one where students are actively involved in doing mathematics. Challenging problems should be posed and students should be expected to work on them individually and in groups, sometimes for extended periods of time, and sometimes on unfamiliar topics. They should be encouraged to develop traits and strategies — such as perseverance, cooperative work skills, decision-making, and risk-taking — which will be key to their success in mathematics.

Cumulative Progress Indicators

Experiences will be such that all students:

1. Demonstrate confidence as mathematical thinkers, believing that they can learn mathematics and can achieve high standards in mathematics, and accepting responsibility for their own learning of mathematics.
2. Recognize the power that comes from understanding and doing mathematics.
3. Develop and maintain a positive disposition to mathematics and to mathematical activity.
4. Participate actively in mathematical activity and discussion, freely exchanging ideas and problem-solving strategies with their classmates and teachers, and taking intellectual risks and defending positions without fear of being incorrect.
5. Work cooperatively with other students on mathematical activities, actively sharing, listening, and reflecting during group discussions, and giving and receiving constructive criticism.
6. Make conjectures, pose their own problems, and devise their own approaches to problem solving.
7. Assess their work to determine the effectiveness of their strategies, make decisions about alternate strategies to pursue, and persevere in developing and applying strategies for solving a problem in situations where the method and path to the solution are not at first apparent.
8. Assess their work to determine the correctness of their results, based on their own reasoning, rather than relying solely on external authorities.

STANDARD 18 All students will be evaluated using a diversity of assessment tools and strategies to provide multiple indicators of the quality of every student's mathematical learning and of overall program effectiveness.

(This "learning environment standard" was developed and approved by the task force that prepared the *Mathematics Standards* and appears in the Introduction to the *Mathematics Standards* chapter of the New Jersey State Department of Education's *Core Curriculum Content Standards*; however, since it was not considered a "content standard," it was not presented to the New Jersey State Board of Education for adoption.)

Descriptive Statement

A variety of assessment instruments should be used to enable the teacher to monitor students' progress in understanding mathematical concepts and in developing mathematical skills. Assessment of mathematical learning should not be confined to intermittent standardized tests. The learning environment should embody the perspective that the primary function of assessment is to improve learning.

Cumulative Progress Indicators

Experiences will be such that all students:

1. Are engaged in assessment activities that function primarily to improve learning.
2. Are engaged in assessment activities based upon rich, challenging problems from mathematics and other disciplines.
3. Are engaged in assessment activities that address the content described in all of New Jersey's *Mathematics Standards*.
4. Demonstrate competency through varied assessment methods including, but not limited to, individual and group tests, authentic performance tasks, portfolios, journals, interviews, seminars, and extended projects.
5. Engage in ongoing assessment of their work to determine the effectiveness of their strategies and the correctness of their results.
6. Understand and accept that the criteria used to evaluate their performance will be based on high expectations.
7. Recognize errors as part of the learning process and use them as opportunities for mathematical growth.
8. Select and use appropriate tools effectively during assessment activities.
9. Reflect upon and communicate their mathematical understanding, knowledge, and attitudes.

Nine Vignettes

This section contains nine vignettes which suggest how New Jersey's *Mathematics Standards* can be effectively implemented in classroom settings.

The table below indicates the content standards and grade levels which each vignette particularly addresses.

The vignettes highlight, using marginal notes, how the learning environment standards and the first five content standards serve as a context for mathematics learning. These reinforce the emphasis that the *why's* and *how's* of mathematics learning must be integrated with the content.

Although these nine vignettes reflect all eighteen standards, they certainly do not fully address all of the cumulative progress indicators that are attached to the standards. They are intended to be illustrations of the way that individual educators have suggested that these standards be implemented. Teachers are encouraged to review and discuss them, to experiment with practices that they exemplify, and to develop their own activities consistent with the standards.

		Content Standard															
Vignette	Page	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Grades K-4																	
Elevens Alive!	46	X	X	X	X	X			X				X				X
Product and Process	48	X		X	X	X	X		X		X						X
Sharing A Snack	51	X	X	X	X	X	X		X						X		X
Grades 5-8																	
The Powers of the Knight	54	X	X	X	X	X	X		X			X		X			X
Short-circuiting Trenton	56	X		X	X	X				X					X		X
Mathematics at Work	57	X	X	X	X	X		X		X	X						X
Grades 9-12																	
On The Boardwalk	60	X	X	X	X			X				X	X		X	X	X
A Sure Thing!?	64	X	X	X	X			X				X					X
Breaking The Mold	66	X	X	X	X	X		X		X	X			X		X	X

Elevens Alive!	***The students:***
While Mr. Johnson is meeting with some of the children in his first-grade class, others are involved in a number of different activities. At the Math Center, pairs of students have cups with eleven chips that are yellow on one side and red on the other. As each pair pours out the chips, they write a number sentence showing how many yellows and how many reds they got, as well as the total. When they have written ten number sentences each, they move on to another activity.	*work on basic facts in the context of a problem and in relation to other areas of mathematics.* *work in pairs with manipulatives.*
Later in the day, as Mr. Johnson begins the math lesson, he asks the students to recall their discussion from the previous day, "What were we talking about yesterday in math?"	*practice their number facts by writing down each result.*
"We were doing numbers that add up to eleven, like $5+6$ and $2+9$," answers Clark.	*share mathematical ideas.*
"Or $3+8$ and $4+7$," adds Sarah.	
"Is there more than one way to get a sum of eleven?"	
Mr. Johnson lists all of the children's responses on the board. He goes on to ask them, "What were you doing at the Math Center earlier today?"	*connect their understanding of one mathematical idea to another.*
Jackie responds, "We were tossing counters and writing number sentences."	
"We were tossing eleven counters!" says Toni.	
"What can you tell me about your results?" asks Mr. Johnson. "Did you get the same number sentences as your partner?"	
"No — we got different ones!"	
"Our answers were always the same — eleven!"	*report and reflect on the differences of their results.*
"I got some number sentences more than once!"	
"I got 5+6 three times!"	
"I didn't get 0+11 or 11+0 at all!"	
"Why do you think you got different answers?" asks Mr. Johnson. He listens as the students talk about fairness, luck, and chance, pointing out that all of the counters are alike. The students agree finally that the different number sentences are a result of chance.	*informally explore the concepts of probability.*

The students continue their discussion of which number sentences appear more often than others. One of the children suggests that maybe they should make a graph to help them see which number sentences occur most often. Mr. Johnson thinks that this is a good idea. He goes through their list of number sentences, asking students to raise one finger if they got that number sentence once, two fingers if they got it twice, and so on. For each finger raised, he puts a tally mark on the board. When they are done, he asks whether there were any other number sentences that anyone got. Then the children look at the general shape of the data, noticing that most of the number sentences were in the middle. Mr. Johnson points out that not all of the number sentences are equally likely to occur. He says that tomorrow they will have a chance to play a game with the counters in which they will need to select which number sentences will be winners. Tomorrow's activity will continue providing opportunities for practicing basic facts while building on the beginning ideas of probability.

The students:

use different methods to display data.

make inferences about their data.

Product and Process

The students:

Mr. Marshall had assigned the following problem from the New Jersey Early Warning Test as a homework assignment for his fourth grade class:

> Use each of the digits 3, 4, 5, 6, 7, and 8 once and only once to form three-digit numbers that will give the largest possible sum when they are added. Show your work.
> Is more than one answer possible? Explain your answer.

are asked to respond to open-ended questions and present and defend their solutions.

The students were to solve the problem and match their response with that of Tilly Tester to see if they agree or disagree with Tilly's response and explain why.

are asked to analyze problems for reasonableness of results and to diagnose errors.

Tilly Tester

876	876	864	853
345	543	753	764
1221	1419	1617	1617

More than one answer is possible.
I Tried several ways and the
last two got the same answer.

As the math class begins, Mr. Marshall allows the students to work in the cooperative learning groups which they have been working with this month to compare the results of their homework assignment. Mr. Marshall visits each group noting who has completed the assignment as well as the direction of the discussion for each group. Homework assignments are important and students are given credit for homework. Strategies such as displaying answers on the overhead projector and working in cooperative learning groups are used to ensure that homework review is no more than 5 to 10 minutes.

work cooperatively to assess their own and each other's work.

Mr. Marshall then asks the students to show the level of their agreement with Tilly's response on a 0-5 scale, with 0 signifying disagreement and 5 signifying total agreement. Most students raise 4 or 5 fingers, and the discussion then focuses on how Tilly's answer could be improved. One group notes that Tilly should have added each pair of numbers and shown the sum for each, while another group explains that Tilly could have also changed the hundreds place to get 754+863 and 763+854.

are willing to take a position without the fear of being incorrect.

At this point, Mr. Marshall discusses Tilly's understanding of place value and uses the opportunity to summarize the students' responses and lead

use their knowledge of numeration to help solve problems.

into the objective of the day which focuses on place value and multiplication.

"Let's work on multiplication today, and to get started, let's do some mental math with multiplication. On the back of your homework, number 1 to 10. Write the answers only for my mental math flashcards."

Individually, the students write answers for 8000x3, 6000x7+50, 300x7, etc. After the ten problems, Mr. Marshall has the students exchange papers, and they correct and discuss the answers. The papers are collected, and Mr. Marshall poses the following problem for his students:

> *Use four of these five digits and construct the multiplication problem that gives the greatest product: 1, 3, 5, 7, 9*

Before allowing the students to start work on the problem, he asks them to estimate what the largest product obtained in this manner might be. Students offer estimates ranging from 3000 to 10,000 and provide explanations for their guesses. When allowed to, the class works in their cooperative learning groups. Calculators are available, and some students start guessing and checking with their calculators.

One group begins to discuss which digits to use, wondering whether there would be a reason not to use the four largest digits. Another group is discussing whether a 2-by-2 or a 1-by-3 arrangement would be the best for getting a large product, an aspect of the problem that some groups have completely missed. Most of the groups get around to trying out sample problems of a variety of sorts to get some parameters worked out. Toward the end of the class session, the groups share the specific answers they have come up with. The three examples that are suggested are:

753	93	953
x 9	x 75	x 7
6777	6975	6671

It is clear to everyone that the 2-by-2 digit problem is the one with the greatest product, but Mr. Marshall is looking for some generalizations that can be made. He points out that none of the groups used the digit "1" in their examples. *Can the lowest digit always be ruled out?* He asks the groups that arrived at the 2-digit problem to explain how they decided where to put the individual digits. *Does it matter where they were placed? Where does the largest one go? The smallest? Do you think it would always work that way regardless of what the individual digits were? How can you check?* The students reflexively pick up their calculators and begin to formulate other versions of the problem that use other digits and to check which arrangements of the digits give the largest product.

The students:

use mental math regularly throughout the curriculum.

demonstrate their understanding of mathematical concepts in a variety of ways, each of which provides valuable assessment information to the teacher.

are encouraged to estimate solutions before actually determining answers.

use calculators to aid in the problem-solving process.

using mathematical reasoning to formulate strategies and solutions.

approach numerical operations from a holistic point of view rather than only through paper-and-pencil manipulation.

One student asks his partners what they think would happen if two or three digits were the same.

For homework, Mr. Marshall asks the students to use the same five digits, but to find the smallest possible product. They are then to write a paragraph describing their solution and the reasoning they used to show it is, indeed, the smallest product. Specifically, they are to consider the question: *Can you just turn your thinking about the way you got the largest product upside down and use it to get the smallest product?*

The students:

write paragraphs describing and justifying their positions.

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Sharing a Snack

The students:

Today is November 12 and Maria, a student in Miss Palmer's second grade class is very excited. Today is Maria's birthday, and as is the custom in her class, she is bringing in a birthday snack to share with her classmates. Maria and her father spent much of the previous evening making a batch of chocolate chip cookies and she proudly walks into class carrying a cannister full to the brim. Miss Palmer realizes that she can use mathematics to help the class divide the cookies.

use mathematics to devise a solution for real-world problems.

Before the afternoon snack time, Miss Palmer poses the problem to the whole class.

Miss Palmer states, "Today is Maria's birthday and she has brought in some delicious chocolate chip cookies for all of us to enjoy at snack time. Maria told me she baked a whole bunch of cookies. I would like us to think about how we could determine the number of cookies each student in the class should get. Discuss it with your partner."

use cooperative work to generate potential solutions.

The students begin to discuss all of their ideas. After a few minutes, Miss Palmer calls on a few of the students. As they share their ideas, the teacher records them on the language experience chart.

regularly share their ideas publicly.

Sarah states, "Well, me and Mario think that the first thing we have to do is count the cookies to find out how many there are."

Jerome adds, "Yeah, and we also need to know how many children are in the class today."

"That's easy. I did the lunch count this morning and there are 22 children in school today," Maria volunteers.

Luis chimes in. "Once we know how many cookies and how many children, then we can figure out a way to solve the problem."

The children all agreed that since they know there are 22 children in class today, the next step was to determine the number of cookies. Miss Palmer highlights that idea on the language experience chart and gives each pair of children a bag of counters which represents the number of cookies.

Miss Palmer says, "Each pair of children received a bag of counters. I want you to pretend that these are Maria's cookies. I've counted the cookies. The number of counters in each bag is equal to the number of cookies Maria brought for a snack. With your partner, use your counters to first decide how many cookies Maria brought to school and then determine how many cookies each student will get if the cookies are to be

use manipulatives to model real-world situations.

shared equally among everyone in the class. When you are finished, each pair will need to write a story which explains how both of you solved the problem."

The students:

The children worked with their regular partners. The first task they all tackled was to count the number of counters in each bag. Most of the pairs of children counted by twos to determine the total number of counters was 62. However, Alex and Laura kept losing count when trying to count all the counters and decided to group the counters by ten. Miss Palmer was delighted to see that most pairs of children had written the total number of counters (62) on a sheet of paper. She had been stressing the importance of collecting data and recording information.

use their knowledge of decimal place value to simplify the task.

As Miss Palmer continued to circulate around the classroom, she noticed the children were solving the sharing problem in various ways.

One pair of students begins by drawing 22 stick figures to stand for the students in the class and then starts to "give out" the cookies by drawing them in their stick figures' hands. Another pair also starts with 22 stick figures but then draws 62 little cookies on another part of the paper and is stumped about where to go from there. Mario and Sarah begin to sort the 62 counters into 22 piles. Another pair, trying to use calculators to solve the problem, starts by adding 22 cookies for everyone to another 22 cookies for everyone to a third 22 cookies for everyone and then realizes that they have exceeded the number of cookies available.

develop their own methods for solving the problem.

use technology as a problem-solving tool.

Miss Palmer, noticing that the students will be unable to finish the problem before they have to go to Physical Education, calls the students back together.

"I want all of you to stop what you are doing, and with your partner write a story to tell me how you are attempting to solve this problem," she directs.

The students eagerly write their stories. Some use pictures to help illustrate their solutions.

draw pictures to model their solutions.

Miss Palmer requests, "I would like some of the pairs to report to the whole class how they were attempting to solve the problem."

Luis states, "Well, Elizabeth and I figured out that each student could have 2 cookies and there will be 18 cookies left. We know this because we drew a picture of the class and put counters on each student. When we couldn't give counters to every kid, we decided those were leftovers and we counted them."

give explanations of their strategies for solving the problem.

Lisa volunteers, "We drew stick figures too. After we gave out 2 cookies to each child, Jerome said we couldn't give out the 18 leftovers. But I think we can break the leftover cookies in half. Then each child would get 2 whole cookies and one half cookie. But I'm not sure how many would be left over then."

"Sarah and I used the calculator to solve the problem. We put in 62 and I counted while Sarah subtracted 22. We got 2 with 18 left over," Mario added.

"Alex and I got a different answer. We used the counters and put them into 22 piles, but we got 17 leftovers," Laura said.

Lisa suggested, "Maybe you and Alex should count them again to make sure you have the right number in each pile."

Laura and Alex recount their piles and discover that one counter fell on the floor.

Vanessa states, "Me and my partner thought of another way of sharing the leftover cookies. Everyone could write their name on a piece of paper, then put all the papers into a bag and have Maria close her eyes and pick out 18 names. Those kids would get the extra cookies."

Sarah protested, "We forgot about Miss Palmer. We should give her 2 cookies and that would leave 16 left over. Maria could give them to the principal and the other ladies in the office."

Miss Palmer wrapped up the discussion. "We've discussed many ideas for sharing the 62 cookies Maria brought for a snack. On the back of the sheet of paper I gave you, I would like you and your partner to decide on how you think we could fairly share the cookies."

The children work on their final summary of the problem and hand their papers in before getting on line for Physical Education. While the children are in Physical Education, Miss Palmer reads the children's solutions. She makes notes on the cards she keeps for each child. This will help her better understand various developmental levels of her students. She notices that Vanessa has really made progress since September. Laura and Alex still like to "rush" to finish their work. She makes a note on their paper encouraging them not to be so concerned about being the first ones finished. Overall, she feels encouraged, not only about the solutions to the problems, but also about the ways in which her class has learned to communicate their ideas both orally and on paper. She decides to let the class choose one of the methods suggested to distribute the cookies at snack time.

The students:

informally explore the uses of fractions and notions of fair sharing.

are mutually supportive and regularly offer feedback to each other.

demonstrate their understanding of mathematical concepts in a variety of ways, each of which provides valuable assessment information to the teacher.

The Powers of the Knight

The students:

Mr. Santos' 6th grade class has just completed a review of place value in the decimal number system and he is preparing to start a unit introducing exponents. He has coordinated the timing of this unit with the language arts teacher whose class is in the midst of a unit on fables. One fable they have read involves a knight who saves a kingdom from a horrendous dragon. Given the opportunity to determine his own reward, he tells the king that he would take one penny on the first square of a chessboard, two pennies on the second, four pennies on the third, and so forth until each square on the chessboard has twice as many as the previous one. Mr. Santos has the students recall the story and then asks the students to determine how much money the knight would make with this method of payment.

are connecting a language arts experience to their mathematics learning.

Mary said, "We need to know how many squares there are on a chessboard before we can do this problem."

are comfortable taking risks.

Lionel stated, "Give me a minute to think. I play on the chess team, but I need to take a moment to picture it. Let's see, I know it's square and there are 1,2,3, ...8 squares along the one side. There are 64 squares!"

use known facts to explain their thinking.

Jerry shouted, "He gets 128 pennies. Two on each square."

"The fable doesn't say he gets two on each square! It says that each square has twice as many as the one before. It has to be more than 128!" corrected Meredith.

react substantively to others' comments.

"We need to examine this situation in some organized fashion. I want you to get in your groups of four and determine the people who will serve the usual roles of leader, recorder, reporter, and analyzer of group interaction," stated Mr. Santos.

use standard cooperative learning strategies.

One group decided to develop a computer program which printed a table listing the number of the square, the number of coins on that square, and a subtotal to that point.

use technology to help solve the problem.

Another group borrowed the class chessboard and began placing play coins on the squares. It soon became obvious to them that they would not have enough play money to complete this attempt. They started to make a table with the information they had constructed and worked to find a pattern which they could extend to the complete board. Their table only included columns showing the number of the square, the number of coins on that square, and a column to list patterns. They discovered that the number of coins could be represented by raising 2 to the power which was one less than the number of the square. Using calculators, they found the number of coins on each square and then the total number of coins.

concretely model the problem before they move on to more symbolic procedures.

use self-assessment to determine the effectiveness of their method.

Another group began making a table similar to the group above, but they also included a column showing the partial sums and another which attempted to find a pattern in the partial sums. Eventually, they discovered that the partial sum at each square was one less than 2 raised to the power equal to the number of the square. They could then quickly utilize the calculators to compute the total.

At the end of the period, Mr. Santos reminded the groups that they were to prepare a report of their methods which included a description of their processes, an explanation of why they chose them, and their evaluation of their processes. He asked each of them to consider the magnitude of their answer and find some way to explain to another person just how large the answer was. Students brainstormed some ideas such as the distance between two known points or objects, the magnitude of the national debt, and the number of people on earth.

The students:

analyze mathematical situations by recognizing and using patterns and relationships.

choose technology to reduce the computational load.

write about their approaches and solutions to problems.

connect their knowledge of mathematics to the real world.

Short-circuiting Trenton

The students:

Ms. Ramirez announces to her seventh grade class that in three weeks they will make a journey to Trenton, the capital of New Jersey. They will be visiting eight sites — the Capitol, the New Jersey Museum, the War Memorial, the Old Graveyard, Trent House, the Old Barracks Museum, the Firehouse, and the Pedestrian Mall. To ensure that they spend as much time at the sites as possible, and do as little walking as possible, the class must find the most efficient walking tour for the trip, starting and ending at the parking lot.

apply mathematical skills to solve a real-world problem.

The first problem that the students must address is finding the walking distance between each pair of sites. Ms. Ramirez supplies each team with a street map and a ruler; the maps identify all the sites to be visited and the routes joining them. She assigns each group the task of finding the distances between one site and all the others. This turns out to be an interesting task, since different groups interpret it differently. Some groups, for example, measure the straight line distance between two sites forgetting that buildings or ponds might render that walk impossible. How to measure the walking distance thus becomes an important topic of discussion, as does the question of appropriate units. These questions are eventually settled and the teacher uses the students' measurements to write a matrix which indicates the walking distance between any two of the eight sites; different groups occassionally have obtained different numbers, but after discussion, they have arrived at a common answer.

use cooperative group work to generate problem-solving strategies.

freely exchange ideas and participate in discussions requiring higher-order thinking.

collect and organize data needed to solve the problem.

Ms. Ramirez selects a sample route for the walking tour and through discussion with the class explains how the total length of the walking tour is obtained from the matrix of information that the students generated — you find the distances between consecutive sites on the tour, and then add up the walking distances along the tour. She now asks her students to work in groups to decide on a strategy that they think will produce an efficient route (which starts and ends at the parking lot), and to assist the group's recorder in writing a short paragraph explaining their strategy. Some groups decide to list all possible routes and calculate how long a walk each route entails. (Ms. Ramirez asks the students how many possible routes do they think they will have to list.) Other groups suggest that the best route is obtained by always going to the nearest site.

recognize there are numerous ways to solve the problem.

work in cooperative groups to develop alternative strategies.

Ms. Ramirez now asks the students to use calculators to carry out their strategy and determine the travel time for the routes they will be considering. After each group presents its results, the class will together compare the various methods that were proposed and the accompanying results. Among the questions which Ms. Ramirez will ask are: "Do the various methods give the same result?", "Which methods result in a most efficient route?", "What other strategies could we have used?" Responses from the students might include: "always use the shortest distance", "never use the longest distance", "put distances in increasing order and use only those that neither make a loop or put a third edge into a vertex."

compare the variety of strategies proposed.

Mathematics at Work

The students:

As a regular feature in his class, Mr. Arbeiter has parents of each student make a presentation about their job and how the various educational disciplines are needed for them to be successful. Today, Emily has asked her Mom, the owner of a heating and air conditioning company, to talk to her class. Mrs. Flinn and Mr. Arbeiter decide to have the students help her solve a problem similar to one which her company faces regularly. She briefly describes her company, the work that she does, and tells the students that they are going to help her determine how large an air conditioner will be needed in the classroom. She poses the following problem: What information about the room would be most important in determining how large an air conditioner is needed? The students quickly agree that the amount of air conditioning would depend on the amount of air in the room, and that in turn, depended on how much space there was in the room. Through suggestions and hints, Mrs. Flinn had them realize that the amount of sunlight entering the room would have an effect as well and they quickly agreed that the area of the windows must be found too.

interact with parents who use mathematics and other disciplines in their daily lives.

have the time to explore a problem situation thoroughly.

Mr. Arbeiter reminded the class that there is a mathematical term which represents the amount of space, and asked each student to write down that term. As was his custom, Mr. Arbeiter asked six students, one quarter of the class, to read the words they had each written. Four read the word "volume" and two read "area." By a show of hands, he found that about one third of the class had written "area" and two thirds had written "volume." In their groups, the students were asked to discuss the difference between area and volume and to write down the differences between them. As the groups discussed these concepts Mrs. Flinn and Mr. Arbeiter circulated among them, making sure that each group had focused on the difference between area and volume; subsequently the groups read the statements they had prepared, and the entire class discussed and commented on the groups' statements. Mrs. Flinn had the class discuss which of the concepts were needed on the two phases; amount of space in the classroom and how much window space there was.

are regularly assessed through a variety of methods.

work in a variety of settings to develop concepts and understanding.

Now that all students agreed on the difference between area and volume and where each applied in this case, the discussion turned to discussion centered on how one obtains the volume of the classroom and the area of the windows. Although familiar with the concept of volume, the class was not able to calculate volume easily, so Mr. Arbeiter suggested that each group build a rectanguar box out of cubes and figure out how many cubes the box contained. Most groups discovered that they could get the answer by multiplying the number of cubes in the bottom layer by the number of layers (the "height"), and agreed with Mr. Arbeiter's conclusion that V = BxH. When Mr. Arbeiter asked them how they calculate the number of cubes in the bottom layer, all agreed that you multiplied length times width; and when the teacher wrote

use concrete materials to develop a model for volume.

V=BxH=(LxW)xH, several other groups recognized that that was how they found the volume of their box.

Mr. Arbeiter asked the class "How does the volume formula help us find the volume of the classroom?" The students agreed that the shape of the classroom was about the shape of a rectangular box, but were quick to point out that to any answer obtained by the formula would have to be considered an estimate, since it would not be taking alcoves and pillars into consideration. They agreed to change the question to "How does this formula help us estimate the volume of the classroom?"

"All we have to do is measure the three quantities — length, width, and height, the three dimensions of the classroom, and multiply the three numbers together" was the prevailing sentiment. Marcia observed that "since we're only going to get an estimate anyway, why should we measure those three amounts exactly?" And Mrs. Flinn noted that her sales people often estimated the size of the room without making any measurements. "How can we estimate the dimensions of a room without making measurements?", she asked. Paula suggested that "maybe the salesperson estimates the three dimensions and multiplies those estimates together." "A great suggestion," Mrs. Flinn responded. "Let's try that ourselves."

"Let's first estimate the *width* of the room. About how many inches wide is this room?" Brian pointed out that inches is an appropriate unit for a piece of paper, but not for a room. After a brief discussion, Mrs. Flinn revised her question to "About how many *feet* wide is this room?"

The students wrote down their estimates and explanations of how they arrived at them. After hearing all of the students estimates and reasons, the students were asked to return to their regular groups and decide as groups what they thought the width of the room was. "Well," said Mr. Arbeiter, "you all gave good reasons for your estimates, but now let's see whose estimate was closest. We'll measure the width of the classroom." Great cheers were heard for the groups whose estimate was closest to the actual measurement. The same process was repeated for length, and width as well as estimating the window area of the classroom. Mrs. Flinn pointed out that estimates were getting closer to the actual measurements each time they did it. She then showed the class a formula used to determine the number of BTUs needed for a room in terms of the volume of the room and the area of the windows. The data obtained by the class for the volume and window area was entered in the formula, and a quick calculation gave the number of BTUs needed for the classroom. Mrs. Flinn wrapped up her presentation by making the connection between the size needed, the cost of the purchase, and the regular expense of running the air conditioner. She emphasized that the success of her business rested on the sales people and their ability to estimate the needs well.

The students:

recognize and apply estimating to geometric situations.

are exposed to a variety of open-ended questions and respond .

feel comfortable identifying errors.

communicate their answers and defend their thought processes.

examine the correctness of their results.

Mr. Arbeiter thanked Mrs. Flinn for her presentation and asked the students how they would like to practice the skills they had discussed today. Feeling confident, the students volunteered to estimate the data for their other classrooms. Mr. Arbeiter agreed to display the results, so long as the students agreed to leave off estimating while their other classes were in session.

The students:

extend their skills through practice in similar problems.

On the Boardwalk

The students:

"It isn't fair!", Jasmine announced to her class one Monday morning. "I used up $10 worth of quarters playing a boardwalk game over the weekend at the shore, and I only won once. And all I got for winning was a lousy stuffed animal!"

Ms. Buffon often told her class that mathematics was all around them, and had encouraged them to see the world with the eyes of a mathematician. So she wasn't surprised that Jasmine shared this incident with the class.

recognize the role that mathematics can play in explaining and describing the world around them.

"Please explain why you thought there was mathematics here," Ms. Buffon asked Jasmine.

"Well, first of all, I threw the quarters onto a platform which was covered with squares, you know, like a tile floor, so that reminded me of geometry. And as I was throwing my quarters away, one after another, I was reminded of all the probability experiments that we did last year, you know, throwing coins and dice. It wasn't exactly the same, but it was like the same."

connect previously learned mathematics to the current situation.

"Those were very good observations, Jasmine," said Ms. Buffon, "you recognized that the situation involved both geometry and probability, but you didn't tell the class what you had to do to win the game."

"Oh, you just had to throw the quarter so that it didn't touch any of the lines!" Ms. Buffon asked Jasmine to go to the board to draw a picture, explaining to her that not everyone will visualize easily the game she was talking about.

use different forms of communication to define a problem and share their insights.

Every other Monday, Ms. Buffon began her geometry class with a sharing session. Sometimes the "mathematics situations" that the students shared did not lead to extended discussions, in which case Ms. Buffon continued with the lesson she had prepared. But she was prepared to use the entire period for the discussion, and even carry it over into subsequent days, if the students got interested in the topic.

are afforded the opportunity to fully explore and resolve mathematical problems.

"Why didn't you think the game was fair?" she asked Jasmine. Jasmine repeated what she had said earlier, that she should have won more often and that the prizes should have been better. Other students in the class were asked to respond to the question, and after a lively interchange, they decided that for the game to be really fair, you should get about $10 in prizes if you play $10 in quarters; but, considering that they were having fun playing the game, and considering that the people running the game should get a profit, they would be satisfied with about $5 in prizes. Jasmine listened to the conversation intently, and chimed in at the end "That lousy bear wasn't worth more than a dollar or two!"

explore questions of fairness, geometry, and probability.

Moving the discussion in another direction, Ms. Buffon said "Now that we understand that it is possible to explain 'fairness' mathematically, let us investigate Jasmine's game to see if it really was unfair. What do you think were Jasmine's chances of winning a prize?"

The students:

This question evoked many responses from the class, and after some discussion the class agreed with Rob's comment that it all depended on the size of the squares. Jasmine did not know the actual size of the squares, so the class agreed that they might as well try to figure out the answer for different size squares. Dalia pointed out that this looked like another example of a function, and Ms. Buffon commended her for making this connection to other topics they had been discussing.

are encouraged to make connections to other topics within mathematics.

Returning to her previous question, Ms. Buffon suggested that the students do some experiments at home to help determine the probability of winning a prize. Each pair of students was asked to draw a grid on poster board, throw a quarter onto the poster board 100 times, and record the number of times the quarter was entirely within the lines; to simplify the problem, quarters that landed off the grid were not counted at all. Different students chose different size grids, ranging from 1.5" to 3.5", at quarter inch intervals.

model problems and conduct experiments to help them solve problems.

After school, Ms. Buffon visited the Math Lab where she spent some time trying to find materials related to this problem. When she looked under "probability" in the indexes of various mathematics education journals, she was led to several articles discussing geometric probability, which she learned is a branch of mathematics which addresses problems like Jasmine's game. With these resources available to her, Ms. Buffon no longer feels that she has to have all the answers, and can entertain discussions about mathematical topics with which she is unfamiliar. Tomorrow she will be able to tell the class what she has learned!

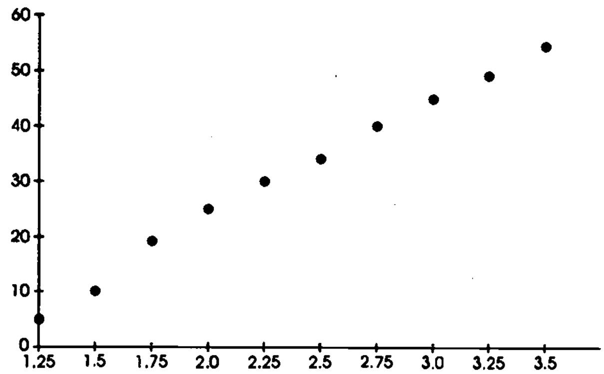
The next day the students reported on their results, and Ms. Buffon tabulated them in the following chart, and, at the same time, plotted their results on a graph:

Size of Squares	Number of Wins
1.25	5
1.5	10
1.75	18
2	25
2.25	30
2.5	34
2.75	40
3	45
3.25	48
3.5	54

collect, analyze, and make inferences from data.

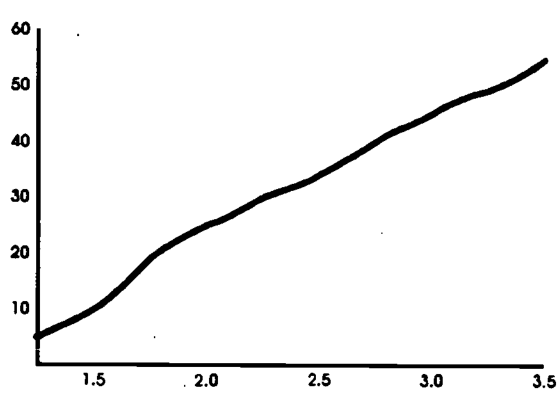
The students:

"Do you see any patterns here?", Ms. Buffon asked the class. They all agreed that, as expected, the larger the size of the squares, the more frequently Jasmine would have won the game. "What do you think the size of the squares were on the boardwalk game?" Ms. Buffon asked next. Everyone agreed that the squares were most likely smaller than



1.25" since her one prize out of forty quarters corresponded to 2.5 wins out of a 100 games, which was lower than obtained for the smallest squares in the experiment. Turning to the graph, Ms. Buffon asked "What would have happened if we tried the experiment with squares smaller than 1.25?" The students laughed, one after another, as they realized that if the size of the squares were small enough, you would never win the game. "Well, then, what would have happened if we tried the experiment with larger and larger squares?" Looking at the graph, the class found this a difficult question, but Fran broke the group's mindset by saying "Yeah, suppose the squares were as big as this room?" Then everyone realized that if the squares were larger and larger, you would become almost certain to win the game. "Dalia, do you remember your comment yesterday, that it sounded like we were working on a function?" Ms. Buffon asked. "Would you sketch the graph of that function for the class?" Ms. Buffon made a mental note to discuss this problem with her precalculus students, since she had many questions to ask them about this graph.

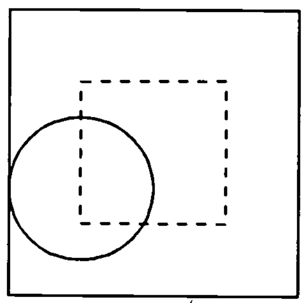
recognize connections between numerical patterns and functions.



"Well, we've gotten a lot of information by using experimental methods about Jasmine's game; let's see if we can figure out the probability theory behind it as well." At this point, Ms. Buffon was eager to tell her class what she had learned at the Math Lab. However, being aware that the students will grasp the solution method better if they have an opportunity to discover it for themselves, she asked the class to discuss the following question in their study groups: "How can you tell from the position of the quarter whether or not you would win the game?" Going from group to group, Ms. Buffon listens to the discussions. When the groups have discovered that to win the game, the center of the quarter must be sufficiently far from the closest border, she gives the groups their next task — to describe where in the square the center of the quarter must be.

The students:

formulate and test mathematical conjectures.



construct a pictorial model to represent the problem.

By reasoning in different ways, the groups all arrived at the same picture involving a smaller square inside the original square, and at the same conclusion — that you win if the center of the quarter lies inside the smaller square. With this information, the students are able to calculate the probability for any particular size of the square, and even to write an equation for the function whose graph they sketched earlier.

Having found the probability of winning the game, Ms. Buffon planned to return to the question that began this whole discussion — whether Jasmine's game was fair. But that was the topic for another day.

Note: Ms. Buffon realized that the graph of the function was not linear, as depicted earlier, even though the data seemed to indicate linear growth. With her precalculus students, she would have them translate the above situation into the equation $y=(x-d)^2/x^2$, where d is the diameter of the quarter. Then she would have them graph the function, enabling them to discover that although the graph appears to be linear, in reality it increases at a decreasing pace, and goes asymptotically to the line $y = 1$.

A Sure Thing!?

The students:

Ms. Jackson is teaching her geometry students to use and identify inductive reasoning.

She asks each student to draw a large triangle on their paper. She then asks the students to hold up their triangles so that they can see the wide variety that have been created. The students observe that all the triangles are different.

use a variety of types of mathematical reasoning to solve problems.

Ms. Jackson then asks the students to cut out their triangle, to tear off the corners of their triangle and to place the corners together so that they are adjacent. She circulates around the room to be sure everyone is on task, and tells students to record a description of what they see in their notebooks.

Ms. Jackson then asks a representative sampling of students to tell the class what they observed after fitting the corners together. The students report that it looks as if the corners form a straight line. Everyone agrees.

are encouraged to form generalizations based on observations they have made.

Ms. Jackson now asks the students to write a generalization about the angles of ANY triangle based upon the class results of this activity. She asks another representative sampling of students to state their generalizations. The students conclude that the sum of the measures of the angles of ANY triangle is 180 degrees.

She gives the students a definition of inductive reasoning. They recognize that they have used induction to reach their generalization about the angles of a triangle. She then asks them to think about when they have used inductive reasoning in the past and write an example in their notebooks.

Volunteers are asked to share their recollections with the rest of the class. Some are funny and some quite poignant. The teacher asks if anyone can see a drawback to inductive reasoning within social as well as mathematical contexts.

are regularly asked to write about their understandings of mathematics and its uses in the real world.

The class decides that one drawback is that you can't check all examples - all triangles cannot be checked to see if the angles always add to 180 degrees. Another is that if you check too few examples you might reach an erroneous conclusion. They discuss how this is the reason for much of the racial and gender stereotyping that they encounter. Ms. Jackson asks students to identify counterexamples for racial and gender stereotypes.

Ms. Jackson then asks the students to do another experiment. They use their compasses to draw 5 circles. On the first circle, the students identify and connect 2 points with a chord. They then state the number of non-overlapping regions into which the circle has been divided. On the second circle, students identify three points and draw all chords connecting these points. Once again, they state the number of non-overlapping regions into which the circle has been divided. They continue this procedure until they find the number of non-overlapping regions formed when 5 points on the circle are fully connected by chords. Students record their data in a table and use inductive reasoning to predict the number of non-overlapping regions produced by fully connecting n points on the circle with chords:

The students:

# of Points	# of non-overlapping regions
2	2
3	4
4	8
5	16
n	$2^{(n-1)}$?????

generate a set of data and use pattern-based thinking to formulate solutions.

They are asked to state their conclusion in narrative form.

The students agree that the number of non-overlapping regions produced by fully connecting n points on a circle with chords is $2^{(n-1)}$. Students then test their conclusion by carrying out the experiment with 6 points. Many find their conclusion is wrong for n=6. They fully expected to find 32 regions but only got 31!

validate conclusions by looking for counterexamples.

As class draws to a close, Ms. Jackson gives a homework assignment in which students will induce as well as produce counterexamples to conclusions. Students leave class somewhat dazed by the last experiment. Many of them tell Ms. Jackson that something must be wrong because they are sure the answer is 32. They tell her that they will prove her wrong by reenacting the experiment at home. She looks delighted and encourages their pursuit.

Breaking the Mold

The students:

Mr. Miller wants his ninth grade mathematics class to review the rectangular coordinate system, reinforce how mathematics is used to model situations, and develop the concept of exponential functions. He decides this would be an excellent opportunity to utilize a real-world situation. He elects to build his effort around an experiment involving mold growth found in an old SMSG book entitled *Mathematics and Living Things*.

incorporate scientific applications in their study of mathematics.

At the beginning of the unit, Mr. Miller presents the class with a packet of required readings, each of which deals with growth patterns of living things. There is an article on the rabbit population of Australia, another on world population, and another on the spread of AIDS. He explains the goals of the unit, gives the expectations for the readings, and describes the purpose of the experiment the class will conduct. Mr. Miller has students distribute the lab directions and materials, and he has them prepare the medium for the mold growth.

LAB DIRECTIONS

Materials:

1 - 9-inch circular aluminum pie plate
2 - sheets of 10x10-squares-to-the-inch graph paper
1 - rubber band
glue
scissors, ruler
saran wrap
mixture of clear gelatin, bouillon, and water

Directions:

Cut one piece of graph paper to fit the bottom of the tin as closely as possible. Draw a set of axes with the origin as near the center as possible. Cement the paper to the bottom of the tin with rubber cement. Pour the mixture into the tin so as to cover the graph paper with a thin layer. Allow the tin to sit 5 minutes, cover with plastic wrap, and hold in place by a rubber band. Place the tin in a dark place where the temperature is fairly uniform.

On each day over the next two weeks, students record an estimate of the area covered by the mold, the increase in the area from the previous day, and the percent of increase. On Fridays, they are asked to extrapolate the growth they expect to occur on Saturday and Sunday and then interpolate the same information from the growth they see on Monday. They are required to maintain a graph of the percent of increase versus the days. The extrapolated and interpolated points are both graphed with special marks such as "X" or "O."

estimate area of irregular figures.

collect and analyze data.

During the period of data-gathering, Mr. Miller develops exponential growth through the concept of compound interest and uses a graphing calculator to illustrate the graph of such growth. Each student is asked to suggest a function which would yield something close to their data, and has the opportunity to put their function into the graphing calculator and revise it until they are satisfied with the estimate. Time is provided to have the students discuss their reactions to the readings.

At the end of the two-week period, Mr. Miller has the students prepare a report relating the graph of their observations to the discussions of the readings and the work on compound interest. To extend the ideas developed in this experiment, students are given different data sets which came from actual measurements of various types of growth. Students work in groups, each group taking one of the sets of data. The groups are expected to make a presentation discussing the exponential function which models the growth, what limiting factors could be involved, and the carrying capacity of the environment.

As a closing activity, students are asked to choose a country from around the world, examine population growth over some period of time, and write a paper for inclusion in their portfolio discussing the mathematical issues and biological issues involved as well as a general discussion of the impact of such growth on the history of that period.

The students:

use technology as a tool of learning.

spend the time needed for mathematical discovery.

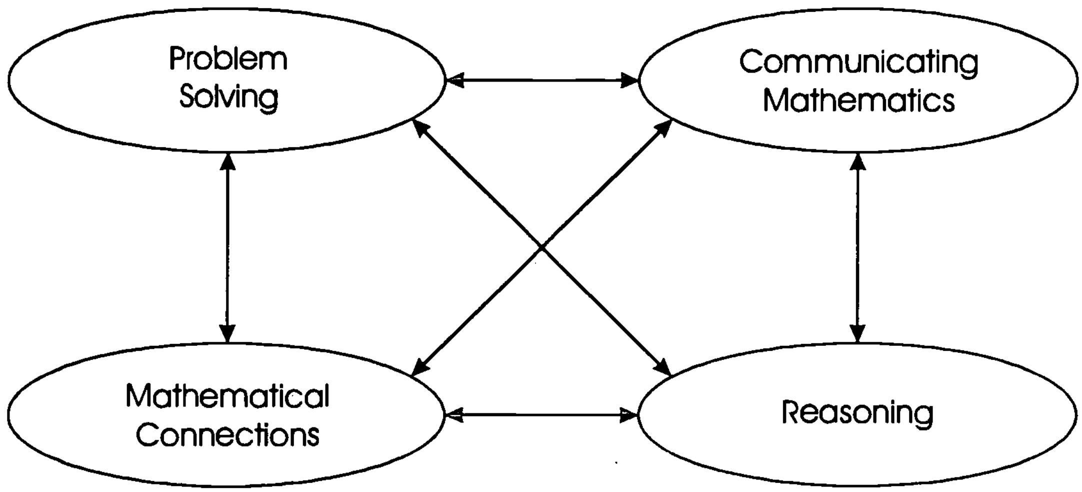
write about their understandings of the connections between mathematics and physical phenomena.

extend their understanding of mathematical concepts through cooperative work and presentation.

are assessed through alternative means.

explore the uses of mathematics in other disciplines.

THE FIRST FOUR STANDARDS



The First Four Standards deal with mathematical processes that apply to every topic in the mathematics curriculum. Students should always be improving their ability to **solve problems**, to **communicate about mathematics**, to **make connections** within mathematics and between mathematics and other subjects, and to **reason mathematically**.

Although the First Four Standards are conceptually separate, in practice they are interwoven. For example, when students are working in groups to solve a mathematics problem that may have arisen in another area, they use communication to explain their solution strategies and reasoning to justify their conclusions. The inter-relation of the First Four Standards is reflected in the graphic that appears above and elsewhere in this chapter.

This *Framework* therefore discusses the First Four Standards in a single chapter. It begins with a separate K-12 Overview for each of the First Four Standards. Following that, as with other standards, there is a grade-level section for each of the grade levels K-2, 3-4, 5-6, 7-8, and 9-12. In the grade-level sections, the First Four Standards are discussed together in a grade-level overview, vignettes are used to illustrate how the First Four Standards can be achieved at that grade-level, and the vignettes are then related to the cumulative progress indicators for each of the First Four Standards at that grade-level.

In the *Curriculum and Evaluation Standards for School Mathematics* of the National Council of Teachers of Mathematics, these four standards are referred to as "process standards." In the *Mathematics Standards* of the New Jersey State Department of Education's *Core Curriculum Content Standards*, they are enumerated among the core curriculum content standards because, like the other content standards, the First Four Standards involve concepts and skills that students must master, and explicit focus on them is necessary in order for that mastery to be achieved.

STANDARD 1 — PROBLEM SOLVING

K-12 Overview

All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.

Descriptive Statement

Problem posing and problem solving involve examining situations that arise in mathematics and other disciplines and in common experiences, describing these situations mathematically, formulating appropriate mathematical questions, and using a variety of strategies to find solutions. By developing their problem-solving skills, students will come to realize the potential usefulness of mathematics in their lives.

Meaning and Importance

Problem solving is a term that often means different things to different people. Sometimes it even means different things at different times for the same people! It may mean solving simple word problems that appear in standard textbooks, applying mathematics to real-world situations, solving nonroutine problems or puzzles, or creating and testing mathematical conjectures that may lead to the study of new concepts. In every case, however, problem solving involves an individual confronting a situation which she has no guaranteed way to resolve. Some tasks are problems for everyone (like finding the volume of a puddle), some are problems for virtually no one (like counting how many eggs are in a dozen), and some are problems for some people but not for others (like finding out how many balloons 4 children have if each has 3 balloons, or finding the area of a circle).

Problem solving involves far more than solving the word problems included in the students' textbooks; it is an approach to learning and doing mathematics that emphasizes questioning and figuring things out. The *Curriculum and Evaluation Standards* of the National Council of Teachers of Mathematics considers problem solving as the central focus of the mathematics curriculum.

> "As such, it is a primary goal of all mathematics instruction and an integral part of all mathematics activity. Problem solving is not a distinct topic but a process that should permeate the entire program and provide the context in which concepts and skills can be learned." (p. 23)

Thus, problem solving involves all students a large part of the time; it is not an incidental topic stuck on at the end of the lesson or chapter, nor is it just for those who are interested in or have already mastered the day's lesson.

Students should have opportunities to pose as well as to solve problems; not all problems considered should be taken from the text or created by the teacher. However, the situations explored must be interesting, engaging, and intellectually stimulating. Worthwhile mathematical tasks are not only interesting to the students, they also develop the students' mathematical understandings and skills, stimulate them to make connections and develop a coherent framework for mathematical ideas, promote communication about mathematics, represent mathematics as an ongoing human activity, draw on their diverse background experiences and inclinations, and promote the development of all students' dispositions to do mathematics (*Professional Standards* of the National Council of Teachers of Mathematics). As a result of such activities, students come to understand mathematics and use it effectively in a variety of situations.

K-12 Development and Emphases

Much of the work that has been done in connection with problem solving stems from George Polya's book, *How to Solve It*. Polya describes four types of activities necessary for problem solving: understanding the problem, making a plan, carrying out the plan, and looking back.

The first step in solving a problem is **understanding the problem**. Suppose that we want to solve the following problem:

> *A farmer had some pigs and chickens. One day he counted 20 heads and 56 legs. How many pigs and how many chickens did he have?*

After reading the problem, we want to be sure we understand it. We might begin by noting that we probably have to use the number of heads and the number of legs in some way. We know that pigs have four legs and chickens have two. We see that there must be 20 animals in all. We might observe that, if the farmer had only chickens, there would be 40 legs. If, on the other hand, he had only pigs, there would be 80 legs.

Some techniques that may help students with this important aspect of problem solving — understanding the problem — include restating the problem in their own words, drawing a picture, or acting out the problem situation. Some teachers have students work in pairs on problems, with one student reading the problem and then, without referring to the written text, explaining what the problem is about to their partner.

A second type of activity relating to problem solving involves **making a plan**. For our pigs and chickens problem, the plan might be to make a chart that shows various combinations of 20 chickens and pigs and how many legs they have altogether. If we have too many legs, we need fewer pigs, and if we have too few legs, we need more pigs.

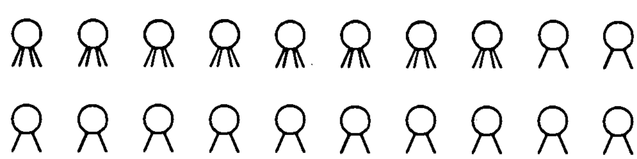
In order to be successful problem solvers, students need to become familiar with a variety of strategies that are used in making a plan for solving problems. Some of the strategies that are especially useful are making a list, making a chart or a table, drawing a diagram, making a model, simplifying the problem, looking for a pattern, using manipulatives, working backwards, eliminating possibilities, using a formula or equation, acting out the problem, using logic, using guess and check, using a spreadsheet, using a computer sketching program like *Geometer's Sketchpad, The Geometry SuperSupposer*, or *Cabri*, writing a computer program, or using a graphing calculator.

Let's **carry out our plan** for using a chart to solve the pigs and chickens problem. If we have 10 pigs (that's 40 legs) and 10 chickens (that's 20 legs), then we have 60 legs — that's too many legs. Let's try 9 pigs and 11 chickens — still too many. How about 8 pigs and 12 chickens? That's just right.

Number of Pigs	Number of Chickens	Number of Legs
10	10	40 + 20 = 60
9	11	36 + 22 = 58
8	12	32 + 24 = 56

Carrying out the plan is sometimes the easiest part of solving a problem. However, many students jump to this step too soon. Others carry out inappropriate plans, or give up too soon and stop halfway through solving the problem. To reinforce the process of making a plan and carrying it out, teachers might use the following technique: Divide a sheet of notebook paper into two columns. On the left side of the page, the student solves the problem. On the right side of the page, the student writes about what is going on in his/her mind concerning the problem. *Is the problem hard? How can you get started? What strategy might work? How did you feel about the problem?*

Let's **look back** at the problem we have just finished. The pigs and chickens problem may remind some of you of other problems you have solved; it's a little bit like some of the algebra problems involving the value of coins. Others may be intrigued by the pattern that we seem to have started in the last column of our chart and seek an explanation for this pattern. Still others may have solved this problem a completely different way; we could discuss all of the different strategies the students used and decide which ones seem most effective. One strategy used by young children is to draw a picture. Twenty circles represent the animals' heads. Each animal gets 2 legs. Additional pairs of legs are drawn on animals, starting at the left, until there are 56 legs.



This looking back activity is where students reflect upon the problem. *Does the answer make sense? Is the question answered completely? How is the problem like others you have seen? How is it different?*

While it might seem most logical to begin problem solving with Polya's first activity and proceed through each activity until the end, not all successful problem solvers do so. Many successful problem solvers begin by understanding the problem and making a plan. But then as they start carrying out their plan, they may find that they have not completely understood the problem, in which case they go back to step one. Or they may find that their original plan is extremely difficult to pursue, so they go back to step two and select another approach. By using these four activities as a general guide, however, students can become more adept at monitoring their own thinking. This "thinking about their thinking" can help them to improve their problem solving skills.

Students move through a continuum of stages in their development as problem solvers (Kantowski, 1980). Initially, they have little or no understanding of what problem solving is, of what a strategy is, or of the mathematical structure of a problem. Such students usually do not know where to begin to solve a problem; the teacher must model the problem solving process for these students. At the second level, students are able to follow someone else's solution and may suggest strategies for similar problems. They may participate actively in group problem solving situations but feel insecure about independent activities, requiring the teacher's continued support. At the third level, students begin to be comfortable with solving problems, suggesting strategies different from those they have seen used before. They understand and appreciate that problems may have multiple solutions or perhaps even no solution at all. Finally, at the last level, students are not only adept at solving problems, they are also interested in finding elegant and efficient solutions and in exploring alternate solutions to the same problem. In teaching problem solving, it is important to address the needs of students at each of these levels within the classroom.

IN SUMMARY, the real test of whether a student knows mathematics is whether she can use it in a problem situation. Students should experience problems as introductions to learning about new topics, as applications of content already studied, as puzzles or non-routine problems that have many solutions, and as situations that have no one best answer. They should not only solve problems but also pose them. They should focus on understanding a problem, making a plan for solving it, carrying out their plan, and then looking back at what they have done.

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STANDARD 2 — COMMUNICATION

K-12 Overview

> All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.

Descriptive Statement

Communication of mathematical ideas will help students clarify and solidify their understanding of mathematics. By sharing their mathematical understandings in written and oral form with their classmates, teachers, and parents, students develop confidence in themselves as mathematics learners and enable teachers to better monitor their progress.

Meaning and Importance

Mathematics can be thought of as a language that must be meaningful if students are to communicate mathematically and apply mathematics productively. Communication plays an important role in making mathematics meaningful; it enables students to construct links between their informal, intuitive notions and the abstract language and symbolism of mathematics. It also plays a key role in helping students make critical connections among physical, pictorial, graphic, symbolic, verbal, and mental representations of mathematical ideas. When students see that one representation, such as an equation, can describe many situations, they begin to understand the power of mathematics. When they realize that some ways of representing a problem are more helpful than others, they begin to understand the flexibility and usefulness of mathematics.

K-12 Development and Emphases

Communication involves a **variety of modes**: speaking, listening, writing, reading, and representing visually (with pictures, graphs, diagrams, videos, or other visual means). Each of these can help students understand mathematics and use it effectively. Students should also use communication to **generate and share ideas.** Communicating with each other, with peers, with parents, with other adults, and with the teacher, orally and in writing, helps students learn mathematics as they clarify their own ideas and listen to those of others. The language of mathematics itself is a thinking tool that facilitates mathematical understanding and connects to natural language and everyday thinking.

Students need to have many experiences in communicating about mathematics in a **variety of settings.** Some experiences will involve working in pairs; for example, kindergartners can sit back-to-back with one giving the other directions about how to make a tower of Unifix cubes. Other experiences will involve working in small groups, such as when tenth-graders combine information from several separate clues to find the distance around a park. Some experiences will involve explaining something to the whole class, while others may involve drawing a picture, making a model, or writing in a journal.

Students need to learn the appropriate **use of mathematical language and symbols**. Most experiences relating to mathematical communication will involve the use of natural language, but some will also involve the use of tables, charts, graphs, manipulatives, equations, computers, and calculators. Students should not only be able to use each of these different media to describe mathematical ideas and solutions to problems, but they should also be able to interrelate the descriptions obtained using different media.

IN SUMMARY, communicating mathematics — orally, in writing, and using symbols and visual representations — is vitally important to learning and using mathematics. Students should use a variety of forms of communication in a variety of settings to generate and share ideas.

STANDARD 3 — MATHEMATICAL CONNECTIONS

K-12 Overview

All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas and the roles that mathematics and mathematical modeling play in other disciplines and in life.

Descriptive Statement

Making connections enables students to see relationships between different topics and to draw on those relationships in future study. This applies within mathematics, so that students can translate readily between fractions and decimals, or between algebra and geometry; to other content areas, so that students understand how mathematics is used in the sciences, the social sciences, and the arts; and to the everyday world, so that students can connect school mathematics to daily life.

Meaning and Importance

Although it is often necessary to teach specific concepts and procedures, mathematics must be approached as a whole; concepts, procedures, and intellectual processes are interrelated. More generally, although students need to learn different content areas, they also should come to see all learning as interwoven. In a very real sense, the whole is greater than the sum of its parts. Thus, the curriculum should include deliberate efforts, through specific instructional activities, to connect ideas and procedures both among different mathematical topics and with other content areas.

K-12 Development and Emphases

One important focus of this standard is that of **unifying mathematical ideas** — major mathematical themes which are relevant in several different strands. They emerge when a higher-level view of content is taken. They tie together individual mathematical topics, revealing general principles at work in several different strands and showing how they are related. Unifying ideas also set priorities, and the curriculum should be designed so that students develop depth of understanding in each of the unifying ideas at their grade level. Since it often takes years to achieve understanding of these unifying ideas, students will encounter them repeatedly in many different contexts.

An example of a unifying idea is the concept of proportional relationships. Proportional relationships play a key role in a wide variety of important topics, such as ratios, proportions, rates, percent, scale, similar geometric figures, slope, linear functions, parts of a whole, probability and odds, frequency distributions and statistics, motion at constant speed, simple interest, and comparison. Awareness of the common principle operating in all these topics is an important part of mathematical understanding. Unifying ideas for grades K-4 include quantification (*how much? how many?*), patterns (finding, making, and describing), and representing quantities and shapes. Unifying ideas for the middle grades include proportional relationships, multiple representations, and patterns and generalization. For the high school, the unifying ideas are mathematical modeling, functions and variation (*how is a change in one thing associated with a change in another?*), algorithmic thinking (developing, interpreting, and analyzing mathematical procedures), mathematical argumentation, and a continued focus on multiple representations.

Another focus of this standard is **mathematical modeling**, developing mathematical descriptions of real-world situations and (usually) predicting outcomes based on that model. Even very young children use mathematics to model real situations, counting candies or cookies or matching cups and saucers on a one-to-one basis. Affirming the ways things make sense outside of school and connecting them to things in school is important for students. Older students use mathematical modeling as they develop the concepts of function and variable, extending ideas they already have about growth, motion, or cause and effect. At various grade levels, students can use their experiences waiting in lines to investigate the general relationships among waiting time, the number of people in the line, the position at the end of the line, and the length of time each person takes to buy a ticket or a lunch. Older students can further develop a mathematical model that can help them predict waiting time and formulate and evaluate their suggestions for solving a real problem — getting everyone through the school lunch line more quickly.

Throughout the grades, students need to develop their understanding of the **relationship of mathematics to other disciplines.** Mathematics is frequently used as a tool in other disciplines. In science, students measure quantities and analyze data. In social studies, they collect and analyze data and make choices using discrete mathematics. In art, they generate designs and show perspective. Other disciplines can also provide interesting contexts to learn about new mathematical ideas. For example, students might learn about symmetry by generating symmetric designs with paint, or they might explore exponential functions by modeling a dying population by repeatedly scattering M&Ms, at each step removing those that have the M showing, so that about half of the population "dies" each time.

Every person views the process of making connections between disciplines differently, and everyone works in different circumstances in which often-competing goals must be balanced. Nevertheless, it is important to establish some means of making explicit the connections between mathematics and other disciplines. There are several ways in which the presentation of content can be approached, ranging from "fragmented" to "integrated" (adapted from Fogarty, 1991):

1. Instruction may be *fragmented*. This is the traditional model of separate and distinct disciplines, each of which is presented in isolation.
2. Each subject may be *connected* within itself; concepts are explicitly connected within each course and from course to course within each subject area, but connections between subjects are not made.
3. Instruction in each subject may have *nested* within it discussion of particular topics, but connections are not made across the disciplines.
4. Teachers may arrange and *sequence* related topics or units of study to coincide with one another. Similar ideas are taught in concert but remain unconnected.
5. *Shared* planning and teaching take place; overlapping concepts or ideas emerge as organizing elements.

6. A fertile theme is *webbed* to each subject area; teachers use the theme to sift out appropriate concepts, topics, and ideas.
7. The *threaded* approach weaves shared topics throughout each subject.
8. The *integrated* approach involves focusing on overlapping topics and concepts, and includes team teaching.

While the first two approaches do not provide for the establishment of appropriate connections between subjects, the remaining six do provide some degree of interaction. Teachers must consider carefully which of these is most appropriate and feasible for their own situation.

This discussion addresses the connections between mathematics and other disciplines. Of special significance because of their many commonalities is **the relationship of mathematics to science**. Berlin and White (1993) have identified six areas in which mathematics and sciences share concepts or skills: ways of learning, ways of knowing, process and thinking skills, conceptual knowledge, attitudes and perceptions, and teaching strategies.

Ways of Learning — The two disciplines have a common perspective on how students experience, organize, and think about science and mathematics. Both disciplines endorse active, exploratory learning with opportunities for students to share and discuss ideas. Students must *do* science and mathematics in order to learn science and mathematics.

Ways of Knowing — Both science and mathematics use patterns to help students develop understanding. Even very young children seek to make sense of patterns in order to make sense of their world. They make generalizations based on what they have observed and apply these generalizations to new situations. Sometimes their guess works, reinforcing the generalization, and sometimes it doesn't, requiring the child to revise the generalization.

An example from chemistry demonstrates the similar ways of knowing used in mathematics and science. Chemist Dmitri Mendeleev proposed a periodic table of the elements based on increasing atomic weights. Sometimes he left open spaces in the table, where he reasoned that unknown elements should go. In 1869, when he arranged his table, the element gallium was unknown; however, Mendeleev predicted its existence. He based his predictions on the properties of aluminum (which appeared directly above gallium in the table). Mendeleev even went so far as to predict the melting point, boiling point, and atomic weight of the then-unknown gallium, which he called eka-aluminum. Six years later, while analyzing zinc ore, the French chemist Lecoq de Boisdaudran discovered the element gallium. Its properties were almost identical to those Mendeleev had predicted.

Process and Thinking Skills — Central to both disciplines are process skills. Mathematics focuses primarily on the following four process skills: problem solving, reasoning, communication, and connections. Basic process skills in science (Tobin and Capie, 1980) include observing, inferring, measuring, communicating, classifying, formulating hypotheses, experimenting, interpreting data, and formulating models.

Conceptual Knowledge — There is considerable overlapping of content between science and mathematics. By examining the concepts, principles, and theories of science and mathematics, those ideas that are unique to one subject and those which overlap both disciplines can be identified. Some of the "big ideas" which are common to both include conservation (of number, volume, etc.), equilibrium, measurement, models (including both concrete and symbolic), patterns (including trends, cycles, and chaos), probability and statistics, reflection, scale (including size, duration, and speed), symmetry, systems, variables, and vectors. An example of a way to interrelate science and mathematics contents is to link population dynamics and genetics in science with sampling and probability in mathematics.

Attitudes and Perceptions — Mathematics and science share certain values, attitudes, and ways of thinking: accepting the changing nature of science and mathematics, basing decisions and actions on data, exhibiting a desire for knowledge, having a healthy degree of skepticism, relying on logical reasoning, being willing to consider other explanations, respecting reason, viewing information in an objective and unbiased manner, and working together cooperatively to achieve better understanding. Both disciplines also value flexibility, initiative, risk-taking, curiosity, leadership, honesty, originality, inventiveness, creativity, persistence, and resourcefulness, as well as being thorough, careful, organized, self-confident, self-directed, and introspective, and valuing science and mathematics (*Science for All Americans*). Students' engagement in personal and social issues and interests may also help to encourage, support, and nurture their confidence in their ability to do science and mathematics.

Teaching Strategies — The shared goals of mathematics and science instructions are, according to *Science for All Americans*, to have students acquire scientific and mathematical knowledge of the world as well as scientific and mathematical habits of mind. Both disciplines support teaching strategies which foster inquiry and problem-solving, promote discourse among students, challenge students to take responsibility for their own learning and to work collaboratively, encourage all students to participate fully, and nurture a community of learners (*National Science Education Standards*).

IN SUMMARY, making connections within mathematics and between mathematics and other subjects not only helps students understand the mathematical ideas more clearly, it also captures their interest and demonstrates how mathematics is used in the real world. Important connections that need to be established include working with unifying mathematical themes, using mathematical modeling, and relating mathematics to other disciplines and to the real world.

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STANDARD 4 — REASONING

K-12 Overview

All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers.

Descriptive Statement

Mathematical reasoning is the critical skill that enables a student to make use of all other mathematical skills. With the development of mathematical reasoning, students recognize that mathematics makes sense and can be understood. They learn how to evaluate situations, select problem-solving strategies, draw logical conclusions, develop and describe solutions, and recognize how those solutions can be applied. Mathematical reasoners are able to reflect on solutions to problems and determine whether or not they make sense. They appreciate the pervasive use and power of reasoning as a part of mathematics.

Meaning and Importance

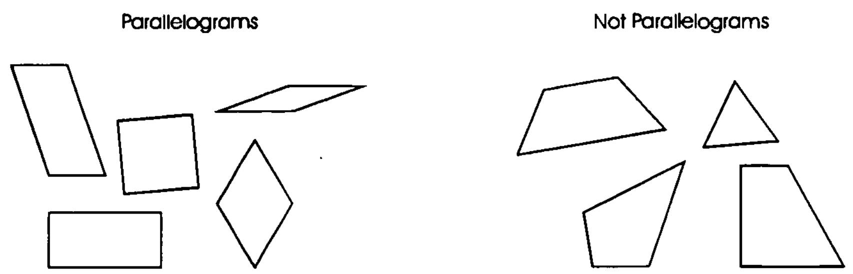
There are various terms used to refer to "reasoning": critical thinking, higher-order thinking, logical reasoning, or simply reasoning. Different subject areas tend to use different terms. Across all of these subject areas, however, there are commonalities. The following phrases often appear in discussions of how reasoning is used (adapted from Resnick, 1987, pp 2-3):

- *Nonalgorithmic* — The route to a solution is not fully specified in advance.
- *Complex* — The complete path to a solution is not fully apparent from any single vantage point.
- *Multiple criteria* — The conditions established in the problem may conflict with one another.
- *Uncertainty* — Not everything that bears on the task at hand is known.
- *Imposing meaning* — The individual must find structure in apparent disorder.
- *Effortful* — There is considerable mental work involved in the elaborations and judgments required.
- *Self-regulation* — The individual monitors his or her own progress, and determines the appropriate course of action.
- *Multiple solutions* — There is no single "best" solution; rather, there are many solutions, each with costs and benefits.
- *Nuanced judgment* — The results must be interpreted.

K-12 Development and Emphases

Every student has potential for higher-order thinking. The key is to unlock the world of mathematics through a student's natural inclination to strive for purpose and meaning. Reasoning is fundamental to the knowing and doing of mathematics. Conjecturing and demonstrating the logical validity of conjectures are the essence of the creative act of *doing* mathematics. To give more students access to mathematics as a powerful way of making sense of the world, it is essential that an emphasis on reasoning pervade all mathematical activity. In order to become confident, self-reliant mathematical thinkers, students need to develop the capability to confront a mathematical problem, persevere in its solution, and evaluate and justify their results.

Inductive reasoning involves looking for patterns and making generalizations. For example, students use this type of reasoning when they look at many different parallelograms, and try to list the characteristics they have in common. The reasoning process is enhanced by also considering figures that are not parallelograms and discussing how they are different.



Students may use inductive reasoning to discover patterns in multiplying by ten or a hundred or in working with exponents. Learning mathematics should involve a constant search for patterns, with students making educated guesses, testing them, and then making generalizations.

Many students use inductive reasoning more frequently than teachers realize, but the generalizations that they form are not always correct. For example, a student may see the examples $16/64 = 1/4$ and $19/95 = 1/5$ and reason inductively that the common digits in a fraction may be canceled. The student must realize that she needs to continue to test her conjecture before making such a generalization, since $17/76 \neq 1/6$, for example. Students must also realize that while inductive reasoning demonstrates the power of mathematics and allows great leaps forward in understanding, it is insufficient in itself. The generalizations that are obtained by using inductive reasoning can only be accepted by "proving" them through deductive reasoning.

Deductive reasoning involves making a logical argument, drawing conclusions, and applying generalizations to specific situations. For example, once students have developed an understanding of "parallelogram," they apply that generalization to new figures to decide whether or not each is a parallelogram. This kind of reasoning also may involve eliminating unreasonable possibilities and justifying answers. Although students as young as first graders can recognize valid conclusions, the ability to use deductive reasoning improves as students grow older. More complex reasoning skills, such as recognizing invalid arguments, are appropriate at the secondary level.

Understanding the power of reasoning to make sense of mathematics is critical to helping students become self-reliant, independent mathematical thinkers. Students must be able to judge for themselves the accuracy of their answers; they must be able to apply mathematical reasoning skills in other subject areas and in their daily lives. They must recognize that mathematical reasoning can be used in many different situations to help them make choices and reach decisions.

IN SUMMARY, mathematical reasoning is the glue that binds together all other mathematical skills. By using inductive and deductive reasoning as they learn mathematical concepts and solve mathematical problems, students come to recognize the extent to which reasoning applies to mathematics and to their world.

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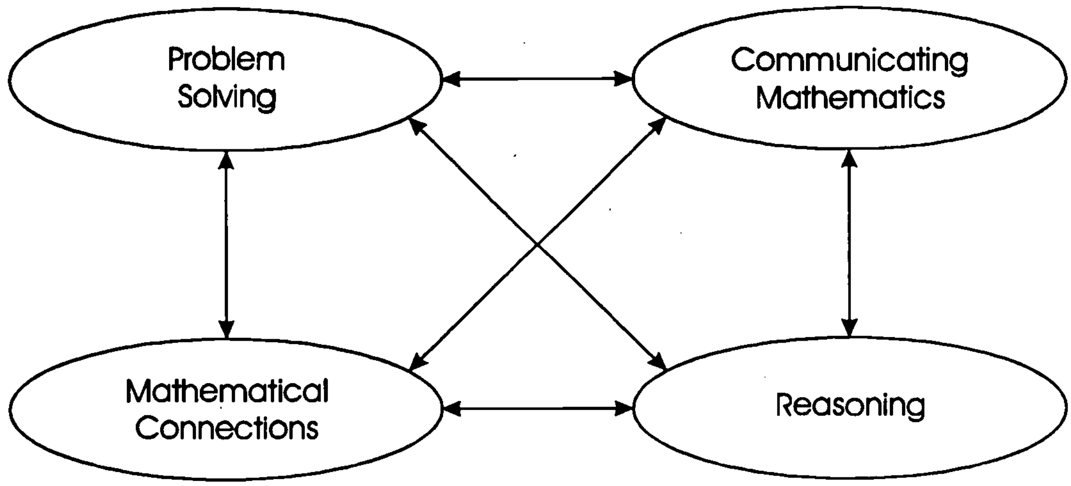
The First Four Standards — Grades K-2



Overview

Young children enter school with informal strategies for solving mathematical problems, communication skills, ideas about how number and shape connect to each other and to their world, and reasoning skills. In grades K-2, students should build upon these informal strategies.

Early instruction in **problem solving** should focus on taking time to understand the problem before rushing to solve it. Kindergartners should begin, for example, by representing problems using physical objects. By second grade, students should begin to move away from dependence on physical objects towards the use of pictures and figures. One of the goals of problem solving in numerical situations is to move students toward the use of more efficient problem solving strategies — from modeling with concrete objects to counting methods to using number facts. Even kindergartners should have experience with multiple-step problems (*Mary has 3 cookies. She eats one. Her mother gives her two more. How many cookies does she have now?*) in order to focus their attention on understanding the problem and developing a plan for its solution. Students should be able to describe how they have solved a problem and justify their answer. They should also develop the habit of comparing problems to each other, noting how they are alike and different.

Communication activities in grades K-2, whether with individuals, small groups, or the whole class, initially emphasize oral (e.g., counting) and pictorial representations. Much time is spent, however, in introducing students to symbolic representations (e.g., numerals and symbols for operations). As students develop written communication skills, they also begin to communicate in writing about mathematics. At first, the teacher may write the students' responses on the board or on sentence strips in order to facilitate this written communication. Students use many concrete representations (e.g., base ten blocks, pattern blocks) and need to learn how to represent their work with these manipulatives through pictures. Students also begin to communicate mathematics using graphs and diagrams.

Many **mathematical connections** begin to be established in kindergarten. Students should connect the number three to triangles, for example, as well as to sets of three objects and the numeral 3. Especially important are quantification (*how much? how many?*), patterns, and representing quantities and shapes. Using children's literature to motivate and set a context for problem solving and learning mathematics is especially appropriate for K-2, as is illustrated in one of the following vignettes. Connections to social studies may involve using graphs to describe characteristics of the class, the school, or the community.

Many connections between science and mathematics can be established, from looking for patterns to developing specific skills in measurement and data collection. Children observe life cycles and cycles in nature, such as the seasons, and the growth and decay of plant forms. Children begin by using words to describe physical characteristics: color intensity (bright or dull), sound volume (loud or quiet), temperature (hot or cold), and size (longest or shortest). This allows them to make simple descriptive comparisons and to place objects in an order. They move on to using numbers to describe such characteristics. For example, students might measure the height of plants at different times, summarize their data in a table, and prepare a graph (bar or line) showing the height over time. They might repeat the experiment with different growing conditions, and then compare their graphs for the different conditions.

Students in grades K-2 should spend a great deal of time on inductive reasoning, looking for patterns, making educated guesses, generating hypotheses, and forming generalizations based on their experiences. They should also begin to develop some skill in drawing logical conclusions and justifying answers (deductive reasoning), perhaps by using manipulatives such as attribute blocks. They should continually strive to make sense of mathematics by using reasoning to predict answers and compare and contrast examples and problem situations.

In grades K-2, students build on what they already know as they develop their skills in problem solving, communication, mathematical connections, and reasoning. They begin to move from informal, intuitive strategies and processes towards more symbolic representations and more explicit recognition of their thinking strategies.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

The First Four Standards — Grades K-2

Vignette — Will a Dinosaur Fit?

Standards: In addition to the First Four Standards, this vignette highlights Standards 6 (Number Sense), 7 (Geometry), 9 (Measurement), and 11 (Estimation).

The problem: The second grade was in the midst of a unit on dinosaurs when the teacher read to her class the book *Danny and the Dinosaur* by Syd Hoff (Harper & Row, 1958). After the first reading, the children re-examined some of the illustrations. One picture depicted the dinosaur larger than a block of homes, another showed the dinosaur almost completely hidden by one house. One picture showed the dinosaur taller than an apartment building and yet another showed the dinosaur not quite as tall as a lamp post. Students were intrigued by the idea that Danny's dinosaur friend did not seem to be of a consistent size. They voiced opinions about the dinosaur's actual size. Since students seemed to have a sustained interest in exploring the sizes of dinosaurs, the teacher presented students with this question: *Do you think that a dinosaur could fit into our classroom?*

The discussion: Brainstorming was encouraged by the teacher as questions such as the following were posed by students and by the teacher. *What does it mean to "fit" in the classroom? What information would we need to get in order to determine if a dinosaur could fit in our classroom? Do you think all of our answers will be the same? Why? What do we know already that might help us? What materials do you think we would need?*

Solving the problem: Students worked in groups of 3 over a period of several days. They began by choosing a specific dinosaur and then they used a variety of books and computer software in the classroom to find the size of their dinosaur. They determined the size of the classroom, choosing to measure with a trundle wheel or a tape, or by using estimation. Then they decided, by comparing the measures found in books with those made of the classroom, whether the dinosaur would fit into the classroom. Each group was responsible for creating a display and making a presentation to the class to answer the question. The displays made use of models, pictures, and text. Students with more than a few sentences to write were encouraged to make use of the word processor available in the classroom.

Summary: Students used their displays to make presentations to the class. There were a variety of answers. Those who had chosen one of the smaller dinosaurs, the velociraptors, for example, found that the dinosaur could walk through the doorway and several dinosaurs would fit in the room. Others, who had chosen larger dinosaurs, the stegosaurus, for example, found that if the dinosaur could have gotten through the doorway, several would have fit in the room. Still others, who had chosen very large dinosaurs, the brachiosaurus, for example, found that the dinosaur would not have fit into the room at all. As the presentations ended, several children suggested further explorations that might be interesting: *Would the dinosaur I chose fit into the multi-purpose room? Was the dinosaur I chose as long as the driveway in front of the school? Was the dinosaur I chose taller than the school building?*

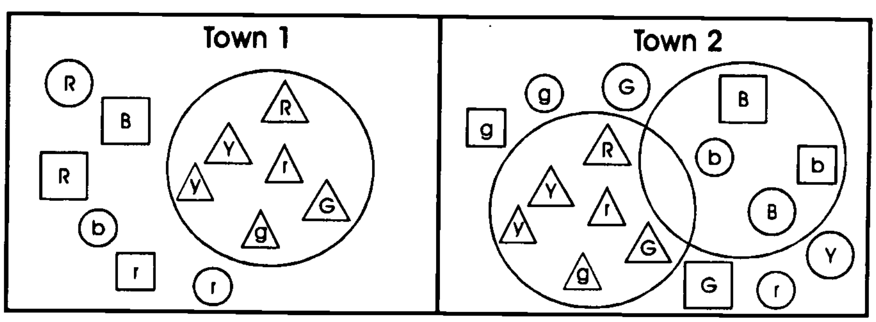
The First Four Standards — Grades K-2

Vignette — Shapetown

Standards: In addition to the First Four Standards, this vignette highlights Standards 7 (Geometry), 11 (Patterns), and 14 (Discrete Mathematics).

The problem: The students in kindergarten had been involved in a unit that allowed them to explore their town. They had been exposed to a variety of activities, including building symmetric and non-symmetric block buildings, drawing neighborhood maps, and using letter-number ordered pairs (like A-2) to locate places on a grid. In this lesson, pairs of students were challenged to build towns with attribute blocks and loops based on a rule or pattern that they made up.

The discussion: With the class sitting on the carpet in a circle, the teacher placed a loop within everyone's sight. She explained that the loop was a town and that the blocks were buildings. Using blocks of different colors, she then placed several triangles inside the loop and several non-triangles outside the loop. Ideas about the rule used to build Town 1 were discussed: *Tell me about the town. Describe a pattern that you see. Put this triangle on the carpet to follow the pattern. Put this circle on the carpet to follow the pattern. How could you tell someone else about our town so they could build one just like it?* The verbalization was then called *the rule* for the town. Town 2 was created with two loops, blocks were placed inside and outside these loops, and similar questions were raised and discussed. Several reasonable rules were suggested. For example, one rule was: triangles in one loop, blue blocks in the other loop, other colors and shapes in the overlapping loop and outside the loop. Another rule was: triangles in one loop, blue blocks in the other loop, blue triangles in the overlapping loop, and all other blocks outside the loops.



Solving the problem: Students were given loops and some attribute blocks. They were challenged to work together to build a town that used a rule. At the end of the working time, each pair of students challenged the class to place other blocks in their town and then to verbalize the rule that was used to create the town.

Summary: Students worked independently to record their town designs using crayons and shapes cut from colored construction paper. Students described the rules that they used to build their towns.

The First Four Standards — Grades K-2

Indicators

The cumulative progress indicators for grade 4 for each of the First Four Standards appear in boldface type below the standard. Each indicator is followed by a brief discussion of how the preceding grade-level vignettes might address the indicator in the classroom in kindergarten and in grades 1 and 2. The Introduction to this *Framework* contains three vignettes describing lessons for grades K-4 which also illustrate the indicators for the First Four Standards; these are entitled *Elevens Alive!*, *Product and Process*, and *Sharing a Snack*.

Standard 1. All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.

Experiences will be such that all students in grades K-2:

1. **Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand mathematical content appropriate to the early elementary grades.**
    - *Will a Dinosaur Fit?* uses the question *Do you think a dinosaur would fit into our classroom?* to launch an investigation involving measurement, geometry, estimation, and large numbers. *Shapetown* develops students' logical (deductive) reasoning skills using shapes (geometry), sorting (discrete mathematics), and pattern analysis.

2. **Recognize, formulate, and solve problems arising from mathematical situations and everyday experiences.**
    - In *Will a Dinosaur Fit?*, students recognize and help to formulate the question they will investigate, based on a book they have read and its illustrations. In *Shapetown*, students develop their own logic problems in connection with a unit in social studies on their community.

3. **Construct and use concrete, pictorial, symbolic, and graphical models to represent problem situations.**
    - Students in *Will a Dinosaur Fit?* begin with a pictorial model (the pictures in the book) and then use numerical models and graphs to represent the problem situation. Students in *Shapetown* use concrete materials (attribute blocks) to represent their problem situation and then record their "rules" using pictures.

4. **Pose, explore, and solve a variety of problems, including non-routine problems and open-ended problems with several solutions and/or solution strategies.**
    - In *Will a Dinosaur Fit?*, each group investigates a different dinosaur, using their own strategies. Different groups have different answers, depending on the size of their

dinosaur. In *Shapetown*, pairs of students pose their own problems for the others to solve.

5. **Construct, explain, justify, and apply a variety of problem-solving strategies in both cooperative and independent learning environments.**

   - The students in *Will a Dinosaur Fit?* work in groups of three, measuring the classroom and collecting data from books. The students in *Shapetown* work in pairs to develop the rules for their towns.

6. **Verify the correctness and reasonableness of results and interpret them in the context of the problems being solved.**

   - Students in *Will a Dinosaur Fit?* present their results to the class for verification. The students in *Shapetown* verify their results by having other students solve their problems.

7. **Know when to select and how to use grade-appropriate mathematical tools and methods (including manipulatives, calculators and computers, as well as mental math and paper-and-pencil techniques) as a natural and routine part of the problem-solving process.**

   - In *Will a Dinosaur Fit?* students select their own measuring tools and some use computers. They decide whether to estimate or measure and how to determine their answers (compare numbers or subtract mentally or with a calculator or with paper-and-pencil). The students in *Shapetown* use manipulatives (attribute blocks) to develop their rules.

8. **Determine, collect, organize, and analyze data needed to solve problems.**

   - The students in *Will a Dinosaur Fit?* determine what information they need to know about their dinosaurs, collect that information, organize it and analyze it. The students in *Shapetown* organize and analyze the placement of objects in the town in accordance with the rules they were given and the rules they generated or discovered.

9. **Recognize that there may be multiple ways to solve a problem.**

   - In their sharing, the students in *Will a Dinosaur Fit?* find out about the many different ways in which students address this problem. The students in *Shapetown* might explain how they figure out the "rules" their classmates use for their own towns.

Standard 2. All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.

Experiences will be such that all student in grades K-2:

1. **Discuss, listen, represent, read, and write as vital activities in their learning and use of mathematics.**

   - In *Will a Dinosaur Fit?*, the students read a story, read information from books about their dinosaurs, represent their results using symbols and words, and explain their results orally. In *Shapetown*, the students listen to the teacher explain how to develop a "rule," discuss their rules in pairs as they develop them, and record their rules with a picture.

2. **Identify and explain key mathematical concepts, and model situations using oral, written, concrete, pictorial, and graphical methods.**

    - The students in *Will a Dinosaur Fit?* model their problem situations using oral and written language. Some groups may also use pictorial and/or graphical methods. The students in *Shapetown* use concrete materials to model their problems and oral methods to solve them.

3. **Represent and communicate mathematical ideas through the use of learning tools such as calculators, computers, and manipulatives.**

    - Some students in *Will a Dinosaur Fit?* use computers; others use trundle wheels or measuring tape. Students in *Shapetown* use manipulatives (attribute blocks).

4. **Engage in mathematical brainstorming and discussions by asking questions, making conjectures, and suggesting strategies for solving problems.**

    - The teacher in *Will a Dinosaur Fit?* begins the discussion of the problem by having students brainstorm what it means for a dinosaur to fit in the classroom. The students in *Shapetown* discuss the problems posed by the teacher and make conjectures as they try to solve them.

5. **Explain their own mathematical work to others, and justify their reasoning and conclusions.**

    - Students in *Will a Dinosaur Fit?* explain their work and justify their reasoning about their group's dinosaur. Students in *Shapetown* explain their work and justify their results as they challenge each other to solve their problem.

Standard 3. All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas and the roles that mathematics and mathematical modeling play in other disciplines and in life.

Experiences will be such that all students in grades K-2:

1. **View mathematics as an integrated whole rather than as a series of disconnected topics and rules.**

    - In both vignettes, the students are investigating problems that involve several content standards.

2. **Relate mathematical procedures to their underlying concepts.**

    - In *Will a Dinosaur Fit?*, students research the size of their dinosaurs, determine the size of their classroom by measuring, and compare the measures to see which is larger. In

Shapetown, students apply the fundamental concepts of Venn diagrams.

3. **Use models, calculators, and other mathematical tools to demonstrate the connections among various equivalent graphical, concrete, and verbal representations of mathematical concepts.**

    - In *Will a Dinosaur Fit?,* students create a display and make a presentation to the class to support their conclusion. In *Shapetown*, the students verbalize the rule used for their town and then create an equivalent representation for their attribute block models using a picture.

4. **Explore problems and describe and confirm results using various representations.**

    - The second-graders in *Will a Dinosaur Fit?* use a variety of representations (symbols and words) to record their results as they investigate the problem. The students in *Shapetown* use a pictorial representation to describe their results.

5. **Use one mathematical idea to extend understanding of another.**

    - The teacher in *Will a Dinosaur Fit?* uses the students' understanding of relative size to extend their understanding of estimation and measurement. The students in *Shapetown* use their understanding of geometric shapes to build their "rules" as they learn more about logical reasoning.

6. **Recognize the connections between mathematics and other disciplines, and apply mathematical thinking and problem solving in those areas.**

    - The dinosaur lesson involves applying mathematics to learn about dinosaurs (science). The *Shapetown* lesson builds upon a social studies unit in which students use mathematics to locate buildings, construct buildings, and draw maps.

7. **Recognize the role of mathematics in their daily lives and in society.**

    - The students in *Will a Dinosaur Fit?* learn how mathematics is involved in the sizes of illustrations in the books that they read. The *Shapetown* students learn how mathematics is used in buildings, in determining locations, and in classifying and characterizing objects.

Standard 4. All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers.

Experiences will be such that all students in grades K-2:

1. **Make educated guesses and test them for correctness.**

    - The students in *Will a Dinosaur Fit?* could address this indicator by predicting whether their dinosaur will fit before measuring the classroom. The students in *Shapetown* are challenged to guess the rule for placing blocks on the carpet, and then to verbalize the rule they think is being used.

2. **Draw logical conclusions and make generalizations.**

    - The students in *Will a Dinosaur Fit?* draw conclusions from the data they collect by measuring and using texts or the computer. They might also make some generalizations about dinosaurs collectively after discussing the results of all the groups. Drawing logical conclusions is the major focus of the *Shapetown* lesson.

3. **Use models, known facts, properties, and relationships to explain their thinking.**

    - The students in *Will a Dinosaur Fit?* use models, known facts (from books and software), and relationships to explain how they know whether their dinosaur will fit. The *Shapetown* students use models to explain their thinking.

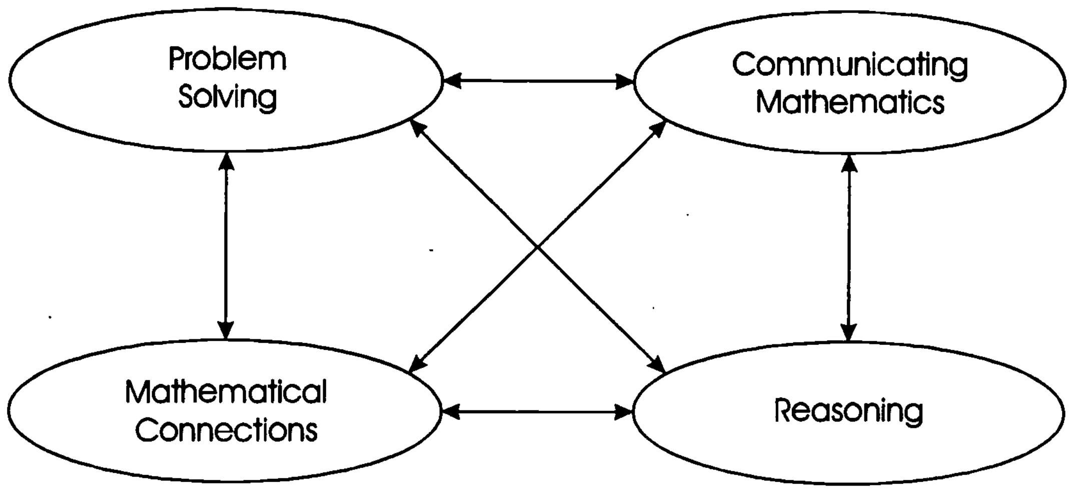
4. **Justify answers and solution processes in a variety of problems.**

    - Students in both vignettes justify their answers and solution processes.

5. **Analyze mathematical situations by recognizing and using patterns and relationships.**

    - The students in *Will a Dinosaur Fit?* solve their problems by comparing the sizes of the various dinosaurs with other sizes, such as the classroom and its doorway. The students in *Shapetown* recognize and use patterns and relationships as they pose and solve their problems involving attribute blocks.

The First Four Standards — Grades 3-4



Overview

In the third and fourth grades, students continue to develop their ability to solve problems, communicate mathematically, make connections within mathematics and between mathematics and other subject areas, and reason mathematically.

Students in grades 3-4 should continue to focus on understanding in their **problem solving** activities but should also begin to develop a repertoire of strategies for solving problems. These should include not only drawing a picture, using concrete objects, and writing a number sentence, but also drawing a diagram, working backwards, solving a simpler problem, and looking for a pattern. Students begin to spend more time developing a problem-solving plan, since they now have a greater variety of strategies to consider and select from. They also focus more on looking back, comparing each problem to ones they have solved previously.

Communication activities become more elaborate in third and fourth grade, as students become more comfortable with symbolic and written representations of ideas. Students should communicate with each other about mathematics on a daily basis, exploring problem situations and justifying their solutions. Different types of writing assignments may be used: keeping journals, explaining solutions to math problems, explaining mathematical ideas, and writing about the reasoning involved in solving a problem. Students continue to use manipulatives to explore new ideas and learn to relate different representations of an idea to each other. For example, after using base ten blocks to solve 7×36, students might provide a pictorial representation of these blocks (at left below) followed by a written explanation of what they did to get $7 \times 36 = 252$. Linking the use of concrete manipulatives to the pictorial and symbolic representations is critical to understanding the mathematical procedures.

|||•••••• |||•••••• |||••••••	36 x 7	I laid out 7 groups of 3 tens and 6 ones.
|||•••••• |||••••••	210	I counted up 7 x 3 = 21 tens and wrote down 210.
|||•••••• |||••••••	42	I counted up 7 x 6 = 42 ones and wrote down 42.
	252	I added those together to get 252.

Children in third and fourth grade continue to build **mathematical connections.** Within mathematics, the major unifying ideas continue to be quantification (how much and how many, especially with larger quantities), patterns, and representing quantities and shapes. For example, students need to see the relationship between the quantification that they do with measurement (using centimeters and meters) and that they do with base ten blocks (representing numbers in the hundreds). Literature and social studies continue to provide opportunities for using mathematics in context. Students are also able to use mathematics more in their study of science, doing computations with the measurements they have made (e.g., averages). Measurement and data analysis, in particular, offer good opportunities for integrating science and mathematics. For example, students might measure the distance a hungry mealworm crawls in 90 seconds and compare it to the distance a well-fed mealworm crawls in the same amount of time.

Third- and fourth-graders use both **inductive reasoning** (looking for patterns, making educated guesses, forming generalizations) and **deductive reasoning** (using logical reasoning, eliminating possibilities, justifying answers). Teachers should create situations in which students may form incorrect generalizations based on only a few examples, and should be prepared to provide counter-examples to those incorrect generalizations. For example, if fourth-graders think that multiplying by 100 always means they add two zeros to the right side of the number, then the teacher should ask them to multiply 0.5 by 100 on their calculators. Instructional activities should continue to emphasize that mathematics makes sense and that mathematical reasoning helps people both to understand their world and to make decisions rationally.

Students in grades 3 and 4 continue to develop more formal and abstract notions of problem solving, communication, mathematical connections, and reasoning. They begin to focus more on what they are thinking as their communication and reasoning skills improve. They solve a wider range of problems and connect mathematics to a greater variety of situations in other subject areas and in life.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

The First Four Standards — Grades 3-4

Vignette — Tiling a Floor

Standards: In addition to The First Four Standards, this vignette highlights Standards 7 (Geometry) and 10 (Estimation).

The problem: The third grade students toured the school and the playground to find and sketch the geometric shapes that they saw. On returning to the classroom, the class discussed names for each shape, compared the shapes, and talked about where each shape had been found. Several of the shapes had been copied from tiles on walls and floors. The teacher used the tiling idea to challenge students to decide which of the shapes could be used to tile a floor or a wall. (The use of shapes to form a tiling pattern is often referred to as "tessellation.")

The discussion: Questions such as these were examined by the teacher and students, to help clarify the task: *What do you know about tiling a floor? Can shapes go on top of each other? Can there be spaces? What are the names of the shapes that we found? How could we check each shape to see if we could tile with it? Do you think we all have to solve this problem the same way? What materials could we use to make the shapes? How many copies of each shape do you think we will need?*

Solving the problem: Students worked in pairs over a two-day period. Each pair selected three shapes to test for tiling. Before copying each shape, students wrote in their journals, naming each shape they selected, predicting whether each shape could or could not be used as a tile, and estimating how many would be needed to cover one sheet of paper. The names of some of the unfamiliar shapes were taken from a poster that was hanging in the classroom. Students selected a variety of materials for making copies of their shapes. Some selected plain paper and used rulers to draw copies of their shapes, others selected grid paper, still others selected square or triangular dot paper. Some pairs recognized their shapes in the container of pattern blocks and used them instead. Several pairs used a computer drawing program and were able to create many copies of their shapes quickly and easily. After making and cutting out 5 or more copies of each shape, they attempted to tile sheets of paper with the shapes. Successful and unsuccessful tilings were glued to construction paper. Students checked their previous predictions and continued their journal entries, reflecting on their predictions.

Summary: At the end of the two-day working period, the tilings were sorted into two groups: successful and unsuccessful. Discussion began with the successful tilings. Each tiling was labeled with the name of the shape. The teacher had students talk about the similarities and differences among the successful tilings. Students noticed that many of the successful tilings were made with four-sided shapes, that all the triangles led to successful tilings, and that there were many more shapes that were unsuccessful in tiling than were successful. Similar ideas were discussed for the unsuccessful tilings. Then students tried to verbalize why some shapes could be used as tiles and others could not. They were able to generalize that shapes that could be used for tilings were able to fit around a point without leaving spaces and without overlapping. To close the activity, students wrote in their journals about this generalization using their own words.

The First Four Standards — Grades 3-4

Vignette — Sharing Cookies

Standards: In addition to The First Four Standards, this vignette highlights Standards 6 (Number Sense) and 8 (Numerical Operations).

The problem: The fourth-grade teacher was ready to introduce students to experiences with fractions. This problem was posed as a way to gather information about the ideas that each student already had about fractions:

You have 8 cookies to share equally among 5 people. How much will each person get?

The discussion: Discussion began when the teacher posed the question *Why is this a problem?* With some prompting, students began to realize that there were not enough cookies to give each person 2 whole cookies, but if they gave each person just 1 cookie, there would be some left over. Students concluded that they would have to give each person 1 whole cookie and some part of another cookie. They had realized that finding that part was the "problem." The next major question for the students was *What would you like to use to solve the problem?* Students made many suggestions: *get cookies and cut them, use linking cubes, draw a picture of 8 circles, use paper circles.* The teacher provided construction paper circles, linking cubes, and cookies with plastic knives.

Solving the problem: Working in groups of 3 or 4, students were told that each group was to decide which materials to use to solve the problem, and that each group would explain its solution using pictures and numbers. Finally, they were told that they would be asked to share their solution with the whole class. Students worked in their groups, most choosing to use the real cookies, until they felt comfortable with their solutions. This was one solution: give each person 1 cookie, divide the rest of the cookies into halves, give each person one of these halves, divide the remaining half of a cookie into 5 equal pieces, and give each person one of those pieces. Another solution was: divide all the cookies into halves, give each person three halves, divide the remaining half into 5 equal pieces, and give each person one of those pieces. Students wrote number sentences describing the amount of each person's share, but most found that they were unable to simplify the number sentences to determine how much cookie each person gets.

Summary: The summary discussion centered on how much cookie each person got. The teacher found that students were able to determine the size of the smallest piece of cookie (1/10), but they were unable to determine how much one cookie, 1/2 of a cookie, and 1/10 of a cookie were altogether. The teacher extended the discussion so that the class was able to explore what made the problem difficult and how the problem could be changed to make it easier.

The First Four Standards — Grades 3-4

Indicators

The cumulative progress indicators for grade 4 for each of the First Four Standards appear in boldface type below the standard. Each indicator is followed by a brief discussion of how the preceding grade-level vignettes might address the indicator in the classroom in grades 3 and 4. The Introduction to this *Framework* contains three vignettes describing lessons for grades K-4 which also illustrate the indicators for the First Four Standards; these are entitled *Elevens Alive!*, *Product and Process*, and *Sharing a Snack*.

Standard 1. All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand mathematical content appropriate to early elementary grades.**
   - In *Tiling a Floor*, students investigate tiling with different geometric shapes in the context of tiling a floor. In *Sharing Cookies*, students begin their study of fractions by considering a real-life problem.
2. **Recognize, formulate, and solve problems arising from mathematical situations and everyday experiences.**
   - In both vignettes, the students begin with a problem that arises from everyday experiences.
3. **Construct and use concrete, pictorial, symbolic, and graphical models to represent problem situations.**
   - In *Tiling a Floor*, the students use concrete materials (copies of shapes) to represent the problem situation. In *Sharing Cookies*, the students use a variety of manipulatives (paper circles and real cookies) to help them understand the problem.
4. **Pose, explore, and solve a variety of problems, including non-routine problems and open-ended problems with several solutions and/or solution strategies.**
   - In *Tiling a Floor*, the students use a variety of shapes in their exploration, although most use similar strategies (either concrete materials or a software program). The students in *Sharing Cookies* solve the same problem in a variety of ways. Although they recognize that the different solution methods lead to the same answer, they are unable to explain what fraction that is.

5. **Construct, explain, justify, and apply a variety of problem-solving strategies in both cooperative and independent learning environments.**

    - The students in the tiling vignette work in pairs, applying different strategies to investigate which shapes would tile the floor. The students in the cookie vignette work in groups of three or four; they construct a strategy for solving the problem, explain it, and justify it.

6. **Verify the correctness and reasonableness of results and interpret them in the context of the problems being solved.**

    - In both vignettes, the students share their results with the whole class. This sharing serves the purpose of verifying the correctness and reasonableness of these results.

7. **Know when to select and how to use grade-appropriate mathematical tools and methods (including manipulatives, calculators and computers, as well as mental math and paper-and-pencil techniques) as a natural and routine part of the problem solving process.**

    - Students in both vignettes are encouraged to select the tools they wish to use to solve the problems.

8. **Determine, collect, organize, and analyze data needed to solve problems.**

    - The students in *Tiling a Floor* decide what shapes they want to examine, collect data about which ones work and which ones do not, organize this information, and begin the analysis of the results.

9. **Recognize that there may be multiple ways to solve a problem.**

    - In *Tiling a Floor,* the students select a variety of materials, including a computer program, to solve the problem. In *Sharing Cookies*, the students all use different methods to solve the same problem.

Standard 2. All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Discuss, listen, represent, read, and write as vital activities in their learning and use of mathematics.**

    - The students in *Tiling a Floor* discuss the problem both initially and at the end, represent the problem situation using manipulatives or the computer, represent their solutions pictorially, and write about the solutions in their journals. The students in *Sharing Cookies* discuss the problem before and after working in groups, represent the problem situation and their solution using manipulatives or a picture and a number sentence, and write about their solution.

2. **Identify and explain key mathematical concepts, and model situations using oral, written, concrete, pictorial, and graphical methods.**

    - The students in the tiling lesson identify and explain the key concept of tiling from geometry. They use oral, written, concrete, and pictorial methods. In the cookie lesson, the students identify and explain the key concept of fractions (from number sense). They use oral, written, concrete, and pictorial methods.

3. **Represent and communicate mathematical ideas through the use of learning tools such as calculators, computers, and manipulatives.**

    - The students in *Tiling a Floor* use manipulatives and computers. The students in *Sharing Cookies* use manipulatives.

4. **Engage in mathematical brainstorming and discussions by asking questions, making conjectures, and suggesting strategies for solving problems.**

    - In both vignettes, the students brainstorm to clarify the task. They also make conjectures and suggest strategies for solving the problem.

5. **Explain their own mathematical work to others, and justify their reasoning and conclusions.**

    - In both vignettes, the students explain their solutions to the class and justify their work.

Standard 3. All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas and the roles that mathematics and mathematical modeling play in other disciplines and in life.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **View mathematics as an integrated whole rather than as a series of disconnected topics and rules.**

    - The students in *Tiling a Floor* tour the school and playground to find and discuss shapes, and then link these to discovering which shapes might be used for tiling a floor or wall. In *Sharing Cookies*, the students examine fractions as an application of a real-life problem rather than as an isolated topic in the book.

2. **Relate mathematical procedures to their underlying concepts.**

    - In *Tiling a Floor* students are able to discover that tiles which fit around a point without leaving spaces can be used as a pattern for tiling the floor. The students in *Sharing Cookies* will, in the near future, be learning how to simplify their answers by using addition to find a single fraction that describes their answer.

3. **Use models, calculators, and other mathematical tools to demonstrate the connections among various equivalent graphical, concrete, and verbal representations of mathematical concepts.**

   - Students in *Tiling a Floor* might use a software program such as *Tesselmania!* to create their own successful tessellations. In *Sharing Cookies*, the students use models to demonstrate that their answers are the same even though the number sentences are different.

4. **Explore problems and describe and confirm results using various representations.**

   - The students in both vignettes use shapes to explore their problems and describe their results in pictures and words.

5. **Use one mathematical idea to extend understanding of another.**

   - The tiling vignette uses the idea of shape to extend understanding of tiling. The cookie vignette uses division to build understanding of fractions.

6. **Recognize the connections between mathematics and other disciplines, and apply mathematical thinking and problem solving in those areas.**

   - The tiling vignette demonstrates the connection between mathematics and art. The cookie vignette connects mathematics to home economics where fractions are often used.

7. **Recognize the role of mathematics in their daily lives and in society.**

   - Both vignettes illustrate the use of mathematics in daily life.

Standard 4. All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Make educated guesses and test them for correctness.**

   - The students in *Tiling a Floor* predict whether each shape can or can not be used as a tile. The students in *Sharing Cookies* might investigate how many different cookie amounts they can find which can be shared equally among five people (or 4 people, or 7, 8, and so on) with none left over. They guess that every multiple of the number of people would work, and use models for several examples to show that each number they select can be separated into piles containing equal amounts with none left over.

2. **Draw logical conclusions and make generalizations.**

   - The students in the tiling vignette make a generalization that shapes which can be used for tilings fit around a point without leaving spaces or being on top of each other. Students in the cookie vignette conclude that 8 cookies are not enough to give each of the 5 people two whole cookies, but that each could have 1 cookie with some left over.

3. **Use models, known facts, properties, and relationships to explain their thinking.**

   - Students in both vignettes use models to explain their thinking.

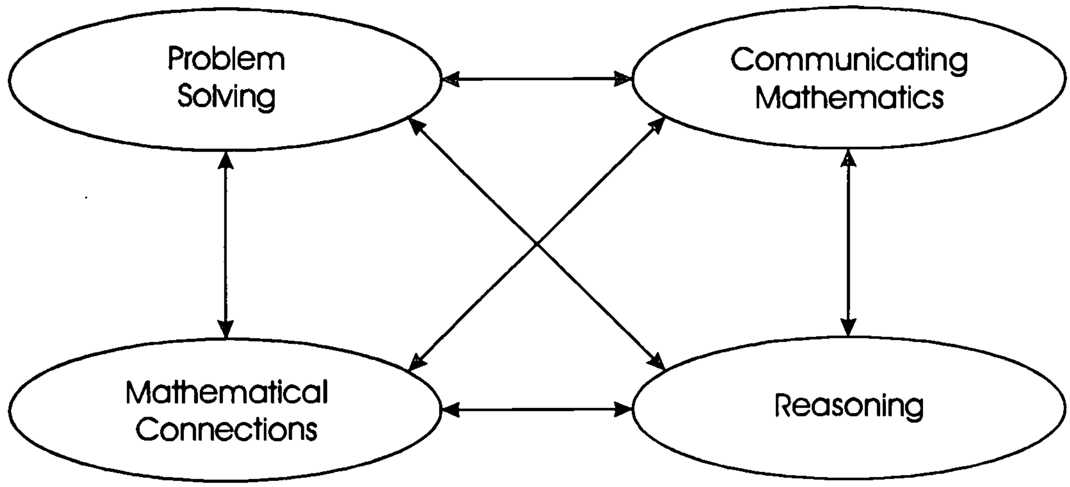
4. **Justify answers and solution processes in a variety of problems.**

   - Students in both vignettes explain their answers and how they got them.

5. **Analyze mathematical situations by recognizing and using patterns and relationships.**

   - Students in the tiling vignette recognize that any triangle can be used as a tile. Students in the cookie vignette realize that each solution names the same fraction with a number sentence.

The First Four Standards — Grades 5-6



Overview

By grades 5-6, students have mastered the basics of problem solving, communication, mathematical connections, and reasoning, and can apply these with reasonable facility to familiar topics. They are ready to consider more complex tasks in each of these areas as well as to apply their skills to more advanced mathematical topics such as probability, statistics, geometry, and rational numbers.

Students in grades 5 and 6 should have experiences with a wide variety of **problem-solving** situations. Some of these should be process problems, in which the strategies students use are of more interest than the solution. For example: *The Boys' Club held its annual carnival last weekend. Admission to the carnival was $3 for adults and $2 for children under 12. Total attendance was 100 people and $232 was collected. How many adults and how many children attended the carnival?* (Kroll & Miller, 1993, p. 60)

Charles & Lester (1982) recommend that students first discuss a problem as a whole class, focusing on understanding the problem and discussing possible strategies to use in solving the problem. While students are working on the problem, the teacher should observe and question students, offering hints or problem extensions as needed. After the students have solved the problem, they should show and discuss their solutions and relate the problem to others previously solved. For example, the problem above is similar in structure to the one involving pigs and chickens that was discussed in the K-12 Overview.

Fifth- and sixth-graders should apply their improved **communication** skills to the new mathematics they are learning in order to help them better understand the concepts and procedures and to help their teachers better assess the students' understanding. Students should not only talk with each other in pairs, small groups, and as a class, they should also use mathematical symbols and language and draw pictures, diagrams, and other visual representations. Writing assignments may include keeping a journal or log, writing about mathematical concepts or procedures, or explaining how they solved a problem.

Students in the middle grades have a wider range of **mathematical connections** to address than do younger students. Within mathematics, multiple representations and patterns and generalizations are particularly important. For example, students might consider the numbers in the table below, looking for patterns and trying to develop a generalization (rule) that describes how the numbers are related.

x	0	1	2	5
y	0	3	6	15

They should consider not only the number sentence $y = 3x$ that describes this pattern but also the verbal rule *y is three times as big as x* and the graph, which involves a straight line. Such activities not only develop communication skills, they also address patterns, numerical operations, and algebra.

Students in grades 5 and 6 also apply mathematics to other subject areas and to real life. Social studies offers many opportunities for using data analysis and discrete mathematics, while art activities require the application of geometry. In science, students should observe and experiment with scientific phenomena and then summarize their observations using graphs, symbols, and geometry; they should translate the patterns they observe into mathematical terms. Following are several examples: *Students might compare the amount of bounce that results from dropping a ball from various heights, develop a rule relating the height of bounce to the height of the drop, and use the rule to predict how high the ball will bounce from any given height. They might repeat the activity for a variety of different balls. Students might learn about how numbers are used as rates by having them measure the distance a toy car goes down an inclined ramp and the amount of time it takes, and then compute the speed. The study of angles might be related to reflections of light in a mirror or the construction of sundials, students might use graphs and charts concerning relative humidity or wind chill, they might examine magnification as a real-world context for multiplication, they might connect volume to studying about water or density, or they might use pulse rates or the mean temperature or rainfall as examples of averaging.* Each of these activities demonstrates for students the usefulness of mathematics in learning about and doing science.

Students in fifth and sixth grade continue to develop their basic **reasoning** skills. Much emphasis at these grade levels should be placed on inductive reasoning in which students look for patterns, make and test conjectures, and form generalizations based on their observations. Their ability to use deductive reasoning — to use logic and justify conclusions — is enhanced at these grade levels by the teacher's frequent use of this type of reasoning. For example: *A square is a kind of rectangle. A rectangle is a kind of parallelogram. So a square is a kind of parallelogram.*

Students in the middle grades must be encouraged constantly to use their reasoning skills not only in mathematics class but also in other subjects and in their daily lives. Only in this way will they come to recognize the power of mathematical reasoning.

As fifth- and sixth-graders expand their understanding of mathematics to include more advanced topics, they also expand their understanding of problem solving, communication, mathematical connections, and reasoning. They apply these skills to more difficult problems concerning topics they have already studied and to problems involving new mathematical concepts and procedures. They further use these skills to gain a better understanding of new ideas.

References

Charles, R., and F. Lester. *Teaching Problem Solving: What, Why, & How.* Palo Alto, CA: Dale Seymour Publications, 1982.

Kroll, D. L., and T. Miller. "Insights from Research on Mathematical Problem Solving in the Middle Grades." In *Research Ideas for the Classroom: Middle Grades Mathematics*. (D. T. Owens, Ed.) Reston, VA: National Council of Teachers of Mathematics, 1993.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

The First Four Standards — Grades 5-6

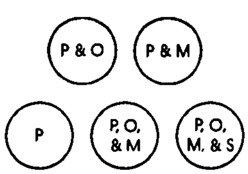
Vignette — Pizza Possibilities

Standards: In addition to The First Four Standards, this vignette highlights Standards 8 (Numerical Operations), 11 (Patterns), and Discrete Mathematics (14).

The problem: In a multi-aged group of fifth- and sixth-graders, the teacher posed this problem. *Pizza Shack has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: pepperoni, sausage, mushrooms, and onions. How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all the possible choices.*

The discussion: To check for understanding, the teacher had various students restate the problem. Students began the discussion thinking that there were only 4 choices. One student suggested that there could be a pizza with both pepperoni and sausage. Students then realized the difficulty of the problem. There was a question about whether a pizza with no toppings was allowed. Students brainstormed about ways to solve the problem. Some suggestions were: draw pictures, make a list, make a chart, use cubes of different colors to represent the different toppings, and use letters or numbers to represent each topping. Before the groups began to solve the problem, the teacher had students predict and record the number of possible choices.

Solving the problem: Students worked in pairs for 15 to 20 minutes. Some groups had divided the work, with one student doing the 1- and 2-topping pizzas and the other the 3- and 4-topping pizzas. Others did the work separately and then compared their results. Still others had one person use cubes to show each combination as the other person recorded the combinations. Students kept track of the different kinds of pizzas in several ways.



P	M	S	O
0	0	0	0
1	0	0	0
0	1	0	0
0	0	1	0

pepperoni and sausage
pepperoni and mushrooms
pepperoni and onions
no toppings

Summary: Pairs of students reported to the class about how they solved the problem, and they compared their predictions with their actual results. They had some success giving convincing arguments, mostly relying on well-organized listing procedures. One pair of students suggested that they knew their solution was correct because it was part of the following pattern which involves powers of 2: if no toppings are available, there is only one pizza; if one topping is available, there are two possible pizzas; if two toppings are available, there are four possible pizzas; if three toppings are available, there are eight possible pizzas; and if four toppings are available, there are sixteen possible pizzas. With each extra topping, they correctly reasoned, the number of possibilities doubled, since each pizza on the previous list could now also be made with the additional topping. They confirmed their reasoning by constructing a chart like that above with 16 different rows, each row representing one possible pizza.

The First Four Standards — Grades 5-6

Vignette — Two-Toned Towers

Standards: In addition to the First Four Standards, this vignette highlights Standards 7 (Geometry), 8 (Numerical Operations), 11 (Patterns), 13 (Probability and Statistics), and 14 (Discrete Mathematics).

The problem: The same multi-aged group which worked on the pizza problem was challenged with this towers problem several weeks later. *Your group's task is to build as many towers as you can that are 4 cubes tall and that use no more than 2 colors. Then you are to convince each other that there are no duplicates and none has been omitted.*

The discussion: Students questioned the meaning of "no more than 2 colors." Some thought that each tower must have 2 colors, other thought that towers of 1 color were also allowed. After some discussion, students agreed, for this problem, to build towers with 1 or 2 colors. The teacher then asked two volunteers to build two towers each — one that satisfied the conditions of the problem and the other that did not. Students discussed each tower, and decided which ones were appropriate. Some students wondered if they had to build the towers, or if they could just draw and label the towers. The group decided that not actually building the towers was okay, as long as each student kept a permanent record of the towers that he "built."

Solving the problem: Students worked in pairs or other small groups to begin building the towers. In most groups, at least one person felt more comfortable actually building the towers, rather than relying on drawing the towers. As the groups worked, the teacher stopped at each group, and encouraged students to talk about how they knew when all the towers were made and how they knew there were no duplicates. Some students working with red and blue cubes were very organized, building towers in this order: RRRR, BRRR, RBRR, RRBR, RRRB, and so on. Others worked very haphazardly, and did not organize their towers until they were all made. Still others used an "opposites" approach, making the RRRB tower followed by BBBR, RBBR followed by BRRB, and so on.

Summary: Students used their solutions to the problem as a focal point for their reports to the class. All of the students agreed that there were 16 towers and were able to convince their classmates by using some variation of the organized list strategy. Then, one student suggested that somehow the pizza problem and the towers problem were alike. Students grappled with the idea for a few minutes. Then the teacher suggested that they list the things that were alike about the problems. Their list included these ideas: there were 4 toppings in the pizza problem and 4 blocks in each tower; 0 toppings was like a tower of all one color and 4 toppings was like a tower of all the other color. After more discussion, students realized that they could better match up the two problems if they put the toppings in a specific order, like pepperoni, mushrooms, sausage, and onions. Then if you ordered a pizza with only pepperoni and mushrooms, it would be like saying "yes, yes, no, no" to the four toppings; and that was like building a tower which was "red, red, blue, blue." A pizza with pepperoni and sausage, or "yes, no, yes, no", would match up to the tower of "red, blue, red, blue." As the discussion went on, students went back to their problem-solving groups to record the connections between the two problems. At the end of the class, the teacher explained that both of these problems belong to an area of mathematics called *combinatorics*, which deals with problems involving combinations and is used in analyzing games of chance.

The First Four Standards — Grades 5-6

Indicators

The cumulative progress indicators for grade 8 for each of the First Four Standards appear in boldface type below the standard. Each indicator is followed by a brief discussion of how the preceding grade-level vignettes might address the indicator in the classroom in grades 5 and 6. The Introduction to this *Framework* contains three vignettes describing lessons for grades 5-8 which also illustrate the indicators for the First Four Standards; these are entitled *The Powers of the Knight*, *Short-circuiting Trenton*, and *Mathematics at Work*.

Standard 1. All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

4*. Pose, explore, and solve a variety of problems, including non-routine problems and open-ended problems with several solutions and/or solution strategies.

- Both vignettes involve problems with multiple solution strategies.

5*. Construct, explain, justify, and apply a variety of problem-solving strategies in both cooperative and independent learning environments.

- In both vignettes, students apply, explain, and justify their choice of problem-solving strategies. In each case, the students work in pairs or small groups.

6*. Verify the correctness and reasonableness of results and interpret them in the context of the problems being solved.

- Students in both vignettes verify their results by sharing them with each other. Since the problems involve single answers, the students decide that they are correct if they all agree and can understand each other's reasoning.

7*. Know when to select and how to use grade-appropriate mathematical tools and methods (including manipulatives, calculators and computers, as well as mental math and paper-and-pencil techniques) as a natural and routine part of the problem-solving process.

- Students in both vignettes use manipulatives as a part of the problem-solving process.

* Reference is made here to Indicators 4, 5, 6, 7, and 8, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

8[*]. Determine, collect, organize, and analyze data needed to solve problems.

- The students in *Pizza Possibilities* determine the different possibilities, organize this information in a table, and analyze their results. The students in *Two-Toned Towers* make organized lists and analyze their results.

10. Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand mathematical content appropriate to the middle grades.

- Both vignettes deal with the same content, determining the number of possible combinations (discrete mathematics). The topic is introduced through two different problem situations which are then compared to each other.

11. Recognize, formulate, and solve problems arising from mathematical situations, everyday experiences, and applications to other disciplines.

- The pizza problem arises from the students' everyday experiences, while the tower problem arises from a more fanciful situation.

12. Construct and use concrete, pictorial, symbolic, and graphical models to represent problem situations and effectively apply processes of mathematical modeling in mathematics and other areas.

- The students in both vignettes use concrete materials, pictures, lists, or tables to represent the problem situation.

13. Recognize that there may be multiple ways to solve a problem, weigh their relative merits, and select and use appropriate problem-solving strategies.

- Students in both vignettes share their different strategies for solving the problem.

14. Persevere in developing alternative problem-solving strategies if initially selected approaches do not work.

- In the pizza vignette, students begin the discussion thinking that there are only 4 choices, but after futher discussion and brainstorming they are able to use various ways to solve the problem. In the towers vignette, some students are not comfortable with drawing pictures and so use concrete materials instead.

Standard 2. All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

[*] Reference is made here to Indicators 4, 5, 6, 7, and 8, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

1*. Discuss, listen, represent, read, and write as vital activities in their learning and use of mathematics.

- Students in *Pizza Possibilities* discuss the problem before and after solving it, listen to each other, represent their solutions in different ways, and write about their solutions. Students in *Two-Toned Towers* also discuss, listen, represent, and write about their solutions.

2*. Identify and explain key mathematical concepts, and model situations using oral, written, concrete, pictorial, and graphical methods.

- Students in both vignettes use oral, written, concrete, and pictorial methods to explain the mathematical ideas involved in their problems. The oral discussion in the towers vignette might be followed up by asking students to write about what they learned concerning combinations.

3*. Represent and communicate mathematical ideas through the use of learning tools such as calculators, computers, and manipulatives.

- Both vignettes involve the use of manipulatives.

4*. Engage in mathematical brainstorming and discussions by asking questions, making conjectures, and suggesting strategies for solving problems.

- In both vignettes, the teacher involves the class in discussion before solving the problem.

5*. Explain their own mathematical work to others, and justify their reasoning and conclusions.

- In both vignettes, students explain their work to their classmates, justifying their solutions.

6. Identify and explain key mathematical concepts and model situations using geometric and algebraic methods.

- In both vignettes, students use cubes of different colors to model the number of possibilities geometrically.

7. Use mathematical language and symbols to represent problem situations, and recognize the economy and power of mathematical symbolism and its role in the development of mathematics.

- Students use symbols to represent the toppings in the pizza vignette and the colors of the cubes in the towers vignette. They also make the connections between these symbols in discussing how the two problems were alike.

8. Analyze, evaluate, and explain mathematical arguments and conclusions presented by others.

- Students in both vignettes analyze each other's arguments and evaluate them.

* Reference is made here to Indicators 1, 2, 3, 4, and 5, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

> Standard 3. All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas and the roles that mathematics and mathematical modeling play in other disciplines and in life.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

1*. View mathematics as an integrated whole rather than as a series of disconnected topics and rules.

- By drawing students' attention to the connection between two seemingly unrelated problems, the teacher encourages the students to look for connections between mathematical topics.

2*. Relate mathematical procedures to their underlying concepts.

- In both vignettes, the students relate the problem to making lists of possibilities and to taking powers of 2.

3*. Use models, calculators, and other mathematical tools to demonstrate the connections among various equivalent graphical, concrete, and verbal representations of mathematical concepts.

- In both vignettes, students use models to represent the situation. After describing and representing the problems concretely, they discover that their models are equivalent and that the problems are essentially the same.

4*. Explore problems and describe and confirm results using various representations.

- In both vignettes, students use a variety of representations to explore the problem situation and to describe their results.

8. Recognize and apply unifying concepts and processes which are woven throughout mathematics.

- Both vignettes deal with the unifying concepts of patterns and the process of making an organized list.

9. Use the process of mathematical modeling in mathematics and other disciplines, and demonstrate understanding of its methodology, strengths, and limitations.

- The students develop a very simple mathematical model when they introduce symbols to represent the ingredients on the pizza and record the different combinations. Once they notice the pattern of powers of two the students develop a numerical representation of their model. In *Two-Toned Towers*, students use essentially the same simple model to

* Reference is made here to Indicators 1, 2, 3, and 4, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

create the different towers. Then they link these representations to the pizza topping problem and also to "yes, no" choices for selecting items.

10. Apply mathematics in their daily lives and in career-based contexts.

- The students in both vignettes apply mathematics to a realistic problem.

11. Recognize situations in other disciplines in which mathematical models may be applicable, and apply appropriate models, mathematical reasoning, and problem solving to those situations.

- Students might investigate in which other disciplines the concept of continual doubling occurs. For example, in science, bacteria cells divide, and thus double, repeatedly.

Standard 4. All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

2*. Draw logical conclusions and make generalizations.

- At the end of the towers vignette, as the students compare the two problems, they begin to draw logical conclusions about the relationship between the problems. The teacher might further extend these generalizations by building upon the numerical pattern observed by students in the pizza problem. For example, a tower one cube high can be built in two ways with two colors, a tower two cubes high can be built in $2 \times 2 = 4$ ways, one three cubes high can be built in $2 \times 2 \times 2 = 8$ ways, and one four cubes high can be built in $2 \times 2 \times 2 \times 2 = 16$ ways. *In how many ways could you build a tower ten cubes high with two colors? In how many ways could you build a tower three cubes high with five colors?*

3*. Use models, known facts, properties, and relationships to explain their thinking.

- In both vignettes, the students use their models to explain their thinking.

5*. Analyze mathematical situations by recognizing and using patterns and relationships.

- In both vignettes, the students look for patterns to help them solve the problem.

6. Make conjectures based on observation and information, and test mathematical conjectures and arguments.

- In both vignettes, the students make conjectures about the problem solution based on observation and test their conjectures.

* Reference is made here to Indicators 2, 3, and 5, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

7. **Justify, in clear and organized form, answers and solution processes in a variety of problems.**

   - Students in both vignettes organize their results and justify their solutions.

8. **Follow and construct logical arguments, and judge their validity.**

   - Students construct logical arguments in explaining their solutions and understand and agree with the arguments presented by other students in both vignettes.

9. **Recognize and use deductive and inductive reasoning in all areas of mathematics.**

   - The students use both types of reasoning. First, they use inductive reasoning as they look for patterns in the problems. They begin to use deductive reasoning in justifying their results and further extend their use of inductive and deductive reasoning as they discuss how the two problems are related.

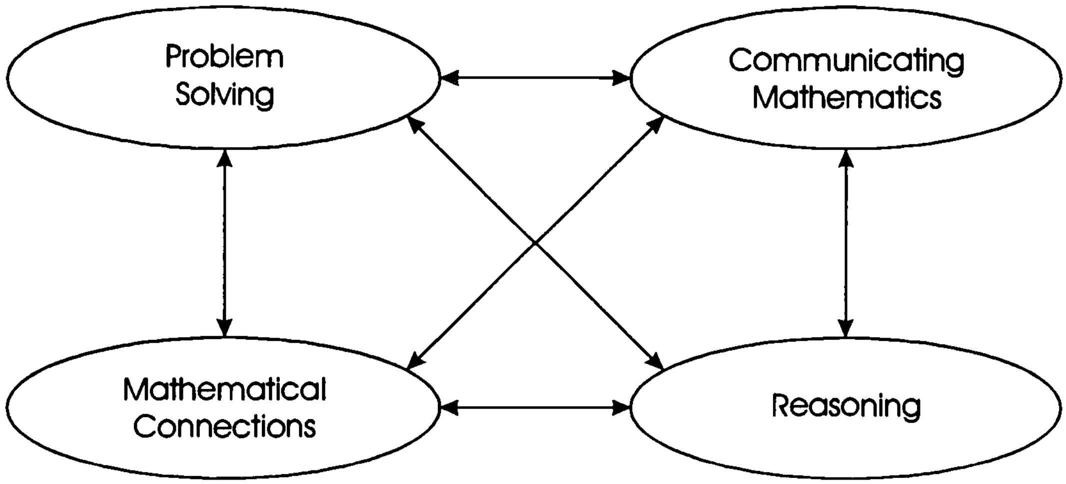
10. **Utilize mathematical reasoning skills in other disciplines and in their lives.**

    - The pizza and related towers vignette demonstrate the use of mathematical reasoning in real life.

11. **Use reasoning rather than relying on an answer-key to check the correctness of solutions to problems.**

    - The students in both vignettes decide whether or not their answers are correct by comparing them and listening to each other's justifications.

The First Four Standards — Grades 7-8



Overview

Seventh- and eighth-graders continue to improve their mastery of problem solving, communication, mathematical connections, and reasoning. They are increasingly able to apply these skills in more formal and abstract situations.

Students in grades 7-8 need to have many opportunities to practice **problem solving**, reflecting upon their thinking and explaining their solutions. They should have developed a considerable repertoire of problem-solving strategies by this time, including working backwards, writing an equation, looking for patterns, making a diagram, solving a simpler problem, and using concrete objects to represent the problem situation. Students should be able to apply the steps of understanding the problem, making a plan, carrying out the plan, and looking back. (See the K-12 Overview at the beginning of this chapter for a discussion of these steps.)

Students in grades 7 and 8 need many opportunities to **communicate** mathematically. Explaining and justifying one's work orally and in writing leads to deeper understanding of concepts and principles. Through discussion, students reach agreement on the meanings of words and recognize the importance of having common definitions. The need for mathematical symbols arises from the exploration of concepts, and these concepts must be firmly connected to the symbols that represent them through frequent and explicit discussion of the relationships between concepts and symbols. Students must also be encouraged to construct connections among concepts, procedures, and approaches by using questions that require more elaborate communication skills. For example, students might give examples of: a rectangle with four congruent sides, a parallelogram with four right angles, a trapezoid with two equal angles, a number between 1/3 and 1/2, a number with a repeating decimal representation, a net for a cube, or an equation for a line that passes through the point (−1,2). (See *Curriculum and Evaluation Standards for School Mathematics*, p. 80.) [A net is a flat shape which when folded along indicated lines will produce a three-dimensional object; for example, six identical squares joined in the shape of a cross can be folded to form a cube. Tabs added to the net facilitate attaching appropriate edges so that the shape remains three-dimensional.]

Many students in the middle grades have in the past viewed mathematics as a collection of isolated skills to be memorized and later forgotten. These students must broaden their perspective, viewing mathematics as an integrated whole and acknowledging its relevance in and out of school. They must improve their understanding of **mathematical connections.** Students must understand, for example, that translations on the number line are fundamentally the same as adding numbers. They should connect the various interpretations of fractions to measurement, ratios, and algebra. They should link Pascal's triangle with counting, exponents, number patterns, algebra, geometric patterns, probability, and number theory. (See Standard 14, Discrete Mathematics.) Such connections can be enhanced by using technology and by exploring the same mathematical ideas in varied contexts. Some of the most important connections for students in grades 7 and 8 include proportional relationships (see Standard 6, Number Sense) and the relationship between data in tables and their algebraic and graphical representations. (See Standard 11, Patterns; Standard 12, Probability and Statistics; and Standard 13, Algebra).

Students in these grades also need many opportunities to discuss the connections between mathematics and other disciplines and the real world. For example, students concerned about traffic at a nearby intersection might design a study to collect data about the situation. Their study might involve first deciding what "traffic" is, how to count it, how to record the data, what the data means, how to remedy the situation, who is responsible for dealing with this type of situation, and how to convince that person that change is needed. Their study might end with a letter to the town council, recommending changes needed at the intersection.

In conjunction with studying about maps in social studies, for example, students might study scaling and its relationship to similarity, ratio, and proportion. Measurement situations arise in social studies, science, home economics, industrial technology, and physical education. Weather forecasting, scientific experiments, advertising claims, chance events, and economic trends offer more opportunities to relate mathematics to the real world, often through the use of statistics and probability. Connections between mathematics and science are plentiful:

- Students use symmetry and perspective to create original artistic work.
- Students might learn about scientific notation for large numbers by studying the solar system or for small numbers by studying viruses and bacteria.
- Different scientific contexts, such as electrical charges or forces in opposite directions, might help students develop multiple representations for integers.
- Students might plot earthquake locations by using a compass and protractor.
- The study of geometric shapes might be related to the study of crystals.
- Area and volume might be considered in conjunction with looking at the number of fish in a pond. *Why would each measure be important to consider? How could the area of a pond be found? How could the volume be found?*
- Students might investigate levers, gears, or pulleys, looking for numerical patterns and generating equations that describe the relationships found.

Students need frequent opportunities to explore and discuss many of these types of relationships in order to develop an appreciation for the power of mathematics.

Students in grades 7 and 8 experience rapid improvements in their ability to use mathematical **reasoning.** Special attention should be paid at these grades to proportional and spatial reasoning and reasoning from graphs. Students need to experience both inductive reasoning (making conjectures, predicting outcomes, looking for patterns, and making generalizations) and deductive reasoning (using logical arguments, justifying answers). Experiences with inductive reasoning need to include situations in which incorrect generalizations based on too few examples are tested. Instructional activities that address two specific types of deductive reasoning are appropriate at these grades. *Class reasoning* involves applying generalizations to a specific situation: for example, all even numbers are divisible by 2; x is an even number; so x is divisible by 2. *Conditional reasoning* involves using if-then statements. These types of reasoning need to be used on a regular basis in class discussions, assignments, and tests in order to help students become familiar with valid reasoning patterns. Students should also be asked to justify and explain solutions to the satisfaction of their peers.

In grades 7 and 8, students apply their problem-solving, communication, and reasoning skills in an increasingly diverse set of situations as they develop better understanding of the connections within mathematics and between mathematics and other subjects and the real world. In doing so, they learn new mathematical concepts and apply familiar mathematics to new situations.

References

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA, 1989.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

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The First Four Standards — Grades 7-8

Vignette — Sketching Similarities

Standards: In addition to the First Four Standards, this vignette highlights Standards 5 (Tools and Technology), 7 (Geometry), 9 (Measurement), and 15 (Building Blocks of Calculus).

The problem: A heterogeneously grouped class of middle school students reported to the computer lab for their mathematics class. Students had used *Geometer's Sketchpad* software (Key Curriculum Press) frequently during the year. They had been studying proportions in everyday and geometric situations during several previous class periods. They had also defined similar figures as those generated by an expansion or a contraction. The teacher presented this task to the class: *Draw several similar figures. What do you notice about the ratios of the measures of corresponding sides? What do you notice about the measures of the corresponding angles? Record your results and write your conclusions.*

The discussion: The teacher asked students to begin brainstorming to generate ideas about the problem. As ideas were suggested, the teacher wrote them on the chalkboard. Ideas included: *Use simple figures to make the problem easier. Use the dilate transformation. Use measure and length on Sketchpad to find the lengths of the sides. Make more than two figures because the problem says several. What is corresponding? Do we have to write complete sentences? Use measure and angle on Sketchpad to find the measures of the angles.* After all the ideas were recorded, the class discussed each one. Some were questions that could be answered by students, by rereading the problem or by checking glossaries in math books. Those answers were recorded next to the appropriate questions. Some of the ideas were eliminated because they were either incorrect or off-task. The teacher then had students restate the problem and, using the list, talk about the steps they would follow to solve the problem.

Solving the problem: Students joined their regular lab partners at the computers and began the investigation. As students worked, some requested teacher help, asking if their work was "finished yet" and if their results were "good enough." The teacher directed those students to reread the problem and the list of ideas developed during the discussion, and determine for themselves which parts of the problem were completed adequately and which were left to be done. Students worked for two class periods, completing the task by preparing diagrams, ratios, and written explanations. The teacher provided each pair with a transparency on which they were to record their diagrams and ratios to share with the class.

Summary: The summary discussion began with a volunteer restating the problem. Another volunteer described the steps she used to solve the problem. Others interjected during the description to tell how their methods differed. Then students presented their findings, using their transparencies and their written conclusions. After lengthy discussion, students wrote a general class conclusion that similar figures have corresponding angles with equal measures and have corresponding sides whose measures form equal ratios.

The First Four Standards — Grades 7-8

Vignette — Rod Dogs

Standards: In addition to the First Four Standards, this vignette highlights Standards 7 (Geometry), 8 (Numerical Operations), 9 (Measurement), 11 (Patterns), and 15 (Building Blocks of Calculus).

The problem: Students in this multi-aged middle school class had been working for several days on a project that gave them opportunities to find the surface area and volume of objects constructed with Cuisenaire rods and tacky putty. Today they were to begin an investigation of the relationship between the surface area and the volume of 3-dimensional objects enlarged with scale factors from 2 to 8. Students, working in groups of two or three, were instructed to build a "dog" using 1 yellow (5cm in length) and 5 red (2cm) rods. The yellow rod represented the dog's body, and a red rod represented each leg and the head. Then students calculated the surface area and volume of the dog. (Note that all rods have cross-section 1cm x 1cm.) Students were then instructed to build a "double dog" that was twice as big in each of the three dimensions. After several false starts, students constructed a dog with 4 orange (10cm) rods and 20 purple (4cm) rods. The 4 orange rods represented the dog's body, and 4 purple rods represented each leg and the head. Then students found and recorded the surface area and volume of this double dog. Each group was to build one other dog, using an assigned scale factor of 3, 4, 5, 6, 7, or 8 times the original dog. Their challenge was to determine how the scale factors were related to the surface area and volume of the enlarged dogs.

The discussion: Students asked questions to clarify the problem, mostly checking the task by saying *You mean, if my scale factor is 7, my dog's body has to be 35 by 7 by 7?* The teacher encouraged other students to confirm or correct each question. Then students focused on how they were to construct the dogs. Many began to think about how many rods they would need to complete the construction and decided that for some of the scale factors, they just would not have enough rods. The teacher then pointed out the centimeter grid paper, scissors, and tape at the front of the room and suggested that the students could build the dogs with those supplies.

Solving the problem: Students worked for two days building their enlarged dogs. Part of the challenge was to lay out a pattern for each part of the dog on a sheet of paper so that it could be folded and then put together with as few taped edges as possible. Those students who were given scale factors of 3 or 4 found the task rather simple and were able to count the surface area and volume without much trouble. Those with greater scale factors such as 7 or 8 had a more difficult task, but were able to complete it successfully.

Summary: The scale factor, surface area, and volume of each dog was listed on the chalkboard. Students discussed the various methods that different groups used to construct the dogs and to find the surface area and volume. The teacher challenged students to find the pattern in the chart, to apply the rule to scale factors other than those already used, and to generalize the rule to a scale factor of n. Students made liberal use of calculators, using a guess and check strategy to find the pattern. Students were able to verbalize that the surface area was the scale factor squared times the original surface area and that the volume was the scale factor cubed times the original volume.

The First Four Standards — Grades 7-8

Indicators

The cumulative progress indicators for grade 8 for each of the First Four Standards appear in boldface type below the standard. Each indicator is followed by a brief discussion of how the preceding grade level vignettes might address the indicator in the classroom in grades 7 and 8. The Introduction to this *Framework* contains three vignettes describing lessons for grades 5-8 which also illustrate the indicators for the First Four Standards; these are entitled *The Powers of the Knight*, *Short-circuiting Trenton*, and *Mathematics at Work*.

Standard 1. All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

4*. Pose, explore, and solve a variety of problems, including non-routine problems and open-ended problems with several solutions and/or solution strategies.

- The students in *Sketching Similarities* explore a problem involving geometry in which they choose their own figures to investigate. The students in *Rod Dogs* explore and solve a non-routine problem.

5*. Construct, explain, justify, and apply a variety of problem-solving strategies in both cooperative and independent learning environments.

- In both vignettes, students use the strategy of looking for patterns. The students in *Sketching Similarities* work in pairs and use inductive reasoning to find a pattern. The groups of students in *Rod Dogs* use concrete materials as their primary strategy, with some using rods and others using grid paper.

6*. Verify the correctness and reasonableness of results and interpret them in the context of the problems being solved.

- In both vignettes, the students verify their results and interpret them through a class discussion following their work in pairs or groups.

7*. Know when to select and how to use grade-appropriate mathematical tools and methods (including manipulatives, calculators and computers, as well as mental math and paper-and-

* Reference is made here to Indicator 4, 5, 6, 7, and 8, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

pencil techniques) as a natural and routine part of the problem-solving process.

- Students in *Sketching Similarities* use computers. Students in *Rod Dogs* use manipulatives and calculators as needed in solving their problems.

8*. Determine, collect, organize, and analyze data needed to solve problems.

- The students in *Sketching Similarities* decide what data to collect in the initial discussion of the problem. They analyze the collected data in the discussion following their work in pairs. The students in *Rod Dogs* collect and organize the data as they work in groups. In their summary discussion, they analyze all the data collected to look for patterns and form a generalization.

10. Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand mathematical content appropriate to the middle ages.

- The students in both vignettes use a problem to investigate new mathematical concepts.

11. Recognize, formulate, and solve problems arising from mathematical situations, everyday experiences, — and applications to other disciplines.

- The *Sketching Similarities* vignette involves examining a problem that arises from a mathematical situation. The *Rod Dogs* vignette involves a problem that arises from everyday experiences — enlarging a three-dimensional object.

12. Construct and use concrete, pictorial, symbolic, and graphical models to represent problem situations and effectively apply processes of mathematical modeling in mathematics and other areas.

- The students in *Sketching Similarities* use graphical (computer diagrams) and symbolic (numbers) models in considering their problem. The students in *Rod Dogs* use concrete materials (rods and grid paper folded to make boxes) and symbolic (numbers and formulas) models.

13. Recognize that there may be multiple ways to solve a problem, weigh their relative merits, and select and use appropriate problem-solving strategies.

- The students in *Sketching Similarities* use brainstorming to generate ideas about solving the problem, and share their methods with the class. In *Rod Dogs*, students approach the problem in different ways, and the class discusses the various methods that different groups use to construct the "dogs" and to solve the problem; some students use a guess and check strategy to find the pattern.

14. Persevere in developing alternative problem-solving strategies if initially selected approaches do not work.

- When some students are at a standstill in *Rod Dogs*, the teacher directs their attention to available manipulatives. In *Sketching Similarities,* the teacher encourages students to

* Reference is made here tp Indicators 4, 5, 6, 7, and 8, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

research the problem and to make use of the list of ideas generated by the class in their continued attempts to solve the problem.

Standard 2. All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

1*. Discuss, listen, represent, read, and write as vital activities in their learning and use of mathematics.

- Students in both vignettes discuss, listen, represent the problem situation concretely and symbolically, and read and write about their solutions.

2*. Identify and explain key mathematical concepts, and model situations using oral, written, concrete, pictorial, and graphical methods.

- Students in *Sketching Similarities* explain the concept of similarity and model the situation using oral, written, concrete, and pictorial methods. Students in *Rod Dogs* explain how the scale factor is related to changes in area and volume and use oral, written, concrete, and pictorial methods.

3*. Represent and communicate mathematical ideas through the use of learning tools such as calculators, computers, and manipulatives.

- Students in *Sketching Similarities* use computers and manipulatives. Students in *Rod Dogs* use manipulatives and calculators.

4*. Engage in mathematical brainstorming and discussions by asking questions, making conjectures, and suggesting strategies for solving problems.

- Students in both vignettes begin their activity by asking questions, making conjectures, and suggesting strategies to be used.

5*. Explain their own mathematical work to others, and justify their reasoning and conclusions.

- Students in both vignettes share their work with the class and justify their solutions.

6. Identify and explain key mathematical concepts and model situations using geometric and algebraic methods.

- In *Sketching Similarities* students recognize that similarity of triangles has an algebraic counterpart, that corresponding sides have the same ratio; the activities for the vignette

* Reference is made here to Indicators 1, 2, 3, 4, and 5, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

could be extended by asking the students to use algebra to determine the lengths of the sides. In *Rod Dogs* the students develop a formula relating the volume and surface area of the enlarged "dogs" to the volume and surface area of the original one.

7. **Use mathematical language and symbols to represent problem situations, and recognize the economy and power of mathematical symbolism and its role in the development of mathematics.**

   - Students in both vignettes use mathematical language and symbols to represent their solutions.

8. **Analyze, evaluate, and explain mathematical arguments and conclusions presented by others.**

   - Students in both vignettes analyze, evaluate, and explain their solutions.

Standard 3. All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas and the roles that mathematics and mathematical modeling play in other disciplines and in life.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

1*. **View mathematics as an integrated whole rather than as a series of disconnected topics and rules.**

   - The *Sketching Similarities* vignette illustrates the connection between similarity and ratios. The *Rod Dogs* vignette illustrates the connection between geometry and measurement (surface area, volume, scale factor) and exponents.

2*. **Relate mathematical procedures to their underlying concepts.**

   - The *Sketching Similarities* vignette relates the mathematical procedure of setting up ratios for similar figures to the underlying concept of similarity. The *Rod Dogs* vignette relates the mathematical procedure of taking powers of a number to the underlying concepts of area and volume.

3*. **Use models, calculators, and other mathematical tools to demonstrate the connections among various equivalent graphical, concrete, and verbal representations of mathematical concepts.**

   - Students in both vignettes use tools (models, computers, calculators) to discover, demonstrate, and verbalize the connections between the concrete geometric models in each problem and specific numerical relationships which result. In *Sketching Similarities*, they might also apply their conclusions to express and solve numerical and algebraic problems

* Reference is made here to Indicators 1, 2, 3, and 4, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

associated with the angles and sides of similar triangles. In *Rod Dogs*, students might use their discoveries regarding the 3-dimensional objects and graph the relationships of scale factor vs. surface area and scale factor vs. volume.

4*. Explore problems and describe and confirm results using various representations.

- Students in both vignettes explore problems and describe their results in various ways; the representations used are generally concrete and written.

8. Recognize and apply unifying concepts and processes which are woven throughout mathematics.

- The students in *Sketching Similarities* recognize and apply the concept of proportion. The students in *Rod Dogs* recognize and apply the concept of powers.

9. Use the process of mathematical modeling in mathematics and other disciplines, and demonstrate understanding of its methodology, strengths, and limitations.

- Students in both vignettes establish a mathematical model for their problem situations and use it to solve their problems. The students in *Rod Dogs*, in particular, also note the limitations of modeling concretely and the need to move to a more symbolic model.

10. Apply mathematics in their daily lives and in career-based contexts.

- The students in both vignettes apply mathematics to a realistic problem.

11. Recognize situations in other disciplines in which mathematical models may be applicable, and apply appropriate models, mathematical reasoning, and problem solving to those situations.

- Students might investigate how the 2-dimensional relationships in *Sketching Similarities* and the 3-dimensional results from *Rod Dogs* could be used in art, architecture, engineering, business, and the building trades.

Standard 4. All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

2*. Draw logical conclusions and make generalizations.

- Students in both vignettes draw logical conclusions and make generalizations based on the data they collect.

* Reference is made here to Indicators 2, 3, and 5, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

3*. **Use models, known facts, properties, and relationships to explain their thinking.**

- Students in both vignettes use models to explain their thinking.

5*. **Analyze mathematical situations by recognizing and using patterns and relationships.**

- Students in both vignettes look for patterns to help them solve the problems.

6. **Make conjectures based on observation and information, and test mathematical conjectures and arguments.**

- Students in both vignettes make conjectures and test them.

7. **Justify, in clear and organized form, answers and solution processes in a variety of problems.**

- Students in both vignettes justify their answers in writing and orally.

8. **Follow and construct logical arguments, and judge their validity.**

- In both vignettes, the students could be challenged to explain why they thought that their generalizations would always work.

9. **Recognize and use deductive and inductive reasoning in all areas of mathematics.**

- The students in both vignettes use inductive reasoning to find a pattern and make a generalization.

10. **Utilize mathematical reasoning skills in other disciplines and in their lives.**

- Both vignettes illustrate the use of mathematical reasoning in real life.

11. **Use reasoning rather than relying on an answer-key to check the correctness of solutions to problems.**

- The students base their judgments about correctness of their results on the explanations given in the summary discussion rather than on an answer key.

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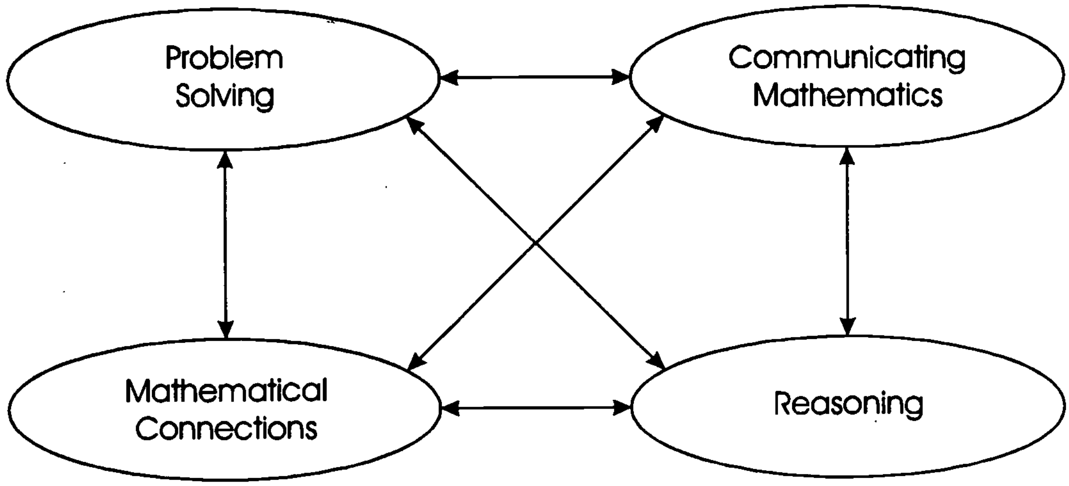
* Reference is made here to Indicators 2, 3, and 5, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

The First Four Standards — Grades 9-12



Overview

New Jersey's *Mathematics Standards* calls for a shift in emphasis from a high school curriculum often dominated by memorization of isolated facts and procedures and by proficiency with paper-and-pencil skills to one that stresses understanding of concepts, multiple representations and connections, mathematical modeling, and mathematical problem solving.

The distinction between mathematical **problem solving** and doing mathematics should begin to blur in the high school grades. The problem-solving strategies learned in the earlier grades should have become increasingly internalized and integrated to form a broad basis for doing mathematics, regardless of the specific topic being addressed. From this perspective, problem solving is much more than solving word problems; it is the process by which mathematical ideas are constructed and reinforced. There is more emphasis in high school on introducing new mathematical concepts and tools as responses to problem situations in mathematics, and on developing students' ability to pose problems themselves.

Through extensive experiences with mathematical **communication**, students improve their understanding of mathematics. Students must be able to describe how they obtain an answer or the difficulties they encounter in trying to solve a problem. Facility with mathematical language enables students to form multiple representations of ideas, express relationships within and among these representations, and form generalizations.

High school students should continue to experience two types of **mathematical connections** — those within mathematics, and those to other areas. First, students should make connections between different mathematical representations of the same concept or process. Students who are able to apply and translate among different representations of the same problem situation or of the same mathematical concept have not only a powerful, flexible set of tools for solving problems but also a deeper appreciation of the consistency and beauty of mathematics. Unifying ideas within mathematics to be emphasized in the high school grades include mathematical modeling, variation (how a change in one thing is associated with a change in another), algorithmic thinking (developing, interpreting, and analyzing mathematical procedures), mathematical argumentation, and a continued focus on multiple representations.

Second, students in high school should regularly discuss the connections between mathematics and other subjects and the real world. Connections between mathematics and science are particularly plentiful. Examples of such activities are:

- Students use computer-aided design (CAD) to produce scale drawings or models of three-dimensional objects such as houses.
- Students use statistical techniques to predict and analyze election results.
- Cooling a cup of coffee provides an opportunity for students to collect and analyze data while learning about the process of cooling: *Why does cooling occur? Does the coffee cool at a constant rate? Does the temperature follow an asymptotically decreasing pattern?* By using temperature probes connected to a computer or graphing calculator (CBL), students can collect data on the temperature of a cup of hot coffee as it cools, plot this data on a coordinate graph, and describe the pattern verbally and with an equation.
- Students might explore models using logarithmic scales by studying the Richter scale used to measure the strength of earthquakes, the decibel scale for sound loudness, or the scale used to describe the brightness of stars.
- Students might develop experiments in which they collect data in order to investigate polynomials of higher degree. They can examine the collected data by using "finite differences" to predict the degree of the polynomial. In addition, this graphic approach helps students develop a better understanding of the concept of "zeros of a function."
- Students at the precalculus level might use science experiments to investigate trigonometry. Students enrolled in physics classes often use the Law of Sines and Law of Cosines prior to their development and discussion in precalculus classes. Thus, there is a need to reexamine the order of presentation of topics both in science and in mathematics.
- Trigonometric functions can be applied not only to modeling terrestrial and astronomical problems requiring indirect measurement but also to describing the motion of water waves, waves in a rope, sound waves, and light waves.
- Students can collect data to model rational functions, including not only $y = 1/x$ but also more complex equations such as $y = (x-1)(x+2)/(x+2)(x-3)$. Students should see why the data excludes the domain value of -2 as well as why there is an asymptote at $x=3$.
- Students might integrate the study of geometric sequences or logarithmic and exponential functions with science topics involving growth and decay. (See *Breaking the Mold* at the end of the Introduction of this *Framework* in which students look at the growth patterns of living things and the vignette involving carbon dating at the beginning of the Introduction.)
- Students might conduct experiments involving velocity with constant acceleration, such as dropping a ball, to study parabolas and quadratic equations.
- Students might study vectors in conjunction with complex numbers.
- Students can relate three-dimensional figures to the geometry of molecules, crystals, and symmetry.
- Students might link the study of solving linear equations in mathematics classes to the

balancing of equations in chemistry.

- The study of direct and inverse variation and different types of functions might be linked to the study of volcanic action and earthquakes.
- The study of calculus and physics might be integrated. In fact, a team-taught course might be more appropriate than the present approach, especially for the applications of calculus. This might prove more appropriate for those students who do not need a theoretical approach to calculus at this time.
- Students might design and construct a container that will sound an alarm when opened. In completing this task, they must use measurement, geometry, numerical operations, algebra, and mathematical reasoning as well as knowledge of electrical circuits, wiring, switches, electricity, insulators, conductors, Ohm's law and its use, and cost estimation.

A student who is doing mathematics often makes a conjecture by generalizing from a pattern of observations made in specific cases (inductive reasoning) and then tests the conjecture by constructing either a logical verification or a counterexample (deductive reasoning). High school students need to appreciate the role of both forms of **reasoning**. They should also learn that deductive reasoning is the method by which the validity of a mathematical assertion is finally established. Much inductive reasoning may take place in algebra, with students looking for patterns that arise in number sequences, making conjectures about general algebraic properties based on their observations, and verifying their conjectures with numerical substitutions. Students can be introduced to deductive reasoning by examining everyday situations, such as advertising, in which logic arises. Logical arguments in mathematical situations need not follow any specific format and may be presented orally or in writing in the student's own words.

High school students focus on mathematical problem solving, using multiple representations of mathematical concepts, mathematical connections, and reasoning throughout all of their mathematics learning. As they learn and do mathematics, they should regularly encounter situations where they are expected to discuss and solve problems, develop mathematical models, explain their results, and justify their reasoning. The content of these four standards is inextricably interwoven with the fabric of mathematics.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

The First Four Standards — Grade 9-12

Vignette — Making Rectangles

Standards: In addition to the First Four Standards, this vignette highlights Standards 7 (Spatial Sense), 11 (Patterns), and 13 (Algebra).

The problem: Yesterday, Mrs. Ellis' class finished a unit in which they used algebra tiles to help them develop procedures for multiplying binomials. Today, Mrs. Ellis began by asking the students to consider the following problem: *Suppose we have a collection of red x^2 tiles, orange x tiles, and yellow unit tiles. Can we put them together to form a rectangle? For example, if we have one red tile, five orange tiles, and four yellow tiles, we can make the rectangle at the right. What combinations of tiles can form a rectangle, and what combination cannot?*

	x	1	1	1	1
x	x^2	x	x	x	x
1	x	1	1	1	1

The discussion: Mrs. Ellis encouraged the students to share their ideas for how they might go about solving this problem. Some of their questions were: *What materials might be helpful in working on this problem? How could we use the algebra tiles? What might be some combinations of tiles that we might try? How could we keep track of what we have tried? Can we use more than one of the red x^2 tiles? Will we be able to try out all of the possible combinations? Do you suppose that we will find some sort of pattern that can help us predict which combinations will work and which will not?*

Solving the problem: The students worked in groups of three or four over a period of three days. They tried different combinations of tiles, recording how many of each tile they used, whether or not that combination could be used to form a rectangle, and, if so, the dimensions of the rectangle. As they worked, they began to notice patterns. Some of their comments were: *This seems like multiplying binomials but in reverse. When it works, the product of the number of 1s on the top times the number of 1s on the sides equals the number of yellow unit tiles. If there's just one of the red tiles, then the sum of the number of 1s on the top plus the number of 1s on the side equals the number of orange tiles.* Each group summarized its conclusions and the patterns they found in a report.

Summary: Mrs. Ellis asked the groups to exchange reports with another group, then read, review, and comment on the other group's report. Each group then had an opportunity to review the comments on their report. Mrs. Ellis asked the students about the patterns they had found. *Are there some patterns that both of your groups found? Are there others that only one of the groups found? She recorded the findings on the board. How can we be sure all of these statements are correct?* The students suggested that, if everyone agreed with a statement and could justify their reasoning, then it should be accepted as correct. They discussed each of the statements, explaining their reasoning and arguing about some of the statements. For homework, Mrs. Ellis asked the students to use their findings to make some predictions about other combinations of tiles and to relate their results to the idea of factoring binomials. Mrs. Ellis expects that in the next classes she will connect the problem of making a rectangle from *a* red x^2 tiles, *b* orange *x* tiles, and *c* yellow unit tiles to the problem of factoring $ax^2 + bx + c$.

The First Four Standards — Grades 9-12

Vignette — Ice Cones*

Standards: In addition to the First Four Standards, this vignette highlights Standards 6 (Number Sense), 7 (Geometry), 8 (Numerical Operations), 9 (Measurement), 11 (Patterns), 13 (Algebra), and 15 (Building Blocks of Calculus).

The problem: Ms. Longhart began class by posing the following problem for her students: *Suppose that you are setting up a water ice stand for the summer and are trying to decide how to make the cones in which you will serve the water ice. You've found some circles of radius 10 cm that are the right type of paper and have figured out that by cutting on a radius, you can make cones. You decide that you would like to make cones that hold as much water ice as possible, so you can charge a higher price. What will be the radius of the base of your cones? What will be the height?*

The discussion: Ms. Longhart asked the students what materials might be useful in helping them solve the problem. Some students suggested making models out of paper circles, while others thought that writing equations and using graphing calculators to find the maximum volume would be best. After some discussion of the relative merits of each approach, Ms. Longhart suggested that they do both and compare their answers.

Solving the problem: The students separated into groups to work on the problem. Most students remembered that the volume of a cone is $1/3\pi r^2h$. They made a variety of paper cones out of circles with radius 10cm, and measured the radius and height of those cones. Using the formula, and a calculator, they generated a table of values, trying to find the maximum volume. They wanted to graph the formula using their calculator, but realized that they needed to solve for h in terms of r. After some initial difficulties, they decided that, since the original circles had a radius of 10 cm, the height of the resulting cone must be $\sqrt{100 - r^2}$. Then they graphed the equation they had generated on the graphing calculator and used the graph to find the maximum volume. Finally, each group summarized its findings in writing.

Summary: Each group listed its actual measurements and its results generated on the graphing calculator on the board and then explained any discrepancies that might have occurred. The class as a whole discussed the accuracy of the solutions. One of the students noticed that, in the cone of maximum volume, the radius was much larger than the height of the cone, and asked why that happened. Ms. Longhart asked the class to think about some possible reasons. To summarize the lesson, Ms. Longhart asked the students to list in their journals all of the mathematical concepts that they used to solve the problem in class that day. For homework, she asked the students to (1) describe how the function they generated would change if the radius of the circle was 8 cm, 9 cm, 11 cm, or 12 cm; (2) find the maximum volumes and corresponding radius for each of the new functions; and (3) determine whether there is a relationship between the radius and the maximum volume for each of the five functions.

* Adapted from Longhart, Karen. "Making Connections." *Eightysomething!* Volume 3, Number 2, Summer 1994.

The First Four Standards — Grades 9-12

Vignette — Building Parabolas

Standards: In addition to the First Four Standards, this vignette highlights Standards 7 (Geometry), 11 (Patterns), and 13 (Algebra).

The problem: Before this session, students in Mr. Evans' class investigated different situations that can be modeled using quadratic functions. They looked at how to maximize the area of a yard given a fixed amount of fencing, and how to predict the path of a rocket. They graphed many quadratic functions, some by plotting points and some on the graphing calculator, and discovered that all of the quadratic functions have graphs that are parabolas. For this session, they went to the computer lab to investigate the relationship of each of the constants in the general form of the parabola to the graph of that equation.

The discussion: Before beginning work on the computers, the students reviewed the general shape of a parabola and discussed the differences between one quadratic function and another: the width of the parabola, how high up or down the vertex is, and whether the parabola opens up or down. Mr. Evans introduced the general form of the equation of a parabola, $y = a\,(x - h)^2 + k$, and asked the students to predict how each of a, h, and k will affect the graph of the parabola. He asked them to explain their reasoning and record their predictions individually in their notebooks, and then he led a discussion of their predictions.

Solving the problem: Each pair of students used a program on *Green Globs* software to predict the equation for a series of parabolas, keeping notes on how the different constants seemed to affect the graphs. There was considerable excitement in the room, as well as some disagreements and some frustration at times. Some pairs of students found that several of the graphs required many attempts before the correct equation was found. At the conclusion of the computer activity, the students compared their results to their predictions and discussed their findings with each other in pairs. They then individually wrote a description of how the values of a, h, and k in the general equation $y = a\,(x - h)^2 + k$ affect the graph of $y = x^2$.

Summary: For homework, Mr. Evans asked the students to use their findings to sketch the graphs of several parabolas without using graphing calculators and then to check their sketches by using graphing calculators. He suggested that they revise their journal entries if they find that some of their hypotheses don't work. Mr. Evans began class the next day by having pairs of students play the computer game *Green Globs*, allowing the students to use only parabolas to hit the globs on the coordinate grid. After about 15 minutes, he led a discussion of the students' findings about parabolas, asking them how they arrived at their hypotheses, what steps they took to verify them, and whether they modified their hypotheses based on their experiences with the homework and the computer game. He then asked the students to think of other areas of mathematics that seem to be related, making connections between their findings about parabolas and what they learned last year about geometric transformations.

The First Four Standards — Grades 9-12

Indicators

The cumulative progress indicators for grade 12 for each of the First Four Standards appear in boldface type below the standard. Each indicator is followed by a brief discussion of how the preceding grade level vignettes might address the indicator in the classroom in grades 9-12. The Introduction to this *Framework* contains three vignettes describing lessons for grades 9-12 which also illustrate the indicators for the First Four Standards; these are entitled *On the Boardwalk*, *A Sure Thing!?*, and *Breaking the Mold.*

> Standard 1. All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

4*. Pose, explore, and solve a variety of problems, including non-routine problems and open-ended problems with several solutions and/or solution strategies.

- In *Making Rectangles*, the students explore and solve an open-ended question. In *Ice Cones*, the students explore and solve a problem using several different solution strategies (concrete materials or writing equations). In *Building Parabolas*, the students explore an open-ended question.

5*. Construct, explain, justify, and apply a variety of problem-solving strategies in both cooperative and independent learning environments.

- In all three vignettes, the students work in small groups to construct, explain, and justify their strategies.

6*. Verify the correctness and reasonableness of results and interpret them in the context of the problems being solved.

- In all three situations, the students verify their results by sharing them and interpreting them in a whole-class discussion.

7*. Know when to select and how to use grade-appropriate mathematical tools and methods (including manipulatives, calculators and computers, as well as mental math and paper-and-pencil techniques) as a natural and routine part of the problem-solving process.

* Reference is made here to Indicators 4, 5, 6, 7, 8, 12, and 14 which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

- Students in *Making Rectangles* use manipulatives to help them solve their problem. Some students in *Investigating Cones* choose to use manipulatives, while others use graphing calculators to help them solve the problem. Students in *Building Parabolas* use computers to help develop their ideas.

8*. Determine, collect, organize, and analyze data needed to solve problems.

- The students in *Making Rectangles* organize and analyze their results. Some of the students in *Ice Cones* decide to approach the problem by looking at some specific cones and finding their volume. (Note that this was not a particularly efficient approach to solving this problem.) The students in *Building Parabolas* decide which parabolas to try out in playing *Green Globs* and then collect, organize, and analyze their data.

12*. Construct and use concrete, pictorial, symbolic, and graphical models to represent problem situations and effectively apply processes of mathematical modeling in mathematics and other areas.

- Students in *Making Rectangles* use concrete, pictorial, and symbolic models to represent their problem situation. Students in *Ice Cones* use concrete, symbolic, and graphical models to represent the problem and develop a mathematical model to describe the height of the desired cone. Students in *Building Parabolas* use symbolic and graphical models to represent the problem situation.

14*. Persevere in developing alternative problem-solving strategies if initially selected approaches do not work.

- The students who experience initial difficulties in *Ice Cones* are often the ones who try a "guess-and-check" strategy. They decide to try a different approach, using an equation, when they encounter problems. The students in *Building Parabolas* must use alternative approaches to predict the correct equations from the *Green Globs* program. There is considerable frustration for some students and perseverance is required as they make numerous attempts to find the correct function.

15. Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand the mathematical content appropriate to the high-school grades.

- The students in *Making Rectangles* are learning about factoring by solving a problem. The students in *Ice Cones* are exploring how to find a maximum in a problem involving both algebra and geometry. In *Building Parabolas*, the students are using a question about parabolas to learn about how the coefficients of the equation affect the graph.

16. Recognize, formulate, and solve problems arising from mathematical situations, everyday experiences, applications to other disciplines, and career applications.

- The problem in *Ice Cones* arises from career applications. The problems in *Building Parabolas* and *Making Rectangles* arise from mathematics itself.

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* Reference is made here to Indicators 4, 5, 6, 7, 8, 12, and 14 which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

17. **Monitor their own progress toward problem solutions.**

- The students in *Making Rectangles* keep a record of the different combinations of tiles they try, whether or not the combination could be used to form a rectangle, and the dimensions of the rectangle; they begin to notice patterns. The students in *Ice Cones* try a different strategy when they find that they are not making progress. In *Building Parabolas* the very use of the computer game *Green Globs*, helps students monitor their own progress towards a solution.

18. **Explore the validity and efficiency of various problem-posing and problem-solving strategies, and develop alternative strategies and generalizations as needed.**

- The students in all three vignettes share the strategies they use to solve their problems. In *Tiling a Floor,* the students select a variety of materials, including a computer program, to solve the problem. In *Sharing Cookies*, students use different methods to solve the same problem.

> Standard 2. All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

1*. **Discuss, listen, represent, read, and write as vital activities in their learning and use of mathematics.**

- Students in all three vignettes use a variety of communication activities: listening, discussing in small and large groups, representing in algebraic and graphical or concrete contexts, and reading and writing about their solutions.

2*. **Identify and explain key mathematical concepts, and model situations using oral, written, concrete, pictorial, and graphical methods.**

- In *Ice Cones*, the teacher asks the students to list all of the mathematical concepts they use in solving the problem in their journals. In all three vignettes, students model situations using different methods: oral, written, concrete, and graphical.

3*. **Represent and communicate mathematical ideas through use of learning tools such as calculators, computers, and manipulatives.**

- Some students in *Ice Cones* use graphing calculators; others use manipulatives (paper circles). The students in *Building Parabolas* use a computer program. Students in *Making Rectangles* use manipulatives.

* Reference is made here to Indicators 1, 2, 3, 4, 5, 6, 7, and 8, which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

4*. Engage in mathematical brainstorming and discussions by asking questions, making conjectures, and suggesting strategies for solving problems.

- In all three vignettes, the students engage in brainstorming before solving the problem.

5*. Explain their own mathematical work to others, and justify their reasoning and conclusions.

- In all three vignettes, the students explain their work to the others in the class.

6*. Identify and explain key mathematical concepts and model situations using geometric and algebraic methods.

- In *Making Rectangles*, most of the students draw pictures of the various tiles they can make and write about the tiles using algebraic notation. For example, they write that they can use one red x^2 tile, two orange x tiles, and one yellow unit tile to make a larger square tile and record this as $x^2 + 2x + 1 = (x + 1)(x + 1)$. The students in *Ice Cones* model the situation using both geometric (paper circles) and algebraic (graphing calculator) methods. In *Building Parabolas*, the students model the situation both algebraically with equations and geometrically with graphs.

7*. Use mathematical language and symbols to represent problem situations, and recognize the economy and power of mathematical symbolism and its role in the development of mathematics.

- The students in all three vignettes use mathematical language and symbols to represent the problem situation.

8*. Analyze, evaluate, and explain mathematical arguments and conclusions presented by others.

- The students in all three vignettes present their results to the class, evaluating and explaining their conclusions.

9. Formulate questions, conjectures, and generalizations about data, information, and problem situations.

- In *Making Rectangles*, the students use their findings to make predictions about combinations of tiles. In *Ice Cones*, the students make conjectures and generalizations, including determining whether there is a general relationship between the radius and the maximum volume for each of five different cones. The students in *Building Parabolas* make conjectures and generalizations about the effects of the coefficients in the general form of the equation on the graph.

10. Reflect on and clarify their thinking so as to present convincing arguments for their conclusions.

- Students in all three vignettes are asked to reflect on and clarify their thinking by sharing with others in small groups and by summarizing their findings in writing.

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* Reference is made here to Indicators 1, 2, 3, 4, 5, 6, 7, and 8, which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

Standard 3. All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas and the roles that mathematics and mathematical modeling play in other disciplines and in life.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

1*. View mathematics as an integrated whole rather than as a series of disconnected topics and rules.

- In *Making Rectangles*, factoring binomials is related to area and multiplication. In *Ice Cones*, the students are drawing from concepts in algebra, geometry, and calculus. In *Building Parabolas*, the students have already investigated different situations that can be modeled by parabolas. At the end, they also relate their work to previous work with geometric transformations.

2*. Relate mathematical procedures to their underlying concepts.

- *Making Rectangles* relates factoring to the underlying concepts of area and multiplication. *Ice Cones* relates finding a maximum to the underlying concepts of volume and solving equations. *Building Parabolas* relates the shape of a graph to its equation.

3*. Use models, calculators, and other mathematical tools to demonstrate the connections among various equivalent graphical, concrete, and verbal representations of mathematical concepts.

- Students in *Making Rectangles* use their models to demonstrate the connection between the geometric topic of area and the algebraic topic of factoring. The students in *Ice Cones* use calculators to help demonstrate the connection between the equation and the maximum value for the volume. Students in *Building Parabolas* use computers to demonstrate the connection between the algebraic and graphical representations for parabolas.

8*. Recognize and apply unifying concepts and processes which are woven throughout mathematics.

- Students in *Making Rectangles* and *Building Parabolas* are focusing on multiple representations. Students in *Ice Cones* are applying mathematical modeling to a real-life situation.

9*. Use the process of mathematical modeling in mathematics and other disciplines, and demonstrate understanding of its methodology, strengths, and limitations.

- The *Ice Cones* vignette illustrates the process of mathematical modeling in real life. In *Making Rectangles* and *Building Parabolas*, students use mathematical modeling to discover patterns which ultimately help them to solve the problems.

* Reference is made here to Indicators 1, 2, 3, 8, 9, 10, and 11, which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

10*. Apply mathematics in their daily lives and in career-based contexts.

- *Making Rectangles* and *Ice Cones* involve applying mathematics in career-based contexts.

11*. Recognize situations in other disciplines in which mathematical models may be applicable, and apply appropriate models, mathematical reasoning, and problem solving to those situations.

- All three vignettes focus on modeling within mathematics. The 9-12 Overview describes a number of situations in other disciplines in which mathematical models are applicable.

12. Recognize how mathematics responds to the changing needs of society, through the study of the history of mathematics.

- The problems discussed in the vignettes are not presented in a social or historical context. However, students can also investigate the role of the quadratic functions, discussed in *Building Parabolas*, in ballistics, and can extend their *Ice Cones* discussions to include other examples of packaging.

Standard 4. All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

2*. Draw logical conclusions and make generalizations.

- Students in all three vignettes draw logical conclusions and make generalizations.

5*. Analyze mathematical situations by recognizing and using patterns and relationships.

- Students in all three vignettes look for patterns and relationships in order to make their generalizations.

8*. Follow and construct logical arguments, and judge their validity.

- In *Making Rectangles*, the students discuss how they will know that their work is correct. They judge the validity of each others' arguments in the subsequent discussion. In *Ice Cones*, the students worked in groups and explained their results and any discrepancies they encountered. The class as a whole discussed the accuracy of solutions. In *Building Parabolas*, also in a class discussion, students explained their hypotheses, how they verified them, and whether they had to modify them based on their experiences.

9*. Recognize and use deductive and inductive reasoning in all areas of mathematics.

- All three vignettes deal primarily with inductive reasoning. However, building on the

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*Reference is made here to Indicators 2, 5, 8, 9, 10, and 11, which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

activities in these vignettes, students can use deductive reasoning to show the effect of increasing k by 3 in *Building Parabolas* or to prove that certain collections of titles can or cannot form rectangles in *Making Rectangles*.

10*. Utilize mathematical reasoning skills in other disciplines and in their lives.

- *Making Rectangles* and *Ice Cones* illustrate the use of mathematics in daily life.

11*. Use reasoning rather than relying on an answer-key to check the correctness of solutions to problems.

- None of the students in any of the vignettes checks answers with an answer key; they all report their answers to the class and explain how they got them.

12. Make conjectures based on observation and information, and test mathematical conjectures, arguments, and proofs.

- The students in *Making Rectangles* make some initial conjectures about which tiles they can make and then test them. The students in *Ice Cones* make and test conjectures in their homework as they begin to generalize their results. The students in *Building Parabolas* make conjectures about the effects of each constant and then test these using the computer.

13. Formulate counter-examples to disprove an argument.

- This indicator is not explicitly addressed in these vignettes. However, in the discussion in any of the vignettes, it is possible that some of the analysis of solutions will lead students to present counterexamples to disprove another student's argument.

* Reference is made here to Indicators 2, 5, 8, 9, 10, and 11, which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

STANDARD 5 — TOOLS AND TECHNOLOGY

K-12 Overview

All students will regularly and routinely use calculators, computers, manipulatives, and other mathematical tools to enhance mathematical thinking, understanding, and power.

Descriptive Statement

Calculators, computers, manipulatives, and other mathematical tools need to be used by students in both instructional and assessment activities. These tools should be used, not to replace mental math and paper-and-pencil computational skills, but to enhance understanding of mathematics and the power to use mathematics. Historically, people have developed and used manipulatives (such as fingers, base ten blocks, geoboards, and algebra tiles) and mathematical devices (such as protractors, coordinate systems, and calculators) to help them understand and develop mathematics. Students should explore both new and familiar concepts with calculators and computers, but should also become proficient in using technology as it is used by adults, that is, for assistance in solving real-world problems.

Meaning and Importance

Both mathematics and the way we do mathematics have changed dramatically in the last few decades. The presence of technology has made more mathematics accessible to us, has allowed us to solve mathematical problems never before solved, and has brought a new and extraordinarily higher level of proficiency with mathematical operations to all members of our society. At the same time, our resulting increased need to truly understand that mathematics has brought on a renewed interest in developing materials and approaches that provide concrete models and that can engage learners. Both of these new directions combine in this one standard which calls for the appropriate and effective use of tools and technology.

Manipulatives are concrete materials that are used for "modeling" or representing mathematical operations or concepts. In much the same way that children make and use models of race cars or the human skeleton so that they can study and learn about them, students can make and learn from models of two-digit numbers or multiplication. The difference between the two situations is that while the car and skeleton models are smaller versions of actual concrete things, the number models or multiplication models are concrete models of abstract concepts and operations.

When students use bundles of sticks and single sticks to represent tens and ones, or algebra tiles to represent polynomials, they are using manipulatives to "model" mathematical ideas. Technically, even when young children count on their fingers, they are concretely modeling numbers.

The mechanism by which concrete modeling aids children in constructing mathematical knowledge is still not completely understood. There is little doubt, however, that it does. There is a good deal of research which shows that the optimal presentation sequence for new mathematical content is *concrete-pictorial-abstract*. Activities with concrete materials should precede those which show pictured relationships and those should, in turn, precede formal work with symbols. Ultimately, students need to reach that final level of symbolic proficiency with many of the mathematical skills they master, but the meanings of those symbols must be firmly rooted in experiences with real objects. Otherwise, their performance of the symbolic operations will simply be rote repetitions of meaningless memorized procedures.

Calculators and **computers** provide still other benefits for students. In the 1994 *Position Statement on The Use of Technology in the Learning and Teaching of Mathematics*, the National Council of Teachers of Mathematics states:

> *Students are to learn how to use technology as a tool for processing information, visualizing and solving problems, exploring and testing conjectures, accessing data, and verifying their solutions. . . . In a mathematics setting, technology must be an instructional tool that is integrated into daily teaching practices, including the assessment of what students know and are able to do.*

The availability of technology requires that we re-evaluate our mathematics curricula. What we teach and how we teach it are now inextricably linked to these new tools. Their presence makes some traditional mathematics topics obsolete — we certainly do not still teach the square root extraction algorithm, but what about hours and hours of paper-and-pencil practice with the long division algorithm? How much is necessary? How much is adequate? How much will our students need that skill when they become adults? The presence of the technology also makes some traditional topics more important than ever — efficient calculator use requires a high level of estimation, place value, and mental math skills so that calculations can be quickly checked for reasonableness and accuracy. And the technology allows some topics to be dealt with that were never accessible to students previously — graphing calculators allow students to instantly see the graphs of complex functions that would have taken a whole class period to graph by hand, and computer-based geometry construction tools allow students to experiment with animation to see the effects of transformations of figures in three-dimensional space.

Stephen Willoughby, a former president of the National Council of Teachers of Mathematics, offers this analogy to those who might be reluctant to change the traditional curriculum:

> *When automobiles first appeared, there were undoubtedly many people who kept a spare horse in the garage lest the automobile fail, but very few of us still do today. Calculators, with and without batteries, have become so inexpensive and reliable that it is more efficient to keep an extra calculator handy than it is to learn to do well everything a calculator does better. Most of us no longer find the ability to shoe a horse or cinch a saddle to be essential skills. Is it not reasonable that in the near future we may feel the same way about multi-digit long division? (Willoughby, 1990, p. 62)*

The New Jersey State Board of Education, recognizing the need for students to develop appropriate and integrated technology skills, requires the use of calculators in the state's mathematics assessment program at the eighth and eleventh grade levels. To be adequately prepared to use calculators on those assessments, it is critical that the students have ample opportunity to use them as a regular and natural part of their mathematics classes and other testing.

We are only just beginning to explore the ways in which technology can be helpful to mathematics learners, but already there are tremendous opportunities at all levels. Two additional tools that have not yet been mentioned in this Overview possibly provide the most futuristic vision of what mathematics classrooms might soon look like. Calculator-Based Laboratories (CBLs) allow measurement probes to be connected to hand-held calculators to collect and analyze scientific data such as light, distance, voltage, temperature, and so on. This direct observation, collection, and display of data allows the student to focus more on the hypothesis, interpretation, and analysis phases of experiments. And the World-Wide Web, or graphical Internet, holds untold amounts of data and information. From geographic and census data, to current information about almost any mathematical or scientific subject, to rich sources of mathematical problems for K-12 students, the Internet will greatly expand the research capability of both teachers and students.

Strategies and teaching approaches which utilize technology have been shown to improve student attitudes, problem solving ability, ability to visualize mathematics, and overall performance. Here, possibly more than anywhere else in this *Framework*, we need to be open to new ideas and receptive to new approaches.

K-12 Development and Emphases

Many specific suggestions for appropriate uses of **calculators, computers,** and **manipulatives** are given in the following pages and, indeed, throughout this *Framework*. The point to be made here, though, is that the frequent, well-integrated use of these tools at all levels is essential. Young children find the use of concrete materials to model problem situations very natural. Indeed they find such modeling more natural than the formal work they do with number sentences and equations. Older students will realize that the adults around them use calculators and computers all the time to solve mathematical problems and will be prepared to do the same. Perhaps more challenging, though, is the task of getting the "reverse" to happen as well, so that technology is also used with *young children*, and the *older students'* learning is enhanced through the use of *concrete models*. Such opportunities do exist, however, and new approaches and tools are being created all the time.

IN SUMMARY, mathematical tools play an ever-more important role in today's mathematics. Students who will be expected to be knowledgeable users of such tools when they leave school must see those tools as a regular and routine part of "doing mathematics" in school.

References

National Council of Teachers of Mathematics. *Position Statement on The Use of Technology in the Learning and Teaching of Mathematics*. Reston, VA, 1994.

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Standard 5 — Tools and Technology — Grades K-2

Overview

This standard addresses the use of calculators, computers and manipulatives in the teaching and learning of mathematics. These tools of mathematics can and should play a vital role in the development of mathematical thought in students of all ages.

In the primary grades, **manipulatives** are the most natural of the three types of tools to use. Primary grade teachers have traditionally used many manipulative materials in their teaching of mathematics because they correctly perceived them to be of great value for young children. Typically, concrete materials are used to model mathematical concepts such as number or shape when those concepts are first introduced to the students.

Young children counting with lima beans, colored chips, linking cubes, smooth stones, or their fingers is a familiar sight in many New Jersey classrooms as they begin to master early counting skills and are introduced to addition and subtraction concepts. More sophisticated models should then be used, though, to begin to explore more sophisticated number concepts. Colored rods in graduated lengths give students a different sense of number than a set of discrete objects. Students should be able to see both a yellow rod **and** five colored chips as representative of the number five — the first being more of a measure model and the second a count model. Ice cream sticks and base ten blocks as well as chip trading activities help students begin to understand the very abstract concepts involved with place value and number base.

Attribute blocks, blocks with different shapes and colors, help students begin to classify and categorize objects and recognize their specific characteristics. Pattern blocks allow them to make patterns and geometric designs as they become familiar with the geometric properties of the shapes themselves. Geoboards allow students to explore the great variety of shapes that can be made and also to deal with issues of properties, attribute, and classification.

A great variety of different materials should be used to explore measurement. Paper clips, shoes, centimeter and decimeter rods, paper cutouts of handspans, and building blocks can all be used as non-standard units of length (even though some of them are really standard). Students place them down one after another to see how many paper clips long the desk is or how many handspans wide the doorway is. The transition can then be made to more standard measures and, following that, to rulers.

This list is, of course, not intended to be exhaustive. Many more suggestions for materials to use and ways to use them are given in the other sections of this *Framework*. The message in this section is a very simple one — concrete materials help children to construct mathematics that is meaningful to them.

Calculators have not been used traditionally in primary classrooms, but there are several appropriate uses for them. It is never too early for students to be introduced to the tool that most of the adults around them use whenever they deal with mathematics. In fact, many students now come to kindergarten having already played with a calculator at home or somewhere else. To ignore calculators completely at this level is to send the harmful message that the mathematics being done at school is different from the mathematics being done at home or at the grocery store.

The use of calculators at this level does not imply that students don't need to develop the arithmetic skills traditionally introduced at the primary level. They certainly do need to develop these skills. This Standard does not suggest that all traditional learning be replaced by calculator use; rather, it calls for the appropriate and effective use of calculators.

One of the most effective uses of the calculator with young children is the use of the constant feature of most calculators to count, forward or backward, or to *skip count*, forward or backward, by twos or threes or other numbers. This process allows children to anticipate what number will come next and then get confirmation of their guess when they see it appear in the display. Students can also greatly enhance their estimation ability through calculator use. *Range-finding games* ask students, for instance, to add a number to 34 that will give them an answer between 80 and 90. After the estimate is made, it is punched into the calculator to see whether or not it did the job. Calculators will prompt young students to be curious about mathematical topics that are not typically taught at their level. For example, when counting back by threes by entering 15 - 3 = = = . . . into the calculator, after the expected sequence of 12, 9, 6, 3, 0, the child will see -3, -6, -9, . . . A curious child will begin to ask questions about what those numbers are, but will also begin to develop an intuitive notion about negative numbers.

Computers are a valuable tool for primary children. As more and more computers find their way into primary classrooms, the software available for them will dramatically improve; however, there are already many good programs that can be used with kindergartners and first and second graders. *MathKeys* links on-screen manipulative materials to standard symbolic representations and to a writing tool for children to use. A number of different counting programs match objects on the screen to a standard symbolic representation of the number and the number is said aloud so that a young student can count along with the program. Many other new programs focus on money skills and help children recognize different coins and determine the values of sets of coins through simulated purchases.

Standard 5 — Tools and Technology — Grades K-2

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Select and use calculators, software, manipulatives, and other tools based on their utility and limitations and on the problem situation.**

   - Students participate in races to complete a set of computation problems between some students who use calculators and others who use mental math. They try to determine what makes the calculator a useful tool in some circumstances (large numbers, harder operations) and not terribly useful in others (basic facts, easy numbers).
   - Students are regularly asked to make their own decisions about what is the right type of linear measuring device for a particular situation: mental estimation, colored rods, ruler, yard or meter stick, or tape measure. Different decisions are made in different circumstances: Estimation is fine when you are deciding whether you will fit through a small doorway, but accurate ruler measurement is important if you are cutting out a frame for a picture.
   - In problem solving situations, students are regularly provided with calculators, manipulatives, and other tools so that they may choose for themselves what will be useful to help solve the problem.

2. **Use physical objects and manipulatives to model problem situations, and to develop and explain mathematical concepts involving number, space, and data.**

   - Students use popsicle sticks to model multi-digit base-ten numbers and then use them to further model operations with the numbers.
   - Students use pipe cleaners and straws to make models of two-dimensional geometric shapes. They then compare, contrast, and sort all the shapes using whatever criteria they think are important, including number of corners, straight or curvy sides, number of sides, and so on.
   - Students work through the *Shapetown* lesson that is described in the First Four Standards of this *Framework*. Students in kindergarten are challenged to build towns with attribute blocks and loops based on a rule or pattern they make up.
   - Kindergarten students each use a cubic inch block to represent himself or herself in a bar graph that describes the favorite flavors of ice cream of all the students in the class. On a table in the front of the room, the teacher has placed mats that say *Vanilla, Chocolate,* and *Strawberry*. One by one, the students walk past the table, dropping their blocks on one of the piles that build up on the mats. When this concrete "bar graph" is complete, the children ask questions that can be answered with the data displayed: *What's the most*

favorite flavor in the class? What's the least favorite? Are there more people who like vanilla than chocolate?

3. **Use a variety of technologies to discover number patterns, demonstrate number sense, and visualize geometric objects and concepts.**

   - Students use the constant function on a calculator to count by ones, twos, tens, fourteens, and other numbers, both forward and backward. As they do so, they try to keep up with the calculator by saying the numbers orally as they come up in the display, and even trying to say them before they come up.
   - Students use a beginner's Logo to explore movement in two-dimensional space. They move the turtle on the computer screen forward and backward with simple commands and also turn the turtle through predetermined angles to the right and to the left with other commands. The turtle leaves a trail of where it's been on the screen so that its movements actually create a drawing of a figure. The students try to have the turtle draw a square, a different rectangle, and a triangle before progressing to harder tasks.
   - Students use a geoboard to make shapes that are composed of unit squares. One challenge they are given is to find as many shapes as possible that are made up of 10 unit squares.

4. **Use a variety of tools to measure mathematical and physical objects in the world around them.**

   - Young students develop meaning for rulers by first measuring with individual paper clips, then a paper clip chain, then taping the clip chain to a paper strip, then marking and numbering the ends of the clips on the strip, and last, removing the clip chain from the paper strip leaving just the marks and the numbers. This leaves the students with *paper clip rulers* with which they can measure the lengths of a variety of objects. The unit of measurement is, of course, a paper clip.
   - Students use a balance scale to determine the weights of a variety of classroom objects in terms of units that are other classroom objects; for example, *How many pennies does a math book weigh? How many paper clips does a pencil weigh?*
   - Students work through the *Will a Dinosaur Fit?* lesson that is described in the First Four Standards of this *Framework*. Second grade students measure the size of their classroom and other places in a variety of ways to determine whether dinosaurs they are studying would fit into them.
   - As part of the morning calendar routine, second graders check each of two thermometers — one Fahrenheit and one Celsius — and make daily recordings of the outside temperature. They record the temperatures in a chart and look for interesting patterns. They notice that, as the school year progresses and the temperatures change, whenever one of the temperatures goes up or down, so does the other.
   - Students regularly use both analog and digital stopwatches to practice timing events that are usually measured in seconds such as: the amount of time it takes a classmate to say the alphabet, how long a classmate goes without blinking, or how long the morning announcements take.

5. **Use technology to gather, analyze, and display mathematical data and information.**

- Students take a survey to determine every child's birth month and then use the *Graph Club* or *Primary Graphing and Probability Workshop* software to display the resulting data in graphs.
- Using a World Wide Web page that reports meteorological data (possibly http://www.rainorshine.com/weather/index/sites/njo/), students find the predicted high temperatures for a variety of cities in different regions around the country, write those numbers on a map of the United States, and then look for patterns and trends in different regions.
- Students use *Table Top* software to make a Venn diagram to show which of them have brothers, which have sisters and which have both (the intersection of the two sets). Students who have no siblings are shown outside the rings. Other attributes of the children are also used to make Venn diagrams.

References

Software

Graph Club. Tom Snyder Productions.

Logo. Many versions of Logo are commercially available.

Primary Graphing and Probability Workshop. Scott Foresman.

TableTop. TERC.

MathKeys. Minnesota Educational Computing Consortium (MECC).

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 5 — Tools and Technology — Grades 3-4

Overview

This standard addresses the use of calculators, computers and manipulatives in the teaching and learning of mathematics. These tools of mathematics can and should play a vital role in the development of mathematical thought in students of all ages.

In grades 3 and 4, **manipulatives** have traditionally not been used as much as they have been in the primary grades. It is fairly common for teachers at this level to think that once initial notions of number and shape have been established with concrete materials in the lower grades, the materials are no longer necessary and a more symbolic approach is preferable. Research shows, however, that concrete materials and the modeling of mathematical operations and concepts is just as useful at these grade levels as it is for younger students. The content being modeled is, of course, different and so the models are different — but no less important.

Third- and fourth-graders can use square tiles to model one-digit multiplication arrays in a manner that makes the operation very meaningful for them, and later use base-ten blocks to model two-digit multiplication arrays. The added advantage to this kind of a model is the degree to which students who have used it can visualize what's happening with the factors in the problem and so can develop much better estimation and mental math skills than students who have simply learned the standard paper-and-pencil algorithms. The relationship between multi-digit multiplication and division is also clearly shown by such models.

Geometry models, both two- and three-dimensional, are an important part of learning about geometry and development of spatial sense in students of this age. Students should use geoboards to explore area and perimeter and to begin to develop procedures for finding the areas of irregular shapes. They can also use construction materials like pipe cleaners and straws to make three-dimensional geometric shapes like cubes and pyramids so that they can study them directly. Such models make it much easier to determine the number of faces or edges in a figure than two-dimensional drawings.

Third- and fourth-graders should also be in the habit of using a variety of materials to help them model problem situations in other areas of the mathematics curriculum. They might use different colored unifix cubes to represent all of the different double-decker ice cream cones that can be made with three different flavors of ice cream. They should be able to use a variety of measurement tools to measure and record the data in a science experiment. They might use coin tosses or dice throws to simulate real-world events that have a one-in-two chance or a one-in-six chance of happening.

This list is, of course, not intended to be exhaustive. Many more suggestions for materials to use and ways to use them are given in the other sections of this *Framework*. The message in this section is a very simple one — concrete materials help children construct mathematics that is meaningful to them.

There are several appropriate uses for **calculators** at these grade levels. It is never too early for students to be introduced to the tool that most of the adults around them will use whenever they deal with mathematics.

The use of calculators at this level does not imply that students don't need to develop arithmetic skills traditionally introduced at the primary level. They certainly do need to develop these skills. This Standard does not suggest that all traditional learning be replaced by calculator use; rather, it calls for the appropriate and effective use of calculators.

One of the most effective uses of the calculator with young children which can be continued in grade three is the use of the constant feature of most calculators to count, forward or backward, or to skip count. This process allows children to anticipate what number will come next and then get confirmation of their guess when they see it come up in the display. Students can greatly enhance their estimation ability through calculator use. *Range-finding games* ask students, for instance, to add a number to 342 that will give them an answer between 800 and 830. After the estimate is made, it is punched into the calculator to see whether or not it did the job.

Calculators will also prompt students to be curious about mathematical topics to which they are about to be introduced. For example, while routinely using calculators in problem solving activities, some students may notice that whenever they add, subtract, or multiply two whole numbers, they get a whole number for an answer. Sometimes that happens for division, too, but sometimes when they divide they get an answer like 3.5. *What does that mean?* These kinds of questions offer a great opportunity for some further exploration and investigation; for example, *Which problems give you answers like those? What happens when you solve those problems using pencil-and-paper?*

Computers are a valuable tool for students in third and fourth grade. As more and more computers find their way into these classrooms, the software available for them will dramatically improve; however, there are already many good programs that can be used with students of this age. *MathKeys* links on-screen manipulative materials to standard symbolic representations and to a writing tool for children. Logo can be used by students to explore computer programming and geometry concepts at the same time. *Tesselmania!* and other programs offer an opportunity to play with geometric transformations on the screen and produce striking designs. *The King's Rule* is a program that asks students to determine the rules that distinguish one set of numbers from another, fostering creative and inductive thinking. The World Wide Web can be an exciting and eye-opening tool for third-and fourth-graders as they retrieve and share information. Specifically, in these grades, they might look for state populations, meteorological data, and updates on current events.

Standard 5—Tools and Technology—Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Select and use calculators, software, manipulatives, and other tools based on their utility and limitations and on the problem situation.**

    - Students participate in races between some students who use calculators and others who use mental math, each working to complete a set of computation problems involving newly learned arithmetic skills. They try to determine what makes the calculator a useful tool in some circumstances (large numbers, harder operations) and not terribly useful in others (basic facts, easy numbers).
    - Students work through the *Tiling a Floor* lesson that is described in the First Four Standards of this *Framework*. Third grade students test various shapes made of a variety of materials to determine which can be used to tessellate an area.
    - Students choose to use a computer spreadsheet on their classroom computer as a neat way to organize tables and charts, but they also use a full-function word processor when there is a good deal of text involved or when using different fonts and text formatting.
    - Students use base ten blocks rather than popsicle sticks when performing operations with large numbers because they can create models more efficiently and more quickly with them.

2. **Use physical objects and manipulatives to model problem situations, and to develop and explain mathematical concepts involving number, space, and data.**

    - Students use base ten blocks to demonstrate the operations of multiplication and division with multi-digit numbers using both repeated subtraction and partition methods.
    - Students work through the *Sharing Cookies* lesson that is described in the First Four Standards of this *Framework*. Fourth grade students use manipulatives to determine how to divide 8 cookies equally among 5 people.
    - Students use a variety of devices such as dice, coin flips, spinners, and decks of cards for generating random numbers and understand the essential equivalence of these devices.
    - Students use pipe cleaners and straws to build and study three-dimensional objects, finding it easier to discuss things like numbers of edges, faces, and vertices and the relationships among them if they have a physical model with which to work.
    - Students use geoboards to solve Farmer Brown's problem. She has 16 meters of fencing and wants to fence in the largest rectangular area possible for her dog to romp around in.

- Students use colored rods or pattern blocks to develop early notions of fractions, using different rods or blocks as the unit and discovering by trial-and-error the resulting fractional values of all of the other pieces.

3. Use a variety of technologies to discover number patterns, demonstrate number sense, and visualize geometric objects and concepts.

- Students play the game *target practice* in the *New Jersey Calculator Handbook*. In it, one student enters a number into a calculator to be used as an operand, enters an operation (addition, subtraction, multiplication, or division) into the calculator by pressing the appropriate sign, and then specifies a "target range" for the answer. For instance, the student may enter: *82* x and specify the range as *2000-3000*. A second student must then enter a second operand into the calculator and press the equals key. If the answer is within the specified target range, the shot was a bull's eye.
- Students play *The Biggest Product*, also from *The New Jersey Calculator Handbook*. In it, four cards are dealt face up from a shuffled deck of cards containing only the cards from ace to nine. The students who are playing then use their calculators to try to compose the multiplication problem that uses only the digits on the cards, each only once, that has the largest possible product. After several rounds, the students begin to notice a pattern in their answers and become much more efficient at finding the correct problems.
- Students begin to use Logo to create geometric figures on the computer screen. They write routines that have the turtle's path describe a square, a rectangle, a triangle, and other standard polygons. As a challenge, they write a routine to have the turtle draw a simple house with windows and a roof.
- Students solve the problems posed in *Logical Journey of the Zoombinis* by using logic and classification and categorization skills. In it, they create Zoombinis, little creatures that have specific characteristics that allow them to accomplish specified tasks.
- After reading *Counting on Frank* by Rod Clement, students practice their estimation skills by using software of the same title.

4. Use a variety of tools to measure mathematical and physical objects in the world around them.

- Students regularly use both analog and digital stopwatches to practice timing events that happen in short time periods such as: the amount of time it takes a classmate to recite the Pledge of Allegiance or count to 60, how long a classmate takes to run a 50 meter dash, or how long the morning announcements take. They begin to record the elapsed time in decimals that include tenths or hundredths of a second.
- Students first estimate and then use a metric trundle wheel to measure long distances such as the distance from the cafeteria doors to the sandbox, the distance from the classroom door to the principal's office door, or the distance all the way around the school on the sidewalk.
- Students read *Counting on Frank* by Rod Clement and repeat some of the estimates made by the boy in the book. *How many peas would it take to fill up the room? How long a line can a pen write?* They make up their own silly things to estimate, and devise ways to make the appropriate measures and estimates.

5. **Use technology to gather, analyze, and display mathematical data and information.**

- Students use the New Jersey State homepage http://www.state.nj.us on the World Wide Web to gather data about the latest reported populations for each of the municipalities in their county. They then enter the collected data into a simple spreadsheet and use its graphing function to produce a bar graph of all of the populations of the towns and cities. They highlight their own town to show where it stands in relationship to the others.
- Students use the *Graphing and Probability Workshop* or similar software to generate large amounts of random data. This software simulates a variety of probability experiments including up to 300 coin tosses, spinner spins, and dice rolls. Discussions focus on whether the simulated outcomes were as expected or were different from what was expected.
- There is always math help available at the Dr. Math World Wide Web site (dr.math@forum.swarthmore.edu). In Dr. Math's words, "Tell us what you know about your problem, and where you're stuck and think we might be able to help you. Dr. Math will reply to you via e-mail, so please be sure to send us the right address. K-12 questions usually include what people learn in the U.S. from the time they're five years old through when they're about eighteen."

References

Association of Mathematics Teachers of New Jersey. *The New Jersey Calculator Handbook.* 1993.

Clement, Rod. *Counting on Frank*. Milwaukee, WI: Gareth Stevens Children's Books, 1991.

Software

Counting on Frank. EA Kids Software.

Graphing and Probability Workshop. Scott Foresman.

Logical Journey of the Zoombinis. Broderbund.

Logo. Many versions of Logo are commercially available.

MathKeys. Minnesota Educational Computing Consortium (MECC).

Tesselmania! Minnesota Educational Computing Consortium (MECC).

The King's Rule. Sunburst Communications.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards.*

Standard 5 — Tools and Technology — Grades 5-6

Overview

This standard addresses the use of calculators, computers and manipulatives in the teaching and learning of mathematics. These tools of mathematics can and should play a vital role in the development of mathematical thought in students of all ages.

Traditionally, by grades 5 and 6, teachers are devoting relatively little time to student modeling with **manipulatives** since they have begun to concentrate on symbolic and abstract approaches to content. It is fairly common for teachers at this level to think that once initial notions of number and shape have been established with concrete materials in the lower grades, such materials are no longer necessary and a more symbolic approach is preferable. Research shows, however, that concrete materials and the modeling of mathematical operations and concepts is just as useful in these grades as it is for younger students. The content being modeled is, of course, different and so the models are different — but they are no less important.

Fifth- and sixth-graders can use a great variety of materials, including colored rods, base-ten blocks, pattern blocks, fraction strips and circles, and tangrams, to develop very rich notions of rational numbers and the operations associated with them. Initial fraction notions are well-modeled with colored rods. Students use different length rods to represent different units and then decide on the fractional or mixed number value of the other rods. Base-ten blocks, with the values 1, 0.1, 0.01, and 0.001 assigned to the different sizes, model operations with decimals just as well as the values normally associated with base-ten blocks model whole number computation. Pattern blocks and tangram pieces are comfortable and familiar tools with which to begin to explore notions of ratio and proportion.

Geometry models, both two- and three-dimensional, are an important part of learning about geometry and development of spatial sense in students of this age. Students can use geoboards to develop procedures for finding the areas of polygons or irregular shapes. They can also use construction materials like pipe cleaners and straws or cut-out cardboard faces to make complex three-dimensional geometric figures which can be studied directly. It is much easier to determine the number of faces or edges in a figure from such a model than from a two-dimensional drawing of the figure.

Fifth- and sixth-graders should also be in the habit of using a variety of materials to help them model problem situations in other areas of the mathematics curriculum. They might use two-colored counters to represent positive and negative integers in initial explorations of addition and subtraction of signed numbers. They should be able to use a variety of measurement tools to measure and record the data in a science experiment. They might use play money to concretely construct solutions to coin problems or to riddles that ask how much an individual actually profited or lost in some complex business dealing.

This list is, of course, not intended to be exhaustive. Many more suggestions for materials to use and ways to use them are given in the other sections of this *Framework*. The message in this section is a very simple one — concrete materials help children to construct mathematics that is meaningful to them.

There are many appropriate uses for **calculators** in these grade levels. In his article, *Using Calculators in* *the Middle Grades*, in *The New Jersey Calculator Handbook*, David Glatzer suggests that there are three major categories of calculator use in the middle grades:

To explore, develop, and extend concepts — for example, when the students use the square and square root keys to try to understand these functions and their relationship to each other.

As a problem solving tool — for example, to see how increasing each number in a set by 15 increases the mean of the set.

To learn and apply calculator-specific skills — for example, to learn how to use the memory function of a calculator to efficiently solve a multi-step problem.

These three categories provide a good framework for thinking about calculators at these grade levels. For another powerful example of the first category, consider using an ordinary four-function calculator to explore and begin to describe the relationships between common fractions and decimals. Entering 2/3 into the calculator by pressing *2*, then the division key, and then *3* gives a result of 0.6666667. Discussion of this result, attempts to create other similar results, and working out some of the problems by hand lead to discoveries about terminating and non-terminating decimals, repeating decimals, and fraction-decimal equivalence. Such explorations also should be used to highlight the limitations of the calculator, which does not always give the answer 1 when 1/3 is added three times.

Computers are a valuable resource for students in fifth and sixth grade, and the software tools available for them are more like adult tools than those available for younger children. The standard computer productivity tools — word processors, spreadsheets, graphing utilities, and databases — can all be used as powerful tools in problem solving situations, and students should begin to rely on them to help in finding and conveying problem solutions.

In terms of specific mathematics education software, there are many good choices. Logo, of course, can be used effectively by students at these grade levels to explore computer programming and geometry concepts at the same time. It is an ideal tool to learn about one of the critical cumulative progress indicators for Standard 14 (Discrete Mathematics) for grades 5-8: the use of iterative and recursive processes. *Oregon Trail II* is a very popular CD-ROM program that effectively integrates mathematics applications with social studies. *How the West Was One + Three x Four* also uses an old west theme to work on arithmetic operations. *Tesselmania!*, the *Teaching and Learning with Computers* series, *Elastic Lines*, and the *Geometry Workshop* all allow students to make geometric constructions on the computer screen and then transform them in a variety of ways in order to experiment with the effects of the transformations.

Graph Power, *Graphing and Probability Workshop*, *AppleWorks*, *TableTop*, *Graphers*, and *MacStat* are some of the many tools available that include database, spreadsheet, or graphing facilities written for students at this age. Many other valuable pieces of software are available.

The World Wide Web can be an exciting and eye-opening tool for fifth- and sixth-graders as they retrieve and share information. Specifically, in these grades, they might look for demographic data about geographic locations in which they are interested, summaries of the vote totals for different precincts in local elections, and home pages from other schools in this country and abroad.

Standard 5 — Tools and Technology — Grades 5-6

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

1*. Select and use calculators, software, manipulatives, and other tools based on their utility and limitations and on the problem situation.

- In problem solving situations, students are no longer provided with instructions concerning which tool to use, but rather are expected to select the appropriate tool from the array of manipulatives, calculators, computers and other tools that are always available to them.
- Students work through the *Pizza Possibilities* lesson that is described in the First Four Standards of this *Framework*. They use a variety of manipulatives to help them visualize and solve the problem.
- Having used paper-and-pencil to develop an interesting geometric shape which tessellates the plane, the students go to the computer to use *Tesselmania!* to reproduce it and then to tile the computer screen with it. They color the printout from the program to produce a unique piece of artwork which is then posted in a class display.
- Students engage in the four activities of *Target Games: Estimation is Essential!* in *The New Jersey Calculator Handbook*. In these activities, students learn the role that estimation plays in effective calculator use and learn to identify reasonable and unreasonable answers in the calculator display.
- Students explore the rich source of problems at the Math Forum World Wide Web site at Swarthmore College (http://forum.swarthmore.edu).

2*. Use physical objects and manipulatives to model problem situations, and to develop and explain mathematical concepts involving number, space, and data.

- Using standard base ten blocks as a model, where the four blocks, as usual, represent one, ten, one hundred, and one thousand, students demonstrate their understanding of place value on an assessment by writing in their journals a verbal description (accompanied by drawings) of what a ten-thousand block and a one-hundredth block might look like.
- Students use gumdrops and toothpicks to build a variety of polyhedra. Using these models, they try to generalize a relationship among the faces, edges, and vertices that works for all solids. (There is one! It's called Euler's formula after its discoverer and is: $F + V = E + 2$.)

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* Activities are included here for Indicators 1, 2, 3, 4, and 5 which are also listed for grade 4, since the Standards specify that students demonstrate continual progress in these indicators.

- Students read *How Much is a Million* by David Schwartz. The book describes how tall a stack of a million children standing on each other's shoulders would be, how long it would take to count to a million, and so on. The students pick some object of their own and try to determine how big a space would be needed to contain a million of them. Typical objects to inquire about include blades of grass, pennies, and dollar bills.

3*. Use a variety of technologies to discover number patterns, demonstrate number sense, and visualize geometric objects and concepts.

- Students are asked to enter the number 6561 into their calculators and then to keep pressing the square root key to try to discover what it does. After more experimentation with the key, the students are asked to predict what would have to be entered into the calculator's display if they wanted to press the square root key 6 times and wind up with the number 4.
- Students work through the *Two-Toned Towers* lesson that is described in the First Four Standards of this *Framework*. Students use manipulatives to determine how many towers can be built which are 4 cubes tall and use no more than 2 colors, and then discuss the pattern that results when the length of the towers can be 4,5,6, or a larger number of cubes. They also relate their answers to the solution of the *Pizza Possibilities* lesson on the following page in the First Four Standards.
- Students use *Elastic Lines* or another version of an electronic geoboard to construct geometric figures and then transform them through rotations, reflections, and translations. Students create a figure and its transformed image on the screen and challenge each other to describe the specific transformation that created the image.
- Students play a computer golf game where they must hit a ball into a hole. The ball and the hole are both visible on the screen, but at opposite sides. Players specify an angular orientation (where 0° is straight up) and a number of units of length which will describe the path of the ball once it is struck. The object, just like in real golf, is to get the ball in the hole in as few strokes as possible. Good estimation of both angle measure and length are critical to success.

4*. Use a variety of tools to measure mathematical and physical objects in the world around them.

- Students divide into groups to make a scale model of their classroom by accurately measuring critical elements of the room, using a standard proportional relationship to convert the actual measurements to the model's measurements, and then measuring again to cut the modeling material (cardboard, balsa wood, or manila paper) to the correct size. Their model of the room should also contain models of the blackboard, the teacher's desk, some student desks, the shelves in the room, and so on. Each student group is responsible for different elements of the room.
- Students measure the volumes of several rectangular boxes by filling them with cubic inch blocks or cubic centimeter blocks. After some thought and discussion, they devise

* Activities are included here for Indicators 1, 2, 3, 4, and 5 which are also listed for grade 4, since the Standards specify that students demonstrate continual progress in these indicators.

formulas to compute the volumes from direct measurement of the three appropriate dimensions.

- Students use ratios and proportions to determine the heights of objects that are too tall to easily measure directly. Measuring the heights of some known objects and the lengths of the shadows they cast, students determine the heights of the school building, the flagpole, and the tallest tree outside the school by measuring their shadows.

5. Use technology to gather, analyze, and display mathematical data and information.

- Students use the New Jersey State homepage http://www.state.nj.us on the World Wide Web to gather data about the latest reported population for each county in the state and about the area of the counties. They enter the collected data into two adjacent columns in a spreadsheet and configure a third column to calculate the population density for each county (population / area). They highlight their own county in the printout of the spreadsheet to show where it stands in relationship to the others.
- Students use HyperStudio to create the reports they write about biographies of mathematicians, about how mathematics is used in real life, or about solutions to problems they've solved. The software allows them to create true multimedia presentations.
- Students measure various body parts such as height, length of forearm, length of thigh, length of hands, and arm span. They enter the data into a spreadsheet and produce various graphs as well as a statistical analysis of the class. They update their data every month and discuss the change, as it relates both to individuals and to the class.
- Students conduct a survey of the population of the entire school to determine the most popular of all of the choices for school lunches. After gathering the data, they enter it into a spreadsheet and use the program to graph it in a variety of ways — as a bar graph, a circle graph, and a pictograph. They discuss which of the graphs best illustrates their data and publish the one they choose in a report distributed to all of the students in the school.
- There is always math help available at the Dr. Math World Wide Web site (dr.math@forum.swarthmore.edu). In Dr. Math's words, "Tell us what you know about your problem, and where you're stuck and think we might be able to help you. Dr. Math will reply to you via e-mail, so please be sure to send us the right address. K-12 questions usually include what people learn in the U.S. from the time they're five years old through when they're about eighteen."

6. Use a variety of technologies to evaluate and validate problem solutions, and to investigate the properties of functions and their graphs.

- Students solve the problems posed in *Logical Journey of the Zoombinis* by using logic and classification and categorization skills. In it, they create Zoombinis, little creatures that have specific characteristics that allow them to accomplish specified tasks.
- Students use their knowledge of theoretical probability to predict the relative frequency of occurrence of each of the possible sums when rolling a pair of dice. They use simulation software like the *Graphing and Probability Workshop* to simulate the rolling of 300 pairs of dice. They examine the simulated frequencies and judge them to either be consistent or inconsistent with their predictions and reexamine their predictions if necessary.

- Students use the data they gathered earlier concerning the heights of objects and the lengths of the shadows they cast at the same time on a sunny day. They enter the data as ordered pairs (height, shadow) into a simple graphing program and notice that the resulting points all lie on a line. They use the line to predict the heights of objects whose shadows they can measure.
- Students make a pattern using square tiles to build increasingly larger squares (a 1x1, a 2x2, a 3x3, and so on). They count the number of tiles it took to build each successive square and plot the resulting ordered pairs ((1,1), (2,4), (3,9), (4,16), . . .) on an x-y plane. The resulting parabola is a non-linear function which is easy to discuss.

7. **Use computer spreadsheets and graphing programs to organize and display quantitative information and to investigate properties of functions.**
   - Students measure each of a variety of objects in both inches and centimeters. They enter the collected data into a spreadsheet as ordered pairs in two adjacent columns, measurements in inches followed by measurements in centimeters. They have the spreadsheet program graph the ordered pairs on an x-y plane. After they discover that all of the points lie on a line, they draw the line and use it to determine the customary measure of an object whose metric measure they know and vice versa.
   - Students configure a simple spreadsheet to assist them in finding magic squares by automatically computing all of the sums. For example, they reserve a three-by-three array of cells for the magic square somewhere in the middle of the spreadsheet. In the cells that are at the end of the rows, they enter formulas that show the sums of the entries in the cells in each row, and enter similar formulas at the end of each column and diagonal. When proposed entries are placed in the magic square cells, their various sums are instantly provided in the adjacent cells, facilitating adjustment of the entries. The students then use their new tool to solve and create magic square puzzles.

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Software

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Elastic Lines. Sunburst Communications.

Geometry Workshop. Scott Foresman.

Graph Power. Ventura Educational Systems.

Graphers. Sunburst Communications.

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MacStat. Minnesota Educational Computing Consortium (MECC).

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Table Top. TERC.

Teaching and Learning with Computers. International Business Machine, Inc. (IBM).

Tesselmania! Minnesota Educational Computing Consortium (MECC).

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 5 — Tools and Technology — Grades 7-8

Overview

This standard addresses the use of calculators, computers and manipulatives in the teaching and learning of mathematics. These tools of mathematics can and should play a vital role in the development of mathematical thought in students of all ages.

Seventh- and eighth-graders can use a variety of **manipulatives** to enhance their mathematical understanding and problem solving ability. For example, new approaches to the teaching of elementary concepts of algebra incorporate concrete materials at many levels. Two-colored counters are used to represent positive and negative integers as students build a sense of operations with integers. Algebra tiles are used to represent variables and polynomials in operations involving literal expressions. Concrete approaches to equation solving are becoming more and more popular as students deal meaningfully with such mathematical constructs as equivalence, inequality, and balance.

In geometry, students can best understand issues of projection, perspective, and shadow by actually building concrete constructions out of blocks or cubes and viewing them from a variety of directions and in different ways. Slicing clay models of various three-dimensional figures convinces students of the resulting planar shape that is the cross-section. Using pipe cleaners and straws, students can build their own version of a Sierpinski tetrahedron.

Seventh- and eighth-graders should also be in the habit of using a variety of materials to help them model problem situations in other areas of the mathematics curriculum. They might use spinners or dice to simulate a variety of real-life events in a probability experiment. They should be able to use a variety of measurement tools to measure and record the data in a science experiment. They might use counters to represent rabbits as they simulate Fibonacci's famous question about rabbit populations.

This list is, of course, not intended to be exhaustive. Many more suggestions for materials to use and ways to use them are given in the other sections of the *Framework*. The message in this section is a very simple one — concrete materials help students to construct mathematics that is meaningful to them.

There are many appropriate uses for **calculators** in these grade levels as well. In his article, *Using Calculators in the Middle Grades*, in *The New Jersey Calculator Handbook*, David Glatzer suggests that there are three major categories of calculator use in the middle grades:

- To explore, develop, and extend concepts — for example, when the students use the square and square root keys to try to understand these functions and their relationship to each other.
- As a problem solving tool — for example, to see how increasing each number in a set by 15 increases the mean of the set.
- To learn and apply calculator-specific skills — for example, to learn how to use the memory function of a calculator to efficiently solve a multi-step problem.

These three categories provide a good framework for thinking about calculators in the seventh and eighth grade. For another powerful example of the first category, consider the question of compounding interest.

When asked how much a bank account might accumulate after 10 years with an initial balance of \$1000 and a simple annual interest rate of 6 percent, most students would first calculate the interest for the first year, add it to the initial balance to get the new balance, multiply that by 0.06 to get the interest for the second year, add that to the previous year's balance, and so forth. After discussion of that iteration, most seventh- and eighth-graders are able to understand that each year's balance is the product of the previous year's balance times *1.06*, so to find the balance after three years, one could simply use the formula: *\$1000* x *1.06* x *1.06* x *1.06*. After still more discussion, most students will transform this into the standard formula, which is easy to apply with a calculator: *\$1000* x 1.06^n. These concepts develop nicely in a classroom where all of the students have calculators and can do the computations easily and quickly. In a traditional classroom without calculators, the progression takes much longer and the resulting formula is much less believable to students.

Students at this level should also have some experience with graphing calculators. Although these tools will be most useful in the high school curriculum, middle school students should be exploring graphs of linear functions and other simple graphs and should be making use of the statistical capabilities of most graphing calculators. They should also be exploring the use of Calculator Based Laboratories (CBL) which enables them to gather data and display the data graphically in the viewing window.

Computers are also an essential resource for students in seventh and eighth grade, and the software tools available for them are more like adult tools than those available for younger children. The standard computer productivity tools — word processors, spreadsheets, graphing utilities, and databases — can all be used as powerful tools in problem solving situations, and students should begin to rely on them to help in finding and conveying solutions to problems.

In terms of specific mathematics education software, there are also many good choices. Logo, of course, can be used effectively by students at these levels to explore computer programming and geometry concepts at the same time. It is an ideal tool to learn about one of the critical cumulative progress indicators for Standard 14 (Discrete Mathematics) for grades 5-8: the use of iterative and recursive processes. *Oregon Trail II* is a very popular CD-ROM program that effectively integrates mathematics applications with social studies. *How the West Was One + Three x Four* also uses an old west theme to work on arithmetic operations.

A variety of computer golf games allow students to play a competitive game while sharpening their estimation ability with angle and length measure. The *Geometric Supposer and Pre-Supposer* series is one of the most popular geometry construction tools for students of this age. With it, students construct geometric figures on the screen, measure them, transform them, and identify a variety of geometric properties of their creations. Discovery-oriented lessons using these types of software are easy to create and very engaging and useful for students.

Graph Power, the *Graphing and Probability Workshop*, *AppleWorks*, *TableTop*, *Graphers*, and *MacStat* are some of the many tools available that include database, spreadsheet, or graphing facilities written for students at this age. Many other valuable pieces of software are available.

The World Wide Web can be an exciting and eye-opening tool for seventh- and eighth-graders as they retrieve and share information. Specifically, in these grades, they might look for good math problems from the Web bulletin boards, biographical data about famous mathematicians, and census data for local towns.

Standard 5 — Tools and Technology — Grades 7-8

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

1*. Select and use calculators, software, manipulatives, and other tools based on their utility and limitations and on the problem situation.

- In problem solving situations, students are no longer provided with instructions concerning which tool to use, but rather are expected to select the appropriate tool from the array of manipulatives, calculators, computers and other tools that are always available to them.
- Students continue with activities used in previous grades and use problems like those of *Target Games: Estimation is Essential!* in *The New Jersey Calculator Handbook,* in which students learn to identify reasonable and unreasonable answers in the calculator display.
- Students use a variety of materials to demonstrate their understandings of basic mathematical properties and relationships. For instance, they are able to use geoboards, dot paper, and *Geometer's Sketchpad* to demonstrate the Pythagorean Theorem.
- Students work through the *Rod Dogs* lesson that is described in The First Four Standards of this *Framework*. They use Cuisenaire rods to model the increase of the dimensions of an object by various scale factors, but when they realize that there are not enough rods to simulate the situation, they find other models which can be used.

2*. Use physical objects and manipulatives to model problem situations, and to develop and explain mathematical concepts involving number, space, and data.

- Students use two-colored counters to model signed numbers and integer operations. On the red side, a counter represents $+1$, on the white side, -1. As sets of counters are combined or separated to model the operations, students look for patterns in the answers so that they can write rules for completing the operations without counters.
- Students use the same counters and also red and white cubes, representing $+x$ and $-x$, to model and solve equations. By setting up counters and cubes to represent the initial equation and then removing equal sets from both sides, students model the essential elements of solving linear equations and develop the appropriate language with which to discuss those elements. Conversion to symbolic processes comes soon after mastery is achieved with the concrete objects.

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* Activities are included here for Indicators 1, 2, 3, 4, and 5 which are also listed for grade 4, since the Standards specify that students demonstrate continual progress in these indicators.

- Students make three-inch cubes of clay and then experiment to see in how many different ways they can slice the cube with a plane to produce different cross-sections. Drawings of the cross-sections and a description of the cuts that created them are displayed on a poster in the classroom.

3*. Use a variety of technologies to discover number patterns, demonstrate number sense, and visualize geometric objects and concepts.

- Students work on this seemingly simple problem from the New Jersey Department of Education's *Mathematics Instructional Guide*: *A copy machine makes 40 copies per minute. How long will it take to make 20,000 copies? A) 5 hours B) 8 hours 20 minutes C) 8 hours 33 minutes D) 10 hours*. Students immediately decide to use their calculators to solve the problem, but then have an interesting discussion regarding the calculator display of the answer. When 20,000 is divided by 40, the display shows 8.33333. *Which of the answer choices is that? Why?*
- Students use the *Geometer's Sketchpad* to create a triangle on the computer screen and simultaneously place on the screen the measures of each of the angles as well as the sum of the three angles. They notice that the sum of the angles is 180 degrees. They then click and drag one of the vertices around the screen to make a whole variety of other triangles. They notice that even though the measure of each of the angles changes in this process, the sum of 180 degrees never changes, thus intuitively demonstrating the triangle sum theorem.
- Students use videotape of a person walking to model integer multiplication. The videotape shows a person walking forward with a sign that says "forward" and then walking backward with a sign that says "backward." When run forward, the video shows the "forward" walker walking forward ($+ \cdot + = +$). When run backward, the video shows the "forward" walker walking backward ($- \cdot + = -$). The other two possibilities also work out correctly to show all of the forms of integer multiplication.

4*. Use a variety of tools to measure mathematical and physical objects in the world around them.

- Groups of students build toothpick bridges in a competition to see whose bridge can hold the most weight in the center of the span. Each group has the same materials with which to work and the bridges must all span the same distance. In the process of building the bridges, the students conduct a good deal of research into bridge designs and about factors that contribute to structural strength.
- Students work through the *Sketching Similarities* lesson that is described in the First Four Standard of this *Framework*. They use the *Geometer's Sketchpad* to measure the length of sides and the angles of similar figures to discover the geometric relationships between corresponding parts.
- Students explore the relationship between the height of a ramp and the length of time it

* Activities are included here for Indicators 1, 2, 3, 4, and 5 which are also listed for grade 4, since the Standards specify that students demonstrate continual progress in these indicators.

takes a matchbox car to roll down it. The teacher provides stopwatches, long wooden boards, and meter sticks. The students use a spreadsheet program to enter their data relating height and time for several different heights, and use the spreadsheet's integrated graphing program to plot the ordered pairs. They then look for the relationship between the height and the time.

- Students are challenged to answer the question: *In how many ways can you measure a ball?* After the obvious spatial characteristics are named (volume, diameter, circumference, and so on), students get more creative and suggest bounceability, density, fraction of its height that it loses if a five pound book is placed on it, weight, number of times it bounces when dropped from one meter, and so on. When the list is complete, different groups of students select several of the possible characteristics and develop ways to measure them.

5*. Use technology to gather, analyze, and display mathematical data and information.

- Students use a temperature probe connected to a graphing calculator to collect data about the rate of cooling of a cup of boiling water. The data is displayed in chart form and in a graph by the calculator after the experiment is performed.
- Students use *HyperStudio* to create the reports they write about biographies of mathematicians, about how mathematics is used in real life, or about solutions to problems they've solved. The software allows them to create true multimedia presentations.
- Students explore the great wealth of mathematical information available at the University of St. Andrews' History of Mathematics World Wide Web site (http://www.groups.dcs.st-and.ac.uk/~history/).
- Students gather data from their fellow students regarding the number of people in their households. They then enter the data into a graphing calculator and learn how to produce a histogram, showing the number of students with each size household, from the data on the calculator.
- Students decide to resolve the debate that two of them were having about which of their favorite baseball players was the better hitter. They find a great deal of numerical data on their respective teams' World Wide Web homepages regarding the number of at bats, singles, doubles, triples, homeruns, and walks each batter had accumulated in his career. The class decides what weight to attribute to each type of hit and then computes a weighted score for each player to decide who the winner is.
- There is always math help available at the Dr. Math World Wide Web site (dr.math@forum.swarthmore.edu). In Dr. Math's words, "Tell us what you know about your problem, and where you're stuck and think we might be able to help you. Dr. Math will reply to you via e-mail, so please be sure to send us the right address. K-12 questions usually include what people learn in the U.S. from the time they're five years old through when they're about eighteen."

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* Activities are included here for Indicators 1, 2, 3, 4, and 5 which are also listed for grade 4, since the Standards specify that students demonstrate continual progress in these indicators.

6. **Use a variety of technologies to evaluate and validate problem solutions, and to investigate the properties of functions and their graphs.**

   - Students use *Geometer's Sketchpad* to work out a solution to this problem from the New Jersey Department of Education's *Mathematics Instructional Guide*: *Two of the opposite sides of a square are increased by 20% and the other two sides are decreased by 10%. What is the percent of change in the area of the original square to the area of the newly formed rectangle? Explain the process you used to solve the problem.* In their solution attempts, they construct a square on the screen with known sides, and then a rectangle with the sides indicated in the parameters of the problem. The program calculates the areas of the two figures and the students are close to a solution.

   - Students use their calculators to solve this problem from the New Jersey Department of Education's *Mathematics Instructional Guide*:

     A set of test scores in Mrs. Ditkof's class of 20 students is shown below.

     62 77 82 88 73 64 82 85 90 75
     74 81 85 89 96 69 74 98 91 85

     Determine the mean, median, mode, and range for the data.

     Suppose each student completes an extra-credit assignment worth 5 points, which is then added to his/her score. What is the mean of the set of scores now if each student received the extra five points? Explain how you calculated your answer.

   - Students play *Green Globs and Graphing Equations*, a computer game in which they score points for writing the equations of lines that will pass through several green globs splattered on the x-y plane.

7. **Use computer spreadsheets and graphing programs to organize and display quantitative information and to investigate properties of functions.**

   - Students use a simple spreadsheet/graphing program to solve this problem from the New Jersey Department of Education's *Mathematics Instructional Guide*:

     VOTING RESULTS

Class Colors	Number of Votes
red and white	10
green and gold	12
blue and orange	5
black and yellow	9

     Rather than using the tools the problem suggests (protractor, compass, and straight edge), the students enter the data into a spreadsheet and construct a circle graph from the spreadsheet.

   - Students measure the temperatures of a variety of differently heated and cooled liquids in both Fahrenheit and Celsius. They then enter the collected data into a spreadsheet as ordered pairs in two adjacent columns, measurements in Fahrenheit followed by measurements in Celsius. They have the spreadsheet program graph the pairs on an x-y plane. After they discover that all of the points lie on a line, they draw the line and use it to determine the Fahrenheit temperature for a given Celsius temperature and vice versa.

- Students configure a spreadsheet to act as an order-processing form for a stationery store (or some other retail operation). They decide on the five items they'd like to sell, enter the prices they'll charge, and then program all of the surrounding cells to compute the prices for the quantities of items ordered, add the tax, and compute the final charge.

Items	Price	Quantity Ordered	Cost
Pencils	.05		
Pens	.29		
Paper Pads	.59		
Tape	.49		
Scissors	1.39		
		Tax:	
		Total:	

References

Association of Mathematics Teachers of New Jersey. *The New Jersey Calculator Handbook.* 1993.

Glatzer, D. "Using Calculators in the Middle Grades," in *The New Jersey Calculator Handbook.* New Jersey: Association of Mathematics Teachers of New Jersey, 1993.

New Jersey Department of Education. *Mathematics Instructional Guide.* D. Varygiannes, Coord. Trenton, NJ: 1996.

Software

AppleWorks. Apple Computer Corp.

Geometer's Sketchpad. Key Curriculum Press.

Geometric Pre-Supposer. Sunburst Communications.

Geometric Supposer. Sunburst Communications.

Geometry Workshop. Scott Foresman.

Graph Power. Ventura Educational Systems.

Graphers. Sunburst Communications.

Graphing and Probability Workshop. Scott Foresman.

Green Globs and Graphing Equations. Sunburst Communications.

How the West Was One + Three x Four. Sunburst Communications.

HyperStudio. Roger Wagner.

Logo. Many versions of Logo are commercially available.

MacStat. Minnesota Educational Computing Consortium (MECC).

Oregon Trail II. Minnesota Educational Computing Consortium (MECC).

TableTop. TERC.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 5 — Tools and Technology — Grades 9-12

Overview

This standard addresses the use of calculators, computers and manipulatives in the teaching and learning of mathematics. These tools of mathematics can and should play a vital role in the development of mathematical thought in students of all ages.

High school students can use a variety of **manipulatives** to enhance their mathematical understanding and problem solving ability. For example, new approaches to the teaching of concepts of algebra incorporate concrete materials at many levels. Two-colored counters are used to represent positive and negative integers as students build a sense of operations with integers. Algebra tiles are used to represent variables and polynomials in operations involving literal expressions. Concrete approaches to equation solving are becoming more and more popular as students deal meaningfully with such mathematical constructs as equivalence, inequality, and balance.

In geometry, students can create solids of revolution by cutting plane figures out of cardboard, attaching a rubber band along an axis of rotation, winding it up, and then letting it unwind by itself, creating a vision of the solid as it does so. Students can use Miras to do reflective geometry — to find the center of a circle or a perpendicular bisector of a line segment. They might build models of a pyramid with a square base and a cube with the same size base to use in an investigation of the relationship of their volumes. Using pipe cleaners and straws, students can build their own version of a Sierpinski tetrahedron.

High school students should also be in the habit of using a variety of materials to help them model problem situations in other areas of the mathematics curriculum. They might use spinners or dice to simulate a variety of real-life events in a probability experiment. Suppose, for instance, they had statistics about the frequency of occurrence of a particular genetic trait in fruit flies and were interested in the probability that they would see it in a given population. A simulation using dice or a spinner might be a useful approach to the problem. They should be able to use a variety of measurement tools to measure and record the data in a science experiment. They might use counters to represent rabbits as they simulate Fibonacci's famous question about rabbit populations.

This list is, of course, not intended to be exhaustive. Many more suggestions for materials to use and ways to use them are given in the other sections of this *Framework*. The message in this section is a very simple one — concrete materials help students to construct mathematics that is meaningful to them.

There are many appropriate uses for **calculators** in these grade levels as well. In his article, "Technology and Mathematics Education: Trojan Horse or White Knight?" in *The New Jersey Calculator Handbook*, Ken Wolff suggests that the availability of calculators, especially graphing calculators, has presented a unique opportunity for secondary mathematics educators. He asserts: "Tedium can be replaced with excitement and wonder. Memorization and mimicry can be replaced with opportunities to explore and discover." Wolff offers some challenging problems for students to try with their calculators to illustrate the scope of what is now possible in secondary classrooms:

> *What happens when we continually square a number close to the value 1? Try continually taking the square root of a number. Does it matter what number you start with?*

Enter the radian measure of an angle and continually take the sine of the resulting values. What happens? Can you explain why it happens? Replace the sine operator with the tangent and repeat the experiment.

Where does the graph of $y = 2 \sin (3x)$ *intersect the graph of* $y = -4x + 3$*?*

He suggests that these are but a few of the problems that students will gladly try if they have a calculator, but would be very reluctant to do without one. Many secondary teachers have had similar experiences with graphing calculators. As graphing calculators pervade the world of secondary mathematics, what we teach and how we teach it will dramatically change. Nowhere more than in these classrooms will the educational impact of this technology be felt. A sample unit on finding regression lines using graphing calculators can be found in Chapter 17 of this *Framework.*

High school students should be using Calculator Based Laboratories (CBL) in conjunction with their graphing calculators, to generate, analyze, and display data obtained using a variety of probes; discussions of these activities should be coordinated with activities in their science classrooms.

Computers are also an essential resource for students in high school, and the software tools available for them are very much like adult tools. The standard computer productivity tools — word processors, spreadsheets, graphing utilities, and databases — can all be used as powerful tools in problem solving situations, and students should begin to rely on them to help in finding and conveying problem solutions. In terms of specific mathematics education software, there are also many good choices. The *Geometric Supposer and Pre-Supposer* series, *Geometer's Sketchpad*, and *Cabri Geometry* are all popular geometry construction tools for students of this age. With them, students construct geometric figures on the screen, measure them, transform them, and identify a variety of geometric properties of their creations. Discovery-oriented lessons using these types of software are easy to create and very engaging and useful for students.

Algebra tools include *Derive*, *Maple*, and *Mathematica*. These tools all can manipulate algebraic symbols and equations, solve a variety of equations, do two- and three-dimensional plotting, and much more. The programs offer significantly more power than the graphing calculators, but are also more expensive. They can be used very effectively for classroom presentations with a projection viewing device.

There is also a good variety of algebra learning programs. *The Function Supposer*, *Green Globs and Graphing Equations*, and *The Algebra Sketchbook* are all popular pieces of software that deal with functions and their graphs. Many other valuable pieces of software are available.

The World Wide Web can be an exciting and eye-opening tool for ninth- through twelfth-graders as they retrieve and share information. Specifically, in these grades, they might look for information about colleges in which they might be interested, the history of mathematics, or ecology experiments in which students are gathering and contributing local data.

Standard 5 — Tools and Technology — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11 and 12.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

1*. Select and use calculators, software, manipulatives, and other tools based on their utility and limitations and on the problem situation.

- Students have a variety of tools available to them in the well-equipped mathematics classroom: a bank of computers loaded with algebraic symbol manipulation and function-plotting programs, spreadsheet and graphing programs, and geometry construction programs; a set of graphing calculators for relevant explorations and computations; and manipulative materials related to the content studied. The students easily move from one type of tool to another, understanding both their strengths and limitations.
- Students work through the *Making Rectangles* lesson that is described in The First Four Standards of this *Framework*. They use algebra tiles organized in rectangular form to help them develop procedures for re-writing binomial expressions as multiplication problems (factoring).
- Students can use *algebra tiles*, *Hands-On Equations* materials, and a variety of equation-manipulating software to simplify and solve equations. They understand and can demonstrate the relationship between various manipulations of tiles or pawns and the corresponding symbolic actions in the software solution procedure.
- Students use both graphing calculator techniques and paper-and-pencil techniques for solving systems of equations. Depending on the complexity of the system and on the degree of accuracy needed in the answer, they may try to locate the intersection of two graphs by tracing and zooming on the calculator screen, by calculating the solution with matrices, or by using a simple addition or substitution paper-and-pencil method.

2*. Use physical objects and manipulatives to model problem situations, and to develop and explain mathematical concepts involving number, space, and data.

- Students use a process described in *Algebra in a Technological World* to construct cones from a circular piece of paper by cutting a wedge-shaped sector from it and then taping together the edges. They then try to find the cone constructed in this manner that has the largest volume. A similar activity is described in The *Ice Cones* lesson in The First Four Standards of this *Framework*.

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* Activities are included here for Indicators 1, 2, 3, 5, and 7, which are also listed for grade 8, since the Standards specify that students demonstrate continued programs in their indicators.

- Students use molds to make cones of clay and then experiment to see in how many different ways they can slice the cone with a plane to produce different cross-sections. Drawings of the cross sections and a description of the cuts that created them are displayed on a poster in the classroom.

- In one of the units of the *Interactive Mathematics Program*, students read *The Pit and the Pendulum* by Edgar Allan Poe and then work in groups to investigate the properties and behavior of pendulums. The ultimate goal, after a good deal of measurement and statistical manipulations of their data, is to determine how much time the prisoner in the story has to escape from the 30-foot, razor-sharp, descending pendulum.

3*. Use a variety of technologies to discover number patterns, demonstrate number sense, and visualize geometric objects and concepts.

- Students play *Green Globs and Graphing Equations*, a computer game in which they score points for writing the equations of functions that will pass through several green globs splattered on the x-y plane. As they gain experience with the game, their ability to hit the targets with more and more creative functions improves.

- Students use a set of spherical materials like the *Lenart Sphere* to study a non-Euclidean geometry. With these materials, students make geometric constructions on the surface of a sphere to realize that, in some geometries, a triangle *can* have three right angles, and to find the spherical equivalent of the line that is the shortest distance between two points.

- Students use calculators to investigate interesting number patterns. For example, they try to determine why this old trick always works: *Enter any three-digit number into the calculator. Without clearing the display, enter the same three digit number again so that you have a six-digit number. Divide the number by 7. Then divide the result by 11. Then divide that result by 13. What is in the display?*

5*. Use technology to gather, analyze, and display mathematical data and information.

- Students use a simulation program to check their predictions regarding the answer to this problem from the New Jersey Department of Education's *Mathematics Instructional Guide*: *Two standard dice are rolled. What is the probability that the sum of the two numbers rolled will be less than 5? A) 1/3 B) 1/6 C) 1/9 D) 1/12.* After determining the probability theoretically, they use a simulation program for 1000 rolls of two dice and check the outcome data to see if their predicted probability was in the right ballpark.

- Students use HyperStudio to create the reports they write about biographies of mathematicians, about how mathematics is used in real life, or about solutions to problems they've solved. The software allows them to create true multimedia presentations.

- Students explore the great wealth of mathematical information available at the University of St. Andrews' History of Mathematics World Wide Web site (http://www.groups.dcs.st-and.ac.uk/~history/).

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* Activities are included here for Indicators 1, 2, 3, 5, and 7, which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

- Students use *Algebra Animator* software to simulate and manipulate the motion of a variety of objects such as cars, projectiles, and even planets. They gather data about the motion and directly visualize both the functions that describe the motion and their graphs.
- Working in small groups, students use a distance probe connected to a graphing calculator to collect data about the rate of approach of a classmate walking toward the calculator. After the walk is finished, the calculator plots the student's position relative to the calculator as a function of time. The group then presents the finished graph to the rest of the class and challenges them to describe the walk that was taken: *What rate of progress was made? Was it steady progress? Where did the student stop? Was there ever any backward walking?*
- Students explore the rich links suggested on the Cornell University Math and Science Gateway World Wide Web Site (http://www.tc.cornell.edu/Edu/MathSciGateway).
- There is always math help available at the Dr. Math World Wide Web site (dr.math@forum.swarthmore.edu). In Dr. Math's words, "Tell us what you know about your problem, and where you're stuck and think we might be able to help you. Dr. Math will reply to you via e-mail, so please be sure to send us the right address. K-12 questions usually include what people learn in the U.S. from the time they're five years old through when they're about eighteen."

7*. Use computer spreadsheets and graphing programs to organize and display quantitative information and to investigate properties of functions.

- Students work through the *Building Parabolas* lesson that is described in The First Four Standards of this *Framework*. They use both the *Green Globs* software and their graphing calculators to investigate how the various coefficients affect the graph of parabolas.
- Students use calculators, a spreadsheet, and an integrated plotter to work on this problem from *Algebra in a Technological World*:

  A new professional team is in the process of determining the optimal price for a special ticket package for its first season. A survey of potential fans reveals how much they are willing to pay for a four-game package. The data from the survey are displayed below.

Price of the Four-Game Package	Number of Packages That Could Be Sold at That Price
$96.25	5,000
90.00	10,000
81.25	15,000
56.25	25,000
50.00	27,016
40.00	30,000
21.25	35,000

* Activities are included here for Indicators 1, 2, 3, 5, and 7, which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

On the basis of the foregoing data, find a relationship that describes the price of a package as a function of the number sold (in thousands). Then determine the selling price which will maximize the revenue, and its number of packages likely to be sold at this price.

- Students investigate the growth of the world's population by researching estimates of the level of population at various times in history and plotting the corresponding ordered pairs in a piece of software called *Data Models*. They then use the software tool to find a line or curve of best fit and use the resulting graph to predict the population in the year 2100. As a last step, they find the predictions made by several social scientists and compare them to their own.
- Students work on a lesson from *The New Jersey Calculator Handbook* which uses graphing calculators to focus on the linear functional relationship between circumference and diameter. The students measure everyday circular objects to collect a sample of diameters and circumferences. They then enter their data into a calculator which plots a scattergram for them and finds a line of best fit. The slope of the line is, of course, an approximation of π.
- Students use the *Geometry Inventor* for constructions which illustrate a proof of the Pythagorean Theorem. With the construction tool, they create a right triangle in the center of the computer screen, and a square on each of the legs. They then make a table of the areas of the three squares. As they manipulate the triangle to adjust the relationship among the lengths of the legs, they notice that the basic additive relationship of the areas of the three squares remains the same.
- Students use *The Geometer's Sketchpad* to create an initial polygon and then apply a series of complex transformations to it resulting in a whole sequence of transformed polygons spread out across the screen. The results are often striking colorful images that the students can preserve as evidence of the connections between geometry and modern artistic design.

8. Use calculators and computers effectively and efficiently in applying mathematical concepts and principles to various types of problems.

- Students quickly determine the appropriate window for finding the intersection of two functions by playing with the zoom and range functions on a graphing calculator.
- Students solve a variety of on-line trigonometry problems posted on the *Trigonometry Explorer* World Wide Web site (http://www.cogtech.com/EXPLORE).
- Having just conducted a science experiment where they collected data about the rates of cooling of a liquid in three different containers, the students quickly and efficiently enter the data into a computer spreadsheet and generate broken-line graphs to represent the three different settings.
- Students solve the following problem by writing a function that describes the volume of the box, plotting the function on a graphing calculator, and searching visually for the peak of the graph. *An open-topped box is made from a six-inch square piece of paper by cutting a square out of each corner, folding up the sides and taping them together. What size square should be cut out of the corners to maximize the volume of the box that is formed?*

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Association of Mathematics Teachers of New Jersey. *The New Jersey Calculator Handbook.* 1993.

Fendel, D., D. Resek, L. Alper, and S. Fraser. *Interactive Mathematics Program.* Key Curriculum Press.

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Lenart Sphere. Key Curriculum Press.

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Wolff, K. "Technology and Mathematics Education: Trojan Horse or White Knight?" in *The New Jersey Calculator Handbook*. Association of Mathematics Teachers of New Jersey, 1993.

Software

Algebra Animator. Logal.

The Algebra Sketchbook. Sunburst Communications.

Cabri Geometry. IBM.

Data Models. Sunburst Communications.

Derive. Soft Warehouse.

The Function Supposer. Sunburst Communications.

Geometer's Sketchpad. Key Curriculum Press.

Geometric Pre-Supposer. Sunburst Communications.

Geometric Supposer. Sunburst Communications.

Geometry Inventor. Logal.

Green Globs and Graphing Equations. Sunburst Communications.

HyperStudio. Roger Wagner.

Maple. Brooks/Cole Publishing Co.

Mathematica. Wolfran Research.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards.*

STANDARD 6 — NUMBER SENSE

K-12 Overview

All students will develop number sense and an ability to represent numbers in a variety of forms and use numbers in diverse situations.

Descriptive Statement

Number sense is defined as an intuitive feel for numbers and a common sense approach to using them. It is a comfort with what numbers represent, coming from investigating their characteristics and using them in diverse situations. It involves an understanding of how different types of numbers, such as fractions and decimals, are related to each other, and how they can best be used to describe a particular situation. Number sense is an attribute of all successful users of mathematics.

Meaning and Importance

Successful users of mathematics have good number sense. When someone chooses to use fractions in one situation and decimals in another because the respective operations are easier to perform or the results are easier to understand, that process is evidence of good number sense. When students continue work on a problem involving numbers until they recognize that their answers are reasonable in the context of the problem, they are using good number sense. When a student is comfortable with using an approximation to a number in certain situations, or understands that an approximation rather than an exact number might have been used, then that reflects good number sense. When students recognize that an answer is off by a factor of ten, or alternately that a decimal point has been misplaced, they are using good number sense.

Our students often do not connect what is happening in their mathematics classrooms with their daily lives. It is essential that the mathematics curriculum build on the sense of number that students bring with them to school. Problems and numbers which arise in the context of the students' world are more meaningful to students than traditional textbook exercises and help them develop their sense of how numbers and operations are used. Frequent use of estimation and mental computation are also important ingredients in the development of number sense, as are regular opportunities for student communication. Discussion of their own invented strategies for problem solutions helps students strengthen their intuitive understanding of numbers and the relationships between numbers.

A "sense-building mode" is best established when students are provided with opportunities to explore number relationships, are encouraged to question and to challenge, and are allowed to experiment to discover strategies and techniques of their own that ease the path to the solution of mathematical problems.

K-12 Development and Emphases

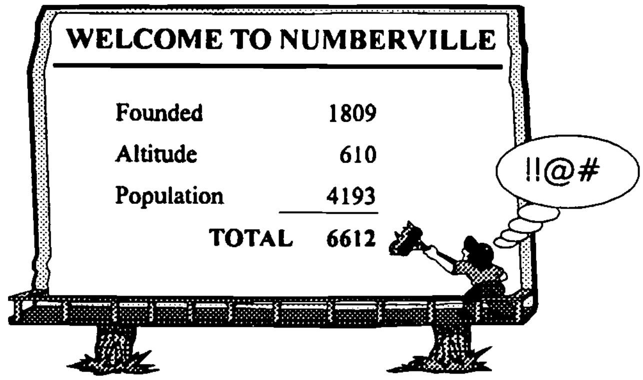
A necessary foundation for a strong number sense is the development of meaning for numbers, beginning with whole numbers, decimals, and fractions. Traditionally, this component of the curriculum has been called **numeration**, and it is vitally important. The K-12 mathematics curriculum should provide the appropriate experiences, physical models, and manipulatives to assist in the construction of these meanings. Appropriate technology should also be regularly used to help students develop their number sense. Through regular and frequent experiences emphasizing the measurement of real objects, the counting and grouping of sets of discrete objects, and the well designed use of calculators, elementary students develop **place value concepts**, a **sense of magnitude (size)**, and **approximation and estimation skills** for whole numbers, decimals, and fractions. Real-world situations should be incorporated into their experiences to help young students become aware of the existence of other numbers, such as negative numbers.

The development of personal meaning for numbers should be reinforced in the middle grades with an extension to other numbers and notations such as integers, percents, exponents, and roots. High school students should extend their meaning of number to the real number system and recognize that still other number systems exist. They should have the opportunity to develop intuitive proofs of the fundamental properties of closure, commutativity, associativity, and distributivity.

Students must develop a facility for working with the different types of numbers we use in day-to-day activities. Statements about one particular quantity might best be expressed with a fraction, a percent, a decimal, a ratio, an approximated whole number, or some other number form; **comparisons** often have to be made among numbers in different forms. Therefore, students will need to be able to transform numbers from one form to **equivalent numbers** in another form, and to intelligently select the right form of numbers to use in a particular situation. The correct choice depends on the context of the situation, and therefore students must possess an adequate understanding of each form and the interrelationships among them.

One way to achieve such an understanding in the classroom is through the identification and description of number patterns and the use of pattern-based thinking. For example, examining and modeling many pairs of fractions with equal numerators help students develop the understanding and generalization that the fractions with larger denominators represent smaller quantities. Activities promoting pattern-based thinking can assist students in making similar generalizations about other number forms and their relationships, as well as build initial notions of still other types of important number concepts such as odd and even, prime numbers, and factors and multiples.

Graduates of our schools must be able to use numbers intelligently and understand them wherever they are encountered in real life. They must develop an **awareness of numbers and their uses**. Numbers are used as counts, measures, labels, and locations, and each use has unique characteristics and restrictions on the appropriate forms and operations. The opportunity to develop the needed familiarity with all of these uses comes through the regular presentation of problem situations which utilize them. Some activities should focus on the explicit uses of the numbers themselves, however. A discussion of why it makes no sense to add the numbers on a town's roadway welcome sign (see diagram on the next page) which lists its population, altitude, and year of founding would serve a dual purpose: to provide an example of some the standard uses of numbers, and to challenge the thoughtless computational manipulation of numbers.



IN SUMMARY, the commitment to develop number sense requires a dramatic shift in the way students learn mathematics. Our students will only develop strong number sense to the extent that their teachers encourage the **understanding** of mathematics as opposed to the memorization of rules and mechanical application of algorithms. Every child has the capability to succeed as a user of mathematics, but the degree of success is directly related to the strength of their number sense. The way to assure that all students acquire a good sense of number is to have them consistently engage in activities which require them to think about numbers and number relationships and to make the connection with quantitative information encountered in their daily lives.

NOTE: *Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

Standard 6 — Number Sense — Grades K-2

Overview

Students can develop a clear sense of number from consistent ongoing experiences in classroom activities where a variety of manipulatives and technology are used. The key components of number sense, as identified in the K-12 Overview, include an **awareness of numbers and their uses** in the world around us, a good sense of **place value concepts, approximation, estimation, and magnitude**, the **concept of numeration**, and an understanding of **comparisons** and the **equivalence** of different representations and forms of numbers.

Kindergarten, first, and second graders are just beginning to develop their concepts of number. They have most likely come to school with some ability to count, but with differing notions of what that activity means. It is in these grades that they begin to attach meaning to the numbers that they hear about and see all around them. One useful activity that can be repeated many times throughout this age range is the keeping of a scrapbook reflecting all the **uses of numbers** that the children can identify. It would probably include telephone numbers, addresses, ages, page numbers, clothing sizes, room numbers, and many others. Discussions of the similarities and differences in all of these uses can provide some interesting insights.

In terms of **numeration**, students in these grades start by constructing meaning for one-digit numbers and build up to formal work with three-digit numbers. The regular and consistent use of concrete models for that development is essential. Kindergartners need a variety of things to count, from poker chips to marbles to beans. Both concrete and rote counting are critically important in developing a sense of number. Adequate attention to counting activities throughout these grades will help to assure both a good sense of **magnitude (size) of numbers** and a real readiness for all four basic operations. (See Standard 8.) Counting by ones should be followed by counting back; skip counting by twos, fives, and tens; counting from a given starting number to a given target number by ones and by other numbers; counting on by tens from non-multiples of ten like 43; and so on.

As students are able to handle larger numbers, place value and base-ten ideas are introduced through grouping activities. Many of the models with which they are comfortable for single units can translate nicely into beginning base-ten models; poker chips can be put in groups of ten into small paper cups; beans can be pasted in tens onto tongue depressors, and so on. These newly enhanced models, along with the single digit units, are then used to represent two-digit numbers. As the next step, of course, groups of ten tens can be made to create hundreds. These first models of base-ten number are the best ones to use with young children who are first encountering these notions because they can actually build larger units from smaller units. Such models are called *bundle-able*. Another property these have is *proportionality*, because the model for a ten is actually ten times as large as the model for a one. A widely used model which is both bundle-able and proportional involves popsicle sticks which are wrapped into tens and hundreds with rubber bands.

The next type of model to be used would be one which is still proportional, but no longer bundle-able. The best examples of this type are the standard base-ten blocks. They require the child to trade ten ones for a ten rather than directly constructing a ten from the ones, and, as a result, are slightly more sophisticated. The last level of sophistication in this sequence of models includes those that are neither proportional nor bundle-able. Two models of this type which are regularly used are chip trading materials and play money. With chip trading materials, there is no inherent concrete ten-to-one relationship between the red chips and the green chips; the red chips are not ten times as large as the green ones. The relationship holds only because of an external rule that is made up and followed. Similarly, there is no inherent concrete ten-to-one relationship that exists between dimes and pennies. The relationship only exists because of a rule that is external to the coins themselves. As a result, these most sophisticated models should be used *after* the underlying concepts are developed with the earlier models.

Children at these grade levels also begin to learn about **equivalence**. When youngsters find as many "names" as they can for the number *7* (such as $2 + 5$, $9 - 2$, and *one more than 6*), they are creating equivalent forms of the same number. Slightly older students should be using similar activities to generate equivalent forms of multi-digit numbers, partly in preparation for operations involving them: *67 = 6 tens and 7 ones = 5 tens and 17 ones = 4 tens and 27 ones.*

Estimation should be a routine part, not only of daily mathematics lessons, but also of the entire school day. Children should be regularly engaged in estimating both quantities and the results of operations. They should respond to questions that arise naturally during the course of the day, like: *About how many kids do you think there are out here in the playground? About how many pieces of construction paper will we need for this project if everyone needs three different colors?* and *How many of your great graphs do you think will fit on the bulletin board without overlapping?* After several children have had chances to make estimates about numbers like these, they should defend their estimates by giving some rationale for thinking they are close to the actual number. These discussions can be invaluable in helping them to develop good number sense.

Technology plays an important role in number sense at these grade levels. Calculators can be wonderful teaching tools when programmed to count forward and backward by some constant. Children can do the programming easily themselves and try to anticipate the calculator display. Appropriate computer software provides environments in which students can first develop a sense of small whole numbers and then build an understanding of place-value and base-ten ideas.

The topics that should comprise the number sense focus of the kindergarten through second grade mathematics program are:

whole number meanings through three digits
place value and number base
counting and grouping

Standard 6 — Number Sense — Grades K-2

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Use real-life experiences, physical materials, and technology to construct meanings for whole numbers, commonly used fractions, and decimals.**

   - Young students make and use a variety of models for "number" ranging from poker chips to dot patterns on a paper plate, to Cuisenaire Rods, to tally marks, to domino and dice combinations. A large component of their early work with number focuses on the various *parts* into which any given number can be broken.
   - Students play the *Broken Key* game on their calculators. Kindergartners try to get the calculator display to show *7* while pretending that the *7* key is broken and cannot be pressed. Second graders might try to get the display to show *45* without pressing the *4* or the *5* key.
   - Students use *5-frames* and *10-frames* to help develop initial ideas of small numbers. By filling up a 5-cell grid with counters first and then putting out 2 more while trying to show "7 in all," the child not only learns about "7" but also about its relationship to "5."
   - Students use numbers throughout the school day as they discuss the date, attendance, time, snacks, money, etc.
   - Students investigate fractions by listening to the story *Gator Pie* by Louise Mathews and by discussing how Alvin and Alice can share their pie with more and more alligators.
   - Second-graders record prices as decimals ($0.39) and use this notation to find totals over $1 on a calculator.
   - Students find half of a sheet of paper by folding horizontally, by folding vertically, and by folding diagonally. They compare the results and discuss how they are alike and how they are different.
   - Students use *Balancing Bear* software to find combinations of numbered weights that will balance a seesaw or that will be greater or less than a given weight.

2. **Develop an understanding of place value concepts and numeration in relationship to counting and grouping.**

   - Calendar activities at the beginning of the school day incorporate a *Daily Count* feature where each day another popsicle stick is added to a collection representing all of the days of school to date. Whenever 10 single sticks are available, they are bundled with a rubber band and are thereafter counted as a *ten*. On the hundredth day of school, the ten *tens* are wrapped together to make a *hundred*, and the class celebrates the event with a party.

- Students progress from a *proportional* and *bundle-able* base ten model like popsicle sticks to a *proportional* but not *bundle-able* model like base-ten blocks to a model that is neither *proportional* nor *bundle-able* like pennies and dimes. (See K-2 Overview)
- Pairs of students play *Race to One Hundred* with base ten blocks. Each, in turn, rolls one or two dice and takes that many unit cubes. Whenever there are ten unit cubes in a player's collection, the player *must* trade for a ten block. The first player able to trade ten ten blocks for a hundred block is the winner.
- Students have 3 dimes and 4 pennies to spend on a variety of items that are displayed in a classroom store. The items have tags ranging from 3¢ to 56¢ and the children are asked: *Which of these items can be bought for exactly the amount of money that you have (requiring no change)? Which items can you buy and have some money left over? Which of these items cannot be bought because you do not have enough money? What items are left?*
- Student understanding of place value for two-digit numbers is assessed by asking each student to represent a different number using popsicle sticks or base 10 blocks.

3. See patterns in number sequences, and use pattern-based thinking to understand extensions of the number system.

- Students find patterns in a hundred number chart. When asked to describe patterns that they see, some children see a counting by ones pattern horizontally, others see the tens digit increasing and the ones digit staying the same as they move down the chart vertically, and still others see in the last column the numbers that they use to count by tens.
- Students use the constant function feature of their calculators to program a *skip count*. They press + 2 = = = to watch the display count by twos, try to anticipate what number comes next and make predictions to each other. Any number can replace the "2" to add difficulty to the activity.
- Students play *Find the Number* on a hundred number chart located at the front of the room, with each of the numbers covered by a Post-it or a small tag. One child calls out a number, like *45*, and a volunteer tries to identify where it is on the chart. The indicated Post-it is then lifted to check the guess.

4. Develop a sense of the magnitudes of whole numbers, commonly used fractions, and decimals.

- Children are presented with four jars of jelly beans — one with 3 beans in it, one with 19 beans in it, one with 52 beans in it, and one with 156. The teacher then asks *Which of these jars do you think has about 50 beans in it?* The students discuss their reasons for believing as they do.
- Second graders are challenged to guess how many sheets of paper are in the ream of paper on the front table. After everyone has made a guess, one student counts out 25 sheets from the top of the pile and places them next to the rest of the pile. Everyone is offered a chance to change their estimates and to discuss the reason for their change. Then students agree on a way to verify their guesses before trying to guess how many such reams it

would take to reach the ceiling!

- Students work through the *Will a Dinosaur Fit?* lesson that is described in the First Four Standards of this *Framework*. They discuss how many dinosaurs of different types might fit into the classroom.
- Students fold paper circles into halves, fourths, and eighths and are asked questions like: *Which would you rather have, a half of a cherry pie or a fourth of the pie? How about three-eighths of a pizza or one-fourth?*
- Students read or listen to a piece of children's literature that has fractions as its theme, such as *Eating Fractions* by Bruce McMillan.

5. Understand the various uses of numbers including counting, measuring, labeling, and indicating location.

- A kindergarten teacher announces to her class: *Boys and Girls! Great News! The principal told me that our class has just won FIVE!* A discussion then ensues regarding the need for that number to exist in some context, to have some unit or label before it makes sense.
- Second graders are given a stack of old magazines. They cut out any information which uses numbers and sort them according to how they are used: as page numbers, as prices, as dates, as addresses, and so on.
- The class takes a walk around the school or neighborhood pointing out to each other the numbers they see, and discuss how they are used.

6. Count and perform simple computations with money.

- Students use play money to show different combinations of coins that can be used to "buy" an object. For example, an 11¢ pencil can be bought with 11 pennies, a dime and a penny, one nickel and six pennies, or two nickels and a penny.
- Students earn 2¢ each day for attendance and 1¢ for good behavior. They keep their play money in a bank, count it regularly and use it to buy objects from a treasure chest.
- Students play *Spend a Dollar*. They each start with $1 (either as a bill or in change) and then roll one or two dice to find out how much they "spend" on that turn. They trade coins as needed. The student who spends all of her money first wins.
- Students play a shopping board game. They each begin with a given amount of money in coins. They roll two dice to determine how far they move each turn. As they land on a space, they must buy whatever is shown. Some spaces may provide refunds. The winner is the first person to go around the board and still have money left.
- Students' abilities to recognize coins and find the value of a group of coins are assessed by having each student select three objects to "buy," identify and name the coins needed to purchase each object, and find the total amount of money required to purchase all three.

7. **Use models to relate whole numbers, commonly used fractions, and decimals to each other, and to represent equivalent forms of the same number.**

   - When modeling 2-digit numbers with base-ten models such as popsicle sticks, base-ten blocks, or pennies and dimes, students are frequently asked to show all the ways they can make a given number. Children then begin to see that *3 tens and 7 ones, 2 tens and 17 ones, 1 ten and 27 ones,* and *37 ones* all represent the same number *37.*
   - Students each develop questions whose answers are all equivalent to some target number. For example, if the target is *8*, students may ask the following questions: *What is 4+4? What is 9-1? What is 8+0? How many hands do four children have? How many days is one more than a week?* or *How much is a nickel and three pennies?*
   - Students use geoboards, pattern blocks, Cuisenaire Rods, paper folding, and tangrams to explore simple common fractions like halves, thirds, and fourths. For instance, they may be challenged to model 1/2 with all of the different models.

8. **Compare and order whole numbers, commonly used fractions, and decimals.**

   - Young students use dot pattern cards or dominoes to practice *more, less,* and *same.* For example, given a card with 6 dots on it, students use counters to make a set that is more, another that is less, and one that is the same. They can then label the sets with cards that show the appropriate words. With dominoes, students work in pairs to compare the dots on the two halves and state which is more and by how much.
   - Students play the old favorite card game *war* with either dot cards or with a deck of regular playing cards minus the face cards. Every now and then, the rule changes so that the student with the card that is *less* wins the play.
   - Students play *Guess the Point*. A long number line with endpoints of 20 and 75, for example, is drawn on the board where all of the intermediary points are labeled above the line. The labels are then covered by a long piece of paper that can be lifted to reveal them. A student places a finger somewhere on the line and others must estimate the numerical label of the point chosen. The paper is then lifted to check the accuracy of their responses.

9. **Explore real-life settings which give rise to negative numbers.**

   - Primary classrooms are equipped with Celsius thermometers, in addition to Fahrenheit ones, so that "below zero" outdoor temperatures can be recorded. Temperature reports, possibly in both scales, become a part of the everyday calendar routine.

References

Baratta-Lorton, Mary. *Mathematics Their Way*. Menlo Park, CA: Addison-Wesley, 1995.

Mathews, Louise. *Gator Pie*. New York: Dodd, Mead, and Co., 1979.

McMillan, Bruce. *Eating Fractions*. New York: Scholastic, 1991.

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA, 1989.

Software

Balancing Bear. Sunburst Communication.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 6 — Number Sense — Grades 3-4

Overview

In third and fourth grade, students continue to develop their number sense by using manipulatives and technology. The key components of number sense, as identified in the K-12 Overview, include an **awareness of numbers and their uses** in the world around us, a good sense of **place value concepts, approximation, estimation, and magnitude**, the **concept of numeration**, and an understanding of **comparisons** and the **equivalence** of different representations and forms of numbers.

Third and fourth graders are refining their understanding of whole numbers and are just beginning to develop understanding of numbers like decimals and fractions that require substantially different ways of thinking about numbers. An excellent activity that can be used to impress upon these students the omnipresence of numbers around them is the keeping of a journal reflecting all of the **uses of numbers** that they can find in magazines and books. In fourth grade, they focus particularly on uses of fractions and decimals. They may include decimal prices in advertisements, fraction-off sales, and decimal or fraction measurements. Discussions of the uses found and the meanings of the numbers involved can provide interesting insights.

Their **numeration** work in earlier grades, having focused on models of number, has enabled students to use relatively sophisticated models like play money or chip trading to represent whole numbers up to three digits. The regular and consistent use of concrete models is essential for the continuing development of their understanding of numeration. They should use base-ten models not only to extend their experiences with whole numbers to four places, and then symbolically beyond that, but also to create meaning for decimals.

In addition, models are essential for the initial explorations of the meaning of fractions. Fraction Circles or Fraction Bars help children establish rudimentary meaning for fractions, but have the drawback of using the same size unit for all of the pieces. Cuisenaire Rods or paper folding can be used to accomplish the same goals without this drawback.

Children at these grade levels also continue their learning about **equivalence**. They should be engaged in activities using concrete models to generate equivalent forms of many different kinds of numbers. For multi-digit numbers, equivalences such as: *367 = 3 hundreds, 6 tens and 7 ones = 3 hundreds, 5 tens and 17 ones = 2 hundreds, 14 tens and 27 ones* are useful in promoting a confident feeling about place value and will help in understanding multi-digit computation. Early explorations of equivalent fractions (1/2 = 2/4) and equivalent decimals (3 tenths = 30 hundredths) can accompany the exploration of the basic equivalences between fractions and decimals (1/2 = 0.5).

Estimation should be a routine part not only of mathematics lessons, but of the entire school day. Children should be regularly engaged in estimating both quantities and the results of operations. They should respond to questions that arise naturally during the course of the day like: *About how many kids do you think there are in the auditorium? About how many paper cranes will each student have to fold if the class needs to make 200 altogether?* and *How many floor tiles do you think are on the floor?* After several children have had chances to make estimates about numbers like these, they should defend their estimates by giving some rationale for thinking they are close to the actual number. These discussions can be invaluable in helping them develop number sense.

Technology plays an important role in number sense at these grade levels. Calculators can be wonderful exploration tools when examining new numbers. Students will themselves raise questions about decimals when someone divides 30 by 60 inadvertently instead of 60 by 30 and wonder what the 0.5 in the display means. Computers provide software that creates environments in which students manipulate base-ten models on-screen and explore initial fraction and decimal concepts.

The topics that should comprise the number sense focus of the third and fourth grade mathematics program are:

whole number meanings through many digits
place value and number base
initial meanings for fractions and decimals

Standard 6 — Number Sense — Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Use real-life experiences, physical materials, and technology to construct meanings for whole numbers, commonly used fractions, and decimals.**

   - Students are comfortable using a full array of base-ten models including money, base-ten blocks, and chip trading materials to represent both whole numbers and decimals.
   - Students use computer software that provides easy pictorial representation of large whole numbers, decimals, and fractions like the *MECC Math Tools*, the *Silver Burdett Math Workshop*, and the *Wasatch Math Construction Tools*.
   - Students use geoboards to model common fractions. For example, they search for multiple ways to show 1/4 on the geoboard.
   - Students use Cuisenaire Rods to model fractions, frequently switching the rod or length used as the *whole* to avoid the misconception that, for instance, the yellow rod is always one-half.

2. **Develop an understanding of place value concepts and numeration in relationship to counting and grouping.**

   - Pairs of students play *Race to Five Hundred* and its opposite, *Race to Zero*, with base ten blocks. In the first game, each student, in turn, rolls a red die and a green die and makes a two digit number from the faces showing (using the red die as the tens digit). He or she then takes that many tens and ones from the bank. Whenever there are ten tens or ten ones in a player's collection, the player must trade for a larger block. The first player to collect 5 hundreds is the winner. In *Race to Zero*, the players *start* with 5 hundreds and *give back* blocks according to their dice rolls.
   - Students use a die to generate random digits from 1 to 6. After each roll, they decide where to place the digit in a 4-digit whole number. The goal is to produce as large a number as possible. If a ten-sided die or spinner with ten equal sectors is available, students should use it to generate random digits from 0 to 9 and repeat the activity.
   - Students work in groups to decide what the next base-ten block after the thousands block would look like.
   - Students read and listen to children's literature that is related to a numeration theme like *Millions of Cats* and *The 500 Hats of Bartholomew Cubbins*.

3. **See patterns in number sequences, and use pattern-based thinking to understand extensions of the number system.**

- Students use the constant function feature of their calculators to program a *skip count*. They press + *12* = = = to watch the calculator display count by twelves, trying to anticipate what number will come next and making predictions to each other. Any number can replace the *12* to change the difficulty level of the activity.
- Students also use their calculators to play *Guess My Rule* games. One student secretly programs the calculator by typing something like x *2* =. Thereafter, every time a number is pressed followed by an equals sign, the original number will be multiplied by two. A second student must guess the rule that was programmed. Rules like + *3* =, ÷ *4* =, and - *2* = also work.
- Students create and solve *arrow puzzles* on a hundred number chart. By naming a number and then giving directions for movement on the chart, instructions are given to arrive at some other number. For example: *72, down, down, right, right, up* leaves the student at the number *84*. After examples of different patterns are demonstrated on the chart, students point out patterns and try to solve puzzles mentally.
- Students make a table to reflect how many handshakes there would be if everyone shook hands in groups of different sizes. For example, for 2 people, 1 handshake; for 3 people, 3 handshakes; for 4 people, 6 handshakes. As they extend the table for larger groups, the students look for a pattern in the emerging numbers.
- Students search for patterns in the addition of even and odd numbers by using unifix cubes to represent the numbers and trying to arrange the sum into two stacks of equal height. (This will work for even numbers, but not for odd ones.)

4. **Develop a sense of the magnitudes of whole numbers, commonly used fractions, and decimals.**

- Students imagine collecting 10,000 of something. They discuss what objects would be reasonable to collect (such as bottle caps, pennies for charity, or pebbles from the beach), how much space this collection might take up, and how much it would weigh?
- Students cut and paste sheets of base-ten graph paper to make models of the different powers of ten: *1, 10, 100, 1000, 10,000.*
- Students locate numbers as points on a number line strung across the room, continuing to attach labels as they learn more about numbers. Paper clips or tape are used to fasten equivalent forms of a number to the same point.
- Students estimate and investigate how long a million seconds is using calculators.
- Students write a Logo or BASIC computer program which will count to 100, printing the numbers to the screen as it runs, and timing how long it takes. Then they predict how long it would take the program to count to one thousand, to one hundred thousand, and to one million. They make the required changes to the program and check their predictions.

5. **Understand the various uses of numbers including counting, measuring, labeling, and indicating location.**

   - Students keep a 24-hour diary recording all of the ways they use or see others use numbers. They pool all of these uses of numbers and classify them into categories that they design.
   - As part of a geography unit, students make a map of a fantasy island, using a Cartesian coordinate system to help describe the location of various places on the island. They use other numbers to describe geographical properties of the sites: elevation, longitude and latitude, population, and the like.
   - The students brainstorm ways to describe their math book in terms of numbers: its width, the number of pages, the publication date, a student-generated "quality rating" of the book, the area of the cover, and so on.

6. **Count and perform simple computations with money.**

   - Students establish a school store and make transactions on a regular basis, with different students assigned as clerk each day.
   - Students read *Dollars and Cents for Harriet* and then decide how they would spend five dollars.
   - Students practice making change with coins by *counting up* to the amount given. For example, if the bill is $1.73, and $2.00 is the amount given, the students would count up to $2.00 by starting with two pennies and saying, "$1.74" and "$1.75"; then they add one quarter to bring the total to $2.00. They would then count this change to find its value of $0.27.
   - Students play *Treasure Math Storm* on the computer or use IBM's *Exploring Measurement, Time and Money*.

7. **Use models to relate whole numbers, commonly used fractions, and decimals to each other, and to represent equivalent forms of the same number.**

   - When modeling 3- and 4-digit numbers with a base-ten model like base-ten blocks or place value chips, the students are frequently asked questions like: *Show all the ways you can make 327.* Children thus begin to see that *3 hundreds, 2 tens, and 7 ones; 2 hundreds, 12 tens, and 7 ones; 2 hundreds, 11 tens, and 17 ones;* and *32 tens and 7 ones* all represent the same number. Students are assessed by asking them to show 327 in two different ways.
   - Students use shadings on ten-by-ten grids to represent fractions and decimals that are equivalent. For example, the representation for 0.4 is the same as that for 4/10.
   - Students develop their own questions, the answers to which are equivalent to some target number. For example, if the target number is *24*, students may ask the following questions: *What is 20 + 4? What is 2 x 12? What is 2 x 2 x 2 x 3? How much is 2 dozen? How many is 3 less than the number of children in our class?* or *How much would something cost if you paid a quarter and got back a penny in change?*
   - Students use geoboards, pattern blocks, Cuisenaire Rods, paper folding, and tangrams to

explore common fractions. They may be challenged to model 3/4, for instance, with all of the different models.

- Students use money to represent decimals. For example, 8 dimes = \$0.80 = .8. They also represent fractional parts of a dollar as a decimal (a quarter = 1/4 = 25¢ = .25).
- Students use graham crackers, candy bars, pizzas, and other food to illustrate fractions.
- Students work through the *Sharing Cookies* lesson that is described in the First Four Standards of this *Framework*. They realize that 8 is not readily divisible by 5 and try to find ways to solve that sharing problem using real cookies.
- Students play *Bowl a Fact* by rolling three dice and using the numbers shown to make number sentences whose answers equal numbers from *1* to *10*. For each different answer, they knock down the bowling pin labeled with that number. For example, if they roll 2, 5, and 3, they can make these number sentences: $2 + 5 + 3 = 10$, $5 + 3 - 2 = 6$, $5 \times 2 - 3 = 7$, $5 - 3 + 2 = 4$, and $3 \times 2 - 5 = 1$, and therefore knock down the *10, 6, 7, 4* and *1* pins. If they cannot knock down all ten pins on the first roll, they roll the dice again and try to get the remaining pins. The students are assessed by giving all of them the same outcomes of two rolls of the three dice to play the game.

8. Compare and order whole numbers, commonly used fractions, and decimals.

- Students use base-ten materials such as, blocks, sticks or money to make models of pairs of 3- or 4-digit numbers like *405* and *450* and compare them to see which is larger. Responses and reasons can be written in a journal.
- Students play *Guess the Point*. A long number line with endpoints of 130 and 470, for example, is drawn on the board with the intermediary points labeled as multiples of ten above the line. The labels are then covered by a long piece of butcher paper that can be lifted to reveal them. A student places a finger somewhere on the line and others must estimate the numerical label of the point chosen. The paper is then lifted to check the accuracy of their responses.
- Pairs of students play *Hi-Lo* with whole numbers and decimals. One student thinks of a number within a given range such as 1 to 1000. The other student tries to guess the number, receiving feedback after each guess as to whether the guess was too high or too low, and keeping a written record of the guesses and the feedback. The goal is to find the number using as few guesses as possible.
- When using Cuisenaire Rods, students choose a *base rod* to represent one whole, and then determine the values of all of the rest of the rods. They then use the rods to model the comparison of the relative sizes of two fractions with different denominators.

9. Explore real-life settings which give rise to negative numbers.

- Students record daily low Celsius temperatures throughout the winter and draw a line graph of those temperatures. Students discuss changes in the graph and the meaning of the line dipping below the zero degree line.
- Students examine a videotape of a section of a football game and record the results of a

series of plays as a series of integers — gains as positive integers and losses as negative integers (for example: −3, +5, +9, first down; −5, −4, +6, punt). They use their record to determine the total yardage gained during the drive.

- Students use an almanac to find the altitudes of selected cities around the country, and discuss what it means for a city to be below sea level.

References

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National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA, 1989.

Seuss, Dr. *The 500 Hats of Bartholomew Cubbins*. New York: Vanguard, 1938.

Software

Exploring Measurement, Time, and Money. IBM.

Math Construction Tools. Wasatch.

Math Tools. Minnesota Educational Computing Consortium (MECC).

Math Workshop. Silver Burdett.

Treasure Math Storm. The Learning Company.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 6 — Number Sense — Grades 5-6

Overview

Fifth and sixth graders should have a good sense of whole numbers and their orders of magnitude and should be focusing mostly on developing number sense with decimals, fractions, and rational numbers, which require substantially different ways of thinking about numbers. They also should be exploring two relatively new topics: ratios and integers. The key components of number sense, as explained in the K-12 Overview, are an **awareness of the uses of numbers** in the world around us, a good sense of **approximation, estimation, and magnitude**, the **concept of numeration**, and an understanding of **comparisons** and the **equivalence** of different representations and forms of numbers.

Students at this age are capable of categorizing all of the ways in which numbers are used in our society. An excellent activity is to have them collect ways in which they see numbers being used during a twenty-four hour period. Their **uses of numbers** would probably include telephone numbers, addresses, ages, page numbers, clothing sizes, library book numbers, room numbers, and many others. Discussions of the similarities and differences among these uses should resolve themselves into some of the standard categorizations: counts, measures, labels, and indicators of location. The students' data can then be graphed according to these categories.

Their **numeration** work in earlier grades, having focused on models of whole numbers, has taken these students to the point where they are able to use relatively sophisticated models like play money or chip trading to represent whole numbers up to three digits. The regular and consistent use of concrete models is essential for the continuing development of their understanding of numeration. The focus now shifts to a real sense of the meanings of decimals and fractions and to providing models which adequately serve that purpose. Money continues to provide a superb setting for the learning of decimal concepts (at least up to two decimal places) because of the students' increasing familiarity with it, because of the vast array of real-world applications that it makes available, and because of its inherent motivational quality. Base-ten blocks are also useful as a slightly more abstract model. They have the added advantage of being able to represent *any number with four digits* in the place value system. A block model of 3274 with which the students are familiar can become a model of the decimal 3.274 if, instead of thinking of the smallest block as one unit, they think of the largest block as one unit.

In addition, models are essential for the continued exploration of fraction meaning and fraction operations. Fraction Circles or Fraction Bars help children establish rudimentary meaning for fractions, but have the drawback of using the same size unit for all the pieces. This is a fairly serious drawback leading to the misconception, for instance, that 1/3 is *always* less than 1/2 without regard to the *units* in which those fractions are expressed; students need to be aware, for example, that 1/3 of a large pizza is frequently larger than 1/2 of a small one. Cuisenaire Rods or paper folding can also be used to accomplish many of the same goals without the same drawback. A sample unit on fractions for the sixth-grade level can be found in Chapter 17 of this *Framework*.

Work with ratios, percents, and integers in grades 5 and 6 should be limited to informal exploration, with no use of more formal, symbolic procedures. Students should use models as they explore these topics.

They might use 2 red tiles and 1 yellow tile to illustrate mixing paint in the ratio 2:1 and extend this pattern in order to make larger (and smaller) quantities. By using base ten blocks and 10 x 10 grids, they can visualize percent more easily. Two-color counters or the number line might be used to model positive and negative numbers (integers).

Students in grades 5 and 6 should begin to understand the ways in which different types of numbers are related. For example, they should understand that every whole number is a rational number, since it can be written as a fraction. Similarly, every decimal is a rational number. By the end of sixth grade, they should have had sufficient experiences with integers to realize that the integers consist of the whole numbers and their opposites (additive inverses).

Students at these grade levels continue their learning about **equivalence**, but there is a significant shift in what that means. As third and fourth graders, they have explored simple fractions and decimals, and their work with equivalence has focused primarily on the multiple ways to represent whole numbers (*8 = 2 + 6 = 9 - 1, 23 = 2 tens + 3 ones = 1 ten + 13 ones*, and so on). Now, as fifth- and sixth-graders, they should begin to focus on the representation of the same quantity with different *types* of numbers. Their work with equivalent fractions (1/2 = 2/4) and equivalent decimals (3 tenths = 30 hundredths), for example, should lead to exploration of the basic equivalences of fractions and decimals (1/2 = 0.5). They should be engaged in activities using concrete models to generate equivalent forms of many different kinds of numbers. They also begin to explore the role of ratios and percents in this mix. Ten-by-ten grid paper helps enormously with these activities, since all forms of a quantity can frequently be represented on it.

Estimation should be a routine part not only of mathematics lessons, but of the entire school day. Children should be regularly engaged in estimating both quantities and the results of operations. They should respond to questions that arise naturally during the course of the day, like: *About what fraction (percentage) of the kids in the playground do you think are wearing gloves? About one-third of our students stay for the after-school program in the afternoon; if there are 500 students in the school about how many of them stay?* After children have had several chances to make estimates about numbers like these, they should defend their estimates by giving some rationale for thinking they are close to the actual number. These discussions can be invaluable in helping them develop good number sense.

Technology plays an important role in number sense at these grade levels. Calculators can be wonderful exploration tools when examining new relationships. Many insights about the relationships between fractions and decimals, for instance, can be achieved by simply dividing the numerators of fractions by their denominators. Generalizations about what kinds of fractions produce what kinds of decimals start to flow very freely in such open-ended explorations. Computer software also creates environments in which students manipulate decimal models on-screen and explore fraction and decimal relationships.

The topics that should comprise the number sense focus of the fifth and sixth grade mathematics program are:

- fractions
- decimals
- equivalence
- integers
- ratio and percent

Standard 6 — Number Sense — Grades 5-6

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

10. Understand money notations, count and compute money, and recognize the decimal nature of United States currency.

- Students try to identify fake "bargains" or misleading information from store advertisements, and try to determine which of several options is the best. For example: *Which of the following offers the cheapest price?*

XYZ Mega-Deal Tuna	*ABC Tuna*	*Aunt Betty's Best Tuna*
10 oz can $2.80	*3 oz can $0.87*	*4 oz can $1.04*

- Students plan a fantasy driving trip to Walt Disney World for their family. They research and consider the variety of expenses to be incurred — lodging along the way and at the park, meals, souvenirs, gasoline, admission, and so on. A reasonable budget for the trip is the centerpiece of a report prepared by each student group. Their reports are assessed using a scoring rubric that includes mathematical correctness as well as creativity.
- Students use play money to model decimal numbers. They use decimal language and find fractional equivalents for each coin. For example: *a dime = 0.1 = 1/10 = 1 tenth.*
- Students first estimate and then use a calculator to find out *how long* it would take to spend one million dollars at a rate of one dollar per second.

11. Extend their understanding of the number system by constructing meanings for integers, rational numbers, percents, exponents, roots, absolute values, and numbers represented in scientific notation.

- Students each develop questions, the answers to all of which are equivalent to some target number. For example, if the target number is *24*, students may ask the following questions: *What is 8 x 3? What is (-25) - (-49)? What is* 5^2 *-1? What is 3 more than the sixth triangular number? What is 1 less than one-fourth of 100? or What is the smallest positive number with 8 factors?*
- Students continue to refine their concepts of fractions using all available models to answer questions like: *Is 1/4 always larger than 1/8? Is 1/4 of every pizza larger than 1/8 of every other pizza?* Issues that point out the importance of defining the unit are special topics for discussion.
- Students use two-color counters to construct models of the set/subset meaning of fraction. You might ask: *Given 3 red counters in a set of 12, what are the equivalent fractions that*

represent the reds as a part of the set?

- Students also use two-color counters to model and begin to make sense of positive and negative integers. In this system, a positive 1 is represented by one color and a negative 1 by the other. Students determine the value of a pile of counters by pairing up counters, one of each color, setting aside all pairs, and counting the remaining counters.
- Students read Shel Silverstein's poem *A Giraffe and a Half* and discuss how to describe an amount that is more than one whole but less than two.
- Students read *The Phantom Tollbooth* and discuss the relationships between decimals and fractions in the book. For example, Milo meets half a child (actually, .58 of a child since the average family has 2.58 children).
- Students construct a time line to scale to show the history of the earth. Significant periods and events are shown along the line with numbers reflecting the number of years since the earth's beginning.

12. Develop number sense necessary for estimation.

- Students imagine collecting a million of something. They discuss what objects would be reasonable to collect (such as toothpicks, punched holes from fan-folded computer paper or pages in telephone books to be recycled), how much space this collection would take up, and how much it would weigh.
- Students make estimates to answer the question: *How much drinking water do you think Columbus' ships carried with them on their trip across the ocean?* Then they gather the data they need to make more informed estimates. (*Addenda Series Grade 4 Book.)*
- Students determine the number of decimal places in a simple decimal multiplication product, not by mechanically adding the number of places in the factors, but by estimating a reasonable range for the product and placing the decimal point so that their computed product falls within that range.
- Students investigate the question: *What size room would be needed to hold one million ping-pong balls?*
- Students read *Counting on Frank* and estimate how many dogs would fill their classroom.
- Students use estimates to compare fractions. For example, $3/7 < 9/16$ since 3/7 is less than half and 9/16 is more than half.

13. Expand the sense of magnitudes of different number types to include integers, rational numbers, and roots.

- Students challenge each other to find target numbers on a number line. First one student asks another to find 3.2. The second target number then must be between 0 and 3.2, say 1.74. The third must then be between 0 and 1.74 and so on.
- Students use their calculators to explore the types of decimal expansions for common fractions. They discover that some such decimals terminate, some repeat, and some appear to do neither. (Actually, if the calculators could exhibit more digits, each such decimal would either terminate or repeat.)

- Students use calculators to come as close as they can to an answer to: *What number multiplied by itself gives an answer of 2?*

14. Understand and apply ratios, proportions, and percents in a variety of situations.

- Students begin to see a ratio as both the comparison of two quantities and as a number in its own right. They are challenged to find ratios that they frequently use like *$0.65 per pound, 55 miles per hour,* and so on.
- In a social studies unit, students use population and area data for countries in South America to compute population densities, and then compare their results to those for other areas of the world.
- Students use two different sizes of grid paper to copy a simple drawing of a house from the smaller grid to the larger grid, investigating and discussing the change from one to the other and exploring ways to represent it numerically. They then copy the same drawing onto a third grid, smaller than the second but larger than the first.
- Students are challenged to use any combination of the digits 3, 4, 5, and 9 to make a ratio as close as possible to 90%. As follow-up, they invent other *closeness* problems for each other.
- Students search for as many uses of percent as they can find over the course of a week. The sources for the uses, however, are to be exclusively within the school setting. Likely entries in the resulting list are: *grades on tests, foul shot success of the basketball team, a measure of how close the PTA is to their fund-raising goal for the new playground equipment,* and so on. For each use found, the students explain what 100% would represent and whether percentages above 100% would make any sense in the given context.

15. Develop and use order relations for integers and rational numbers.

- Students use a deck of *fraction cards* for a variety of tasks. A deck consists of one card containing each of a number of fractions, for example, 1/4, 9/10, 2/19, 4/7, 7/9, 1/3, 12/15, 2/5, and 5/8. They are asked to: *Find the smallest fraction in the set. Sort into two groups more than 1/2 and less than 1/2. Determine which pairs have a value close to 1.*
- Students use a number line, including both positive and negative integers, to graph inequalities stated verbally. For example: *Show all of the numbers larger than* −2.
- Students gain understanding of the order relationships among fractions and integers by comparing them with similar ones for whole numbers. *How is the comparison of 4/7 to 5/7 like the comparison of 4 to 5? How is comparing 4/7 to 4/8 like comparing 7 to 8? How is comparing −4 to −7 like comparing 4 to 7?* They answer similar questions on their test.
- Students use 4 digits, say *2, 3, 4,* and *5* to construct as many true fraction sentences as they can. For example, $2/3 < 5/4$.

16. **Recognize and describe patterns in both finite and infinite number sequences involving whole numbers, rational numbers, and integers.**

- Given the first four rows, students formulate a rule for generating succeeding rows of Pascal's triangle. They look for other patterns in the triangle.
- Students explore the well-known problem of taking a long walk by first doing half of it, then half of what remains, half again of what remains, and so on. They write the series as 1/2 + 1/4 + 1/8 + ... *What happens to the walker?*
- Students solve this classic problem: *Which would you choose as the method for getting your allowance next month: $1.00 every day; or 1 cent the first day, 2 cents the second, 4 cents the third, 8 cents the fourth, and so on?*

17. **Develop and apply number theory concepts, such as, primes, factors, and multiples, in real-world and mathematical problem situations.**

- Students build rectangular arrays with square tiles to determine which of the first fifty counting numbers are *rectangular* (composite) and which are *non-rectangular* (prime).
- Students use the Sieve of Eratosthenes to generate a list of all the primes in the first 100 counting numbers.
- Students use common multiples to solve problems like this: *Hot dog buns come in packages of 8. Hot dogs come in packages of 6. What is the smallest number of packages of each that can be bought so that there are no extra buns or hot dogs?*

18. **Investigate the relationships among fractions, decimals, and percents, and use all of them appropriately.**

- Students address the questions *How are 0.50 and 40/100 alike?* and *How are they different?* Answers can be written in their math journals.
- Students use shadings on a ten by ten grid to discuss all of the different equivalences. For example, the same shading can be named 3/10, 30/100, 0.3, 0.30, .3, .30, and 30%. Thinking of the grid as $1.00 leads to some interesting insights about two-place decimals.
- Students explore the density property of numbers by addressing problems like: *Find 4 decimals between 0.456 and 0.457. Find 3 fractions between 3/5 and 4/5.*

19. **Identify, derive, and compare properties of numbers.**

- Students use Venn diagrams to explore the multiple sets to which particular number belong. For example, a Venn diagram is created for these three sets of numbers less than 25: multiples of 3, factors of 24, primes; the Venn diagram is used to answer questions like: *How many numbers are in exactly two of these sets*? A similar question is used on their test.
- Students explore the property of *closure* for a variety of sets of numbers under various operations. For example: *Using subtraction, is there always an answer within the set of positive whole numbers for any member of the set minus any other?* (no); *Is there always an answer within the set of integers?* (yes); *Is there always an answer within the set of even integers?* (yes); *within the set of odd integers?* (no).

- Students explore the properties of odd and even numbers under various operations. For instance: *What can always be said about the sum of two even numbers? of two odd numbers? of an even and an odd number?*
- Students create a book about *zero* for second-graders.
- Students explore the concepts of place value and zero by learning about other number systems. For example, they might use the computer program *Maya Math* to learn about the Mayan number system.

References

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Silverstein, Shel. *A Giraffe and a Half*. New York: Harper & Row, 1964.

Software

Maya Math. Sunburst Communications.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 6 — Number Sense— Grade 7-8

Overview

Seventh and eighth graders should have a good sense of whole numbers and their orders of magnitude and should be focusing on further developing number sense with decimals and fractions. They should be extending their understanding of whole numbers to negative numbers, including comparison and ordering. They also should be working on incorporating ratio, proportion and percent, powers and roots, and scientific notation into their conception of the number system. The key components of number sense, as explained in the K-12 Overview, are an **awareness of the uses of numbers** in the world around us, a good sense of **approximation, estimation, and magnitude**, the **concept of numeration**, and an understanding of **comparisons** and the **equivalence** of different representations and forms of numbers.

Students at this age are capable of categorizing the ways in which numbers are used in our society. One interesting activity is to have them collect data on how numbers appear in a portion of a newspaper, categorize these uses, and then graph their results.

In their work with **numeration**, seventh- and eighth-graders should begin to see mathematics as a coherent body of knowledge. They should begin to see the integers and the rationals as logical and necessary extensions of the whole number system. Only with these extensions can expressions like *3 - 7* and *4 divided by 3* have answers.

In grades 7 and 8, students are focusing on ratios, proportions, and percents, topics which they were just beginning to consider in grades 5 and 6. Their work with these concepts and the relationship of these numbers to fractions, decimals, and whole numbers form the foundation for a very powerful problem solving skill: proportional reasoning. New topics at these grade levels are exponents, roots, and scientific notation. Students should also explore irrational numbers, such as π, square roots of numbers which are not perfect squares, and other decimals which neither end nor repeat.

Students at these grade levels need to continue learning about **equivalence**, but there is a vast array of kinds of equivalence to be considered here. Students focus on the representation of the same quantity using different *types* of numbers and the selection of the appropriate number type given a particular problem context. It is particularly important that students understand the difference between the exact value of a fraction, such as 2/3, and its approximation of .667, especially since they now use calculators routinely. Relationships among decimals, fractions, ratios, and percents comprise the largest emphasis, but work with exponents and roots and their relationship to scientific notation is also a focus in these grades. Number theory provides a rich context for interesting problems in this area. Questions about infinity, division by zero, and primes and composites, combine with discussions about finite and infinite sequences and series and searches for patterns to open up the full richness of a mathematical world.

Estimation should be a routine part of mathematics classes. Students should be regularly engaged in estimating both quantities and the results of operations. They should respond to questions that arise naturally during the course of the class with answers which demonstrate confident and well-conceived use of estimation strategies and sense of number.

Technology plays an important role in number sense at these grade levels. Calculators can be wonderful exploration tools when examining numerical relationships. Many insights about the relationships between fractions and decimals, for instance, can be attained by simply dividing the numerators of fractions by their denominators. Generalizations about what kinds of fractions produce what kinds of decimals start to flow very freely in such open-ended explorations. Computer software can also be very useful. Spreadsheets, for example, can show a great many ratios on the screen at the same time. For example, the five ingredients of a waffle recipe that makes 4 waffles can be listed across the top row of the spreadsheet, with following rows showing how the recipe changes to make 2, 8, and 12 waffles (*Curriculum and Evaluation Standards*, 1989, p. 89).

The topics that should comprise the number sense focus of the seventh and eighth grade mathematics program are:

rational numbers (both positive and negative)
equivalence
integers
ratio, proportion, and percent
exponents, roots, and scientific notation

Standard 6 — Number Sense— Grades 7-8

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

10. Understand money notations, count and compute money, and recognize the decimal nature of United States currency.

- Students can solve a variety of real-world money problems such as: *If you make $750.00 a month, would you rather have a 12% raise or an $85 a month raise?* or *Which sale is better on a $17.00 sweater, a 1/3 off sale, a $5.00 discount, or a 30% discount?*
- Students use time-cards and pay rates to compute weekly wages and deductions for various workers. Issues such as, time-and-a-half for overtime, double time for holidays, and percentages of total wages to be deducted for various taxes all come into play.
- Students investigate the details and make plans to dispose of the proceeds of the $27,000,000 lottery which they just won.

11. Extend their understanding of the number system by constructing meanings for integers, rational numbers, percents, exponents, roots, absolute values, and numbers represented in scientific notation.

- Students develop a *scientific notation Olympics* by creating events like the 9.144×10^3 *centimeter sprint* (100 yard dash) or the 7.272×10^6 *milligram hurl* (shot put).
- Students use a bookkeeping simulation to explore the effect of bills and credits coming into, or going out of, their business. These financial activities are recorded as various actions on positive and negative integers, all affecting the net worth of the business.
- Students view the *Powers of Ten* video, developed to show how one's view of the world is affected by changes in the order of magnitude of one's position.
- Students construct a hypothetical stock portfolio, using $10,000 to "buy" shares of stock, and track the performance of the portfolio and each individual stock day by day.
- Students explore absolute values as the distance between two points on a number line and compare this to subtraction.
- Students measure the circumference and diameter of 15 to 20 round objects, recording the results in a table. They make a scatterplot (with diameter on the horizontal axis) and use a piece of spaghetti to draw the line of best fit. They discuss why π is represented in the drawing as the slope of the line.
- Students construct line segments of varying irrational lengths on a geoboard or dot paper.

For example, the diagonal of a unit square has length $\sqrt{2}$, the diagonal of a 1 x 2 rectangle has length $\sqrt{5}$, and the circumference of a circle with unit diameter has length π. Through exploring these common irrational numbers that arise in problem situations, students learn that not all numbers can be represented as a ratio of two integers.

- Students construct their own graph for the square roots of the numbers from 1 to 25, using trial-and-error to approximate each root to the nearest tenth. They plot the numbers on the horizontal scale, and their square roots on the vertical scale.
- Students work on traditional "systems of equations" problems, involving two unknowns, by devising non-algebraic solution strategies for them. Some samples are: *Two numbers have a sum of 32; they have a product of 240. What are the numbers?* or *Sally is 22 years younger than her Dad. In 3 years, her Dad will be 3 times as old as she. How old is Sally?*

12. Develop number sense necessary for estimation.

- Students wrestle with this classic problem: *After spending most of the day looking for her missing pet cat, Whiskers, the eccentric billionaire, Ms. Money Bags, received a ransom demand. The caller said she was to bring a suitcase packed with $1,000,000 in one- and five-dollar bills to the bus station and leave it in Locker #26-C. Then her pet would be returned to her. How did she respond?*
- Students describe a scale model of the solar system built on the premise that the earth is represented by a ping-pong ball.
- Students make estimates of the number of times various events happen in an average lifetime, discuss their strategies for estimation, and then check their estimates against some reference. A good reference for this activity is *In an Average Lifetime* by Tom Heymann. Among other things, in an average lifetime, an American consumes 10,231 gallons of beverages, spends $1,331 on home-delivered food, and spends 911 hours brushing his or her teeth.

13. Expand the sense of magnitudes of different number types to include integers, rational numbers, and roots.

- Students play *Locate the Point*. A number line with end points -5 and 5 is suspended in the classroom, using a long string with tabs to indicate the positions of the integers between the two end numbers. Students are given cards with different types of numbers on them. (For example: $-12/3$, 1.1, 1.01, $\sqrt{2}$, π, -2^2, $(-2)^2$, $\sqrt{3}$, $\sqrt{8}$, $1.\overline{9}$, 2, $\sqrt[3]{8}$, etc.) They take turns and attach their card on the appropriate spot on the number line. Classmates decide whether the position is correct. If more than one expression is used for the same number, the cards with those numbers are attached by tape.
- Students use only the multiplication and division functions on their calculators to perform a series of successive approximations to find acceptable values for several roots: the square roots of 2, 3, 7, and 10, and the cube roots of 10 and 100.
- Pairs of students play *Hi-Lo* with decimals as a way to emphasize the density of the rational numbers. One student thinks of a number between 0 and 10 with up to 4 decimal

places. The other student tries to guess the number, receiving feedback after each guess as to whether the guess was too high or too low. Written records of the guesses and the feedback are kept. The goal is to find the number using as few guesses as possible.

14. Understand and apply ratios, proportions, and percents in a variety of situations.

- Students take consumer price data from 10 and 25 years ago and figure out the percentage increase or decrease in the prices of various products over those periods of time. They discuss questions such as: *What makes a price go up? What would make it go down?*
- Students predict, and then determine, which body part ratios are fairly constant from person to person. Some interesting ones are height/arm span, wrist circumference/hand span, and waist/neck circumference.
- Students make a three-dimensional model of the classroom with different groups taking responsibility for modeling different objects in the class. First the desired size of the model is discussed and a scale factor agreed upon. Then each of the groups measures and applies that scaling factor to their objects, determines appropriate materials and means of construction, and builds the models.
- Students examine whether it is better to take a discount of 20% and then add a 6% sales tax or add the sales tax and then take a 20% discount. (The answer may surprise the students!)
- Students examine different statements involving proportions and discuss which ones make sense and which do not. For example: *If one girl can mow the yard in 30 minutes, then two girls can mow the yard in 15 minutes. If one boy can walk to school in 20 minutes, then two boys can walk to school in 10 minutes.*
- Students compare magazine subscription prices for 6, 9, and 12 months in order to decide which is the better buy.
- Students estimate what percent of plain M&M's are red, green, yellow, blue, brown, and orange. They test their guesses by counting the number of each color in a small bag and finding the percentages. They also discuss whether they improve their estimates by combining their data.
- Students simulate running a business using the computer program *The Whatsit Corporation* or *Survival Math*.

15. Develop and use order relations for integers and rational numbers.

- Students use concrete and pictorial models to develop order relations among fractions and integers. Using Cuisenaire Rods and varying the *unit*, students demonstrate that one fraction is larger than another. Similar arguments and conclusions are made on a number line for integers.
- Students' abilities to order rational numbers (both positive and negative) are assessed by asking them to identify points on a number line between, say, -3 and -5.
- Students are each given a rational number on a large card (-1.2, 4, 3/4, -2 1/4, -1, 3.14, 22/7, and so on). They then order themselves from least to greatest along the front or side of the classroom. They also respond to instructions like: *Hold up your card if it is*

between -2.5 and +0.7.

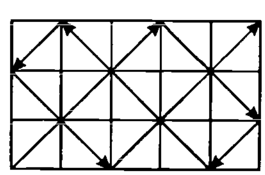
16. Recognize and describe patterns in both finite and infinite number sequences involving whole numbers, rational numbers, and integers.

- Students formulate a description of the *nth* row of Pascal's triangle.
- Students investigate the Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, 21, ...) to see how it is generated and then do library research to find theories about its startling occurrences in nature.
- Students explore the *golden ratio* discovered and used by the ancient Greeks. They find examples of golden rectangles (whose sides are in the golden ratio) in everyday objects (3 x 5 cards, bricks, cereal boxes), and in architecture (the Parthenon).
- Students discuss and predict the sum of this well-known series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

17. Develop and apply number theory concepts, such as primes, factors, and multiples, in real-world and mathematical problem situations.

- Students write a Logo or a BASIC computer program to find all the factors of any number that is provided as input. They can then use the same program to determine if any input number is prime.
- Students explore Goldbach's conjecture (a mathematical hunch which has never been proved nor disproved) which states that: *Any even number greater than two can be written as the sum of two prime numbers. For example: 14 = 11 + 3, 24 = 11 + 13, and 56 = 3 + 53. Can you find one that cannot be written this way?*
- Students develop rules of divisibility for all one-digit numbers and explain and apply these rules on a test.
- Students investigate the path of a ball on a billiard table with sides of whole number length when the ball starts in a corner and always travels at a 45 degree angle. For example, a ball on the 3 x 5 table in the diagram starts in the lower left corner and takes the path shown, hitting the perimeter eight times (including the first and last corners) and going through all 15 squares, before ending at the top right corners. They make a table which records the length and width of the billiard table, the number of hits against the perimeter, and the number of squares passed through, for billiard tables of various sizes, and look for relationships. The number of hits against the perimeter of the table, including the first and last corners, is the sum of the width and the length of the billiard table divided by their greatest common factor.



18. **Investigate the relationships among fractions, decimals, and percents, and use all of them appropriately.**

- Given a circle graph of some interesting data, students estimate the size of each section of the graph as a fraction, a percent, and as a decimal. Students also create their own circle graphs.
- Students use two different-color interlocking paper circles (each has a cut along a radius so they fit together), each marked off in wedges that are one hundredth of the circle, to show fractions that have denominators of 10 or 100, decimals to hundredths, and whole number percents to 100%.
- Students explore patterns in particular families of decimal expansions, such as those for the fractions, *1/7, 2/7, 3/7, ...* or *1/9, 2/9, 3/9, ...* .

19. **Identify, derive, and compare properties of numbers.**

- Students work on this problem from the *Curriculum and Evaluation Standards for School Mathematics* (p. 93): *Find five examples of numbers that have exactly three factors. Repeat for four factors, and then again for five factors. What can you say about the numbers in each of your lists?*
- Students explore *perfect numbers,* those numbers that are equal to the sum of all of their factors including 1 but excluding themselves. Six is the first perfect number, where $6 = 1 + 2 + 3$. Interestingly, the next one has 2 digits, the third has 3 digits and the fourth has 4 digits. The pattern breaks down there, though, since the fifth perfect number has 8 digits. Students who have worked on a computer program to find all of the factors of numbers (see Indicator 17 on the previous page) may want to revise their program to see how many perfect numbers they can find.

References

Heyman, Tom. *On an Average Day*. New York: Fawcett Columbine, 1989.

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA, 1989.

Software

Logo. Many versions of Logo are commercially available.

Survival Math. Sunburst Communications.

The Whatsit Corporation. Sunburst Communications.

Video

Powers of Ten. Philip Morrison, Phylis Morrison, and the office of Charles and Roy Eames. New York: Scientific American Library, 1982.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 6 — Number Sense — Grades 9-12

Overview

High school students build upon their knowledge of rational numbers as they increase their understanding of irrational numbers and generalize number relationships through their work with algebra. The key components of number sense, as identified in the K-12 Overview, are an **awareness of the uses of numbers** in the world around us, a good sense of **approximation, estimation, and magnitude**, the **concepts of numeration**, and an understanding of the **equivalence** of different representations and forms of numbers.

In their work with **numeration**, ninth- through twelfth-graders should view mathematics as a coherent body of knowledge. They should see the integers, the rational numbers, and the real numbers as logical and necessary extensions of the whole number system. Only with these extensions can expressions like *3 minus 7*, *4 divided by 3*, and *the ratio of the circumference of a circle to its diameter* have values. High school students should understand how the integers, rational numbers, irrational numbers, and real numbers are related to each other and what properties are true for the numerical operations on these number systems.

Students at these grade levels continue their learning about **equivalence**, but with an increasing focus on approximation, particularly for irrational numbers (see Standard 14 — Building Blocks of Calculus). High school students should understand the difference between an exact value of an irrational number, such as $\sqrt{3}$, and its approximation (1.728). They should also be familiar with the use of scientific notation as an equivalent form of decimal number.

Estimation continues to be a regular part of mathematics classes, both estimation of quantities and estimation of the results of operations. Students should respond to questions that arise naturally during the course of the class with answers that illustrate confident and well-conceived use of estimation strategies and number sense.

Technology also plays an important role in number sense at these grade levels, particularly since calculators and computers use approximations for some fraction-to-decimal conversions and for irrational numbers. Calculators can be wonderful exploration tools when examining numerical relationships, and computer software which allows exploration of number relationships through conversion utilities and graphs opens up even more possibilities.

The topics that should comprise the number sense focus of the ninth through twelfth grade mathematics program are:

the real number system
exponents, roots, and scientific notation
properties of number systems

Standard 6 — Number Sense — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11, and 12.

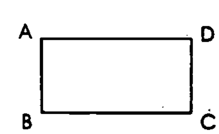
Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

20. Extend their understanding of the number system to include real numbers and an awareness of other number systems.

- Students explore alternative number bases and note their advantages and disadvantages. Binary and hexadecimal systems are of primary interest because of their use in computer programming.
- Students read about and understand the historical relationship between the square root of 2 and the Pythagoreans' attempt to express the length of the diagonal of a square in terms of its sides. They use geoboards to illustrate how the hypotenuse of a right triangle can represent a length of $\sqrt{2}$, $\sqrt{3}$, or $\sqrt{5}$, and so on.
- Students' understanding of real numbers is assessed by asking them to locate rational and irrational numbers on a number line.
- Students find two numbers between any two given real numbers, such as $.4\overline{2}$ and $.\overline{42}$.
- Students can give examples of irrational numbers such as 3.010010001 … or .4323323332 … .

21. Develop conjectures and informal proofs of properties of number systems and sets of numbers.

- Students discuss whether the transitive and reflexive properties hold for different relationships, such as "is a friend of", "is perpendicular to," or "is a factor of."
- Students make up a number system using the symbols ▲, ■, ★, ♥ and ●. They develop algorithms for adding and multiplying within their system and decide whether these operations are commutative and associative.
- Students explore the properties of clock arithmetic or a modular arithmetic system.
- Students examine properties involving addition of matrices, scalar multiplication, and matrix multiplication. They demonstrate that matrix multiplication is not commutative by providing a counter-example.
- Students investigate transformations of the rectangle ABCD: reflection about its the horizontal line of symmetry (H), reflection about its vertical line of symmetry (V), rotation by 180° (R), and rotation by 360° (the identity, I). They construct an operations table



(see below) which tells what happens if one of these transformations is followed by another. Thus, for example, if you reflect about the vertical line of symmetry (V) and then rotate by 180° (R), the result is the same as reflecting about the horizontal line of symmetry (H); this is indicated in the table by placing H as the entry in the row for V and column for R representing the conclusion that *V followed by R is H*. Students investigate the properties of this operation "followed by."

	I	**H**	**V**	**R**
I	**I**	**H**	**V**	**R**
H	**H**	**I**	**R**	**V**
V	**V**	**R**	**I**	**H**
R	**R**	**V**	**H**	**I**

22. Extend their intuitive grasp of number relationships, uses, and interpretations and develop an ability to work with rational and irrational numbers.

- Students create computer spreadsheets to help assess real-world and purely mathematical numerical situations and to ask *what if* questions regarding complex data.
- Students use calculators and the formula for compound interest to answer specific questions regarding the amount of money that will be in a particular bank account after 1, 10, and 100 years.
- Students informally solve maximum/minimum problems with the help of graphing calculators.
- Students use formulas for projectile motion to solve problems regarding distance traveled, time in flight, maximum height, and so on.
- Students compare different representations for π, including 3.14, 22/7, and the value given by their calculators. They discuss the accuracy of each approximation, suggesting appropriate circumstances for its use.
- Students work through the *Ice Cones* lesson that is described in the First Four Standards of this *Framework*. They use graphing calculations to determine the maximum volume of a cone created from a 10 inch circle which is cut along a radius.

23. Explore a variety of infinite sequences and informally evaluate their limits.

- Students explore the value of .99 ... (or $4.9\overline{9}$ or $3.2\overline{9}$) as an infinite series (9/10 + 9/100 + ...), and conclude that its value is exactly 1 (or 5 or 3.3).
- Students analyze and discuss the sums of infinite series such as the following:

  $1 - 1 + 1 - 1 + 1 - 1 + \ldots$
  $(1 - 1) + (1 - 1) + (1 - 1) + \ldots$
  $1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots$

- Students informally find the limits of real-world series such as the total vertical distance traveled by a ball dropped from a height of 10 meters which always bounces back to 3/4 of its original height.
- Students investigate the sums of series such as:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots \quad or \quad \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

- Students estimate the area of a circle with radius of 10 by informally judging the limiting value of the sequence produced by the areas of inscribed regular polygons as their sides increase by 1; that is, they calculate the area of an inscribed equilateral triangle, an inscribed square, an inscribed pentagon, and so on.
- Students discuss *pyramid schemes* and create a mathematical model to determine how many people would have to participate in the scheme for everyone at the fifth level to be paid, for everyone at the tenth level to be paid, and then for everyone who participated at any level to be paid. For example, if each person pays four others, then there are 4^n people at the nth level.

References

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA, 1989.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

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STANDARD 7 — GEOMETRY AND SPATIAL SENSE

K-12 Overview

All students will develop spatial sense and an ability to use geometric properties and relationships to solve problems in mathematics and in everyday life.

Descriptive Statement

Spatial sense is an intuitive feel for shape and space. It involves the concepts of traditional geometry, including an ability to recognize, visualize, represent, and transform geometric shapes. It also involves other, less formal ways of looking at two- and three-dimensional space, such as paper-folding, transformations, tessellations, and projections. Geometry is all around us in art, nature, and the things we make. Students of geometry can apply their spatial sense and knowledge of the properties of shapes and space to the real world.

Meaning and Importance

Geometry is the study of spatial relationships. It is connected to every strand in the mathematics curriculum and to a multitude of situations in real life. Geometric figures and relationships have played an important role in society's sense of what is aesthetically pleasing. From the Greek discovery and architectural use of the golden ratio to M. C. Escher's use of tessellations to produce some of the world's most recognizable works of art, geometry and the visual arts have had strong connections. Well-constructed diagrams allow us to apply knowledge of geometry, geometric reasoning, and intuition to arithmetic and algebra problems. The use of a rectangular array to model the multiplication of two quantities, for instance, has long been known as an effective strategy to aid in the visualization of the operation of multiplication. Other mathematical concepts which run very deeply through modern mathematics and technology, such as symmetry, are most easily introduced in a geometric context. Whether one is designing an electronic circuit board, a building, a dress, an airport, a bookshelf, or a newspaper page, an understanding of geometric principles is required.

K-12 Development and Emphases

Traditionally, elementary school geometry instruction has focused on the categorization of shapes; at the secondary level, it has been taught as the prime example of a formal deductive system. While these perspectives of the content are important, they are also limiting. In order to develop spatial sense, students should be exposed to a broader range of geometric activities at all grade levels.

By virtue of living in a three-dimensional world, having dealt with space for five years, children enter school with a remarkable amount of intuitive geometric knowledge. The geometry curriculum should take advantage of this intuition while expanding and formalizing the students' knowledge. In early elementary school, a rich, qualitative, hands-on study of geometric objects helps young children develop spatial sense and a strong intuitive grasp of geometric properties and relationships. Eventually they develop a comfortable vocabulary of appropriate geometric terminology. In the middle school years, students should begin to use their knowledge in a more analytical manner to solve problems, make conjectures, and look for patterns and generalizations. Gradually they develop the ability to make inferences and logical deductions based on geometric relationships and to use spatial intuition to develop more generic mathematical problem-solving skills. In high school, the study of geometry expands to address coordinate, vector, and transformational viewpoints which utilize both inductive and deductive reasoning. Geometry instruction at the high school level should not be limited to formal deductive proof and simple measurement activities, but should include the study of geometric transformations, analytic geometry, topology, and connections of geometry with algebra and other areas of mathematics.

At all grade levels, the study of geometry should make abundant use of experiences that require active student involvement. Constructing models, folding paper cutouts, using mirrors, pattern blocks, geoboards, and tangrams, and creating geometric computer graphics all provide opportunities for students to learn by doing, to reflect upon their actions, and to communicate their observations and conclusions. These activities and others of the same type should be used to achieve the goals in the seven specific areas of study that constitute this standard and which are described below.

In their study of **spatial relationships**, young students should make regular use of concrete materials in hands-on activities designed to develop their understanding of objects in space. The early focus should be the description of the location and orientation of objects in relation to other objects. Additionally, students can begin an exploration of symmetry, congruence, and similarity. Older students should study the two-dimensional representations of three-dimensional objects by sketching shadows, projections, and perspectives.

In the study of **properties of geometric figures**, students deal explicitly with the identification and classification of standard geometric objects by the number of edges and vertices, the number and shapes of the faces, the acuteness of the angles, and so on. Cut-and-paste constructions of paper models, combining shapes to form new shapes and decomposing complex shapes into simpler ones are useful exercises to aid in exploring shapes and their properties. As their studies continue, older students should be able to understand and perform classic constructions with straight edges and compasses as well as with appropriate computer software. Formulating good mathematical definitions for geometric shapes should eventually lead to the ability to make hypotheses concerning relationships and to use deductive arguments to show that the relationships exist.

The standard **geometric transformations** include translation, rotation, reflection, and scaling. They are central to the study of geometry and its applications in that these movements offer the most natural approach to understanding congruence, similarity, symmetry, and other geometric relationships. Younger children should have a great deal of experience with *flips, slides,* and *turns* of concrete objects, figures made on geoboards, and drawn figures. Older students should be able to use more formal terminology and procedures for determining the results of the standard transformations. An added benefit of experience gained with simple and composite transformations is the mathematical connection that older students can make to functions and function composition.

Coordinate geometry provides an important connection between geometry and algebra. Students can work informally with coordinates in the primary grades by finding locations in the room, and by studying simple maps of the school and neighborhood. In later elementary grades, they can learn to plot figures on a coordinate plane, and still later, study the effects of various transformations on the coordinates of the points of two- and three-dimensional figures. High-school students should be able to represent geometric transformations algebraically and interpret algebraic equations geometrically.

Measurement and geometry are interrelated, and an understanding of the **geometry of measurement** is necessary for the understanding of measurement. In elementary school, students should learn the meaning of such geometric measures as length, area, volume and angle measure and should be actively involved in the measurement of those attributes for all kinds of two- and three-dimensional objects, not simply the standard ones. Throughout school, they should use measurement activities to reinforce their understanding of geometric properties. All students should use these experiences to help them understand such principles as the quadratic change in area and cubic change in volume that occurs with a linear change of scale. Trigonometry and its use in making indirect measurements provides students with another view of the interrelationships between geometry and measurement.

Geometric modeling is a powerful problem-solving skill and should be used frequently by both teachers and students. A simple diagram, such as a pie-shaped graph, a force diagram in physics, or a dot-and-line illustration of a network, can illuminate the essence of a problem and allow geometric intuition to aid in the approach to a solution. Visualization skills and understanding of concepts will both improve as students are encouraged to make such models.

The relationship between geometry and **deductive reasoning** originated with the ancient Greek philosophers, and remains an important part of the study of geometry. A key ingredient of deductive reasoning is being able to recognize which statements have been justified and which have been assumed without proof. This is an ability which all students should develop in all areas, not just geometry, or even just mathematics! At first, deductive reasoning is informal, with students inferring new properties or relationships from those already established, without detailed explanations at every step. Later, deduction becomes more formal as students learn to use all permissible assumptions in a proof and as all statements are systematically justified from what has been assumed or proved before. The idea of deductive proof should not be confused with the specific two-column format of proof found in most geometry textbooks. The reason for studying deductive proof is to develop reasoning skills, not to write out arguments in a particular arrangement. Note that proof by mathematical induction is another deductive method that should not be neglected.

Much of the current thinking about the development of geometric thinking in students comes from the work of a pair of Dutch researchers, Pierre van Hiele and Dina van Hiele-Geldof. Their model of geometric thinking identifies five levels of development through which students pass when assisted by appropriate instruction.

- Visual recognition of shapes by their appearances as a whole (level 0)
- Analysis and description of shapes in terms of their properties (level 1)
- Higher "theoretical" levels involving informal deduction (level 2)
- Formal deduction involving axioms and theorems (level 3)
- Work with abstract geometric systems (level 4).

(Geddes & Fortunato, 1993)

Although the levels are not completely separate and the transitions are complex, the model is very useful for characterizing levels of students' thinking. Consistently, the research shows that appropriately targeted instruction is critical to children's movement through these levels. Stagnation at early levels is the frequent result of a geometry curriculum that never moves beyond identification of shapes and their properties. The discussion in this K-12 Overview draws on this van-Hiele model of geometric thinking.

IN SUMMARY, students of all ages should recognize and be aware of the presence of geometry in nature, in art, and in human-built structures. They should realize that geometry and geometric applications are all around them and, through study of those applications, come to better understand and appreciate the role of geometry in life. Carpenters use triangles for structural support, scientists use geometric models of molecules to provide clues to understanding their chemical and physical properties, and merchants use traffic-flow diagrams to plan the placement of their stock and special displays. These and many, many more examples should leave no doubt in students' minds as to the importance of the study of geometry.

NOTE: Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.

Reference

Geddes, Dorothy, & Fortunato, Irene. "Geometry: Research and Classroom Activities," in D. T. Owens, Ed., *Research Ideas for the Classroom: Middle Grades Mathematics*. New York: Macmillan, 1993.

Standard 7 — Geometry and Spatial Sense — Grades K-2

Overview

Students can develop strong spatial sense from consistent experiences in classroom activities that use a variety of manipulatives and technology. The key components of spatial sense, as identified in the K-12 Overview, are spatial relationships, properties of geometric figures, geometric transformations, coordinate geometry, geometry of measurement, geometric modeling, and reasoning.

In kindergarten through second grade, the emphasis is on qualitative, not quantitative, properties of geometric objects. Students are at the visualization level of geometric thinking, where they perceive figures as "wholes". They recognize squares and rectangles, but perhaps not that squares are a special case of rectangles. To enrich and develop their geometric thinking, children at these grade levels need to explore geometry using a variety of physical objects, drawings, and computer tools. They work with solids, pattern blocks, templates, geoboards, and computer drawing tools to develop their understanding of geometric concepts and their spatial sense. They construct models and drawings to experiment with orientation, position, and scale, and to develop visualization skills. Students begin to develop a geometric vocabulary. A sample unit on geometry for the second-grade level can be found in Chapter 17 of this *Framework*.

In their study of **spatial relationships**, students focus on developing their understanding of objects in space. They discuss and describe the relative positions of objects using phrases like "in front of" and "on top of." They describe and draw three-dimensional objects in different relative locations. They compare and contrast shapes, describing the shapes of the faces and bases of three-dimensional figures. They discuss symmetry and look for examples of symmetry in their environment. They look for shapes that are the same size and shape (congruent) or the same shape but different sizes (similar). They use mirrors to explore symmetry.

In beginning their study of **properties of geometric figures**, students look for shapes in the environment, make models from sticks and clay or paper and glue, and draw shapes. They sort objects according to shape. They recognize, classify, sort, describe, and compare geometric shapes such as the sphere, cylinder, cone, rectangular solid, cube, square, circle, triangle, rectangle, hexagon, trapezoid, and rhombus. They describe the angle at which two edges meet in different polygons as being smaller than a right angle, a right angle, or larger than a right angle. They discuss points, lines, line segments, intersecting and non-intersecting lines, and midpoints of lines.

Students begin looking at **geometric transformations** by using concrete materials such as paper dolls to model slides (translations), flips (reflections), and turns (rotations). Students put shapes together to make new shapes and take shapes apart to form simpler shapes. Students work on spatial puzzles, often involving pattern blocks or tangrams. They look for plane shapes in complex drawings and explore tilings. They divide figures into equal fractional parts, for example, by folding along one, two, or three lines.

Coordinate geometry in grades K-2 involves describing the motion of an object. Students make maps of real, imaginary, or storybook journeys. They describe the location of an object on a grid or a point in a plane using numbers or letters. They give instructions to an imaginary "turtle" to crawl around the outline of a figure.

Students in these grades also begin to explore the **geometry of measurement**. In kindergarten, students discuss and describe quantitative properties of objects using phrases like "bigger" or "longer." They order objects by length or weight. In first and second grade, they quantify properties of objects by counting and measuring. They determine the areas of figures by cutting them out of grid paper and counting the squares. They measure the perimeter of a polygon by adding the lengths of all of the sides.

Students begin to explore **geometric modeling** by constructing shapes from a variety of materials, including toothpicks and clay, paper and glue, or commercial materials. They use templates to draw designs, and record what they have constructed out of pattern blocks and tangrams. They fold, draw, and color shapes. They copy geoboard figures, and construct them both from memory and by following oral or written instructions. They may also use geometric models, such as the number line, for skip counting or repeated addition.

Geometry provides a rich context in which to begin to develop students' **reasoning** skills. Students apply thinking skills in geometric tasks from identifying shapes to discovering properties of shapes, creating geometric patterns, and solving geometric puzzles and problems in a variety of ways. They create, describe, and extend geometric patterns. They use attribute blocks to focus on the similarities and differences of objects.

Geometry provides a unique opportunity to focus on the First Four Standards, especially Standard 2 which stresses the importance of making connections to other mathematical topics. For example, students often use their understanding of familiar shapes to help build an understanding of fractions. Teachers in grades K-2 need to plan classroom activities that involve several mathematical processes and relate geometry to other topics in mathematics. Geometry should not be taught in isolation; it should be a natural and integrated part of the entire curriculum.

Standard 7 — Geometry and Spatial Sense — K-2

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Explore spatial relationships such as the direction, orientation, and perspectives of objects in space, their relative shapes and sizes, and the relations between objects and their shadows or projections.**

    - Blindfolded students are given real objects to touch and then, with the blindfolds removed, select the objects from a collection of visible objects.
    - Students work through the *Will a Dinosaurs Fit?* lesson that is described in the First Four Standards of this *Framework*. They discuss the size of the different dinosaurs and arrange them from smallest to largest.
    - Students predict what shape will result when a small piece is cut out of a folded piece of paper in different ways (along a diagonal, across a fold or a corner, or in the center) and the paper is then unfolded.
    - Students compare the sequence of objects seen from different points of view. For example, from the classroom window, the swings are to the left of the monkeybars, but the relationship is reversed if the objects are viewed from the blacktop facing the classroom.
    - Students predict and draw what the shadow of an object placed between a light and a screen will look like.

2. **Explore relationships among shapes, such as congruence, symmetry, similarity, and self-similarity.**

    - Students look for examples of congruent figures (same size and shape) in the environment.
    - Students explore symmetry by using mirrors with pattern blocks or by folding paper or by making inkblot designs. Students find the lines of symmetry in the letters of the alphabet and in numerals. They fold paper and cut out symmetric designs. They identify the symmetry in wallpaper or giftwrap designs.
    - Students use different size dolls and action figures as an introduction to the concept of similarity (same shape, different size).

3. **Explore properties of three- and two-dimensional shapes using concrete objects, drawings, and computer graphics.**

    - Students predict what shape they will see when they make various impressions of 3-dimensional objects in sand. For example, the top of a cylinder forms a circle, its side forms a rectangle.

- Students outline a triangle, a square, and a circle on the floor with string or tape. Then they walk around each figure, chanting a rhyme, such as "Triangle, triangle, triangle, 1, 2,3, I can walk around you as easy as can be," and counting the sides as they walk.
- Students work through the *Shapetown* lesson that is described in the First Four Standards of this *Framework*. They explore properties of two-dimensional shapes by applying the fundamental concepts of Venn diagrams.
- Some students use *Muppet™ Math* to work with Kermit's geometric paintings, while others use *Shape Up!* to compare everyday objects to geometric shapes.

4. Use properties of three- and two-dimensional shapes to identify, classify, and describe shapes.

- Students make shapes with their fingers and arms.
- Students listen to and look at the book *The Shapes Game* by Paul Rogers. Each page shows a different shape and many of the things in the world that have that shape. As each page is read, the children find other objects in the room that have the same shape.
- Students listen to and draw illustrations for the story *The Greedy Triangle* by Marilyn Burns.
- A good open-ended assessment for this critical indicator is to ask students to sort a collection of shapes into groups, explaining their reasoning. Some groups they might consider include "all right angles" or "four-siders." The teacher should encourage the students to invent appropriate group names and to use informal language to describe the properties, and should record the students' responses to look for progress over time.
- A more traditional, but still useful, assessment strategy is to ask students to sort pictures cut from magazines according to shape. This more focused task will generate information about the students' ability to recognize and differentiate among shapes.
- Students make class books shaped like a triangle, a rectangle, a square, and a circle. They fill each book with pictures of objects that have the shape of the book.
- Students turn a geometric shape into a picture. For example, a triangle might become a tower, a clown face, or the roof of a house.

5. Investigate and predict the results of combining, subdividing, and changing shapes.

- Students use tangram pieces to construct triangles, rectangles, squares, and other shapes.
- Students investigate which pattern block shapes can be formed from the equilateral triangles, recording their results in pictures and on a chart.
- Students work in groups to decide how to divide a rectangular candy bar among four people. The students then compare the various ways that each group solved the problem.

6. Use tessellations to explore properties of geometric shapes and their relationships to the concepts of area and perimeter.

- Students use Unifix cubes or pattern blocks to create colorful designs. They then discuss how many blocks they used (area) and the distance around their design (perimeter). They also discuss why these polygon shapes fit together like a puzzle.

- Students use different shapes to make quilt patterns.
- During free play time, students use pattern blocks to make different space-filling designs. They record any patterns that they especially like, using templates or drawing around the blocks.

7. Explore geometric transformations such as rotations (turns), reflections (flips), and translations (slides).

- Students look at the world around them for examples of changes in position that do not change size or shape. For example, a child going down a slide illustrates a slide, a merry-go-round or hands on a clock illustrate a turn, and a mirror illustrates a flip.
- Students look through and discuss the no-text book *Changes, Changes* by Pat Hutchins. In it, a man and a woman use the same set of building blocks to transform a house into a fire engine, then a boat, a truck, and back to a house. The students tell the story and then draw pictures to show how the blocks changed from one object to another.
- Students investigate the shapes that they can see when they place a mirror on a square pattern block.

8. Develop the concepts of coordinates and paths, using maps, tables, and grids.

- Students use maps of their community to find various ways to get from school to the park. They use letters and numbers to describe the location of the school and that of the park.
- Students create a map based on the familiar story of *The Little Gingerbread Man*, showing where each of the people in the story lives.
- Students describe how to get from one point in the school to another and try to follow each others' directions.

9. Understand the variety of ways in which geometric shapes and objects can be measured.

- In connection with a unit on dinosaurs in science, students discuss the different ways in which the size of dinosaurs can be described. They decide to measure the size of a dinosaur's footprint in two ways: by using string to measure the distance around it and by using base ten blocks to measure the space inside it.
- Pairs of students investigate the many different designs that they can make using unit squares and 1/2-unit right triangles. They record their results on dot paper.

10. Investigate the occurrence of geometry in nature, art, and other areas.

- Students take a "geometry walk" through their school or their neighborhood, looking for examples of specific shapes and concepts.
- Students create geometric patterns using potato prints.
- Students decorate their classroom for the winter holidays using geometric shapes.
- Students examine and discuss geometric patterns found in works of art.

References

Burns, Marilyn. *The Greedy Triangle*. New York: Scholastic, Inc., 1994.

Hutchins, Pat. *Changes, Changes*. New York: MacMillan, 1987.

Rogers, Paul. *The Shapes Game*. New York: Henry Holt and Company, 1989.

Software

Muppet™ Math. Jim Henson Productions. Sunburst Communications.

Shape Up! Sunburst Communications.

General Reference

Burton, G. and T. Coburn. *Curriculum and Evaluation Standards for School Mathematics: Addenda Series: Kindergarten Book*. Reston, VA: National Council of Teachers of Mathematics, 1991.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 7 — Geometry and Spatial Sense — Grades 3-4

Overview

Students can develop strong spatial sense from consistent experiences in classroom activities that use a wide variety of manipulatives and technology. The key components of spatial sense, as identified in the K-12 Overview, are spatial relationships, properties of geometric figures, geometric transformations, coordinate geometry, geometry of measurement, geometric modeling, and reasoning.

In third and fourth grade, students are beginning to move beyond recognizing whole shapes to analyzing the relevant properties of a shape. They continue to use their own observations about shapes and the relations among these shapes in the physical world to build understanding of geometric concepts. Thus, using manipulative materials to develop geometric concepts and spatial sense remains important at these grade levels. Exploring concepts in a number of different contexts helps students to generalize. Students are extending their understanding of cause and effect and their ability to make conjectures. They are particularly interested in *Why?* Questions such as *Why are most rooms shaped like rectangles?* offer interesting points of departure for studying geometric concepts. Connections among geometry, spatial sense, other areas of mathematics, and other subject areas provide many opportunities for students to see how mathematics fits into their lives.

With respect to **spatial relationships**, students in these grade levels continue to examine direction, orientation, and perspectives of objects in space. They are aware of the relative positions of objects; you might ask *Which walls are opposite each other? What is between the ceiling and the floor?* Students also expand their understanding of congruence, similarity, and symmetry. They can identify congruent shapes, draw and identify a line of symmetry, and describe the symmetries found in nature. They search for examples in nature where each part of an item looks like a miniature version of the whole (self-similar).

Students are extending their understanding of **properties of geometric figures**. Now they are ready to discuss these more carefully and to begin relating different figures to each other. By experimenting with concrete materials, drawings, and computers, they are able to discover properties of shapes and to make generalizations like *all squares have four equal sides*. They use the language of properties to describe shapes and to explain solutions for geometric problems, but they are not yet able to deduce new properties from old ones or consider which properties are necessary and sufficient for defining a shape. They recognize the concepts of point, line, line segment, ray, plane, intersecting lines, radius, diameter, inside, outside, and on a figure. They extend the shapes they can identify to include ellipses, pentagons, octagons, cubes, cylinders, cones, prisms, pyramids, and spheres.

Students continue to explore **geometric transformations**. Using concrete materials, pictures, and computer graphics, they explore the effects of transformations on shapes.

Using coordinate geometry students create and interpret maps, sometimes making use of information found in tables and charts. Some grids use only numbers at these grade levels, while others use a combination of letters and numbers.

The geometry of measurement begins to take on more significance in grades 3 and 4, as students focus more on the concepts of perimeter and area. Students learn different ways of finding the perimeter of an object: using string around the edge and then measuring the length of the string, using a measuring tape, measuring the length of each side and then adding the measures together, or using a trundle wheel. They also develop non-formula-based strategies for finding the area of a figure.

Geometric modeling allows students to approach topics visually. For example, geometric shapes allow students to build an understanding of fraction concepts as they cut the shapes into congruent pieces. They can use the problem solving skill of drawing geometric diagrams, such as a polygon with its diagonals, to find out how many matches are played in a round robin tournament. They continue to build three-dimensional models of shapes, to draw two- and three-dimensional shapes with increasing accuracy, and to use computers to help them analyze geometric properties.

Students' use of **reasoning** continues to provide opportunities to connect geometry to Standards 1 - 4, to other areas of mathematics, to other disciplines, and to the real world. Students explain how they have approached a particular problem, share results with each other, and justify their answers.

Students in third and fourth grade are still dealing with geometry in a qualitative way but are beginning to adopt more quantitative points of view. They are able to use their natural curiosity about the world to expand their understanding of geometric concepts and spatial sense.

Standard 7 — Geometry and Spatial Sense — Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Explore spatial relationships such as the direction, orientation, and perspectives of objects in space, their relative shapes and sizes, and the relations between objects and their shadows or projections.**

   - Students compare the sizes of the many shapes found in the classroom, such as the heights of students or the areas of their hands.
   - The teacher holds up a shape or describes a shape. Students locate this shape hidden in a box or bag containing a number of shapes, without looking at the shapes.
   - Students explore what happens to the shadow of a square when it is held at various angles to a beam of light. They continue their investigation with other two- and three-dimensional figures.
   - At half-hour intervals, students measure the length of the shadow of a stick stuck vertically into the ground.
   - Students trace the faces of a solid on a transparency and then challenge each other to identify the solid. They check their guess by bringing the solid to the overhead projector and placing it on each face in turn.
   - Students read *Ellipse* by Mannis Charosh. This one-concept book illustrates ellipses in all of their possible orientations and describes a variety of experiments that the students can perform to better understand the role of perspective in geometry.
   - Students predict the positions of three students from different points of view (perspective). For example, from the front of the room, they might see Joe on the left, Rhonda in the middle, and Carly on the right. From the back of the room, the positions would be reversed. Students find a perspective from which Rhonda is on the left, Carly is in the middle and Joe is on the right.

2. **Explore relationships among shapes, such as congruence, symmetry, similarity, and self-similarity.**

   - Students make a collection of natural shapes, including a wide variety of three-dimensional shapes such as fruits and vegetables, shells, flowers, and leaves. They describe the symmetry found in these shapes.
   - Students find objects that exhibit self-similarity, i.e., that contain copies of a basic motif which is repeated at smaller sizes of the same shape. Examples of such objects are

feathers, the shape of a coastline, chambered nautilus shells, and plants which branch out such as cauliflower, broccoli, Queen Anne's lace, and ferns.

- Students look for examples of congruent figures (same size and shape) in the environment.
- Students use scale models of cars and airplanes to study similarity. They recognize that figures that have the same shape but different sizes are similar.

3. **Explore properties of three- and two-dimensional shapes using concrete objects, drawings, and computer graphics.**

   - Students look for a "Shape of the Day" throughout the school day, recording the number of times that the shape is seen.
   - Students look for lines in the classroom, identifying pairs of lines that are parallel, that intersect, or that are perpendicular.
   - Students use the computer language Logo to describe the path made by a turtle as it goes around different geometric shapes.

4. **Use properties of three- and two-dimensional shapes to identify, classify, and describe shapes.**

   - Students make a chart or bar graph showing how many squares, rectangles, triangles, etc., they find in their classroom.
   - Students "walk" a shape and have other students guess the shape.
   - Students classify shapes according to whether they contain right angles only, all angles smaller than a right angle, or at least one angle larger than a right angle.
   - One student thinks of a shape. The others ask questions about its properties, trying to guess it. For example, *Does it have a right angle?*

5. **Investigate and predict the results of combining, subdividing, and changing shapes.**

   - Students investigate the shapes found in their lunches and then discuss how the shapes change as they nibble away. For example: *Can you change a four-sided sandwich into a triangle?*
   - Students investigate how to use four triangles from the pattern blocks to make a large triangle, a four-sided figure, and a six-sided figure.
   - Students combine tangram pieces to create a variety of shapes.

6. **Use tessellations to explore properties of geometric shapes and their relationships to the concepts of area and perimeter.**

   - Students use square, triangular, and hexagonal grid paper to create colorful designs. They discuss why these polygon shapes fit together like a puzzle.
   - Students use Unifix cubes or pattern blocks to create designs. They then discuss how many blocks they used (area) and the distance around their design (perimeter).
   - Students work through the *Tiling a Floor* lesson that is described in the First Four

Standards of this *Framework*. They discover that the shapes which can be used for tiling must be able to fit around a point without leaving spaces and without overlapping.

7. **Explore geometric transformations such as rotations (turns), reflections (flips), and translations (slides).**

   - Students use stuffed animals or two-sided paperdolls to show movements in the plane: slides, flips, and turns. They discuss how all slides (or flips or turns) are alike.
   - Students create borders from a single simple design element which is repeated using slides, flips, and turns.
   - Students study and describe the use of transformations in Pennsylvania Dutch hex signs, and then they design their own.
   - Students discuss transformations found in nature, such as the symmetry in the wings of a butterfly (a flip), the way a honeycomb is formed (slides of hexagons), or the petals of a flower (turns).
   - Students create quilt designs by using geometric transformations to repeat a basic pattern.

8. **Develop the concepts of coordinates and paths, using maps, tables, and grids.**

   - Students create Logo procedures for drawing rectangles or other geometric figures.
   - A good interdisciplinary assessment in both reading and mathematics is to have students draw maps for stories they have read, using coordinates to identify the locations of critical events or objects in the story.
   - Students find the lengths of paths on a grid, such as the distance from Susan's house to school.

9. **Understand the variety of ways in which geometric shapes and objects can be measured.**

   - Students discuss how to describe the size of a truck. Some suggestions include the length of the truck, its height (very important to know when it passes under another road), its cargo capacity (volume), or its weight (important for assessing taxes).
   - Each pair of students is given a pattern to cut out of oaktag and fold up into a three-dimensional shape. They are asked to measure the shape in as many ways as they can. They report their findings to the class.

10. **Investigate the occurrence of geometry in nature, art, and other areas.**

    - Students investigate the natural shapes that are produced by the processes of growth and physical change. They identify some of the simple basic shapes that occur over and over again in more complex structures. Students bring examples to class and describe the process in writing. Some interesting examples are honeycombs, pinecones, and seashells.
    - Students make a bulletin board display of "Shapes in the World Around Us."
    - Students read the beautifully illustrated book *Listen to a Shape* by Marcia Brown. The color photographs in the book move from the occurrence in nature of simple shapes to more complex ones. Children can be asked to describe and draw their favorite shapes in

nature as a follow-up.

- Students read *Shapes* by Phillip Yenawine. This carefully selected collection of works from the Museum of Modern Art is analyzed to show how shapes contribute to the images on the canvas. An interesting open-ended assessment activity would be to ask the students to create their own works of art, combining the geometric shapes they know to make similar striking images.

References

Brown, Marcia. *Listen to a Shape*. New York: Franklin Watts, 1979.

Charosh, Mannis. *Ellipse*. New York: Thomas Y. Crowell, 1971.

Yenawine, Phillip. *Shapes*. New York: Delacourte Press, 1991.

Software

Logo. Many versions of Logo are commercially available.

General reference

Burton, G. et al. *Curriculum and Evaluation Standards for School Mathematics: Addenda Series: Third-Grade Book*. Reston, VA: National Council of Teachers of Mathematics, 1992.

On-Line Resources

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Standard 7 — Geometry and Spatial Sense — Grades 5-6

Overview

Students can develop strong spatial sense from consistent experiences in classroom activities that use a wide variety of manipulatives and technology. The key components of spatial sense, as identified in the K-12 Overview, are spatial relationships, properties of geometric figures, geometric transformations, coordinate geometry, geometry of measurement, geometric modeling, and reasoning.

Informal geometry and spatial visualization are vital aspects of a mathematics program for grades 5 and 6. Middle school students experience the fun and challenge of learning geometry through creating plans, building models, drawing, sorting and classifying objects, and discovering, visualizing, and representing concepts and geometric properties. Students develop the understanding needed to function in a three-dimensional world through explorations and investigations organized around physical models.

Studying geometry also provides opportunities for divergent thinking and creative problem solving while developing students' logical thinking abilities. Geometric concepts and representations can help students better understand number concepts while being particularly well-suited for addressing the First Four Standards: problem solving, reasoning, making connections, and communicating mathematics.

Students' experiences in learning geometry should help them perceive geometry as having a dynamically important role in their environment and not merely as the learning of vocabulary, memorizing definitions and formulas, and stating properties of shapes. Students, working in groups or independently, should explore and investigate problems in two and three dimensions, make and test conjectures, construct and use models, drawings, and computer technology, develop spatial sense, use inductive and deductive reasoning, and then communicate their results with confidence and conviction. They should be challenged to find alternative approaches and solutions.

In their study of **spatial relationships**, students in grades 5 and 6 further develop their understanding of projections (e.g., top, front, and side views), perspectives (e.g., drawings made on isometric dot paper), and maps. They also consolidate their understanding of the concepts of symmetry (both line and rotational), congruence, and similarity.

Students expand their understanding of **properties of geometric figures** by using models to develop the concepts needed to make abstractions and generalizations. They focus on the properties of lines and planes as well as on those of plane and solid geometric figures. Students at this age begin to classify geometric figures according to common properties and develop informal definitions.

Still using models, drawings, and computer graphics, students expand their understanding of **geometric transformation**, including translations (slides), reflections (flips), rotations (turns), and dilations (stretchers/shrinkers). At these grade levels, the connections between transformations and congruence, similarity, and symmetry are explored. Students also begin to use **coordinate geometry** to show how figures change orientation but not shape under transformations. For these investigations they use all four quadrants of the coordinate plane (positive and negative numbers).

Students develop greater understanding of the **geometry of measurement** as they develop strategies for finding perimeters, areas (of rectangles and triangles), volumes, surface areas, and angle measures. The emphasis at this level should be on looking for different ways to find an answer, not simply on using formulas. Students use models for many problems, look for patterns in their answers, and form conjectures about general methods that might be appropriate for certain types of problems. Students apply what they are learning about areas to help them develop an understanding of the Pythagorean Theorem.

Students continue to use **geometric modeling** to help them solve a variety of problems. They explore patterns of geometric change as well as those involving number patterns. They use geometric representations to assist them in solving problems in discrete mathematics. They use concrete materials, drawings, and computers to help them visualize geometric patterns.

Students in these grade levels are beginning to develop more sophisticated **reasoning** skills. In studying geometry, they have many opportunities to make conjectures based on data they have collected and patterns they have observed. This inductive reasoning can then be related to what they already know; students should be encouraged to explain their thinking and justify their responses.

Throughout fifth and sixth grade, students use concrete materials, drawings, and computer graphics to increase the number of geometric concepts with which they are familiar and to explore how these concepts can be used in geometric reasoning. Students' natural curiosity about the world provides ample opportunities for linking mathematics with other subjects. The continued experience with two- and three-dimensional figures provided at these grade levels helps students build the firm foundation needed for the more formal geometry of the secondary school.

Standard 7 — Geometry and Spatial Sense — Grades 5-6

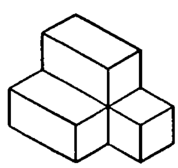
Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

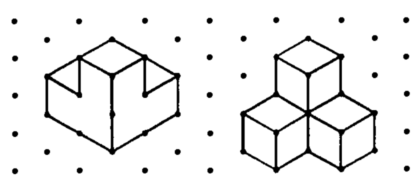
11. Relate two-dimensional and three-dimensional geometry using shadows, perspectives, projections, and maps.

- Students use centimeter cubes to construct a building such as the one pictured below. They then represent their building by drawing the base and telling how many cubes are stacked in each portion of building.

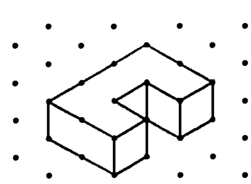


2	2	1
1	1	

- Students put three or four cubes together to make a solid and draw two different projective views of the solid on triangle dot paper, such as those shown below.



- Students copy pictures of solids drawn on triangle dot paper such as the one below, build the solids, and find their volumes.



- Students use circles and rectangles to make 3-dimensional models of cylinders, cones, prisms, and other solids.

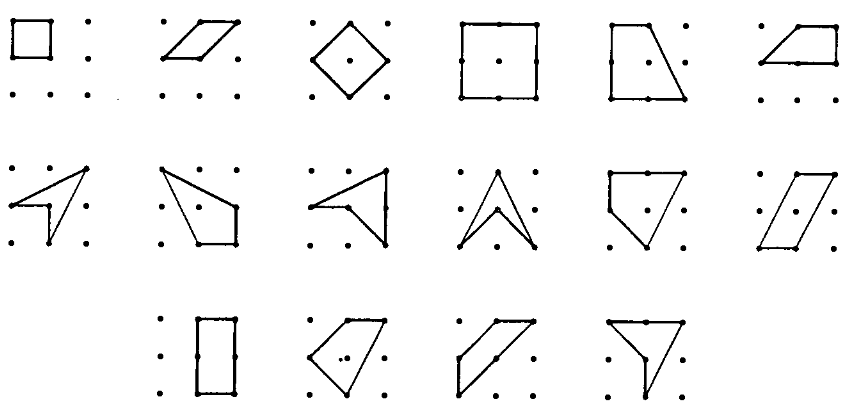
- Students predict and sketch the shapes of the faces of a pyramid, or, given a flat design for a box, predict what it will look like when put together.

12. **Understand and apply the concepts of symmetry, similarity, and congruence.**

- Students compare different Logo procedures for drawing similar rectangles.
- Students look for examples of congruent figures (same size and shape) in the environment.
- Students explore symmetry by looking at the designs formed by placing a mirror on a pattern block design somewhere other than the line of symmetry, or by folding paper more than one time. They identify the symmetry in wallpaper or giftwrap designs. They also identify the rotational symmetry found in a pinwheel (e.g., 90°). (The figure matches itself by turning rather than by flipping or folding.)
- Students build scale models to investigate similarity. They recognize that figures which have the same shape but different sizes are similar.

13. **Identify, describe, compare, and classify plane and solid geometric figures.**

- Students are given a sheet of 3 x 3 dot paper grids. They find and draw as many noncongruent quadrilaterals as they can, using a different set of nine dots for each figure; altogether sixteen different quadrilaterals (pictured below) can be formed.



A nice open-ended approach to assessment of their understanding and comfort with properties of geometric figures is to ask them to sort these quadrilaterals in different ways, including concave vs. convex, by angle sizes, by area, by symmetry, and so on. See how many ways they can devise.

- Students trace a figure onto several transparencies; figures such as squares, rectangles, parallelograms, rhombuses, trapezoids, kites, and arrowheads can be used. Then they draw the lines of symmetry for the figure. They rotate, translate, and flip the transparencies and compare them to an original transparency to investigate such properties as: the number of congruent sides in the figure, the number of parallel sides in the figure, whether the diagonals are congruent, whether the diagonals bisect each other, whether the diagonals are perpendicular, and whether the figure has half-turn symmetry (180°). They write about their findings and explain their reasoning.

- Students use Logo to investigate the sum of the measures of the exterior angles of any polygon (360°) and the angle measure of each exterior angle of a regular polygon.
- Students select straws cut to five different lengths (for example, from one inch to five inches) and form as many different triangles as they can, recording the results. They sort the triangles into groups with 0, 2, or 3 equal sides and label the groups as scalene, isosceles, and equilateral triangles.

14. Understand the properties of lines and planes, including parallel and perpendicular lines and planes, and intersecting lines and planes and their angles of incidence.

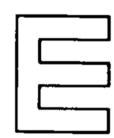
- Students use index cards with slits cut in them to build models of two planes that are parallel or two planes that intersect (in a line).
- Students use toothpicks to explore how two lines might be related to each other (parallel, intersecting, perpendicular, the same line).
- Students find examples of parallel lines and planes, perpendicular lines and planes, and intersecting lines and planes with different angles in their environment.

15. Explore the relationships among geometric transformations (translations, reflections, rotations, and dilations), tessellations (tilings), and congruence and similarity.

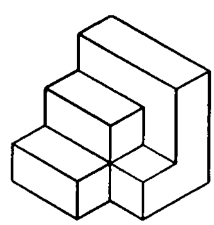
- Students read and examine *The World of M.C. Escher* or any other collection of M.C. Escher's work to find and describe the tessellations in them. Transformations of tessellating polygons are then performed by the students to make their own artwork.
- Students create a design on a geoboard, sketch their design, move the pattern to a new spot by using a specified transformation, and sketch the result.
- Students investigate wallpaper, fabric, and gift wrap designs. They create a template for a unit figure which they will use to create individual border designs for their classroom. Each student presents her/his design to the class, describing the transformations used to create the design.
- Working in small groups, students tile a portion of their desktop using oak-tag copies they have cut of a shape they have created by taping together two pattern blocks. Each group presents its results. The teacher then asks the students to compare the results of the different groups and identify examples of the different transformations used.
- Students investigate how transformations affect the size, shape, and orientation of geometric figures. A reflection or flip is a mirror image. A translation or slide moves a figure a specified distance and direction along a straight line. A rotation or turn is a turning motion of a specified amount and direction about a fixed point, the center. These transformations do not change the size and shape of the original figure. However, a dilation enlarges (stretches) or reduces (shrinks) a figure, producing a new figure with the same shape but a different size.
- Students use *Tesselmania!* software to manipulate and transform colorful shapes on the computer screen and create complex tessellations.
- Students continue to look for and report on transformations they find in the world around them.

16. **Develop, understand, and apply a variety of strategies for determining perimeter, area, surface area, angle measure, and volume.**

- Students are given a transparent square grid to place over a worksheet with triangles drawn on it. Using the grid to measure, they find the base, height, and area of each triangle, recording their findings in a table. They discuss patterns that they see, developing their own formula to find areas of triangles.
- Students find the perimeter of a figure by taping a string around it and then untaping and measuring the string. For something large, like the classroom, they might construct and use a trundle wheel.
- Students first estimate the perimeter (or area, volume, or surface area) of a classroom object, then measure it, determine its perimeter, and compare their answers to their estimates. Objects which might be used include books, desks, closets, doors, or windows.
- Students use various same-shape pattern blocks and arrange as many as are needed around a point to complete a circle. They discover the size of each angle since there are 360° in one circle. For example, if it takes six (green) triangles, then each angle must be 60° (360° ÷ 6).
- Students are given a sheet with rectilinear figures (only right angles) on it, such as the letter "E" at the right, and a transparent square centimeter grid that they can place over each of the figures. By counting the squares, they can find the area of each figure; by counting the number of units around it, they can determine its perimeter.



- Students use centimeter cubes to build a structure such as the one shown below and then count the cubes to find the volume of the structure.



- Students bring cereal boxes from home, cut them apart, and determine their surface areas.
- Students find the volumes of different backpacks by using familiar solids to approximate their shape. They compare their results and write about which backpack they think would be "best" and why.

17. **Understand and apply the Pythagorean Theorem.**

- Students construct squares on each side of a right triangle on a geoboard and find the area of each square. They repeat this process using several different triangles, recording their results in a table. Then they look for patterns in the table.
- Students measure the distance diagonally from first to third base on a baseball field and compare it to the distance run by a player who goes from first to second to third. They

note that it is a shorter distance diagonally across the field than it is along the two sides. They repeat this type of measuring activity for other squares and rectangles, noting their results in a table and discussing any patterns they see. They calculate the square of each of the three sides of each triangle, record their results in a table, and look for patterns.

18. Explore patterns produced by processes of geometric change, relating iteration, approximation, and fractals.

- Students use the reducing and/or enlarging feature on a copier to explore repeated reductions/enlargements by the same factor (iteration).
- Students investigate the natural shapes that are produced by growth. They look at how nature produces complex structures in which basic shapes occur over and over. For example, spider webs, honeycombs, pineapples, pinecones, nautilus shells, and snowflakes grow larger in a systematic way (iteration).

19. Investigate, explore, and describe the geometry in nature and real-world applications, using models, manipulatives, and appropriate technology.

- Students design a three-dimensional geometric sculpture. Some may want to find plans for making a geodesic dome and construct it out of gumdrops and toothpicks.
- Students work through the *Two-Toned Towers* lesson that is described in the First Four Standards of this *Framework*. They use models to determine how many different towers can be built using four blocks of two different colors.
- Groups of students working together design a doghouse to be built from a 4' x 8' sheet of plywood. They construct a scale model of their design from oaktag.
- Students use computer programs like *The Geometry PreSupposer* to explore the relationships of sides of polygons or properties of quadrilaterals.
- Assessments that make use of manipulatives and computer software allow students to demonstrate their knowledge and understanding of geometry. The results of performance tasks such as the following would be appropriate for a portfolio: *Make as many different sized squares as you can on a five-by-five geoboard. Create a tessellation pattern with pattern blocks or Tessellmania! software that uses slides, flips, and turns.*
- Students select a country or culture, research the use of specific geometric patterns in that culture, and make a report to the class.
- Specific manipulatives that may be helpful for geometry include pattern blocks, color tiles, linking cubes, centimeter cubes, tangrams, geoboards, links, and templates. Computer programs such as *Logo, Shape Up!, Elastic Lines, Building Perspective,* or *The Factory* may also be helpful.

References

Looher, J. L, Ed. *The World of M. C. Escher*. New York: Abradale Press, Harry N. Abrams, Inc., 1971, 1988.

Software

Building Perspective. Sunburst Communications.

Elastic Lines. Sunburst Communications.

Logo. Many versions of Logo are commercially available.

Shape Up! Sunburst Communications.

Tesselmania! Minnesota Educational Computing Consortium (MECC).

The Factory. Sunburst Communications.

The Geometry PreSupposer. Sunburst Communications.

General references

Diggins, Julia. *String, Straightedge, and Shadow: The Story of Geometry*. New York: Viking Press, 1965.

Geddes, D. *Curriculum and Evaluation Standards for School Mathematics: Addenda Series: Geometry in the Middle Grades*. Reston, VA: National Council of Teachers of Mathematics, 1992.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 7 — Geometry and Spatial Sense — Grades 7-8

Overview

Students can develop strong spatial sense from consistent experiences in classroom activities that use a wide variety of manipulatives and technology. The key components of spatial sense, as identified in the K-12 Overview, are spatial relationships, properties of geometric figures, geometric transformations, coordinate geometry, geometry of measurement, geometric modeling, and reasoning.

Students in grades 7 and 8 learn geometry by: engaging in activities and spatial experiences organized around physical models, modeling, mapping, and measuring; discovering geometric relationships by using mathematical procedures such as drawing, sorting, classifying, transforming, and finding patterns; and solving geometric problems.

Building explicit linkages among mathematical topics is especially important with respect to geometry, since geometric concepts contribute to students' understanding of other topics in mathematics. For example, the number line provides a way of representing whole numbers, fractions, decimals, integers, lengths, and probability. Regions are used in developing understanding of multiplication, fraction concepts, area, and percent. The coordinate plane is used to relate geometry to algebra and functions. Similar triangles are used in connection with ratio and proportion.

Students continue to develop their understanding of **spatial relationships** by examining projections (viewing objects from different perspectives), shadows, perspectives, and maps. They apply the understanding developed in earlier grades to solve problems involving congruence, similarity, and symmetry.

Students begin to explore the logical interrelationships among previously-discovered **properties of geometric figures** at these grade levels. They extend their work with two-dimensional figures to include circles as well as special quadrilaterals. They continue to work with various polygons, lines, planes, and three-dimensional figures such as cubes, prisms, cylinders, cones, pyramids, and spheres.

The study of **geometric transformations** continues as well at these grade levels, becoming more closely linked to the study of algebraic concepts and **coordinate geometry** in all four quadrants. Students begin to represent transformations and/or their results symbolically. They also continue to analyze the relationships between figures and their transformations, considering congruence, similarity, and symmetry.

The geometry of measurement is extended to circles, cylinders, cones, and spheres in these grades. Students learn about π and use it in a variety of contexts. They explore different ways to find perimeters, circumferences, areas, volumes, surface areas, and angle measures. They also develop and apply the Pythagorean Theorem. The emphasis is on understanding the processes used and on recording the procedures in a formula; students should not simply be given a formula and be expected to use it.

Students continue to use **geometric modeling** to represent problem situations in different areas. Drawings of various types are particularly useful to students in understanding the context of problems. Number lines, coordinate planes, regions, and similar triangles help students to visualize numerical situations.

Especially important are the patterns produced by change processes, including growth and decay.

Students further develop their **reasoning** skills by making conjectures as they explore relationships among various shapes and polygons. For example, as students learn about the midpoints of line segments, they can make guesses about the shapes produced by connecting midpoints of consecutive sides of quadrilaterals. By testing their hypotheses with drawings they make (by hand or using a computer), the students come to actually see the possibilities that can exist. The informal arguments that students develop at these grade levels are important precursors to the more formal study of geometry in high school.

The emphasis in grades 7 and 8 should be on investigating and using geometric ideas and relationships, not on memorizing definitions and formulas. A special feature of these grade levels is that students are preparing to take the New Jersey Early Warning Test (EWT). Many of the items in the Measurement and Geometry Cluster of the EWT will ask students to use those geometric ideas and relationships to solve problems - not simply to recall formulas. Indeed, the formulas needed for the problems are given to them on the Reference Sheet that accompanies the test. In their general classroom activity, as well as in preparation for the EWT, students should use a variety of concrete materials to model and analyze situations in two and three dimensions. They should use drawings that they make, either by hand or with the aid of a computer, to further examine geometric situations or to record what they have done. Geometry approached in this way can be fun and challenging for students.

Standard 7 — Geometry and Spatial Sense — Grades 7-8

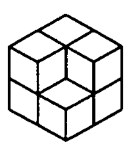
Indicators and Activities

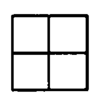
The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

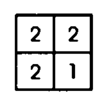
Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

11. Relate two-dimensional and three-dimensional geometry using shadows, perspectives, projections, and maps.

- Students build and draw solids made of cubes. They learn to build solids from drawings and to make their own drawings of solids. Among the drawings with which they should be familiar are the two-dimensional flat view from top, front, and side; the three-dimensional corner view; and the map view showing the base of the building with the number of cubes in each stack. For example, they can build the solid below; presented here are a three-dimensional corner view, a flat view, and a map view.







- Students predict what the intersections of a plane with a cylinder, cone, or sphere will be. Then they slice clay models to verify their predictions.
- Students use cubes made of clear plexiglas and partially filled with colored water to investigate cross sections of a plane with a cube. They try to tilt the cube so that the surface of the water forms various shapes, such as a square, a rectangle that is not a square, a trapezoid, a hexagon, and others.

12. Understand and apply the concepts of symmetry, similarity, and congruence.

- Students create three-dimensional symmetric designs using cubes, cylinders, pyramids, cones, and spheres.
- Students build scale models of the classroom, using similarity to help them determine the appropriate measures of the models.
- Students use compasses and straight-edges to construct congruent line segments and angles.
- Students work through the *Sketching Similarities* lesson that is described in the First Four Standards of this *Framework*. Students use a computer program and various similar figures to discover that corresponding angles have equal measures and corresponding sides have equal ratios.

13. **Identify, describe, compare, and classify plane and solid geometric figures.**

- Students use toothpicks to construct as many different types of triangles as possible, where each side of the triangles consists of between one and five toothpicks. They record their findings in a table, showing how many triangles are scalene, isosceles, equilateral, right, and obtuse. They also indicate which combinations of sides are impossible.

- Students sort collections of quadrilaterals according to the number of lines of symmetry that each has.

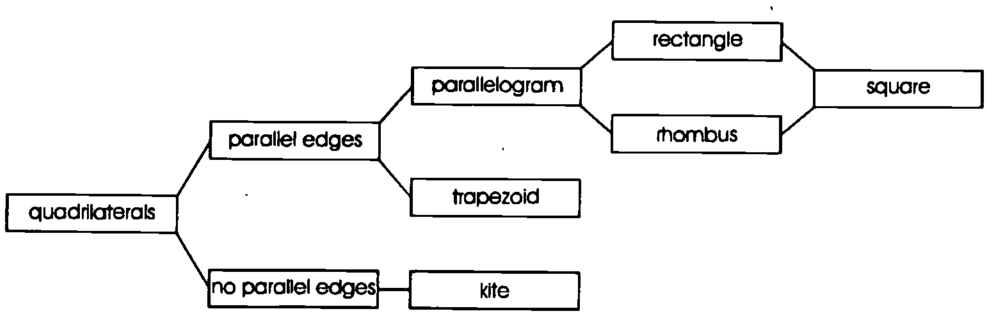
- Students play clue games designed to help them distinguish between necessary and sufficient conditions in describing a shape. For example: *If you want to challenge your friend to identify a square by giving a set of clues, which minimum set of clues would you select from the list below? Explain your selection. Is it possible to select a different minimum set of clues? Explain.*

— 4 right angles	— all angles are 90°
— 4 sides	— all sides of equal length
— all angles congruent	— opposite angles congruent
— opposite sides parallel	— simple closed curve

- Open-ended assessment items like those used on the Early Warning Test can always be used to provoke discussion and classroom activity. One of the sample items in the New Jersey Department of Education's *Mathematics Instructional Guide* (MG3) shows several figures and asks which of them can be put together to form a square. The developmental and extension activities provided there offer good suggestions for manipulative and transformation tasks.

- Students work through the *A Sure Thing !?* lesson that is described in the Introduction to this *Framework*. They investigate the relationship among the measures of the interior angles of a triangle by cutting out arbitrary triangles, tearing them into three pieces so that each corner is intact, and fitting the corners around a single point to make a straight angle.

- Students use diagrams to demonstrate the relationships among properties. For example, they might draw a diagram to show the logical relationship of ideas leading to the angle sum for a quadrilateral, or, as below, to clarify the relationship among different types of quadrilaterals.

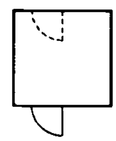


14. **Understand the properties of lines and planes, including parallel and perpendicular lines and planes, and intersecting lines and planes and their angles of incidence.**

- Students build a model of a cube, connect a midpoint of an edge with a midpoint of another edge, and then connect two other midpoints of edges to each other. They describe the relationships of the segments they have constructed. They change one of the line segments to another location and repeat the activity.
- Students identify congruent angles on a parallelogram grid, using their results to develop conjectures about alternate interior angles and corresponding angles of parallel lines and about opposite angles of a parallelogram.
- Working in groups, students review geometric vocabulary by sorting words written on index cards into groups and explaining their reasons for setting up the groups in the way that they did.

15. **Explore the relationships among geometric transformations (translations, reflections, rotations, and dilations), tessellations (tilings), and congruence and similarity.**

- Students use the "nibble" technique to create a shape which will tessellate the plane, that is, copies of this shape will fit together to cover a planar surface like a sheet of white oak-tag. Start with a square, cut off a "nibble" along the top or bottom edge of the square and translate the nibble vertically to the opposite edge of the square; the "nibble" will then be outside the boundary of the original square. Take a "nibble" from the right or left edge of the square and translate it horizontally to the opposite edge of the square. Trace this shape repeatedly onto a sheet of white oak-tag, by interlocking the pattern, and decorate the copies of the shape. Attempt this process several times until a pleasing shape is created.



- Students analyze the patterns found in Arabic designs such as tiled floors and walls in Spain, identifying figures that represent translations, reflections, and rotations. Then they generate their own tile designs using basic geometric shapes. They can create their own tile patterns using *Tesselmania!* software.
- Students apply transformations to figures drawn on coordinate grids, record the coordinates of the original figure and its image, and look for patterns. They express these patterns verbally and symbolically. For example, flipping a point across the x-axis changes the sign of the y-coordinate so that the point (x,y) moves to $(x,-y)$.
- Students practice doing geometric transformations mentally by using the *Geometric Golfer* or similar computer software. These programs present a series of puzzles in which there is an *object* shape and a *target* shape. The task is to use the fewest transformations possible to change the *object* shape so that it is congruent to the *target* shape. In the golf game, the object is a ball and the target is a hole.

16. Develop, understand, and apply a variety of strategies for determining perimeter, area, surface area, angle measure, and volume.

- Students use a paper fastener to connect two models of rays to form angles of different sizes. They estimate the correct position, then measure their guess with a protractor to see how close they were.
- Students are given a parallelogram-shaped piece of oak-tag and asked to cut it apart and arrange the parts so that it is easy to find its area. Their solutions are expressed verbally and symbolically. This same process is repeated for a trapezoid.
- Students bring cans from home, arrange them by estimated volume from smallest to largest, determine the actual volumes by measuring and computing, and compare these results to their estimates.
- Good conceptual assessment items designed to measure students' understanding of area frequently ask the students to find the area remaining in one figure after the area of another figure has been removed. One sample item from the New Jersey Department of Education's *Mathematics Instructional Guide*, for example, asks students to find the area of a circular path that surrounds a circular flower garden (MG1). Problems like this one are not only good practice for tests like the Early Warning Test but can also be used as informal assessments by teachers who listen carefully to their students' discussions about them.
- Students work through the *Rod Dogs* lesson that is described in the First Four Standards of this *Framework*. Students determine the effects of increasing the dimensions of an object on its surface area and volume.

17. Understand and apply the Pythagorean Theorem.

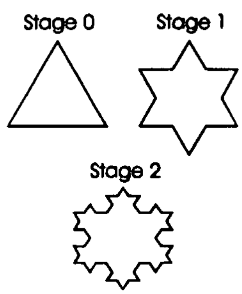
- Students draw right triangles on graph paper with legs of specified lengths and measure the lengths of their hypotenuses. They record their results in a chart and look for patterns.
- Students create a small right triangle in the middle of a ten-by-ten geoboard or on dot paper and then build squares on each side of the triangle. They record the areas of the squares and look for a relationship involving these areas.
- Students use tangram pieces to build squares on each side of the middle-sized triangular tangram piece. They then describe the relationship among the areas of the three squares.

18. Explore patterns produced by processes of geometric change, relating iteration, approximation, and fractals.

- Students use the reducing and/or enlarging feature on a copier to explore repeated reductions/enlargements by the same factor (iteration).
- Students learn about the natural shapes that are produced by growth. They investigate how nature produces complex structures in which basic shapes occur over and over. For example, spider webs, honeycombs, and snowflakes grow larger in a systematic way (iteration). Students measure the age of a tree by looking at its rings; this illustrates approximation. Students produce geometric designs that illustrate these principles, as well as fractals, where miniature versions of the entire design are evident within a small

portion of the design.

- Students view the slides which accompany the activity book, *Fractals for the Classroom, Vol. 1*, and determine why each picture might have been included in a book about fractals.
- Students make a table showing the perimeter of a Koch snowflake (a type of fractal) and its area at each stage. They discuss the patterns in the table. This is an example where the perimeter increases without bound but the area approaches a limit.



Stage	Perimeter	Area
0	3	1
1	4	4/3
2	16/3	40/27
3		
4		

19. Investigate, explore, and describe geometry in nature and real-world applications, using models, manipulatives, and appropriate technology.

- Some students read and prepare a report and presentation to the class on *String, Straightedge, and Shadow: The Story of Geometry* by Julia Diggins. Starting with a chapter about the presence of geometry in nature, this story traces the history of geometric discoveries from the invention of early measuring instruments.
- Students model decay in a bacterial culture by cutting a sheet of grid paper in half repeatedly and recording the area of each rectangle in a table. They then graph the number of cuts versus the area to see an example of exponential decay.
- Students investigate the golden ratio $((1+\sqrt{5})/2)$ and its application to architecture (such as the Parthenon), designs of everyday objects such as index cards and picture frames, and its occurrence in pinecones, pineapples, and sunflower seeds.
- Students write about why manufacturers make specially designed containers for packaging their products, indicating how the idea of tessellations might be important in the designs.
- Students use a computer program such as *The Geometry PreSupposer* to investigate the relationship between the lengths of the sides and the measures of the angles in isosceles, scalene, and equilateral triangles.
- Groups of students prepare slide shows using slides from *Geometry in Our World* to illustrate the connections between geometry, science, and art.
- Pairs of students build kites of different shapes, explaining to the class why they chose a particular shape. Each student predicts which kite will fly highest, writing the prediction in his/her journal. The class flies all of the kites, recording the heights of each by using a clinometer and similar triangles.
- Students watch the video *Donald in Mathmagic Land*. Although getting a bit dated, this

video still thrills most viewers as Donald Duck encounters many animated applications of geometry. Students then form teams which focus on an aspect of the video and do further research on that application.

- Students read and choose projects to make from the book *Origami, Japanese Paper Folding* by Florence Sakade or some other origami instruction books. The detailed instructions usually given in such books are rich in mathematical language and discussions among the students should provide a setting for the use of much geometric terminology.

References

Diggins, Julia. *String, Straightedge, and Shadow: The Story of Geometry.* New York: Viking Press, 1965.

Engelhardt, J., Ed. *Geometry in Our World.* Reston, VA: National Council of Teachers of Mathematics, 1992.

New Jersey Department of Education. *Mathematics Instructional Guide*. D. Varygiannes, Coord. Trenton: 1996.

Peitgen, Heinz-Otto, et al. *Fractals for the Classroom: Strategic Activities, Vol. 1.* Reston, VA: National Council of Teachers of Mathematics, and New York: Springer-Verlag, 1991.

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Software

Geometric Golfer. Minnesota Educational Computing Consortium (MECC).

The Geometry PreSupposer. Sunburst Communications.

Tesselmania! Minnesota Educational Computing Consortium (MECC).

Video

Donald in Mathmagic Land. Walt Disney Studios. Los Angeles: The Walt Disney Company, 1959.

General references

Geddes, D. *Curriculum and Evaluation Standards for School Mathematics: Addenda Series: Geometry in the Middle Grades*. Reston, VA: National Council of Teachers of Mathematics, 1992.

Owens, D. T., Ed. *Research Ideas for the Classroom: Middle Grades Mathematics*. New York: MacMillan, 1993.

On-Line Resources

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Standard 7 — Geometry and Spatial Sense — Grades 9-12

Overview

Students can develop a strong spatial sense from consistent experiences in classroom activities which use a wide variety of manipulatives and technology. The key components of this spatial sense, as identified in the K-12 Overview, are spatial relationships, properties of geometric figures, geometric transformations, coordinate geometry, geometry of measurement, geometric modeling, and reasoning.

Geometry has historically held an important role in high school mathematics, primarily through its focus on deductive reasoning and proof; developing skills in deductive reasoning, learning how to construct proofs, and understanding geometric properties are important outcomes of the high school geometry course. Equally important, however, is the continued development of visualization skills, pictorial representations, and applications of geometric ideas since geometry helps students represent and describe the world in which they live and answer questions about natural, physical, and social phenomena.

Deductive reasoning is highly dependent upon understanding and communication skills. In fact, mathematics can be considered as a language — a language of patterns. This language of mathematics must be meaningful if students are to discuss mathematics, construct arguments, and apply geometry productively. Communication and language play a critical role in helping students to construct links between their informal, intuitive geometric notions and the more abstract language and symbolism of high school geometry.

Geometry describes the real world from several viewpoints. One viewpoint is that of standard Euclidean geometry — a deductive system developed from basic axioms. Other widely used viewpoints are those of coordinate geometry, transformational geometry, and vector geometry. The interplay between geometry and algebra strengthens the students' ability to formulate and analyze problems from situations both within and outside mathematics. Although students will at times work separately in synthetic, coordinate, transformational, and vector geometry, they should also have many opportunities to compare, contrast, and translate among these systems. Further, students should learn that certain types of problems are often solved more easily in one specific system than another specific system.

Visualization and pictorial representation are also important aspects of a high school geometry course. Students should have opportunities to visualize and work with two- and three-dimensional figures in order to develop spatial skills fundamental to everyday life and to many careers. By using physical models and other real-world objects, students can develop a strong base for geometric intuition. They can then draw upon these experiences and intuitions when working with abstract ideas.

The goal of high school geometry includes applying geometric ideas to problems in a variety of areas. Each student must develop the ability to solve problems if he or she is to become a productive citizen. Instruction thus must begin with problem situations — not only exercises to be accomplished independently but also problems to be solved in small groups or by the entire class working cooperatively.

Applications of mathematics have changed dramatically over the last twenty years, primarily due to rapid advances in technology. Geometry has, in fact, become more important to students because of computer graphics. Thus, calculators and computers are appropriate and necessary tools in learning geometry.

Students in high school continue to develop their understanding of **spatial relationships.** They construct models from two-dimensional representations of objects, they interpret two- and three-dimensional representations of geometric objects, and they construct two-dimensional representations of actual objects.

Students formalize their understanding of **properties of geometric figures**, using known properties to deduce new relationships. Specific figures which are studied include polygons, circles, prisms, cylinders, pyramids, cones, and spheres. Properties considered should include congruence, similarity, symmetry, measures of angles (especially special relationships such as supplementary and complementary angles), parallelism, and perpendicularity.

In high school, students apply the principles of **geometric transformations** and **coordinate geometry** that they learned in the earlier grades, using these to help develop further understanding of geometric concepts and to establish justifications for conclusions inferred about geometric objects and their relationships. They also begin to use vectors to represent geometric situations.

The **geometry of measurement** is extended in the high school grades to include formalizing procedures for finding perimeters, circumferences, areas, volumes, and surface areas, and solving indirect measurement problems using trigonometric ratios. Students should also use trigonometric functions to model periodic phenomena, establishing an important connection between geometry and algebra.

Students use a variety of geometric representations in **geometric modeling** at these grade levels, such as graphs of algebraic functions on coordinate grids, networks composed of vertices and edges, vectors, transformations, and right triangles to solve problems involving trigonometry. They also explore and analyze further the patterns produced by geometric change.

Deductive reasoning takes on an increasingly important role in the high school years. Students use inductive reasoning as they look for patterns and make conjectures; they use deductive reasoning to justify their conjectures and present reasonable explanations.

Standard 7 — Geometry and Spatial Sense — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11 and 12.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

16*. **Develop, understand, and apply a variety of strategies for determining perimeter, area, surface area, angle measure, and volume.**

- Students find volumes of objects formed by combining geometric figures and develop formulas describing what they have done. For example, they might generate a formula for finding the volume of a silo composed of a cylinder of specified radius and height topped by a hemisphere of the same radius.
- Students construct models to show how the volume of a pyramid with a square base and height equal to a side of the base is related to the volume of a cube with the same base.
- Students develop and use a spreadsheet to determine what the dimensions should be for a cylinder with a fixed volume, in order to minimize the surface area. Similarly, they investigate what should be the dimensions for a rectangle having a fixed perimeter in order to maximize the enclosed area. They discuss how the symmetry of these figures relates to the solutions.

19*. **Investigate, explore, and describe geometry in nature and real-world applications, using models, manipulatives, and appropriate technology.**

- Students use a computer-aided design (CAD) program to investigate rotations of objects in three dimensions.
- Students use *The Geometric SuperSupposer* to measure components of shapes and make observations. For example, they might construct parallelograms and measure sides, angles, and diagonals, observing that opposite sides are congruent, as are opposite angles, and that diagonals bisect each other.
- Students use *The Geometer's Sketchpad* to investigate the effects of rotating a triangle about a fixed point.
- Students use commercial materials such as GeoShapes or Polydrons to construct three-dimensional geometric figures. They make tables concerning the number of vertices, edges, and faces in each solid. They measure the figures to determine their surface areas and volumes. They lay the patterns out flat to examine the nets of each solid. [A net is a

* Activities are included here for Indicators 16 and 19, which are also listed for grade 8, since the Standards specify that students demonstrate continued progress in these indicators.

flat shape which when folded along indicated lines will produce a three-dimensional object; for example, six identical squares joined in the shape of a cross can be folded to form a cube. Tabs added to the net facilitate attaching appropriate edges so that the shape remains three-dimensional.]

- Students work through the *Ice Cones* lesson that is described in the First Four Standards of this *Framework*. Students create a variety of paper cones out of circles with radius 10 inches which are cut along a radius. They use graphing calculators to find the maximum volume of such cones.
- Students copy geometric designs using compass and straightedge, and generate their own designs.
- Students investigate wallpaper patterns, classifying them according to the transformations used. They study the structure of crystals from a geometric perspective.

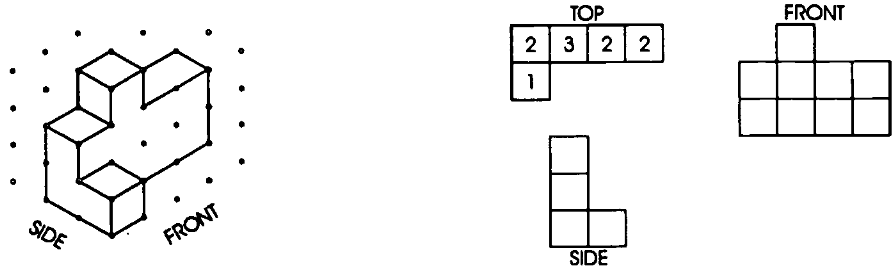
20. Understand and apply properties involving angles, parallel lines, and perpendicular lines.

- Students make tessellations with an assortment of different triangles, noting the variety of geometric patterns that are formed, including parallel lines, congruent angles, congruent triangles, similar triangles, parallelograms, and trapezoids.
- Students identify congruent angles on a parallelogram grid, and use their results to develop conjectures about alternate interior angles, corresponding angles of parallel lines, and opposite angles of a parallelogram.
- Working together, students review geometric vocabulary by sorting words written on index cards into groups and explaining their reasons for creating the groups they did. For example, they might place "parallelogram," "rhombus," "square," and "rectangle" in one group (since they are all parallelograms) and place "kite" and "trapezoid" in another group (since they are not parallelograms).
- Students find a variety of strategies for demonstrating that the sum of the measures of the angles of a triangle is 180°. Some use protractors and measure a pencil-and-paper figure, others create a triangle with *Geometer's Sketchpad* software and post the measures of the angles before dragging it from a vertex to notice that the sum always remains the same, and still others use a method that requires tearing each of the corners from an oaktag triangle and then fitting them together to make a line.

21. Analyze properties of three-dimensional shapes by constructing models and by drawing and interpreting two-dimensional representations of them.

- Pairs of students work together to describe and draw geometric figures. One student is given a picture involving one or more geometric figures and must describe the drawing to the other student without using the names of the figures. The second student, without seeing the figure, must visualize and represent the picture.
- Students create wind-up posterboard models of rotational three-dimensional solids. They cut out a plane figure such as a circle or a rectangle from posterboard, punch two holes in it near its edges, thread a cut rubberband through the holes, and attach the ends of the rubberband to the ends of a coathanger from which the horizontal wire has been removed. They then twist the rubber band to wind up the figure and release to "show" the solid.

- Students use isometric dot paper to sketch figures made up of cubes. They also sketch top, front, and side views (projections) of the figure.



- One long-term project that some high school teachers use for assessment is the following: Using a variety of means and materials, students begin by constructing models of the Platonic solids and other three dimensional geometric figures. They are then challenged to work in teams to find a relationship among the number of faces, vertices, and edges that holds for all of the solids (Euler's Formula: $F + V - E = 2$).
- Students read *Flatland: A Romance of Many Dimensions* by Edwin Abbott, a fascinating and imaginative story about life in a two-dimensional world.
- Students use a computer-aided design (CAD) program to investigate rotations of objects in three dimensions.

22. Use transformations, coordinates, and vectors to solve problems in Euclidean geometry.

- Students construct a polygon that outlines the top view of their school. They are asked to imagine that they are architects who need to send this outline by computer to a builder who has no graphics imaging capabilities. They develop strategies for sending this information to the builder. One group locates one corner of the building at the origin and determines the coordinates for the other vertices. Another group uses vectors to tell the builder what direction to proceed from the initial corner located at the origin.
- Students work on the question of where a power transformer should be located on a line so that the length of the cable needed to run to two points not on that line is minimized. They find that if the two points are on the same side of the line, then by using reflections they can construct a straight line that crosses the given line at the desired location.
- Students first determine the coordinates for the vertices of a parallelogram, a rhombus, a rectangle, an isosceles trapezoid, and a square with one vertex at the origin and a side along the x-axis. They then work in groups to determine where the coordinate system should be placed to simplify the coordinate selection for a kite, a rhombus, and a square.
- Students draw two congruent triangles anywhere in the plane and determine the minimum number of reflections needed to map one onto the other.
- Students draw a triangle on graph paper and then find the image of the triangle when the coordinates of each vertex are multiplied by various constants. They draw each resulting triangle and determine its area. They make a table of their results and look for relationships between the constants used for dilation and the ratios of the areas.

- Students use a Mira (Reflecta) to find the center of a circle, to draw the perpendicular bisectors of a line segment, or to draw the medians of a triangle.
- Students apply transformations to figures drawn on coordinate grids, record the coordinates of the original figure and its image, and look for patterns. They express these patterns verbally and symbolically. For example, flipping a point across the x-axis changes the sign of the y-coordinate so that the point (x,y) moves to (x, -y).
- Given the equation of a line, students plot the line on a coordinate grid and then shift the line according to a given translation. They then determine the equation of the resulting line. After doing several such problems, students identify patterns that they have found and write conjectures.
- Students work through the *Building Parabolas* lesson that is described in the First Four Standards of this *Framework*. They investigate the effects of various coefficients on the general shape of a parabola and connect these to geometric transformations.

23. Use basic trigonometric ratios to solve problems involving indirect measurement.

- Students use trigonometric ratios to determine distances which cannot be measured directly, such as the distance between two points on opposite sides of a chasm.
- Students investigate how the paths of tunnels are determined so that people digging from each end wind up in the same place.
- Students use trigonometry to determine the cloud ceiling at night by directing a light (kept in a narrow beam by a parabolic reflector) toward the clouds. An observer at a specified distance measures the angle of elevation to the point at which the light is reflected from the cloud.
- Students plot the average high temperature for each month over the course of five years to see an example of a periodic function. They discuss what types of functions might be appropriate to represent this relationship.

24. Solve real-world and mathematical problems using geometric models.

- Students visit a construction site where the "framing" step of a building process is taking place. They note where congruence occurs (such as in the beams of the roof, where angles must be congruent). They write about why congruence is essential to buildings and other structures.
- Students use paper fasteners and tagboard strips with a hole punched near each end to investigate the rigidity of various polygon shapes. For shapes that are not rigid, they determine how to make the shape more rigid.
- Students draw a geometric representation and develop a formula to solve the problem of how many handshakes will take place if there are *n* people and each person shakes hands with each other person exactly once.
- Students work through the *On the Boardwalk* lesson that is described in the Introduction to this *Framework*. They determined the probability of winning a prize when tossing a coin onto a grid by having the coin avoid all of the grid lines.

- Students use graph models to represent a situation in which a large company wishes to install a pneumatic tube system that would enable small items to be sent between any of ten locales, possibly by relay. Given the cost associated with possible tubes (edges), the students work in groups to determine optimal pneumatic tube systems for the company. They report their results in letters written individually to the company president.
- Students work through the *Making Rectangles* lesson that is described in the First Four Standards of this *Framework*. They use combinations of algebra tiles which they try to arrange into rectangle shapes to help them develop procedures for multiplying binomials and factoring polynomials.

25. Use inductive and deductive reasoning to solve problems and to present reasonable explanations of and justifications for the solutions.

- In a computer-based, open-ended, assessment, groups of students use computer software to draw parallelograms, make measurements, and list as many properties of parallelograms and their diagonals as they can.
- Students prove deductively that a parallelogram is divided into congruent triangle by a diagonal. They also prove that any angle inscribed in a semi-circle is a right angle. (An angle ABC is inscribed in a semi-circle if AC is a diameter and B is any other point on the circle.)
- Students explain in writing to a friend what the formula is for the measure of each interior angle in a regular polygon with n sides and how it is derived.
- Students build staircases from cubes, recording the number of steps and the total number of cubes used for each construction. They look for patterns, expressing them in words and symbolically in equations. They then try to justify their results using deductive reasoning.
- Students use *Cabri* software to investigate what happens when consecutive midpoints of a quadrilateral are connected in order. They state a conjecture based on their investigation and explain why they think it is true.
- Students investigate the relationship between the number of diagonals that can be drawn from one vertex of a polygon and the number of sides of that polygon. They write about their findings in their journals.
- Students work through the *A Sure Thing!?* lesson in the Introduction to this *Framework*. They investigate the number of non-overlapping regions that can be created if they draw all the chords joining n points on the circumference of a circle.

26. Analyze patterns produced by processes of geometric change and express them in terms of iteration, approximation, limits, self-similarity, and fractals.

- Students duplicate the beginning stages of a fractal construction in the plane and analyze the sequences of their perimeters and their areas.
- Students use the reduction and enlargement capabilities of a copy machine to investigate the effects on area. They make a table showing the linear rate of reduction/enlargement and the resulting area for each successive reduction/enlargement. Then they graph the results — an exponential function showing either decay or growth.

- Students use the slides and appropriate activities from *Fractals for the Classroom, Vol. 1* to analyze patterns produced by changes in geometric shapes.
- Students model decay in a bacterial culture by cutting a sheet of grid paper in half repeatedly and recording the area of each rectangle in a table. They then graph the number of cuts versus the area to see an example of exponential decay.
- Students plot the relationship between body height and arm length for people from one year of age through adulthood on coordinate grid paper and on log-log paper. They see that the graph is not a straight line on the coordinate grid paper; it is actually a logarithmic function. They find that the function appears as a straight line on log-log paper.

27. Explore applications of other geometries in real-world contexts.

- Students represent lines using string and pins on styrofoam balls (spheres). They analyze the properties of lines (e.g., all lines intersect) and triangles (e.g., it is possible to have a triangle with three 90° angles). They apply their results to finding the shortest route between two points on the earth.
- Students investigate the angel and devil drawings of M. C. Escher as examples of geometries in which there may be many "lines" through a given point that do not intersect a given "line." In this case, a "line" is an arc of a circle that is perpendicular to the outside circle of the drawing.
- Students explore another geometry using *Non-Euclidean Adventures on the Lénárt Sphere*.
- Students determine how many people are needed on a committee if there are to be four subcommittees, with each person on two subcommittees and each pair of subcommittees having one person in common. Most groups use letters to represent the individuals, and represent the four subcommittees by collections of letters, as in the following proposed solution {ABC, ADE, BDF, CEF}. The teacher asks the students to make a diagram of their solution, using "points" for people and "lines" for subcommittees, so that each subcommittee is a line whose points are its members. The rules for subcommittees become axioms about these points and lines; for example, "each person is on two subcommittees" becomes "each point is on two lines." The resulting geometry is an example of a finite geometry.

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On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
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STANDARD 8 — NUMERICAL OPERATIONS

K-12 Overview

All students will understand, select, and apply various methods of performing numerical operations.

Descriptive Statement

Numerical operations are an essential part of the mathematics curriculum. Students must be able to select and apply various computational methods, including mental math, estimation, paper-and-pencil techniques, and the use of calculators. Students must understand how to add, subtract, multiply, and divide whole numbers, fractions, and other kinds of numbers. With calculators that perform these operations quickly and accurately, however, the instructional emphasis now should be on understanding the meanings and uses of the operations, and on estimation and mental skills, rather than solely on developing paper-and-pencil skills.

Meaning and Importance

The wide availability of computing and calculating technology has given us the opportunity to significantly reconceive the role of computation and numerical operations in our school mathematics programs. Up until this point in our history, the mathematics program has called for the expenditure of tremendous amounts of time in helping children to develop proficiency with paper-and-pencil computational procedures. Most people defined *proficiency* as a combination of speed and accuracy with the standard algorithms. Now, however, adults who need to perform calculations quickly and accurately have electronic tools that are more accurate and more efficient than any human being. It is time to re-examine the reasons to teach paper-and-pencil computational algorithms to children and to revise the curriculum in light of that re-examination. Mental mathematics, however, should continue to be stressed; students should be able to carry out simple computations without resort to either paper-and-pencil or calculators. Fourth-graders must know the basic facts of the multiplication table, and seventh-graders must be able to evaluate in their heads simple fractions, such as *What's two-thirds of 5 tablespoons?*

K-12 Development and Emphases

At the same time that technology has made the traditional focus on paper-and-pencil skills less important, it has also presented us with a situation where some numerical operations, skills, and concepts are much more important than they have ever been. **Estimation** skills, for example, are critically important if one is to be a competent user of calculating technology. People must know the range in which the answer to a given problem should lie before doing any calculation, they must be able to assess the reasonableness of the results of a string of computations, and they should be able to be satisfied with the results of an estimation when an exact answer is unnecessary. They should also be able to work quickly and easily with changes in order of magnitude, using powers of ten and their multiples. **Mental mathematics** skills also play a more important role in a highly technological world. Simple two-digit computations or operations that involve powers of ten should be performed mentally by a mathematically literate adult. Students should have enough confidence in their ability with such computations to do them mentally rather than using either a calculator or paper and pencil. Most importantly, a student's **knowledge of the meanings and uses of the various arithmetic operations** is essential. Even with the best of computing devices, it is still the human who must decide which operations need to be performed and in what order to answer the question at hand. The construction of solutions to life's everyday problems, and to society's larger ones, will require students to be thoroughly familiar with when and how the mathematical operations are used.

The major shift in this area of the curriculum, then, is one away from drill and practice of paper-and-pencil procedures and toward real-world applications of operations, wise choices of appropriate computational strategies, and integration of the numerical operations with other components of the mathematics curriculum. *So what is the role of paper-and-pencil computation in a mathematics program for the year 2000? Should children be able to perform any calculations by hand? Are those procedures worth any time in the school day?* Of course they should and of course they are.

Most simple **paper-and-pencil procedures** should still be taught and one-digit **basic facts** should still be committed to memory. We want students to be proficient with two- and three-digit addition and subtraction and with multiplication and division involving two-digit factors or divisors, but there should be changes both in the way we teach those processes and in where we go from there. The focus on the learning of those procedures should be on understanding the procedures themselves and on the development of accuracy. There is no longer any need to concentrate on the development of speed. To serve the needs of understanding and accuracy, non-traditional paper-and-pencil algorithms, or algorithms devised by the children themselves, may well be better choices than the standard algorithms. The extensive use of drill in multi-digit operations, necessary in the past to enable people to perform calculations rapidly and automatically, is no longer necessary and should play a much smaller role in today's curriculum.

For procedures involving larger numbers, or numbers with a greater number of digits, the intent ought to be to bring students to the point where they understand a paper-and-pencil procedure well enough to be able to extend it to as many places as needed, but certainly not to develop an old-fashioned kind of proficiency with such problems. In almost every instance where the student is confronted with such numbers in school, **technology** should be available to aid in the computation, and students should understand how to use it effectively. Calculators are the tools that real people in the real world use when they have to deal with similar situations and they should not be withheld from students in an effort to further an unreasonable and antiquated educational goal.

IN SUMMARY, numerical operations continue to be a critical piece of the school mathematics curriculum and, indeed, a very important part of mathematics. But, there is perhaps a greater need for us to rethink our approach here than to do so for any other component. An enlightened mathematics program for today's children will empower them to use all of today's tools rather than require them to meet yesterday's expectations.

***NOTE:** Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

Standard 8 — Numerical Operations — Grades K-2

Overview

The wide availability of computing and calculating technology has given us the opportunity to significantly reconceive the role of computation and numerical operations in our elementary mathematics programs, but, in kindergarten through second grade, the effects will not be as evident as they will be in all of the other grade ranges. This is because the numerical operations content taught in these grades is so basic, so fundamental, and so critical to further progress in mathematics that much of it will remain the same. The approach to teaching that content, however, must still be changed to help achieve the goals expressed in the *New Jersey Mathematics Standards*.

Learning the **meanings of addition and subtraction,** gaining facility with **basic facts**, and mastering some **computational procedures for multi-digit addition and subtraction** are still the topics on which most of the instructional time in this area will be spent. There will be an increased conceptual and developmental focus to these aspects of the curriculum, though, away from a traditional drill-and-practice approach, as described in the K-12 Overview; nevertheless, students will be expected to be able to respond quickly and easily when asked to recall basic facts.

By the time they enter school, most young children can use counters to act out a mathematical story problem involving addition or subtraction and find a solution which makes sense. Their experiences in school need to build upon that ability and deepen the children's understanding of the **meanings of the operations**. School experiences also need to strengthen the children's sense that modeling such situations as a way to understand them is the right thing to do. It is important that they be exposed to a variety of different situations involving addition and subtraction. Researchers have separated problems into categories based on the kind of relationships involved (Van de Walle, 1990, pp. 75-6); students should be familiar with problems in all of the following categories:

Join problems

- Mary has 8 cookies. Joe gives her 2 more. *How many cookies does Mary have in all?*
- Mary has some cookies. Joe gives her 2 more. Now she has 8. *How many cookies did Mary have to begin with?* (Missing addend)
- Mary has 8 cookies. Joe gives her some more. Now Mary has 10. *How many cookies did Joe give Mary?* (Missing addend)

Separate problems

- Mary has 8 cookies. She eats 2. *How many are left?* (Take away)
- Mary has some cookies. She eats 2. She has 6 left. *How many cookies did Mary have to begin with?*
- Mary has 8 cookies. She eats some. She has 6 left. *How many cookies did Mary eat?* (Missing addend)

Part-part-whole problems

- Mary has 2 nickels and 3 pennies. *How many coins does she have?*
- Mary has 8 coins. Three are pennies, the rest nickels. *How many nickels does Mary have?*

Compare problems

- Mary has 6 books. Joe has 4. *How many more books does Mary have than Joe?*
- Mary has 2 more books than Joe. Mary has 6 books. *How many books does Joe have?*
- Joe has 2 fewer books than Mary. He has 4 books. *How many books does Mary have?*

Basic facts in addition and subtraction continue to be very important. Students should be able to quickly and easily recall one-digit sums and differences. The most effective way to accomplish this has been shown to be the focused and explicit use of basic fact strategies—conceptual techniques that make use of the child's understanding of number parts and relationships to help recover the appropriate sum or difference. By the end of second grade, students should not only be able to use *counting on*, *counting back, make ten,* and *doubles* and *near doubles* strategies, but also explain why these strategies work by modeling them with counters. Building on their facility with learning doubles like $7 + 7 = 14$, children recast $7 + 8$ as $7 + 7 + 1$, which they then recognize as 15 (*near doubles*). *Make ten* involves realizing that in adding $8 + 5$, you need two to make ten, and recasting the sum as $8 + 2 + 3$ which is $10 + 3$ or *13*. *Counting on* involves starting with the large number and counting on the smaller number so that adding $9 + 3$, for example, would involve counting on *10, 11*, and then *12*. *Counting back* is used for subtraction, so that finding $12 - 4$, the child might count *11*, *10*, *9*, and then *8*.

Students must still be able to perform **multi-digit addition and subtraction** with paper and pencil, but the widespread availability of calculators has made the particular procedure used to perform the calculations less important. It need no longer be the single fastest, most efficient algorithm chosen without respect to the degree to which children understand it. Rather, the teaching of multi-digit computation should take on more of a problem solving approach, a more conceptual, developmental approach. Students should first use the models of multi-digit number that they are most comfortable with (base ten blocks, popsicle sticks, bean sticks) to explore the new class of problems. Students who have never formally done two-digit addition might be asked to use their materials to help figure out how many second graders there are in all in the two second grade classes in the school. Other similar real-world problems should follow, some involving regrouping and others not. After initial exploration, students share with each other all of the strategies they've developed, the best ways they've found for working with the tens and ones in the problems, and their own approaches (and names!) for regrouping. Most students can, with direction, take the results of those discussions and create their own paper-and-pencil procedures for addition and subtraction. The discussions can, of course, include the traditional approaches, but these ought not to be seen as *the only right way* to do these operations.

Kindergarten through second grade teachers are also responsible for setting up an atmosphere where **estimation** and **mental math** are seen as reasonable ways to do mathematics. Of course students at these grade levels do almost exclusively mental math until they reach multi-digit operations, but estimation should also comprise a good part of the activity. Students regularly involved in real-world problem solving should begin to develop a sense of when estimation is appropriate and when an exact answer is necessary.

Technology should also be an important part of the environment in primary classrooms. Calculators provide a valuable teaching tool when used to do student-programmed skip counting, to offer estimation and mental math practice with *target games*, and to explore operations and number types that the students have not formally encountered yet. They should also be used routinely to perform computation in problem solving situations that the students may not be able to perform otherwise. This use prevents the need to artificially contrive the numbers in real-world problems so that their answers are numbers with which the students are already comfortable.

The topics that should comprise the numerical operations focus of the kindergarten through second grade mathematics program are:

addition and subtraction basic facts
multi-digit addition and subtraction

Standard 8 — Numerical Operations — Grades K-2

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Develop meaning for the four basic arithmetic operations by modeling and discussing a variety of problems.**

   - Students use unifix cube towers of two colors to show all the ways to make *"7"* (for example: *3+4, 2+5, 0+7*, and so on). This activity focuses more on developing a sense of *"sevenness"* than on addition concepts, but a good sense of each individual number makes the standard operations much easier to understand.
   - Kindergartners and first graders use workmats depicting various settings in which activity takes place to make up and act out story problems. On a mat showing a vacant playground, for instance, students place counters to show 3 kids on the swings and 2 more in the sandbox. *How many kids are there in all? How many more are on the swings than in the sandbox? What are all of the possibilities for how many are boys and how many are girls?*
   - Students work through the *Sharing a Snack* lesson that is described in the Introduction to this *Framework*. It challenges students to find a way to share a large number of cookies fairly among the members of the class, promoting discussion of early division, fraction, and probability ideas.
   - Students learn about addition as they read *Too Many Eggs* by M. Christina Butler. They place eggs in different bowls as they read and then make up addition number sentences to find out how many eggs were used in all.
   - Kindergarteners count animals and learn about addition as they read *Adding Animals* by Colin Hawkins. This book uses addends from one through four and shows the number sentences that go along with the story.
   - Students are introduced to the take-away meaning for subtraction by reading *Take Away Monsters* by Colin Hawkins. Students see the partial number sentence (e.g., *5 - 1 =*), count to find the answer, and then pull the tab to see the result.
   - Students explore subtraction involving missing addend situations as they read *The Great Take-Away* by Louise Mathews. This book tells the story of one lazy hog who decides to make easy money by robbing the other pigs in town. The answers to five subtraction mysteries are revealed when the thief is captured.
   - Students make booklets containing original word problems that illustrate different addition or subtraction situations. These may be included in a portfolio or evaluated independently.

2. **Develop proficiency with and memorize basic number facts using a variety of fact strategies (such as "counting on" and "doubles").**

   - Students play *one more than* dominoes by changing the regular rules so that a domino can be placed next to another only if it has dots showing *one more than* the other. Dominoes of any number can be played next to others that show *6* (or *9* in a set of double nines). *One less than* dominoes is also popular.
   - Students work through the *Elevens Alive* lesson that is described in the Introduction to this *Framework*. It asks them to consider the parts of eleven and the natural, random, occurrence of different pairs of addends when tossing eleven two-colored counters.
   - Second graders regularly use the *doubles* and *near doubles*, the *make ten*, and the *counting on* and *counting back* strategies for addition and subtraction. Practice sets of problems are structured so that use of all of these strategies is encouraged and the students are regularly asked to explain the procedures they are using.
   - Students play games like *addition war* to practice their basic facts. Each of two children has half of a deck of playing cards with the face cards removed. They each turn up a card and the person who wins the trick is the first to say the sum (or difference) of the two numbers showing. Calculators may be used to check answers, if necessary.
   - Students use the calculator to count *one more than* by pressing + *1*= = =. The display will increase by one every time the student presses the = key. Any number can replace the *1* key.
   - Students use two dice to play board games (*Chutes and Ladders* or home-made games). These situations encourage rapid recall of addition facts in a natural way. In order to extend practice to larger numbers, students may use 10-sided dice.
   - Students use computer games such as *Math Blaster Plus* or *Math Rabbit* to practice basic facts.

3. **Construct, use, and explain procedures for performing whole number calculations in the various methods of computation.**

   - Second graders use popsicle sticks bundled as tens and ones to try to find a solution to the first two-digit addition problem they have formally seen: *Our class has 27 children and Mrs. Johnson's class has 26. How many cupcakes will we need for our joint party?* Solution strategies are shared and discussed with diversity and originality praised. Other problems, some requiring regrouping and others not, are similarly solved using the student-developed strategies.
   - Students use calculators to help with the computation involved in a first-grade class project: to see how many books are read by the students in the class in one month. Every Monday morning, student reports contribute to a weekly total which is then added to the monthly total.
   - Students look forward to the hundredth day of school, on which there will be a big celebration. On each day preceding it, the students use a variety of procedures to determine how many days are left before day 100.

- As part of their assessment, students explain how to find the answer to an addition or subtraction problem (such as *18 + 17*) using pictures and words.
- Students find the answer to an addition or subtraction problem in as many different ways as they can. For example, they might solve *28 + 35* in the following ways:

  8 + 5 = 13 and 20 + 30 = 50, so 13 + 50 = 63

  28 + 30 = 58. Two more is 60, and 3 more is 63

  25 + 35 = 60 and 3 more is 63.
- Students use estimation to find out whether a package of 40 balloons is enough for everyone in the class of 26 to have two balloons. They discuss the strategies they use to solve this problem and decide if they should buy more packages.

4. Use models to explore operations with fractions and decimals.

- Kindergartners explore part/whole relations with pattern blocks by seeing which shapes can be created using other blocks. You might ask: *Can you make a shape that is the same as the yellow hexagon with 2 blocks of some other color? with 3 blocks of some other color? with 6 blocks of some other color?* and so on.
- Students use paper folding to begin to identify and name common fractions. You might ask: *If you fold this rectangular piece of paper in half and then again and then again, how many equal parts are there when you open it up?* Similarly folded papers, each representing a different unit fraction, allow for early comparison activities.
- Second graders use fraction circles to model situations involving fractions of a pizza. For example: *A pizza is divided into six pieces. Mary eats two pieces. What fraction of the pizza did Mary eat? What fraction is left?*
- Students use manipulatives such as pattern blocks or Cuisenaire rods to model fractions. For example: *If the red rod is one whole, then what number is represented by the yellow rod?*

5. Use a variety of mental computation and estimation techniques.

- Students regularly practice a variety of oral counting skills, both forward and backward, by various steps. For instance, you might instruct your students to: *Count by ones — start at 1, at 6, at 12, from 16 to 23; Count by tens — start at 10, at 30, at 110, at 43, at 67, from 54 to 84,* and so on.
- Students estimate sums and differences both before doing either paper-and-pencil computation or calculator computation and after so doing to confirm the reasonableness of their answers.
- Students are given a set of index cards on each of which is printed a two-digit addition pair (*23+45, 54+76, 12+87,* and so on). As quickly as they can they sort the set into three piles: *more than 100, less than 100,* and *equal to 100.*
- Students play "*Target 50*" with their calculator. One student enters a two-digit number and the other must add a number that will get as close as possible to *50.*

6. **Select and use appropriate computational methods from mental math, estimation, paper-and-pencil, and calculator methods, and check the reasonableness of results.**

   - The daily *calendar routine* provides the students with many opportunities for computation. Questions like these arise almost every day: *There are 27 children in our class. Twenty-four are here today. How many are absent? Fourteen are buying lunch; how many brought their lunch?* or *It's now 9:12. How long until we go to gym at 10:30?* The students are encouraged to choose a computation method with which they feel comfortable; they are frequently asked why they chose their method and whether it was important to get an exact answer. Different solutions are acknowledged and praised.
   - Students regularly have *human vs. calculator races*. Given a list of addition and subtraction basic facts, one student uses mental math strategies and another uses a calculator. They quickly come to realize that the human has the advantage.
   - Students regularly answer multiple choice questions like these with their best guesses of the most reasonable answer: *A regular school bus can hold: 20 people, 60 people, 120 people? The classroom is: 5 feet high, 7 feet high, 10 feet high?*
   - As part of an assessment, students tell how they would solve a particular problem and why. They might circle a picture of a calculator, a head (for mental math), or paper-and-pencil for each problem.

7. **Understand and use relationships among operations and properties of operations.**

   - Students explore three-addend problems like $4 + 5 + 6 =$. First they check to see if adding the numbers in different orders produces different results and, later, they look for pairs of compatible addends (like *4* and *6*) to make the addition easier.
   - Students make up humorous stories about adding and subtracting zero. *I had 27 cookies. My mean brother took away zero. How many did I have left?*
   - Second graders, exploring multiplication arrays, make a *4* x *5* array of counters on a piece of construction paper and label it: *4 rows, 5 in each row* = *20*. Then they rotate the array 90° and label the new array, *5 rows, 4 in each row* = *20*. Discussions follow which lead to intuitive understandings of commutativity.

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On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

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Standard 8 — Numerical Operations — Grades 3-4

Overview

The widespread availability of computing and calculating technology has given us the opportunity to reconceive the role of computation and numerical operations in our third and fourth grade mathematics programs. Traditionally, tremendous amounts of time were spent at these levels helping children to develop proficiency and accuracy with paper-and-pencil procedures. Now, adults needing to perform calculations quickly and accurately have electronic tools that are both more accurate and more efficient than those procedures. At the same time, though, the new technology has presented us with a situation where some numerical operations, skills, and concepts are much more important than they used to be. As described in the K-12 Overview, **estimation, mental computation,** and **understanding the meanings of the standard arithmetic operations** all play a more significant role than ever in the everyday life of a mathematically literate adult.

The major shift in the curriculum that will take place in this realm, therefore, is one away from drill and practice of paper-and-pencil procedures with symbols and toward real-world applications of operations, wise **choices of appropriate computational strategies**, and integration of the numerical operations with other components of the mathematics curriculum.

Third and fourth graders are primarily concerned with cementing their understanding of addition and subtraction and developing new **meanings for multiplication and division.** They should be in an environment where they can do so by modeling and otherwise representing a variety of real-world situations in which these operations are appropriately used. It is important that the variety of situations to which they are exposed include all the different scenarios in which multiplication and division are used. There are several slightly different taxonomies of these types of problems, but minimally students at this level should be exposed *to repeated addition and subtraction, array, area,* and *expansion* problems. Students need to recognize and model each of these problem types for both multiplication and division.

Basic facts in multiplication and division continue to be very important. Students should be able to quickly and easily recall quotients and products of one-digit numbers. The most effective approach to enabling them to acquire this ability has been shown to be the focused and explicit use of basic fact strategies—conceptual techniques that make use of the child's understanding of the operations and number relationships to help recover the appropriate product or quotient. *Doubles* and *near doubles* are useful strategies, as are discussions and understandings regarding the regularity in the *nines* multiplication facts, the roles of *one* and *zero* in these operations, and the roles of *commutativity and distributivity.*

Students must still be able to perform **two-digit multiplication and division** with paper and pencil, but the widespread availability of calculators has made the particular procedure used to perform the calculations less important. It need no longer be the single fastest, most efficient algorithm chosen without respect to the degree to which children understand it. Rather, the teaching of two-digit computation should take on more of a problem solving approach, a more conceptual, developmental approach. Students should first use the models of multi-digit numbers that they are most comfortable with (base ten blocks, money) to explore this new class of problems. Students who have never formally done two-digit multiplication might be asked to use their materials to help figure out how many pencils are packed in the case just received in the school office. There are 24 boxes with a dozen pencils in each box. *Are there enough for every* *student in the school to have one?* Other, similar, real-world problems would follow, some involving regrouping and others not.

After initial exploration, students share with each other all of the strategies they've developed, the best ways they've found for working with the tens and ones in the problem, and their own approaches to dealing with the place value issues involved. Most students can, with direction, take the results of those discussions and create their own paper-and-pencil procedures for multiplication and division. The discussions can, of course, include the traditional approaches, but these ought not to be seen as *the only right way* to perform these operations.

Estimation and mental math become critically important in these grade levels as students are inclined to use calculators for more and more of their work. In order to use that technology effectively, third and fourth graders must be able to use estimation to know the range in which the answer to a given problem should lie before doing any calculation. They also must be able to assess the reasonableness of the results of a computation and be satisfied with the results of an estimation when an exact answer is unnecessary. Mental mathematics skills, too, play a more important role in third and fourth grade. Simple two-digit addition and subtraction problems and those involving powers of ten should be performed mentally. Students should have enough confidence in their ability with these types of computations to do them mentally instead of relying on either a calculator or paper and pencil.

Technology should be an important part of the environment in third and fourth grade classrooms. Calculators provide a valuable teaching tool when used to do student-programmed repeated addition or subtraction, to offer estimation and mental math practice with *target games*, and to explore operations and number types that the students have not yet formally encountered. Students should also use calculators routinely to find answers to problems that they might not be able to find otherwise. This use prevents the need to artificially contrive real-world problems so that their answers are numbers with which the students are already comfortable.

The topics that should comprise the numerical operations focus of the third and fourth grade mathematics program are:

- multiplication and division basic facts
- multi-digit whole number addition and subtraction
- two-digit whole number multiplication and division
- decimal addition and subtraction
- explorations with fraction operations

Standard 8—Numerical Operation—Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Develop meaning for the four basic arithmetic operations by modeling and discussing a variety of problems.**

    - Students broaden their initial understanding of multiplication as repeated addition by dealing with situations involving arrays, expansions, and combinations. Questions of these types are not easily explained through repeated addition: *How many stamps are on this 7 by 8 sheet? How big would this painting be if it was 3 times as big? How many outfits can you make with 2 pairs of pants and 3 shirts?*
    - Students use counters to model both repeated subtraction (*There are 12 cookies. How many bags of 3?*) and sharing (*There are 12 cookies and 3 friends. How many cookies each?*) meanings for division and write about the difference in their journals.
    - Students work through the *Sharing Cookies* lesson that is described in the First Four Standards of this *Framework*. They investigate division by using 8 cookies to be shared equally among 5 people, and discuss the problem of simplifying the number sentence which describes the amount of each person's share.
    - From the beginning of their work with division, children are asked to make sense out of remainders in problem situations. The answers to these three problems are different even though the division is the same: *How many cars will we need to transport 19 people if each car holds 5? How many more packages of 5 ping-pong balls can be made if there are 19 balls left in the bin? How much does each of 5 children have to contribute to the cost of a $19 gift?*
    - Students explore division by reading *The Doorbell Rang* by Pat Hutchins. In this story, Victoria and Sam must share 12 cookies with increasing numbers of friends. Students can use counters to show how many cookies each person gets.
    - Students learn about multiplication as an array by reading *One Hundred Hungry Ants* by Elinor Pinczes, *Lucy and Tom's 1, 2, 3* by Shirley Hughes or *Number Families* by Jane Srivastava.
    - Students make books showing things that come in *3's, 4's, 5's, 6's, or 12's.*

2. **Develop proficiency with and memorize basic number facts using a variety of fact strategies (such as "counting on" and "doubles").**

    - Students use *streets and alleys* as both a mental model of multiplication and a useful way to recover facts when needed. It simply involves drawing a series of horizontal lines (streets) to represent one factor and a series of vertical lines (alleys) crossing them to represent the other. The number of intersections of the *streets* and *alleys* is the product!

- Students use a *double maker* on a calculator for practice with doubles. They enter x *2* = on the calculator. Any number pressed then, followed by the equal sign, will show the number's double. Students work together to try to say the double for each number before the calculator shows it.

- Students regularly use *doubles, near doubles,* and *use a related fact* strategies for multiplication; they are using the *near doubles* strategy when they calculate a sum like $15 + 17$ by recognizing that it is 2 more than double *15*. More generally, they are using the *use a related fact* strategy when they use any fact they happen to remember, like $8 + 4 = 12$, to make a related calculation like $8 + 5 = 12 + 1 = 13$. They also recover facts by knowledge of the role of zero and one in multiplication, of commutativity, and of the regular patterned behavior of multiples of nines. Practice sets of problems are structured so that use of all these strategies is encouraged and the students are regularly asked to explain the procedures they are using.

- Pairs of students play *Circles and Stars* (Burns, 1991). Each student rolls a die and draws as many circles as the number shown, then rolls again and puts that number of stars in every circle, and then writes a multiplication number sentence and records how many stars there are all together. Each student takes seven turns, and adds the total. The winner is the student with the most stars.

- Students use color tiles to show how a given number of candies can be arranged in a rectangular box.

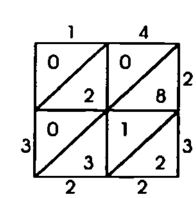
- Students play *multiplication war,* using a deck of cards with kings and queens removed. All of the cards are dealt out. Each player turns up two cards and multiplies their values (Jacks count as *0*; aces count as *1*). The "general" draws a target number from a hat. The player closest to the target wins a point. The first player to get *10* points wins the game.

- Students use computer programs such as *Math Workshop* to practice multiplication facts.

3. Construct, use, and explain procedures for performing whole number calculations in the various methods of computation.

- Students work through the *Product and Process* lesson that is described in the Introduction to this *Framework*. It challenges students to use calculators and four of the five digits *1, 3, 5, 7,* and *9* to discover the multiplication problem that gives the largest product.

- Students explore lattice multiplication and try to figure out how it works. For example, the figure at the right shows $14 \cdot 23 = 322$.



- Students use the skills they've developed with *arrow puzzles* (See Standard 6—Number Sense—Grades 3-4—Indicator 3) to practice mental addition and subtraction of 2- and 3-digit numbers. To add *23* to *65*, for instance, they start at *65* on their "mental hundred number chart," go down twice and to the right three times.

- Students use base ten blocks to help them decide how many blocks there would be in each group if they divided 123 blocks among 3 people. The students describe how they used the blocks to help them solve the problem and compare their solutions and solution strategies.

4. **Use models to explore operations with fractions and decimals.**

- Students use *fraction circle* pieces (each unit fraction a different color) to begin to explore addition of fractions. Questions like: *Which of these sums are greater than 1?* and *How do you know?* are frequent.
- Students use the base ten models that they are most familiar with for whole numbers and relabel the components with decimal values. Base ten blocks represent 1 whole, 1 tenth, 1 hundredth, and 1 thousandth. Coins, which had represented a whole number of cents, now represent hundredths of dollars.
- Students operate a school store with school supplies available for sale. Other students, using play money, decide on purchases, pay for them, receive and check on the amount of change.
- In groups, students each roll a number cube and use dimes to represent the decimal rolled. For example, a student rolling a *4* would take 4 dimes to represent 4 tenths of a dollar. When a student gets 10 dimes, he turns them in for a dollar. The first student to get $5 wins the game.
- Students use money to represent fractions. For example, a quarter and a quarter equals half a dollar.
- Students demonstrate equivalent fractions using pattern blocks. For example, if a yellow hexagon is one whole, then three green triangles (*3/6*) is the same size as one red trapezoid (*1/2*). Pattern blocks may also be used to represent addition and subtraction of fractions.

5. **Use a variety of mental computation and estimation techniques.**

- Students frequently do warm-up drills that enhance their mental math skills. Problems like: *3,000* x *7* = , *200* x *6* = , and *5,000* x *5* + *5* = are put on the board as individual children write the answers without doing any paper-and-pencil computation.
- Students make appropriate choices from among *front-end*, *rounding*, and *compatible numbers* strategies in their estimation work depending on the real-world situation and the numbers involved. *Front end* strategies involve using the first digits of the largest numbers to get an estimate, which of course is too low, and then adjusting up. *Compatible numbers* involves finding some numbers which can be combined mentally, so that, for example, *762* + *2,444* + *248* is about (*750* + *250)* + *2,500*, or *3,500*.
- Students use money and shopping situations to practice estimation and mental math skills. *Is $20.00 enough to buy items priced at $12.97, $4.95, and 3.95? About how much would 4 cans of beans cost if each costs $0.79?*
- Students explore estimation involving division as they read *The Greatest Guessing Game: A Book about Dividing* by Robert Froman. A little girl and her three friends solve a variety of problems, estimating first and discussing what to do with remainders.

6. **Select and use appropriate computational methods from mental math, estimation, paper-and-pencil, and calculator methods, and check the reasonableness of results.**

   - Students play *addition max out*. Each student has a 2 x 3 array of blanks (in standard 3-digit addition form) into each of which will be written a digit. One student rolls a die and everyone must write the number showing into one of their blanks. Once the number is written in, it can not be changed. Another roll — another number written, and so on. The object is to be the player with the largest sum when all six digits have been written. If a player has the largest possible sum that can be made from the six digits rolled, there is a bonus for *maxing out*.

   - Students discuss this problem from the NCTM Standards (p. 45): *Three fourth grade teachers decided to take their classes on a picnic. Mr. Clark spent $26.94 for refreshments. He used his calculator to see how much the other two teachers should pay him so that all three could share the cost equally. He figured they each owed him $13.47. Is his answer reasonable?* As a follow-up individual assessment, they write about how they might find an answer.

7. **Understand and use relationships among operations and properties of operations.**

   - Students take *7x8* block rectangular grids printed on pieces of paper. They each cut along any one of the *7* block-long segments to produce two new rectangles, for example, a *7x6* and a *7x2* rectangle. They then discuss all of the different rectangle pairs they produced and how they are all related to the original one.

   - Students write a letter to a second grader explaining why *2+5* equals *5+2* to demonstrate their understanding of commutativity.

   - Students explore modular, or clock, addition as an operation that behaves differently from the addition they know how to do. For example: *6 hours after 10 o'clock in the morning is 4 o'clock in the afternoon, so 10 + 6 = 4 on a 12-hour clock. How is clock addition different from regular addition? How is it the same? How would modular subtraction and multiplication work?*

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On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

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Standard 8 — Numerical Operations — Grades 5-6

Overview

As indicated in the K-12 Overview, the widespread availability of computing and calculating technology has given us the opportunity to significantly reconceive the role of computation and numerical operations in our fifth and sixth grade mathematics programs. Some skills are less important while others, such as **estimation, mental computation,** and **understanding the meanings of the standard arithmetic operations,** all play a more significant role than ever in the everyday life of a mathematically literate adult.

The major shift in the curriculum that will take place in grades 5 and 6, therefore, is one away from drill and practice of paper-and-pencil symbolic procedures and toward real-world applications of operations, wise **choices of appropriate computational strategies**, and integration of the numerical operations with other components of the mathematics curriculum. At these grade levels, students are consolidating their understanding of whole number operations (especially multiplication and division) and beginning to develop computational skills with fractions and decimals. A sample unit on fractions for the sixth-grade level can be found in Chapter 17 of this *Framework*.

Much research in the past decade has focused on students' understandings of operations with large whole numbers and work with fractions and decimals. Each of these areas requires students to restructure their simple conceptions of number that were adequate for understanding whole number addition and subtraction.

Multiplication requires students to think about different meanings for the two factors. The first factor in a multiplication problem is a "multiplier." It tells how many groups one has of a size specified by the second factor. Thus, students need different understandings of the roles of the two numbers in the operation of multiplication than their earlier understandings of addition, in which both addends meant the same thing.

A similar restructuring is necessary for dealing appropriately with operations involving fractions and decimals. This restructuring revolves around the role of the "unit" in these numbers. In earlier grades, students thought about *5* or *498* as numbers that represented that many *things*. The understood unit, one, is the number which was used to count a group of objects. With fractions and decimals, though, the unit, still one and still understood, is a harder concept to deal with because its essential use is to help define the fraction or decimal rather than as a counter. When we speak of *5 poker chips* or *35 students*, our message is reasonably clear to elementary students. But when we speak of *2/3 of the class* or *0.45 of the price of the sweater*, the meaning is significantly less clear and we must be much more explicit about the role being played by the unit.

The topics that should comprise the numerical operations focus of the fifth and sixth grade mathematics program are:

multi-digit whole number multiplication and division
decimal multiplication and division
fraction operations
integer operations

Standard 8 — Numerical Operations — Grades 5-6

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

6*. Select and use appropriate computational methods from mental math, estimation, paper-and-pencil, and calculator methods, and check the reasonableness of results.

- Fifth and sixth grade students have calculators available to them at all times, but frequently engage in competitions to see whether it is faster to do a given set of computations with the calculators or with the mental math techniques they've learned.
- Fifth graders make rectangular arrays with base-ten blocks to try to figure out how to predict how many square foot tiles they will need to tile a *17' by 23'* kitchen floor.
- Students are challenged to answer this question and then discuss the appropriate use of estimation when an exact answer is almost certain to be wrong: *The Florida's Best Orange Grove has 15 rows of 21 orange trees. Last year's yield was an average of 208.3 oranges per tree. How many oranges might they expect to grow this year? What factors might affect that number?*
- Students play *multiplication max out*. Each student has a 2 x 2 array of blanks (in the standard form of a 2-digit multiplication problem) into each of which a digit will be written. One student rolls a die and everyone must write the number showing into one of the blanks. Once a number is written, it cannot be moved. Another roll—another number written, and so on. The object is to be the player with the largest product when all four digits have been written. If a player has the largest possible product that can be made from the four digits rolled, there is a bonus for *maxing out*.

8. Extend their understanding and use of arithmetic operations to fractions, decimals, integers, and rational numbers.

- Students work in groups to explore fraction multiplication and division. They use *fraction circles* and *fraction strips* to solve problems like: *How can you divide four cakes among five people evenly?* They solve the problems and then write in their math journals about the methods they used and the reasons they believe their answers to be good ones.
- Students complete their study of fraction addition and subtraction by reading about Egyptian fractions. The Egyptians wrote every fraction as a unit fraction or the sum of a series of unit fractions with different denominators, for example, $7/8 = 1/2 + 1/4 + 1/8$.

* Activities are included here for Indicator 6, which is also listed for grade 4, since the Standards specify that students demonstrate continued progress in this indicator.

They try to find Egyptian fractions for *2/3 (1/2 + 1/6); 2/5 (1/3 + 1/15);* and *4/5 (1/2 + 1/5 + 1/10).*

- Students demonstrate their understanding of division of fractions on a test by drawing a picture to show that *"1½ ÷ ½"* means: *How many halves are there in 1½?*
- Students use two-color chips to explore addition of integers. They each take ten chips and toss them ten times. Each time, the students record the number of yellow chips as a positive number (points earned) and the number of red chips as a negative number (points lost). For each toss, the student writes a number sentence, such as $6 + -4 = 2$ for *6* points earned and *4* lost. Students may also keep a running total of points overall.
- Students read and discuss *If You Made a Million* by David Schwartz, relating money to decimals.
- Students read and discuss "Beasts of Burden" in *The Man Who Counted: A Collection of Mathematical Adventures* by Malba Tahan. In this story, three brothers must divide their Father's 35 camels so that one gets 1/2 of the camels, another 1/3, and the last 1/9. The narrator and a wise mathematician help them solve the problem by adding their camel to the 35, making 36. One brother then gets 18, another 12, and the third receives 4 making a total of 34. The narrator and mathematician take back their camel as well as the one left over.
- Students demonstrate their understanding of addition and subtraction of unlike fractions on a test by finding the errors made by a fictitious student and explaining to that student what he/she did wrong.
- Students read Shel Silverstein's poem *A Giraffe and a Half* and make up stories about mixed fractions.
- Students discuss whether *2/3* x *5/4* is more or less than *2/3*. They explain their reasoning.
- Students listen to John Ciardi's poem "Little Bits" and discuss the concept that a whole can be described by an infinite number of equivalent names, such as *1/2 + 1/4 + 1/8 + 1/8* or *3/3*.
- Students solve *missing link* problems, like the one below, in which they must find number(s) and/or symbols that will make a true sentence:

$$\_\_\_\_ + 3.1 - \_\_\_\_ - 5.4 = 8.7$$

9. Extend their understanding of basic arithmetic operations on whole numbers to include powers and roots.

- Students explore the exponent key, the x^2 key, and the square root key on their calculators. The groups are challenged to define the function of each key, to tell how each works, and to create a keypress sequence using these keys, the result of which they predict before they key it in.
- Students work through *The Powers of the Knight* lesson that is described in the Introduction to this *Framework*. It introduces a classic problem of geometric growth which engages them as they encounter notions of exponential notation.
- Students work through the *Pizza Possibities* lesson that is described in the First Four

Standards of this *Framework*. In it, students discover that the number of pizzas possible doubles every time another choice of topping is added. They work through the *Two-Toned Towers* lesson that is also described in the First Four Standards and note the similarities in the problems and in their solutions.

- Students join the midpoints of the sides of a 2 x 2 square on a geoboard to form a smaller square. They determine the area of the smaller square and explore the lengths of its four sides.

10. Develop, apply, and explain procedures for computation and estimation with whole numbers, fractions, decimals, integers, and rational numbers.

- Students working in groups develop a method to estimate the products of two-digit whole numbers and decimals by using the kinds of base-ten block arrays described in Indicator 6 above. Usually just focusing on the "flats" results in a reasonable estimate.
- Students follow up a good deal of experience with concrete models of fraction operations using materials such as *fraction bars* or *fraction squares* by developing and defending their own paper-and-pencil procedures for completing those operations.
- Students develop rules for integer operations by using *postman stories*, as described in Robert Davis' *Discovery in Mathematics*. The teacher plays the role of a postman who delivers mail to the students. Sometimes the mail delivered contains money (positive integers) and sometimes bills (negative integers). Sometimes they are delivered to the students (addition) and sometimes picked up from them (subtraction).
- Students model subtraction with two-color chips by adding pairs of red and yellow chips. (First, they must agree that an equal number of red and yellow chips has a value of *0*.) For example, to show $4 - (-2)$, they lay out 4 red chips, add 2 pairs of red and yellow chips (whose value is *0*), and then take away 2 yellow chips. They note that $4 - (-2)$ and $4 + 2$ give the same answer and try to explain why this is so.

11. Develop, apply, and explain methods for solving problems involving proportions and percents.

- Students develop an estimate of π by carefully measuring the diameter and circumference of a variety of circular objects (cans, bicycle tires, clocks, wooden blocks). They list the measures in a table and discuss observations and possible relationships. After the estimate is made, π is used to solve a variety of real-world circle problems.
- Students use holiday circulars advertising big sales on games and toys to comparison shop for specific items between different stores. *Is the new Nintendo game, Action Galore, cheaper at Sears where it is 20% off their regular price of $49.95 or at Macy's where it's specially priced at $41.97?*
- One morning, as the students arrive at school, they see a giant handprint left on the blackboard overnight. They measure it and find it to be almost exactly one meter long. *How big was the person who left the print? Could she have fit in the room to make the print, or did she have to reach in through the window? How could you decide how much she weighs?*
- Students read *Jim and the Giant Beanstalk* by Raymond Briggs. Jim helps the aging giant

by measuring his head and getting giant eyeglasses, false teeth and a wig. The students use the measures given in the book to find the size of the giant's hand and then his height.

- Students develop a sampling strategy and use proportions to determine the population of *Bean City* (*NCTM Addenda Booklet, Grade 6*), whose inhabitants consist of three types of beans.
- Students discuss different ways of finding "*easy*" percents, such as *50%* of *30* or *15%* of *25*. They then generate percent exercises that can be solved mentally and share them with their classmates.

12. Understand and apply the standard algebraic order of operations.

- Students bring in calculators from home to examine their differences. Among other activities, they each key in "*6 + 2* x *4 =* " and then compare their calculator displays. Some of the displays show *32* and others show *14*. *Why? Which is right? Are the other calculators broken?*
- Students play *rolling numbers*. They use four white dice and one red one to generate four working numbers and one target number. They must combine all of the working numbers using any operations they know to formulate an expression that equals the target number. For example, for *2, 3, 4, 5* with target number *1*, the following expression works: $(2+5)/(3+4)=1$. Questions about order of operations and about appropriate use of parentheses frequently arise.

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Standard 8 — Numerical Operations — Grades 7-8

Overview

Traditionally, tremendous amounts of time were spent at these grade levels helping students to finish their development of complex paper-and-pencil procedures for the four basic operations with whole numbers, fractions, and decimals. While some competency with paper-and-pencil computation is necessary, **estimation, mental computation**, and **understanding the meanings of the standard arithmetic operations** all play a more significant role than ever in the everyday life of a mathematically literate adult.

As indicated in the K-12 Overview, then, the major shift in the curriculum that will take place at these grade levels is one away from drill and practice of paper-and-pencil symbolic procedures and toward **real-world applications of operations**, wise **choices of appropriate computational strategies**, and **integration of the numerical operations with other components of the mathematics curriculum.**

Seventh- and eighth-graders are relatively comfortable with the unit shift discussed in this standard's Grades 5-6 Overview. Operations on fractions and decimals, as well as whole numbers, should be relatively well developed by this point, allowing the focus to shift to a more holistic look at operations. "Numerical operations" becomes less a specific object of study and more a process, a set of tools for problem setting. It is critical that teachers spend less time focused on numerical operations, per se, so that the other areas of the *Standards*-based curriculum receive adequate attention.

One important set of related topics that needs to receive some significant attention here, however, is **ratio, proportion,** and **percent**. Seventh and eighth graders are cognitively ready for a serious study of these topics and to begin to incorporate proportional reasoning into their set of problem solving tools. Work in this area should start out informally, progressing to the student formulation of procedures that make proportions and percents the powerful tools they are.

Two other topics that receive greater attention here, even though they have been informally introduced earlier, are **integer operations** and **powers and roots**. Both of these types of operations further expand the students' knowledge of the types of numbers that are used and the ways in which they are used.

Estimation, mental math, and **technology use** begin to mature in seventh and eighth grades as students use these strategies in much the same way that they will as adults. If earlier instruction in these skills has been successful, students will be able to make appropriate choices about which computational strategies to use in given situations and will feel confident in using any of these in addition to paper-and-pencil procedures. For example, students should evaluate simple problems involving fractions, such as *what's two-thirds of 5 tablespoons?* using mental math. Students need to continue to develop alternatives to paper-and-pencil as they learn more about operations on other types of numbers, but the work here is primarily on the continuing use of all of the strategies in rich real-world problem solving settings.

The topics that should comprise the numerical operations focus of the seventh and eighth grade mathematics program are:

rational number operations	powers and roots
integer operations	proportion and percent

Standard 8 — Numerical Operations — Grades 7-8

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

6*. Select and use appropriate computational methods from mental math, estimation, paper-and-pencil, and calculator methods, and check the reasonableness of results.

- Students choose a stock from the New York Stock Exchange and estimate and then compute the net gain or loss each week for a $1,000 investment in the company.
- Students use spreadsheets to "program" a set of regular, repeated, calculations. They might, for example, create a prototype on-line order blank for a school supply company that lists each of the ten items available, the individual price, a cell for each item in which to place the quantity ordered, the total computed price for each item, and the total price for the order.
- Students regularly have *human vs. calculator races*. Given a list of specially selected computation exercises (e.g., *53* x *20, 40* x *10, 95 + 17 +5*), one student uses mental math strategies and another uses a calculator. They quickly come to realize that the human has the advantage in many situations.

8. Extend their understanding and use of arithmetic operations to fractions, decimals, integers, and rational numbers.

- Given a decimal or a fractional value for a piece of a tangram puzzle, the students determine a value for each of the other pieces and a value for the whole puzzle.
- Students use *fraction squares* to show why the multiplication of two fractions less than one results in a product that is less than either.
- Students demonstrate their understanding of operations on rational numbers by formulating their own reasonable word problems to accompany given number sentences such as *3/4 divided by 1/2 = 1½*.

9. Extend their understanding of basic arithmetic operations on whole numbers to include powers and roots.

- Students play *powers max out*. Each student has a set of 5 blanks, into each of which will be written a digit. They are in the form $VW^X + YZ$. One student rolls a die and everyone must write the *number* showing into one of their blanks. Once written, a

* Activities are included here for Indicator 6, which is also listed for grade 4, since the Standards specify that students demonstrate continued progress in this indicator.

number can not be moved. Another roll — another number written, and so on. The object is to be the player with the largest-valued expression when all five digits have been written. If a player has the largest possible value that can be made from the five digits rolled, there is a bonus for *maxing out*.

- Students develop their own "rules" for operations on numbers raised to powers by rewriting the expressions without exponents. For example, $7^2 \times 7^4 = (7 \times 7) \times (7 \times 7 \times 7 \times 7) = 7 \times 7 \times 7 \times 7 \times 7 \times 7 = 7^6$. *You just add the exponents!*

- Students read *The King's Chessboard, The Rajah's Rice: A Mathematical Folktale from India*, or *A Grain of Rice*. All of these stories involve a situation in which a quantity is doubled each day. Students use the story to discuss powers of 2 and to look for patterns in the sums of the powers of 2.

- Students use the relationship between the area of a square and the length of one of its sides to begin their study of roots. Starting with squares on a geoboard with areas of *1*, *4*, *9*, and *16*, they then are asked to find squares whose areas are *2*, *5*, and *13*.

- Students work through the *Rod Dogs* lesson that is described in the First Four Standards of this *Framework*. They investigate how the surface area and volume of an object changes as it is enlarged by various scale factors.

10. Develop, apply, and explain procedures for computation and estimation with whole numbers, fractions, decimals, integers, and rational numbers.

- Students use a videotape of a youngster walking forward and backward as a model for multiplication of integers. The "product" of running the tape forward (+) with the student walking forward (+) is walking forward (+). The "product" of running it backward (-) with the student walking forward (+) is walking backwards (-). The other two combinations also work out correctly.

- Students use base ten blocks laid out in an array to show decimal multiplication. *How could the values of the blocks be changed to allow it to work? What new insights do we gain from the use of this model?*

- Students judge the reasonableness of the results of fraction addition and subtraction by "rounding off" the fractions involved to *0, 1/2, or 1*.

- Students explore the equivalence between fractions and repeated decimals by finding the decimal representations of various fractions and using the resulting patterns to find the fractional equivalents of some repeated decimals.

11. Develop, apply, and explain methods for solving problems involving proportions and percents.

- Students use *The Geometer's Sketchpad* software to draw a geometric figure on a computer screen, scale it larger or smaller, and then compare the lengths of the sides of the original with those of the scaled image. They also compare the areas of the two.

- Students are comfortable using a variety of approaches to the solution of proportion problems. Example: *If 8 pencils cost 40¢, how much do 10 pencils cost?* This problem can be solved by:

unit-rate method	*8 pencils for 40¢ means 5¢/pencil or 10* x *5=50¢ for 10*
factor-of-change method	*10 pencils is 10/8 of 8 pencils, so cost is (10/8)*x*40=50¢*
cross multiplication method	*8/40 = 10/*x, *8*x *= 400, so* x*=50¢.*

- Students set up a part/whole proportion as one method of solving percent problems.
- Students spend $100 by selecting items from a catalog. They must compute sales tax and consider it in deciding what they will buy.

12. **Understand and apply the standard algebraic order of operations.**
    - Students bring in calculators from home to examine their differences. Among other activities, they each key in "*3 + 15 ÷ 3*" and then compare their calculator displays. Some of the displays show *6* and others show *8*. *Why? Which is right? Are the other calculators broken?* Students decide what key sequence would work for the calculators that do not use order of operations.
    - Students play with the software *How the West was One + Three* x *Four*, which requires them to construct numerical expressions that use the standard order of operations.
    - Students use the digits *1, 2, 3,* and *4* to find expressions for each of the numbers between *0* and *50*. For example, *7 = (3*x*4)/2 + 1*.

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Barry, David. *The Rajah's Rice: A Mathematical Folktale from India.* San Francisco, CA: W. J. Freeman, 1995.

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Software

Geometer's Sketchpad. Key Curriculum Press.

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On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 8 — Numerical Operations — Grades 9-12

Overview

In the ninth through twelfth grades, the themes described in the K-12 Overview — **estimation, mental computation,** and **appropriate calculator and computer use**—become the focus of this standard. What is different about this standard at this level when compared to the traditional curriculum is its mere presence. In the traditional academic mathematics curriculum, work on numerical operations was basically finished by eighth grade and focus then shifted exclusively to the more abstract work in algebra and geometry. But, in the highly technological and data-driven world in which today's students will live and work, strong skills in numerical operations have perhaps even more importance than they once did. By giving older students a variety of approaches and strategies for the computation that they encounter in everyday life, approaches with which they can confidently approach numerical problems, they will be better prepared for their future.

The major work in this area, then, that will take place in the high school grades, is continued opportunity for **real-world applications of operations**, wise **choices of appropriate computational strategies**, and **integration of the numerical operations with other components of the mathematics curriculum.**

The new topics to be introduced in this standard for these grade levels involve factorials, matrices, operations with polynomials, and operations with irrational numbers as useful tools in problem solving situations.

Estimation, mental math, and **technology use** should fully mature in the high school years as students use these strategies in much the same way that they will as adults. If earlier instruction in these skills has been successful, students will be able to make appropriate choices about which computational strategies to use in given situations and will feel confident in using any of these in addition to paper-and-pencil techniques. Students need to continue to develop alternatives to paper-and-pencil as they learn about operations with matrices and other types of number, but the work here is almost exclusively on the continuing use of all of the strategies in rich, real-world, problem solving settings.

The topics that should comprise the numerical operations focus of the ninth through twelfth grade mathematics program are:

- operations on real numbers
- translation of arithmetic skills to algebraic operations
- operations with factorials, exponents, and matrices

Standard 8 — Numerical Operations — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11 and 12.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

6*. Select and use appropriate computational methods from mental math, estimation, paper-and-pencil, and calculator methods, and check the reasonableness of results.

- Students frequently use all of these computational strategies in their ongoing mathematics work. Inclinations to over-use the calculator, in situations where other strategies would be more appropriate, are overcome with five minute "contests," speed drills, and warm-up exercises that keep the other skills sharp and point out their superiority in given situations.
- Numerical problems in class are almost always worked out in "rough" form before any precise calculation takes place so that everyone understands the "ballpark" in which the computed answer should lie and which answers would be considered unreasonable.
- Students use estimation in their work with irrational numbers, approximating the results of operations such as $\sqrt{15} + \sqrt{17}$ or $\sqrt{32}\ \sqrt{8}$, and developing general rules.
- Students discuss the advantages and disadvantages of using graphing calculators or computers to perform computations with matrices.
- Students demonstrate their ability to select and use appropriate computational methods by generating examples of situations in which they would choose to use a calculator, to estimate, or to use mental math.
- Students solve given computational problems using an assigned strategy and discuss the advantages and disadvantages of using that particular strategy with that particular problem.

13. Extend their understanding and use of operations to real numbers and algebraic procedures.

- Students work on the painted cube problem to enhance their skill in writing algebraic expressions: *A 3-inch cube is painted red. It is then cut into 1-inch cubes. How many of them have 3 red faces? 2 red faces? 1-red face? No red faces? Repeat the problem using an original 4-inch cube, then a five-inch cube, then an n-inch cube.*
- Students develop a procedure for binomial multiplication as an extension of their work with 2-digit whole number multiplication arrays. Using *algebra tiles*, they uncover the parallels between 23×14 (which can be thought of as $(20+3)(10+4)$) and $(2x+3)(x+4)$.
- Students work through the *Ice Cones* lesson that is described in the First Four Standards of

* Activities are included here for Indicator 6, which is also listed for grade 8, since the Standards specify that students demonstrate continued progress in this indicator.

this *Framework*. They discover that in order to graph the equation to determine the maximum volume of the cones, they need to use algebraic procedures to solve for h in terms of r.

- Students devise their own procedures and "rules" for operations on variables with exponents by performing trials of equivalent computations on whole numbers.
- Students use *algebra tiles* to develop procedures for adding and subtracting polynomials.
- Students use compasses and straightedges to construct a Golden Rectangle and find the ratio of the length to the width $(1 + \sqrt{5})/2$.
- Students consider the ratios of successive terms of the Fibonacci sequence *(1, 1, 2, 3, 5, 8, ...)*, where each term after the first two is the sum of the two preceding terms, finding that the ratios get closer and closer to the Golden Ratio $(1 + \sqrt{5})/2$.

14. Develop, apply, and explain methods for solving problems involving factorials, exponents, and matrices.

- Students work through the *Breaking the Mold* lesson that is described in the Introduction to this *Framework*. It uses a science experiment with growing mold to involve students in discussions and explorations of exponential growth.
- Students use their graphing calculators to find a curve that best fits the data from the population growth in the state of New Jersey over the past 200 years.
- Students discover the need for a factorial notation and later incorporate it into their problem solving strategies when solving simple combinatorics problems like: *How many different five card poker hands are there? In how many different orders can four students make their class presentations? In how many different orders can six packages be delivered by the letter carrier?*
- Students compare 2^{101}, $(2^{50})^2$, and $3x2^{100}$ to decide which is largest. They explain their reasoning.
- Students read "John Jones's Dollar" by Harry Keeler and discuss how it demonstrates exponential growth. They check the computations in the story, determining their accuracy.

References

Keeler, Harry Stephen. "John Jones's Dollar," in Clifton Fadiman, Ed. *Fantasia Mathematica.* New York: Simon and Schuster, 1958.

On-Line Resources

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STANDARD 9 — MEASUREMENT

K-12 Overview

All students will develop an understanding of and will use measurement to describe and analyze phenomena.

Descriptive Statement

Measurement helps describe our world using numbers. We use numbers to describe simple things like length, weight, and temperature, but also complex things such as pressure, speed, and brightness. An understanding of how we attach numbers to those phenomena, familiarity with common measurement units like inches, liters, and miles per hour, and a practical knowledge of measurement tools and techniques are critical for students' understanding of the world around them.

Meaning and Importance

Measurement is important because it helps us to quantify the world around us. Although it is perfectly natural to think about length, area, volume, temperature, and weight as attributes of objects that we measure, a little reflection will produce many other measurable quantities: speed, loudness, pressure, and brightness, to name just a few. An understanding of the processes of measurement, the concept of a unit, and a familiarity with the tools and common units of measurement, are all critical for students to develop an understanding of the world around them.

This standard is also, in many ways, the prototypical "integrated" standard because of its strong and essential ties to almost every one of the other content standards. Measurement is an ideal context for dealing with numbers and with numerical operations of all sorts and at all levels. Fractions and decimals appear very naturally in real-world measurement settings. Metric measures provide perhaps the most useful real-world model of a base-ten numeration system we can offer to children. Similarly geometry and measurement are almost impossible to consider separately. For instance, treatments of area and perimeter are called "measurement" topics in some curricula and "geometry" topics in others because they are, quite simply, measurements of geometric figures. Another of the content standards which is inextricably linked to measurement is estimation. Estimation of measures should be a focus of any work that students do with measurement. Indeed, the very concept that any continuous measurement is inexact — that it is at best an "estimate" — is a concept that must be developed throughout the grades.

Think about how many different content standards are incorporated into one simple measurement experience for middle school students: the measurement of a variety of circular objects in an attempt to explore the relationship between the diameter and circumference of a circle. Clearly involved are the *measurement* and *geometry* of the situation itself, but also evident are opportunities to deal with *patterns* in the search for regularity of the relationship, *estimation* in the context of error in the measurements, and *number sense* and *numerical operations* in the meaning of the ratio that ultimately emerges.

K-12 Development and Emphases

Throughout their study and use of measurement, students should be confronted explicitly with the important **concept of a measurement unit.** Its understanding demands the active involvement of the learner; it is simply not possible to learn about measurement units without measuring things. The process of measurement can be thought of as *matching* or *lining up* a given unit, as many times as possible, with the object being measured. For instance, in its easiest form, think about lining up a series of popsicle sticks, end to end, to see how many it takes to cover the width of the teacher's desk; or consider how many pennies it takes to balance the weight of a small box of crayons on a pan balance. At a slightly more sophisticated level, multiple units and more standard units might be used to add precision to the answers. The desk might be measured with as many orange Cuisenaire Rods as will fit completely and then with as many centimeter cubes as will fit in the space remaining; the crayons, with as many ten-gram weights as can be used and then one-gram weights to get a better estimate of its weight. These types of activities — this active iteration of units — make the act of measurement and the relative sizes of units significantly more meaningful to children than simply reading a number from a measurement instrument like a meter stick or a postal scale. Of course, as the measures themselves become the focus of study, rather than the act of measurement or the use of actual physical units, students should become knowledgeable in the use of a variety of instruments and processes to quickly and accurately determine measurements.

Much research has dealt with the development of children's understanding of measurement concepts, and the general agreement in the findings points to a need for coherent sequencing of curriculum. Young children start by learning to identify the attributes of objects that are measurable and then progress to direct comparisons of those attributes among a collection of objects. They would suggest, for instance, that this stick is *longer than* that one or that the apple is *heavier than* the orange. Once direct comparisons can consistently be made, informal, non-standard units like pennies or "my foot" can be used to quantify how heavy or how long an object is. Following some experiences which illustrate the necessity of being able to replicate the measurements regardless of the measurer or the size of the measurer's foot, these non-standard units quickly give way to **standard, well-defined units** like inches and grams.

Older students should continue to develop their notions of measurement by delving more deeply into the process itself and by measuring more complex things. Dealing with various measurement instruments, they should consider questions concerning the inexact nature of their measures, and to adjust for, or account for, the inherent **measurement error** in their answers. Issues of the **degree of precision** should become more important in their activities and discussion. They need to appreciate that no matter how accurately they measure, more precision is always possible with smaller units and better instrumentation. Decisions about what level of precision is necessary for a given task should be discussed and made before the task is begun, and revised as the task unfolds.

Older students should also begin to develop procedures and formulas for determining the measures of attributes like area and volume that are not easily measured directly, and to develop **indirect measurement techniques** such as the use of similar triangles to determine the height of a flagpole. Their universe of measurable attributes should expand to include measures of a whole variety of physical phenomena (sound, light, pressure) and a consideration of rates as measures (pulse, speed, radioactivity).

Connections are another strong focus of students' work with measurement. The growth of technology in schools opens up a wide range of new possibilities for students of all ages. Inexpensive instruments attached to appropriately programmed graphing calculators and computers are capable of making and recording measurements of temperature, distance, sound and light intensity, and many other physical phenomena. The calculators and computers may then be used to graph those measurements with respect to time or any other measure, to present them in tabular form, or to manipulate them in other ways. These opportunities for scientific data collection and analysis using this technology are unlike any that have been available to mathematics and science teachers in the past and hold great promise for real-life investigations and for the integration of these two disciplines.

IN SUMMARY, measurement offers us the challenge to actively and physically involve students in their learning as well as the opportunity to tie together seemingly diverse components of their mathematics curriculum like fractions and geometry. It is also one of the major vehicles by which we can bring the real worlds of other disciplines such as the natural and social sciences, health, and physical education into the mathematics classroom.

NOTE: *Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

Standard 9 — Measurement — Grades K-2

Overview

Students can develop a strong understanding of measurement and measurement systems from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding, as identified in the K-12 Overview, are: **the concept of a measurement unit; standard measurement units; connections to other mathematical areas and to other disciplines; indirect measurement**; and, for older students, **measurement error** and **degree of precision.**

Students in the early grades encounter measurement in many situations, from their daily work with the calendar, to situations in stories that they are reading, to describing how quickly they are growing. Hands-on science activities often require students to measure objects or compare them directly. Daily calendar activities frequently offer work with temperature, time, and money, in addition to number. Thus, many opportunities for **connections** present themselves in a natural way.

The study of measurement also encourages students to develop their number sense and to practice their counting skills. By using measures, students can recognize that numbers are often used to describe and compare properties of physical objects. Students in the early grades should make estimates not only of discrete objects like marbles or seeds but also of continuous properties like the length of a jumprope or the number of children's feet which might fit in a dinosaur's footprint.

Students need to focus on identifying the property that they wish to measure. They should understand what is meant by the length of an object or its weight or its capacity. Concrete experiences in describing the properties of objects, in sorting objects, and in comparing and contrasting objects provide them with opportunities to develop these concepts.

Students begin by making direct comparisons. *Which string is longer? Which child is taller? Which rock is heavier? Which glass holds more?* Making comparisons will help children better to understand the properties which they are discussing. They also begin to make some **indirect measurements.** For example, in order to compare the height of the blackboard with the height of a window, they might measure both objects using links and then compare the number of links used for each. In this way, they start to see a need for a **measurement unit,** a unit that they can use over and over to compare to a variety of objects.

In grades K-2, students should use a variety of non-standard units to measure objects. *How many links long is a desk? How many erasers high are you? How many pennies balance a Unifix cube?* In each case, students should first be asked to make an estimate and then proceed to actually measure the object. Students should also use different units to measure the same object. They should begin to understand that when the size of a measuring unit increases, the number of units needed to measure the object decreases.

In these grades, students also begin to use **standard measurement units** and standard measurement devices such as rulers and scales. It is important that the students see the use of the standard devices as simply an extension of their earlier activities. For example, the use of an inch ruler is just a more efficient procedure than lining up a series of cubic inch blocks. Students should explore length using inches, feet, centimeters and meters; liquid capacity using quarts, pints, cups, and liters; mass/weight using pounds, ounces, grams, and kilograms; time using days, weeks, months, years, seconds, minutes, and hours; and temperature using degrees Fahrenheit and Celsius.

Whether making direct comparisons, using non-standard units, or using standard measurement units, students in the early grades should always estimate a measure first and then perform the measurement. In this way, their estimation and number sense skills will be reinforced.

Standard 9 — Measurement — Grades K-2

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Use and describe measures of length, distance, capacity, weight, area, volume, time, and temperature.**

   - Students find out how many cubes long their hand is. The class can then generate a graph showing the results.
   - Using a large map of the school community, students estimate and then use paper clips or links to measure who lives farthest from school. This type of activity might be related to a specific story that was used in the Social Studies unit on community.
   - Students name objects big enough to hold a football or too small to hold a soccer ball.
   - Students lay out a model zoo with several toy animals, using boxes of different sizes for their cages or yards. They also cut doors of appropriate sizes in the boxes for the animals.
   - Students listen to and look at the book *Let's Find Out about What's Light and What's Heavy* by Martha and Charles Shapp. The simple text and humorous illustrations lead to the conclusion that weighing things using a standard unit of weight helps answer the question in the title.
   - Students name objects they can lift and ones that they cannot lift.
   - Students estimate and then use balances to find out how many pennies balance a small familiar object.
   - Students cut strips of paper to fit around a pumpkin or to make Santa's belt.
   - Students fill a large bottle with water using first a 4 ounce cup and then an 8 ounce cup. They then compare the results.
   - Students make their own measuring jug using a large plastic jar. They pour in one cupful of water and mark the water level on the jar with a marker; they repeat this procedure with one cupful after another until no further cupfuls will fit inside the jar.
   - Students read *The Little Gingerbread Man* and make a gingerbread village. In doing so, they measure lengths and capacities.
   - Students make their own paper clip ruler. First they make a paper clip chain and then paste it down on a long cardboard strip. They draw a small vertical line where each paper clip ends.

- Students estimate and measure the distance around an object using Unifix cubes or a paper clip chain.
- Students conduct experiments using timers: *How many times can you bounce a ball before all the water runs out of the can? How many times can you clap your hands before the sand runs out of the timer? How many times can you blink your eyes before the second hand goes all the way around the clock?*
- Students read or listen to *The Very Hungry Caterpillar* by Eric Carle which shows the time of day at which various activities occur.
- Students make a book describing their day at school. On each page, they stamp a clock face and write underneath a time that the teacher has written on the board. They then draw the hands on the face to show the time. When the actual time of day on the classroom clock matches a time in their book, students draw a picture of what they are doing next to the correct clock face.
- Students line up Cuisenaire Rods in different combinations to measure the width of a sheet of paper.

2. Compare and order objects according to some measurable attribute.

- Kindergarten students listen to and look at the book *Big Friend, Little Friend* by Eloise Greenfield. In it a young boy and his two friends explore situations that clearly demonstrate what it means to be big and what it means to be little. As a nice follow-up assessment activity to this and other manipulative activities, the teacher has the students draw pictures which illustrate "big and little."
- Students compare the lengths of pencils to find out which is longest. They arrange a set of pencils in order from longest to shortest.
- Students use water, rice, or sand to fill different objects, pouring from one object into another to find out which object holds more. They explain how the shape of each object plays a role in the amount it holds.
- Students line up in order, from tallest to shortest.
- Children make stick drawings of a family: father, mother, school-aged child, and baby. They discuss which stick drawings are taller or shorter than others, and relate these to the relative size of the individuals in the family.
- Students collect a large variety of cardboard boxes and arrange them in order from smallest to largest.
- Each group of students is given a cup and several containers of different sizes, plain white paper, some uncooked rice, and 1 inch graph paper. They find out how many cups of rice will fit in each container and show the number of cups as a bar on the graph paper under a picture of the container. After completing the graph, they arrange the containers in order from largest to smallest.
- Students work through the *Will a Dinosaur Fit?* lesson that is described in the First Four Standards of this *Framework*. They arrange the dinosaurs in order from tallest to smallest according to their height, and from longest to shortest according to their length.

3. **Recognize the need for a uniform unit of measure.**

- Students measure the width of their desks by counting how many widths of their hands it would take to go from one end of the desk to the other. They compare their results and discuss what would happen to the number of hands if the teacher's hand were used instead.
- Students read and discuss *How Big Is a Foot?* by Rolf Myllar. The king wishes to give the queen a special bed for her birthday and measures the size using his own foot. He gives the measurements to the carpenter, who gives them to the little apprentice. The bed that he makes is too small, but the apprentice solves the problem and everyone lives happily ever after. The students use their own feet to measure the width or length of the hallway and compare their results. Finally, they measure the hallway using meter sticks.
- As an assessment of the students' understanding of units, the teacher has the students measure the length of their math book using paper clips, unifix cubes, and yellow Cuisenaire Rods. They write about their results and explain why they are different.

4. **Develop and use personal referents for standard units of measure (such as the width of a finger to approximate a centimeter).**

- Students identify parts of their body that are the same length as the unit cube from a base tens block set (1 centimeter).
- Students make a list of foods or drinks that come in quarts and others that come in liters.
- Students find out that ten pennies weigh about an ounce.
- Students find that first-graders are a little taller or a little shorter than a meter.

5. **Select and use appropriate standard and non-standard units of measurement to solve real-life problems.**

- Students decide whether they should use paper clips or pennies to measure the weight of a pencil.
- Students discuss whether they should use links or meter sticks to measure the length of the gym.
- Students write about how they might measure the distance from the cafeteria to their classroom.

6. **Understand and incorporate estimation and repeated measures in measurement activities.**

- Students estimate how many of their shoes will fit in a giant's footprint (left conveniently on the classroom blackboard!) and write their estimates. They trace around their shoes and cut out the tracings. After the teacher has pasted a few shoes onto the giant's footprint, the students revise their estimate. They then check the accuracy of their estimates by pasting as many shoes as will fit into the footprint.
- Students estimate the weight of various objects in beans and then use a balance scale to check the accuracy of their measurements.

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On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

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Standard 9 — Measurement — Grades 3-4

Overview

Students can develop a strong understanding of measurement and measurement systems from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding, as identified in the K-12 Overview, are: **the concept of a measurement unit; standard measurement units; connections to other mathematical areas and to other disciplines; indirect measurement;** and, for older students, **measurement error** and **degree of precision.**

Students in grades 3 and 4 continue to encounter measurement situations in their daily lives and in their schoolwork. They investigate how much weight different structures will support or make a model of the solar system in science class, they make maps in social studies, and they read and discuss stories in which people measure objects. Measurement continues to provide opportunities for making mathematical **connections** among subject areas.

Measurement also helps students make **connections** within mathematics. For example, as students begin to develop their understanding of fraction concepts, they extend their understanding of measurement to include fractions of units as well. Measurement is interwoven with developing understanding of the geometric concepts of perimeter, area, and volume. Furthermore, students develop their estimation skills as they develop their understanding of measurement.

Students also continue to learn about more attributes of objects that can be measured. In addition to length, distance, capacity, weight, area, volume, time, and temperature, they now are able to discuss the size of angles and the speed of a car or a bike. Students begin to make more **indirect measurements** as well. For example, they will measure a desk to find out whether it will fit through a door, or measure how far a toy car goes in a minute and divide to find its speed in feet per second.

The emphasis in these grades is on building on the students' earlier experience with non-standard units and their developing **concept of measurement unit** to the use of more sophisticated **standard units of measurement.** They solidify their understanding of the basic units introduced in the earlier grades and begin to use fractional units. Students use half-inches, quarter-inches, and eighths of an inch, for example, in measuring the lengths of objects. Students also begin to use some of the larger units: miles, kilometers, and tons.

Some students may begin to discover formulas to help count units. For example, students may use shortcuts to find out how many squares cover a rectangle, multiplying the number of rows times the number of squares in each row. Or they may find the distance around an object by measuring each side and then adding the measures.

In summary, in grades 3 and 4, it is important that all students get extensive hands-on experience with measuring the properties of a variety of physical objects. They will learn to measure by actually doing so with an appropriate measuring instrument.

Standard 9 — Measurement — Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Use and describe measures of length, distance, capacity, weight, area, volume, time, and temperature.**
    - Students find out how many inches long their hand is. The class then generates a graph showing the results.
    - Students use rulers to measure the length of the room in feet and inches and then in metric units.
    - Students move thermometers to different parts of the school, recording the temperature at each location. For example, it may be hot in the cafeteria and cold in the gym. They learn to identify appropriate reference points on both Celsius and Fahrenheit scales (e.g., 30° C is a hot day).
    - Students investigate truth-in-packaging by reading labels, estimating weights, and then using balances to weigh foods.
    - Students investigate how many cups in a pint, how many pints in a quart, and how many quarts in a gallon by making lemonade and filling various sizes of containers.
    - Students make their own rulers, marking off intervals equal in length to one centimeter.
    - Students estimate and measure the distance around an object using a length of string which they then measure with centimeter cubes.
    - Students conduct experiments using timers: *how many times can you bounce a ball, clap your hands, or blink your eyes in one minute?* They discuss how many times each would occur in 10 minutes, in an hour, or in a day, if they continued at the same rate, and why their answers might be different.
    - Students measure all sorts of performances in their physical education class: the time it takes to run 100 meters, the length of a long jump in inches, and the length of a softball throw in meters.
    - Students read *Time to . . .* by Bruce McMillan. In this book about a farm boy and his daily activities, clock faces are always there to remind the reader what time it is.
    - Students use calculators to help them find out how many days old they are.
    - When going on a field trip, students determine how much time they will have available at a museum by considering when they will arrive and when they must leave.

- Students use cubes to fill rectangular boxes of various sizes as they explore the concept of volume.

2. **Compare and order objects according to some measurable attribute.**

   - Students compare the areas of different leaves and order them from smallest to largest. They use a variety of strategies; some students cover the leaves with centimeter cubes, others make a copy of the leaf on grid paper, and still others just "eyeball" it. They discuss the different strategies used, comparing their advantages and disadvantages.
   - Students bring in a variety of cereal boxes from home and estimate their order from smallest volume to largest volume. They then check their accuracy by filling the boxes with cubic inch blocks, with cubic centimeter blocks, and with sand, and discuss the reasons for the differences in their results.
   - Students build bridges using straws and pipe cleaners, estimate how many round metal washers their bridge will hold, and then place the washers on their bridge until it buckles or breaks. They compare different types of bridges to determine what type is strongest.
   - Students estimate and then weigh objects, putting them in order from heaviest to lightest.

3. **Recognize the need for a uniform unit of measure.**

   - Students measure the length of their classroom using their paces and compare their results. They discuss what would happen if the teacher measured the room with his or her pace.
   - Students read and study the illustrations in the book *Long, Short, High, Low, Thin, Wide* by James Fey. There are many activities in this attractive book that take the students through an historical account of the development of standard units.

4. **Develop and use personal referents for standard units of measure (such as the width of a finger to approximate a centimeter).**

   - Students identify parts of their body that are the same length as ten centimeters and use them to measure the length of their pencil.
   - Students find things in their environment that weigh about an ounce.
   - Students use a meter stick to identify a personal referent which is approximately a meter. For example, for one child it may be an armspan, for another it might be the distance between a kneecap and the top of the head.
   - Students measure the length of their pace in inches and use that information, along with a measurement of the length of the room in paces, to find the length of the room in inches.

5. **Select and use appropriate standard and non-standard units of measurement to solve real-life problems.**

   - Students decide what units they should use to measure the weight of a textbook.
   - Students discuss what units they should use to measure the length of the hallway outside their classroom.

- Students write about how they might measure the distance from the cafeteria to their classroom or the area of the gym.
- A nice approach to assessment of students' skills with this topic is to make a list of items that they can measure, such as the length of a piece of notebook paper, the weight of a teacher, the amount of water a bucket can hold, and the distance between Trenton and Newark, and ask them to name a measurement unit with which it would be appropriate to measure the given item. They discuss their choice of unit, estimate the measure of each item, and then actually measure it and compare it to their estimate and to the results of other students.

6. Understand and incorporate estimation and repeated measures in measurement activities.

- Students read and laugh about the pictures in *Counting on Frank* by Rod Clement. Frank is a dog whose young owner, challenged by his father to use his brain, estimates and imagines all sorts of measurements leading to some pretty silly situations. Possible extensions are plentiful and easy to devise. As an open-ended assessment follow-up to the story, the teacher asks each of the students to make a "Counting-on-Frank-like" estimate of how many of something (television, little sister, car, dog) would fit into their bedroom and to draw a picture showing all of them there.
- Students estimate the weight of various objects in grams and then use a balance scale to check the accuracy of their measurements.
- Students estimate the weight of, and then weigh the contents of a box of animal crackers, graphing their results and comparing them to the weights indicated on the packages.

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Standard 9 — Measurement — Grades 5-6

Overview

Students can develop a strong understanding of measurement and measurement systems from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding, as identified in the K-12 Overview, are: **the concept of a measurement unit; standard measurement units; connections to other mathematical areas and to other disciplines; indirect measurement;** and, for older students, **measurement error** and **degree of precision**.

Why teach measurement? The ability to measure enables students to **connect** mathematics to the environment and offers opportunities for interdisciplinary learning in social studies, geography, science, music, art, and other disciplines. In addition, measurement tools and skills have a variety of uses in everyday adult life.

However, in the most recent international assessment of mathematics achievement, 13-year-olds in the United States performed very poorly in comparison to other nations. The results of this study indicated that, while students are given instruction on measurement, they do not learn the concepts well. For example, some students have difficulty recognizing two fundamental ideas of measurement, the concept of a unit and the iteration of units. A common error is counting number marks on a ruler rather than counting the intervals between the marks. Another difficult concept is that the size of the unit and the number of units needed to measure an object are inversely related; as one increases the other decreases. In the fifth and sixth grades, students begin to encounter both very small and very large **standard measurement units** (such as milligrams or tons), and these ideas become increasingly critical to understanding measurement.

Students must be involved in the act of measurement; they must have opportunities to use measurement skills to solve real problems if they are to develop understanding. Textbooks by themselves can only provide symbolic activities. Middle grade teachers must take responsibility for furnishing hands-on opportunities that reinforce measurement concepts with all common measures.

Using measurement formulas as a more efficient approach to some types of direct measurement is an important part of fifth and sixth grade mathematics. It represents the first formal introduction to **indirect measurement**. Multiplying the length by the width of a rectangle is certainly an easier way to find its area than laying out square units to cover its surface. The formulas should develop, however, as a result of the students' exploration and discovery; they should be seen as efficient ways to count iterated units. Having students memorize formulas that, for them, have no relation to reality or past direct measurement experiences will be unsuccessful. Fewer than half of the U.S. seventh grade students tested in the international competition could figure out the area of a rectangle drawn on a sheet of graph paper, and only slightly more than half could compute the area given the dimensions of length and height. Often length and width are taught separately and how the two measurements combine to form the square units of area is not emphasized in instruction. In addition, area and perimeter are often confused with each other by middle grade students. Limiting students' experiences with measurement to the printed pages of textbooks restricts flexibility so that their understanding cannot be developed or generalized.

In order to further strengthen students' understanding of measurement concepts, it is important to provide **connections** of measurement to other ideas in mathematics and to other areas of learning. Students should measure objects, represent the information gathered visually (e.g., in a graph), model the situation with symbols (e.g., with formulas), and apply what they have learned to real-world events. For example, they might collect information about waste in the school lunchroom and present their results to the principal with suggestions for reducing waste. Integrating across mathematical topics helps to organize instruction and generates useful ideas for teaching the important content of measurement.

In summary, measurement activities should require a dynamic interaction between students and their environment, as students encounter measurement outside of school as well as inside school. Students should use each measuring instrument until its use becomes second nature. The curriculum should focus on the development of understanding of measurement rather than on the rote memorization of formulas. This approach can be reinforced by teaching students to estimate and to be aware of the context whenever they make an estimate. For example, when buying carpeting it is advisable that the estimate be too high rather than too low. Students must be given the opportunity to extend their learning to new situations and new applications.

Standard 9 — Measurement — Grades 5-6

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

7. Use estimated and actual measurements to describe and compare phenomena.

- Students estimate the number of square centimeters in a triangle. Then they enclose the triangle in a rectangle and use centimeter cubes or a transparent square centimeter grid to find the area of the rectangle. They also count squares to find the area of the triangle and that of any other triangles formed by the rectangle. They look for a pattern in their results and compare their results to their estimates.
- Students explain why the following is or is not reasonable: An average person can run one kilometer in one minute.
- Students measure how long it takes to go 10 meters, first using "baby steps," then using normal steps, and finally using "giant steps." They then compare their rates.
- Students measure the area of their foot by tracing around it on centimeter graph paper and counting the number of squares covered. To ease the counting task, students can color the squares completely inside the outline blue, those that are one-half inside green, those that are one-third inside yellow, and those that are one-fourth inside orange. Then all of the like-colored squares can be counted more easily and the various totals added to each other.
- An interesting open-ended group assessment project to use after the previous foot-tracing activity has been completed is to tell the students that Will Perdue (of the Chicago Bulls) wears a size 18 ½ shoe which measures 21 ½ inches long. Students are asked to use what they know about the areas of their own feet to estimate the area of his foot. Students who make a good estimate will deal with several issues: the fact that they have information about the length of their own feet, but only about the length of Will's shoe; the fact that as the foot gets longer, it also gets wider; and the issue of how to set up a proportion between appropriate quantities.

8. Read and interpret various scales, including those based on number lines and maps.

- Students use a given scale to compute the actual length of a variety of illustrated dinosaurs.
- Students make a scale drawing of their classroom and use two-dimensional scale models of its furniture in order to propose new ways of arranging the classroom so that they can work more efficiently in cooperative groups.
- Students use a map to find the distance between two cities.

- Students work through the *Short-Circuiting Trenton* lesson that is described in the Introduction to this *Framework*. Using a map of Trenton and a ruler, students determine the distances between various sites, and then find the most efficient walking tour for their class trip.

9. Determine the degree of accuracy needed in a given situation and choose units accordingly.

- Students plan a vegetable garden, determining the unit of measure appropriate for the garden, estimating its size, and then computing the perimeter (for fencing) and area (for fertilizer).
- Groups of students use a scale drawing of an apartment (1 cm = 1 foot) to find out how many square yards of carpeting are needed for the rectangular (9' x 12') living room and other rooms.
- Statements work through the *Mathematics at Work* lesson that is described in the Introduction to this *Framework*. A parent discusses a problem which her company faces regularly: to determine how large an air conditioner will be needed for a particular room. To solve this problem, the company has to estimate the size of the room, in terms of its volume and the areas of any windows, after determining the appropriate units.

10. Understand that all measurements of continuous quantities are approximate.

- Students measure a specific distance in the room and compare their results, focusing on the idea that any measurement is approximate.
- Before a cotton ball toss competition, students discuss what units should be used to measure the tosses. They decide that measuring to the nearest centimeter should be close enough, even though the actual tosses will probably be slightly more or slightly less.

11. Develop formulas and procedures for solving problems related to measurement.

- Students complete a worksheet showing several rectangles on grids that are partially obscured by inkblots. In order to find the area of each rectangle, they must use a systematic procedure involving multiplying the length of the rectangle by its width.
- Students develop the formula for finding the volume of a rectangular prism by constructing and filling boxes of various sizes with centimeter or inch cubes and looking for patterns in their results. As a journal entry, they describe the "shortcut" way to find the volume.

12. Explore situations involving quantities which cannot be measured directly or conveniently.

- Students work in groups to estimate the number of bricks needed to build the school building. They explain their results in a class presentation, describing the strategies they used.
- Students are asked to estimate how many heads tall they are. Then they work in groups to develop a procedure for finding out how many heads tall each student is.
- Students construct a measuring tool that they can use to find the height of trees, flagpoles, and buildings when they are standing a fixed distance from the object to be measured.

13. **Convert measurement units from one form to another, and carry out calculations that involve various units of measurement.**

- Students are asked to find how many pumpkin seeds there are in a kilogram. They decide to measure how much 50 seeds weigh and use this result to help them find the answer.

- Students use approximate "rules of thumb" to help them convert units. For example:

  1 km is about 6/10 of a mile
  1 liter is a little bigger than a quart
  1 meter is a little bigger than a yard
  1 kg is about 2 pounds
  20° C is about 70° F (room temperature)
  1000 ml of water normally weighs about 1 kg

14. **Understand and apply measurement in their own lives and in other subject areas.**

- Students measure the heights of bean plants at regular intervals under different conditions. Some are in sunlight and some are not. The students discuss their results and make a graph of their findings.

- Students estimate and weigh cups filled with jellybeans, raisins, dried beans, peanuts, and sand to find out that equal volumes of different objects do not always weigh the same.

- Students learn how much water is in different foods by first trimming pieces of 5 different foods to a standard 15 grams, then measuring their weights again the next day. *Where did the water go?*

- Students estimate what fraction of an orange is edible, then weigh oranges, peel them and separate the edible parts. They weigh the edible part and then compute what fraction is actually edible and compare that fraction to their estimate.

- Students create their own food recipes.

- An ice cube is placed on a plastic tray in five different parts of the classroom. One group of students is assigned to each ice cube and tray. The students are asked to estimate how long it will take each ice cube to melt. They then observe the ice cube at five-minute intervals, recording their observations. After the ice cubes have melted, the groups share their observations and compare the length of time it took for the ice cubes to melt. They make a conjecture about the warmest spots in the classroom and then measure the temperature in each location to confirm their conjecture.

- Students read *Anno's Sundial* by Mitsumasa Anno. This sophisticated three-dimensional pop-up book presents an extraordinary amount of information about the movement of the earth and the sun, the relationship between those movements, and how people began to tell time. It is an ideal kick-off for an integrated, multi-disciplinary unit with upper-elementary students, incorporating reading, social studies, science, and mathematics.

- Students estimate and then develop a plan to find out how many pieces of popped popcorn will fit in their locker.

- Students work in pairs to design a birdhouse that can be made from a single sheet of wood (posterboard) that is 22" x 28". The students use butcher paper to lay out their plans so that the birdhouse is as large as possible. Each pair of students must show how the pieces can be laid out on the posterboard before cutting.
- Students compare the measurements of an object to those of its shadow on a wall as the distance between the object and the wall increases.

15. Understand and explain the impact of the change of an object's linear dimensions on its perimeter, area, or volume.

- Students use pattern blocks to see how the area of a square changes when the length of its side is doubled. They repeat the experiment using equilateral triangles.
- Students use cubes to explore how the volume of a cube changes when the length of one side is doubled, then when the lengths of two sides are doubled, and, finally, when the lengths of all three sides are doubled.
- Students use graph paper to draw as many rectangles as they can that have a perimeter of 16 units. They find the area of each rectangle, look for patterns, and summarize their results.

16. Apply their knowledge of measurement to the construction of a variety of two- and three-dimensional figures.

- Students use paper fasteners and tagboard strips with a hole punched in each to investigate the rigidity of various polygon shapes. For shapes that are not rigid, they determine how they can be made so.
- Students design and carry out an experiment to see how much water is wasted by a leaky faucet in an hour, a day, a week, a month, a year.
- Students use straw and string to construct models of the two simplest regular polyhedra, the cube and the tetrahedron.

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On-Line Resources

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Standard 9 — Measurement — Grades 7-8

Overview

Students can develop a strong understanding of measurement and measurement systems from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding, as identified in the K-12 Overview, are: **the concept of a measurement unit; standard measurement units; connections to other mathematical areas and to other disciplines; indirect measurement;** and, for older students, **measurement error** and **degree of precision.**

In grades seven and eight, students begin to look at the measurement process more abstractly while continuing to develop their actual measurement skills and using measurement in connection with other subjects and other topics in mathematics.

All measurement activities should involve both estimation and actual measurement at these grade levels. Estimation strategies should include (1) having a model or referent (e.g., a doorknob is about one meter from the floor), (2) breaking an object to be estimated into parts that are easier to measure (*chunking*), and (3) dividing the object up into a number of equal parts (*unitizing*). Students should also discuss when an estimate is appropriate and when an actual measurement is needed and they should have opportunities to select appropriate measuring tools and units.

Especially in the context of making measurements in **connection with other disciplines**, the approximate nature of measure is an aspect of number that needs particular attention. Because of students' prior experience with counting and with using numerical operations to obtain exact answers, it is often difficult for them to develop the concept of the approximate nature of measuring. Only after considerable experience do they recognize that when they correctly measure to the nearest "unit," the maximum possible error would be one-half of that unit. Teachers must help students to understand that the **error of a measurement** is not a mistake but rather a result of the limitations of the measuring device being used. Only through measurement activities can students discover and discuss how certain acts, such as the selection and use of measuring tools, can affect the **degree of precision** and accuracy of their measurements.

Students in grades seven and eight should expand their understanding of measurement to include new types of measures, especially those involving **indirect measurement**. For example, they learn about density and force and how these characteristics are measured in science class. Middle school students also should develop a deeper understanding of the concept of rate, by experiencing and discussing different rates. Constructing scale drawings and scale models or relating biological growth and form provide excellent opportunities for students to use proportions to solve problems, as does using a variety of measuring tools to find the measures of inaccessible objects. Such personal experiences help students recognize and appreciate the use of measurement concepts in other real-world settings.

Standard 9 — Measurement — Grades 7-8

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

7. **Use estimated and actual measurements to describe and compare phenomena.**
   - Students estimate the number of square centimeters in a trapezoid. Then they use a transparent grid and count squares to find the area. They compare that result to the area of a rectangle whose base is the average of the two bases of the trapezoid and whose height is the same as that of the trapezoid. They look for a pattern in their results and compare their results to their estimates.
   - Students build a bridge out of paper to connect two bricks and place weights on the bridge until it breaks, noting how much weight it held.
   - Students read and discuss sections of *This Book Is About Time* by Marilyn Burns. Its many engaging activities and experiments are interspersed with an historical treatment of time and the instruments designed to measure it.

8. **Read and interpret various scales, including those based on number lines and maps.**
   - Students use objects shown in a movie poster for *King Kong* to determine how tall the ape is.
   - As a long-term assessment project, students make a three-dimensional scale model of their classroom.
   - Students use a map to plan an auto trip across the United States, finding the distance traveled each day and the amount of time required to drive each day's route.

9. **Determine the degree of accuracy needed in a given situation and choose units accordingly.**
   - Students plan a school garden, determining the unit of measure appropriate for the garden, estimating its size, and then computing the perimeter (for fencing) and area (for fertilizer).
   - Groups of students make and use a scale drawing (1/4 inch = 1 foot) of an apartment and use scale models of the furniture to furnish the living room and dining room.
   - Students make a floor plan for a small restaurant furnished with round tables.

10. **Understand that all measurements of continuous quantities are approximate.**
    - Students measure a given hallway in school and compare their results, noting that their results are different because any measurement is approximate. They discuss how accurate

their individual measurements are (degree of precision) and, after reviewing all of their measurements, determine the likely errors in their individual measurements. They also discuss how more precise measures may be obtained and what degree of precision is needed in this situation.

- Each student in a group measures the circumferences and diameters of several round objects using a tape measure or ruler and string. They compare their measurements and decide what the most accurate set of measurements is for each of the objects. They use a calculator to find the ratio of the circumference to the diameter for each object.

11. Develop formulas and procedures for solving problems related to measurement.

- Students develop a formula for finding the surface area of a rectangular prism by constructing boxes of various sizes using graph paper, finding the area of each side and adding them, and looking for patterns in their results. They describe their findings in their journals.
- Students construct different parallelograms whose base and height have the same length on their geoboards. They sketch each parallelogram and record its area (found by counting squares). They discuss their results.
- Students work through the *Sketching Similarities* lesson that is described in the First Four Standards of this *Framework*. They use a computer program and measure the sizes of the corresponding sides and corresponding angles of similar figures. They conclude that similar figures have equal corresponding angles and their corresponding sides have the same ratio.
- Students use plastic models of a pyramid and a prism, each having the same height and polygonal base, to investigate the relationship between their volumes.

12. Explore situations involving quantities which cannot be measured directly or conveniently.

- Students and parents in the McKinley School have been dropping pennies into the very large plastic cylinder in the school lobby in an effort to raise money for new playground equipment. The students are challenged to devise a method to estimate how much money is in the cylinder as a function of the height of the pennies at any given time.
- Students construct a measuring tool that they can use to find the height of trees, flagpoles, and buildings, using cardboard, graph paper, straws, string, and washers.
- Students use proportions to find the height of the flagpole in front of the school.

13. Convert measurement units from one form to another, and carry out calculations that involve various units of measurement.

- Students are given a ring and asked to find the height of the person who lost the ring. They measure their own fingers and their heights, plotting the data on a coordinate graph. They use a piece of spaghetti to fit a straight line to the plotted points and make a prediction about the height of the person who owns the ring, based on the data they have collected.

- Students plan the *weird name Olympics* by renaming standard events in different measurement units. For example, "the hundred meter dash" becomes the "100,000 millimeter marathon" and the "ten meter dive" could become the "one-hundredth of a kilometer splash."
- Students continue to use approximate "rules of thumb" learned in earlier grades to help them convert units. For example:

  1 km is about 6/10 of a mile
  1 liter is a little bigger than a quart
  1 meter is a little bigger than a yard
  1 kg is about 2 pounds
  1 inch is about 2.5 centimeters
  20° C is about 70° F (room temperature)
  1000 ml of water normally weighs about 1 kg

14. Understand and apply measurement in their own lives and in other subject areas.

- Students design and carry out an experiment to see how much water is wasted by a leaky faucet in an hour, a day, a week, a month, and a year.
- Students are told that when people get out of a bath, a film of water about 0.05 cm thick clings to their skin. They are then challenged to find what volume of water clings to the skin of an average eighth grader. In order to make the estimate, of course, they need to estimate the surface area of the body. They can do so by considering a collection of cylinders and spheres that approximates a human.
- Students investigate the concept of density by finding objects for which they can find both the volume and the weight, measuring both, and dividing the latter by the former. Interesting objects to use include an orange, a block of wood, a textbook, a rubber sponge ball, and an air-filled rubber ball. Discussions of the results should lead to interesting conjectures about density which can then be confirmed with additional experimentation.

15. Understand and explain the impact of the change of an object's linear dimensions on its perimeter, area, or volume.

- Students use the computer program *The Geometric SuperSupposer* to explore the relationship in similar triangles between corresponding sides and the perimeters of the triangles. They also analyze the relationship between corresponding sides and the areas of the triangles.
- Students build a "staircase" using wooden cubes. Then they double all of the dimensions and compare the number of cubes used in the second staircase to the number used in the original staircase.
- Students work through the *Rod Dogs* lesson that is described in the First Four Standards of this *Framework*. They discover that if an object is enlarged by a scale factor, its new surface area is the scale factor squared times the original area, and its new volume is the scale factor cubed times the original volume.

16. Apply their knowledge of measurement to the construction of a variety of two- and three-dimensional figures.

- Students use straw and string to construct models of the five regular polyhedra: the cube, the tetrahedron, the octahedron, the icosahedron, and the dodecahedron.
- Students use cardboard and tape to construct a model that demonstrates that the volume of a pyramid is one-third that of a prism with the same base and height.
- Students build scale models of the classroom, the school, or a monument.

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Software

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Standard 9 — Measurement — Grades 9-12

Overview

Students can develop a strong understanding of measurement and measurement systems from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding, as identified in the K-12 Overview, are: **the concept of a measurement unit; standard measurement units; connections to other mathematical areas and to other disciplines; indirect measurement;** and, for older students, **measurement error** and **degree of precision**.

Building upon the measurement skills and understandings developed in grades K-8, high school students move to a more routine use of measurement. They examine measurement as a more abstract process, focusing on **measurement error** and **degree of precision.** They spend much more time on **indirect measurement techniques** than they did in earlier grades, expanding their repertoire to include not only the use of proportions and similarity but also the use of the Pythagorean Theorem and basic right triangle trigonometric relationships.

Students at the high school level will frequently use measurement to help develop **connections** to other mathematical concepts. For example, students may use a computer program that measures angles to help them discover the relationship between the measures of two vertical angles formed by intersecting lines or the measures of inscribed angles intercepting the same arc of a circle. They may also develop algebraic techniques to help them find measures, as, for example, when they develop a formula for finding the distance between two points in the coordinate plane.

High school students also use measurement frequently in **connection** with other subject areas. Science experiments require a precise use of measurement. Social studies activities often require students to read and interpret maps and/or scale drawings. In technology classes, woodshop, drafting, sewing, and cooking, students must also use a variety of measuring tools and techniques; and in physical education, students frequently will need to measure distances and rates.

Standard 9 — Measurement — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11, and 12.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

17. Use techniques of algebra, geometry, and trigonometry to measure quantities indirectly.

- Students use coordinate geometry techniques to determine the distance between two points.
- Students use similar figures and proportions to measure the height of a tree or a flagpole.
- Students use the Pythagorean Theorem to determine how long a ladder is needed to climb a wall, including a determination of a safe angle at which to place the ladder.
- Students use right-triangle trigonometry to measure the width of a canyon or the height of a waterfall.
- Students work through the *Ice Cones* lesson that is described in the First Four Standards of this *Framework*. They use the formula for the volume of a cone and a graphing calculator to determine the maximum volume of a cone made from a paper circle of radius 10 cm which is cut along a radius.

18. Use measurement appropriately in other subject areas and career-based contexts.

- Students investigate how the volume of a cereal box changes with its area by finding the volume and surface area of a box of their favorite kind of cereal. They also discuss how the shape of the box affects its volume and surface area and why the volume of the box is so large for the amount of cereal it contains.
- As a question on their take-home final exam in Algebra, students are asked to measure their own and gather data about the length of other people's femurs and overall height in an effort to determine whether there is a relationship between the two lengths. They plot the resulting ordered pairs on a coordinate plane and find a line of best fit for their data. For extra credit, they make a prediction of the height of a male with a 22 inch femur.
- Students discuss the possible meaning of "light year" as a unit of measure and find units equivalent to it.

19. Choose appropriate techniques and tools to measure quantities in order to achieve specified degrees of precision, accuracy, and error (or tolerance) of measurements.

- Students use significant digits appropriately in measuring large distances, such as the distance from one school to another, from one city to another, and from one planet to another.

- Students find the distance between two cities by adding the numbers given on a road map for the segments of the trip, by measuring the segments and using the mileage scale, and by referring to a published mileage table. They explain the different results by referring to the degrees of prevision of the different measurements.
- In making a scale drawing of a house, students discuss the degree of accuracy of their measurements.
- Students read and discuss the photographs in *Powers of Ten* by Phillip and Phylis Morrison and the office of Charles and Ray Eames, and view the associated videotape. This well-known book takes the reader on a trip through perspectives representing forty-two powers of ten, from the broadest view of the universe to the closest view of the nucleus of an atom. The measurement units used and the progression from one to another highlight the range and power of our system of measurement.
- Students use computer drawing and measuring utilities to discover geometric concepts. They also discuss the limitations of such a program. For example, a program may give 14.7 for the length of the base of a triangle and 7.3 for its midline (the segment joining the midpoints of the other two sides); however, because of the program's measurement limitations, its answer for the length of the midline may not be exactly half the length of the base, as is the case in reality.
- Students determine what kind of measuring instrument needs to be used to measure ingredients for pain-relievers, for cough syrup, for a cake, and for a stew. They bring to class a variety of empty bottles and packages and note how the ingredients are measured: What does 325 mg (of acetaminophen) mean, or one fluid ounce (of cough syrup) as opposed to 1/4 cup of oil (for a cake). They discuss the accuracy, error, and tolerance of each measurement.

References

Morrison, Philip, Phylis Morrison, and the office of Charles and Ray Eames. *Powers of Ten*. Scientific American, 1988. (Revised, 1994. See also reference under Video.)

General reference

Froelich, G. *Curriculum and Evaluation Standards for School Mathematics: Addenda Series: Connecting Mathematics*. Reston, VA: National Council of Teachers of Mathematics, 1991.

Video

Powers of Ten. Philip Morrison, Phylis Morrison, and the office of Charles and Roy Eames. New York: Scientific American Library, 1991.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

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STANDARD 10 – ESTIMATION

K-12 Overview

All students will use a variety of estimation strategies and recognize situations in which estimation is appropriate.

Descriptive Statement

Estimation is a process that is used constantly by mathematically capable adults, and that can be mastered easily by children. It involves an educated guess about a quantity or a measure, or an intelligent prediction of the outcome of a computation. The growing use of calculators makes it more important than ever that students know when a computed answer is reasonable; the best way to make that decision is through estimation. Equally important is an awareness of the many situations in which an approximate answer is as good as, or even preferable to, an exact answer.

Meaning and Importance

As used in this standard, estimation is the process of determining approximate values in a variety of situations. Estimation strategies are used universally throughout daily life, but an examination of the mathematics curriculum of the past leads to the view that the strength of mathematics lies in its exactness, in the ability to determine the "right" answer. The growing use of calculators in the classroom requires greater emphasis on determining whether the answer given by a calculator or paper-and-pencil method is reasonable, a process that requires estimation ability, but efforts in support of this goal have been minimal compared to the time devoted to getting that one right answer. As a result, students have developed the notion that exactness is always preferred to estimation and their potential development of intuition may have been hindered with unnecessary calculations and detail.

People who use mathematics in their lives and careers find estimation to be preferable to the use of exact numbers in many circumstances. Frequently, it is either impossible to obtain exact answers or too expensive to do so. *An air conditioning salesperson preparing a bid would be wasting time and money by measuring rooms exactly. Astronomers attempting to determine movements of celestial objects cannot obtain precise measurements.* Many people use approximations because it is easier than using exact numbers. *Shoppers, for example, use approximations to determine whether they have sufficient funds to purchase items. Travelers use rough estimates of time, distance, and cost when planning trips.* Commonly reported data often use levels of precision which have been accepted as appropriate, even though they may not be considered "exact." *Astronomers usually report information to two significant digits, and batting averages for baseball players are always reported as three-place decimals.*

K-12 Development and Emphases

Part of being functionally numerate requires expertise in using **estimation with computation**. Such facility demands a strong sense of number as well as a mastery of the basic facts, an understanding of the properties of the operations as well as their appropriate uses, and the ability to compute mentally. As these skills and understandings are developed throughout the mathematics curriculum, students should have frequent opportunities to develop methods for obtaining estimates, and to recognize that estimation is useful. Estimation can help determine the correct answer from a set of possible answers, and establish the **reasonableness of answers**. Ideally, students should have an idea of the approximate size of an answer; then, if they recognize that the result they have obtained is incorrect, they can immediately rework the problem. This ability becomes increasingly important as students use calculators more and more.

Instruction in estimation has traditionally focused on the use of rounding. There are times when rounding is an appropriate process for finding an estimate, but this standard emphasizes that it is only one of a variety of processes. Computational **estimation strategies** are a new and important component in the curriculum. *Clustering, front-end digits, compatible numbers*, and other strategies are all helpful to the skillful user of mathematics, and can all be mastered by young students. Selection of the appropriate strategy to use depends on the setting, and the numbers and operations involved. Students should realize that there is not necessarily a "right" answer when estimating; different techniques may yield different estimates, and that is quite acceptable.

The foregoing discussion describes a new emphasis on the use of estimation in computational settings, but students should also be thoroughly comfortable with the use of **estimation in measurement**. Students should develop the ability to estimate measures such as length, area, volume, and angle size visually as well as through the use of personal referents, such as the width of a finger or the length of a pace. Measurement is rich with opportunities to develop an understanding that estimates are often used to determine approximate values which are then used in computations, and that results so obtained are not exact but fall within **a range of tolerance**.

Estimation should be emphasized in many other areas of the mathematics curriculum in addition to the obvious uses in numerical operations and measurement. Within statistics, for example, it is often useful to estimate measures of central tendency for a set of data; estimating probabilities can help a student determine when a particular course of action would be advisable; problem situations related to algebraic concepts provide opportunities to estimate rates such as slopes of lines and average speed; and working with sequences in algebra and increasing the number of sides of a regular polygon in geometry yield opportunities to estimate limits.

IN SUMMARY, estimation is a combination of content and process. Students' abilities to use estimation appropriately in their daily lives develops as they have regular opportunities to explore and construct estimation strategies and as they acquire an appreciation of its usefulness in the solution of problems.

NOTE: *Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

Standard 10 — Estimation — Grades K-2

Overview

As indicated in the K-12 Overview, students' ability to use estimation appropriately in their daily lives develops as they focus on the **reasonableness of answers**, explore and construct **estimation strategies**, and **estimate measurements, quantities, and the results of computation.**

One of the estimation emphases for very young children is the development of the idea that guessing is an important and exciting part of mathematics. The teacher must employ sound management practices which ensure that everyone's guess is important and which encourage student risk-taking and sharing of ideas about how their guesses were determined. When first asked to guess an answer, many students will give nonsense responses until they establish appropriate experiences, build their sense of numbers, and develop informal strategies for creating a guess. Children begin to make reasonable estimates when the situations involved are relevant to their immediate world. Building on comparisons of common objects and using personal items to build a sense of lengths, weights, or quantities helps children gain confidence in their guessing. As children communicate with each other about how guesses are formulated they begin to develop informal **strategies for estimation.**

Estimation with computation is as important at these early grade levels as it is at all the other grade levels. Estimation of sums and differences should be a part of the computational process from the very first activity with any sort of computation. Children should regularly be asked *About how many do you think there will be in all?* or *About what do you think the difference is?* or *About how many do you think will be left?* in the standard addition and subtraction settings. These questions are appropriate whether or not exact computations will be done. Children should understand that, sometimes, the estimate will be an accurate enough number to serve as an answer. At other times, an exact computation will need to be done, either mentally, with paper-and-pencil, or with a calculator to arrive at a more precise answer. The particular procedure to be used is dependent on the setting and the problem.

One of the most useful computational **estimation strategies** at these grade levels also reinforces an important place value idea. Students should understand that in two-digit numbers the tens digit is much more meaningful than the ones digit in contributing to the overall value of the number. A reasonable approximation, then, of a two-digit sum or difference can always be made by considering only the tens digits and ignoring the ones. This strategy is referred to as *front end estimation* and is used with larger numbers as well, although then the first two digits may be used. It is the main estimation strategy that many adults use.

Standard 10 — Estimation — Grades K-2

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and in grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Judge without counting whether a set of objects has less than, more than, or the same number of objects as a reference set.**

   - Students place various amounts of counters or other small objects in individual plastic bags. Working in groups of four students, the children choose one bag to be the reference set and judge whether each of the other bags has more than, less than, or the same as the reference set. Initially, they should try to make the judgments without counting. The teacher observes the groups as they work, making notes about the students' progress.
   - Young children benefit from frequently comparing sets of objects to some given number. For example, given sets of colored chips arranged on a table, they should name which sets have more than five and which have less than five.
   - Students play the card game *War* with a set of cards without numerals (i.e., cards which only show sets of hearts, clubs, diamonds, or spades). Students will easily distinguish between the 7's and the 3's, but will be reluctant to make judgments about closer numbers like the 4's and 5's without counting. As they play more often, however, their ability to distinguish will visibly improve. They may also begin to notice patterns involving even and odd numbers on their own.
   - Students learn to recognize certain arrangements of dots or stars as representing certain numbers. Using flashcards, they estimate the number of dots or stars, and then count to check their estimates.
   - After reading *Ten Black Dots* by Donald Crews, students make up their own uses for 1-10 black dots. They use adhesive dots to create their own books that include uses for each of the numbers from 1 through 10. They then estimate and count the total number of dots they actually use for their book. (The total might surprise them: 55.)
   - As an assessment of students' ability to judge without counting, the teacher puts some counters (more than five) on the overhead projector, turns it on for a few seconds, and then asks the students to write whether the number of counters shown is closer to 10 or to 20.

2. **Use personal referents, such as the width of a finger as one centimeter, for estimations with measurement.**

   - Students estimate lengths of pieces of spaghetti, yarn, paper, pencils, paper clips, etc., using suggested non-standard personal units such as width of thumb, length of a foot, and so on. They note that different students get different "right" answers.
   - As standard units like foot and centimeter are introduced, students are challenged to find

some part of their body or some personal action that is about that size at this point in their growth. For instance, they may decide that the width of their little finger is almost exactly one centimeter or the length of one baby step is one foot.

- Students use their self-discovered personal body referents to estimate the measures of various classroom objects like the length of the blackboard or the width of a piece of paper. They compare their answers, noting that when larger units are used, the estimated answer is a smaller number; and when smaller units are used, the estimated answer is a larger number.

3. Visually estimate length, area, volume, or angle measure.

- Students look at a quantity of sand, salt, flour, water, macaroni, corn, or popcorn and estimate how many times it could fill up a specified container.
- Students estimate how many pieces of notebook paper it would take to cover a given area such as the blackboard, or a portion of the classroom floor.
- Students regularly estimate lengths using a variety of non-standard units such as *my feet, Unifix cubes, paper clips,* and *orange Cuisenaire Rods.* They then measure to verify or revise their estimates.
- Students begin to develop an understanding of angle measure by making right-hand (or left-hand) turns repeatedly to turn completely around. They also compare angles to right-angle "corners," and decide whether an angle is more than or less than a "corner."
- Students note that there are 12 numbers on a clock face and discuss how far each hand moves in an hour. They note that each hand moves in a circle, but that the hour hand moves much more slowly.
- Students work through the *Will a Dinosaur Fit?* lesson that is described in the First Four Standards of this *Framework*. They determine the size of the room, hear their classmates' presentations about the dinosaurs, and then as a whole class activity estimate which dinosaurs, and how many of them, might fit into the room.

4. Explore, construct, and use a variety of estimation strategies.

- Students are asked if a sixty-seat bus will be adequate to take the two first grade classes on their field trip. After it is known that there are 23 children in one class and 27 in the other, individuals volunteer their answers and give a rationale to support their thinking; *front end estimation* should lead to the conclusion that the total number of students is between 40 and 60. A discussion might be directed to the question of whether an exact answer to the computation was needed for the problem.
- Students are shown a glass jar filled with about eighty marbles and asked to estimate the number in the jar. In small groups, they discuss various approaches to the problem and strategies they can use. Each group shares one strategy with the class, and the estimate that resulted. The teacher makes notes about students' work throughout the activity.
- Second grade students can be challenged to estimate the total number of students in the school. They will need to talk informally about the average number of students in each class, the number of classes in a grade level, and the number of grade levels in the school.

They might then use calculators to get an answer, but the result, even though the exact answer to a computation, is still an estimate to the original problem. They discuss why that is so.

- Primary-grade students explore the meanings of comparison words by listening to *How Many is Many?* by Margaret Tuten. They compare *big* and *small, long* and *short,* a *lot* and a *few*. They list how many pieces of candy would be a few and how many pieces would be many, eventually reaching general agreement, perhaps on 5 as a few. Then they consider whether 5 teaspoons of medicine would be a few.

5. **Recognize when estimation is appropriate, and understand the usefulness of an estimate as distinct from an exact answer.**

   - Given a pair of real-life situations, students determine which situation in the pair is the one for which estimation is a good approach and which is the one that probably requires an exact answer. One such pair, for example, might be: *sharing a bag of peanuts among 3 friends* and *paying for 3 tickets at the movie theater.*
   - Given a set of cartoons with home-made mathematical captions, first graders decide which of the cartoon characters arrived at exact answers and which got estimates. Two of the cartoons might show an adult and a child looking at a jar of jellybeans and the captions might read: *Susie guessed that there were 18 jellybeans left in the jar* and *Susie's mom counted the 14 jellybeans left in the jar.*
   - Students read or listen to newspaper headlines and discuss which involve exact numbers and which might be estimates.

6. **Determine the reasonableness of an answer by estimating the result of operations.**

   - Students are regularly asked if their answer makes sense in the context of the problem they were solving. They respond with full sentences explaining what they were asked to find and why the numerical answer they found fits the context reasonably, that is, why it *could be* the answer. For example, first graders might be asked to decide if their answer to the following problem makes sense: *Mary made 27 cookies and Jose made 15. How many cookies did they make in all?* Some responses might indicate that the answer should be more than $20 + 10 = 30$ and less than $30 + 20 = 50$. Other students might say that they know that *25* and *15* is *40*, so the answer should be a little more.
   - Students estimate *reasonable* numbers of times that particular physical feats can be performed in one minute. For example: *How many times can you bounce a basketball in a minute? How many times can you hop on one foot in a minute? How many times can you say the alphabet in a minute?* and so on. Other students judge whether the estimates are reasonable or unreasonable and then the tasks are performed and the actual counts made.
   - Second-grade students are given a set of thirty cards with two-digit addition problems on them. In one minute, they must sort the cards into two piles: those problems whose answers are greater than 100 and those less than 100. The correct answers can be on the backs of the cards to allow self-checking after the task is completed.
   - Second-grade students are given a page of addition or subtraction problems in a multiple choice format with 4 possible answers for each problem. Within some time period which

is much too short for them to do the computations, students are asked to choose the most reasonable answer from each set of four.

7. **Apply estimation in working with quantities, measurement, time, computation, and problem solving.**

   - Students have small pieces of yarn of slightly different lengths ranging from 2 to 6 inches. Each student first estimates the number of his or her pieces it would take to match a much longer piece — about 30 inches long — and then actually counts how many. Then they use their individual pieces to measure other objects in the room. Each child is responsible for estimating the lengths in terms of his or her own yarn, but they can use evidence from other children's measuring to help make their own estimates.
   - Students regularly estimate in situations involving classroom routines. For example, at snack time, they may guess how many cups can be filled by each can of juice or how many crackers each student will get if all of the crackers in the box are given out.
   - Kindergartners always have fun deciding which color is best represented in a group of multi-colored objects. Good examples of such an activity would be choosing the color that shows up most often (or least often) in bags of M&M's, in handfuls of small squares of colored paper, or in a jar full of marbles. After everyone has committed to a guess, the children can sort the objects and count each color. They can then make bar graphs to show the distribution of the different colors.
   - Students use Tana Hoban's photographs in *Is It Larger? Is It Smaller?* as a starting point for investigating and comparing quantities and measures in their classroom. For example, on one page, three vases are shown filled with three different kinds of flowers. The reader must decide which objects to compare, such as the vases, before ordering them — from tallest to shortest and/or by volume.

References

Crews, Donald. *Ten Black Dots.* New York: Greenwillow, 1986.

Hoban, Tana. *Is It Larger? Is It Smaller?* New York: Greenwillow, 1985.

Tuten, Margaret. *How Many is Many?* Chicago: Children's Press, 1970.

On-Line Resources

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Standard 10 — Estimation — Grades 3-4

Overview

As indicated in the K-12 Overview, students' ability to use estimation appropriately in their daily lives develops as they focus on the **reasonableness of answers**, explore and construct **estimation strategies,** and **estimate measurements, quantities, and the results of computation.**

For this type of development to occur, the atmosphere established in the classroom ought to assure everyone that their estimates are important and valued. Children should feel comfortable taking risks, and should understand that an explanation and justification of estimation strategies is a regular part of the process. Third- and fourth-graders, for the most part, should be beyond just "guessing." As children communicate with each other about how their estimates are formulated, they further develop their personal bank of **strategies for estimation.**

Students should already feel comfortable with estimation of sums and differences from their work in earlier grades. Nonetheless, they should regularly be asked *About how many do you think there will be in all?* or *About what do you think the difference is?* or *About how many do you think will be left?* in the standard addition and subtraction settings. These questions are appropriate whether or not exact computations will be done. As the concepts and the related facts of multiplication and division are introduced through experiences that are relevant to the child's world, **estimation with computation** must again be integrated into the development and practice activities.

One of the most useful computational **estimation strategies** in these grade levels also reinforces an important place-value idea. Students should understand that in multi-digit whole numbers the larger the place value, the more meaningful the digit in that position is in contributing to the overall value of the number. A reasonable approximation, then, of a multi-digit sum or difference can always be made by considering only the leftmost places and ignoring the others. This strategy is referred to as *front end estimation* and is the main estimation strategy that many adults use. In third and fourth grades, it should accompany the traditional *rounding* strategies.

Children should understand that, sometimes, the estimate will be accurate enough to serve as an answer. At other times, an exact computation will need to be done, either mentally, with paper-and-pencil, or with a calculator to arrive at a more precise answer. The particular procedure to be used is dependent on the setting and the problem. Also at this level, estimation must be an integral part of the development of concrete, algorithmic, or calculator approaches to multi-digit computation. Students must be given experiences which clearly indicate the importance of formulating an estimate *before* the exact answer is calculated.

In third and fourth grades, students are developing the concepts of a thousand and then of a million. Many opportunities arise where estimation of quantity is easily integrated into the curriculum. Many *what if* questions can be posed so that students continue to use estimation skills to determine practical answers.

Standard 10 — Estimation — Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Judge without counting whether a set of objects has less than, more than, or the same number of objects as a reference set.**

   - Students estimate the numbers represented by groups of base ten blocks or bundles of popsicle sticks. For example, one set might consist of 1 hundred, 6 tens, and 3 ones, and the other 0 hundreds, 17 tens, and 7 ones. Students first estimate which is more without arranging the blocks or counting them and then they determine the correct answer. These kinds of proportional models allow the "quantity of wood" to be proportional to the actual size of the number.

   - Students read *The Popcorn Book* by Tomie dePaola and make estimates with popcorn. For example, they might consider two quantities of popcorn, one popped and one unpopped. *Which contains the largest number of kernels?* They might also predict how many measuring cups the unpopped popcorn will fill once it is popped and find out the result after popping the popcorn. (The total number of cups made may be surprising.)

2. **Use personal referents, such as the width of a finger as one centimeter, for estimations with measurement.**

   - Students estimate the height of a classmate in inches or centimeters by standing next to him or her and using their own known height for comparison.

   - As standard units like yard and half-inch are introduced, students are challenged to find some part of their body or some personal action that is about that size at this point in their growth. For instance, they may decide that the width of their little finger is almost exactly one half-inch or the length of two *giant steps* is one yard.

   - Students measure the width of their handspan in centimeters (from thumb tip to little finger tip with the hand spread as far as possible) and then use the knowledge of its width to estimate the metric measures of various classroom objects by counting the number of handspans across and multiplying by the number of centimeters.

3. **Visually estimate length, area, volume, or angle measure.**

   - Students estimate the number of 3" x 5" cards it would take to cover their desktops, a floor tile, and the blackboard. They describe the process they used in writing, which is then read by the teacher to determine the students' progress.

   - Students work through the *Tiling a Floor* lesson that is described in the First Four

Standards of this *Framework*. They estimate how many of their tiles it would take to cover one sheet of paper, and compare their answers to the actual number needed.

- Students estimate the capacities of containers of a variety of shapes and sizes, paying careful attention to the equal contributions of width, length, and height to the volume. They sort a series of containers from smallest to largest and then check their arrangements by filling the smallest with uncooked rice, pouring that into the second, verifying the fact that it all fits and that more could be added, pouring all of that into the third, and so on.
- Students estimate the angle formed by the hands of the clock. They can also be challenged to find a time when the hands of the clock will make a 90° angle, a 120° angle, or a 180° angle.

4. Explore, construct, and use a variety of estimation strategies.

- Students develop and use the *front end estimation* strategy to obtain an initial estimate of the exact answer. For example, to estimate the total mileage in a driving trip where 354 miles were driven the first day, 412 the second day, and 632 the third, simply add all digits in the hundreds' place: $3 + 4 + 6 = 13$ *hundred* or 1300.
- Students learn to *adjust* the *front end estimation* strategy to give a more accurate answer. To do so, the second number from the left (in the problem above, the tens) is examined. Here, the estimate would be adjusted up one hundred because the $5 + 1 + 3$ *tens* are almost another hundred. This would give a better estimate of 1400.
- Students use *rounding* to create estimates, especially in multi-digit addition and subtraction. They do so flexibly, however, rather than according to out-of-context rules. In a grocery store, for example, when a person wants to be sure there is enough money to pay for items that cost \$1.89, \$2.95, and \$4.45, the best strategy may be to *round each price up to the next dollar*. In this case then, the actual sum of the prices is definitely less than \$10.00 ($2 + 3 + 5$). On the other hand, to be sure that the total requested by the cashier is approximately correct, the best strategy may be to *round each price to the nearest dollar* and get $2 + 3 + 4$ which is \$9.
- Students are shown a glass jar filled with about two hundred marbles and are asked to estimate the number in the jar. In small groups, they discuss various approaches to the problem and the strategies they can use. They settle on a strategy to share with the class along with the estimate that resulted.
- Students write about how they might find an estimate for a specific problem in their journals.

5. Recognize when estimation is appropriate, and understand the usefulness of an estimate as distinct from an exact answer.

- Given pairs of real-life situations, students determine which situation in the pair is the one for which estimation is the best approach and which is the one for which an exact answer is probably needed. One such pair, for example, might be: *deciding how much fertilizer is needed for a lawn* and *filling the bags marked "20 pounds" at the fertilizer company.*
- Given a set of cartoons with home-made mathematical captions, third graders decide which of the cartoon characters arrived at exact answers and which got estimates. One of

the cartoons might show an adult standing in the checkout line at a supermarket and another might show the checkout clerk. The captions would read: *Mr. Harris wondered if he had enough money to pay for the groceries he had put in the cart* and *Harry used the cash register to total the bill.* Students make up their own similar cartoon.

- Students share with each other various situations within the past week when they and their families had to do some computation and describe when an exact answer was necessary (and why) and when an estimate was sufficient (and why).

6. Determine the reasonableness of an answer by estimating the result of operations.

- Students are regularly asked if their answer makes sense in the context of the problem they were solving. They respond with full sentences explaining what they were asked to find and why the numerical answer they found fits the context reasonably, that is, why it *could be* the answer.
- Fourth graders might be asked to decide if their estimated answer to the following problem is reasonable. *The band has 103 students in it. They line up in 9 rows. How many students are there in each row?* The students' responses might indicate, for example, that there should be about 10 students in each row, since 103 is close to 100 and 9 is close to 10.
- Students estimate *reasonable* numbers of times that particular physical feats can be performed in one minute. For example: *How many times can you skip rope in a minute? How many times can you hit the = button on the calculator in a minute? How many times can you blink in a minute? How many times can you write your full name in a minute?* and so on. Other students judge whether the estimates are reasonable or unreasonable and then the tasks are performed and actual counts made. (To determine the number of times the = button is hit in a minute, press +1= so that each time the = button is pressed, the display increases by 1.)
- Third-grade students are given a set of thirty cards with three-digit subtraction problems on them. In one minute, they must sort the cards into two piles: those problems whose answers are greater than 300 and those whose answers are less than 300. The correct answers can be on the backs of the cards to allow self-checking after the task is completed.
- For assessment, fourth-grade students might be given a page of one-digit by multi-digit multiplication problems in a multiple choice format with four possible answers for each problem. Within some time period which is much too short for them to perform the actual computations, students are asked to choose the most reasonable estimate from each set of four answers.

7. Apply estimation in working with quantities, measurement, time, computation, and problem solving.

- Students work through the *Product and Process* lesson that is described in the Introduction to this *Framework*. It challenges the students to form two three-digit numbers using 3, 4, 5, 6, 7, 8 which have the largest product; estimation is used to determine the most reasonable possible choices.

- Students learn about different strategies for estimating by reading *The Jellybean Contest* by Kathy Darling or *Counting on Frank* by Rod Clement.
- Students regularly try to predict the numerical facts presented in books like *In an Average Lifetime . . .* by Tom Heymann. Using knowledge they have and a whole variety of estimation skills, they predict answers to: *What is the number of times the average American eats in a restaurant in a lifetime? (14,411) What is the total length each human fingernail grows in a lifetime? (77.9 inches)* and *What is the average number of major league baseball games an American attends in a lifetime? (16)*
- Students regularly estimate in situations involving classroom routines. For example, they may estimate the total amount of money that will be collected from the students who are buying lunch on Pizza Day or the number of buses that will be needed to take the whole third and fourth grade on the class trip.
- Students investigate environmental issues using estimation. One possible activity is for them to estimate how many gallons of water are used for various activities each week in their home. (See *Healthy Environment — Healthy Me.*)

Activity	Total # in 1 week		Each time	Total # of gallons used
Take shower or bath	_______	x	18 gallons	=
Flush toilet	_______	x	7 gallons	=
Wash dishes	_______	x	10 gallons	=
Wash clothes	_______	x	40 gallons	=
			Total # of gallons used in 1 week	=

Students discuss possible reasons for differences among their estimates, and they compute the class total for the number of gallons consumed during that week.

References

Clement, Rod. *Counting on Frank.* Milwaukee, WI: Gareth Stevens Children's Books, 1991.

Darling, Kathy. *The Jellybean Contest.* Champaign, IL: Garrard, 1972.

de Paola, Tomie. *The Popcorn Book.* New York: Holiday House, 1978.

Environmental and Occupational Health Sciences Institute. *Healthy Environment — Healthy Me. Exploring Water Pollution Issues: Fourth Grade.* New Jersey: Rutgers University, 1991.

Heymann, Tom. *In an Average Lifetime ...* New York: Random House, 1991.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

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Standard 10 — Estimation — Grades 5-6

Overview

In grades 5 and 6, students extend estimation to new types of numbers, including fractions and decimals. As indicated in the K-12 Overview, they continue to work on determining the **reasonableness of results,** using **estimation strategies**, and **applying estimation to measurement, quantities,** and **computation.**

In fifth and sixth grade, estimation and number sense are even more important skills than algorithmic paper-and-pencil computation with multi-digit whole numbers. Students should become masters at applying estimation strategies so that answers displayed on a calculator are instinctively compared to a reasonable range in which the correct answer lies.

The new estimation skills that are also important in fifth and sixth grade are skills in estimating the results of fraction and decimal computations. Even though the study of the concepts and arithmetic operations involving fractions and decimals begins before fifth grade, a great deal of time will be spent on them here. A sample unit on fractions for the sixth-grade level can be found in Chapter 17 of this *Framework*. As students develop an understanding of fractions and decimals and perform operations with them, estimation ought always to be present. Estimation of quantities in fraction or decimal terms and of the results of operations on those numbers is just as important for the mathematically literate adult as the same skills with whole numbers.

Children should understand that, sometimes, an estimate will be an accurate enough number to serve as an answer. At other times, an exact computation will need to be done, either mentally, with paper-and-pencil, or with a calculator to arrive at a more precise answer. Which procedure should be used is dependent on the setting and the problem. Even in cases where exact answers are to be calculated, however, students must understand that it is always a good idea to have an estimate in mind before the actual exact computation is done so that the computed answer can be checked against the estimated one.

Standard 10 — Estimation — Grades 5-6

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skill gained in the preceding grades, experiences in grades 5-6 will be such that all students:

5*. Recognize when estimation is appropriate, and understand the usefulness of an estimate as distinct from an exact answer.

- Given pairs of real-life situations, students determine which is the one in which estimation is the best approach and which is the one needing an exact answer. For example, one such pair, might be: *planning how long it would take to drive from Boston to New York* and *submitting a bill for mileage to your boss.*
- Students collect data for a week on various situations when they and their families had to do some computation and describe when an exact answer was necessary (and why) and when an estimate was sufficient (and why).
- When doing routine problems, the students are always reminded to consider whether their answers make sense. For instance, in the following problem, an estimate makes much more sense than an exact computation. *Molly Gilbert is the owner of a small apple orchard in South Jersey. She has 19 rows of trees with 12 trees in each row. Last year the average production per tree was 761.3 apples. At that rate, what can she expect the total yield to be this year?* For this problem, an exact computation is certain to be wrong and will also be a number that is very hard to remember or use in further planning.
- Students look for examples of estimation language in their reading and/or in the newspapers.
- Students investigate what decisions at the school are based on estimates (e.g., quantity of food for lunch, ordering of textbooks or supplies, making up class schedules, organizing bus routes) by interviewing school employees and making written and/or oral reports.

6. Determine the reasonableness of an answer by estimating the result of operations.

- Students estimate the size of a crowd at a rock concert from a picture. They share all of their various strategies with the rest of the class.
- Students demonstrate their understanding by estimating whether or not they can buy a set of items with a given amount of money. For example: *You have only $10. Explain how you can tell if you have enough money to buy: 4 cans of tuna @ 79¢ each, 2 heads of lettuce @ 89¢ each, and 2 lbs. of cheese @ $2.11 per lb.*

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* Activities are included here for Indicators 5 and 6, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

- Students place decimal points in multi-digit numbers to make absurd statements more *reasonable.* Some typical statements could be *Mr. Brown averages 2383 miles per gallon when he drives to work, My dog weighs 5876 pounds,* or *Big Burger charges $99 for a large order of french fries.*

8. Develop, apply, and explain a variety of different estimation strategies in problem situations involving quantities and measurement.

- Students develop the concept of a billion by estimating its size relative to one hundred thousand, one million, and so on. For instance, they explore questions like *If a calculator is programmed to repeatedly add 1 to the previous display and it takes about forty hours to reach a million, how long would it take to reach a billion?* Students might relate this to the size of the national debt (now $5 trillion).

- Students estimate the area inside a closed curve in square centimeters and then check the estimate with centimeter graph or grid paper.

- After reading Shel Silverstein's poem "How Many, How Much" and Tom Parker's *Rules of Thumb*, students write their own rules of thumb (e.g., *You should never have homework higher than one inch.*).

- Students make estimates about things that happen in one day at school after reading about some of the data in *In One Day,* by Tom Parker. For example, they estimate: *How much pizza is eaten? How much milk is drunk? How many students go home sick?* or *How many students forget something?* They then interview people around the school or conduct a survey to check their estimates.

- Students develop strategies for estimating sums and differences of fractions as they work with them. For example, a fifth grade class is asked to determine which of the following computation problems have answers greater than 1 without actually performing the calculation; many students' strategies will hinge on comparisons of the given fractions to 1/2.

$$\frac{1}{2} + \frac{3}{4} = \qquad \frac{1}{3} + \frac{5}{6} = \qquad 1\frac{5}{16} - \frac{1}{2} =$$

- Students make use of strategies like *clustering* and *compatible numbers* in estimating the results of computations. They recognize that a sum of numbers that are approximately the same, such as 37, 39, and 42, can be replaced in an estimate by the product 3 x 40 (*clustering*). They also know that other computations can be performed easily by changing the numbers to numbers that are closely related to each other, such as changing 468 divided by 9 to 450 divided by 9 (*compatible numbers*).

- Students work through the *Mathematics at Work* lesson that is described in the Introduction to this *Framework*. A parent discusses a problem which her company faces regularly: to determine how large an air conditioner is needed for a particular room. To solve this problem, the company has to estimate the size of the room.

9. **Use equivalent representations of numbers such as fractions, decimals, and percents to facilitate estimation.**

- Students use fractions, decimals, or mixed numbers interchangeably when one form of a number makes estimation easier than another form. For example, rather than estimating the product 3/5 x 4, students consider 0.6 x 4 which yields a much quicker estimate.
- In their beginning work with percent, students master the common fraction equivalents for familiar percentages and use fractions for estimation in appropriate situations. For example, an estimate of 65% of 63 can be easily obtained by considering 2/3 of 60.

10. **Determine whether a given estimate is an overestimate or an underestimate.**

- Students decide, as they discuss each new estimation strategy they learn, whether the strategy is likely to give an overestimate, an underestimate, or neither. For instance, using front-end digits will always give an underestimate; rounding everything up (as one might do to make sure she has enough money to pay for items selected in a grocery store) always gives an overestimate; and ordinary rounding may give either an overestimate or an underestimate.
- Students frequently use *guess, check, and revise* as a problem solving strategy. With this strategy, the answer to a problem is estimated, then calculations are made using the estimate to see whether this estimate meets the conditions of the problem, and the estimate is then revised upwards or downwards as a result. For example, students are asked to find two consecutive pages in a book the product of whose page numbers is 1260. An initial *guess* might be 30 and 31, the *check* might involve concluding that this product is a little over 900, and, as a result, they might *revise* their estimate upwards.

References

Parker, Tom. *In One Day*. Boston: Houghton Mifflin, 1984.

Parker, Tom. *Rules of Thumb*. Boston: Houghton Mifflin, 1983.

Silverstein, Shel. "How Many, How Much," in *A Light in the Attic*. New York: Harper and Row, 1981.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

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Standard 10 — Estimation — Grades 7-8

Overview

Estimation, as described in the K-12 Overview includes three primary themes: determining the **reasonableness of answers**, using a variety of **estimation strategies** in a variety of situations, and **estimating the results of computations**.

In seventh and eighth grade, estimation and number sense are much more important skills than algorithmic paper-and-pencil computation with whole numbers. Students should become masters at applying estimation strategies so that an answer displayed on a calculator is instinctively compared to a reasonable range in which the correct answer lies. It is critical that students understand the displays that occur on the screen and the effects of calculator rounding either because of the calculator's own operational system or because of user-defined constraints. Issues of the number of significant figures and what kinds of answers make sense in a given problem setting create new reasons to focus on **reasonableness of answers**.

The new estimation skills begun in fifth and sixth grade are still being developed in the seventh and eighth grades. These include skills in estimating the results of fraction and decimal computations. As students deepen their understanding of these numbers and perform operations with them, estimation ought always to be present. Estimation of quantities in fraction or decimal terms as a result of operations on those numbers is just as important for the mathematically literate adult as the same skills with whole numbers.

In addition, the seventh and eighth grades present students with opportunities to develop strategies for estimation with ratios, proportions, and percents. Estimation and number sense must play an important role in the lessons dealing with these concepts so that students feel comfortable with the relative effects of operations on them. Another new opportunity here is estimation of roots. It should be well within every eighth grader's ability, for example, to estimate the square root of 40.

Students should understand that sometimes, an estimate will be accurate enough to serve as an answer. At other times, an exact computation will need to be done, either mentally, with paper-and-pencil, or with a calculator. Even in cases where exact answers are to be calculated, however, students must understand that it is almost always a good idea to have an estimate in mind so that the computed answer can be checked against it.

Standard 10 — Estimation — Grades 7-8

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

5*. Recognize when estimation is appropriate, and understand the usefulness of an estimate as distinct from an exact answer.

- Students regularly tackle problems for which estimation is the only possible approach. For example: *How many hairs are on your head?* or *How many grains of rice are in this ten-pound bag?* Solution strategies are always discussed with the whole class.
- Students create a plan to "win a contract" by bidding on projects. For example: *Your class has been given one day to sell peanuts at Shea Stadium. Prepare a presentation that includes the amount of peanuts to order, the costs of selling the peanuts, the profits that will be made, and the other logistics of selling the peanuts. Organize a schedule with estimated times for completion for the entire project.*
- Students use estimation skills to run a business using *Hot Dog Stand* or *Survival Math* software.
- Students apply estimation skills to algebraic situation as they try to guess the equations to hit the most globs in *Green Globs* software.

6. Determine the reasonableness of an answer by estimating the result of operations.

- Students estimate whether or not they can buy a set of items with a given amount of money. For example: *I have only $50. Can I buy a reel, a rod, and a tackle box during the sale advertised below?*

  ALL ITEMS 1/3 OFF AT JAKE'S FISHING WORLD!

ITEM	REGULAR PRICE
Daiwa Reels	$29.95 each
Ugly Stick Rods	$20.00 each
Tackle Boxes	$17.99 each

  To assess students' performance, the teacher asks them to write about how they can answer this question without doing any exact computations.

  If 6% sales tax is charged, can you tell whether $50 is enough by estimating? Explain. Calculate the exact price including tax.

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* Activities are included here for Indicators 5 and 6, which are also listed for grade 4, since the Standards specify that students demonstrate continued progress in these indicators.

- Students evaluate various statements made by public figures to decide whether they are reasonable. For example: *The Phillies' center fielder announced that he expected to get 225 hits this season. Do you think he will?* In order to determine what confidence to have in that prediction, a variety of factors need to be estimated: number of at-bats, lifetime batting average, likelihood of injury, whether a baseball strike will occur, and so on.

- Students discuss events in their lives that might have the following likelihoods of occurring:

  100%, 0.5 %, 3/4, 95%

- Students simulate estimating the number of fish in a lake by estimating the number of fish crackers in a box using the following method. Some of the fish are removed from the box and "tagged" by marking them with food coloring. They are "released" to "swim" as they are mixed in with the other crackers in the box. Another sample is drawn and the number of tagged fish and the total number of fish are recorded. This data is used to set up a proportion (# tagged initially: total # fish = # tagged caught: total # caught) to predict the total number of fish in the box. The estimate is further improved by taking additional samples, making predictions based on each sample of the total number of fish in the box, and finally averaging all the predictions.

8. Develop, apply, and explain a variety of different estimation strategies in problem situations involving quantities and measurement.

- Students regularly have opportunities to estimate answers to straight-forward computation problems and to discuss the strategies they use in making the estimations. Even relatively routine problems generate interesting discussions and a greater shared number sense within the class:

  23% of 123, *5* x *38,* *28* x 425, 486 x 2004, $423 \div 71$

- Given a ream of paper, students work in small groups to estimate the thickness of one sheet of paper. Answers and strategies are compared across groups and explanations for differences in the estimates are sought. To assess student understanding, the teacher asks each student to write about how his or her group solved the problem.

- Students develop strategies for estimating the results of operations on fractions as they work with them. For example, a seventh grade class is asked to determine which of these computation problems have answers greater than 1 without actually performing the calculation:

  $\frac{1}{2} \times \frac{3}{4} =$ $\frac{5}{6} \div \frac{1}{3} =$ $\frac{1}{2} \div 1\frac{1}{2} =$

- Students make estimates of the number of answering machines, cellular phones, and fax machines in the United States and check their results against data from *America by the Numbers* by Les Krantz.

- Students estimate the total number of people attending National Football League games by determining how many teams there are, how many games each plays, and what the average attendance at a game might be. They then use these estimates to determine the overall answer (17,024,000 according to *America by the Numbers*, p. 194).

- Students determine the amount of paper thrown away at their school each week (or month or year) by collecting the paper thrown away in their math class for one day and multiplying this by the number of classes in the school and then by five (or 30 or 50).

9. Use equivalent representations of numbers such as fractions, decimals, and percents to facilitate estimation.

- Students use fractions, decimals, or mixed numbers interchangeably when one form of the number makes estimation easier than another. For example, rather than estimating 33⅓% of $120, students consider 1/3 x 120 which yields a much quicker estimate.
- Similarly, in their work with percents, students master the common fraction equivalents for familiar percentages and use fractions for estimation in appropriate situations. For example, an estimate of 117% of 50 can most easily be obtained by considering 6/5 of 50.
- Students collect and bring to class sales circulars from local papers which express the discounts on sale items in a variety of ways including percent off, fraction off, and dollar amount off. For items chosen from the circular, the students discuss which form is the easiest form of expression of the discount, which is most understandable to the consumer, and which makes the sale seem the biggest bargain.

10. Determine whether a given estimate is an overestimate or an underestimate.

- Using calculators, but without using the square root key, students try to find good approximations for a few square roots, for example, the square root of 40. Through a series of approximations, they make a guess, perform the multiplication on the calculator, determine whether the approximation was too large or too small, adjust it, and begin again. This series of approximations, in itself a very useful strategy, continues until an approximation is reached that is satisfactory.
- Students compare three different rounding strategies:

  Round everything up: for example, 2345, rounded to the nearest ten, would be 2350. To the nearest hundred, it would be 2400.

  Round up if 5 or more, down if less than 5: for example, 2345, rounded to the nearest ten, would be 2350. To the nearest hundred, it would be 2300.

  Round up if more than 5, down if less than 5; and for the special case when the digit equals five make the preceding digit even. That is, if the number before the five is odd, round up; if it is even, round down. For example, 2345, rounded to the nearest ten, would be 2340. To the nearest hundred, it would be 2300.

  They discuss whether each strategy would be more likely to yield an overestimate or an underestimate when adding up a total and then apply all these strategies to several situations.

References

Krantz, Les. *America by the Numbers*. New York: Houghton Mifflin, 1993. (Note: Some of the entries in this book are unsuitable for seventh- and eighth-graders.)

Software

Green Globs and *Graphing Equations*. Sunburst Communications.

Hot Dog Stand. Sunburst Communications.

Survival Math. Sunburst Communications.

On-Line Resources

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Standard 10 — Estimation — Grades 9-12

Overview

Estimation is a combination of content and process. Students' abilities to use estimation appropriately in their daily lives develop as they have regular opportunities to explore and construct estimation strategies and as they acquire an appreciation of its usefulness through using estimation in the solution of problems. At the high school level, estimation includes focusing on the **reasonableness of answers** and using various **estimation strategies for measurement, quantity, and computations.**

In the high school grades, estimation and number sense are much more important skills than algorithmic paper-and-pencil computation. Students need to be able to judge whether answers displayed on a calculator are within an acceptable range. They need to understand the displays that occur on the screen and the effects of calculator rounding either because of the calculator's own operational system or because of user-defined constraints. Issues of the number of significant digits and what kinds of answers make sense in a given problem setting create new reasons to a focus on **reasonableness of answers.**

Measurement settings are rich with opportunities to develop an understanding that estimates are often used to determine approximate values which are then used in computations and that results so obtained are not exact but fall within a **range of tolerance**. Appropriate issues for discussion at this level include acceptable limits of tolerance, and assessments of the degree of error of any particular measurement or computation.

Another topic appropriate at these grade levels is the estimation of probabilities and of statistical phenomena like measures of central tendency or variance. When statisticians talk about "eyeballing" the data, they are explicitly referring to the process where these kinds of measures are estimated from a set of data. The skill to be able to do that is partly the result of knowledge of the measures themselves and partly the result of experience in computing them.

Standard 10 — Estimation — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11, and 12.

Building upon knowledge and skills gained in the preceding grades, experience in grades 9-12 will be such that all students:

6*. Determine the reasonableness of an answer by estimating the result of operations.

- Students are routinely asked if the answers they've computed make sense. *Latisha's calculator displayed 17.5 after she entered 3 times the square root of 5. Is this a reasonable answer?*
- Students are sometimes presented with hypothetical scenarios that challenge both their estimation and technology skills: *During a test, Paul entered* $y = .516x - 2$ *and* $y = .536x + 5$ *in his graphics calculator. After analyzing the two lines displayed on the "standard" screen window setting [-10,10, -10,10], he decided to indicate that the lines were parallel and that there was no point of intersection. Was Paul's answer reasonable?*
- On a test, students are asked the following question: *Jim used the zoom feature of his graphics calculator and found the solution to the system* $y = 2x + 3$ *and* $y = -2x - 1$ *to be (-.9997062,1.0005875). When he got his test back his teacher had taken points off. What was wrong with Jim's answer?*

11. Estimate probabilities and predict outcomes from real-world data.

- Students use tables of data from an almanac to make estimates of the means and medians of a variety of measures such as the average state population or the average percentage of voters in presidential elections. Any table where a list of figures (but no mean) is given can be used for this kind of activity. After estimates are given, actual means and medians can be computed and compared to the estimates. Reasons for large differences between the means and medians ought to also be explored.
- Students collect data about themselves and their families for a statistics unit on standard deviation. After everyone has entered data in a large class chart regarding number of siblings, distance lived away from school, oldest sibling, and many other pieces of numerical data, the students work in groups to first estimate and then compute means and medians as a first step toward a discussion of variation.
- Students each track the performance of a particular local athlete over a period of a few weeks and use whatever knowledge they have about past performance to predict his or her performance for the following week. They provide as detailed and statistical a prediction

* Activities are included here for Indicator 6, which is also listed for grade 8, since the Standards specify that students demonstrate continued progress in this indicator.

as possible. At the end of the week, predictions are compared to the actual performance. Written reports are evaluated by the teacher.

12. Recognize the limitations of estimation, assess the amount of error resulting from estimation, and determine whether the error is within acceptable tolerance limits.

- Students in high school learn methods for estimating the magnitude of error in their estimations at the same time as they learn the actual computational procedures. Discussions regarding the acceptability of a given magnitude of error are a regular part of classroom activities when estimation is being used.

- Students work in small groups to carefully measure the linear dimensions of a rectangular box and determine its volume using measures to the nearest 1/8 inch or the smallest unit on their rulers. After their best measurements and computation, the groups share their estimates of the volume and discuss differences. Each group then constructs a range in which they are sure the exact answer lies by first using a measure for each dimension which is clearly short of the actual measure and multiplying them, and then by finding a measure for each dimension which is clearly longer than the actual measure and multiplying those. The exact answer then lies between those two products. Each group prepares a written report outlining their procedures and results.

- Students are presented with these two solutions to the following problem and discuss the error associated with each approach: *How many kernels of popcorn are in a cubic foot of popcorn?*

    1. *There are between 3 and 4 kernels of popcorn in 1 cubic inch. There are 1728 cubic inches in a cubic foot. Therefore there are 6048 kernels of popcorn in a cubic foot. [3.5 (the average number of kernels in a cubic inch) x 1728 = 6048]*
    2. *The diameter of a kernel of popcorn is approximately 9/16 of an inch. The volume of this "sphere" is 0.09314 cubic inches. Therefore (1728/0.09314) = 18552.71634 or 18,000 pieces of popcorn.*

- Students write a computer program to round any number to the nearest hundredth.

- Students analyze the error involved in rounding to any value. For example, a number rounded to the nearest ten, say 840, falls into this range: $835 \leq X < 845$. The error involved could be as large as 5. Similarly, a number rounded to the nearest hundredth, say .84, falls into the range: $0.835 \leq X < 0.845$. The error could be as large as .005.

- Students discuss what is meant by the following specifications for the diameter of an 0-ring: 2.34 ± 0.005 centimeters.

- Students act as *quality assurance officers* for mythical companies and devise procedures to keep errors within acceptable ranges. One possible scenario:

    In order to control the quality of their product, Paco's Perfect Potato Chip Company guarantees that there will never be more than 1 burned potato chip for every thousand that are produced. The company packages the potato chips in bags that hold about 333 chips. Each hour 9 bags are randomly taken from the production line and checked for burnt chips. If more than 15 burnt chips are found within a four hour shift, steps are taken to reduce the number of burnt chips in each batch of chips produced. Will this plan ensure the company's guarantee?

- Students regularly review statistical claims reported in the media to see whether they accurately reflect the data that is provided. For example, did the editor make appropriate use of the data given below? (Based on an example in *Exploring Surveys and Information from Samples* by James Landwehr.)

  The March, 1985 Gallup Survey asked 1,571 American adults "Do you approve or disapprove of the way Ronald Reagan is handling his job as president?" 56% said that they approved. For results based on samples of this size, one can say with 95% confidence that the error attributable to sampling and other random effects could be as much as 3 percentage points in either direction. A newspaper editor read the Gallup survey report and created the following headline: BARELY ONE-HALF OF AMERICA APPROVES OF THE JOB REAGAN IS DOING AS PRESIDENT.

References

Landwehr, James. *Exploring Surveys and Information from Samples.* Palo Alto, CA: Dale Seymour, 1987.

On-Line Resources

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STANDARD 11 – PATTERNS, RELATIONSHIPS, AND FUNCTIONS

K-12 Overview

All students will develop an understanding of patterns, relationships, and functions and will use them to represent and explain real-world phenomena.

Descriptive Statement

Patterns, relationships, and functions constitute a unifying theme of mathematics. From the earliest age, students should be encouraged to investigate the patterns that they find in numbers, shapes, and expressions, and, by doing so, to make mathematical discoveries. They should have opportunities to analyze, extend, and create a variety of patterns and to use pattern-based thinking to understand and represent mathematical and other real-world phenomena. These explorations present unlimited opportunities for problem solving, making and verifying generalizations, and building mathematical understanding and confidence.

Meaning and Importance

Mathematics is often regarded as the *science of patterns*. When solving a complex problem, we frequently suggest to students that they try to work on simpler versions of the problem, observe what happens in a few specific cases — that is, *look for a pattern* — and use that pattern to solve the original problem. This *pattern-based thinking*, using patterns to analyze and solve problems, is an extremely powerful tool for doing mathematics. Students who are comfortable looking for patterns and then analyzing those patterns to solve problems can also develop understanding of new concepts in the same way. Most of the major principles of algebra and geometry emerge as generalizations of patterns in number and shape. For example, one important fact in geometry is that: *For a given perimeter, the figure with the largest possible area that can be constructed is a circle.* This idea can be discovered informally by students in the middle grades by examining the pattern that comes from a series of constructions and measurements. Students can be given a length, say 24 centimeters, for the perimeter of all figures to be created. Then they can construct and measure or compute the areas of a series of regular polygons: an equilateral triangle, a square, and a regular hexagon, octagon, and dodecagon (12 sides). The pattern that clearly emerges is that as the number of sides of the polygon increases — that is, as the polygon becomes more "circular"— the area increases.

All of the content standards are interconnected, but this standard is one that is particularly closely tied to all of the others. This is because pattern-based thinking is regularly applied to content in numeration, geometry, operations, discrete mathematics, and the fundamentals of calculus. There is a very special relationship, though, between patterns and algebra. Algebra provides the language in which we communicate the patterns in mathematics. Early on in their mathematical careers, students must begin to make generalizations about patterns that they find, and they should learn to express those generalizations in mathematical terms.

K-12 Development and Emphases

Children become aware of patterns very early in their lives — repetitive daily routines and periodic phenomena are all around them. Breakfast is followed by lunch which is followed by dinner which is followed by bedtime and then the whole thing is repeated again the next day. Each one of the three little pigs says to the wolf, at exactly the expected moment, *Not by the hair on my chinny-chin-chin!* In the primary grades, children need to build on those early experiences by **constructing, recognizing, and extending patterns** in a variety of contexts. Numbers and shapes certainly offer many opportunities, but so do music, language, and physical activity. Young children love to imitate rhythmic patterns in sound and language and should be encouraged to create their own. In addition, they should construct their own patterns with manipulatives such as pattern blocks, attribute blocks, and multilink cubes and should be challenged to extend patterns begun by others. Identifying attributes of objects, and using them for **categorization and classification**, are skills that are closely related to the ability to create and discover patterns and need to be developed at the same time.

Young students should frequently play games which ask them to follow a sequence of rules or to **discover a rule** for a given pattern. Sequences which begin as counting patterns soon develop into rules involving arithmetic operations. Children in the primary grades, for example, will make the transition from *2, 4, 6, 8, ...* as a counting by twos pattern to the rule *Add 2* or "+2." The calculator is a very useful tool for making this connection since it can be used for counting up or counting down by any constant amount. Students can be challenged to guess the number that will come up next in the calculator's display and then to explain the pattern, or rule, to the class.

At a slightly higher level, **input-output** activities which require recognition of relationships between one set of numbers (the "IN" values) and a second set (the "OUT" values) provide an early introduction to **functions**. One of these kinds of activities, the *function machine* games, is a favorite among first through fifth graders. In these, one student has a rule in mind to transform any number suggested by another student. The first number is inserted into the imaginary *function machine* and another number comes out the other side. The rule might be *plus 7*, or, *times 4 then minus 3*, or even *the number times itself.* The class's task is to discover the rule by an examination of the input-output pairs. In the intermediate grades, students can simulate the *function machine* with a computer spreadsheet *secretly* programmed to take the number typed in the first column and *transform* it into another number that is placed in the second column.

Slightly older students begin to work with patterns that can be used to **solve problems** within mathematics and from the real world. There should also be a more deliberate focus on **relationships involving two variables**. An exploration of the relationship between the number of teams in a round robin tournament and the total number of games that must be played, or between a number of coins to be flipped and the total number of possible outcomes, provides a real-world context for pattern-based thinking and informal work with functions. Graphing software and graphing calculators are extremely valuable at this level to help students visualize the relationships they discover.

At the secondary level, students are able to bring more of the tools of algebra to the task of analyzing and representing patterns and relationships. Thus we expect all students to be able to construct as well as to recognize symbolic representations such as $y = f(x) = 4x+1$. They should also develop an understanding of the many other representations and applications of functions as well as of a greater variety of functional relationships. Their work should extend to quadratic, polynomial, trigonometric, and exponential functions in addition to the linear functions they worked with in earlier grades. They should be comfortable with the symbols f, representing a rule, and $f(x)$, representing the value which f assigns to x.

The use of functions in modeling real-life and real-time observations also plays a central role in the high school mathematics experience. Line- and curve-fitting as approaches to the explanation of a set of experimental data help make mathematics come alive for students. Technology must play an important role in this process, since students are now able to graphically explore relationships more easily than ever before. Graphing calculators and computers must be made available to all students for use in these types of investigations.

IN SUMMARY, an important task for every teacher of mathematics is to help students recognize, generalize, and use patterns that exist in numbers, in shapes, and in the world around them. Students who have such skills are better problem solvers, have a better sense of the uses of mathematics, and are better prepared for work with algebraic functions than those who do not.

NOTE: Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.

Standard 11 — Patterns, Relationships, and Functions — Grades K-2

Overview

The development of *pattern-based thinking*, using patterns to analyze and solve problems, is an extremely powerful tool for doing mathematics, and leads in later grades to an appreciation of how functions are used to describe relationships. The key components of *pattern-based thinking* at the early grade levels, as identified in the K-12 Overview, are **recognizing, constructing, and extending patterns, categorizing and classifying objects,** and **discovering rules.**

> "Looking for patterns trains the mind to search out and discover the similarities that bind seemingly unrelated information together in a whole. . . . A child who expects things to 'make sense' *looks* for the *sense* in things and from this sense develops understanding. A child who does not see patterns often does not *expect* things to make sense and sees all events as discrete, separate, and unrelated."
>
> — Mary Baratta-Lorton (cited on p.112 of *About Teaching Mathematics* by Marilyn Burns)

Children in the primary grades develop an awareness of patterns in their environment. Those who are successful in mathematics expand this awareness into understanding and apply it to learning about the number system. Children who do not look for patterns as a means of understanding and learning mathematics often find mathematics to be quite difficult. Thus, it is critical in the early grades to establish an early predisposition to looking for patterns, creating patterns, and extending patterns.

Children should **recognize, construct and extend patterns** with pattern blocks, cubes, toothpicks, beans, buttons and other concrete objects. Children in kindergarten can recognize patterns in motion, color, designs, sound, rhythm, music, position, sizes, and quantities. They are very aware of sound and rhythm, and can clap out patterns that repeat, such as clap-clap-clap-pause, clap-clap-clap-pause, etc. They can sit in a circle and wear colored hats which make a pattern, such as red-white-blue, red-white-blue. One child can walk around the circle and tap successive children in an arm-shoulder-head pattern. The teacher may ask the class who the next person to be tapped on the head would be if the pattern were to be continued. In addition to repeating patterns, students should have experiences with expanding patterns. They can indicate such a pattern by using motion: skip-jump-turn around, skip-jump-jump-turn around, skip-jump-jump-jump-turn around, and so on. Songs are excellent examples of repetition of melody or of words, as well as of rhythmic patterns. Children's literature abounds with stories which rely on rhythm, rhyming, repetition and sequencing. As students move on to first and second grade, they should start to create their own patterns and develop pictorial and symbolic representations of those patterns. The transition will be from working with patterns using physical objects to using pictures, letters, and geometric figures in two and three dimensions, and then to using symbols, such as words and numbers, to represent patterns.

Categorization and classification are also important skills for students in the primary grades. Kindergartners should have numerous opportunities to sort, classify, describe, and order collections of many different types of objects. For example, students might be asked to sort attribute shapes, buttons, or boxes into two groups and explain why they sorted them as they did. This area offers an excellent opportunity for students to verbalize their thought processes and to integrate learning in mathematics and science as they sort natural objects such as shells, rocks, or leaves.

Discovering a rule and **input-output** games are two other settings in which primary children can enhance their work and their skills with patterns. The children might be asked to solve the *mystery of the crackers* as the teacher slowly and deliberately gives every boy two crackers and every girl four crackers one day during snack time. The inequity is addressed, of course, as soon as the children solve the mystery by discovering the rule that the teacher was using. On a different day, first graders can be told that they may request between 3 and 5 crackers for snack. But then each child is actually given two crackers less than his or her request. Again, as soon as the children verbalize the relationship between the request (input) and the portion allotted (output), they receive the missing crackers.

Establishing the habit of looking for patterns is exceedingly important in the primary grades. By studying patterns, young children develop necessary tools to become better learners of mathematics as well as better problem solvers. In addition, patterns help students to appreciate the beauty of mathematics and to make connections within mathematics and among mathematics and other subject areas.

Standard 11 — Patterns, Relationships, and Functions — Grades K-2

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and in grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Reproduce, extend, create, and describe patterns and sequences using a variety of materials.**
   - Students make a collage with examples of patterns in nature.
   - Students create visual patterns with objects, colors, or shapes using materials such as buttons, macaroni, pattern blocks, links, cubes, attrilinks or attribute blocks, toothpicks, beans, or teddy bear counters. They challenge other students to describe or extend their patterns.
   - Students sort objects such as leaves, buttons, animal pictures, and blocks, using categories corresponding to characteristics like number of holes, number of sides, shapes, or thickness.
   - One child walks around the outside of a circle and taps successive children in a head-shoulder-shoulder-head pattern. The teacher asks who the next person to be tapped on the head would be if the pattern were to be continued. The children sing and act out the song, *Head, shoulders, knees and toes*.
   - Students describe patterns made from circles, triangles, and squares, and select the next shape in the pattern.
   - Students make patterns with letters and extend the sequence.
   - As an assessment task, students use letters to translate patterns they have created with objects — for example, RRBRRB for a Unifix pattern of red-red-blue-red-red-blue, or ABBCABBC for a shape pattern of square - circle - circle - triangle - square - circle - circle - triangle.
   - Students connect the dots to make a picture by following a number sequence, such as 2, 4, 6, 8, ...
   - Students create *one more* and *one less* patterns.
   - Students create patterns with the calculator. They enter any number such as 10, and then add 1 for 10+1= = = ... The calculator will automatically repeat the function and display 11, 12, 13, 14, etc. Some calculators may need to have the pattern entered twice: 10+1=+1= = = ... Other calculators will need 1++10= = = ... Students may repeatedly add or subtract any number.
   - Students name things that come in pairs (or 4s or 5s): eyes, ears, hands, arms, legs, mittens, shoes, bicycle wheels, etc. They work in pairs to find how many people there are if there are 20 eyes.

- Students count by 2, 5, or 10 using counters or creating color patterns with Unifix or Linker cubes; they repeat this using skip counting on a number line.
- Students use skip counting or calculators to find multiples of numbers and then color them on the hundreds chart. Linking cubes or Unifix cubes can be used to build towers or trains with every other cube or every third cube a certain color to illustrate, recognize, and practice skip counting patterns.
- Students write their first name repeatedly on a 10x10 grid, and then color the first letter of their name to create a pattern. They discuss the patterns formed.
- Students identify the same pattern in a variety of contexts. For example, black-white-black-white is like sit-stand-sit-stand and ABAB and up-down-up-down and straight-curve-straight-curve.
- Students identify patterns on a calendar using pictures or numerals. For example, in November, even dates might be marked with a snowflake, and odd dates with a picture of a turkey. Or, they might mark each date with the day of the week.
- Students create a pattern using various rubber stamp blocks or picture designs.
- Students use or create patterns with geometric figures (circles, triangles, squares, pentagons, hexagons, etc.) and record how many of each shape exist after each repeating cluster.
- Students create a mosaic design (tessellation) made of different shapes using objects such as pattern blocks. They color congruent shapes of a mosaic design with the same color.

2. Use tables, rules, variables, open sentences, and graphs to describe patterns and other relationships.

- Students complete a table given several starting numbers and a verbal rule.
- Kindergartners look at *Anno's Counting House* by Mitsumasa Anno to see if they can figure out the pattern that is used in moving from one set of pages to the next. The people in this book move, one by one, from one house to another.
- Students describe the pattern illustrated by the numbers in a table by using words (e.g., one more than), and then the teacher helps them to represent it with symbols in an open sentence ($\square = \heartsuit + 1$).
- Students use colored squares to make a graph showing the multiples of 3 and relate this to a table and an expression involving a variable, such as $3 \times \square$.

3. Use concrete and pictorial models to explore the basic concept of a function.

- Students study the pictures in *Anno's Math Games II* by Mitsumasa Anno. As they do, they try to figure out what happens to the objects as the elves put them into the magic machine. Sometimes the number of objects doubles, sometimes the objects grow eyes, and sometimes the objects turn into circles.
- Students put numbers into *Max the Magic Math Machine* and read what comes out. (The teacher acts as Max.) Then they describe what Max is doing to each number. The teacher pays careful attention to the students' responses to assess their levels of understanding.

- Students investigate a hole-making machine that puts 4 holes into buttons. They make a table that shows the number of buttons put into the machine and the total number of holes that must be made in them. Then they write a sentence that describes how the total number of holes changes as new buttons are added.
- Students play *Guess my Rule*. The teacher gives them a starting number and the result after using the rule. She continues giving examples until students discover the rule.
- Students count the number of pennies (or nickels) in 1 dime, 2 dimes, 3 dimes and record their results in chart form. They study the patterns and discuss the *rules* observed.
- Students consider the cost of two or three candies if one candy costs one dime. They make a chart using the information.
- Students count the number of lifesavers in an assorted pack. They make a table showing the number of each color and the total number in one pack. Then, assuming all of the packs are the same, they make a table showing the total number of each color for 2 packs, 3 packs, 4 packs, and so on. They check their results with packs of lifesavers, which in general, have the same number of each color.

4. Observe and explain how a change in one physical quantity can produce a corresponding change in another.

- Students discuss how ice changes to water as it warms. They talk about how it snows in January or February but rains in April or May.
- Students plant seeds and watch them grow. They write about what they see and measure the height of their plants as time passes. They discuss how changes in time bring about changes in the height of the plants. They also talk about how other factors might affect the plants, such as light and water.

5. Observe and recognize examples of patterns, relationships, and functions in other disciplines and contexts.

- Students go on a *pattern hunt* around the classroom and the school, discussing the patterns they find.
- Students sing and act out songs like "Rattlin' Bog" (Bird on the leaf, and the leaf on the tree, and the tree in the hole, and the hole in the ground, . . .) and "Old MacDonald Had a Farm."
- In reading, students recognize patterns in rhythm, in rhyming, in syllables and in sequencing. Stories such as *Ten Black Dots* by Donald Crews, *Five Little Monkeys Jumping on a Bed* by Eileen Christelow, *Jump, Frog, Jump* by Robert Kalan, *The Little Red Hen*, and Dr. Seuss books offer such opportunities. Visual patterns can be shown using picture representations for children's books such as *1 Hunter* by Pat Hutchins, *Rooster's Off to See the World* by Eric Carle, *The Patchwork Quilt* by Valerie Flournoy, and *The Keeping Quilt* by Patricia Polacco.
- Students identify every third letter of the alphabet; every fourth letter, etc. They use those sets of letters to see what words they can make.
- Students choose a day. Using a calendar, they identify the name of the next day, of the

previous day, and also the name of the day two days (or more) before and after. They select a date, and give the date of the next day and of the previous day, the name of the month, of the next month, and of the previous month. They give the name of the date two days before and after, and three days (or more) before and after.

- Students graph daily weather patterns, showing sunny, cloudy, rainy or snowy days. Then they discuss monthly or seasonal patterns.
- In social studies, students identify traffic patterns such as how many cars, trucks, or buses pass the front of the school during five minutes at different times of the day. They keep records for five days, organizing the information in chart form.
- In art, students observe patterns in pictures, mosaics, tessellations, and Escher-like drawings, as well as in wallpaper, fabric, and floor tile designs.

6. Form and verify generalizations based on observations of patterns and relationships.

- Students draw pictures of faces and make a table that shows the number of faces and the number of eyes. The teacher writes a sentence on the board that the class composes, describing the patterns that they find.
- Students observe that there are 12 eggs in a carton of eggs. These are called a dozen. They explain how to find the number of eggs in 2 cartons, 3 cartons, and so on.
- Students write a sentence or more telling about the patterns they have observed in a particular activity. They may use pictures to describe or generalize what they have observed. For example, after students have colored multiples of a certain number on the hundreds chart, they write about the geometric pattern they observe on the chart.

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Kalan, Robert. *Jump, Frog, Jump*. New York: Greenwillow Books, 1981.

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Seuss, Dr. Most Dr. Seuss books exhibit appropriate patterns.

The Little Red Hen. Many versions are available.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 11 — Patterns, Relationships, and Functions — Grades 3-4

Overview

The development of *pattern-based thinking*, using patterns to analyze and solve problems, is an extremely powerful tool for doing mathematics, and leads in later grades to an appreciation of how functions are used to describe relationships. The key components of *pattern-based thinking* at the early grade levels, as identified in the K-12 Overview, are **recognizing, constructing, and extending patterns, categorizing and classifying objects discovering rules,** and **working with input—output situations.**

In grades 3 and 4, students begin to learn the importance of investigating a pattern in an organized and systematic way. Many of the activities at these grade levels focus on creating and using tables as a means of analyzing and reporting patterns. In addition, students in these grades begin to move from learning about patterns to learning with patterns, using patterns to help them make sense of the mathematics that they are learning.

Students in grades 3-4 continue to **construct, recognize, and extend patterns**. At these grade levels, pictorial or symbolic representations of patterns are used much more extensively than in grades K-2. In addition to studying patterns observed in the environment, students should use manipulatives to investigate what happens in a pattern as the number of terms is extended or as the beginning number is changed. Students should also study patterns that involve multiplication and division more extensively than in earlier grades. Students continue to investigate what happens with patterns involving money, measurement, time, and geometric shapes. They should use calculators to explore patterns.

Students in these grades continue to **categorize and classify** objects. Now categories can become more complex, however, with students using two (or more) attributes to sort objects. For example, attribute shapes can be described as red, large, red and large, or neither red nor large. Classification of naturally-occurring objects, such as insects or trees, continues to offer an opportunity for linking the study of mathematics and science.

Students in grades 3 and 4 are more successful in playing **discover a rule** games than younger students and can work with a greater variety of operations. Most students will still be most comfortable, however, with one-step rules, such as *multiplying by 3* or *dividing by 4.*

Third and fourth graders also continue to work with **input-output situations.** While they still enjoy putting these activities in a story setting (such as *Max the Magic Math Machine* which takes in numbers and hands out numbers according to certain rules), they are now able to consider these situations in more abstract contexts. Students at this age often enjoy pretending to be the machine themselves and making up rules for each other.

In grades 3 and 4, then, students expand their study of patterns to include more complex patterns based on a greater variety of numerical operations and geometric shapes. They also work to organize their study of patterns more carefully and systematically, learning to use tables more effectively. In addition, they begin to apply their understanding of patterns to learning about new mathematics concepts, such as multiplication and division.

Standard 11 — Patterns, Relationships, and Functions — Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4..

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Reproduce, extend, create, and describe patterns and sequences using a variety of materials.**

   - Students make a pattern book that shows examples of patterns in the world around them.
   - Students use pattern blocks, attribute blocks, cubes, links, buttons, beans, toothpicks, counters, crayons, magic markers, leaves, and other objects to create and extend patterns. They might describe a pattern involving the number of holes in buttons, the number of sides in a geometric figure, the shape or the thickness of objects.
   - Students use sequences of letters or numbers to identify the patterns they have created.
   - Students investigate the sum of the dots on opposite faces of an ordinary die and find they always add up to 7.
   - Students solve two-dimensional attribute block patterns where, for instance, each column is a different shape and each row is a different color. They should be able to choose the block that fills in the missing cell in such patterns.
   - Students count by 2, 3, 4, 5, 6, 10 and 12 on a number line, on a number grid, and on a circle design.
   - Students begin with numbers between 50 and 100 and count backwards by 2, 3, 5, or 10.
   - Students create patterns with the calculator: They enter any number such as 50, and then repeatedly add or subtract 1 or 2 or 3 etc. If, for example, they enter 50+1=== ... , the calculator will automatically repeat the function and display 51, 52, 53, 54, ... Some calculators may need to have the pattern entered twice: 50+1=+1=== ... . Others may need 1++ 50=== ... .
   - Students begin with a number less than 10, double it, and repeat the doubling at least five times. They record the results of each doubling in a table and summarize their observations in a sentence.
   - Students read *Anno's Magic Seeds* by Mitsumasa Anno. In it, a wizard gives Jack two seeds and tells him that if he eats one, he won't be hungry for a year and if he plants the other one, two new seeds will be formed. Jack continues in this way for awhile and then tries other schemes that produce even more new seeds. The students work in groups to make charts and tables to show how many seeds Jack has at given points in time. As an individual assessment assignment, students are asked to find how many seeds Jack has after ten years using one of the discovered patterns and to support their answers in writing and with tables.

- Students supply the missing numbers on a picture of a ruler which has some blanks. Then they explore how to find the missing numbers between any two given numbers on a number line. They extend this to larger numbers; they might label each of five intervals from 200 to 300 or each of four intervals from 1,000 to 2,000.

- Students investigate number patterns using their calculators. For example, they might begin at 30, repeatedly add 6, and record the first 10 answers, making a prediction about what the calculator will show before they hit the equals key. Or they might begin at 90 and repeatedly subtract 9.

2. Use tables, rules, variables, open sentences, and graphs to describe patterns and other relationships.

- As a regular assessment activity, done during the year whenever new numerical operations have been explored, students fill in *guess my rule* tables like those shown below. Sometimes they are given the rule and sometimes they are asked to find the rule.

times 2	
6	?
9	?
2	?
7	?

plus 12	
34	?
58	?
?	37
?	12

?????	
12	4
27	9
9	3
15	?

- Students describe the pattern illustrated by the numbers in a table by using words (e.g., twice as much as) and then represent it with symbols in an open sentence (□ = 2 x ♡).

- On a coordinate grid, students plot coordinate pairs consisting of a number and the product of the number times 3. They join them with a line, making a line graph. They relate this to a table, and write the rule as an expression involving a variable, such as 3 x □.

- Students repeatedly add (or subtract) multiples of 10 to (from) a 3-digit starting number. They describe the pattern orally and write it symbolically as, for example, 357, 337, 317, 297, ...

- Students work in groups to solve problems that involve organizing information in a table and looking for a pattern. For example, *If you have 12 wheels, how many bicycles can you make? How many tricycles? How many bicycles and tricycles together?* Using objects or pictures, children make models and organize the information in a table. They discuss whether they have looked at all of the possibilities systematically and describe in words the patterns they have found. They write about the patterns in their journals and, with some assistance, develop some symbolic notation (e.g., 2 wheels for each bike and 3 wheels for each trike to get 12 wheels all together might become 2xB + 3xT = 12).

3. Use concrete and pictorial models to explore the basic concept of a function.

- Students use buttons with two or four holes and describe how the total number of holes is related to the number of buttons.

- Students use multilink cubes or base ten blocks to build rectangular solids. They count how many cubes tall their structure is, how many cubes long it is, and how many cubes wide it is. Then they count the total number of cubes in their structure. They record all of this information in a table and look for patterns.

- Students take turns putting numbers into *Max the Magic Math Machine*, reading what comes out, and finding the rule that tells what Max is doing to each number. A student acts as Max each time. Appropriate rules to use in grades 3 and 4 involve multiplication and division.

4. Observe and explain how a change in one physical quantity can produce a corresponding change in another.

- Students use cubes to build a one-story "house" and count the number of cubes used. They add a story and observe how the total number of cubes used changes. They explain how changing the number of stories changes the number of cubes used to build the house.
- Students measure the temperature of a cup of water with ice cubes in it every fifteen minutes over the course of a day. They record their results (time passed and temperature) in a table and plot this information on a coordinate grid to make a broken line graph. They discuss how the temperature changes over time and why.
- Students plant seeds in vermiculite and in soil. They observe the plants as they grow, measuring their height each week and recording their data in tables. They examine not only how the height of each plant changes as time passes but also whether the seeds in vermiculite or soil grow faster.

5. Observe and recognize examples of patterns, relationships, and functions in other disciplines and contexts.

- Students go on a *scavenger hunt* for patterns around the classroom and the school. They are given a list of verbal descriptions of specific patterns to look for, such as a *pattern using squares* or *an ABAB pattern*. They use cameras to make photographs of the patterns that they find.
- Students read *The Twelve Days of Summer* by Elizabeth Lee O'Donnell and Karen Lee Schmidt. Using the same pattern as the song *The Twelve Days of Christmas*, the authors tell the story of a young girl on vacation by the ocean. On the first day, she sees "a little purple sea anemone," on the eighth, "eight crabs a-scuttling," and so on. Since she sees everything that she has previously seen on every succeeding day, the book offers the obvious question *How many things did the little girl see today?*
- Students learn about the different time zones across the country. They describe the number patterns found in moving from east to west, and vice versa.
- Students read books such as *Six Dinner Sid* by Inga Moore or *The Greedy Triangle* by Marilyn Burns. They explore the patterns and relationships found in these books.
- Students study patterns in television programming. For example, they might look at the number of commercials on TV in an hour or how many cartoon shows are on at different times of the day. They discuss the patterns that they find as well as possible reasons for those patterns.

6. **Form and verify generalizations based on observations of patterns and relationships.**

   - Students measure the length of one side of a square in inches. They find the perimeter of one square, two squares (not joined), three squares, and so on. They make a table of values and describe a rule which relates the perimeter to the number of squares. They predict the perimeter of ten squares.
   - Students use their calculators to find the answers to a number of problems in which they multiply a two-digit number by 10, 100, or 1000. Looking at their answers, they develop a "rule" that they think will help them do this type of multiplication without the calculator. They test their rule on some new problems and check whether their rule works by multiplying the numbers on the calculator.

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On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 11 — Patterns, Relationships, and Functions — Grades 5-6

Overview

In grades K-4 students have been encouraged to view patterns in the world around them and to use their observations to explore numbers and shapes. In grades 5-6 students will expand their use of patterns, incorporating variables and using patterns to help them solve problem. The key components of *pattern-based thinking*, as identified in the K-12 Overview, involve **exploring, analyzing, and generalizing patterns**, and **viewing rules and input/output situations as functions.**

Patterns, relationships and functions will become a powerful problem solving strategy. In many routine problem solving activities, the student is taught a rote method which will lead to a solution. Thus, when faced with a problem of the same type he or she just uses that method to get the solution. In the real world, however, real problems are not usually packaged as nicely as textbook problems. The information is vague or fuzzy, some of the information needed to solve the problem might be missing, or there might be extraneous information on hand. In fact, a rote method might not exist to solve the problem. Such problems are generally referred to as non-routine problems.

Students who are faced with non-routine problems, and have no standard method for solving them, often simply give up, because they do not know how to get started. The ability to discover and analyze patterns becomes an important tool to help students move forward. When the students start to collect data and look for a pattern in order to solve a problem, they are often uncertain about what they are looking for. As they organize their information into charts or tables and start to analyze their data, sometimes, almost like magic, patterns begin to appear, and students can use these patterns to solve the problem.

Patterns help students develop an understanding of mathematics. Whenever possible, students in grades 5-6 should be encouraged to use manipulatives to **create, explore, discover, analyze, extend and generalize patterns** as they encounter new topics throughout mathematics. By dealing with more sophisticated patterns in numerical form, they begin to lay a foundation for more abstract algebraic concepts. Looking for patterns helps students tie concepts together, gain a greater conceptual understanding of the world of mathematics, and become better problem solvers.

Students in the middle grades should also continue to work with **categorization and classification**, particularly in the context of new mathematical topics, although much less emphasis should be given to these activities. For example, as students learn about fractions and mixed numbers, they must identify fractions as being less than one or more than one, as being in lowest terms or not. They also apply categorization and classification skills in geometry, as they distinguish between different types of geometric figures and learn more about the properties of these figures.

Students in grades 5-6 should begin using letters to represent variables as they do activities in which they are asked to **discover a rule**. They should also begin working with rules that involve more than one operation. Students at this level describe patterns that they see by using diagrams and pictorial representations of a mathematical relationship; some students will be more comfortable starting with manipulatives and then using a pictorial representation. Students should record their findings in words, in tables, and in symbolic equations.

Students in grades 5-6 should begin thinking of **input/output** situations as **functions.** They should recognize that a function machine takes in a number (or shape), operates with a consistent rule, and provides a predictable outcome. They should begin to use letters to represent the number going in and the number coming out, although considerable assistance from the teacher may be needed.

Throughout their work with patterns, students in grades 5-6 should use the calculator as a tool to facilitate computation and allow time for higher level thinking. Teachers should explore its capabilities with the students and encourage its use, so that students become proficient.

Standard 11 — Patterns, Relationships, and Functions — Grades 5-6

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

7. **Represent and describe mathematical relationships with tables, rules, simple equations, and graphs.**

   - Students let the length of one side of a square be 1 unit. They then find the perimeter of one square, two squares connected along an edge, three squares connected along their edges, and so forth, as shown below.

     They make a table of values and use it to determine a function rule which describes the pattern. They understand the rule $P = 2 \times s + 2$, where s is the number of squares, and they use it to predict the perimeter of ten squares.

   - Students use a geoboard to model squares with sides of 1, 2, 3, and 4 units. They determine the area and discover the rule for the area, $A = s^2$, given that s is the length of the side of a square.

   - Students use multilink cubes or base ten blocks to build rectangular solids. They find the volume of the rectangular solids (either by counting the cubes or by developing their own shortcuts), and record the length, width, and height of each solid along with its volume in a table. They use this information to discover a rule or formula for finding the volume of a rectangular solid.

   - Students find the number of primes between 1 and 100 using a hundreds chart and applying the process of the Sieve of Eratosthenes. That is, they first cross out all multiples of 2, then all multiples of 3, then all multiples of 5, and so on; the numbers which remain are the primes.

   - Students play the *secret number* calculator game. One student enters a secret number into the calculator by dividing the number by itself (e.g., 17 ÷ 17). She then asks her opponent to guess the number. Each guess is entered into the calculator and then the equals sign is pressed; the calculator shows the result of dividing the guess by the secret number. For example, if 17 is the secret number and 34 is guessed, then the student enters 34 = and sees 2 on the calculator. Play continues until 1 is shown on the calculator, so that the opponent has guessed the secret number. (Some calculators will need to have different keys pressed for the same result, such as 17 ÷ ÷ 34 =.)

   - Students look for the numbers which are palindromes (remain the same value when the digits are reversed) between any given pair of numbers. They decide which years in the

21st century will be the first 5 palindromes.

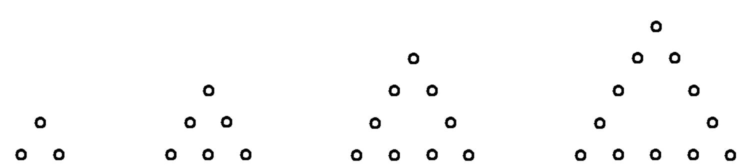
- Students determine how much money is earned hourly for a job mowing lawns or babysitting. They find the amount earned for working different numbers of hours. They organize the data in chart or table form. They look for a pattern and write simple equations; for example, the sentence *For babysitting or mowing lawns, I get $5 per hour.* translates into the equation $E = 5 \times h$ (Earnings equal five times the number of hours worked).

- Students use patterns to help them find the value of a point on a number line between two whole numbers when the number line is divided into fractional or decimal parts.

- Students investigate patterns involving arithmetic operations that can be generalized to a mathematical expression with a variable. For example:

1. Choose any number	☆	n
2. Multiply your number by 6	☆☆☆☆☆☆	$6 \times n$
3. Add 12 to the result	☆☆☆☆☆☆ ||||||||||||	$6 \times n + 12$
4. Take half	☆☆☆ ||||||	$3 \times n + 6$
5. Subtract 6	☆☆☆	$3 \times n$
6. Divide by 3	☆	n
7. Write your answer		n

- Students use lima beans or other counters to create *trinumbers*, the number of beans used for the triangles below. They try to predict what the tenth trinumber will be and then use this result to develop the expression ($3 \times n$) for the nth trinumber.

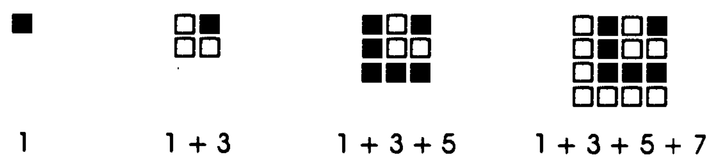


8. Understand and describe the relationships among various representations of patterns and functions.

- Using a 4x4 geoboard or dot paper, students create various sized squares. For each square, they record the length of its side and its area in a table. Then they show their results as a bar graph (side vs. area), as ordered pairs (side, area), as a verbal rule ("the area of a square is the length of its side times itself"), and as an equation ($A = s \times s$, where s is the length of the side). They repeat this for rectangles of varying sizes, recording the length and width and corresponding area in a table. Students discover the pattern and develop the formula for the area of a rectangle, $A = l \times w$, by inspecting the numbers in the table.

- Students cut out squares from graph paper, recording the length of the side of the square and the number of squares around the border of the square. They look for a pattern that will allow them to predict the number of unit squares in the border of a 10 x 10 square and then a 100 x 100 square. They describe their pattern in words. The teacher then helps them to develop a formula $(4 \times n) - 4$ for finding the number of unit squares in the border of an $n \times n$ square.

- Students explore patterns involving the sums of consecutive odd integers (1, 1 + 3, 1 + 3 + 5, 1 + 3 + 5 + 7, ...) by using unit squares to make Ls to represent each number and then nesting the Ls, as in the diagram below:



Then they make a table that shows how many Ls are nested and the total number of unit squares used. They look for a pattern which will help them predict how many squares will be needed if 10 Ls are nested (i.e., if the first 10 odd numbers are added together). They make a prediction and describe how they generated their prediction (e.g., when you add the first 3 odd numbers, it makes a square that is three units on a side, so when you add the first 10 odd numbers, it should make a square that is ten units on a side and you will need 100 squares). They share their solution strategies with each other and develop one (or more) expressions, like $n \times n$, that can be used to find the sum of the first n odd numbers.

9. Use patterns, relationships, and functions to model situations and to solve problems, in mathematics and in other subject areas.

- Students use calculators to study fractions whose decimal expressions repeat, and predict what digit is in any given place.
- Students look at what happens when a ball is hit at a 45° angle on a rectangular pocket billiards table. They make a table that shows dimensions of various tables, initial positions of the ball, and which pocket, if any, the ball eventually goes into. They look for patterns which will help them predict which pocket the ball will go into for other situations.
- Students work in groups to decide how to make a supermarket display of boxes of SuperCrunch cereal. The boss wants the boxes to be in a triangular display which is 10 boxes high and one box deep. Each box is 12 inches high and 8 inches wide. The students use patterns to help them decide how many boxes they would need and whether this is a practical way to display the cereal.
- Students look at the fee structure for crossing a toll bridge near them, such as the Walt Whitman bridge into Pennsylvania or the George Washington bridge into New York, and use patterns to help them decide whether it makes sense for someone who works on the other side of the bridge for 12 days each month to buy a commuter sticker.

10. Analyze functional relationships to explain how a change in one quantity results in a change in another.

- Students predict what size container is needed to hold pennies if, on the first day of a 30-day month, they put in one penny and double the number of pennies each succeeding day After making their predictions, they calculate how many days it will take to fill that container, and how many containers like that they would actually need for the whole month.

- Students make a chart that helps them understand the charges for a taxi ride when the taxi charges $2.75 for the first 1/4 mile and $.50 for each additional 1/4 mile. They look at rides of different lengths and figure out how much each trip would cost. Then they write an explanation of how they found the cost.
- Students look for a pattern between the temperature in degrees Fahrenheit and degrees Celsius and write an explanation of that relationship.

11. Understand and describe the general behavior of functions.

- As a regular assessment exercise, students fill in *Function Machine* tables like those shown below. Sometimes they are given the rule and sometimes they are asked to find the rule.

x	y
6	12
9	18
2	?
7	?
y = 2x	

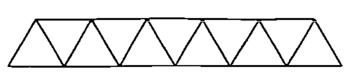
x	y
5	18
7	?
?	38
3	?
y = 4x−2	

x	y
1	4
10	31
?	16
7	?
???	

- Students investigate graphs without numbers. For example, they may study a graph that shows how far Yasmin has walked on a trip from home to her friend's house and back, where time is shown on the horizontal axis and the distance covered is on the vertical axis. Students tell a story about her trip, noting that where the graph is horizontal, she has stopped for some reason.

12. Use patterns, relationships, and linear functions to model situations in mathematics and in other areas.

- Students start with a single equilateral triangle with side of length 1 and find its perimeter. Then they add a second such triangle, matching sides exactly, to make a train or a wall and find its perimeter. They continue adding triangles, as in the diagram below, and find the perimeters of the resulting figures. The students try to predict the perimeter of ten triangles in a wall and then look for a function rule which describes the pattern — for example, if n is the number of triangles, then $P = n + 2$.



- Students look for as many different ways to make change for 50¢ as they can find. They make a table showing their results listed in an organized fashion and explain why they think they have found all of the possibilities.
- Students investigate what happens when they do arithmetic on a 12-hour clock. They find that 3 + 11 = 2 and that 4 − 6 = 10. They understand that 5, 17, and 29 are all equivalent to 5, and connect this to the remainder obtained when dividing each by 12.

- Students develop a patchwork quilt design using squares and isosceles right triangles to make a 12 inch by 12 inch patch. They use patterns to help them decide how many pieces of each size are needed in order to complete a 3 foot by 5 foot quilt.
- Students are given the following open-ended assessment problem: *Who will win the 100 meter race between Pat and his older sister, Terry? Pat runs at an average of 3 meters per second, while Terry runs at an average of 5 meters per second. Since Pat is slower, he gets a 25-meter head start. Use a table or a graph to help you find out who will be the winner. Then write an explanation of how you solved the problem and explain what head-start you think Pat should have.*

13. Develop, analyze, and explain arithmetic sequences.

- Students explore and try to explain the sequence made by the numbers of diagonals in a series of polygons with increasing numbers of sides. For example, a triangle has no diagonals, a square has 2 diagonals, a pentagon has 5 diagonals, a hexagon has 9 diagonals, and so on. They examine the sequence 0, 2, 5, 9, try to extend it, and justify their conclusion.
- Students read *The King's Chessboard* by David Birch. In this old folk tale, which has been told many times in many languages, a king is undecided about a gift to give to one of his advisors. Finally he decides that, on one day, the advisor will receive a single grain of rice on the first square of a chessboard. On the next day, that amount will be doubled and two grains will be placed on the second square. On day three, four grains will be put on the third square, and so on, doubling every day until the entire board is filled. The students use calculators to figure out the king's indebtedness on the tenth day, the twentieth day, and so on.

References

Bird, David. *The King's Chessboard.* Puffin Pied Piper Books, 1988.

On-Line Resources

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Standard 11 — Patterns, Relationships, and Functions — Grades 7-8

Overview

The key components of *pattern-based thinking*, as identified in the K-12 Overview, involve **exploring, analyzing, and generalizing patterns**, and **viewing rules and input/output situations as functions.** In grades 7-8, the importance of studying patterns continues with an emphasis on representing and describing relationships with tables and graphs and on the development of rules using variables. Patterns also become important now in the analysis of statistics and the development of geometric relationships. Graphing calculators and computers are helpful in illustrating the usefulness of symbols and in making symbolic relationships more tangible. Although the symbolism and notation used become more algebraic at these grade levels (e.g., $A = 4s$ instead of $A = 4 \times s$), students should still be encouraged to model many patterns with concrete materials. Engineers, scientists, architects, and other researchers all build working models of projects for analysis and demonstration.

Students in these grades should also be given ample opportunity to analyze patterns, to discover the relevant features of the patterns, and to construct understandings of the concepts and relationships involved in the patterns. From these investigations, students should develop the language necessary to communicate their ideas about the patterns and should learn to differentiate among the variety of patterns they have studied (that is, to **categorize** and to **classify** them). They will apply their understanding of patterns as they learn about such topics as exponents, rational numbers, measurement, geometry, probability, and functions.

Seventh and eighth graders continue to **discover rules** for mathematical relationships and for quantifiable situations from other subject areas. In particular, students should focus on **relationships involving two variables.** Students should analyze how a change in one quantity results in a change in another. They further need to develop their understanding of the general behavior of **functions** and use these to model a variety of phenomena.

Students should be encouraged to solve problems by looking for patterns that involve words, pictures, manipulatives, and number descriptions. These situations naturally lead to the use of variables and informal algebra in solving problems.

As seen in prior grades, computers and graphing calculators provide many benefits for students in investigating mathematical concepts and problems. These tools make mathematics accessible to more students because they enable the students to analyze what they can see rather than requiring them to develop mental images or manipulate situations symbolically from the outset. Furthermore, technology enables students to calculate rapidly and to investigate conclusions immediately, freeing them from the limitations imposed by cumbersome and time-consuming computations.

Standard 11 — Patterns, Relationships, and Functions — Grades 7-8

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

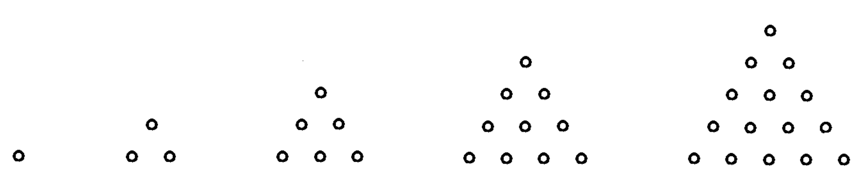
Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

7. **Represent and describe mathematical relationships with tables, rules, simple equations, and graphs.**

   - Students use calculators to investigate which fractions have decimal equivalents that terminate and which repeat. They summarize their findings in their math journals.
   - Students study patterns made by the units digit in the expansion of powers of a number. For example, what is the units digit of 9^{18}? The pattern 9^1, 9^2, 9^3, . . . yields either a 1 or a 9 in the units place. Students record their findings in a table or as a graph on rectangular coordinate paper. They write a paragraph justifying their answer. They then similarly investigate the patterns made by the units digit in the expansion of the powers of other one-digit numbers.
   - Students consider what happens if you start with two bacteria on a kitchen counter and the number of bacteria doubles every hour. They make a table and graph their results, noting that the graph is not linear.

8. **Understand and describe the relationships among various representations of patterns and functions.**

   - Students arrange bowling pins in the shape of equilateral triangles of various sizes, as shown in the diagram below. They make a table showing the number *n* of rows in each triangle and the number *b* of bowling pins in each triangle. The numbers in the second column — 1, 3, 6, 10, ... — are called the *triangular numbers*. They find a rule expressing this relationship $p = n(n + 1) / 2$, by putting two triangles of the same size side by side, counting the total number of bowling pins, and dividing by 2.



   - Using a 5x5 geoboard or dot paper, students create various sized parallelograms. For each parallelogram, they record the length of the base (b), the height of the parallelogram (h), and the area of the parallelogram (A), found by counting squares. The students look for a relationship among the numbers in the three columns of their table, express this relationship as a verbal rule, and then write the rule in symbolic form.

- Students investigate how many stools with three legs and how many chairs with four legs can be made using 48 legs. They may use objects or draw pictures to make models of the solutions. They look for patterns in the numbers and display their results in a table, as ordered pairs graphed on the rectangular coordinate plane, as a rule like $3s + 4c = 48$, and as an equation like $s = 16 - 4c/3$, which gives the number of stools as a function of the number of chairs. They describe the pattern and how they found it in writing.

- Students create their own designs using iteration. They may use patterns such as spirolaterals or write a program in Logo on the computer. They use simple equations to iterate patterns. For example, they use the equation $y = x + 1$ and start with any x value, say 0. The resulting y value is 1. Using this as the new x value yields a 2 for y. Using this as the next x gives a 3, and so on. The related values can be organized in a table and the ordered pairs graphed on a rectangular coordinate system. Students note that the graph is a straight line and use this to predict other values. Then students use a slightly different equation, $y = .1x + .3$. Again, starting with an x value of 0 they find the resulting y value of .3. Using this as the new x value gives a value of .33 for y. Repeating this process yields the series of y values .3, .33, .333, ... which get closer and closer to 1/3.

9. Use patterns, relationships, and functions to model situations and to solve problems in mathematics and in other subject areas.

- Students analyze a given series of terms and fill in the missing terms. Patterns include various arithmetic (repeating patterns) and geometric (growing patterns) sequences and other number and picture patterns. Students develop an awareness of the assumptions they are making. For example, given the sequence 0, 10, 20, 30, 40, 50, one might expect 60 to be next; but not on a football field, where the numbers now decrease!

- Students compare different pay scales, deciding which is a better deal. For example, is it better to be paid a salary of $250 per week or to be paid $6 per hour? They create a table comparing the pay for different numbers of hours worked and decide at what point the hourly rate becomes a better deal.

- Students supply missing fractions between any two given numbers on a number line. They might label each of eight intervals between 1 and 2, or they might label the next 16 intervals from 23 1/2 to 24. They extend this to decimals, labeling each missing number in increments of .1 or .01. For example, students might label each of five intervals between 59.34 and 59.35.

- Students decide how many different double-dip ice cream cones can be made from two flavors, three flavors, and so on up to Baskin and Robbins' 31 flavors. They arrange the information in a table. They discuss whether one flavor on top and another on the bottom is a different arrangement from the other way around, and how that would change their results. They also discuss a similar problem (see Standard 14 and the 5-6 Vignette *Pizza Possibilities* in the First Four Standards): *How many different types of pizzas can be made using different toppings?*

- Students predict how many times they will be able to fold a piece of paper in half. Then they fold a paper in half repeatedly, recording the number of sections formed each time in a table. They find that the number of folds physically possible is surprisingly small (about 7). The students try different kinds of paper: tissue paper, foil, etc. They describe in writing any patterns they discover and generate a rule for finding the number of sections after *10, 20,* or *n* folds. They also graph the data on a rectangular coordinate plane using integral values. They extend this problem to a new situation by finding the number of ancestors each person had ten generations ago and also to the problem of telling a secret to two people who each tell two people, etc.

10. Analyze functional relationships to explain how a change in one quantity results in a change in another.

- Students investigate how increasing the temperature measured in degrees Celsius affects the temperature measured in degrees Fahrenheit and vice versa. They collect data using water, ice, and a burner. They use their data to develop a formula relating Celsius to Fahrenheit, summarize the formula in a sentence, and graph the values they have generated.
- Students investigate how the temperature affects the number of chirps a cricket makes in a minute.
- Students investigate the effect of changing the radius or diameter of a circle upon its circumference by measuring the radius (or diameter) and the circumference of circular objects. They graph the values they have generated, notice that it is close to a straight line, and describe the relationship they have found in a paragraph. Then they develop a symbolic expression that describes that relationship.
- Students investigate the effect on the perimeters of given shapes if each side is doubled or tripled. They summarize their findings.
- Students investigate how the areas of rectangles change as the length is doubled, or the width is doubled, or both are doubled. They discuss their findings.
- Students work on problems like this one from the New Jersey Department of Education's *Mathematics Instructional Guide (p. 7-69): Two of the opposite sides of a square are increased by 20% and the other two sides are decreased by 10%. What is the percent of change in the area of the original square to the area of the newly formed rectangle? Explain the process you used to solve the problem.*
- Students investigate how the areas of triangles change if the base is kept the same, but the height is repeatedly increased by one unit.
- Students stack a given number of unit cubes in various ways and find the surface areas of the structures they have built. They sketch their figures and discuss which of the figures has the largest surface area and which has the smallest, and justify their conclusions.
- Students make models of cubes using blocks or other manipulatives, and investigate how the volume changes if the length, width, and height are all doubled.
- Using a spreadsheet, students investigate how adding (or subtracting) values to given data can affect the mean, median, mode, or range of the data. They discuss how various other changes to the data would affect the mean, the median, the mode, or the range.

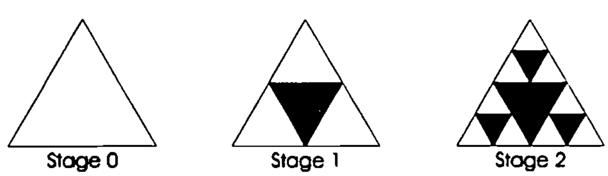
11. Understand and describe the general behavior of functions.

- Students investigate graphs without numbers. For example, they may study a graph that shows how far Olivia has walked on a trip from home to the store and back, where time is shown on the horizontal axis and the distance covered is on the vertical axis. Students tell a story about her trip, noting that where the graph is horizontal, she has stopped for some reason. In addition, their stories account for those parts of the graph that are steeper, by explaining why Olivia is walking faster (e.g., she is running from a dog), and those parts of the graph that are not as steep, by explaining why Olivia is walking slower (e.g., she is going up a hill).

- Students use probes and graphing calculators or computers to collect data involving two variables for several different science experiments (such as measuring the time and distance that a toy car rolls down an inclined plane, or the temperature of a beaker of water when ice cubes are added). They look at the data that has been collected in tabular form and as a graph on a coordinate grid. They classify the graphs as straight or curved lines and as increasing (direct variation), decreasing (inverse variation), or mixed. For those graphs that are straight lines, the students try to match the graph by entering and graphing a suitable equation.
- Given several non-linear functions, such as $y = x^2$, $y = 3x^2$, $y = x^2 + 1$, $y = x^3$, or $y = 16/x$, students create a table of values for each and use graphing calculators to graph them.

12. Use patterns, relationships, and linear functions to model situations in mathematics and in other areas.

- Groups of students pretend that they work for construction companies bidding on a federal project to build a monument. The monument is to be built from marble cubes, with each cube being one cubic foot. The monument is to have a "triangular" shape, with one cube on top, then two cubes in the row below, then three cubes, four cubes, and so on. The monument is to be 100 feet high. The students make a chart and look for a pattern to help them predict how many cubes they will need to buy so that they can include the cost of the cubes in their bid.
- Students look at the Sierpinski triangle as an example of a fractal. Stage 0 is an unshaded triangle. To get Stage 1, you take the three midpoints of the sides of the unshaded triangle, connect them, and shade the new triangle in the middle. To get Stage 2, you repeat this process for each of the unshaded triangles in Stage 1. This process continues an infinite number of times. The students make a table that records the number of unshaded triangles at each stage, look for a pattern, and use their results to predict the number of unshaded triangles there will be at the tenth (3^9) and twentieth (3^{19}) stages.



- Students use the constant function on the calculator to determine when an item will be on sale for half price. If the price goes down by a constant dollar amount each week, then they record successive prices, such as 95 - 15 = = = . . . (or 15 - - 95 = = = on other calculators). If the price is reduced by a certain percent each week, then they use the constant function on the calculator to obtain successive discounts as percents by multiplying. For example, if a $95 item is reduced by 10% each week, they key in 95 x .9 = = = . . . (or as .9 x x 95 = = = . . . on other calculators).

- Using a temperature probe and a graphing calculator or computer, students measure the temperature of boiling water in a cup as it cools. They make a table showing the temperature at five-minute intervals for an hour. Then they graph the results and make observations about the shape of the graph, such as *the temperature went down the most in the first few minutes* or *it cooled more slowly after more time had passed,* or *it's not a linear relationship*. The students also predict what the graph would look like if they continued to collect data for another twelve hours.
- Students use coins to simulate boys (tails) and girls (heads) in a family with five children. They make a list of all of the possible combinations, use patterns to help them organize all of the possibilities, and find the probability that all five children are girls or that exactly three are girls. As a question on a test, they are asked to react to an argument between Pam and Jerry, a couple who want to have four children. Jerry thinks that they will probably end up with two boys and two girls, while Pam thinks that they will probably wind up with an unequal number of boys and girls.
- Students make Ferris wheel models from paper plates, with notches representing the cars. They use the models to make a table showing the height above the ground of a person on a ferris wheel at specified time intervals, determined by the time needed for the next chair to move to loading position. After collecting data through two or three complete turns of the wheel, they make a graph of time versus height. In their math notebooks, they respond to questions about their graphs: *Why doesn't the graph start at zero? What is the maximum height? Why does the shape of the graph repeat?* The students learn that this graph represents a periodic function.

13. Develop, analyze, and explain arithmetic sequences.

- Students use the following chart of postal rate history to make a graph of the increases and then to try to predict what the cost will be to mail a one-ounce letter in the year 2001.

Cost to Mail a One-Ounce Letter Since 1917

Date	Cost	Date	Cost
1917	3 cents	1975	13 cents
1919	2 cents	1978	15 cents
1932	3 cents	1981	18 cents
1958	4 cents	1981	20 cents
1963	5 cents	1985	22 cents
1968	6 cents	1988	25 cents
1971	8 cents	1991	29 cents
1974	10 cents	1995	32 cents

- Students describe, analyze, and extend the Fibonacci sequence 1, 1, 2, 3, 5, 8, ... , where each term is the sum of the two preceding terms. They investigate applications of this sequence in nature, such as sunflower seeds, the fruit of the pineapple, and the rabbit problem. They create their own Fibonacci-like sequences, using different starting numbers.
- Students read Isaac Asimov's short story *Endlessness* and write book reports to convey their reactions.

References

Asimov, Isaac. *Endlessness.* (in *Literature: Bronze*, 2nd Ed.) Englewood Cliffs, NJ: Prentice Hall, 1991.

New Jersey Department of Education. *Mathematics Instruction Guide.* D. Varygiannes, Coord. January, 1996.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 11 — Patterns, Relationships, and Functions — Grades 9-12

Overview

Patterns, relationships, and functions continue to provide a unifying theme for the study of mathematics in high school. Pattern-based thinking throughout the earlier grades, as described in the K-12 Overview, and the informal investigations begun in the middle grades have prepared students to make extensive use of both the concept of a function and functional notation. Students should describe the relationships found in concrete situations with algebraic expressions, formulas and equations, as well as with tables of input-output values, with graphs, and with written statements.

Students in high school **construct, recognize, and extend patterns** as they encounter new areas of the mathematics curriculum. For example, students in algebra recognize patterns when multiplying binomials, and students in geometry utilize patterns in similar triangles. Students in high school should also analyze a variety of different types of sequences, including both arithmetic and geometric sequences, and express their behavior using functional notation.

High school students continue to **categorize and classify** objects, especially in the context of learning new mathematics. For example, in studying geometry, they classify various lines and line segments as chords or secants or tangents to a given circle. In studying algebra, they distinguish linear relationships from non-linear relationships.

The **function** concept is one of the most fundamental unifying ideas of modern mathematics. Students begin their study of functions in the primary grades, as they observe and study patterns in nature and create patterns using concrete models. As students grow and their ability to abstract matures, students investigate patterns using concrete models, and then abstract them to form rules, display information in a table or chart, and write equations which express the relationships they have observed. In high school, students move to expand their knowledge of functions as a natural outcome of the earlier discussion of patterns and relationships. Concepts such as domain and range are formalized and the $f(x)$ notation is introduced as a natural extension of initial informal experiences.

Students frequently have difficulty with the concept of a function, possibly because of its many interpretations. The *formal ordered-pair definition* of a function, while perhaps the most familiar to many teachers, is also the least understood and possibly the most abstract way of approaching functions (Wagner and Parker, 1993). Looking at functions as *correspondences between two sets* seems to be more easily grasped while facilitating the introduction of the concepts of domain and range. Visualizing functions as *graphs* which satisfy the vertical line test provides an extremely accessible way of representing functions, especially when graphing calculators and computers are used. Students entering high school should already be familiar with functions as *input-output processes* through the use of function machines. They should also have encountered functions given by *rules or formulas* involving independent and dependent variables. Students moving on to calculus also need to view functions as *objects of study* in themselves.

The correspondence between all of these interpretations of the concept of a function may not be very clear to students, and so attention should be drawn explicitly to the different ways of understanding functions, and how together they provide a more complete understanding of the concept. For example, while discussing sequences, students should explore how they can be considered as functions using the correspondence model, the rule model, the input-output model, the graph model, and the ordered pairs model.

High school students should spend considerable time in analyzing relationships involving two variables, and should understand how dependent and independent variables are used. Beginning with concrete situations (possibly involving social studies or science concepts), students should collect and graph data (often using graphing calculators or computers), discover the relationship between the two variables, and express this relationship symbolically. Students need to have experiences with situations involving linear, quadratic, polynomial, trigonometric, exponential, and rational functions as well as piecewise-defined functions and relationships that are not functions at all.

High school students should use functions extensively in solving problems. They should frequently be asked to analyze a real-world situation by using patterns and functions. They should extend their understanding of relationships involving two variables to using functions with several dependent variables in mathematical modeling.

Throughout high school, students continue to work with patterns by collecting and organizing data in tables, by graphing the relationships among variables, and by discovering and describing these relationships in formal, written, and symbolic form.

Reference

Wagner, S., and S. Parker. "Advancing Algebra" in *Research Ideas for the Classroom: High School Mathematics*, P. Wilson, Ed. New York: Macmillan Publishing Company, 1993.

Standard 11 — Patterns, Relationships, and Functions — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11, and 12.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

14. Analyze and describe how a change in the independent variable can produce a change in a dependent variable.

- Students investigate the relationship between stopping distance and speed of travel in a car. The students gather data from the driver's education manual, graph the values they have found, note that the relationship is linear, and look for an equation that fits the data.
- Students investigate the effect on the perimeters of given shapes if each side is doubled or tripled. They summarize their findings in writing and symbolically.
- Students investigate how the area of a parallelogram changes as the length of the base is doubled, or the height is doubled, or both are doubled. They repeat the experiment for tripling and quadrupling each measurement. They discuss their findings and represent them symbolically.
- Students compare two fare structures for taxis: one in which the taxi charges $2.75 for the first 1/4 mile and $.50 for each additional 1/4 mile, and one in which $4.25 is charged for the first 1/4 mile and $.20 for each additional 1/8 mile. They develop tables, graph specific points, and generate equations to describe each situation. They find which trips cost more for each fare structure and when both will result in the same cost.
- Students investigate patterns of growth, such as compound interest or bacterial growth, with a calculator. They make a table showing how much money is in a savings account (if none is withdrawn) after one quarter, two quarters, and so on, for ten years. They represent their findings graphically, note that this is not a linear relationship (although simple interest is linear), and write an equation describing the relationship between the amount P deposited initially, the interest rate r, the number n of times that interest is paid each year, the number of years y, and the total T available at the end of that time period: $T = (1 + r/n)^{ny}(P)$.

15. Use polynomial, rational, trigonometric, and exponential functions to model real world phenomena.

- Students model population growth and decline of people, animals, bacteria and decay of radioactive materials, using the appropriate exponential functions.
- Students use a sound probe and a graphing calculator or computer to collect data on sound waves or voice patterns, and graph these data noting that these patterns are represented by trigonometric functions. They use a light probe to collect data on the relationship between the brightness of the light and its distance from the light source, and analyze the graph

provided by the calculator.

- Students use M&Ms to model decay. They spill a package of M&Ms on a paper plate and remove those with the M showing, recording the number of M&Ms removed. They put the remaining M&Ms in a cup, shake, and repeat the process until all of the M&Ms are gone. They plot the trial number versus the number of M&Ms remaining and note that the graph represents an exponential function. They try different equations until they find one that they think fits the data pretty well. They verify their results using a graphing calculator.

- Students work on this HSPT-like problem from the New Jersey Department of Education's *Mathematics Instructional Guide*: *The Granda Theater has a special rate for groups of 10 or more people: $40 for the first 10 people and $3 for each additional person. Which of the following expressions tells the amount that a group of 10 or more will have to pay if n represents the number of people in the group, where* $n \geq 10$*: a.* $40 + 3n$, *b.* $(40 + 3)n$,
 c. $40 + 3(n + 10)$, *or d.* $40 + 3(n - 10)$*?*

- Students learn about the Richter Scale for measuring earthquakes and about the pH measurement of a solution, noting how exponents are built into these measurements. For example, a pH of 4 is 10 times more acidic than a pH of 5 and 100 times more acidic than a pH of 6.

- Students work in groups to investigate what size square to cut from each corner of a rectangular piece of cardboard in order to make the largest possible open-top box. They make models, record the size of the square and the volume for each model, and plot the points on a graph. They note that the relationship is not linear and make a conjecture about the maximum volume, based on the graph. The students also generate a symbolic expression describing this situation and check to see if it matches their data by using a graphing calculator.

16. Recognize that a variety of phenomena can be modeled by the same type of function.

- Different groups of students work on problems with different settings but identical structures. For example, one group determines the number of collisions possible between two, three and four bumper cars at an amusement park and develops an equation to represent the number of possible collisions among *n* bumper cars (assuming that no two bumper cars collide more than once). Another group investigates the number of possible handshakes between 2, 3, and 4 people, and develops an equation to represent the number of handshakes for *n* people. A third group discusses the total number of sides and diagonals possible in a triangle, a quadrilateral, and a pentagon, and develops an equation that gives the total number of sides and diagonals for an *n*-sided polygon. A fourth group looks at the number of games required for a tournament if each team plays every other team only once, while a fifth considers connecting telephone lines to houses. Each group presents its problem, its approach to solving the problem, and its solution. Then the teacher leads the class in a discussion of the similarities and differences among the problems. Students note the similarities between the approaches used by the different groups and that they all came up with the general expression $n(n-1)/2$.

- Students investigate a number of situations involving the equation $y = 2^x$. They look at how much money would be earned by starting out with a penny on the first day and doubling the amount on each successive day. They discuss what happens if they start with two bacteria and the number of bacteria doubles every half hour. They consider the total number of pizzas possible as more and more toppings are added. They consider the number of subsets for a given set. They fold a sheet of paper repeatedly in half and look at how many sections are created after each fold.

- Students look for connections among problem situations involving temperature in Celsius and Fahrenheit, the relationship of the circumference of a circle to its diameter, the relationship between stopping distance and car speed, between money earned and hours worked, between distance and time if the rate is kept constant, and between profit and price per ticket.

17. Analyze and explain the general properties and behavior of functions, and use appropriate graphing technologies to represent them.

- As regular parts of their assessments, students make up graphs to represent specific problem situations, such as the cost of pencils that sell at two for a dime, the temperature of an oven as a function of the length of time since it was turned on, their height from the ground as they ride a ferris wheel as a function of the amount of time since they got on, the time it takes to travel 100 miles as a function of average speed, or the cost of mailing a first-class letter based on its weight in ounces.

- Students use a string of constant length, say 30 inches, and list all possible lengths and widths of rectangles with integral sides which have this perimeter. They determine the perimeter and area for each rectangle. Then they make three graphs from their data: length vs. width, length vs. perimeter, and length vs. area. They look for equations to describe each graph, determine an appropriate range of values for each variable, and then graph the functions using graphing calculators or computers. The rectangle of maximum area, a square, does not have integral values, but can be found using the trace function or algebraic procedures. Students also investigate the area of a circle made with the same string and compare it to the areas of the rectangles.

- Students take on the role of "forensic mathematicians," trying to determine the height of a person whose femur was 17 inches long. They measure their own femurs and their heights, entering the class data into a graphing calculator or computer and creating a scatterplot. They note that the data are approximately linear, so they use the built-in linear regression procedures to find the line of best fit and then make their prediction.

18. Analyze the effects of changes in parameters on the graphs of functions.

- Students investigate the characteristics of linear functions. For example, in $y = kx$, how does a change in k affect the graph? In $y = mx + b$, what is the role of b? Does k in the first equation serve the same purpose as m in the second? Students use the graphing calculator to investigate and verify their conclusions.

- Students investigate the effects of a dilation and/or a horizontal or vertical shift on the algebraic expression of various types of functions. For example, how does moving a graph up 3 units affect its equation?

- Students look at the effects of changing the coefficients of a quadratic equation on its graph. For example, how is the graph of $y = 4x^2$ different from that of $y = x^2$? How is $y = .2x^2$ different from $y = x^2$? How are $y = x^2 + 4$, $y = x^2 - 4$, $y = x^2 - 4x$, and

$y = x^2 - 4x + 4$ each different from $y = x^2$? How is $y = \sin 4x$ different from $y = 4 \sin x$? Students use graphing calculators to look at the graphs and summarize their conjectures in writing.

- Students study the behavior of functions of the form $y = ax^n$. They investigate the effect of a on the curve and the characteristics of the graph when n is even or odd. They use the graphing calculator to assist them and write a sentence summarizing their discoveries.

19. Understand the role of functions as a unifying concept in mathematics

- Students in all mathematics classes use functions, making explicit connections to what they have previously learned about functions. As students encounter a new use or meaning for functions, they relate it to their previous understandings.
- Students use recursive definitions of functions in both geometry and algebra. For example, they define $n!$ recursively as $n! = n \cdot (n-1)!$ They use recursion to generate fractals in studying geometry. They may use patterns such as spirolaterals, the Koch snowflake, the Monkey's Tree curve, the Chaos Game, or the Sierpinski triangle. They may use Logo or other software to iterate patterns, or they may use the graphing calculator. In studying algebra, students consider the equation $y = .1x + .6$, starting with an x-value of .6, and find the resulting y-value. Using this y-value as the new x-value, they then calculate its corresponding y-value, and so on. (The resulting values are .6, .66, .666, .6666, ... — providing closer and closer approximations to the decimal value of 2/3!) Students investigate the results of iterations which are other starting values for the same function; the results are surprising! They use other equations and repeat the procedure. They graph the results with a graphing calculator, adjusting the range values to permit viewing the resulting y-values. (See *Fractals for the Classroom* by H.-O. Peitgen, et al.)

References

Peitgen, Heinz-Otto, et al. *Fractals for the Classroom: Strategic Activities, Volume One and Two*. Reston, VA: NCTM and New York: Springer- Verlag, 1992.

Wagner, S. and S. Parker. "Advancing Algebra" in *Researching Ideas for the Classroom: High School Mathematics*, P. Wilson, Ed. New York: Macmillan Publishing Co., 1993.

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STANDARD 12 — PROBABILITY AND STATISTICS

K-12 Overview

All students will develop an understanding of statistics and probability and will use them to describe sets of data, model situations, and support appropriate inferences and arguments.

Descriptive Statement

Probability and statistics are the mathematics used to understand chance and to collect, organize, describe, and analyze numerical data. From weather reports to sophisticated studies of genetics, from election results to product preference surveys, probability and statistical language and concepts are increasingly present in the media and in everyday conversations. Students need this mathematics to help them judge the correctness of an argument supported by seemingly persuasive data.

Meaning and Importance

Probability is the study of random events. It is used in analyzing games of chance, genetics, weather prediction, and a myriad of other everyday events. Statistics is the mathematics we use to collect, organize, and interpret numerical data. It is used to describe and analyze sets of test scores, election results, and shoppers' preferences for particular products. Probability and statistics are closely linked because statistical data are frequently analyzed to see whether conclusions can be drawn legitimately about a particular phenomenon and also to make predictions about future events. For instance, early election results are analyzed to see if they conform to predictions from pre-election polls and also to predict the final outcome of the election.

Understanding probability and statistics is essential in the modern world, where the print and electronic media are full of statistical information and interpretation. The goal of mathematical instruction in this area should be to make students sensible, critical users of probability and statistics, able to apply their processes and principles to real-world problems. Students should not think that those people who did not win the lottery yesterday have a greater chance of winning today! They should not believe an argument merely because various statistics are offered. Rather, they should be able to judge whether the statistics are meaningful and are being used appropriately.

K-12 Development and Emphases

Statistics and probability naturally lend themselves to plenty of fun, hands-on cooperative learning and group activities. Activities with spinners, dice, and coin tossing can be used to investigate chance events. Students should discuss the theoretical probabilities of different events such as the possible sums of a pair of dice, and check them experimentally. They can choose topics to investigate, such as how much milk and juice the cafeteria should order each day, gather statistics on current orders and student preferences, and make predictions on future use. Connections between these topics and everyday experiences provide motivation and a sense of relevance to students.

In the area of probability, young children start out simply learning to use **probability terms** correctly. Words like *possibly*, *probably*, and *certainly* have definite meanings, referring to the increasing likelihood of an event happening, and it takes children some time to begin to use them correctly. Beyond that, though, elementary age children are certainly able to understand the **probability of an event**. Starting with phrases like *once in six tosses*, children progress to more sophisticated probability language like *chances are one out of six*, and finally to standard fractional, decimal, and percent notation for the expression of a probability. To motivate and foster that maturation, students should be regularly engaged in **predicting and determining probabilities**.

Experiments leading to discussions about the difference between experimental and theoretical probability should be done by older elementary and middle school students. **The theoretical probability** is the probability based on a mathematical analysis of the physical properties and behavior of the objects involved in the event. For instance, when a fair die is rolled each face is equally likely to wind up on top, and so the probability of any particular face showing is one-sixth. **Experimental probabilities** are determined by data gathered through experiments. For example, students may be able to compare the experimental probabilities of rolling a sum of seven vs. a sum of four with two dice long before they can explain why the first is twice as likely from a theoretical point of view.

Older students should understand the difference between **simple and compound events,** like rolling one die vs. rolling two dice, and the difference between **independent and dependent events,** like picking five marbles out of a bag of blue and green marbles one at a time with replacement vs. without replacement. Again, the best way to approach this content is with open-ended investigations that allow the students to arrive at their own conclusions through experimentation and discussion. Eventually, students should feel comfortable representing real-life events using **probability models**.

In statistics, young children can start out as early as kindergarten with **data collection, organization, and graphing**. The focus on those skills, with obviously increasing sophistication, should last throughout their schooling. Students must be able to understand the tables, charts, and graphs used to present data, and they must be able to organize their own data into formats which make them easier to understand. While young students can do exhaustive surveys about some interesting question for *all* of the members of the class, older students should focus some time and energy on the questions involved with **sampling**, where information is obtained from only *some* of the members of a group. Identifying and obtaining data from a well-defined sample of the population is one of the most challenging tasks of a professional pollster.

As students progress through the elementary grades, an increased focus on **central tendency** and later, on **variance and correlation,** are appropriate. Students should be able to use the *average* or *mean*, the *median*, and the *mode* and understand the differences in their uses. Measures of the variance from the center of a set of data, or dispersion, also provide useful insights into sets of numbers. These can be introduced early with the *range* for the early grades, *box-and-whisker plots* showing quartiles of a data distribution for upper elementary school students, and progress to measures like *standard deviation* for older students.

The reason statistics grew as a branch of mathematics, however, was to provide tools that are helpful in

analysis and inference in situations of uncertainty, and that focus should permeate everything students do in this area. Whenever they look at data, they should be trying to answer a question, support a position, or discover a pattern. Students at all grade levels should have many opportunities to look for patterns, draw conclusions, and make predictions about the outcomes of future experiments, polls, surveys, and so on. They should examine data to see whether they are consistent with some hypotheses that a classmate may already have made, and learn to judge whether the data are reliable or whether the hypothesis might need revision.

IN SUMMARY, probability and statistics hold the key for enabling our students to better understand, process, and interpret the vast amounts of quantitative data that exist all around them, and to have a probabilistic sense in situations of uncertainty. To be able to judge the validity of a data-supported argument presented to them, to discern the believability of a persuasive advertisement that talks about the results of a survey of all of the users of a particular product, or to be knowledgeable consumers of the data-intensive government and electoral statistics that are ever-present, students need the skills that they can learn in a well-conceived probability and statistics curriculum strand.

NOTE: *Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

Standard 12 — Probability and Statistics — Grades K-2

Overview

Students can develop a strong understanding of probability and statistics from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in probability for early elementary students, as identified in the K-12 Overview, are **probability terms, the concept of the probability of an event,** and **predicting and determining probabilities.** In statistics they key components for early elementary students are **data collection, organization, and representation.**

The understanding of probability and statistics begins with their introduction and use at the earliest levels of schooling. Children are natural investigators and explorers — curious about the world around them, as well as about the opinions and the habits of their classmates, teachers, neighbors and families. Thus, a fertile setting already exists in children for the development of statistics and probability skills and concepts. As with most of the curriculum at these grade levels, the dominant emphasis should be experiential with numerous opportunities to use the concepts in situations which are real to the students. Statistics and probability can and should provide rich experiences to develop other mathematical content and relate mathematics to other disciplines.

Kindergarten students can **gather data** and **make simple graphs** to organize their findings. These experiences should provide opportunities to look for patterns in the data, to answer questions related to the data, and to generate new questions to explore. By playing games or conducting experiments related to chance, children begin to develop an understanding of probability terms.

First- and second-grade children should continue to **collect and organize data**. These activities should provide opportunities for students to have some beginning discussions on **sampling**, and to **represent their data** in charts, tables, or graphs which help them **draw conclusions,** such as *most children like pizza* or *everyone in the class has between 0 and 4 sisters and brothers,* and raise new questions suggested by the data. As they move through this level, they should be encouraged to design data collection activities to answer new questions. They should be encouraged to see how frequently statistical claims appear in their life by collecting and discussing appropriate items from advertising, newspapers, and television reports.

Students in these grades should experience probability at a variety of levels. Numerous children's games are played with random chance devices such as spinners and dice. Students should have opportunities to play games using such devices. Games where students can make decisions based upon their understanding of probability help to raise their levels of consciousness about the significance of probability. Gathering data can lead to issues of probability as well. Students should experience **probability terms** such as *possibly, probably,* and *certainly* in a variety of contexts. Statements from newspapers, school bulletins, and their own experiences should highlight their relation to probability. In preparation for later work, students need to have experiences which involve systematic listing and counting of possibilities, such as all the possible outcomes when three coins are tossed (see Standard 14, Discrete Mathematics.)

Learning probability and statistics provides an excellent opportunity for connections with the rest of the mathematics standards as well as with other disciplines. Probability provides a rich opportunity for children to begin to gain a sense of fractions. Geometry is frequently involved through use of student-made spinners of varying-sized regions and random number generating devices such as dice cubes or octahedral (eight-sided) shapes. The ability to explain the results of data collection and attempts at verbal generalizations are the foundations of algebra. Making predictions in both probability and statistics provides students opportunities to use estimation skills. Measurement using non-standard units occurs in the development of histograms using pictures or objects and in discussions of how the frequency of occurrence for the various options are related. Even the two areas of this standard are related through such things as the use of statistical experiments to determine estimates of the probabilities of events as a means for solving problems such as how many blue and red marbles are in a bag.

The topics that should comprise the probability and statistics focus of the kindergarten through second grade mathematics program are:

collecting data
organizing and representing data with tables, charts and graphs
beginning analysis of data using concepts such as range and "most"
drawing conclusions based on data
using probability terms correctly
predicting and determining probability of events

Standard 12 — Probability and Statistics — Grades K-2

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Formulate and solve problems that involve collecting, organizing, and analyzing data.**
    - Students collect objects such as buttons, books, blocks, counters, etc. which can be sorted by color, shape, or size. They classify the objects and color one square of a bar graph for each item using different colors for each category. Then they compare the categories and discuss the relationships among them.
    - As an assessment following activities such as the one described above, young students are given a sheet of picture stickers and a blank sheet of paper. They sort the stickers according to some classification scheme and then stick them onto the paper to form a pictograph showing the number in each category.
    - At the front of the room is a magnetic board and, for every child in the class, a magnet with that child's picture. At the start of each day, the teacher has a different question on the board and the children place their magnet in the appropriate area. It might be a bar graph tally for whether they prefer vanilla, chocolate or strawberry ice cream or a Venn diagram where students place their magnet in the appropriate area based on whether they have at least one brother, at least one sister, at least one of both, or neither.
    - Students survey their classmates to determine preferences for things such as food, flavors of ice cream, shoes, clothing, or toys. They analyze the data collected to develop a cafeteria menu or to decide how to stock a store.
    - Second graders record and graph the times of sunrise and sunset one day a week over the entire year. They calculate the time from sunrise to sunset, make a graph of the amount of daylight, and interpret these weekly results over the year.
    - A second grader, upset because she had wanted to watch a TV show the night before but had to go to bed instead, asks the teacher if the class can do a survey to find out when most children her age go to bed.

2. **Generate and analyze data obtained using chance devices such as spinners and dice.**
    - Students roll a die, spin a spinner, or reach blindly into a container to select a colored marble, with replacement, a dozen times. They then color the appropriate square in a bar graph for each pick. *Did some results happen more often or less often than others? Do you think some results are more likely to happen than others?* They repeat the experiment, this time without replacement, and compare the results.
    - Students spill out the contents of cups containing five two-colored counters and record the number of red sides and the number of yellow sides. They perform the experiment twenty

times, examine their data, and then discuss questions such as *Does getting four red sides happen more often than two red sides?* They explain their reasoning.

- Each student has a 4-section spinner. Working in pairs, the students spin their spinners simultaneously and together they record whether they have a match. After doing this several times, they predict how many times they would have a match in 20 spins. Then they compare their prediction with what happens when they actually spin the spinners 20 times. They repeat the activity with a different number of equal sections marked on their spinners. Students in the second grade combine the results of all the students in the class, and compare their predictions with the class total.

3. Make inferences and formulate hypotheses based on data.

- Students roll a pair of dice 100 times and make a frequency bar graph of the sums. They compare their results with those of their classmates. *Do your graphs look essentially alike? Which sum or sums came up the most? Does everyone have a 'winning' sum? Is it the same for everyone? Why do some sums come up less than others?*
- Children are regularly asked to think about their data. *Is there a pattern in the dice throws, bean growth, weather, temperature, or other data? What causes the patterns? Are the patterns in their data the same as those of their classmates?*

4. Understand and informally use the concepts of range, mean, mode, and median.

- When performing experiments, children are regularly asked to find the largest and smallest outcomes (range) for numerical data and the outcome that appeared most often (mode). They are asked to compare the mode they obtained for an experiment with the modes found by their classmates.

5. Construct, read, and interpret displays of data such as pictographs, bar graphs, circle graphs, tables, and lists.

- After collecting and sorting objects, children develop a pictograph or histogram showing the number of objects in each category.
- Students design and make tallies and bar graphs to display data on information such as their birth months.
- Students list all possible outcomes of probability experiments, such as tossing a penny, nickel, and dime together.
- Working in cooperative groups, students are given six sheets of paper each containing an outline of a circle which has been divided into eight equal sectors. The students color each whole circle a different color and then cut their circles into individual sectors so each group has 8 sectors in each of 6 colors. Then they roll a die eight times keeping a tally of the results using orange for rolls of 1, blue for rolls of 2, and so on. They use these eight colored sectors to record their results in a circle graph, which they put aside. They repeat this twice and get two other circle graphs. Finally, as a whole class activity, they gather the circle graphs from all the groups, and rearrange the sectors to make as many solid color circles as they can. They discuss the results.
- Students regularly read and interpret displays of data; they also read information from their classmates' graphs and discuss the differences in their results.

6. **Determine the probability of a simple event, assuming equally likely outcomes.**

- Children roll a die ten times and record the number of times each number comes up. They combine their tallies and discuss the class results.
- Children predict how often heads and tales come up when a coin is tossed. They toss a coin ten times and tally the number of heads and tails. *Are there the same number of heads and tails?* They combine their tallies and compare their class results with their predictions. (See *Making Sense of Data,* in the *Addenda Series,* by Mary Lindquist.)

7. **Make predictions that are based on intuitive, experimental, and theoretical probabilities.**

- Second graders are presented with a bag in which they are told are marbles of two different colors, twice as many of one color as the other. They are asked to guess the probability for drawing each color if a single marble is drawn. *Is this the same as flipping a coin? Will one color be picked more often than the other?* The experiment is performed repeatedly and tallies are recorded. The chosen marble is returned to the bag each time before a new marble is drawn. The children discuss whether their estimates of the probabilities made sense in light of the outcome.
- Students are told that a can contains ten beads, some red ones, some yellow ones, and some blue ones. They are asked to predict how many beads of each color are in the can. The students attempt to determine the answer by doing a statistical experiment. One at a time, each child in the class draws a bead, records the color with a class tally, and replaces it. At various times in the process, the teacher asks the children to return to their prediction to determine if they want to modify it.
- As an informal assessment of the students' understanding of these concepts, they are presented with a bag in which they are told there are 10 yellow marbles and 2 blue ones. They are asked to predict what color marble they will pick out of the bag if they pick without looking, and about how many students in the class will pick a blue marble.

8. **Use concepts of certainty, fairness, and chance to discuss the probability of actual events.**

- Students work through the *Elevens Alive!* lesson that is described in the Introduction to this *Framework*. They make number sentences adding up to 11 by dropping 11 chips which are yellow on one side and red on the other, and writing $11 = 4+7$ when four chips land yellow-side-up and seven chips land red-side-up. They notice that they are writing some number sentences more frequently than others, and these observations lead into a discussion of probability.
- Each child plants five seeds of a fast growing plant. They count the number of seeds which sprout and discuss how many seeds might sprout if they had each planted ten, or twenty, or a hundred seeds. They explain their reasoning. (The numbers can be adjusted for different grade levels.)
- Students predict how many M&Ms of each color are in a large unopened mystery bag. To help make these predictions, cooperative groups are given a handful of M&Ms from the bag; they tally the count of the colors, report their results, and prepare graphs of their results. Students refine their predictions by looking at the class totals. The mystery bag is then opened and the colors counted. Students discuss how their prediction matches the

actual count and how the experiment helped them make their prediction.

- Students examine various types of raisin bran cereal. They experiment with scoops of cereal and determine the number of raisins that appear in each scoop. They make inferences about which brand might be the "raisiniest."

References

Lindquist, M., et al. *Making Sense of Data. Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades K-6.* Reston, VA: National Council of Teachers of Mathematics, 1992.

General References

Burton, G., et al. *First Grade Book. Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades K-6.* Reston, VA: National Council of Teachers of Mathematics, 1991.

Burton, G., et al. *Kindergarten Book. Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades K-6.* Reston, VA: National Council of Teachers of Mathematics, 1991.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards.*

Standard 12 — Probability and Statistics — Grades 3-4

Overview

Students can develop a strong understanding of probability and statistics from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in probability for elementary school students, as identified in the K-12 Overview, are: **probability terms, the concept of the probability of an event, predicting and determining probabilities,** and **the relationship between theoretical and experimental probabilities.** In statistics the key components for elementary school students are **data collection, organization, and representation, central tendency,** and **analysis and inference.**

Based on their earlier experiences with data, third- and fourth-graders should strengthen their ability to **collect, organize, and represent data**. They should build on their informal discussions of data by developing their ability to **analyze data, formulate hypotheses, and make inferences from the data.** As their numerical skills increase, they should begin to understand and to use the **mean** and **median**, as well as range and mode, as **measures of central tendency.** Frequent probability experiments should help students extend their ability to **make predictions** and understand probability as it relates to events around them, and should provide the intuition they will need in order to **determine probabilities** in simple situations.

As in the previous grade levels, probability and statistics understanding is best developed through frequent opportunities to perform experiments and gather and analyze data. Such activities are most valuable when students choose a topic to investigate based on a real problem or based on an attempt to answer a question of interest to them. Children should experience new activities, but they should have the opportunity to revisit problems introduced in grades K-2 when doing so would allow them to practice or develop new understandings.

Probability and statistics are closely related. Students should use known data to predict future outcomes and they should grapple with the concept of uncertainty using **probability terms** such as *likely,* not *likely, more likely,* and *less likely.* Developing an understanding of randomness in probability is crucial to acquiring a more thorough understanding of statistics.

Third and fourth grade is a wonderful time for students to see connections among subjects. Most science programs at this level involve collection and analysis of data as well as a focus on the likelihood of events. Social studies programs usually ask children to begin to develop ideas of the world around them. Discussions might focus on their school, neighborhood, and community. Such explorations can be enhanced through analysis and discussion of data such as population changes over the last century. Third- and fourth-graders are more attuned to their environment and are more sensitive to media information than early elementary school children. Discussions about such things as the claims in TV advertisements or commercials, or newspaper articles on global warming, help students develop the ability to use their understandings in real situations.

At all grade levels, probability and statistics provide students with rich experiences for practicing their skills in content areas such as number sense, numerical operations, geometry, estimation, algebra, patterns and functions, and discrete mathematics.

The topics that should comprise the probability and statistics focus of the mathematics program in grades three and four are:

- collecting, organizing, and representing data
- analyzing data using the concepts of range, mean, median, and mode
- making inferences and formulating hypotheses from their analysis
- determining the probability of a simple event assuming outcomes are equally likely
- making valid predictions based on their understandings of probability

Standard 12 — Probability and Statistics — Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. Formulate and solve problems that involve collecting, organizing, and analyzing data.

- Students wish to study the differences in temperature between their hometown and a school they have connected with in Sweden through the Internet. They exchange highs and lows for each Monday over the three-month period from January through March. They note whether the temperatures are given in degrees Celsius or Fahrenheit, and use a thermometer with both markings to change from one to the other if necessary. They organize and represent the data and develop questions about possible differences in lifestyle that are prompted by the temperature. They then exchange their questions with their sister school to learn more about their culture.
- While studying about garbage and recycling, children notice the amount of waste generated in the cafeteria each day. A variety of questions begin to surface such as: *What types of waste are there? How much of each? Can we measure it? How? How often should we measure it to get an idea of the average amount of waste generated each day? How can we help make less waste?* The class considers how it can find answers to these questions, designs a way to obtain the data, and finds answers to their questions.
- Students perform experiments such as rolling a toy car down a ramp and measuring the distance the car rolled beyond the bottom of the ramp. This experiment is repeated, holding the top of the ramp at various heights above the ground. Students discuss the patterns and relationships they see in the data and use their discoveries to predict the distances obtained for ramps of other heights.

2. Generate and analyze data obtained using chance devices such as spinners and dice.

- Each child in the class rolls a die 20 times and records the outcomes in a frequency table. The class combines the results in a class frequency table. They discuss which outcome occurred most often and least often and then whether the class results differ from their individual results and why that might be.
- Students make their own cubes from cardstock and label the sides 1, 2, 2, 3, 4, 5. They roll their cubes 20 times each, recording the results. After combining their results, the class discusses the experiment and the reasons the results differ from the results obtained when using a regular die.
- As a question on a class test, students are told that Sarah rolled a die 20 times and she got twelve *1s*, two *2s*, three *3s*, and three *6s*. They are asked what they would conclude about Sarah's experiment and what might have accounted for her results.

3. **Make inferences and formulate hypotheses based on data.**

   - Students read *A Three Hat Day* by Laura Geringer. They use concrete objects (different colored beans, hats, or pattern blocks) to show different orders for wearing three different hats. They investigate how many different ways there are to wear four different hats.
   - After collecting, organizing, and analyzing data on the favorite sport of the fourth graders in their school, third graders are asked to interpret the findings. *Why do you suppose soccer was chosen as the favorite sport? How close were other sports? What if we collected data on the same question from fourth graders in another county or another state? Do you think first graders would answer similarly? Why?*
   - Students read *Mr. Archimedes' Bath* and *Who Sank the Boat* by Pamela Allen and discuss what happens to the water level in a container as things are added and why.
   - The fourth grade class is planning a walking tour of a local historic district in February. They want to take hot chocolate but don't know which type of cup to take so that it stays warm as long as possible after being poured. In the science unit on the cooling of liquids, the students discussed notions of variables and constants. They set up an experiment using cups of the same size but of different materials and measure the temperatures in each at equal intervals over a 30-minute period. They plot the data and use their graphs to discuss which cup would be best.

4. **Understand and informally use the concepts of range, mean, mode, and median.**

   - Before counting the number of raisins contained in each of 24 individual boxes of raisins, students are asked to estimate the number of raisins in each box. They count the raisins and compare the actual numbers to their estimates. Students discover that the boxes contain different numbers of raisins. They construct a frequency chart on the blackboard and use the concepts of range, mean, median, and mode to discuss the situation.
   - In a fourth grade assessment, students are asked to prepare an argument to convince their parents that they need a raise in their allowance. Students discuss what type of data would be needed to support their argument, gather the data, and use descriptive measures as a basis for their argument. In a cooperative effort, sixth grade students play the part of parents and listen to the arguments. The sixth graders provide feedback as to whether the students had enough information to convince them to raise the allowance and, if not, what more they might use.

5. **Construct, read, and interpret displays of data such as pictographs, bar graphs, circle graphs, tables, and lists.**

   - Presented with a display of data from *USA TODAY*, students generate questions which can be answered from the display. Each child writes one question on a 3x5 card and gives it to the teacher. The cards are shuffled and redistributed to the students. Each student then answers the question he or she has been given and checks the answer with the originating student. Disagreements are presented to the class as a whole for discussion.
   - Following a survey of favorite TV shows of students in the entire third grade, groups of students develop their own pictographs using symbols of their choosing to represent multiple children.

6. Determine the probability of a simple event assuming equally likely outcomes.

- Children toss a coin fifty times and record the results as a sequence of Hs and Ts. They tally the number of heads and tails. *Are there the same number of heads and tails?* The children discuss situations that often lead to misconceptions such as *If three tosses in a row come up heads, what is the chance that the next toss is a head? Is there a better chance than there would have been before the other tosses took place?* After what is a lively discussion, the children review their sequence of Hs and Ts to see what happened on the next toss each time that three consecutive heads appeared. This analysis should demonstrate that each result does not depend upon the previous ones.

- Students discuss the probability that a particular number will come up when a die is thrown, and predict how many times that number will appear if the die is rolled 50 times. They then toss a die 50 times and compare the results with their predictions.

7. Make predictions that are based on intuitive, experimental, and theoretical probabilities.

- Fourth-graders are presented with a bag in which there are marbles of three different colors, the same number of two of the colors, and twice as many of the third. They are asked what they would expect to happen if a marble were drawn twelve times and placed back in the bag after each time. The experiment is performed and the children discuss whether their estimates of the outcome made sense in light of the actual outcome.

- During an ecology unit, students discuss the capture-recapture method of counting wildlife in a local refuge. A number of animals, say 30 deer, are captured, tagged, and released; later another group of deer is captured. If five of the twenty-five recaptured deer are tagged, then you might conclude that about one in five deer have been tagged, and therefore that the total number of deer in the refuge is about 5 x 30 or 150. The students perform a capture-recapture experiment using a large bag of lollipops to determine the number of lollipops in the bag.

8. Use concepts of certainty, fairness, and chance to discuss the probability of actual events.

- Students discuss the probability of getting a zero or a seven on the roll of one die or picking a blue bead from a bag full of blue beads, and use this as an introduction to a discussion about the probability of certain events and impossible events.

- Students discuss the relationship between events such as flipping a coin, a newborn baby being a girl, guessing on a true-false question, and other events which have an approximately equal chance of occurring.

References

Allen, Pamela. *Mr. Archimedes' Bath*. New York: Lothrop, Lee, and Shepard Books, 1980.

Allen, Pamela. *Who Sank the Boat.* New York: Putnam, 1990.

Lindquist, M., et al. *Making Sense of Data. Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades K-6.* Reston, VA: National Council of Teachers of Mathematics, 1992.

General References

Burton, G., et al. *Third Grade* Book. *Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades K-6.* Reston, VA: National Council of Teachers of Mathematics, 1992.

Geringer, Laura. *A Three Hat Day.* New York: Harper and Row, 1985

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
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Standard 12 – Probability and Statistics – Grades 5-6

Overview

Students can develop a strong understanding of probability and statistics from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in probability for middle school students, as identified in the K-12 Overview, are: **probability terms, the concept of the probability of an event, predicting and determining probabilities, the relationship between theoretical and experimental probabilities,** and **compound events.** In statistics, the key components are: **data collection, organization, and representation, sampling, central tendency, variance and correlation,** and **analysis and inference.**

In grades K-4, students explored basic ideas of statistics by gathering data, organizing data, and representing data in charts and graphs, and then using this information to arrive at answers to questions and raise further questions. Students in grades 5 and 6 are keenly interested in movies, fashion, music, and sports. These areas provide a rich source of real problems for students at this age. The students should make the decision on how to **sample** and then **collect and organize data.** They should determine how best to **represent** the data and begin to develop a more formal understanding of summary statistics of **central tendency** such as the mean, the median, and the mode. They should recognize that for certain types of data, such as height, the mean is an appropriate measure, but it is inappropriate for other types of data, such as hair color. These activities should provide opportunities for students to **analyze data** and to **make inferences** regarding the data and to communicate their inferences in a convincing manner. They should further develop their understanding of statistics through the evaluation of arguments by others, whether they come from classmates, advertising, political rhetoric, or news sources.

While statistical investigations can be similar to those in earlier grades, fifth- and sixth-graders should have access to statistical software on computers or calculators which have statistical capability. This will allow them to carry out statistical work using real data without becoming mired in tedious calculations. The technology will be used to do the manipulation of the data and the students will focus on developing their skills in interpreting the data.

Students enter these grades having participated in a wide variety of activities designed to help them understand the nature of probability and chance. The emphasis in grades K-4 was primarily on simple events such as the roll of a die or the flip of one coin. Even when compound events such as the roll of two dice were considered, the outcomes were looked upon as a simple event. In grades 5 and 6, students begin to experiment with **compound events** such as flips of several coins and rolls of dice and to predict and evaluate their **theoretical and experimental probabilities.** As they develop their understanding of fractions, ratios, and percents, they should use them to represent probabilities in place of phrases such as "three out of four." They begin to model probability situations and to use these models to **predict** events which are meaningful to them.

At all grade levels, probability and statistics provide students with rich experiences for practicing their skills in content areas such as number sense, numerical operations, geometry, estimation, algebra, patterns and functions, and discrete mathematics. Because most of the activities are hands-on and students are constantly dealing with numbers in a variety of ways, it assists the development of number sense as well.

The topics that should comprise the probability and statistics focus of the mathematics program in grades 5 and 6 are:

- collecting, organizing, and representing data
- analyzing data using range and measures of central tendency
- making inferences and hypotheses from their analysis of data
- evaluating arguments based upon data analysis
- interpolating and/or extrapolating from data using a line of best fit
- representing probability situations in a variety of ways
- modeling probability situations
- predicting events based on real-world data

Standard 12 — Probability and Statistics — Grades 5-6

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

9. Generate, collect, organize, and analyze data and represent this data in tables, charts, and graphs.

- Students recognize that this is a time of growth for many of them. The class measures various lengths associated with a person, such as height, length of forearm, length of thigh, handspan, length of foot, and armspan. They enter the data into a spreadsheet and produce various graphs as well as statistical analyses of their measures. They update their data every month and discuss the change, both individually and as a class.
- Students survey another class to determine data of interest, such as the last movies seen, and then organize the data and produce reports discussing the interests of the grade level.
- Students work on problems like this one from the New Jersey State Department of Education's *Mathematics Instructional Guide (p. 7-95)*:

  A fair spinner with 4 congruent regions labeled A, B, C, and D is spun 20 times by each member of a class of 23 students. Assume that your class conducted the experiment and obtained the expected results. Make a bar graph illustrating the combined class results. Explain why you drew your bar graph the way you did. Explain why an individual student's results might be different from the class results.

10. Select and use appropriate graphical representations and measures of central tendency (mean, mode, and median) for sets of data.

- Students demonstrate understanding of measures of central tendency by writing a letter to a fictional classmate explaining how the mean, the median, and the mode each help to describe data. They then extend their discussion by presenting a picture of an "average student" in their grade. The picture discusses height, color of hair, preference in movies, etc. In creating the picture, the students must choose the appropriate measures of central tendency based upon the type of data and justify their choice. (For example, the mean is not appropriate in discussing hair color.) They will likely want to present pictures of both an "average boy" and an "average girl" in the class.
- During a social studies unit, students determine a method to ascertain the value of the homes in their community. They determine the mean, the median, and the mode for the data and decide which provides the most accurate picture of the community. They include in their study homes from different sections of the town.
- Students perform an experiment where one group is given 10 words in a jumbled order while another group is given them in a sequence which facilitates remembering them.

After giving each group one minute to study the words, the students are asked to turn their papers over and write as many of the words as they remember. The papers are graded by fellow students and the scores reported. After considering various graphing methods, the students determine that a *box-and-whiskers* plot would be the best way to illustrate the results and compare the two groups.

11. Make inferences and formulate and evaluate arguments based on data analysis and data displays.

- Students are asked to develop a generalization about their classmates. They are allowed to make any hypothesis which is appropriate. For example, some boys might suggest that boys are stronger than girls or others might say that girls are taller than boys. They should determine how they would determine the validity of their hypotheses by designing a data collection activity related to it.
- The teacher in one fifth-grade class is especially alert for generalizations that students make about any topic. She writes them on slips of paper, and keeps them in a box. As an assessment of the students' ability to develop statistical activities to validate hypotheses, groups of students pull slips from the box, develop data collection activities, collect the data, analyze it, and make reports to the class about the validity of the generalizations originally made.
- Students are shown a newspaper article which states that 25% of fifth graders have smoked a cigarette. They discuss their reaction by indicating whether they believe the figure to be correct, too high, or too low. They then design a survey which they use to poll their fellow fifth graders in an effort to check the validity of the claim for the population of their school. They also send a letter to the newspaper requesting the sources of data for the article and compare the data in the article with their data.

12. Use lines of best fit to interpolate and predict from data.

- Given a jar with straight sides and half filled with water, students drop marbles in five at a time. After each group of five, they measure the height of the water and record in a table the number of marbles in the jar and the height of the water. The students then represent their data in a scatterplot on an x-y plane and find that the points lie almost exactly in a straight line. They draw a line through the data and use it to determine answers to questions like: *How high will the water be after 25 marbles have been added?* and *How many marbles will it take to have the water reach the top?* Activities like this one form the foundation for understanding graphs in algebra.

13. Determine the probability of a compound event.

- Students create a table to show all possible results of rolling two dice. At the left of the rows are the possible rolls of the first die and at the top of the columns are the possible rolls of the second die. They complete the table by putting in each cell the appropriate sum of the number in the top row and the left column. Counting the number of times each sum appears in the table, they determine the probability of getting each possible sum. They then roll two dice 100 times and compare the sums they get with the sums predicted from the table.
- Students make a list of all possible outcomes when four coins are tossed and determine the

theoretical probability of having exactly two heads and two tails.

14. **Model situations involving probability, such as genetics, using both simulations and theoretical models.**

- Students examine the probability of a family with four children having two boys and two girls by simulating the situation using four coins. They first choose which side of the coin will represent males and which will represent females. They toss the set of coins 50 times and record their results as the number of boys and the number of girls in each "family." They compare the results of their experiment with the prediction based on probability. They also survey a large sample of students in the school and record the family composition of all families with four children. All of these are used to discuss the likelihood of an evenly-matched family.

- A 25¢ "prize" machine in the grocery store contains an equal number of each of six plastic containers with Power Ranger tattoos. Students are asked to determine how many containers they need to buy to have a good chance of getting all six. They simulate this situation with a bag containing an equal number of six different colored marbles. They draw out, record, and replace one marble at a time until they have drawn marbles of all six colors, recording the number of times that took. They repeat the simulation three times. The class results are gathered and discussed. One issue discussed is whether the model is a good one for the situation or whether it should be modified in some way to better represent reality.

- Students read *Caps for Sale* by Esphyr Slobodkina. The peddler in the story sells caps and wears his entire inventory on his head: a checked cap and four each of identical blue, gray, and brown hats. Students use concrete objects to model some of the different orders in which the hats can be worn. They come to realize that there are many ways and try to discover the total number of different ways. They search for an efficient way to determine the number of permutations.

- Students work through the *Two-Toned Towers* and *Pizza Possibilities* lessons that are described in the First Four Standards of the *Framework*. They make a systematic list of all the towers built out of four red and blue cubes (or of all the pizza combinations) and calculate the probability that a tower has three or four blue cubes.

15. **Use models of probability to predict events based on actual data.**

- Students examine weather data for their community from previous years, and then use their analysis of the data to predict the weather for the upcoming month. They compare the actual results with their predictions after the month has passed and then use the comparison to determine ways to improve their predictions.

- Using data from previous years, students determine the number of times their favorite professional football team scored a number of points in each of six ranges of scores (0-5, 6-10, 11-15, 16-20, 21-25, and 26 or more). They determine the fraction or percentage of games the score was in each of those ranges and make a spinner whose areas are divided the same way. Each Friday during football season, they spin their spinners to predict how many points the team will score and who will win the game. Toward the end of the season, they discuss the success or failure of their efforts and the probable causes.

16. Interpret probabilities as ratios and percents.

- The students are introduced to the game *Pass The Pigs* (Milton Bradley) where two small hard-rubber pigs are rolled. Each pig can land on a side where there is a dot showing, a side where the dot does not show, on its hooves, on its back, leaning forward balancing itself on its snout, and balancing itself on its left foreleg, snout, and left ear. The students determine the fairness of the distribution of points on the sides of the pig by rolling the pig numerous times, recording the results, and using the ratios of successes for each, divided by the total number of rolls, to represent the probability of obtaining each situation.
- Students examine uses of probability expressed as percentages in such situations as weather forecasting, risks in medical operations, and reporting the confidence interval of surveys.

References

New Jersey State Department of Education, *Mathematics Instructional Guide: Linking Classroom Experiences to Current Statewide Assessments.* D. Varygiannes, Coord. Trenton, N.J., 1996.

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On-Line Resources

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Standard 12 — Probability and Statistics — Grades 7-8

Overview

Students can develop a strong understanding of probability and statistics from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in probability for middle school students, as identified in the K-12 Overview, are: **probability terms, the concept of the probability of an event, predicting and determining probabilities, the relationship between theoretical and experimental probabilities,** and **compound events.** In statistics, the key components are: **data collection, organization, and representation, sampling, central tendency, variance and correlation,** and **analysis and inference.**

Students should enter the seventh grade with a strong intuitive understanding of probability and statistics as a result of their activities in grades K-6, and should have a basic understanding of the more formal methods which were introduced in grades 5-6. They will build on this foundation in grades 7 and 8.

Students in grades 7 and 8 present unique challenges. They are turning to their peer group for leadership and support and, at the same time, placing a strain on the relationships between themselves and significant adults in their lives. Some students begin to experiment with things they associate with being an adult: smoking, alcohol, drugs, and sex. The quantity of statistics in all of these areas provides an ideal opportunity to weave together statistical activities which dovetail with information provided by the the health and physical education department.

Students at these ages also become more aware of community issues. Integrating statistics activities with topics in the social studies curriculum can enhance their work in both areas as well as fit in with their growing interests. Hands-on science activities require good statistical methods and understanding in order to develop accurate and appropriate conclusions. At the same time, students need to understand how often statistics and probability statements are incomplete, misunderstood, or purposely used to mislead. Having students read books such as *How to Lie with Statistics* by Darryl Huff or *Innumeracy* by John Allen Paulos provides excellent opportunities to discuss how statistics and probability are misused.

In statistics, students continue to **collect, organize, and represent data** and to use various **measures of central tendency** to describe their data. But they should now become more focused on **sampling** techniques that justify making **inferences** about entire populations. Examples of this appear frequently in the news media. They also begin to explore **variance and correlation** as additional tools in describing sets of data.

Many of the probability experiments should continue to be related to games and other fun activities. Students in these grades should continue to develop their understanding **of compound events** and their related probabilities, and should continue to consider and compare **experimental and theoretical probabilities**. Furthermore, the connection between probability and statistics should help them understand issues such as **sampling** and **reliability.** Students need to develop a sense of the application of probability to the world around them as well. Everyday life is rich with "coincidences" which are actually likely to occur. For example, they should examine the probability that two people in their class, or any group of 25 or more people, have the same birthday. The results always stir up considerable interest and disbelief.

At all grade levels, probability and statistics provide students with rich experiences for practicing their skills in content areas such as number sense, numerical operations, geometry, estimation, algebra, patterns and functions, and discrete mathematics. Because most of the activities are hands-on and students are constantly dealing with numbers in a variety of ways, it assists the development of number sense as well.

The topics that should comprise the probability and statistics focus of the mathematics program in grades 7 and 8 are:

collecting, organizing, and representing data
analyzing data using range and measures of central tendency
making inferences and hypotheses from their analysis of data
evaluating arguments based upon data analysis
interpolating and/or extrapolating from data using a line of best fit
representing probability situations in a variety of ways
modeling probability situations
analyzing probability situations theoretically
predicting events based on real-world data

Standard 12 — Probability and Statistics — Grades 7-8

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

9. **Generate, collect, organize, and analyze data and represent this data in tables, charts, and graphs.**

   - Most students in grades 7 and 8 have major physical growth activity. Students can continue to maintain the statistics related to their body that they began to collect in the fifth and sixth grades. They should continually update what the average person in the grade would look like in terms of this data.
   - In the spring, the social studies teacher and the mathematics teacher plan a unit on the school board elections. Students are broken into groups to study questions such as *What percent of the registered voters can be expected to vote? Will the budget pass?* and *Who will be elected to the board of education?* Students plan their survey, how they will choose the sample, how best to gather the data, and how best to report the information to the class.

10. **Select and use appropriate graphical representations and measures of central tendency (mean, mode, and median) for sets of data.**

    - Students study the sneakers worn by students in the school. They form into a human histogram based upon their brand of sneakers. The data is recorded and a discussion is encouraged about the distribution of sneakers throughout the school. Students discuss in their journals which of the measures of central tendency a sporting goods store would use in determining which brands to stock and in what proportion. The students gather prices for a variety of brands and styles and enter the data into a spreadsheet. They respond in their journal as to whether and why the mean, median, or mode would be most useful to discuss sneaker prices.
    - Presented with a list of OPEC countries and their estimated crude oil production in a recent year, students determine how best to report the data. Some present their graphs as box plots, others use histograms, and others use circle graphs. They use the three measures of central tendency and discuss what each result would mean in this situation and which would be best to use in other situations.
    - Students work on problems like this one from the New Jersey Department of Education's *Mathematics Instructional Guide (p. 7-99)*:

      A set of test scores in Mrs. Ditkof's class of 20 students is shown below.

62	*77*	*82*	*88*	*73*	*64*	*82*	*85*	*90*	*75*
74	*81*	*85*	*89*	*96*	*69*	*74*	*98*	*91*	*85*

Determine the mean, median, mode, and range for the data. Suppose each student completes an extra-credit assignment worth 5 points, which is then added to his/her score. What is the mean of the set of scores now if each student received the extra five points? Explain how you calculated your answer.

11. Make inferences and formulate and evaluate arguments based on data analysis and data displays.

- Students are presented with data from *The World Almanac* showing the number of cigarettes smoked per year per adult and the rate of coronary heart disease in 21 countries. They produce a scatterplot and recognize a relatively high correlation between the two factors. They write an essay on the possible causes of this relationship and their interpretation of it.
- Students are asked to predict how many drops of water will fit on a penny. They write their prediction on a post-it note along with an explanation of their reasoning. The predictions are collected and displayed on bar graphs or stem-and-leaf plots. Students perform the experiment and record their results on another post-it note. They compare their hypotheses with the conclusions. A science lesson on surface tension can easily be integrated with this lesson.
- Students are studying their community's recycling efforts in an integrated unit. In getting ready for discussion in this area, the mathematics teachers ask the students to predict how many pounds of junk mail comes in to their community in a month. The students collect all junk mail sent to their house over the course of a month. They weigh the junk mail weekly and record the results. At the end of the month, all the students bring in their data. The class determines the mean, median, and mode for the collected data, decides which of these measures would be the best to use, and agrees on a method to use to estimate the amount of junk mail for the entire community.

12. Use lines of best fit to interpolate and predict from data.

- Presented with the problem of determining how long it would take *the wave* to go around Giants Stadium, students design an experiment to gather data from various numbers of students. They produce a scatterplot and use it to determine a line of best fit. They pick two points on the line and determine the equation for that line. Last, they estimate the number of people around the stadium and answer the question.
- Given some of the winning times for the Men's and Women's Olympic 100 meter freestyle events during the past century, students plot the data and produce a line of best fit for each event. They use their equations to estimate the winning times in those years for which the information was not recorded, and they predict when the women's winning times will equal the men's current winning times.
- Students are presented with an article that states that police have discovered a human radius bone which is 25 centimeters long. Students perform measurements of the lengths of radius bones of various-sized people and their heights, produce a scatterplot, fit a line to the data, and determine their prediction of the height of the person whose bone was found. They write a letter to the chief of police, predicting the height of the person, with justifications for their conclusion.

13. Determine the probability of a compound event.

- Students watch the long-range weekend weather forecast and learn that the probability of rain is 40% on Saturday and 50% on Sunday. They determine that the probability that it will rain on both days is 20% by multiplying the two percentages together (.40 x .50 = .20 or 20%), and similarly then find that the probability that it will not rain on either day is 30%. Following the weekend, they discuss the success or failure of their prediction methods.
- Two teams are in a playoff for the division title. If the probability of the Eagles defeating the Falcons in an individual game is 40%, what is the probability that they will win a three game playoff? What about a five-game playoff?

14. Model situations involving probability, such as genetics, using both simulations and theoretical methods.

- During an integrated unit with their science and health classes, students discuss the various gender possibilities for children within a family. For each large family, that is, number of children, up to 6, they calculate the probability of each possible gender mix. Three groups of students conduct simulations — one with coins, one with dice (1, 2, or 3 on a die represent a female) and one with spinners. They also collect this data for all of the students in their school. They report their findings and compare the theoretical possibilities, the simulated probabilities, and the actual outcomes, and discuss the differences and similarities.
- Students study the chances of winning the New Jersey Pick 3 lottery. They model the problem by using spinners with 10 numbers and calculate the theoretical probability. They may also use a computer program to randomly generate a million 3-digit numbers and see how close to 1 out of 1000 times their favorite number comes up.

15. Use models of probability to predict events based on actual data.

- Students are presented with data collected by an ecologist tallying the number of deer of one species that died at ages from 1 to 8 years. Students use the data to discuss the probability of living to various given ages and what they would expect the life expectancy of this species to be.

16. Interpret probabilities as ratios and percents.

- Students examine uses of probability expressed as percentages in such things as weather forecasting, risks in medical operations, and reporting the confidence interval of surveys.
- Students work on problems like this one from the New Jersey State Department of Education's *Mathematics Instructional Guide* (p. 7-103):

  A dart board is composed of three concentric circles with radii 2 cm, 10 cm, and 20 cm [as indicated in an accompanying diagram]. A grand prize is earned if a dart is thrown in the 2 cm circle (bulls-eye). Given that a person is blindfolded and throws a dart somewhere on the board, find the probability that the grand prize will be won when the first dart is thrown. Explain the process you used to solve the problem.

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Huff, D. *How to Lie with Statistics*. New York: Norton, 1954.

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Zawojewski, Judith, et al. *Dealing with Data and Chance*. A Component of the *Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades 5-8*. Reston, VA: National Council of Teachers of Mathematics, 1991.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards.*

Standard 12 — Probability and Statistics — Grades 9-12

Overview

Students can develop a strong understanding of probability and statistics from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in probability for middle school students, as identified in the K-12 Overview, are: **probability terms, the concept of the probability of an event, predicting and determining probabilities, expected value, the relationship between theoretical and experimental probabilities,** and **compound events.** In statistics, the key components are: **data collection, organization, and representation, sampling, central tendency, variance and correlation,** and **analysis and inference.**

The field of statistics is relatively new. Beyond the work of scientists, Florence Nightingale was the great pioneer in gathering and analyzing statistical data for public health questions. During the great cholera epidemic of 1854 in London, England, statistics on the prevalence of cholera cases in various London neighborhoods were used to deduce that the cholera originated with a single well. In our own century statistics touches all of us through such diverse means as statistical quality control in industry, advertising claims, pre-election polls, television show ratings, and weather forecasts. To be successful members of present day society, high-school graduates need an understanding of statistics and probability which formerly was rare even among college graduates.

By the time students enter high school, they should have mastered basic descriptive statistical methods. On the basis of their varied experience, they should be able to set up a study, gather the data, and appropriately analyze and report their findings. Throughout grades 9 to 12, students should have numerous opportunities to continue to practice these skills in a variety of ways, and also to extend these skills, in connection with their growth in other mathematical areas. As students learn new algebraic functions, they might revisit a problem they had previously modeled linearly and apply a different model. For example, they may have linearly modeled the series of winning times of the men's Olympic marathon but now understand that there would probably be a limiting time and so attempt to fit a quadratic or logarithmic curve instead. Where appropriate, the content should be developed through a problem-centered approach. For example, if students are required to generate a report on two sets of data which have the same measures of **central tendency** only to find later they have very different **variance**, they should recognize the need for some way to identify that difference.

John Allen Paulos, in his book, *Innumeracy,* cites numerous problems associated with a lack of understanding of probability. If people are to make appropriate decisions, then they must understand the relationship of probability to real situations and be able to weigh the consequences against the odds. As with statistics, probability needs to be experienced, not memorized. Work done at this level should provide insight into the use of probability and probability distributions in a variety of real-world situations. The normal curve presents interesting opportunities to examine uses and abuses of mathematics.

Students should have access to appropriate technology for their work in probability and statistics, not only to simplify calculation and display charts and graphs, but also to generate appropriate data for activities and projects. They should make use of data taken from the Internet and CD-ROMs, and simulate experiments with Calculator Based Laboratories. Whenever possible, real data gathered from school, the community, or cooperating businesses should be used.

Probability and statistics offers a rich opportunity to integrate with other mathematics content and other disciplines. This content provides the opportunity to generate the numbers and situations which should be used in other areas such as geometry, algebra, functions, and discrete mathematics. The goal to have students become effective members of a democratic society requires them to practice and participate in decision-making experiences. The ability to make intelligent decisions rests on an understanding of statistics and probability, and students should regularly integrate this content with their experiences in social studies, science, and other disciplines.

The topics that should comprise the probability and statistics focus of the mathematics program in grades 9 through 12 are:

designing, conducting, and interpreting statistical work to solve problems
analyzing data using range, measures of central tendency, and dispersion
applying probability dispersions in real situations
evaluating arguments based upon their knowledge of sampling and data analysis
interpolating and/or extrapolating from data using curve fitting
using simulations to estimate probabilities
determining expected values
using the law of large numbers

Standard 12 — Probability and Statistics — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11, and 12.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

17. Estimate probabilities and predict outcomes from actual data.

- In a standard class test, students are asked to compute the probability that a given raffle ticket for a senior class raffle to raise money for the senior trip will win a prize. The class will be printing 500 tickets that they will sell for $1 each. First prize is a stereo worth $150. Second prize is a $100 shopping spree in the local Gap store. Third prize is a $50 gift certificate to The Golden Goose restaurant. There are ten fourth prizes of a commemorative T-shirt worth $8 each. Students also compute the expected value of each ticket.
- Students determine the area of an irregular closed figure drawn on a large sheet of paper using the Monte Carlo method: Each person in the group drops a handful of pennies over their shoulder (without looking) onto the paper containing the figure. They count the number of coins on the paper (total shots) and the number within the figure (hits). They thus produce the ratio of hits to total shots and multiply this fraction by this area of the paper to estimate the area of the figure.
- Students work through the *On the Boardwalk* lesson that is described in the Introduction to the *Framework*. In this lesson they explore the probability that a quarter thrown onto a rectangular grid will land entirely within one of the squares on the grid, and then discuss how changing the size of the squares will affect the probability.
- A point P inside a square is selected at random and is used to form a triangle with vertices A and B of the square. Students determine the probability that the triangle is acute using a simulation and a theoretical calculation.

18. Understand sampling and recognize its role in statistical claims.

- While studying United States history, students read about the prediction in the 1936 presidential race that Alfred Landon would defeat incumbent President Franklin Delanor Roosevelt. They raise questions as to why that prediction was so far off and research how TV stations can forecast winners of some elections with a very small percentage of the voting results reported. Students contact local radio and TV stations and newspapers to discover how they determine their population sample.

19. Evaluate bias, accuracy, and reasonableness of data in real-world contexts.

- After reading the chapter on sampling in the book *How to Lie With Statistics* by Darryl Huff, students bring in ads, graphs, charts, and articles from newspapers which all make statements or claims allegedly based on data. Students examine the articles for information about the sample and identify those claims which may have little or no substantiation. They also discuss how the sample populations chosen could have influenced the outcomes.

- Students take statements such as "50% of the students failed the test," and "4 out of 5 dentists recommend" and discuss what data they would need to know in order to judge if the conclusions were reasonable. How many students took the test? How many dentists were queried? How were the students or dentists selected? What factors can be identified which would bias the results?

20. Understand and apply measures of dispersion and correlation.

- Students are presented with data gathered by an archaeologist at several sites. The data identifies the number of flintstones found at each site and the number of charred bones. The archaeologist claimed that the data showed that the flintstones were used to light the fires that charred the bones. Students produce a scatterplot, find the correlation between the two sets of figures, and use their work to support or criticize the claim.

- As an assessment activity using their journals, students respond to the claim that children with bigger feet spell better. They discuss whether they believe the claim is true, how statistics might have led to this claim, and whether it has any importance to a philosophy of language teaching.

21. Design a statistical experiment to study a problem, conduct the experiment, and interpret and communicate the outcomes.

- Based on a discussion among some members of the class, a question arises as to which are the most popular cars in the community. The students work in cooperative groups to design an experiment to gather the data, analyze the data, and design an appropriate report format for their results.

- Intrigued by the question *How long would it take dominoes set up one inch apart all the way across the room to fall?*, the class designs an experiment to gather data on smaller sets of dominoes and then extrapolates to estimate the answer.

- Students have just finished a unit in which they discussed the capture-recapture method for estimating the population of wildlife. Part of their assessment for the unit is a project where they work in groups to design and conduct a simulation of the capture-recapture method. One group uses the method to determine the number of lollipops in a large bag.

22. Make predictions using curve fitting and numerical procedures to interpolate and extrapolate from known data.

- Students are presented with this data comparing a student's test grade to the number of hours each studied.

Hours	1	2	3	4	4	6	8	9	10	10	12	12
Grade	60	55	65	65	77	80	83	80	75	90	72	80

Earlier in the year they had produced a line of best fit for the data, but they had recognized that it was not a good model for the data. Now the students use calculators to help them fit a quadratic curve to the data and discuss the advantages it has over the straight line.

- Students conduct an experiment where they suspend a weight on a string from a hook in the doorway. They swing their homemade pendulum and time how long it takes for it to swing 10 times. They had performed this experiment in 8th grade and used the median-median line fit method to model the data. In this revisitation of the problem, the teacher insists they use very short lengths and very long lengths in addition to various ones in between. When the data is graphed, it becomes apparent that the data is not linear and would fit a quadratic curve better. (The median-median line, available on many calculators which have statistics capabilities, is found by dividing the data points on the x-y plane into three equal sets, grouped by x-value, finding a single point for each set whose coordinates are the medians of the respective coordinates of the points in the set, connecting the first and third points by a straight line, and shifting this line 1/3 of the way toward the second point. See *Contemporary Precalculus Through Applications*.)

- Students perform an Introductory Physical Science experiment where water is cooled by adding ice cubes and stirring. A Calculator Based Lab temperature probe is attached to a graphing calculator which is programmed to gather the data. Students (or each group of students) link their calculators to the original one to transfer the data to their calculators. They use the statistics functions to perform a quadratic fit, an exponential fit, and a logarithmic fit, and use the function graph capabilities to determine which is the best model.

- Students work through the *What's My Line* unit described in the *Keys to Success in the Classroom* chapter of the *Framework*. They use median–median and regression lines to estimate the height of a person whose thigh bone was found in a dig.

23. Use relative frequency and probability, as appropriate, to represent and solve problems involving uncertainty.

- After a unit where dependent and independent events were detailed, students are challenged by a problem containing this excerpt from The Miami Herald of May 5, 1983.

  An airline jet carrying 172 people between Miami and Nassau lost its engine oil, power, and 12,000 feet of altitude over the Atlantic Ocean before a safe recovery was made.

  When all three engines' low oil pressure warning lights all lit up at nearly the same time, the crew's initial reaction was that something was wrong with the indicator system, not the oil pressure.

  They considered the possibility of a malfunction in the indication system because it's such an unusual thing to see all three with low pressure indications. The odds are so great that you won't get three indications like this. The odds are way out of sight, so the first thing you would suspect is a problem with the indication system.

  Aviation records show that the probability of an engine failure in any particular hour is about 0.00004. If the failures of three engines were independent, what would the probability be of them failing within one hour? Discuss why the speaker in the article would refer to such a probability as "way out of sight." Discuss situations which might make the failures of three engines not independent events.

- Students keep a record of their trips through the town and whether or not they have to stop at each of the four traffic lights. After one month, the data is grouped and studied. They use their data to determine whether the timing of the lights is independent or not.

- While discussing the issue of mandatory drug testing in social studies, students examine the probability of misdiagnosing people as having AIDS with a test that would identify 99% of those who are true positives and misdiagnose 3% of those who don't have AIDS. They examine situations where the prevalence of the disease is 50%, 10%, and 1% using 100,000 people as a base. They discuss the fact that, at the 1% level, 75% of the people identified as having AIDS would be false positives, the implications that fact has on mandatory testing, and potential ways to improve the predictive value of testing.

24. Use simulations to estimate probabilities.

- Students derive the theoretical probability of winning the New Jersey Pick 6 lottery and then write a computer program to simulate the lottery. The students enter the winning numbers and the computer generates sets of 6 numbers until it hits the winning combination. The computer prints out the number of sets generated including the winning one. Students run the program several times, attempting to verify experimentally the theoretical probability they derived.

25. Create and interpret discrete and continuous probability distributions, and understand their application to real-world situations.

- Students work on a project where they pick one form of insurance (life, car, home), and determine the variables which affect the premiums they would need to pay for this type of insurance and what it would cost for them to obtain it. Using their research, they write an essay summarizing how insurance companies use statistics and probabilities to determine their rates.
- An article in *Consumer Reports* indicates that 25% of 5-lb bags of sugar from a particular company are underweight. The class works with the local supermarket to develop and perform a consumer research project. Each group is given a commodity to study (e.g., potato chips, sugar). They design a method for randomly selecting and testing whether the product matches the claimed specifications or not. They use their data to determine the probability that a randomly selected bag would be underweight.
- Students repeatedly extracted five marbles from a bag containing 10 red and 10 blue marbles, and each time record the number of marbles of each color obtained. They combine the data for the entire class, tabulating the number of times there were 0, 1, 2, 3, 4, and 5 red marbles, and the percentages for each number. They compared their percentages to the theoretical percentages for this binomial distribution, and make the connection to the fifth row of Pascal's triangle.

26. Describe the normal curve in general terms, and use its properties to answer questions about sets of data that are assumed to be normally distributed.

- Students describe a typical student in the school. To do this, they first select a random sample of 30 students in their school. They then survey their sample for information they believe necessary to identify what would be "typical." Finally, they use appropriate displays and descriptive statistics to support their representation of a typical student.
- Students are introduced to the "central limit theorem" through this problem:

  A worker on the assembly line at Western Digital is involved in industrial sabotage by *weakening a soldering joint that causes a hard drive to fail after 5 hours of use. At his station, he actually comes in contact with 30% of the drives produced. The other 70% will last 100 hours. If they are packed randomly in boxes of 36, what would be the average expected lifespan of the drives in the box?*

Students prepare simulations of the problem by repeatedly extracting 36 cubes at random from a bag containing 30 yellow and 70 red cubes and calculating the average expected lifespan for each selection. They discover that the answers fall into a normal curve with a mean of approximately 70 hours.

- Students are given the administrator's summary of the school's standardized tests. Each group is given one area on which to focus. They prepare a presentation they would give to the Board of Education discussing the comparisons between local norms, national norms, suburban norms, urban norms, and independent norms using their understanding of normal distribution, percentile ranks, and graphical displays.

27. Understand and use the law of large numbers (that experimental results tend to approach theoretical probabilities after a large number of trials).

- Students are given two dice, each a different color and roll them repeatedly. For each roll, they record the result for each individual die as well as the total. After a large number of rolls they compare their relative frequencies to the expected outcomes. Then they combine the totals for the entire class and compare the experimental results with the theoretical predictions.
- Students are presented with a paper containing the following gambler's formula: *When playing roulette, bet red. If red does not win, double the bet on red. Continue in this manner.* They evaluate whether the formula makes sense, identify potential problems, and limitations, and discuss the fallacy that the odds improve for red to appear on the next roll every time red doesn't win.

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On-Line Resources

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STANDARD 13 — ALGEBRA

K-12 Overview

> All students will develop an understanding of algebraic concepts and processes and will use them to represent and analyze relationships among variable quantities and to solve problems.

Descriptive Statement

Algebra is a language used to express mathematical relationships. Students need to understand how quantities are related to one another, and how algebra can be used to concisely express and analyze those relationships. Modern technology provides tools for supplementing the traditional focus on algebraic techniques, such as solving equations, with a more visual perspective, with graphs of equations displayed on a screen. Students can then focus on understanding the relationship between the equation and the graph, and on what the graph represents in a real-life situation.

Meaning and Importance

Algebra is the language of patterns and relationships through which much of mathematics is communicated. It is a tool which people can and do use to model real situations and answer questions about them. It is also a way of operating with concepts at an abstract level and then applying them, often leading to the development of generalizations and insights beyond the original context. The use of algebra should begin in the primary grades and should be developed throughout the elementary and secondary grades.

The algebra which is appropriate for all students in the twenty-first century moves away from a focus on manipulating symbols to include a greater emphasis on conceptual understanding, on algebra as a means of representation, and on algebraic methods as problem-solving tools. These changes in emphasis are a result of changes in technology and the resulting changes in the needs of society.

The vision proposed by this *Framework* stresses the need to prepare students for a world that is rapidly changing in response to technological advances. Throughout history, the use of mathematics has changed with the growing demands of society as human interaction extended to larger groups of people. In the same way that increased trade in the fifteenth century required businessmen to replace Roman numerals with the Hindu system and teachers changed what they taught, today's education must reflect the changes in content required by today's society. More and more, the ability to use algebra in describing and analyzing real-world situations is a basic skill. Thus, this standard calls for *algebra for all students*.

What will students gain by studying algebra? In a 1993 conference on *Algebra for All*, the following points were made in response to the commonly asked question, *"Why study algebra?"*

- Algebra provides methods for moving from the specific to the general. It involves discovering the patterns among items in a set and developing the language needed to think about and communicate it to others.
- Algebra provides procedures for manipulating symbols to allow for understanding of the world around us.
- Algebra provides a vehicle for understanding our world through mathematical models.
- Algebra is the science of variables. It enables us to deal with large bodies of data by identifying variables (quantities which change in value) and by imposing or finding structures within the data.
- Algebra is the basic set of ideas and techniques for describing and reasoning about relations between variable quantities.

Standard 8 (Numerical Operations) addressed the need for us to rethink our approach to paper-and-pencil computation in light of the availability of calculators; the need to examine the dominance of paper-and-pencil symbolic manipulation in algebra is just as important. The development of manipulatives, graphing calculators, and computers have made a more intuitive view of algebra accessible to all students, regardless of their previous mathematical performance. These tools permit and encourage visual representations which are more readily understood. No longer need students struggle with abstract concepts presented with very few ties to real-life situations. Rather, the new view of algebra offers real situations for students to examine, to generalize, and to represent in ways which facilitate the asking and answering of meaningful questions. Moreover, inexpensive symbolic processors perform algebraic manipulations, such as factoring, quickly and easily, reducing the need for drill and mastery of paper-and-pencil symbol manipulation.

K-12 Development and Emphases

Algebra is so significant a part of mathematics that its foundation must begin to be built in the very early grades. It must be a part of an entire curriculum which involves creating, representing, and using quantitative relationships. In such a curriculum, algebraic concepts should be introduced in conjunction with the study of patterns and developed throughout each student's mathematical education. The earlier students are exposed to informal algebraic experiences, the more willing they will be to use algebra to represent **patterns**.

The concept of representing **unknown quantities** begins with using symbols such as pictures, boxes, or blanks (i.e., $3 + \square = 7$). It is vital that students recognize that the symbol that is used to represent an unknown quantity has meaning. The only way this can be accomplished is to consistently relate the use of unknowns to actual situations; otherwise, students lack the ability to judge whether their answers make sense.

As students develop their understanding of arithmetic operations, they need to investigate the patterns which arise. Some of these patterns (which are commonly called **properties**) should be initially expressed in words. As the students develop more facility with variables, the properties can be expressed in symbolic form.

In the middle grades, problem situations should provide opportunities to generalize patterns and use additional symbols such as names and literal variables (letters). This development should continue throughout the remainder of the program, ensuring that the relationship between the variables (unknowns) and the quantities they represent is consistently stressed. Middle school students should extend their ability to use algebra to generalize patterns by exploring different types of relationships and by formalizing some of those relationships as **functions**. They should explore and generalize patterns which arise from nature, including non-linear relationships. As students move into the secondary grades, the graphing calculator and graphing software provide tools for examining relationships between x-intercepts and roots, between turning points and maximum or minimum values, and between the slope of a curve and its rate of change. As the student continues through high school, similar experiences should be provided for other functions, such as exponential and polynomial functions; these functions should be introduced using situations to which students can relate.

The use of algebra as a tool to **model real world situations** requires the ability to represent data in tables, pictures, graphs, equations or inequalities, and rules. Through exploration of problems and patterns, students are provided with opportunities to develop the ability to use concrete materials as well as the representations mentioned above. Having students use multiple representations for the same situation helps them develop an understanding of the connections among them. The opportunity to verbally explain these different representations and their connections provides the foundation for more formal expressions.

A fundamental skill in algebra is the **evaluation of expressions** and **the solution of equations and inequalities**. This process will be easier to understand if it is related to situations which give them meaning. Expressions, equations, and inequalities should arise from students' exploration in a variety of areas such as statistics, probability, and geometry. Elementary students begin constructing and solving open sentences. The use of concrete materials and calculators allow them to explore solutions to real-life situations. Gradually, students are led to expand these informal methods to include graphical solutions and formal methods. The relationship between the solutions of equations and the graphs of the related functions must be stressed regularly.

IN SUMMARY, there are algebraic concepts and skills which all students must know and apply confidently regardless of their ultimate career. To assure that all children have access to such learning, algebraic thinking must be woven throughout the entire fabric of the mathematics curriculum.

NOTE: *Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

Standard 13 — Algebra — Grades K-2

Overview

Students can develop a strong understanding of algebraic concepts and processes from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in algebra, as identified in the K-12 Overview, are: **patterns, unknown quantities, properties, functions, modeling real-world situations, evaluating expressions** and **solving equations and inequalities.**

Students begin their study of algebra in grades K-2 by learning about the use of pictures and symbols to represent variables. They look at **patterns** and describe those patterns. They begin to look for **unknown numbers** in connection with addition and subtraction number sentences. They **model** the relationships found in real-world situations by writing number sentences that describe those situations. At these grade levels, the study of algebra is very much integrated with the other content standards. Children should be encouraged to play with concrete materials, describing the patterns they find in a variety of ways.

People tend to learn by identifying patterns and generalizing or extending them to some conclusion (which may or may not be true). A major emphasis in the mathematics curriculum in the early grades should be the opportunity to experience numerous **patterns**. The development of algebra as a language should build on these experiences. The ability to extend patterns falls under Standard 11 (Patterns and Functions), but having students communicate their reasoning is also an algebra expectation. Initially, ordinary language and concrete materials should be used for communication. As students grow older and patterns become more complex, students should develop the ability to use tables and pictures or symbols (such as triangles or squares) to represent numbers that may change or are **unknown** (variable quantities).

The primary grades provide an ideal opportunity to lay the foundation for the development of the ability to represent situations using **equations or inequalities** (open sentences) and solving them. Students can be asked to communicate or represent relationships involving concrete materials. For example, two students might count out eight chips and place them on a mat. One of the students then places a margarine tub over some of the counters and challenges the other student to figure out how many chips are hidden under the tub. A more complex situation might involve watching the teacher balance a box and two marbles with six marbles. The students draw a picture of the situation, and try to decide how many marbles would balance the box by physically removing two marbles from each side of the balance. In a problem involving an inequality, students might be asked to find out how many books Jose has if he has more than three books but fewer than ten. Situations from the classroom and the students' real experiences should provide ample opportunities to construct and solve such open sentences.

As operations are developed, students need to examine **properties** and make generalizations. For example, giving students a set of problems which follow the pattern $3 + 4$, $4 + 3$, $1 + 2$, $2+1$, etc. should provide the opportunity to develop the concept that order does not affect the answer when adding (the commutative property). After students understand that these properties are not necessarily true for all operations (e.g., $5 - 2$ is not equal to $2 - 5$), the teacher should mention that the properties are important enough to be given names. However, the focus of this work should be on using the properties of operations to make work easier rather than on memorizing the properties and their names.

Students in grades K-2 spend a great deal of time developing meaning for the arithmetic operations of addition, subtraction, multiplication, and division. As they work toward understanding these concepts, they focus on developing **mathematical models** for concrete problem situations. The number sentences that they write to describe these problem situations form a foundation for more sophisticated mathematical models.

Standard 13 — Algebra — Grades K-2

Indicators and Activities

The cumulative progress indicators for grades K-2 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and in grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Understand and represent numerical situations using variables, expressions, and number sentences.**

   - Students represent a problem situation with an open sentence. For example: *If there are 25 students in the class and Marie brought 26 cookies for snack, how many will be left over? (26 - 25 = ?)* Another example might be: *We have 10 cups left in the package and there are 25 children in the class, so how many more cups do we need to get? (10 + ? = 25)*
   - Students read *The Doorbell Rang* by Pat Hutchins. They act out the story and realize that many different combinations of students can share 12 cookies equally.
   - Students make a table relating the number of people and the number of eyes. They use a symbol such as a stick figure to represent the number of people and a cartoon drawing of an eye to represent the number of eyes and then express the relationship between them.

     ♀ + ♀ → ☼ + ☼ + ☼ + ☼

2. **Represent situations and number patterns with concrete materials, tables, graphs, verbal rules, and number sentences, and translate from one to another.**

   - Students in groups are given a container to which they add water until its height is 5 centimeters, measured with Cuisennaire rods. They add marbles to the container until the height of the water is 6 centimeters. They continue adding marbles, recording each time the number of marbles it takes to raise the water level one centimeter. They describe the relationship between the number of marbles added and the height of the water.
   - In regular assessment activities, students look at a series of pictures which form a pattern. They draw the next shape, describe the pattern in words, and explain why they chose to draw that shape.
   - Using a calculator, students play *Guess My Rule.* The lead student enters an expression such as *5+4* and presses the = key; she shows only the answer to her partner. The second student tries to guess the rule by entering different numbers, one at a time, pressing the = key after each number. The calculator, after each = is pressed, should show the sum of the entered number and the second addend (in this case, 4). (Some calculators perform this function differently; see the user's manual for instructions.) When the second student thinks she knows the pattern (in this case, *adding 4*), she makes a guess. The pattern is written in words and then as a rule using a picture or symbol for

the variable (the number which the second student enters).

- Placing four different-colored cubes in a can, students predict which color would be drawn out most often if each child draws one cube without looking. The teacher helps the students keep track of their results by making a chart with the colors on the horizontal axis and the number of times a color is drawn on the vertical axis. As students select cubes, an "x" is placed above the color drawn, forming a frequency diagram. After several turns, the students describe the patterns they see in the graph.
- Students read *Ten Apples Up on Top!* by Theo Le Sieg and discuss the mathematical comparisons and equations that appear in the story.

3. Understand and use properties of operations and numbers.

- Students are given five computational problems to solve. They are permitted to use the calculator on only two of them. Two of the problems are related to another two by operation properties (e.g., $3 + 2$ and $4 + 6$ are related to $2 + 3$ and $6 + 4$ by the commutative property) and the last involves a property of number such as adding *0*. Students share their thought processes in a follow up discussion.
- The second grade teacher has a box containing slips of paper with open sentences such as $25 - 8 = \square$ or $15 + \square = 23$. Students draw out a slip and tell or write a story which would involve a situation modeled by the sentence.
- Students discover that, since the order of the numbers when adding them is not important, they can solve a problem like $3 + 8$ by starting with 8 and counting up 3, as well as by starting with 3 and counting up 8.
- In their math journals, students write their reactions to the following situation:

  Sally just used her calculator to find out that 324 + 486 was equal to 810. In another problem, she must find the answer to 486 + 324. What should she do? Why?

4. Construct and solve open sentences (example: $3 + \square = 7$) that describe real-life situations.

- Kindergarten students play the *hide the pennies* game. The first player places a number of pennies (say 7) on the table and lets the other player count them. The first player covers up a portion of the pennies, and the second player must determine how many are covered. They may represent the situation with markers or pictures to help them. Some second-grade students are ready to write a number sentence that describes the situation.
- Students are given a bag with Unifix cubes. They are told that the bag and 2 cubes balance 7 cubes. They use a balance scale to find how many cubes are in the bag.

References

Hutchins, Pat. *The Doorbell Rang*. Mulberry Books, 1986.

Le Sieg, Theo. *Ten Apples Up on Top!* New York, NY: Random House, 1961.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 13 — Algebra — Grades 3-4

Overview

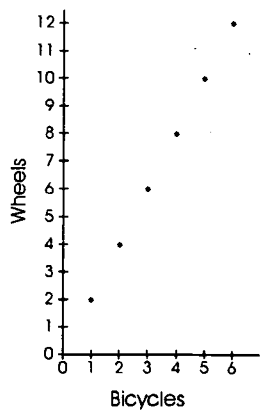
Students can develop a strong understanding of algebraic concepts and processes from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in algebra, as identified in the K-12 Overview, are: **patterns, unknown quantities, properties, functions, modeling real-world situations, evaluating expressions** and **solving equations and inequalities.**

In grades K-2, students use pictures and symbols to represent variables, generalize patterns verbally and visually, and work with properties of operations. Although the formality increases in grades 3 and 4, it is important not to lose the sense of play and the connection to the real world that were present in earlier grades. As much as possible, real experiences should generate situations and data which students attempt to generalize and communicate using ordinary language. Students should explain and justify their reasoning orally to the class and in writing on assessments using ordinary language. When introducing a more formal method of communicating, such as the language of algebra, it is helpful to revisit some of the situations used in previous grades.

Since algebra is the language of patterns, the mathematics curriculum at this level needs to continue to focus on **patterns**. The use of letters to represent **unknown quantities** should gradually be introduced as a replacement for pictures and symbols. The use of **function** machines permits the introduction of letters without the need to move to formal symbolic algebra. Since they have had the opportunity to experience real function machines such as the calculator or a gum bank, where one penny yields two pieces of gum, the notation of function machines should make sense.

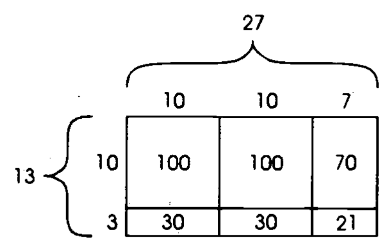
Here the box is thought of as the function machine *times 2* which takes in a number "a" and produces a number "b" which is *twice* "a." Students can use such symbols to communicate their generalization of patterns. They put two or more machines together making a composite function; for example, they can follow the *times 2* machine with an *add 3* machine. They determine not only what each input produces but also what input would produce a given output.

Students should continue to communicate their generalizations of patterns through ordinary language, tables, and concrete materials. Graphs should be introduced as a method for quickly and efficiently representing a pattern or function. They should develop graphs which **represent real situations** and be able to describe patterns of a situation when shown a graph. For example, when given the graph at the right which shows the relationship between the number of bicycles and wheels in the school yard, they should be able to describe the relationship in words.



Students in grades 3 and 4 should continue to use **equations and inequalities** to represent real situations. While variables can be introduced through simple equations such as $35 \div n = 5$, students should be viewing variables as place holders similar to the open boxes and pictures they have already used. At these grade levels, they need not use variables in more complicated situations. Given a situation such as determining the cost of each CD if 5 of them plus $3 tax is $23, they should be permitted to represent it in whatever way they feel comfortable. Students should be able to use, explain, and justify whatever method they wish to solve equations and inequalities. Some may continue to use concrete materials for some situations; they might count out 23 counters, set aside 3 for the tax, and divide the remainder into 5 equal piles of 4. Others might try different numbers until they find one that works. Some students may write $23 - 3 = 20$ and $20 \div 5 = 4$. Still others may want to relate this to function machines and figure out what had to go in for $23 to come out. It is important for students to see the diversity of approaches used and to discuss their interrelationships.

Students should continue to examine the **properties of operations** and use them whenever they would make their work easier. There are some excellent opportunities for providing a foundation for algebraic concepts in these grades. For example, explaining two-digit multiplication by using the area of a rectangle (see figure below illustrating 13 x 27) provides the student with a foundation for multiplication of binomials, the distributive property, and factoring. While the teacher at this grade level should focus on the development of the multiplication algorithm, the teacher of algebra several years later will be able to build on this experience of the student.



Standard 13 — Algebra — Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. **Understand and represent numerical situations using variables, expressions, and number sentences.**

   - Students do comparison shopping based on items that are for sale in multiples. For instance: *If chewing gum is sold at 3 packs for 85 cents (p = 85/3), is that a better or a worse buy than a single pack for 30 cents (p = 30)?* They sort their examples into groups where the multiple buy is a better deal, the same deal, or a worse deal than the single package deal.
   - One student has been folding origami cranes to send to Hiroshima for Peace Day in August. He brings the 47 cranes that he has folded so far to class and asks for help to fold many more. The class decides to have each of the 26 students fold one crane each week for the rest of the school year. The teacher asks groups of students to find a way to determine how many cranes will be in the collection after some given number of weeks. She starts off the discussion by having students list the numbers for the first few weeks:

     47,
47 + 26,
47 + 26 + 26,
47 + 26 + 26 + 26,
and so on.

     They figure out whether they can reach 500 cranes by the end of the year.

2. **Represent situations and number patterns with concrete materials, tables, graphs, verbal rules, and number sentences, and translate from one to another.**

   - Students compare two allowance plans. Plan A provides an allowance of $5 the first week and adds $1 each week. Plan B starts with 1¢ and doubles the allowance each week. Using calculators, students make tables listing the number of the week, the amount of allowance under Plan A and the amount of the allowance under plan B. They complete several rows of the table so that they understand what is happening with each plan and see that Plan B soon overcomes Plan A. They might want to try to use physical objects such as centimeter cubes to demonstrate the behavior of the two plans visually.
   - Each student is given an even number of square tiles and asked to use them all to make a rectangle with two columns. Students are asked to notice that the heights of the rectangles are different for different starting numbers of tiles. They collect the data into a table, giving each student's name, the number of tiles used, and the height of the rectangle.

They understand that the number of tiles is the area and can figure out the height of the rectangle if they know the number of tiles that are used — that is, they can verbalize that the height of the rectangle is half the total number of tiles.

- Students read *1 Hunter* by Pat Hutchins, wherein a determined hunter looks and looks and looks for animals but sees none, even though the reader can clearly see 2 elephants, 3 giraffes, 4 ostriches, and so on, up to 10 parrots. They are asked how many animals in all the hunter was unable to see. Students use graphs, concrete materials, pictures, and number sentences to express their understanding of the situation.

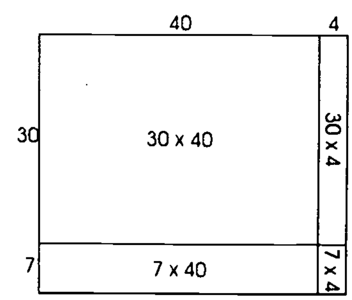
- Students play *Guess My Rule* by suggesting inputs and having the *rule-maker* (the teacher or a student) put the corresponding outputs into a table like this one:

Input	Output
3	7
1	4
16	19
.	.
.	.

Students should always be challenged to show they understand the rule by giving a verbal explanation of it. Partially filled-in *Guess My Rule* tables are a good assessment technique to evaluate the students' inductive reasoning power and their level of comfort with arithmetic operations.

3. Understand and use properties of operations and numbers.

- When the students are introduced to two-digit by two-digit multiplication, they begin with a problem of finding the area of a rectangular field which is 37 feet by 44 feet. They know they need to multiply the numbers to find the area, but they don't know how to multiply without calculators. The teacher draws a rectangle and uses a line to divide the width into two regions which are 30 feet and 7 feet. She does the same with the length, cutting it into lengths 40 feet and 4 feet. This divides the rectangle into four smaller rectangles (*30×40, 30×4, 7×40, 7×4*) all of which are multiplications the students can do.



37 x 44 = (30 + 7)(40 + 4)

- Lea and Suzanne discovered a method for multiplying even numbers by six easily. Their method, applied to the example *6×24*, is:

Cut the other even number in half	12
Add a zero	120
Add the number	120+24=144

When they told their classmates their discovery, they were stumped when they were asked why it worked. The teacher, grasping the *teachable moment*, divided the class into groups and challenged them to do a few examples using the girls' method and try to figure out and explain why it worked.

- Students and teacher together work through Robert Froman's book, *The Greatest Guessing Game: A Book about Dividing* to reinforce their notions of division.

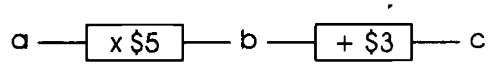
- Students explain that they solved a problem like *300 - 56* mentally by first subtracting *50* and then subtracting *6*, since that is the same as subtracting *56*. They also do $25 \times 7 \times 4$ by first multiplying 25×4 and then multiplying by *7*. Such simplifications will give a good foundation for later work in algebra.

4. Construct and solve open sentences (example: 3 + □ = 7) that describe real-life situations.

- In an assessment situation, groups of students are asked to describe in words the situation of four people sharing a five dollar bill found on the way to school, and then to transform it to symbolic form using pictures, symbols or letters.

- Students want to help the New Jersey environment and raise money at the same time. They discover that in two bordering states (New York and Delaware), plastic soda bottles can each be turned in for a 5¢ refund. They write an equation which represents the amount of money they will receive for *b* bottles. Students answer questions such as *How much money will we get for 25 bottles?* and *How many bottles will we need to make $10?*

- Students are presented with a function machine representing the situation of buying music tapes for $5 each through the mail and paying a $3 shipping and handling charge for the order. They answer questions such as *How much would it cost for 5 tapes?* and *How many tapes were bought if the bill was $43?*



References

Fromer, Robert. *The Greatest Guessing Game: A Book About Dividing.* New York, NY: Thomas Y. Crowell Publishers, 1978.

Hutchins, Pat. *1 Hunter.* New York, NY: Greenwillow Books, 1982.

On-Line Resources

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Standard 13 — Algebra — Grades 5-6

Overview

Students can develop a strong understanding of algebraic concepts and processes from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in algebra, as identified in the K-12 Overview, are: **patterns, unknown quantities, properties, functions, modeling real-world situations, evaluating expressions** and **solving equations and inequalities.**

It is important that students continue to have *informal* algebraic experiences in grades 5 and 6. Students have previously had the opportunity to generalize patterns, work informally with open sentences, and represent numerical situations using pictures, symbols, and letters as variables, expressions, equations, and inequalities. At these grade levels, they will continue to build on this foundation.

Algebraic topics at this level should be integrated with the development of other mathematical content to enable students to recognize that algebra is not a separate branch of mathematics. Students must understand that algebra is an expansion of the arithmetic and geometry they have already experienced and a tool to help them describe situations and solve problems.

Students should use algebraic concepts to investigate situations and solve interesting mathematical and **real world** problems. There should be numerous opportunities for collaborative work. Since algebra is the language for describing **patterns**, students should have regular and consistent opportunities to discuss and explain their use of these concepts. They should write generalizations of situations in words as well as in symbols. To provide such opportunities, the activities should move from a concrete situation or representation to a more abstract setting. Students at this level can begin using standard algebraic notation to represent known and **unknown quantities and operations**. This should be developed gradually, moving them from the previous symbols in such a way that they can appreciate the power and elegance of the new notation.

Students need to learn how variables are different from numbers (a variable can represent many numbers simultaneously, it has no place value, it can be selected arbitrarily) and how they are different from words (variables can be defined in any way we want and can be changed without affecting the values they represent). Students need to see variables (letters) used as names for numbers or other objects, as unknown numbers in equations, as a range of unknown values in inequalities, as generalizations in pattern rules or formulas, and as characteristics to be graphed (independent and dependent variables).

An **algebraic expression** involves numbers, variables, and operations such as $2b$, $3x - 2$, or $c - d$. In fifth and sixth grade, students should begin to become familiar with the common notational shortcut of omitting the operation sign for multiplication, so that when $b=3$, $2b$ equals 6 and not 23. Thus they recognize that there are slightly different rules for reading expressions involving variables than those involving only numbers.

Students in grades 5 and 6 should focus on understanding the role of the equal sign. Because it is so often used to signal the answer in arithmetic, students may view it as a kind of operation sign — a "write the answer" sign. They need to come to see its role as a relation sign, balancing two equal quantities.

Students should develop the ability to **solve simple linear equations** using manipulatives and informal methods. With the appropriate background, students at grades 5 and 6 have the ability to find the solution of an equation, such as *7* for $x+5=12$, whether they use manipulatives, a graph, or any other method. It is imperative that in the discussion of the solution of an equation, the many methods in obtaining that solution are described.

Students in grades 5 and 6 should use concrete materials, such as algebra tiles, to help them develop a visual, geometric understanding of algebraic concepts. For example, students can represent the expression $3x - 2$ by using three strips and two units. They should make graphs on a rectangular coordinate system from data tables, analyze the shape of the graphs, and make predictions based on the graphs. Students should have opportunities to plot points, lines, geometric shapes, and pictures. They should use variables to generalize the formulas they develop in studying geometry (e.g., $p = 4s$ *for a square or* $A = l \times w$ *for a rectangle*). Students should be able to describe movements of objects in the plane through horizontal and vertical slides (translations). They should experiment with probes which generate the graphs of experimental data on computers or graphing calculators. The majority of this work will be with graphs that are straight lines (linear functions), but students should have some experience seeing other shapes of graphs as well; in particular, when dealing with real data and probes, many times the graph will not be linear.

Standard 13 — Algebra — Grades 5-6

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

5. Understand and use variables, expressions, equations, and inequalities.

- Students find the perimeter of one square, two squares connected along an edge, three squares connected along their edges, and so forth, as shown below. The length of one side of a square is assumed to be one unit.

  They make a table of values and use it to determine a function rule which describes the pattern. They understand the rule $P = 2 \times s + 2$, and use it to predict the perimeter of ten squares.

- Students write a Logo program to draw a rectangle of any dimensions using the variables :LENGTH and :WIDTH.

- In a health unit, students are studying the dietary needs for maintaining healthy bodies. The teacher provides guidelines for the maximum amounts of fat and cholesterol at any meal. Each group of students chooses two foods for a meal. They determine the fat content per unit of each of the foods and the cholesterol amount per unit and make inequalities which relate these unit values, the number of units, and the given maximum amounts. The students determine possible combinations of the amounts of the two foods whose fat and cholesterol would still be acceptable. The teacher uses a function graphing computer program or graphing calculator to represent visually the acceptable amounts.

6. Represent situations and number patterns with concrete materials, tables, graphs, verbal rules, and standard algebraic notation.

- Each group of students is given a *Mr.* or *Mrs. Grasshead* (i.e., a sock filled with dirt and grass seed which sits in a dish of water). They create a name for their grasshead and begin a diary, recording the number of days that have passed and the height of the grass. At the end of specified time periods, they discuss the changes in the height, the average rate of change over the time period, and the overall behavior of the grass growth. Each group makes a graph of height versus the number of days. The students note whether the graph is close to a straight line.

- Students find the number of tiles around the border of a floor 10 tiles long and 10 tiles wide by looking at smaller square floors, making a table, and identifying a pattern. They describe their pattern in words and, with assistance from the teacher, develop the expression $(4 \times n) + 4$ for the number of border tiles needed for an $n \times n$ floor.

- Students play *Guess My Rule* by suggesting input and having the *rule-maker* (the teacher or a student) put the corresponding output into a table like this one:

Input	Output
3	9
1	5
16	35
.	.
.	.
.	.
n	2n+3

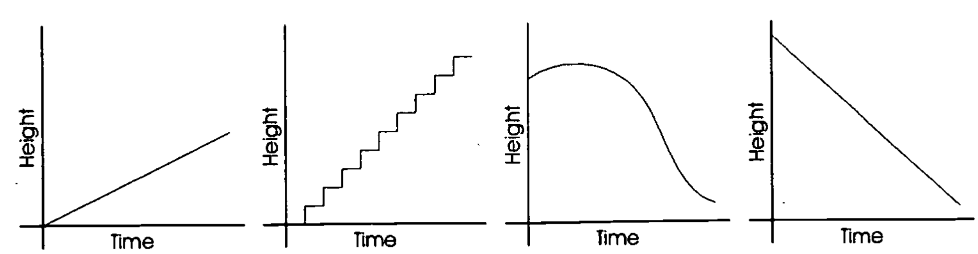
Students should always be challenged to show they understand the rule by filling in the last row for an input of *n*. Partially filled-in *Guess My Rule* tables are also a good assessment technique to evaluate students' inductive reasoning power and their ability to use standard algebraic notation to express relationships.

- Students use play money to act out the following situation and solve the problem.

  A man wishes to purchase a pair of slippers marked $5. He gives the shoe salesman a $20 bill. The salesman does not have change for the bill so he goes to the pharmacist next door and gets a $10 and two $5 bills. He gives the customer his change and the man leaves. The pharmacist enters shortly after and complains the $20 was counterfeit. The shoe salesman gives her $20 and gives the counterfeit bill to the FBI. How much did the shoe salesman lose?

- Students place 8 two-color chips in a paper cup and toss them ten times, recording the number of red and yellow sides showing on each toss. For each red chip that shows, they lose $1. For each yellow, they win $1. For each toss, the students write a number sentence that shows their win or loss for that toss. For example, after tossing 3 yellows and 5 reds, their sentence would read $3 - 5 = -2$. Afterwards, the students look for patterns in the number sentences that they have written. They discuss these patterns and then write about them in their notebooks.

- In both classroom and assessment situations, students interpret simple non-numeric graphs and decide what kinds of relationships they demonstrate. For example: *Which of the following graphs would show the relationship between the height of a flag and time as a boy scout raised the flag on a flagpole?*



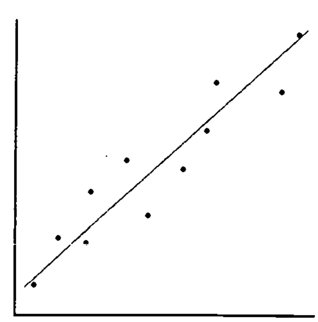
- Students work through the *Powers of the Knight* lesson that is described in the Introduction to this *Framework*. They learn that doubling the number of coins on consecutive squares of a chessboard results in a rapidly increasing sequence of numbers — the powers of 2.
- Students read *Anno's Mysterious Multiplying Jar* by Mitsumasa Anno and try to analyze and represent the numerical patterns shown using variables.

7. Use graphing techniques on a number line to model both absolute value and arithmetic operations.

- Students use number lines to demonstrate addition of integers. They point to the number representing the amount currently in the bank and then slide their finger in the appropriate direction (right for deposits and left for withdrawals) over the distance indicated by the second amount. As they slide their finger, they use arrows to track their movements over the number line, and the teacher keeps track of the operations using positive and negative integers. Through this dual representation, students begin to understand the relationship between the addition of integers and movement along the number line.
- Students are given a variety of objects whose dimensions they must determine. They are given a number line marked from 5 to 27 to simulate a broken yardstick. Students work in pairs to develop a process for determining the lengths of each of the objects to the nearest unit. After they have a workable method and have written an explanation of it in their journal, the teacher replaces the tape with another which is marked from -10 to 10. The students repeat their effort. This process helps them develop an understanding of subtraction of integers and the relationship between the operation of subtraction and distance on the number line.
- Addition and subtraction of signed numbers is explored using two-colored disks and a number line. Red is used to represent positive numbers and yellow is used to represent negative numbers. When given a problem such as $-3 + 5$ the students place 3 yellows and 5 reds on the table. They pair up as many red and yellow disks as they can and remove them from the table. In the case of the example, 3 red and yellow disks would be paired and removed, leaving 2 red disks which represents the sum, +2.

8. Analyze tables and graphs to identify properties and relationships.

- Students use tables or two-color chips to help them solve the following problem: *A classroom has 25 lockers in a row. The first person to enter the room opened every locker. The second person closed every other locker beginning with the second locker. The third person started with the third locker and changed every third locker from open to closed or closed to open. This continued until 25 people had passed through the room. Which lockers would be open after the 25th person walked into the room?*



- A plastic rectangular shape is exhibited on the overhead. The lengths of both sides of the image, and the distance from the screen to the overhead are measured. The overhead is moved and the process is repeated so that measurements are taken at six to ten different distances. One group of students is responsible for determining the relationship between the distance from the screen and the length of one side of the image. A second group is responsible for studying the relationship between the distance from the screen and the area of the image. Each group makes a scatterplot of its data and eyeballs a line of best fit using a piece of spaghetti. They then use the graph to answer questions about the relationship between distance and length or distance and area. They also develop a summary statement describing the relationship.

- Students make pendulums using strings of length 64, 32, 16, and 8 cm with a washer at one end and a screw eye or ruler at the other. The strings are swung from a constant height and the number of swings in 30 seconds is recorded. A graph is made plotting the number of swings against the string length. Students study the results and determine if there might be a pattern they could continue. They attempt to answer questions such as: *Will the number of swings ever reach zero?* (This activity is a good one to repeat at later grades since the relationship appears linear but when very short lengths and very long lengths are used, it becomes clear that it is actually a quadratic relationship.)
- Students are given the times of the Olympic 100-meter freestyle swimming winners both in the men's event and the women's event. Using different colors for the two genders, they produce a scatterplot and use a piece of spaghetti to eyeball a line of best fit for each set of data. They use their lines to determine times in the years not given (when no Olympics were held) and to predict times in the years beyond those they were given. They also determine if the data supports the assertion that the women will some day swim as fast as the men and predict from their lines when that would happen.

9. Understand and use the rectangular coordinate system.

- Students are paired to play a game similar to battleship in which they attempt to determine where the two lines their opponent has drawn intersect. Both students draw axes which go from -10 to 10 in both the x- and y-directions. They sit so that neither can see the other's paper. The first player draws two lines which intersect at a point with integer coordinates and colors the four regions different colors. The second player gives the coordinates of a point. The first player responds with the color of the region the point is in, or that the point is on a line, or that it is *the* point! The second player keeps a record of his guesses on his axes and continues guessing until the chosen point is determined.
- Students keep track of the high and low temperatures for a month in two different colors on a graph. The horizontal axis represents the day of the month and the vertical axis represents the temperature. At the end of the month, they connect the points making two broken-line graphs. They use their graphs to discuss the temperature variations of that month and to determine the overall "high" and "low" for the month.
- Students consider what happens if they start with two bacteria and the number of bacteria doubles every hour. They make a table showing the number of hours that have passed and the number of bacteria and then plot their results on a coordinate graph.
- Students draw broken-line pictures in a cartesian plane and identify the coordinates of critical points in the pictures. Their partners attempt to re-create the picture using the coordinates of the critical point and verbal descriptions of how the critical points should be connected.

10. **Solve simple linear equations using concrete, informal, and graphical methods, as well as appropriate paper-and-pencil techniques.**

- Students use algebra tiles to solve an equation. For example, they represent the equation $3x + 2 = 5x$ by placing three strips and two units on one side of a picture of a balance beam and five strips on the other side. They decide that the balance will stay even if they take the same number of objects off both sides, so they take three strips off both sides and have two units balanced with two strips. Then they correctly decide that 1 unit must balance one strip.

- Students want to use their class fund as a donation to a town in Missouri devastated by the summer floods. They agree that each of the 26 students in the class is going to contribute 25¢ a week. The fund already contains $7. The students develop the expression $\$6.50 \times W + \$7$ as the amount of money in the fund at the end of W weeks. The teacher asks them how much they would like to send to the town, and the students agree on $100. The teacher then asks them to write an equation which would say that the amount of money after W weeks was $100. The students write $\$6.50 \times W + \$7 = \$100$. Finally, the students try a number of different strategies for finding out what W should be. Some of the students use calculators in a guess-and-check method. Some students go to the computer and use a spreadsheet to generate the amounts for different weeks until the total is more than $100. Others express the relationship as a composite function using function machines and then use the inverse operations to subtract $7 and then divide by $6.50. They use their calculators to carry out the division. The teacher discusses all these methods and introduces the traditional algebraic shorthand method for solving the problem.

11. **Explore linear equations through the use of calculators, computers, and other technology.**

- Using a motion sensor connected to a graphing calculator or computer, the class experiments with generating graphs which represent the distance from the sensor against time. They discover that if they walk at a fixed rate the graph is a straight line. They try walking away from the sensor at different fixed rates of speed to determine what effect the speed has on the line. They start at different distances from the sensor to see the effect that has. They try walking toward the sensor and standing still. Students discuss the relationships between the lines they are generating and the physical activity they do. As an assessment, the teacher has individual students walk so as to generate a straight line. The students are then asked to write in their journals what someone would have to do to produce a line which was less steep. After closing their journals, individual students are provided an opportunity to verify their conclusions using the graphing calculator.

- After measuring several students' heights and the length of the shadows they produce, the data is entered in a spreadsheet, computerized statistics package, or graphing calculator. A scatterplot is formed from the data and the students see that the plot is approximately linear. The technology is used to produce a line of best fit which the class uses to determine heights of unknown objects (such as a flagpole) and the length of the shadow of objects with known heights.

12. **Investigate inequalities and nonlinear equations informally.**

- Students explore patterns involving the sums of the odd integers $(1, 1+3, 1+3+5, \ldots)$ by using small squares to make Ls to represent each odd number and then nesting the Ls. They make a table that shows how many Ls are nested and the total number of squares used.

They look for a pattern that will help them predict how many squares will be needed if 10 Ls are nested (i.e., if the first 10 odd numbers are added together). They make a prediction and describe the basis for their prediction (e.g., when you added the first 3 odd numbers, and placed the three Ls together, they formed a square that was three units on a side, so when you add the first 10 odd numbers, that should make a square that is ten units on a side and whose total area is 100 squares.) They share their solution strategies with each other and develop an expression that can be used to find the sum of the first n odd numbers (i.e., $n \times n$).

- Students set up a table listing the length of the sides of various squares (x) and their areas (y). Some students use the centimeter blocks to help them find the values. The teacher completes a table of values in a function graphing computer package on the class computer which has an LCD panel for overhead projection or on an overhead version of a graphing calculator. When the students have finished completing the table, the teacher turns on the overhead and displays her table. The students check their answers and ask questions. The teacher graphs the data on the computer or calculator, and the students use the graph to answer questions such as *If the side was 3.5 cm, what would the area be?* and *If the area was 60 square centimeters, what would the side be?* The teacher uses the trace function to identify the points being discussed.
- Students explore inequality situations such as: *I have $150. How many more weeks would I need to save my $15 allowance to buy a stereo that costs $200?* They represent the relationship as an inequality, both in words and in symbols, and use play money, base ten blocks, graphs, or trial and error to solve the problem.

13. Draw freehand sketches of, and interpret, graphs which model real phenomena.

- Students keep track of how far they are from home during one specified day. They draw a graph which represents the distance from home against the time of day and write an explanation of their graph in relation to their actual activities on that day.
- Students are presented with a graph representing a student's monthly income from performing lawn care for people over the past year. The graph shows no income during the months of November, December, and March. They write a story which explains the behavior of the graph in terms of the need for services over the course of the year.

References

Anno, Mitsumasa. *Anno's Mysterious Multiplying Jar.* Philomel Books, 1983.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 13 — Algebra — Grades 7-8

Overview

Students can develop a strong understanding of algebraic concepts and processes from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in algebra, as identified in the K-12 Overview, are: **patterns, unknown quantities, properties, functions, modeling real-world situations, evaluating expressions** and **solving equations and inequalities.**

Students in grades 7 and 8 continue to explore algebraic concepts in an informal way. By using physical models, data, graphs, and other mathematical representations, students learn to generalize number **patterns** to model, represent, or describe observed physical patterns, regularities, and problems. These informal explorations help students gain confidence in their ability to abstract relationships from contextual information and use a variety of representations to describe those relationships. Manipulatives such as algebra tiles provide opportunities for students with different learning styles to understand algebraic concepts and manipulations. Graphing calculators and computers enable students to see the behaviors of **functions** and study concepts such as slope.

Students need to continue to see algebra as a tool which is useful in describing mathematics and solving problems. The algebraic experiences should develop from **modeling situations** where students gather data to solve problems or explain phenomena. It is important that all concepts are presented within a context, preferably one meaningful to students, rather than through traditional symbolic exercises. Once a concept is well-understood, the students can use traditional problems to reinforce the algebraic manipulations associated with the concept.

Many activities which are used in earlier grades should be revisited as students become more sophisticated in their use of algebra. At the same time, activities used in later grades can be incorporated on an informal basis. For example, students in earlier grades might have gathered the heights and armspans and attempted to generalize the relationship between them in words. As students became familiar with the rectangular coordinate system, they might have generalized the relationship using a scatterplot and fitting a line to the data. In seventh and eighth grade, students might be taught how to find the median-median line to determine the line of best fit and use that line to solve problems. In later grades, when students have learned to find the slope of a line through two points symbolically, they can determine the equation of the median-median line. (The median-median line, available on many calculators which have statistics capabilities, is found by dividing the data points on the x-y plane into three equal sets, grouped by x-value, finding a single point for each set whose coordinates are the medians of the respective coordinates of the points in the set, connecting the first and third points by a straight line, and shifting this line 1/3 of the way toward the second point.)

Students should have numerous opportunities to develop an understanding of the relationship between a function and its graph. A limited number of functions should be plotted by hand, but students should also use technology to graph functions. While linear relationships should be the focus, inequalities and nonlinear functions should be explored as well. Students should develop an understanding of the relationship between solutions of equations and graphs of functions. For example, the solution of the equation $3x - 4 = 5$ can be found by plotting $y = 3x - 4$, tracing along the function until a y-value of 5 is found, and then determining the corresponding x-value. Students should develop the ability to find solutions using the trace function of graphing calculators and computer graphing programs and discuss how it assists in solving equations. They should also have opportunities to use spreadsheets as a method for representing and solving problems.

Students should be able to **evaluate expressions** using all forms of real numbers when calculators are available. They should have developed an understanding of the importance of the algebraic order of operations and be able to correctly evaluate expressions using it. It is imperative that students understand that they cannot blindly accept answers produced on the calculator. They should recognize that a standard four-function calculator does not use the standard order of operations. They should recognize that even with a scientific calculator, operations such as the division of two binomial quantities requires the use of parentheses.

Students should refine their ability to solve **simple linear equations** (i.e., $ax+b=cx+d$). Students may continue to use informal, concrete, and graphic methods but should begin to link these methods to more formal symbolic methods. As students have opportunities to explore interesting problems, applications, and situations, they need to be encouraged to reflect on their explorations and summarize concepts, relationships, processes, and facts that have emerged from their discussions. Developing a suitable notation to describe these conclusions leads naturally to a more formal, more symbolic view of algebra.

Standard 13 — Algebra — Grades 7-8

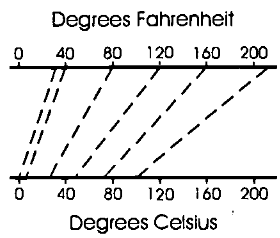
Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

5. Understand and use variables, expressions, equations, and inequalities.

- Students make a model of the relationship between Celsius and Fahrenheit temperatures. They represent the relationship as a formula, and check the formula against two known data points — 0°C = 32°F and 100°C = 212°F. Students use their formula to convert between Celsius and Fahrenheit temperatures.



- Students examine the following situation, making a table for the first few months to gain an understanding of the pattern involved in the problem.

  Juanita opened a checking account and deposited $500. She works as a part-time engineer's assistant in a local firm and will receive a check for $130 on the 1st and 15th of each month. She intends to take $40 from each paycheck for cash expenses and then deposit the remainder. On the 15th of each month, she will write a check for $220 to cover the cost of her car payment.

  The students develop an equation that describes how much money is in the account on the 1st and the 15th of each month. They use their equation to determine the amount for various months in the future as well as to find out when Juanita will overdraw the account. Students use their equations and the information they have found to write a letter to Juanita explaining why her plan is not financially sound and what she might do to correct it.

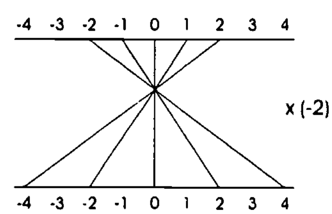
6. Represent situations and number patterns with concrete materials, tables, graphs, verbal rules, and standard algebraic notation.

- The seventh grade is preparing for a skiing trip. The interdisciplinary team has decided to integrate the planning for the trip into all the courses. One of the items being discussed in math class is the number of buses that will be required. Since the actual number of people is not yet determined, the situation is best modeled with variables and unknowns. Knowing that each bus holds 35 people, students develop a table based on 5 or 6 different numbers of people on the trip, in an attempt to find a pattern. Some students who require more concrete operations to develop a sense of the pattern use unit cubes and decimal rods to represent the situation with different numbers of people. The group works together to develop a graph based on their findings. The discussion begins with one person suggesting graphing the points *(35,1)*, *(70,2)*, and *(105,3)* and then connecting them with a straight line. The teacher does this (in a way that can be readily erased later) and asks the class if there is any problem with this solution. More discussion leads to understanding that the graph would be made up of discrete points since there cannot be fractions of people and the graph would not be a straight line but a series of steps 35 people long and going up by 1 bus. Students are able to verbalize the rule but are curious as to how it would be represented symbolically. The teacher shows them the symbol for the step function, $\lceil x \rceil$.

- Students construct squares on each side of right triangles on their geoboards, then find the area of each square. They record their results in a table and look for a pattern, leading them to "discover" the Pythagorean Theorem.

7. Use graphing techniques on a number line to model both absolute value and arithmetic operations.

- Students use a graphic representation of the mapping resulting from a multiplication by a negative number (such as -2 as shown below) to explain why the order of the inequality reverses when both sides are multiplied by that negative number.



They explain that while the original sequence of inputs is ascending *{-2,-1,0,1,2}*, the images are descending *{4,2,0,-2,-4}*.

- Presented with an absolute value equation such as $|x - 5| \leq 3$, the students use the idea that this means identifying all the points which are 3 units or less from 5. They represent the solution set with dots at 2 and 8 and a line connecting them.

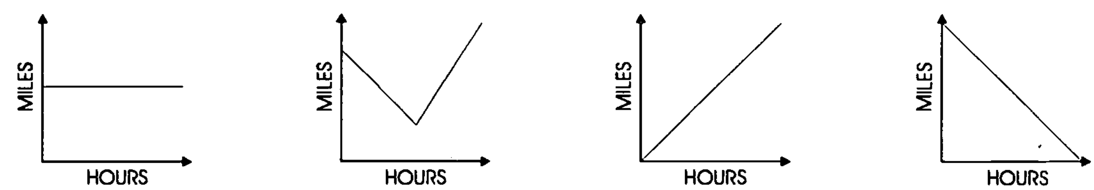
- Students represent multiplication of integers on the number line as repeated addition. They use the idea that 3×4 is $4 + 4 + 4$, and, on the number line, it would be represented by three segments, each four units in length, placed end to end starting at zero. The students write in their journals how they would picture $3 \times (-4)$. After reviewing their responses and clarifying concerns, they discuss that $(-3) \times (-4)$ would be the opposite of $3 \times (-4)$.

8. Analyze tables and graphs to identify properties and relationships.

- Students work on the handshake problem (*How many handshakes would there be in a group of 25 people if each person shook hands with every other person exactly once?*) by considering smaller groups of people and recording the results for these smaller groups in a table. Students identify the pattern and develop an expression which relates the number of people to the number of handshakes. Some students place 25 points on paper, forming a polygon, and begin drawing the segments which connect the points to generate a geometric pattern.

- Students explore the relationship between the number of sides of a regular polygon and the total number of diagonals that can be drawn in that polygon. They organize their work in a table, graph the data, and write a general rule that could be applied to the *n*th polygon. They connect this problem to the handshake problem.
- Students work in groups on the following EWT-like problem from the *Mathematics Instructional Guide* published by the New Jersey State Department of Education:

  The Hiking Club takes a long walk every Saturday. If they hike at a constant speed, which graph shows the relationship between the distance they cover and the time it takes them to cover the distance?



- Students perform an experiment in which they determine how far a toy car rolls from the end of a ramp as the height of the ramp changes. They gather the data, make a scatterplot, and fit a line to the data using the median-median line method. They use the graph to answer additional questions regarding the situation. Some students find the slope and y-intercept of the graph and use these to determine the equation of the line. The class then uses the equation to check its answers to the questions.

9. Understand and use the rectangular coordinate system.

- Students draw the quadrilateral ABCD, where the coordinates of the vertices are A(-3,2), B(4,7), C(2,-3), and D(-5,-6). They produce the figure that results from a size change of ½ and then slide that quadrilateral left 3 units and down 5.
- Students understand that the point (5,-3) is the intersection of two lines which have the equations $x=5$ and $y=-3$. They can identify quickly lines with equations such as $x=5$ and $y=-3$. They can identify the half planes and intersections of half planes identified by inequalities such as $x>5$ and $y< -3$.
- Students study the relationship between perimeters and areas of rectangles. Some students keep the perimeter constant and study the changes in area while others keep the area constant and study the changes in perimeter. Both groups plot their results as graphs and look for patterns.
- Students use squares or grid paper to study the relationship between the radius of a circle and its area (found by counting squares on centimeter grid paper). They graph their data and use it to predict the area of a circle of radius *r*.
- Students play *Green Globs* on the computer, entering equations and trying to hit as many globs as possible with them.

10. Solve simple linear equations using concrete, informal, and graphical methods, as well as appropriate paper-and-pencil techniques.

- Presented with a picture of a balance scale showing objects with unknown weights on both sides as well as known weights (e.g., $3x + 12 = 7x + 4$), students identify the standard algebraic equation related to the picture, describe in words how it would be solved using the concrete objects on the balance scale, and record their actions symbolically.
- Students use algebra tiles to solve an equation like $3(2x + 5) = 21$. They first place 21 units on the right and three groups of two strips and five units on the left. They then note that this is the same as saying 6 strips and 15 units balance 21 units $(6x + 15 = 21)$. Then they take 15 units off both sides, leaving 6 strips balanced with 6 units $(6x = 6)$. They conclude that one strip must equal one unit $(x = 1)$.
- As part of a regular exam, students write a verbal explanation describing the relationship between the function $y=2x+4$, its graph, and the equation $2=2x+4$.

11. Explore linear equations through the use of calculators, computers, and other technology.

- The students perform an experiment to answer the question about how long it would take a *wave* to go around Veterans' Stadium. They gather data by timing *waves* done by a various numbers of people, plot the data, and determine the line of best fit using the median-median line method. They determine the number of people that could sit around Veterans Stadium, but they discover that, unlike the previous data sets, they cannot use the graph to answer the question directly. The teacher explains that they need to determine an equation for the line. She has the students investigate functions of the form $y=mx+b$ using a graphing calculator in order to develop the idea that m represents the slope and b represents the y-intercept.
- Using a motion detector connected to a computer or graphing calculator, the teacher has students walk so that the distance from the detector plotted against time is a straight line. The teacher gives directions to students such as: *Walk so that the line has a positive slope, Walk so that the slope is steeper than the last line, Walk so that the line has a slope of 0,* or *Walk so that the line has a negative slope.*

12. Investigate inequalities and nonlinear equations informally.

- Presented with the information that the Cape May-Lewes Ferry has space for 20 cars, and a bus takes up the space of 3 cars, students are asked to draw a graph which represents how many cars and how many buses can be taken across on one trip. Students use variables to represent the unknowns *(x for cars and y for buses)* and develop the inequality $x+3y \leq 20$ as a model for the situation. Recognizing that the solutions have to be whole numbers, they identify the points whose coefficients are non-negative integers and in the first quadrant on or below the line.
- Students use a motion detector, connected to a calculator or computer, to record the motion of a ping-pong ball tossed by a small catapult. The motion detector is on the floor below the trajectory of the ball. Students note that the graph of distance against time is not linear. They experiment with different initial velocities and different release points to see how these affect the graph.

13. **Draw freehand sketches of, and interpret, graphs which model real phenomena.**

- Students are asked to draw a sketch of the graph which would describe a person's distance off the ground during a ride on a ferris wheel which had a radius of 60 feet. Some students just draw a curve that looks similar to a sine curve. Others put more detail into their drawing showing the step function behavior which occurs as people get on and get off and that there are limited revolutions permitted.
- Presented with a graph showing the population of frogs in a local marsh over the past ten years, students generate hypotheses for why the curve has the shape it does. They check their hypotheses by talking with a local biologist who has studied the marsh over this time period.

References

New Jersey State Department of Education. *Mathematics Instructional Guide*. D. Varygiannes, Coord. Trenton: 1996.

Software

Green Globs and Graphing Equations. Sunburst Communications.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 13 — Algebra — Grades 9-12

Overview

Students can develop a strong understanding of algebraic concepts and processes from consistent experiences in classroom activities where a variety of manipulatives and technology are used. The key components of this understanding in algebra, as identified in the K-12 Overview, are: **patterns, unknown quantities, properties, functions, modeling real-world situations, evaluating expressions,** and **solving equations and inequalities.**

With the foundation developed in the K-8 program, students should be able to be successful in most secondary algebra programs. However, instructional strategies should continue to focus on algebra as a means for representing and **modeling real situations** and answering questions about them. The traditional methods of teaching algebra have been likened to teaching a foreign language, focusing on grammar and not using the language in real conversation. Algebra courses and programs must encourage students to "speak the language" as well as use "proper grammar."

Algebraic understanding is necessary for all students regardless of the structure of the 9-12 program. Students in mathematics programs from technical/basic through college preparatory programs should learn a common core of algebra, with the remainder of the program based on their particular needs. All students should learn the same basic ideas. All students benefit from instructional methods which provide context for the content. Such an approach makes algebra more understandable and motivating.

Techniques for manipulating algebraic expressions remain important, especially for students who may continue into a calculus program. These can be woven into the curriculum or they might all be combined into separate courses labeled "algebra" taken by students who intend to pursue a mathematics-related career. No matter how this instruction is organized, however, instruction must produce students who understand the logic and purposes of algebraic procedures.

Students should be comfortable with **evaluating expressions** and with **solving equations and inequalities**, by whatever means they find most appropriate. They should understand the relationship between the graphs of functions and their equations. Prior to high school, they have focused predominantly on linear functions. In high school, students should gain more familiarity with nonlinear functions. They should develop the ability to solve equations and inequalities using appropriate paper-and-pencil techniques as well as technology. For example, they should be able to understand and solve quadratic equations using factoring, the quadratic formula, and graphing, as well as with a graphing calculator. They should recognize that the methods they use can be generalized to be used when functions look different but are actually composite functions using a basic type (e.g., $\sin^2 x + 3 \sin x + 2 = 0$ is like $x^2 + 3x + 2 = 0$); this method is sometimes called "chunking." This use of **patterns** to note commonalities among seemingly different problems is an important part of algebra in the high school.

Algebraic instruction at the secondary level should provide the opportunity for students to revisit problems. Traditional school problems leave students with the impression that there is one right answer and that once an answer is found there is no need to continue to think about the problem. Since algebra is the language of generalization, instruction in this area should encourage students to ask questions such as *Why does the solution behave this way?* They should develop an appreciation of the way algebraic representation can make problems easier to understand. Algebraic instruction should be rich in problems which are meaningful to students.

Algebra is the gatekeeper for the future study of mathematics and of science, social sciences, business, and a host of other areas. In the past, algebra has served as a filter, screening people out of these opportunities. For New Jersey to be part of a global society, it is important that 9-12 instruction in algebra play a major role in the culminating experiences of a twelve-year program that opens these gates for all.

Standard 13 — Algebra — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11, and 12.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

14. Model and solve problems that involve varying quantities using variables, expressions, equations, inequalities, absolute values, vectors, and matrices.

- Students take data involving two variables in an area of interest to them from the World Almanac, construct a scatterplot, and predict the type of equation or function which would best model the data. They use a computerized statistics package or a calculator to fit a function of this type to the data. They choose from linear, quadratic, exponential or logarithmic methods and discuss how well the model fits as well as the limitations.

- Students use matrices to represent tabular information such as the print runs below for each of two presses owned by a book company. They then calculate a third matrix that indicates the growth in production of each press from 1987 to 1988 and discuss the meaning of the data contained in it.

1988	Textbooks	Novels	Nonfiction
Press 1	250,000	125,000	312,000
Press 2	60,000	48,000	90,000

1987	Textbooks	Novels	Nonfiction
Press 1	190,000	100,000	140,000
Press 2	45,000	60,000	72,000

  Some students perform the operations by hand while others explain how they would do it and then use their graphing calculator or a spreadsheet, or write a computer program, which accomplishes the task.

- Students use vectors to determine the path of a plane that was flying due north at 300 miles per hour while a wind was blowing from the southwest at a speed of 15 miles per hour. Students draw a diagram and use an algebraic approach.

- Students work in groups on the following HSPT-like problem from the *Mathematics Instructional Guide* published by the New Jersey State Department of Education:

  As you ride home from a football game, the number of kilometers you are away from home depends (largely) on the number of minutes you have been riding. Suppose that you are 13 km from home when you have been riding for 10 minutes, and 8 km from home when you have been riding for 15 minutes. (Assume that the distance varies linearly with time.) Make a graph with the vertical axis representing distance home and the horizontal axis representing time. Label your graph. Plot the data given as two points on your graph. About how long did it take (on average) to travel 1 km? About how far was the football game from your home? Explain your answer.

- Students work through the *Breaking the Mold* lesson that is described in the Introduction to this *Framework*. They use a science experiment involving growing a mold to learn about exponential growth of populations and compound interest.

- Students work through the *What's My Line* unit that is described in the Keys to Success chapter of this *Framework*. In this unit, students find a linear relationship between the length of a person's thigh bone and his height, and use this to estimate the height of a person whose thigh bone has been found.

- Students work through the *Ice Cones* lesson that is described in the First Four Standards of this *Framework*. In this lesson, an ice-cream vendor's problem of folding a circle into a cone of maximum volume is solved by expressing the volume as a function which is then displayed on an graphing calculator.

15. Use tables and graphs as tools to interpret expressions, equations, and inequalities.

- On a test, students are asked to determine the truth value of the statement "log $x > 0$ for all positive real numbers x." One student remembers that log x is the exponent to which 10 must be raised to get x, and to get a number less than 1 would require a negative exponent. Another student picks some trial points, develops a chart of the points and their logarithms and discovers that when $x < 1$, $\log x < 0$. A third student graphs the function on his graphing calculator and sees that when $x < 1$, the graph of log x is below the x-axis.

- Faced with the problem of solving the inequality $x^2 - 3x \leq 4$, some students use the equation $x^2 - 3x - 4 = 0$ to determine the boundary points of the interval that satisfies the inequality. They factor the equation and find that these endpoints are -1 and 4. They place dots on the number line at those points, since they know the endpoints are included in the solution set, and then substitute 0 for x in the original inequality. When they find that the resulting statement is true, they shade the interval connecting the two points to obtain their solution $-1 \leq x \leq 4$. Some students determine the endpoints in the same way but roughly sketch a graph of the parabola $y = x^2 - 3x - 4$ and determine that since the inequality was $\leq$, the problem was asking when the parabola was below the x-axis. Their graph indicates that this happens in the region between the two endpoints. Other students used their graphing calculators to graph the function $x^2 - 3x$ and used the trace function to determine when the graph was on or below the line $y=4$. All students, however, are able to use all these methods to solve the problem.

16. Develop, explain, use, and analyze procedures for operating on algebraic expressions and matrices.

- Students use algebra tiles to develop procedures for multiplying binomials and factoring trinomials. They summarize these procedures in their math notebooks, applying them to the solution of real-world problems. They work through the *Making Rectangles* lesson described in the First Four Standards of this *Framework,* where they discuss which combinations of tiles can be formed into rectangles, and relate this question to factoring trinomials.

- Students work in groups on the following HSPT-like problem from the *Mathematics Instructional Guide* (p. 7-153) published by the New Jersey State Department of Education:

  Which of the following is NOT another name for 1?

  A. $\frac{x}{x}$, $x \neq 0$ **B.** $\frac{x+2}{2+x}$, $x \neq -2$

  C. $\frac{x \div 2}{x \div 2}$, $x \neq 0$ **D.** $\frac{x-2}{2-x}$, $x \neq 2$

- After many experiences with trying to determine appropriate windows for graphing functions on computers or graphing calculators, students develop an understanding of the need to know what the general behavior of a function will be before they use the technology. Students are then asked to explain how they could determine the behavior of the graph of the function below.

  $$F(x) = \frac{x^3 + 5x^2 - 6x}{x^3 - 36x}$$

  Students factor the numerator and denominator to determine the values which make them zero and use those values to identify the x-intercepts and vertical asymptotes, respectively. They discuss the fact that the factors x and $x+6$ appear in both places and lead to a removable discontinuity represented by a hole in the graph. They discuss the end behavior of the function as approaching $y = 1$ and the behavior near the vertical asymptote of $x = 6$.

- Following a unit on combinations and binomial expansion, students make a journal entry discussing the power of Pascal's triangle in expanding powers of binomial expressions as compared to the traditional multiplication algorithm.

17. Solve equations and inequalities of varying degrees using graphing calculators and computers as well as appropriate paper-and-pencil techniques.

- Students are asked to find the solutions to $2^x = 3x^2$. Some students use a spreadsheet to develop a table of values. Once they find an interval of length 1 which contains a solution, they refine their numbers to develop the answer to the desired precision. Other students graph both $y = 2^x$ and $y = 3x^2$ using graphing calculators or computers and use the trace function to determine where they intersect. Other students graph the function $y = 3x^2 - 2^x$ and use the trace function to find the zeroes. Other students enter the function in their graphing calculator and check the table of values.

- As a portion of a final assessment, students are given one opportunity to place a cup which is supported 6 inches off the ground in such a position as to catch a marble rolled down a ramp. They perform the roll without the cup to locate the point where the marble strikes the ground. They measure the height of the end of the ramp above the ground and the distance from the point on the ground directly beneath the end of the ramp to the point where the marble struck the ground. They generate the quadratic function which models the path of the marble. Several students use different methods to ensure they have the correct function. Then they decide where they will have to place the cup by substituting 6 for the function value and determining the corresponding x-value. Students solve the equation using the quadratic formula, and the trace function on a graphing calculator, and proceed to place the cup and roll the ball only when the solutions produced by all of the methods agree.

18. Understand the logic and purposes of algebraic procedures.

- Students use matrices as arrays of information, so that the matrix below, for example, is recognized as representing the four vertices {(1,4), (5,6), (3, - 2), (- 2, - 2)} of a polygon. Reducing the polygon by ½ can then be represented by multiplying the matrix by the scalar ½ and moving the polygon to the right one unit can be represented by adding to it a 2 x 4 matrix whose top row consists of 1s and its bottom row of 0s.

$$\begin{bmatrix} 1 & 5 & 3 & -2 \\ 4 & 6 & -2 & -2 \end{bmatrix}$$

- Students read *Mathematics in the Time of the Pharaohs* by Richard Gillings to understand the development of Egyptian mathematics including computational procedures for dealing with direct and inverse proportions, linear equations, and trigonometric functions.
- Students explain in their journals how the identity matrix is like the number one.

19. Interpret algebraic equations and inequalities geometrically, and describe geometric objects algebraically.

- Students investigate the characteristics of the linear functions. For example: *In $y = kx$, how does a change in k affect the graph? In $y = mx + b$, what does b do? Does k in the first equation serve the same purpose as m in the second?* Students use the graphing calculator to investigate and verify their conclusions.
- Students investigate the effects of a dilation and/or a horizontal or vertical shift on the coefficients of a quadratic function. *For example: How does moving the graph up 3 units affect the equation? How does moving the graph right 3 unites affect the equation?*
- Students look at the effects of changing the coefficients of a quadratic equation on the graph. For example: *How is the graph of $y = 4x^2$ different from that of $y = x^2$? How is $y = .2x^2$ different from $y = x^2$? How are $y = x^2 + 4$, $y = x^2 - 4$, $y = x^2 - 4x$, and $y = x^2$*

$-4x + 4$ each different from $y = x^2$? Students use graphing calculators to look at the graphs and summarize their conjectures in writing. They also work through the *Building Parabolas* lesson described in the First Four Standards of this *Framework,* where students discuss the general equation of a parabola and use *Green Globs* software to find equations of parabolas that pass through specified points.

- Students study the behavior of functions of the form $y = ax^n$. They investigate the effect of a on the curve and the characteristics of the graph when n is even or odd. They use the graphing calculator to assist them and write a sentence summarizing their discoveries.

- Students are asked to consider the following situation:

  A landscaping contractor uses a combination of two brands of fertilizers, each containing a different amount of phosphates and nitrates. In a package, brand A has 4 lb. of phosphates and 2 lb. of nitrates. Brand B contains 6 lb. of phosphates and 5 lb. of phosphates. On her current job, the lawn requires at least 24 lb. of phosphates and at least 16 lb. of nitrates. How much of each fertilizer does the contractor need?

  Students represent the given conditions as inequalities and use the intersection of their regions as the set of feasible answers.

- Students recognize that solving two equations simultaneously like $2x+y=5$, $4x-y=1$ amounts to finding the point of intersection of the two lines with equations $y=-2x+5$, $y=4x-1$. Similarly they recognize that solving a quadratic equation like $2x^2 -3x-5=0$ amounts to finding where the parabola $y=2x^2-3x-5$ crosses the x-axis.

References

Gillings, Richard. *Mathematics in the Time of the Pharoahs.*

New Jersey State Department of Education. *Mathematics Instructional Guide.* D. Varygiannes, Coord. Trenton: 1996.

Software

Green Globs and Graphing Equations. Sunburst Communications.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards.*

STANDARD 14 — DISCRETE MATHEMATICS

K-12 Overview

All students will apply the concepts and methods of discrete mathematics to model and explore a variety of practical situations.

Descriptive Statement

Discrete mathematics is the branch of mathematics that deals with arrangements of distinct objects. It includes a wide variety of topics and techniques that arise in everyday life, such as how to find the best route from one city to another, where the objects are cities arranged on a map. It also includes how to count the number of different combinations of toppings for pizzas, how best to schedule a list of tasks to be done, and how computers store and retrieve arrangements of information on a screen. Discrete mathematics is the mathematics used by decision-makers in our society, from workers in government to those in health care, transportation, and telecommunications. Its various applications help students see the relevance of mathematics in the real world.

Meaning and Importance

During the past 30 years, discrete mathematics has grown rapidly and has evolved into a significant area of mathematics. It is the language of a large body of science and provides a framework for decisions that individuals will need to make in their own lives, in their professions, and in their roles as citizens. Its many practical applications can help students see the relevance of mathematics to the real world. It does not have extensive prerequisites, yet it poses challenges to all students. It is fun to do, is often geometry based, and can stimulate an interest in mathematics on the part of students at all levels and of all abilities.

K-12 Development and Emphases

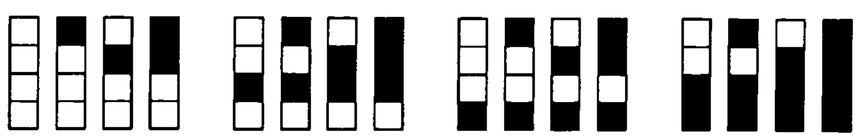
Although the term "discrete mathematics" may seem unfamiliar, many of its themes are already present in the classroom. Whenever objects are counted, ordered, or listed, whenever instructions are presented and followed, whenever games are played and analyzed, teachers are introducing themes of discrete mathematics. Through understanding these themes, teachers will be able to recognize and introduce them regularly in classroom situations. For example, when calling three students to work at the three segments of the chalkboard, the teacher might ask *In how many different orders can these three students work at the board?* Another version of the same question is *How many different ways, such as ABC, can you name a triangle whose vertices are labeled A, B, and C?* A similar, but slightly different question is *In how many different orders can three numbers be multiplied?*

Two important resources on discrete mathematics for teachers at all levels are the 1991 NCTM Yearbook *Discrete Mathematics Across the Curriculum K-12* and the 1997 DIMACS Volume *Discrete Mathematics in the Schools: Making an Impact*. The material in this chapter is drawn from activities that have been reviewed and classroom-tested by the K-12 teachers in the Rutgers University Leadership Program in Discrete Mathematics over the past nine years; this program is funded by the National Science Foundation.

Students should learn to recognize examples of discrete mathematics in familiar settings, and explore and solve a variety of problems for which discrete techniques have proved useful. These ideas should be pursued throughout the school years. Students can start with many of the basic ideas in concrete settings, including games and general play, and progressively develop these ideas in more complicated settings and more abstract forms. Five major themes of discrete mathematics should be addressed at all K-12 grade levels — **systematic listing, counting, and reasoning; discrete mathematical modeling using graphs (networks) and trees; iterative (that is, repetitive) patterns and processes; organizing and processing information;** and **following and devising lists of instructions, called "algorithms," and using algorithms to find the best solution to real-world problems**. These five themes are discussed in the paragraphs below.

Students should use a variety of strategies to **systematically list and count** the number of ways there are to complete a particular task. For example, elementary school students should be able to make a list of all possible outcomes of a simple situation such as the number of outfits that can be worn using two coats and three hats. Middle school students should be able to systematically list and count the number of different four-block-high towers that can be built using blue and red blocks (see example below), or the number of possible routes from one location on a map to another, or the number of different "words" that can be made using five letters. High school students should be able to determine the number of possible orderings of an arbitrary number of objects and to describe procedures for listing and counting all such orderings. These strategies for listing and counting should be applied by both middle school and high school students to solve problems in probability.

Following is a list of all four-block-high towers that can be built using clear blocks and solid blocks. The 16 towers are presented in a systematic list — the first 8 towers have a clear block at the bottom and the second 8 towers have a solid block at the bottom; within each of these two groups, the first 4 towers have the second block clear, and the second 4 towers have the second block solid; etc.

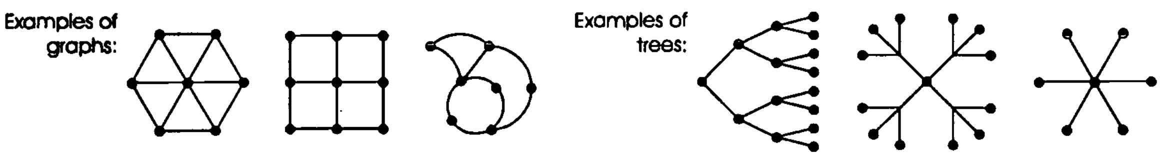


If each tower is described alphabetically as a sequence of C's and S's, representing "clear" and "solid" — the tower at the left, for example, would be C-C-C-C, and the third tower from the left would be C-C-S-C, reading from the bottom up — then the sixteen towers would be in alphabetical order:

C-C-C-C	C-S-C-C	S-C-C-C	S-S-C-C
C-C-C-S	C-S-C-S	S-C-C-S	S-S-C-S
C-C-S-C	C-S-S-C	S-C-S-C	S-S-S-C
C-C-S-S	C-S-S-S	S-C-S-S	S-S-S-S

There are other ways of systematically listing the 16 towers; for example, the list could contain first the one tower with no solid blocks, then the four towers with one solid block, then the six towers with two solid blocks, then the four towers with three solid blocks, and finally the one tower with four solid blocks.

Discrete mathematical models such as graphs (networks) and trees (such as those pictured below) can be used to represent and solve a variety of problems based on real-world situations.



In the left-most graph of the figures above, all seven dots are linked into a network consisting of the six line segments emerging from the center dot; these six line segments form the tree at the far right which is said to "span" the original graph since it reaches all of its points. Another example: if we think of the second graph as a street map and we make the streets one way, we can represent the situation using a directed graph where the line segments are replaced by arrows.

Elementary school students should recognize that a street map can be represented by a graph and that routes can be represented by paths in the graph; middle school students should be able to find cost-effective ways of linking sites into a network using spanning trees; and high school students should be able to use efficient methods to organize the performance of individual tasks in a larger project using directed graphs.

Iterative patterns and processes are used both for describing the world and in solving problems. An iterative pattern or process is one which involves repeating a single step or sequence of steps many times. For example, elementary school students should understand that multiplication corresponds to repeatedly adding the same number a specified number of times. They should investigate how decorative floor tilings can often be described as the repeated use of a small pattern, and how the patterns of rows in pine cones follow a simple mathematical rule. Middle school students should explore how simple repetitive rules can generate interesting patterns by using spirolaterals or Logo commands, or how they can result in extremely complex behavior by generating the beginning stages of fractal curves. They should investigate the ways that the plane can be covered by repeating patterns, called tessellations. High school students should understand how many processes describing the change of physical, biological, and economic systems over time can be modeled by simple equations applied repetitively, and use these models to predict the long-term behavior of such systems.

Students should explore different methods of **arranging, organizing, analyzing, transforming, and communicating information**, and understand how these methods are used in a variety of settings. Elementary school students should investigate ways to represent and classify data according to attributes such as color or shape, and to organize data into structures like tables or tree diagrams or Venn diagrams. Middle school students should be able to read, construct, and analyze tables, matrices, maps and other data structures. High school students should understand the application of discrete methods to problems of information processing and computing such as sorting, codes, and error correction.

Students should be able to **follow and devise lists of instructions, called "algorithms," and use them to find the best solution to real-world problems** — where "best" may be defined, for example, as most cost-effective or as most equitable. For example, elementary school students should be able to carry out instructions for getting from one location to another, should discuss different ways of dividing a pile of snacks, and should determine the shortest path from one site to another on a map laid out on the classroom floor. Middle school students should be able to plan an optimal route for a class trip (see the vignette in the Introduction to this *Framework* entitled *Short-circuiting Trenton*), write precise instructions for adding two two-digit numbers, and, pretending to be the manager of a fast-food restaurant, devise work schedules for employees which meet specified conditions yet minimize the cost. High school students should be conversant with fundamental strategies of optimization, be able to use flow charts to describe algorithms, and recognize both the power and limitations of computers in solving algorithmic problems.

IN SUMMARY, discrete mathematics is an exciting and appropriate vehicle for working toward and achieving the goal of educating informed citizens who are better able to function in our increasingly technological society; have better reasoning power and problem-solving skills; are aware of the importance of mathematics in our society; and are prepared for future careers which will require new and more sophisticated analytical and technical tools. It is an excellent tool for improving reasoning and problem-solving abilities.

***NOTE:** Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

References

Kenny, M. J., Ed. *Discrete Mathematics Across the Curriculum K-12*. 1991 Yearbook of the National Council of Teachers of Mathematics (NCTM). Reston, VA, 1991.

Rosenstein, J. G., D. Franzblau, and F. Roberts, Eds. *Discrete Mathematics in the Schools: Making an Impact*. Proceedings of a 1992 DIMACS Conference on "Discrete Mathematics in the Schools." DIMACS Series on Discrete Mathematics and Theoretical Computer Science. Providence, RI: American Mathematical Society (AMS), 1997. (Available online from this chapter in http://dimacs.rutgers.edu/nj_math_coalition/framework.html/.)

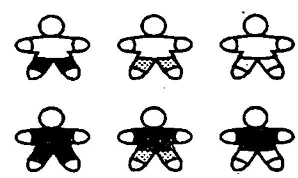
Standard 14 — Discrete Mathematics — Grades K-2

Overview

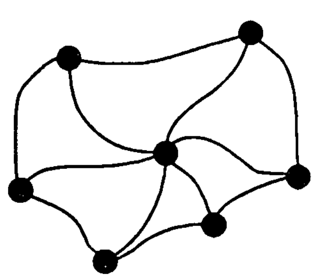
The five major themes of discrete mathematics, as discussed in the K-12 Overview, are **systematic listing, counting, and reasoning; discrete mathematical modeling using graphs (networks) and trees; iterative (that is, repetitive) patterns and processes; organizing and processing information;** and **following and devising lists of instructions, called "algorithms," and using them to find the best solution to real-world problems.**

Despite their formidable titles, these five themes can be addressed with activities at the K-2 grade level which involve purposeful play and simple analysis. Indeed, teachers will discover that many activities they already are using in their classrooms reflect these themes. These five themes are discussed in the paragraphs below.

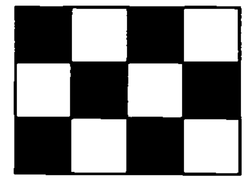
Activities involving **systematic listing, counting, and reasoning** can be done very concretely at the K-2 grade level. For example, dressing cardboard teddy bears with different outfits becomes a mathematical activity when the task is to make a list of all possible outfits and count them; pictured on the right are the six outfits that can be arranged using one of two types of shirts and one of three types of shorts. Similarly, playing any game involving choices becomes a mathematical activity when children reflect on the moves they make in the game.

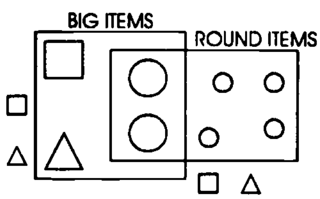


An important **discrete mathematical model** is that of a **network** or **graph**, which consists of dots and lines joining the dots; the dots are often called *vertices (vertex* is the singular) and the lines are often called *edges*. (This is different from other mathematical uses of the term "graph.") The two terms "network" and "graph" are used interchangeably for this concept. An example of a graph with seven vertices and twelve edges is given below. You can think of the vertices of this graph as islands in a river and the edges as bridges. You can also think of them as buildings and roads, or houses and telephone cables, or people and handshakes; wherever a collection of things are joined by connectors, the mathematical model used is that of a network or graph. At the K-2 level, children can recognize graphs and use life-size models of graphs in various ways. For example, a large version of this graph, or any other graph, can be "drawn" on the floor using paper plates as vertices and masking tape as edges. Children might select two "islands" and find a way to go from one island to the other island by crossing exactly four "bridges." (This can be done for any two islands in this graph, but not necessarily in another graph.)



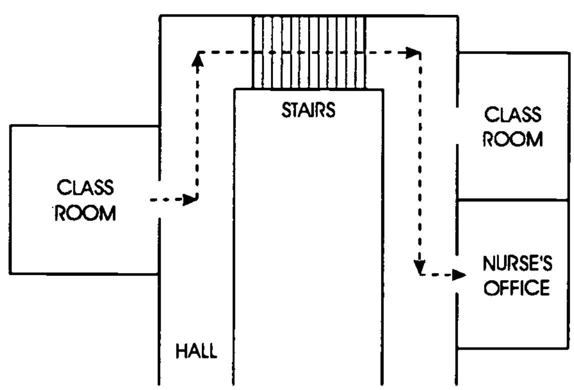
Children can recognize and work with **repetitive patterns and processes** involving numbers and shapes, using objects in the classroom and in the world around them. For example, children at the K-2 level can create (and decorate) a pattern of triangles or squares (as pictured here) that cover a section of the floor (this is called a "tessellation"), or start with a number and repeatedly add three, or use clapping and movement to simulate rhythmic patterns.





Children at the K-2 grade levels should investigate ways of **sorting items** according to attributes like color, shape, or size, and ways of **arranging data** into charts, tables, and family trees. For example, they can sort attribute blocks or stuffed animals by color or kind, as in the diagram, and can count the number of children who have birthdays in each month by organizing themselves into birthday-month groups.

Finally, at the K-2 grade levels, children should be able to **follow and describe simple procedures and** determine and discuss **what is the best solution** to a problem. For example, they should be able to follow a prescribed route from the classroom to another room in the school (as pictured below) and to compare various alternate routes, and in the second grade should determine the shortest path from one site to another on a map laid out on the classroom floor.



Two important resources on discrete mathematics for teachers at all levels are the 1991 NCTM Yearbook *Discrete Mathematics Across the Curriculum K-12* and the 1997 DIMACS Volume *Discrete Mathematics in the Schools: Making An Impact.* Another important resource for K-2 teachers is *This Is MEGA-Mathematics!*

Standard 14 — Discrete Mathematics — Grades K-2

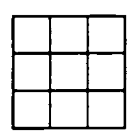
Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Explore a variety of puzzles, games, and counting problems.**

   - Students use teddy bear cut-outs with, for example, shirts of two colors and shorts of three colors, and decide how many different outfits can be made by making a list of all possibilities and arranging them systematically. (See illustration in K-2 Overview.)
   - Students use paper faces or Mr. Potato Head type models to create a "regular face" given a nose, mouth, and a pair of eyes. Then they use another pair of eyes, then another nose, and then another mouth (or other parts) and explore and record the number of faces that can be made after each additional part has been included.
   - Students read *A Three Hat Day* and then try to create as many different hats as possible with three hats, a feather, a flower, and a ribbon as decoration. Students count the different hats they've made and discuss their answers.
   - Students count the number of squares of each size (1 x 1, 2 x 2, 3 x 3) that they can find on the square grid below. They can be challenged to find the numbers of small squares of each size on a larger square or rectangular grid.



   - Students work in groups to figure out the rules of addition and placement that are used to pass from one row to the next in the diagram below, and use these rules to find the numbers in the next few rows.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

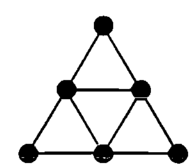
     In this diagram, called Pascal's triangle, each number is the sum of the two numbers that are above it, to its left and right; the numbers on the left and right edges are all 1.

   - Students cut out five "coins" labeled 1¢, 2¢, 4¢, 8¢, and 16¢. For each number in the counting sequence 1, 2, 3, 4, 5, ... (as far as is appropriate for a particular group of students), students determine how to obtain that amount of money using a combination of different coins.

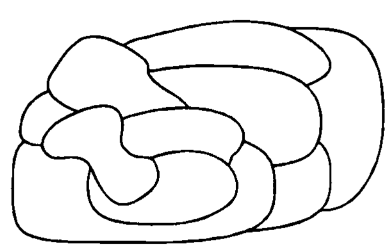
- Students play simple games and discuss why they make the moves they do. For example, two students divide a six-piece domino set (with 0-0, 0-1, 0-2, 1-1, 1-2, and 2-2) and take turns placing dominoes so that dominoes which touch have the same numbers and so that all six dominoes are used in the chain.

2. Use networks and tree diagrams to represent everyday situations.

- Students find a way of getting from one island to another, in the graph described in the K-2 Overview laid out on the classroom floor with masking tape, by crossing exactly four bridges. They make their own graphs, naming each of the islands, and make a "from-to" list of islands for which they have found a four-bridge-route. (Note: it may not always be possible to find four-bridge-routes.)
- Students count the number of edges at each vertex (called the **degree** of the vertex) of a network and construct graphs where all vertices have the same degree, or where all the vertices have one of two specified degrees.
- On a pattern of islands and bridges laid out on the floor, students try to find a way of visiting each island exactly once; they can leave colored markers to keep track of islands already visited. Note that for some patterns this may not be possible! Students can be challenged to find a way of visiting each island exactly once which returns them to their starting point. Similar activities can be found in *Inside, Outside, Loops, and Lines* by Herbert Kohl.



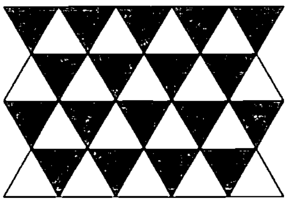
- Students create a map with make-believe countries (see example below), and color the maps so that countries which are next to each other have different colors. *How many colors were used? Could it be done with fewer colors? with four colors? with three colors? with two colors?* A number of interesting map coloring ideas can be found in *Inside, Outside, Loops and Lines* by Herbert Kohl.



3. Identify and investigate sequences and patterns found in nature, art, and music.

- Students use a calculator to create a sequence of ten numbers starting with zero, each of which is three more than the previous one; on some calculators, this can be done by pressing 0 + 3 = = = . . . , where = is pressed ten times. As they proceed, they count one 3, two 3s, three 3s, etc.

- Students "tessellate" the plane, by using groups of squares or triangles (for example, from sets of pattern blocks) to completely cover a sheet of paper without overlapping; they record their patterns by tracing around the blocks on a sheet of paper and coloring the shapes.

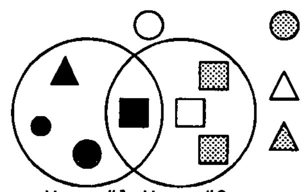


- Students listen to or read *Grandfather Tang's Story* by Ann Tompert and then use tangrams to make the shape-changing fox fairies as the story progresses. Students are then encouraged to do a retelling of the story with tangrams or to invent their own tangram characters and stories.
- Students read *The Cat in the Hat* or *Green Eggs and Ham* by Dr. Seuss and identify the pattern of events in the book. Students could create their own books with similar patterns.
- Students collect leaves and note the patterns of the veins. They look at how the veins branch off on each side of the center vein and observe that their branches are smaller copies of the original vein pattern. Students collect feathers, ferns, Queen Anne's lace, broccoli, or cauliflower and note in each case how the pattern of the original is repeated in miniature in each of its branches or clusters.
- Students listen for rhythmic patterns in musical selections and use clapping, instruments, and movement to simulate those patterns.
- Students take a "patterns walk" through the school, searching for patterns in the bricks, the play equipment, the shapes in the classrooms, the number sequences of classrooms, the floors and ceilings, etc.; the purpose of this activity is to create an awareness of all the patterns around them.

4. Investigate ways to represent and classify data according to attributes, such as shape or color, and relationships, and discuss the purpose and usefulness of such classification.

- Students sort themselves by month of birth, and then within each group by height or birth date. (Other sorting activities can be found in *Mathematics Their Way*, by Mary Baratta-Lorton.)
- Each student is given a card with a different number on it. Students line up in a row and put the numbers in numerical order by exchanging cards, one at a time, with adjacent children. (After practice, this can be accomplished without talking.)
- Students draw stick figures of members of their family and arrange them in order of size.
- Students sort stuffed animals in various ways and explain why they sorted them as they did. Students can use *Tabletop, Jr.* software to sort characters according to a variety of attributes.
- Using attribute blocks, buttons, or other objects with clearly distinguishable attributes such as color, size, and shape, students develop a sequence of objects where each differs from the previous one in only one attribute. *Tabletop, Jr.* software can also be used to create such sequences of objects.

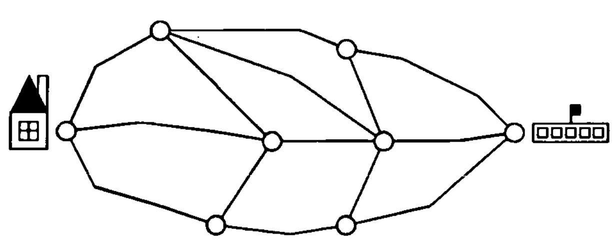
- Students use two Hula Hoops (or large circles drawn on paper so that a part of their interiors overlap) to assist in sorting attribute blocks or other objects according to two characteristics. For example, given a collection of objects of different colors and shapes, students are asked to place them so that all red items go inside hoop #1 and all others go on the outside, and so that all square items go inside hoop #2 and all others go on the outside. *What items should be placed in the overlap of the two hoops? What is inside only the first hoop? What is outside both hoops?*



This is an example of a Venn diagram. Students can also use Venn diagrams to organize the similarities and differences between the information in two stories by placing all features of the first story in hoop #1 and all features of the second story in hoop #2, with common features in the overlap of the two hoops. A similar activity can be found in the *Shapetown* lesson that is described in the First Four Standards of this *Framework*. *Tabletop, Jr.* software allows students to arrange and sort data, and to explore these concepts easily.

5. Follow, devise, and describe practical lists of instructions.

- Students follow directions for a trip within the classroom — for example, students are asked where they would end up if they started at a given spot facing in a certain direction, took three steps forward, turned left, took two steps forward, turned right, and moved forward three more steps.
- Students follow oral directions for going from the classroom to the lunchroom, and represent these directions with a diagram. (See K-2 Overview for a sample diagram.)
- Students agree on a procedure for filling a box with rectangular blocks. For example, a box with dimensions 4"x4"x5" can be filled with 10 blocks of dimensions 1"x2"x4". (Linking cubes can be used to create the rectangular blocks.)
- Students explore the question of finding the shortest route from school to home on a diagram like the one pictured below, laid out on the floor using masking tape, where students place a number of counters on each line segment to represent the length of that segment. (The shortest route will depend on the placement of the counters; what appears to be the most direct route may not be the shortest.)



- Students find a way through a simple maze. They discuss the different paths they took and their reasons for doing so.
- Students use Logo software to give the turtle precise instructions for movement in specified directions.

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Software

Logo. Many versions of Logo are commercially available.

Tabletop, Jr. Broderbund Software. TERC.

On-Line Resources

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The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 14 — Discrete Mathematics — Grades 3-4

Overview

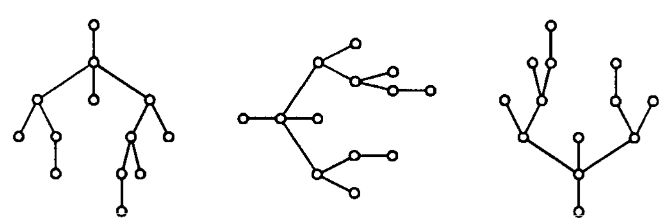
The five major themes of discrete mathematics, as discussed in the K-12 Overview, are **systematic listing, counting, and reasoning; discrete mathematical modeling using graphs (networks) and trees; iterative (that is, repetitive) patterns and processes; organizing and processing information;** and **following and devising lists of instructions, called "algorithms," and using them to find the best solution to real-world problems.**

Despite their formidable titles, these five themes can be addressed with activities at the 3-4 grade level which involve purposeful play and simple analysis. Indeed, teachers will discover that many activities that they already are using in their classrooms reflect these themes. These five themes are discussed in the paragraphs below.

The following discussion of activities at the 3-4 grade levels in discrete mathematics presupposes that corresponding activities have taken place at the K-2 grade levels. Hence 3-4 grade teachers should review the K-2 grade level discussion of discrete mathematics and use might use activities similar to those described there before introducing the activities for this grade level.

Activities involving **systematic listing, counting, and reasoning** should be done very concretely at the 3-4 grade levels, building on similar activities at the K-2 grade levels. For example, the children could systematically list and count the total number of possible combinations of dessert and beverage that can be selected from pictures of those two types of foods they have cut out of magazines or that can be selected from a restaurant menu. Similarly, playing games like Nim, dots and boxes, and dominoes becomes a mathematical activity when children systematically reflect on the moves they make in the game and use those reflections to decide on the next move.

An important **discrete mathematical model** is that of a **graph**, which is used whenever a collection of things are joined by connectors — such as buildings and roads, islands and bridges, or houses and telephone cables — or, more abstractly, whenever the objects have some defined relationship to each other; this kind of model is described in the K-2 Overview. At the 3-4 grade levels, children can recognize and use models of graphs in various ways, for example, by finding a way to get from one island to another by crossing exactly four bridges, or by finding a route for a city mail carrier which uses each street once, or by constructing a collaboration graph for the class which describes who has worked with whom during the past week. A special kind of graph is called a "tree." Three views of the same tree are pictured in the diagram below; the first suggests a family tree, the second a tree diagram, and the third a "real" tree.



At the 3-4 grade levels, students can use a tree diagram to organize the six ways that three people can be arranged in order. (See the Grades 3-4 Indicators and Activities for an example.)

Students can recognize and work with **repetitive patterns and processes** involving numbers and shapes, with classroom objects and in the world around them. Children at the 3-4 grade levels are fascinated with the Fibonacci sequence of numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... where every number is the sum of the previous two numbers. This sequence of numbers turns up in petals of flowers, in the growth of populations (see the activity involving rabbits), in pineapples and pine cones, and in lots of other places in nature. Another important sequence to introduce at this age is the doubling sequence 1, 2, 4, 8, 16, 32, ... and to discuss different situations in which it appears.

Students at the 3-4 grade levels should investigate ways of **sorting items** according to attributes like color or shape, or by quantitative information like size, **arranging data** using tree diagrams and building charts and tables, and **recovering hidden information** in games and encoded messages. For example, they can sort letters into zip code order or sort the class alphabetically, create bar charts based on information obtained experimentally (such as soda drink preferences of the class), and play games like hangman to discover hidden messages.

Students at the 3-4 grade levels should **describe and discuss simple algorithmic procedures** such as providing and following directions from one location to another, and should in simple cases determine and discuss **what is the best solution** to a problem. For example, they might follow a recipe to make a cake or to assemble a simple toy from its component parts. Or they might find the best way of playing tic-tac-toe or the shortest route that can be used to get from one location to another.

Two important resources on discrete mathematics for teachers at all levels are the 1991 NCTM Yearbook *Discrete Mathematics Across the Curriculum K-12* and the 1997 DIMACS Volume *Discrete Mathematics in the Schools: Making An Impact.* Another important resource for 3-4 teachers is *This Is MEGA-Mathematics!*

Standard 14 — Discrete Mathematics — Grades 3-4

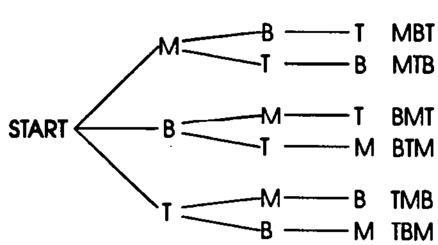
Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences will be such that all students in grades 3-4:

1. **Explore a variety of puzzles, games, and counting problems.**

   - Students read *One Hundred Hungry Ants* by Elinor Pinczes and then illustrate and write their own story books (perhaps titled *18 Ailing Alligators* or *24 Furry Ferrets*) in a style similar to the book using as many different arrangements of the animals as possible in creating their books. They read their books to students in the lower grades.
   - Students count the number of squares of each size (1x1, 2x2, 3x3, 4x4, 5x5) that they can find on a geoboard, and in larger square or rectangular grids.
   - Students determine the number of possible combinations of dessert and beverage that could be selected from pictures of those two types of foods they have cut out of magazines. Subsequently, they determine the number of possible combinations of dessert and beverage that could be chosen from a restaurant menu, and how many of those combinations could be ordered if they only have $4.
   - Students find the number of different ways to make a row of four flowers each of which could be red or yellow. They can model this with Unfix cubes and explain how they know that all combinations have been obtained.
   - Students determine the number of different ways any three people can be arranged in order, and use a tree diagram to organize the information. The tree diagram below represents the six ways that Barbara (B), Maria (M), and Tarvanda (T), can be arranged in order. The three branches emerging from the "start" position represent the three people who could be first; each path from left to right represents the arrangement of the three people listed to the right.



   - Each student uses four squares to make designs where each square shares an entire side with at least one of the other three squares. Geoboards, attribute blocks or Linker cubes can be used. *How many different shapes can be made?* These shapes are called "tetrominoes."

- Each group of students receives a bag containing four colored beads. One group may be given 1 red, 1 black and 2 green beads; other groups may have the same four beads or different ones. Students take turns drawing a bead from the bag, recording its color, and replacing it in the bag. After 20 beads are drawn, each group makes a bar graph illustrating the number of beads drawn of each color. They make another bar graph illustrating the number of beads of each color actually in the bag, and compare the two bar graphs. As a follow-up activity, students should draw 20 or more times from a bag containing an unknown mixture of beads and try to guess, and justify, how many beads of each color are in the container.

- Students determine what amounts of postage can and cannot be made using only 3¢ and 5¢ stamps.

- Students generate additional rows of Pascal's triangle (at right). They color all odd entries one color and all even entries another color. They examine the patterns that result, and try to explain what they see. They discuss whether their conclusions apply to a larger version of Pascal's triangle.

  1
  1 1
  1 2 1
  1 3 3 1
  1 4 6 4 1

- Students make a table indicating which stamps of the denominations 1¢, 2¢, 4¢, 8¢, 16¢, 32¢ would be used (with no repeats) to obtain each amount of postage from 1¢ to 63¢. For the table, they list the available denominations across the top and the postage amounts from 1¢ to 63¢ at the left; they put a checkmark in the appropriate spot if they need the stamp for that amount, and leave it blank otherwise. They try to find a pattern which could be used to decide which amounts of postage could be made if additional stamps (like 64¢ and 128¢) were used.

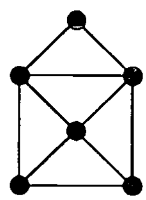
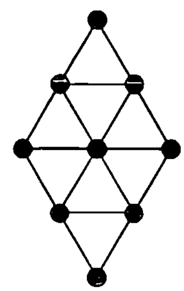
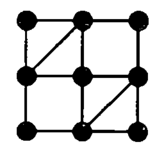
- Students play games like Nim and reflect on the moves they make in the game. (See *Math for Girls and Other Problem Solvers*, by D. Downie et al., for other games for this grade level.) In Nim, you start with a number of piles of objects — for example, you could start with two piles, one with five buttons, the other with seven buttons. Two students alternate moves, and each move consists of taking some or all of the buttons from a single pile; the child who takes the last button off the table wins the game. Once they master this game, students can try Nim with three piles, starting with three piles which have respectively 1, 2, and 3 buttons.

- Students play games like *dots and boxes* and systematically think about the moves they make in the game. In dots and boxes, you start with a square (or rectangular) array of dots, and two students alternate drawing a line which joins two adjacent dots. Whenever all four sides of a square have been drawn, the student puts her or his initial in the square and draws another line; the person with initials in more squares wins the game.

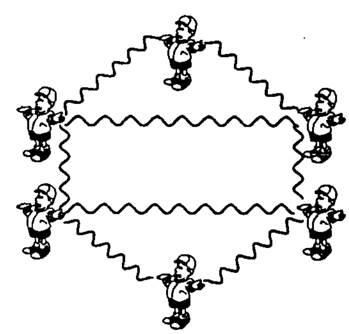
2. Use networks and tree diagrams to represent everyday situations.

- Students make a collaboration graph for the members of the class which describes who has worked with whom during the past week.

- Students draw specified patterns on the chalkboard without retracing, such as those below. Alternatively, they may trace these patterns in a small box of sand, as done historically in African cultures. (See *Ethnomathematics*, *Drawing Pictures With One Line*, or *Insides, Outsides, Loops, and Lines*.) Alternatively, on a pattern of islands and bridges laid out on the floor with masking tape, students might try to take a walk which involves crossing each bridge exactly once (leaving colored markers on bridges already crossed); note that for some patterns this may not be possible. The patterns given here can be used, but students can develop their own patterns and try to take such a walk for each pattern that they create.

- Students create "human graphs" where they themselves are the vertices and they use pieces of yarn (several feet long) as edges; each piece of yarn is held by two students, one at each end. They might create graphs with specified properties; for example, they might create a human graph with four vertices of degree 2, or, as in the figure at the right, with six vertices of which four have degree 3 and two have degree 2. (The **degree** of a vertex is the total number of edges that meet at the vertex.) They might count the number of different shapes of human graphs they can form with four students (or five, or six).



- Students use a floor plan of their school to map out alternate routes from their classroom to the school's exits, and discuss whether the fire drill route is in fact the shortest route to an exit.

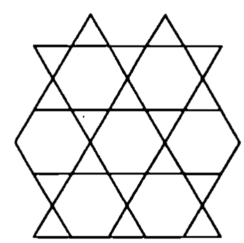
- Students draw graphs of their own neighborhoods, with edges representing streets and vertices representing locations where roads meet. *Can you find a route for the mail carrier in your neighborhood which enables her to walk down each street, without repeating any streets, and which ends where it begins? Can you find such a route if she needs to walk up and down each street in order to deliver mail on both sides of the street?*

- Students color maps (e.g., the 21 counties of New Jersey) so that adjacent counties (or countries) have different colors, using as few colors as possible. The class could then share a NJ cake frosted accordingly. (See *The Mathematician's Coloring Book*.)

- Students recognize and understand family trees in social and historical studies, and in stories that they read. Where appropriate, they create their own family trees.

3. **Identify and investigate sequences and patterns found in nature, art, and music.**

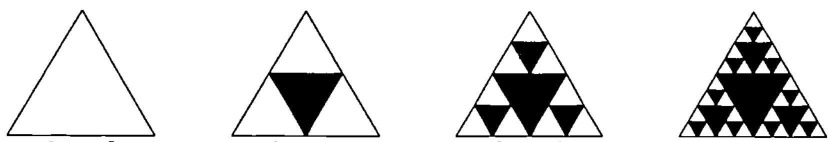
- Students read *A Cloak for a Dreamer* by A. Friedman, and make outlines of cloaks or coats like those worn by the sons of the tailor in the book by tracing their upper bodies on large pieces of paper. Students could use pattern blocks or pre-cut geometric shapes to cover (tessellate) the paper cloaks with patterns like those in the book or try to make their own cloth designs.
- Students read *Sam and the Blue Ribbon Quilt* by Lisa Ernst, and by rotating, flipping, or sliding cut-out squares, rectangles, triangles, etc., create their own symmetrical designs on quilt squares similar to those found in the book. The designs from all the members of the class are put together to make a patchwork class quilt or to form the frame for a math bulletin board.
- Students take a "pattern walk" through the neighborhood, searching for patterns in the trees, the houses, the buildings, the manhole covers (by the way, *why are they always round?*), the cars, etc.; the purpose of this activity is to create an awareness of the patterns around us. *By Nature's Design* is a photographic journey with an eye for many of these natural patterns.
- Students "tessellate" the plane using squares, triangles, or hexagons to completely cover a sheet of paper without overlapping. They also tessellate the plane using groups of shapes, like hexagons and triangles as in the figure at the right.



- Students might ask if their parents would be willing to give them a penny for the first time they do a particular chore, two pennies for the second time they do the chore, four pennies for the third time, eight pennies for the fourth time, and so on. Before asking, they should investigate, perhaps using towers of Unifix cubes that keep doubling in height, how long their parents could actually afford to pay them for doing the chore.
- Students cut a sheet of paper into two halves, cut the resulting two pieces into halves, cut the resulting four pieces into halves, etc. *If they do this a number of times, say 12 times, and stacked all the pieces of paper on top of each other, how high would the pile of paper be?* Students estimate the height before performing any calculations.
- Students color half a large square, then half of the remaining portion with another color, then half of the remaining portion with a third color, etc. *Will the entire area ever get colored? Why, or why not?*
- Students count the number of rows of bracts on a pineapple or pine cone, or rows of petals on an artichoke, or rows of seeds on a sunflower, and verify that these numbers all appear in the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, ... of Fibonacci numbers, where each number is the sum of the two previous numbers on the list. Students find other pictures depicting Fibonacci numbers as they arise in nature, referring, for example, to *Fibonacci Numbers*

in Nature. In *Mathematical Mystery Tour* by Mark Wahl, an elementary school teacher provides a year's worth of Fibonacci explorations and activities.

- Using a large equilateral triangle provided by the teacher, students find and connect the approximate midpoints of the three sides, and then color the triangle in the middle. (See Stage 1 picture.) They then repeat this procedure with each of the three uncolored triangles to get the Stage 2 picture, and then repeat this procedure again with each of the nine uncolored triangles to get the Stage 3 picture. These are the first three stages of the Sierpinski triangle; subsequent stages become increasingly intricate. *How many uncolored triangles are there in the Stage 3 picture? How many would there be in the Stage 4 picture if the procedure were repeated again?*



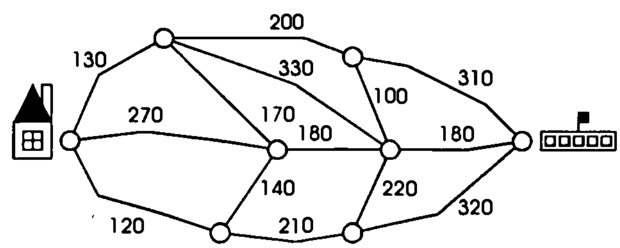
4. Investigate ways to represent and classify data according to attributes, such as shape or color, and relationships, and discuss the purpose and usefulness of such classification.

- Students are provided with a set of index cards on each of which is written a word (or a number). Working in groups, students put the cards in alphabetical (or numerical) order, explain the methods they used to do this, and then compare the various methods that were used.
- Students bring to class names of cities and their zip codes where their relatives and friends live, paste these at the appropriate locations on a map of the United States, and look for patterns which might explain how zip codes are assigned. Then they compare their conclusions with post office information to see whether they are consistent with the way that zip codes actually are assigned.
- Students send and decode messages in which each letter has been replaced by the letter which follows it in the alphabet (or occurs two letters later). Students explore other coding systems described in *Let's Investigate Codes and Sequences* by Marion Smoothey.
- Students collect information about the soft drinks they prefer and discuss various ways of presenting the resulting information, such as tables, bar graphs, and pie charts, displayed both on paper and on a computer.
- Students play the game of *Set* in which participants try to identify three cards from those on display which, for each of four attributes (number, shape, color, and shading), all share the attribute or are all different. Similar ideas can be explored using *Tabletop, Jr.* software.

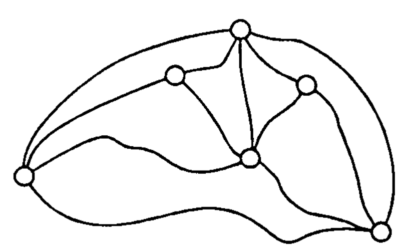
5. Follow, devise, and describe practical lists of instructions.

- Students follow a recipe to make a cake or to assemble a simple toy from its component parts, and then write their own versions of those instructions.

- Students give written and oral directions for going from the classroom to another room in the school, and represent these directions with a diagram drawn approximately to scale.
- Students read *Anno's Mysterious Multiplying Jar* by Mitsumasa Anno. During a second reading they devise a method to record and keep track of the increasing number of items in the book and predict how that number will continue to grow. Each group explains its method to the class.
- Students write step-by-step directions for a simple task like making a peanut butter and jelly sandwich, and follow them to prove that they work.
- Students find and describe the shortest path from the computer to the door or from one location in the school building to another.
- Students find the shortest route from school to home on a map (see figure at right), where each edge has a specified numerical length in meters; students modify lengths to obtain a different shortest route.



- Students write a program which will create specified pictures or patterns, such as a house or a clown face or a symmetrical design. Logo software is well-suited to this activity. In *Turtle Math*, students use Logo commands to go on a treasure hunt, and look for the shortest route to complete the search.
- Working in groups, students create and explain a fair way of sharing a bagful of similar candies or cookies. (See also the vignette entitled *Sharing A Snack* in the Introduction to this *Framework*.) For example, if the bag has 30 brownies and there are 20 children, then they might suggest that each child gets one whole brownie and that the teacher divide each of the remaining brownies in half. Or they might suggest that each pair of children figure out how to share one brownie. *What if there were 30 hard candies instead of brownies? What if there were 25 brownies? What if there were 15 brownies and 15 chocolate chip cookies?* The purpose of this activity is for students to brainstorm possible solutions in the situations where there may be no solution that *everyone* perceives as fair.
- Students devise a strategy for never losing at tic-tac-toe.
- Students find different ways of paving just enough streets of a "muddy city" (like the street map at the right, perhaps laid out on the floor) so that a child can walk from any one location to any other location along paved roadways. In "muddy city" none of the roads are paved, so that whenever it rains all streets turn to mud. The mayor has asked the class to propose different ways of paving the roads so that a person can get from any one location to any other location on paved roads, but so that the fewest number of roads possible are paved.



- Students divide a collection of Cuisenaire rods of different lengths into two or three groups whose total lengths are equal (or as close to equal as possible).

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Software

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Tabletop, Jr. Broderbund Software. TERC.

Turtle Math. LCSI.

On-Line Resources

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Standard 14 — Discrete Mathematics — Grades 5-6

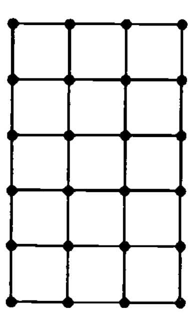
Overview

The five major themes of discrete mathematics, as discussed in the K-12 Overview, are **systematic listing, counting, and reasoning; discrete mathematical modeling using graphs (networks) and trees; iterative (that is, repetitive) patterns and processes; organizing and processing information; and following and devising lists of instructions, called "algorithms," and using them to find the best solution to real-world problems.** Two important resources on discrete mathematics for teachers at all levels are the 1991 NCTM Yearbook *Discrete Mathematics Across the Curriculum K-12* and the 1997 DIMACS Volume *Discrete Mathematics in the Schools: Making an Impact.*

Despite their formidable titles, these five themes can be addressed with activities at the 5-6 grade level which involve both the purposeful play and simple analysis suggested for elementary school students and experimentation and abstraction appropriate at the middle grades. Indeed, teachers will discover that many activities that they already are using in their classrooms reflect these themes. These five themes are discussed in the paragraphs below.

The following discussion of activities at the 5-6 grade levels in discrete mathematics presupposes that corresponding activities have taken place at the K-4 grade levels. Hence 5-6 grade teachers should review the K-2 and 3-4 grade level discussions of discrete mathematics and might use activities similar to those described there before introducing the activities for this grade level.

Activities involving **systematic listing, counting, and reasoning** at K-4 grade levels can be extended to the 5-6 grade level. For example, they might determine the number of possible license plates with two letters followed by three numbers followed by one letter, and decide whether this total number of license plates is adequate for all New Jersey drivers. They need to become familiar with the idea of **permutations**, that is, the different ways in which a group of items can be arranged. Thus, for example, if three children are standing by the blackboard, there are altogether six different ways, call permutations, in which this can be done; for example, if the three children are Amy (A), Bethany (B), and Coriander (C), the six different permutations can be described as ABC, ACB, BAC, BCA, CAB, and CBA. Similarly, the total number of different ways in which three students out of a class of thirty can be arranged at the blackboard is altogether 30x29x28, or 24,360 ways, an amazing total!



An important **discrete mathematical model** is that of a **network** or **graph**, which consists of dots and lines joining the dots; the dots are often called *vertices (vertex* is the singular) and the lines are often called *edges*. (This is different from other mathematical uses of the term "graph"; the two terms "network" and "graph" are used interchangeably for this concept.) An example of a graph with 24 vertices and 38 edges is given at the left. Graphs can be used to represent islands and bridges, or buildings and roads, or houses and telephone cables; wherever a collection of things are joined by connectors, the mathematical model used is that of a graph. At the 5-6 level, students should be familiar with the notion of a graph and recognize situations in which graphs can be an appropriate model. For example, they should be familiar with problems involving routes for garbage pick-ups, school buses, mail deliveries, snow removal, etc.; they should be able to model such problems by using graphs, and be able to solve such problems by finding suitable paths in these graphs, such as in the town whose street map is the graph above.

Students should recognize and work with **repetitive patterns and processes** involving numbers and shapes, with objects found in the classroom and in the world around them. Building on these explorations, fifth- and sixth-graders should also recognize and work with **iterative and recursive processes.** They explore iteration using Logo software, where they recreate a variety of interesting patterns (such as a checkerboard) by iterating the construction of a simple component of the pattern (in this case a square). As with younger students, 5th and 6th graders are fascinated with the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... where every number is the sum of the previous two numbers. Although the Fibonacci sequence starts with small numbers, the numbers in the sequence become large very quickly. Students can now also begin to understand the Fibonacci sequence and other sequences recursively — where each term of the sequence is described in terms of preceding terms.

Students in the 5th and 6th grade should investigate **sorting items** using Venn diagrams, and continue their explorations of **recovering hidden information** by decoding messages. They should begin to **explore how codes are used to communicate information**, by traditional methods such as Morse code or semaphore (flags used for ship-to-ship messages) and also by current methods such as zip codes, which describe a location in the United States by a five-digit (or nine-digit) number. Students should also explore modular arithmetic through applications involving clocks, calendars, and binary codes.

Finally, at grades 5-6, students should be able to **describe, devise, and test algorithms for solving a variety of problems**. These include finding the shortest route from one location to another, dividing a cake fairly, planning a tournament schedule, and planning layouts for a class newspaper.

Two important resources on discrete mathematics for teachers at all levels is the 1991 NCTM Yearbook *Discrete Mathematics Across the Curriculum K-12* and the 1997 DIMACS Volume *Discrete Mathematics in the Schools: Making An Impact*. Another important resource for 5-6 teachers is *This Is MEGA-Mathematics!*

Standard 14 — Discrete Mathematics — Grades 5-6

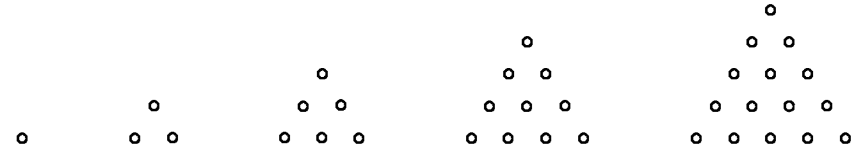
Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences will be such that all students in grades 5-6:

6. Use systematic listing, counting, and reasoning in a variety of different contexts.

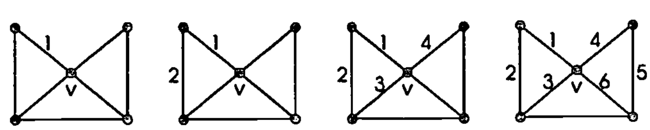
- Students determine the number of different sandwiches or hamburgers that can be created at local eateries using a combination of specific ingredients.
- Students find the number of different ways to make a row of flowers each of which is red or yellow, if the row has 1, 2, 3, 4, or 5 flowers. Modeling this with Unifix cubes, they discover that adding an additional flower to the row doubles the number of possible rows, provide explanations for this, and generalize to longer rows. Similar activities can be found in the *Pizza Possibilities* and *Two-Toned Towers* lessons that are described in the First Four Standards of this *Framework*.
- Students find the number of ways of asking three different students in the class to write three homework problems on the blackboard.
- Students understand and use the concept of permutation. They determine the number of ways any five items can be arranged in order, justify their conclusion using a tree diagram, and use factorial notation, 5!, to summarize the result.
- Students find the number of possible telephone numbers with a given area code and investigate why several years ago the telephone company introduced a new area code (908) in New Jersey, and why additional area codes are being introduced in 1997. *Is the situation the same with zip codes?*
- Students estimate and then calculate the number of possible license plates with two letters followed by three numbers followed by one letter. They investigate why the state license bureau tried to introduce license plates with seven characters and why this attempt might have been unsuccessful.
- Students explore the sequence of triangular numbers 1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, ... which represent the number of dots in the triangular arrays below, and find the location of the triangular numbers in Pascal's triangle.



- Students look for patterns in the various diagonals of Pascal's triangle, and in the differences between consecutive terms in these diagonals. *Patterns in Pascal's Triangle Poster* is a nice resource for introducing these ideas.
- Students analyze simple games like the following: Beth wins the game whenever the two dice give an even total, and Hobart wins whenever the two dice give an odd total. They play the game a number of times, and using experimental evidence, decide whether the game is fair, and, if not, which player is more likely to win. They then try to justify their conclusions theoretically, by counting the number of combinations of dice that would result in a win for each player.
- Students create a table in the form of a grid which indicates how many of each of the coins of the fictitious country "Ternamy" — in denominations of 1, 3, 9, 27, and 81 "terns" — are needed to make up any amount from 1 to 200. They list the denominations in the columns at the top of the table and the amounts they are trying to make in the rows at the left. They write the number of each coin needed to add up to the desired amount in the appropriate squares in that row. The only "rule" to be followed is that the least number of coins must be used; for example, three 1's should always be replaced by one 3. This table can be used to introduce base 3 ("ternary") numbers, and then numbers in other bases.

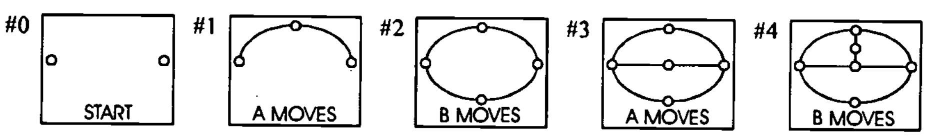
7. Recognize common discrete mathematical models, explore their properties, and design them for specific situations.

- Students experiment with drawing make-believe maps which can be colored with two, three, and four colors (where adjacent countries must have different colors), and explain why their fictitious maps, and real maps like the map of the 50 states, cannot be colored with fewer colors. Note that it was proven in 1976 that no map can be drawn on a flat surface which requires more than four colors. *The Mathematician's Coloring Book* contains a variety of map-coloring activities, as well as historical background on the map coloring problem.
- Students play games using graphs. For example, in the strolling game, two players stroll together on a path through the graph which never repeats itself; they alternate in selecting edges for the path, and the winner is the one who selects the last edge on the path. *Who wins?* In the game below, Charles and Diane both start at V, Charles picks the first edge (marked 1) and they both stroll down that edge. Then Diane picks the second edge (marked 2) and the game continues. Diane has won this play of the game since the path cannot be continued after the sixth edge without repeating itself. *Does Diane have a way of always winning this game, or does Charles have a winning strategy? What if there was a different starting point? What if a different graph was used? What if the path must not cross itself (instead of requiring that it not repeat itself)?* Students should try to explain in each case why a certain player has a winning strategy.



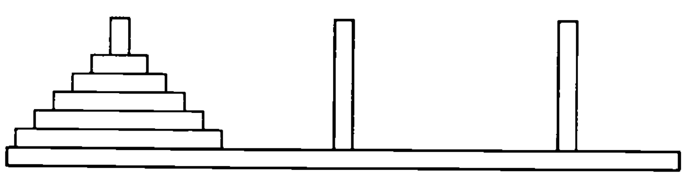
- Students find paths in graphs which utilize each edge exactly once; a path in a graph is a sequence of edges each of which begins where the previous one ends. They apply this idea by converting a street map to a graph where vertices on the graph correspond to intersections on the street map, and by using this graph to determine whether a garbage truck can complete its sector without repeating any streets. See the segment *Snowbound: Euler Circuits* on the videotape *Geometry: New Tools for New Technologies*; the module *Drawing Pictures With One Line* provides a strong background for problems of this kind.

- Students plan emergency evacuation routes at school or from home using graphs.
- All of the students together create a "human graph" where each child in the class is holding two strings, one in each hand. This can be accomplished by placing in the center of the room a number of pieces of yarn (each six feet long) equal to the number of students, and having each student take the ends of two strings. The children are asked to untangle themselves, and discuss or write about what happens.
- Students play the game of Sprouts, in which two students take turns in building a graph until one of them (the winner!) completes the graph. The rules are: start the game with two or three vertices; each person adds an edge (it can be a curved line!) joining two vertices, and then adds a new vertex at the center of that edge; no more than three edges can occur at a vertex; edges may not cross. In the sample game below, the second player (B) wins because the first player (A) cannot draw an edge connecting the only two vertices that have degree less than three without crossing an existing edge.



8. Experiment with iterative and recursive processes, with the aid of calculators and computers.

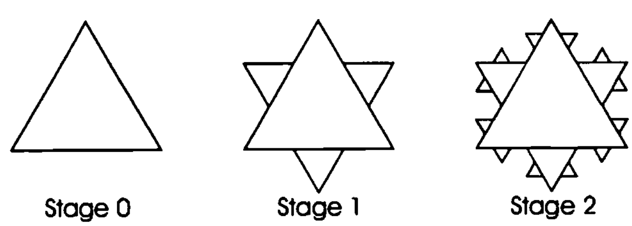
- Students develop a method for solving the Tower of Hanoi problem: There are three pegs, on the first of which is stacked five disks, each smaller than the ones underneath it (see diagram below); the problem is to move the entire stack to the third peg, moving disks, one at a time, from any peg to either of the other two pegs, with no disk ever placed upon a smaller one. *How many moves are required to do this?*



- Students use iteration in Logo software to draw checkerboards, stars, and other designs. For example, they iterate the construction of a simple component of a pattern, such as a square, to recreate an entire checkerboard design.
- Students use paper rabbits (prepared by the teacher) with which to simulate Fibonacci's 13th century investigation into the growth of rabbit populations: *If you start with one pair*

of baby rabbits, how many pairs of rabbits will there be a year later? Fibonacci's assumption was that each pair of baby rabbits results in another pair of baby rabbits two months later — allowing a month for maturation and a month for gestation. Once mature, each pair has baby rabbits monthly. (Each pair of students should be provided with 18 cardboard pairs each of baby rabbits, not-yet-mature rabbits, and mature rabbits.) *The Fascinating Fibonaccis* by Trudi Garland illustrates the rabbit problem and a number of other interesting Fibonacci facts. In *Mathematics Mystery Tour* by Mark Wahl, an elementary school teacher provides a year's worth of Fibonacci explorations and activities.

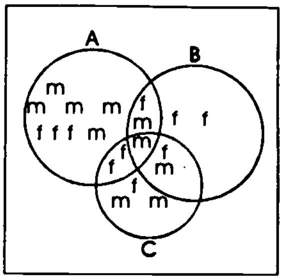
- Students use calculators to compare the growth of various sequences, including counting by 4's (4, 8, 12, 16, ...), doubling (1, 2, 4, 8, 16, ...), squaring (1, 4, 9, 16, 25, ...), and Fibonacci (1, 1, 2, 3, 5, 8, 13, ...).
- Students explore their surroundings to find rectangular objects whose ratio of length to width is the "golden ratio." Since the golden ratio can be approximated by the ratio of two successive Fibonacci numbers, students should cut a rectangular peephole of dimensions 21mm x 34 mm out of a piece of cardboard, and use it to "frame" potential objects; when it "fits," the object is a golden rectangle. They describe these activities in their math journals.
- Students study the patterns of patchwork quilts, and make one of their own. They might first read *Eight Hands Round.*
- Students make equilateral triangles whose sides are 9", 3", and 1" (or other lengths in ratio 3:1), and use them to construct "Koch snowflakes of stage 2" (as shown below) by pasting the 9" triangle on a large sheet of paper, three 3" triangles at the middle of the three sides of the 9" triangle (pointing outward), and twelve 1" triangles at the middle of the exposed sides of the twelve 3" segments (pointing outward). To get Koch snowflakes of stage 3, add forty-eight 1/3" equilateral triangles. *How many 1/9" equilateral triangles would be needed for the Koch snowflake of stage 4? Fractals for the Classroom* is a valuable resource for these kinds of activities and explorations.



- Students mark one end of a long string and make another mark midway between the two ends. They then continue marking the string by following some simple rule such as "make a new mark midway between the last midway mark and the marked end" and then repeat this instruction. Students investigate the relationship of the lengths of the segments between marks. *How many marks are possible in this process if it is assumed that the marks take up no space on the string? What happens if the rule is changed to "make a new mark midway between the last two marks?"*

9. **Explore methods for storing, processing, and communicating information.**

- After discussing possible methods for communicating messages across a football field, teams of students devise methods for transmitting a short message (using flags, flashlights, arm signals, etc.). Each team receives a message of the same length and must transmit it to members of the team at the other end of the field as quickly and accurately as possible.
- Students devise rules so that arithmetic expressions without parentheses, such as 5 x 8 - 2 / 7, can be evaluated unambiguously. They then experiment with calculators to discover the calculators' built-in rules for evaluating these expressions.
- Students explore binary arithmetic and arithmetic for other bases through applications involving clocks (base 12), days of the week (base 7), and binary (base 2) codes.
- Students assign each letter in the alphabet a numerical value (possibly negative) and then look for words worth a specified number of points.
- Students send and decode messages in which letters of the message are systematically replaced by other letters. *The Secret Code Book* by Helen Huckle shows these coding systems as well as others.
- Students use Venn diagrams to sort and then report on their findings in a survey. For example, they can seek responses to the question, *When I grow up I want to be a) rich and famous, b) a parent, c) in a profession I love,* where respondents can choose more than one option. The results can be sorted into a Venn diagram like that at the right, where entries "m" and "f" are used for male and female students. The class can then determine answers to questions like *Are males or females in our class more likely to have a single focus? Tabletop, Jr.* software can be used to sort and explore data using Venn diagrams.



10. **Devise, describe, and test algorithms for solving optimization and search problems.**

- Students use a systematic procedure to find the total number of routes from one location in their town to another, and the shortest such route. (See *Problem Solving Using Graphs*.)
- In *Turtle Math*, students use Logo commands to go on a treasure hunt, and look for the shortest route to complete the search.
- Students discuss and write about various methods of dividing a cake fairly, such as the "divider/chooser method" for two people (one person divides, the other chooses) and the "lone chooser method" for three people (two people divide the cake using the divider/chooser method, then each cuts his/her half into thirds, and then the third person takes one piece from each of the others). *Fair Division: Getting Your Fair Share* can be used to explore methods of fairly dividing a cake or an estate.
- Students conduct a class survey for the top ten songs and discuss different ways to use the information to select the winners.

- Students devise a telephone tree for disseminating messages to all 6th grade students and their parents.
- Students schedule the matches of a volleyball tournament in which each team plays each other team once.
- Students use flowcharts to represent visually the instructions for carrying out a complex project, such as scheduling the production of the class newspaper.
- Students develop an algorithm to create an efficient layout for a class newspaper.

References

Bennett, S., et al. *Fair Division: Getting Your Fair Share.* Consortium for Mathematics and Its Applications (COMAP). Module #9, 1987.

Casey, Nancy, and Mike Fellows. *This is MEGA-Mathematics! —Stories and Activities for Mathematical Thinking, Problem-Solving, and Communication.* Los Alamos, CA: Los Alamos National Laboratories, 1993. (A version is available online at http://www.c3.lanl.gov/mega-math)

Chavey, D. *Drawing Pictures With One Line*. Consortium for Mathematics and Its Applications (COMAP). Module #21, 1992.

Cozzens, M., and R. Porter. *Problem Solving Using Graphs*. Consortium for Mathematics and Its Applications (COMAP). Module #6, 1987.

Francis, R. *The Mathematician's Coloring Book*. Consortium for Mathematics and Its Applications (COMAP). Module #13.

Garland, Trudi. *The Fascinating Fibonaccis*. Palo Alto, CA: Dale Seymor Publications, 1987.

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Kenney, M. J., Ed. *Discrete Mathematics Across the Curriculum K-12*. 1991 Yearbook of the National Council of Teachers of Mathematics (NCTM). Reston, VA, 1991.

Paul, A. *Eight Hands Round*. New York: Harper Collins, 1991.

Peitgen, Heinz-Otto, et al. *Fractals for the Classroom: Strategic Activities Volume One & Two.* Reston, VA: NCTM and New York: Springer- Verlag, 1992.

Rosenstein, J. G., D. Franzblau, and F. Roberts, Eds. *Discrete Mathematics in the Schools: Making an Impact*. Proceedings of a 1992 DIMACS Conference on "Discrete Mathematics in the Schools." DIMACS Series on Discrete Mathematics and Theoretical Computer Science. Providence, RI: American Mathematical Society (AMS), 1997. (Available online from this chapter in http://dimacs.rutgers.edu/nj_math_coalition/framework.html/.)

Wahl, Mark. *Mathematical Mystery Tour: Higher-Thinking Math Tasks*. Tucson, AZ: Zephyr Press, 1988.

Software

Logo. Many versions of Logo are commercially available.

Tabletop, Jr. Broderbund, TERC.

Turtle Math. LCSI.

Video

Geometry: New Tools for New Technologies, videotape by the Consortium for Mathematics and Its Applications (COMAP). Lexington, MA, 1992.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 14 — Discrete Mathematics — Grades 7-8

Overview

The five major themes of discrete mathematics, as discussed in the K-12 Overview, are **systematic listing, counting, and reasoning; discrete mathematical modeling using graphs (networks) and trees; iterative (that is, repetitive) patterns and processes; organizing and processing information; and following and devising lists of instructions, called "algorithms," and using them to find the best solution to real-world problems.**

Despite their formidable titles, these five themes can be addressed with activities at the 7-8 grade level which involve both the purposeful play and simple analysis suggested for elementary school students and experimentation and abstraction appropriate at the middle grades. Indeed, teachers will discover that many activities that they already are using in their classrooms reflect these themes. These five themes are discussed in the paragraphs below.

The following discussion of activities at the 7-8 grade levels in discrete mathematics presupposes that corresponding activities have taken place at the K-6 grade levels. Hence 7-8 grade teachers should review the K-2, 3-4, and 5-6 grade level discussions of discrete mathematics and might use activities similar to those described there before introducing the activities for this grade level.

Students in 7th and 8th grade should be able to use **permutations and combinations and other counting strategies in a wide variety of contexts**. In addition to working with permutations, where the order of the items is important (see Grades 5-6 Overview and Activities), they should also be able to work with combinations, where the order of the items is irrelevant. For example, the number of different three digit numbers that can be made using three different digits is 10x9x8 because each different ordering of the three digits results in a different number. However, the number of different pizzas that can be made using three of ten available toppings is (10x9x8)/(3x2x1) because the *order* in which the toppings are added is irrelevant; the division by 3x2x1 eliminates the duplication.

An important **discrete mathematical model** is that of a **network** or **graph**, which consists of dots and lines joining the dots; the dots are often called *vertices (vertex* is the singular) and the lines are often called *edges*. (This is different from other mathematical uses of the term "graph.") Graphs can be used to represent islands and bridges, or buildings and roads, or houses and telephone cables; wherever a collection of things are joined by connectors, the mathematical model used is that of a graph. Students in the 7th and 8th grades should be able to **use graphs to model situations and solve problems using the model**. For example, students should be able to use graphs to schedule a school's extracurricular activities so that, if at all possible, no one is excluded because of conflicts. This can be done by creating a graph whose vertices are the activities, with two activities joined by an edge if they have a person in common, so that the activities should be scheduled for different times. Coloring the vertices of the graph so that adjacent vertices have different colors, using a minimum number of colors, then provides an efficient solution to the scheduling problem — a separate time slot is needed for each color, and two activities are scheduled for the same time slot if they have the same color.

Students can recognize and work with **iterative and recursive processes**, extending their earlier explorations of **repetitive patterns and procedures.** In the 7th and 8th grade, they can combine their understanding of exponents and iteration to solve problems involving compound interest with a calculator or spreadsheet. Topics which before were viewed iteratively — arriving at the present situation by repeating a procedure *n* times — can now be viewed recursively — arriving at the present situation by modifying the previous situation. They can apply this understanding to Fibonacci numbers, to the Tower of Hanoi puzzle, to programs in Logo, to permutations and to other areas.

Students in the 7th and 8th grades should **explore how codes are used to communicate information**, by traditional methods such as Morse code or semaphore (flags used for ship-to-ship messages) and also by current methods such as zip codes. Students should investigate and report about various codes that are commonly used, such as binary codes, UPCs (universal product codes) on grocery items, and ISBN numbers on books. They should also **explore how information is processed.** A useful metaphor is how a waiting line or queue is handled (or "processed") in various situations; at a bank, for example, the queue is usually processed in first-in-first-out (FIFO) order, but in a supermarket or restaurant there is usually a pre-sorting into smaller queues done by the shoppers themselves before the FIFO process is activated.

In the 7th and 8th grade, students should be able to **use algorithms to find the best solution in a number of situations** — including the shortest route from one city to another on a map, the cheapest way of connecting sites into a network, the fastest ways of alphabetizing a list of words, the optimal route for a class trip (see the *Short-Circuiting Trenton* lesson in the Introduction to this *Framework*), or optimal work schedules for employees at a fast-food restaurant.

Two important resources on discrete mathematics for teachers at all levels are the 1991 NCTM Yearbook *Discrete Mathematics Across the Curriculum K-12* and the 1997 DIMACS Volume *Discrete Mathematics in the Schools: Making an Impact*. Teachers of grades 7-8 would also find useful the textbook *Discrete Mathematics Through Applications*.

Standard 14 — Discrete Mathematics — Grades 7-8

Indicators and Activities

The cumulative progresses indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

Building upon knowledge and skills gained in the preceding grades, experiences will be such that all students in grades 7-8:

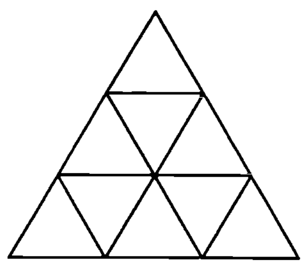
6. Use systematic listing, counting, and reasoning in a variety of different contexts.

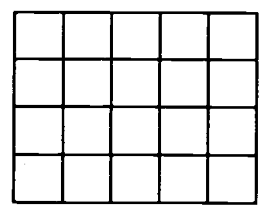
- Students determine the number of possible different sandwiches or hamburgers that can be created at local eateries using a combination of specified ingredients. They find the number of pizzas that can be made with three out of eight available toppings and relate the result to the numbers in Pascal's triangle.
- Students determine the number of dominoes in a set that goes up to 6:6 or 9:9, the number of candles used throughout Hannukah, and the number of gifts given in the song "The Twelve Days of Christmas," and connect the results through discussion of the triangular numbers. (Note that in a 6:6 set of dominoes there is exactly one domino with each combination of dots from 0 to 6.)
- Students determine the number of ways of spelling "Pascal" in the array below by following a path from top to bottom in which each letter is directly below, and just to the right or left of the previous letter.

```
          P
        A   A
      S   S   S
    C   C   C   C
  A   A   A   A   A
L   L   L   L   L   L
```

- Students design different license plate systems for different population sizes; for example, *how large would the population be before you would run out of plates which had only three numbers, or only five numbers, or two letters followed by three numbers?*
- Students find the number of different ways of making a row of six red and yellow flowers; organize and tabulate the possibilities according to the number of flowers of the first color, and explain the connection with the numbers in the sixth row of Pascal's triangle. (See also *Visual Patterns in Pascal's Triangle*.)
- Students pose and act out problems involving the number of different ways a group of people can sit around a table, using as motivation the scene of the Mad Hatter at the tea party. (See *Mathematics, a Human Endeavor*, p. 394.)

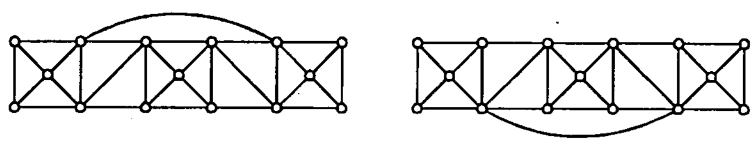
- Students count the total number of different cubes that can be made using either red or green paper for each face. (To solve this problem, they will have to use a "break up the problem into cases" strategy.)
- Students determine the number of handshakes that take place if each person in a room shakes hands with every other person exactly once, and relate this total to the number of line segments joining the vertices in a polygon, to the number of two-flavor ice-cream cones, and to triangular numbers.
- Students count the number of triangles or rectangles in a geometric design. For example, they should be able to count systematically the number of triangles (and trapezoids) in the figure below to the left, noting that there are triangles of three sizes, and the number of rectangles in the 4x5 grid pictured below to the right, listing first all dimensions of rectangles that are present.



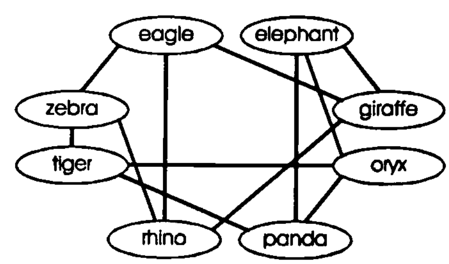


7. **Recognize common discrete mathematical models, explore their properties, and design them for specific situations.**

- Students find the minimum number of colors needed to assign colors to all vertices in a graph so that any two adjacent vertices are assigned different colors and justify their answers. For example, students can explain why one of the graphs below requires four colors while for the other, three colors are sufficient.



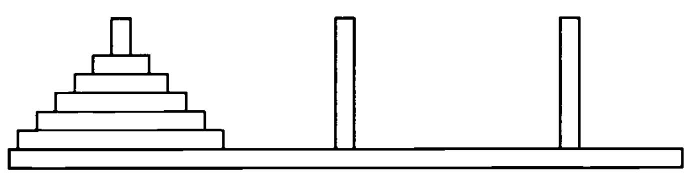
- Students use graph coloring to solve problems which involve avoiding conflicts such as: scheduling the school's extra curricular activities; scheduling referees for soccer games; determining the minimum number of aquariums needed for a specified collection of tropical fish; and assigning channels to radio stations to avoid interference. In the graph at the right an edge between two animals indicates that they cannot share a habitat. The videotape, *Geometry: New Tools for New Technologies* has a segment *Connecting the Dots: Vertex Coloring* which discusses the minimum number of habitats required for this situation.



- Students use tree diagrams to represent and analyze possible outcomes in counting problems, such as tossing two dice.
- Students determine whether or not a given group of dominoes can be arranged in a line (or in a rectangle) so that the number of dots on the ends of adjacent dominoes match. For example, the dominoes (03), (05), (12), (14), (15), (23), (34) can be arranged as (12), (23), (30), (05), (51), (14), (43); and if an eighth domino (13) is added, they can be formed into a rectangle. *What if instead the eighth domino was (24) — could they then be arranged in a rectangle or in a line?*
- Students determine the minimum number of blocks that a police car has to repeat if it must try to patrol each street exactly once on a given map. *Drawing Pictures With One Line* contains similar real-world problems and a number of related game activities.
- Students find the best route for collecting recyclable paper from all classrooms in the school, and discuss different ways of deciding what is the "best." (See *Drawing Pictures With One Line*.)
- Students make models of various polyhedra with straws and string, and explore the relationship between the number of edges, faces, and vertices.

8. Experiment with iterative and recursive processes, with the aid of calculators and computers.

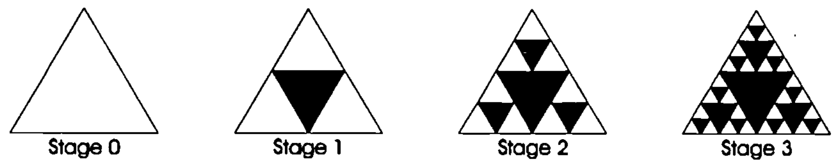
- Students develop a method for solving the Tower of Hanoi problem: There are three pegs, on the first of which five disks are stacked, each smaller than the ones underneath it (see diagram below); the problem is to move the entire stack to the third peg, moving disks, one at a time, from any peg to either of the other two pegs, with no disk ever placed upon a smaller one. *How many moves are required to do this? What if there were 6 disks? How long would it take to do this with 64 disks?* (An ancient legend predicts that when this task is completed, the world will end; should we worry?)

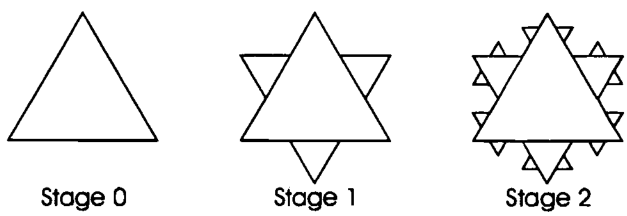


  Students view recursively Tower of Hanoi puzzles with various numbers of disks so that they can express the number of moves needed to solve the puzzle with one more disk in terms of the number of moves needed for the puzzle with the current number of disks.
- Students attempt to list the different ways they could travel 10 feet in a straight line if they were a robot which moved only in one or two foot segments, and then thinking recursively determine the number of different ways this robot could travel n feet.
- Students develop arithmetic and geometric progressions on a calculator.
- Students find square roots using the following iterative procedure on a calculator. Make an estimate of the square root of a number B, divide the estimate into B, and average the result with the estimate to get a new estimate. Then repeat this procedure until an

adequate estimate is obtained. *For example, if the first estimate of the square root of 10 is 3, then the second would be the average of 3 and 10/3, or 19/6 =* $3.166\overline{6}$. *What is the next estimate of the square root of 10? How many repetitions are required to get the estimate to agree with the square root of 10 provided by the calculator?*

- Students develop the sequence of areas and perimeters of iterations of the constructions of the Sierpinski triangle (top figures) and the Koch snowflake (bottom figures), and discuss the outcome if the process were continued indefinitely. (These are discussed in more detail in the sections for earlier grade levels. See Unit 1 of *Fractals for the Classroom* for related activities.)





- Students recognize the computation of the number of permutations as a recursive process — that is, that the number of ways of arranging 10 students is 10 times the number of ways of arranging 9 students.

9. Explore methods for storing, processing, and communicating information.

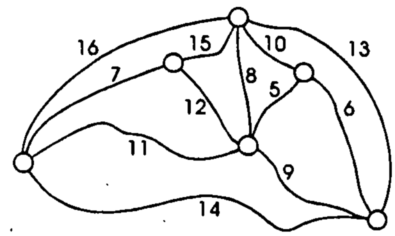
- Students conjecture which of the following (and other) methods is the most efficient way of handing back corrected homework papers which are already sorted alphabetically: (1) the teacher walks around the room handing to each student individually; (2) students pass the papers around, each taking their own; (3) students line themselves up in alphabetical order. Students test their conjectures and discuss the results.
- Students investigate and report about various codes that are commonly used, such as zip codes, UPCs (universal product codes) on grocery items, and ISBN numbers on books. (A good source for information about these and other codes is *Codes Galore* by J. Malkevitch, G. Froelich, and D. Froelich.)
- Students write a Logo procedure for making a rectangle that uses variables, so that they can use their rectangle procedure to create a graphic scene which contains objects, such as buildings, of varying sizes.
- Students are challenged to guess a secret word chosen by the teacher from the dictionary, using at most 20 yes-no questions. *Is this always, or only sometimes possible?*

- Students use Venn diagrams to solve problems like the following one from the New Jersey Department of Education's *Mathematics Instruction Guide* (p. 7-13). *Suppose the school decided to add the springtime sport of lacrosse to its soccer and basketball offerings for its 120 students. A follow-up survey showed that: 35 played lacrosse, 70 played soccer, 40 played basketball, 20 played both soccer and basketball, 15 played both soccer and lacrosse, 15 played both basketball and lacrosse, and 10 played all three sports. Using this data, complete a Venn diagram and answer the following questions: How many students played none of the three sports? What percent of the students played lacrosse as their only sport? How many students played both basketball and lacrosse, but not soccer?*
- Students keep a scrapbook of different ways in which information is stored or processed. For example a list of events is usually stored by date, so the scrapbook might contain a picture of a pocket calendar; a queue of people at a bank is usually processed in first-in-first-out (FIFO) order, so the scrapbook could contain a picture of such a queue. (*How is this different from the waiting lines in a supermarket, or at a restaurant?)*
- Students determine whether it is possible to have a year in which there is no Friday the 13th, and the maximum number of Friday the 13th's that can occur in one calendar year.
- Students predict and then explore the frequency of letters in the alphabet through examination of sample texts, computer searches, and published materials.
- Students decode messages where letters are systematically replaced by other letters without knowing the system by which letters are replaced; newspapers and games magazines are good sources for "cryptograms" and students can create their own. They also explore the history of code-making and code-breaking. The videotape *Discrete Mathematics: Cracking the Code* provides a good introduction to the uses of cryptography and the mathematics behind it.

10. Devise, describe, and test algorithms for solving optimization and search problems.

- Students find the shortest route from one city to another on a New Jersey map, and discuss whether that is the best route. (See *Problem Solving Using Graphs*.)
- Students write and solve problems involving distances, times, and costs associated with going from towns on a map to other towns, so that different routes are "best" according to different criteria.
- Students use binary representations of numbers to find winning strategy for Nim. (See *Mathematical Investigations* for other mathematical games.)
- Students plan an optimal route for a class trip. (See the *Short-circuiting Trenton* lesson in the Introduction to this *Framework*.)
- Students devise work schedules for employees of a fast-food restaurant which meet specified conditions yet minimize the cost.
- Students compare strategies for alphabetizing a list of words, and test to see which strategies are more efficient.

- Students find a network of roads which connects a number of sites and involves the smallest cost. *In the example at the right, what roads should be built so as to minimize the total cost, where the number on each road reflects the cost of building that road (in hundreds of thousands of dollars)?*



- Students develop a precise description of the standard algorithm for adding two two-digit integers.
- Students devise strategies for dividing up the work of adding a long list of numbers among the members of the team.

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On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 14 — Discrete Mathematics — Grades 9-12

Overview

The five major themes of discrete mathematics, as discussed in the K-12 Overview, are **systematic listing, counting, and reasoning; discrete mathematical modeling using graphs (networks) and trees; iterative (that is, repetitive) patterns and processes; organizing and processing information;** and **following and devising lists of instructions, called "algorithms," and using them to find the best solution to real-world problems.**

The following discussion of activities at the 9-12 grade levels in discrete mathematics presupposes that corresponding activities have taken place at the K-8 grade levels. Hence high school teachers should review the discussions of discrete mathematics at earlier grade levels and might use activities similar to those described there before introducing the activities for these grade levels.

At the high school level, students are becoming familiar with algebraic and functional notation, and their understanding of all of the themes of discrete mathematics and their ability to generalize earlier activities should be **enhanced by their algebraic skills and understandings**. Thus, for example, they should use formulas to express the results of problems involving permutations and combinations, relate Pascal's triangle to the coefficients of the binomial expansion of $(x+y)^n$, explore models of growth using various algebraic models, explore iterations of functions, and discuss methods for dividing an estate among several heirs.

At the high school level, students are particularly interested in applications; they ask *What is all of this good for?* In all five areas of discrete mathematics, students should **focus on how discrete mathematics is used to solve practical problems**. Thus, for example, they should be able to apply their understanding of counting techniques, to analyze lotteries; of graph coloring, to schedule traffic lights at a local intersection; of paths in graphs, to devise patrol routes for police cars; of iterative processes, to analyze and predict fish populations in a pond or concentration of medicine in the bloodstream; of codes, to understand how bar-code scanners detect errors and how CD's correct errors; and of optimization, to understand the 200 year old debates about apportionment and to find efficient ways of scheduling the components of a complex project.

Two important resources on discrete mathematics for teachers at all grade levels are the 1991 NCTM Yearbook, *Discrete Mathematics Across the Curriculum K-12* and the DIMACS Volume, *Discrete Mathematics in the Schools: Making an Impact* edited by J. Rosenstein, D. Franzblau, and F. Roberts. Useful resources at the high school level are *Discrete Mathematics Through Applications* by N. Crisler, P. Fisher, and G. Froelich; *For All Practical Purposes: Introduction to Contemporary Mathematics*, by the Consortium for Mathematics and its Applications; and *Excursions in Modern Mathematics* by P. Tannenbaum and R. Arnold.

Standard 14 — Discrete Mathematics — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11, and 12.

Building upon knowledge and skills gained in the preceding grades, experiences will be such that all students in grades 9-12:

11. **Understand the basic principles of iteration, recursion, and mathematical induction.**

    - Students relate the possible outcomes of tossing five coins with the binomial expansion of $(x+y)^5$ and the fifth row of Pascal's triangle, and generalize to values of *n* other than 5.
    - Students develop formulas for counting paths on grids or other simple street maps.
    - Students find the number of cuts needed in order to divide a giant pizza so that each student in the school gets at least one piece.
    - Students develop a precise description, using iteration, of the standard algorithm for adding two integers.

12. **Use basic principles to solve combinatorial and algorithmic problems.**

    - Students determine the number of ways of spelling "mathematics" in the array below by following a path from top to bottom in which each letter is directly below, and just to the right or left of the previous letter.

```
          M
        A   A
      T   T   T
    H   H   H   H
  E   E   E   E   E
M   M   M   M   M   M
  A   A   A   A   A
    T   T   T   T
      I   I   I
        C   C
          S
```

    - Students determine the number of ways a committee of three members could be selected from the class, and the number of ways three people with specified roles could be selected. They generalize this activity to finding a formula for the number of ways an *n* person committee can be selected from a class of *m* people, and the number of ways *n* people with specified roles can be selected from a class of *m* people.

- Students find the number of ways of lining up thirty students in a class, and compare that to other large numbers; for example, they might compare it to the number of raindrops (volume = .1 cc) it would take to fill a sphere the size of the earth (radius = 6507 KM).
- Students determine the number of ways of dividing 52 cards among four players, as in the game of bridge, and compare the number of ways of obtaining a flush (five cards of the same suit) and a full house (three cards of one denomination and two cards of another) in the game of poker.
- Students play Nim (and similar games) and discuss winning strategies using binary representations of numbers.

13. Use discrete models to represent and solve problems.

- Students study the four color theorem and its history. (*The Mathematicians' Coloring Book* provides a good background for coloring problems.)
- Students using graph coloring to determine the minimum number of guards (or cameras) needed for museums of various shapes (and similarly for placement of lawn sprinklers or motion-sensor burglar alarms).
- Students use directed graphs to represent tournaments (where an arrow drawn from A to B represents "A defeats B") and food webs (where an arrow drawn from A to B represents "A eats B"), and to construct one-way orientations of streets in a given town which involve the least inconvenience to drivers. (A directed graph is simply a graph where each edge is thought of as an arrow pointing from one endpoint to the other.)
- Students use tree diagrams to analyze the play of games such as tic-tac-toe or Nim, and to represent the solutions to weighing problems. Example: Given 12 coins one of which is "bad," find the bad one, and determine whether it is heavier or lighter than the others, using three weighings.
- Students use graph coloring to schedule the school's final examinations so that no student has a conflict, if at all possible, or to schedule traffic lights at an intersection.
- Students devise graphs for which there is a path that covers each edge of the graph exactly once, and other graphs which have no such paths, based on an understanding of necessary and sufficient conditions for the existence of such paths, called "Euler paths," in a graph. *Drawing Pictures With One Line* provides background and applications for Euler path problems.
- Students make models of polyhedra with straws and string, and explore the relationship between the numbers of edges, faces, and vertices, and generalize the conclusion to planar graphs.
- Students use graphs to solve problems like the "fire-station problem": *Given a city where the streets are laid out in a grid composed of many square blocks, how many fire stations are needed to provide adequate coverage of the city if each fire station services its square block and the four square blocks adjacent to that one?* The Maryland Science Center in Baltimore has a hands-on exhibit involving a fire-station problem for 35 square blocks arranged in a six-by-six grid with one corner designated a park.

14. Analyze iterative processes with the aid of calculators and computers.

- Students analyze the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ... as a recurrence relation $A_{n+2} = A_n + A_{n+1}$ with connections to the golden ratio. *Fascinating Fibonaccis* illustrates a variety of connections between Fibonacci numbers and the golden ratio.
- Students solve problems involving compound interest using iteration on a calculator or on a spreadsheet.
- Students explore examples of linear growth, using the recursive model based on the formula $A_{n+1} = A_n + d$, where d is the common difference, and convert it to the explicit linear formula, $A_{n+1} = A_1 + n \cdot d$.
- Students explore examples of population growth, using the recursive model based on the formula $A_{n+1} = A_n \times r$, where r is the common multiple or growth rate, convert it to the explicit exponential formula $A_{n+1} = A_1 r^n$, and apply it to both economics (such as interest problems) and biology (such as concentration of medicine in blood supply).
- Students explore logistic growth models of population growth, using the recursive model based on the formula $A_{n+1} = A_n \times (1 - A_n) \times r$, where r is the growth rate and A_n is the fraction of the carrying capacity of the environment, and apply this to the population of fish in a pond. Using a spreadsheet, students experiment with various values of the initial value A_1 and of the growth rate, and describe the relationship between the values chosen and the long term behavior of the population.
- Students explore the pattern resulting from repeatedly multiplying $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ by itself.
- Students use a calculator or a computer to study simple Markov chains, such as weather prediction and population growth models. (See Chapter 7.3 of *Discrete Mathematics Through Applications*.)
- Students explore graphical iteration by choosing a function key on a calculator and pressing it repeatedly, after choosing an initial number, to get sequences of numbers like 2, 4, 8, 16, 32, ... or 2, $\sqrt{2}$, $\sqrt{\sqrt{2}}$, $\sqrt{\sqrt{\sqrt{2}}}$, ... They use the graphs of the functions to explain the behavior of the sequences obtained. They extend these explorations by iterating functions they program into the calculator, such as linear functions, where slope is the predictor of behavior, and quadratic functions $f(x) = ax(1 - x)$, where $0 < x < 1$ and $1 < a < 4$, which exhibit chaotic behavior.
- Students explore iteration behavior using the function defined by the two cases

  $f(x) = x + 1/2$ for x between 0 and 1/2
  $f(x) = 2 - 2x$ for x between 1/2 and 1

  They use the initial values 1/2, 2/3, 5/9, and 7/10, and then, with a calculator or computer, the initial values .501, .667, and .701 (which differ by a small amount from the first group of "nice" initial values). They compare the behavior of the sequences generated by these values to the sequences generated by the previous initial values.
- Students play the *Chaos Game*. Each pair of students is provided with an identical transparency on which have been drawn the three vertices L, T, and R of an equilateral

triangle. Each team starts by selecting any point on the triangle. They roll a die and create a new point halfway to L if they roll 1 or 2, halfway to R if they roll 3 or 4, and halfway to T if they roll 5 or 6. They repeat 20 times, each time using the new point as the starting point for the next iteration. The teacher overlays all of the transparencies and out of this chaos comes ... the familiar Sierpinski triangle. (The Sierpinski triangle is discussed in detail in the sections for earlier grade levels. Also see Unit 2 in *Fractals for the Classroom*. *The Chaos Game* software allows students to try variations and explore the game further.)

15. Apply discrete methods to storing, processing, and communicating information.

- Students discuss various algorithms used for sorting large numbers of items alphabetically or numerically, and explain why some sorting algorithms are substantially faster than others. To introduce the topic of sorting, give each group of students 100 index cards each with one word on it, and let them devise strategies for efficiently putting the cards into alphabetical order.
- Students discuss how scanners of bar codes (zip codes, UPCs, and ISBNs) are able to detect errors in reading the codes, and evaluate and compare how error-detection is accomplished in different codes. (See the COMAP Module *Codes Galore* or Chapter 9 of *For All Practical Purposes*.)
- Students investigate methods of error correction used to transmit digitized pictures from space (Voyager or Mariner probes, or the Hubble space telescope) over noisy or unreliable channels, or to ensure the fidelity of a scratched CD recording. (See Chapter 10 of *For All Practical Purposes*.)
- Students read about coding and code-breaking machines and their role in World War II.
- Students research topics that are currently discussed in the press, such as public-key encryption, enabling messages to be transmitted securely, and data-compression, used to save space on a computer disk.

16. Apply discrete methods to problems of voting, apportionment, and allocations, and use fundamental strategies of optimization to solve problems.

- Students find the best route when a number of alternate routes are possible. For example: *In which order should you pick up the six friends you are driving to the school dance? In which order should you make the eight deliveries for the drug store where you work? In which order should you visit the seven "must-see" sites on your vacation trip?* In each case, you want to find the "best route," the one which involves the least total distance, or least total time, or least total expense. Students create their own problems, using actual locations and distances, and find the best route. For a larger project, students can try to improve the route taken by their school bus.
- Students study the role of apportionment in American history, focusing on the 1790 census (acting out the positions of the thirteen original states and discussing George Washington's first use of the presidential veto), and the disputed election of 1876, and discuss the relative merits of different systems of apportionment that have been proposed and used. (This activity provides an opportunity for mathematics and history teachers to work together.) They also devise a student government where the seats are fairly apportioned

among all constituencies. (See the COMAP module *The Apportionment Problem* or Chapter 14 of *For All Practical Purposes*.)

- Students analyze mathematical methods for dividing an estate fairly among various heirs. (See Chapter 2 of *Discrete Mathematics Through Applications*, Chapter 3 of *Excursions in Modern Mathematics*, or Chapter 13 of *For All Practical Purposes*.)

- Students discuss various methods, such as preference schedules or approval voting, that can be used for determining the winner of an election involving three or more candidates (for example, the prom king or queen). With preference schedules, each voter ranks the candidates and the individual rankings are combined, using various techniques, to obtain a group ranking; preference schedules are used, for example, in ranking sports teams or determining entertainment awards. In approval voting, each voter can vote once for each candidate which she finds acceptable; the candidate who receives the most votes then wins the election. (See the COMAP module *The Mathematical Theory of Elections* or Chapter 11 of *For All Practical Purposes*.)

- Students find an efficient way of doing a complex project (like preparing an airplane for its next trip) given which tasks precede which and how much time each task will take. (See Chapter 8 of *Excursions of Modern Mathematics* or Chapter 3 of *For All Practical Purposes*.)

- Students find an efficient way of assigning songs of various lengths to the two sides of an audio tape so that the total times on the two sides are as close together as possible. Similarly, they determine the minimal number of sheets of plywood needed to build a cabinet with pieces of specified dimensions.

- Students apply algorithms for matching in graphs to schedule when contestants play each other in the different rounds of a tournament.

- Students devise a strategy for finding a "secret number" from 1 to 1000 using questions of the form *Is your number bigger than 837?* and determine the least number of questions needed to find the secret number.

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On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

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STANDARD 15 — CONCEPTUAL BUILDING BLOCKS OF CALCULUS

K-12 Overview

All students will develop an understanding of the conceptual building blocks of calculus and will use them to model and analyze natural phenomena.

Descriptive Statement

The conceptual building blocks of calculus are important for everyone to understand. How quantities such as world population change, how fast they change, and what will happen if they keep changing at the same rate are questions that can be discussed by elementary school students. Another important topic for all mathematics students is the concept of infinity — what happens as numbers get larger and larger and what happens as patterns are continued indefinitely. Early explorations in these areas can broaden students' interest and understanding of an important area of applied mathematics.

Meaning and Importance

Calculus is the mathematics used to describe processes evolving in space or time. **How quantities change** — the velocity of a car as its position changes over time or the area of a square as its sides lengthen — and **what happens in the long run** are central themes of mathematics and its application to the real world. Calculus is used to describe an exact result as the **limit of a sequence of approximations.** Calculus is essential for understanding the physical world and indispensable in economics and industry. Engineers and physicists use calculus to calculate motion in response to forces. Businessmen and economists use calculus to find optimal solutions to pricing and production. An intuitive feel for the mathematics of infinity, limit, and change are accessible and necessary for all students.

Although some students will go on to study these concepts in a formal calculus course, this standard does **not** advocate the formal study of calculus in high school for all students or even for all college-intending students. Rather, it calls for providing opportunities for all students to informally investigate the central ideas of calculus. Considering these concepts will contribute to a deeper understanding of the function concept and its usefulness in representing and answering questions about real-world situations.

K-12 Development and Emphases

Students at the elementary level should understand the concept of **linear growth** (constant increments). For instance, a savings account grows linearly if equal deposits are made at regular intervals and the account bears no interest. This idea and its extensions can be introduced and mastered without using the mathematical formalism of functions, which is introduced later in the middle grades. Beginning in elementary school, children can accumulate records of processes which exhibit linear growth. Some examples are mileage accumulated by traveling between home and school, cumulative expenses for school lunches, and cumulative volume of cereal consumed if everyone in the class eats a bowl every morning. Children should learn to recognize linear growth and compare it to the more irregular pattern of increases in their own height, the height of a bean plant, cumulative rainfall, class consumption of paper, and real expenditures. Children in elementary school can be introduced to **exponential growth** (ever-increasing increments) through the discovery that if every pair of rabbits produces two rabbits each month (one new rabbit for every existing rabbit) then in less than two years there would be more than a million rabbits!

Middle school students should be moving beyond the concrete and pictorial representations used in the elementary grades to more symbolic ones, involving functions and equations. They should use graphing calculators and computers to develop and analyze graphical representations of the changes represented in the tables, and to produce linear and quadratic regression models of the data. They should apply their knowledge of decimals to solving problems involving compound interest, making use of a calculator to determine, for example, the yield of a given investment or the length of time it would take for an investment to double. In high school, students can apply their knowledge of exponents, algebra, and functions to solve these and other more difficult problems, with applications to growth in economics and biology (e.g. population explosion), algebraically and graphically.

Throughout their school years, students should be examining a variety of situations where populations and other quantities **change over time**, and use the mathematical tools at their disposal to describe and analyze this change. As they progress, the situations considered should become more complex; students who experiment with constant motion in their early years will be able to understand the motion of projectiles (a ball thrown into the air, for example) by the time they complete high school.

Similarly, students should be aware how changing the linear dimensions of an object — such as its height, length, or diameter — affects its area and volume. In the early years children should be involved in hands-on experiments which illustrate this; for example, they might find that doubling the diameter of a circular can (of fixed height) increases the volume four-fold by filling the smaller can with water or rice and emptying it into the larger one. By the time students are familiar with variables, this intuition will provide them with the information they need to understand formulas such as those involving volume.

In many settings, the kind of change that takes place over time is repetitive and cumulative, and an important question that should be discussed is what happens in the long run. The principal tool for understanding and discussing such questions is the concept of infinite sequences and the types of patterns that emerge from them. Thus a second central theme is that of **infinity.**

Students are fascinated with the mysteries of large numbers and "infinity," and that excitement should be nourished and be used, as with other "teachable moments," to motivate the learning of more mathematics. Primary students enjoy naming their "largest" number or proudly declaring that there is no largest! In the early years, large numbers and their significance should be discussed, as should the idea that one can extend simple processes forever (e.g., keep adding 2, keep multiplying by 3).

Once students have familiarity with fractions and decimals, these notions can be extended. *What happens when you keep dividing by 2? By 10? Can you find a fraction between 0.999 and 1? What decimal comes just before 1?* Students should explore and experiment with infinitely repeating decimals and other infinite series, where they can make tables and look for patterns. They should learn that by repeated iteration of simple processes you can get better and better answers in both arithmetic (with increased decimal accuracy) and in geometry (with more accurate estimates of the area and volume of irregular objects).

Although the concept of a limiting value (or a limit) may appear inaccessible to K-8 students, this basic notion of calculus can be explored through the process of measuring the area of a region. Students can be provided with diagrams of a large circular (or irregular) region, say a foot in diameter, and a large supply of tiles of different square sizes. By covering the space inside the region (with no protrusions!) with 4" tiles, then with 2" tiles, then with 1" tiles, then with .5" tiles, students can gain an appreciation that the smaller the unit used, the larger the area obtained. They will recognize that the space cannot be filled completely with small tiles, yet, at the same time, the sum of the areas of the smaller tiles gets closer and closer to the actual area of the region.

IN SUMMARY, these kinds of experiences will provide a good foundation for the notions of limits, infinity, and changes in quantities over time. Such concepts find many applications in both science and mathematics, and students will feel much more comfortable with them if they begin to develop these concepts in the early grades.

NOTE: *Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

Standard 15 — Conceptual Building Blocks of Calculus — Grades K-2

Overview

Students in the early primary grades bring to the classroom intuitive notions of the meaning of such terms as *biggest, largest, change*, and so forth. While they may not know the names of large numbers, they certainly have a sense of "largeness." The cumulative process indicators related to this standard for grades K-2 deal primarily with investigating patterns of growth and change over time.

Students in grades K-2 should investigate many different types of patterns. For some of these patterns, such as 2, 4, 6, 8, ... , the same number is added (or subtracted) to each number to get the next number in the sequence. When these patterns are represented with a bar graph, the tops of the bars can be connected by a straight line, so the pattern represents **linear growth.** Older students should also see patterns that grow more rapidly, such as 2, 4, 8, ... These growing patterns involve **exponential growth**; each number in the series is multiplied (or divided) by the same number to get the next one. In this situation, when the tops of the bars on a graph are connected, they do not form a straight line. These types of patterns can be investigated very easily by using calculators to do the computation; students enjoy making the numbers bigger and bigger by using a constant addend (e.g., 2 + 2 = = =) or a constant multiplier (e.g., 2 x 2 = = =). (Note that some calculators require different keystrokes to achieve this effect.) By relating these problems to concrete situations, such as the growth of a plant, students begin to develop a sense of **change over time.**

Students also begin to develop a sense of change with respect to **measurement**. Students begin to measure the length of objects by using informal units such as paperclips or Unifix cubes; they should note that it takes more small objects to measure a given length than large ones. By the end of second grade, they begin to describe the area of objects by counting the number of squares that cover a figure. Again, they should note that it takes more small squares to cover an object than it does large ones. They should also begin to investigate what happens to the area of a square when each side is doubled. Students also need to develop volume concepts by filling containers of different sizes. They might use two circular cans, one of which is twice as high and twice as wide as the other, to find that the large one holds eight times as much as the small one. Measurement may also lead to the beginnings of the idea of **a limiting value** for young children. For example, the size of a dinosaur footprint might be measured by covering it with base ten blocks. If only the 100 blocks are used, then one estimate of the size of the footprint is found; if unit blocks are used, a more precise estimate of the size of the footprint can be found.

Students in grades K-2 should also begin to look at concepts involving **infinity**. As they learn to count to higher numbers, they begin to understand that, no matter how high they count, there is always a bigger number. By using calculators, they can also begin to see that they can continue to add two to a number forever and the result will just keep getting bigger.

The conceptual underpinnings of calculus for students in grades K-2 are closely tied to their developing understanding of number sense, measurement, and pattern. Additional activities relating to this standard can be found in the chapters discussing these other standards.

Standard 15 — Conceptual Building Blocks of Calculus — Grades K-2

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in kindergarten and grades 1 and 2.

Experiences will be such that all students in grades K-2:

1. **Investigate and describe patterns that continue indefinitely.**

    - Students model repeating patterns with counters or pennies. For example, they repeatedly add two pennies to their collection and describe the results.
    - Students create repeating patterns with the calculator. They enter any number such as 10, and then add *1* for *10 + 1 = = = ...* The calculator will automatically repeat the function and display 11, 12, 13, 14, etc. each time the = key is pressed. (Some calculators may need to have the pattern entered twice: *10 + 1 = + 1 = = = . . .* etc. Others may use a key sequence such as *1 + + 10 = = =. . . .*) Students may repeatedly add (or subtract) any number.
    - Second graders create a pattern with color tiles. They start with one square and then make a larger square that is two squares long on each side; they note that they need four tiles to do this. Then they make a square that is three squares long on each side; they need nine tiles to do this. They make a table of their results and describe the pattern they have found.
    - Students investigate a doubling (growing) pattern with Unifix cubes. They begin with one cube and then "win" another cube. Then they have two cubes and "win" two more. They continue this pattern, each time "winning" as many cubes as they already have. Repeating this process, they begin to see how quickly the number of cubes grows. They investigate this further using a calculator.
    - Students start with a rectangular sheet of paper that represents a cake. They simulate eating half of the cake by cutting the sheet in half and removing one of the halves. They eat half of what is left and continue this process. They describe the pattern, noting that after they repeat this about ten times, the cake is essentially gone.

2. **Investigate and describe how certain quantities change over time.**

    - Students keep a daily record of the temperature both inside and outside the classroom. They graph these temperatures and look at the patterns.
    - Students keep a monthly record of their height and record the data collected on a bulletin board. At the end of the school year, they describe what happened over time.
    - Students play catch with a ball in the school playground. One person counts out the number of times the ball is thrown, the other counts out the distance that it travels, a third person adds that distance to the total, and a fourth person records the totals. Afterwards they discuss how the total distance changes over time; they recognize that the same

amount is added repeatedly.

- Students study the changes in the direction and length of the shadow of a paper groundhog at different times of the day. They relate these observations to the position of the sun (e.g., as the sun gets higher, the shadow gets shorter).
- Students discuss how ice changes to water as it gets hotter. They talk about how it snows in January or February but rains in April or May.
- Students plant seeds and watch them grow. They write about what they see and measure the height of their plants at regular time intervals. They discuss how changes in time result in changes in the height of the plant. They also talk about how other factors might affect the growth of the plant, such as light and water.

3. Experiment with approximating length, area, and volume, using informal measurement instruments.

- Students measure the width of a bookcase using the 10-rods from a base ten blocks set. They record this length (perhaps as 6 rods or 60 units). Then they measure the bookcase using ones cubes; some of the students decide that it is easier just to add some ones cubes to the 10-rods that they have already used. They find that the bookcase is actually closer to 66 units long. They decide that they can get a better estimate of length when they use smaller units.
- Students use pattern blocks to cover a picture of a turtle. They count how many of each type of block (green triangle, yellow hexagon, etc.) they used. They make a bar graph that shows how many blocks each student used. They discuss why some students used more blocks than others and what they could do to increase or decrease the number of blocks used.
- Students play with containers of various sizes, transferring water from one container to another. They note that it takes two cups of water to fill a small milk carton. A pitcher holds three milk cartons of water, but four milk cartons overflow the pitcher. Then they find that it takes seven cups to fill the pitcher even though three milk cartons is only six cups. They decide that the smaller container gives a better idea of how much the pitcher will hold.
- Students find the area of huge dinosaur footprints that they find taped to the classroom floor. They first try to fit as many green 4" tiles as possible into a footprint without any overlapping, and without any tiles sticking out of the footprint. Before removing the green tiles, they cover them with blue 2" tiles, and count the number of blue 2" tiles used. Then they remove the green tiles and try to fit more blue 2" tiles into the footprint without overlapping; they discover that they can fit more and discuss why that is the case. They repeat this, using red 1" tiles. They notice that with smaller tiles, less of the footprint is uncovered, so that the smaller tiles provide a better estimate of the footprint's size.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 15 — Conceptual Building Blocks of Calculus — Grades 3-4

Overview

Students in grades 3 and 4 continue to develop the conceptual building blocks of calculus primarily through their work with patterns and changes over time. Students investigate a variety of patterns, using physical materials and calculators as well as pictures. In some of the patterns investigated, a constant is added to, or subtracted from, each number to get the next number in the sequence. These patterns, involving repeated operations, show **linear growth**; when these patterns are represented with a bar graph, the tops of the bars can be connected by a straight line. Examples of such patterns include skip counting, starting the week with $5 and paying 75¢ each day for lunch, or enumerating the multiples of 9. Other patterns should involve multiplying or dividing a number by a constant to get the next number in the sequence. These growing patterns illustrate **exponential growth**, as is the pattern which results when you start with two guppies (one male and one female) and the number of guppies doubles each week. Patterns should also include looking at **changes over time**, since these types of patterns are extremely important not only in mathematics but also in science and social studies. Students might chart the height of plants over time, the number of teeth lost each month throughout the school year, or the temperature outside the classroom over the course of several months.

Students continue to develop their understanding of **measurement**, gaining a greater understanding of the approximate nature of measurement. Students can guess at the length of a stick that is between 3 and 4 inches long, saying it is about 3½ inches long and recognizing when this is a better approximation than either 3 or 4. They can use grids of different sizes to approximate the area of a puddle, recognizing that the smaller the grid the more accurate the measurement. They can begin to consider how one might measure the amount of water in a puddle, coming up with alternative strategies and comparing them to see which would be more accurate. As they develop a better understanding of volume, they may use cubes to build a solid, build a second solid whose sides are all twice as long as the first, and then compare the number of cubes used to build each solid. The students may be surprised to find that it takes eight times as many cubes to build the larger solid!

Students continue to develop their understanding of **infinity** in grades 3 and 4. Additional work with counting sequences, skip counting, and calculators further reinforces the notion that there is always a bigger number. Taking half of something (like a rectangular cake or a sheet of paper) repeatedly suggests that there is always a number that is still closer to zero.

As students develop the conceptual underpinnings of calculus in third and fourth grades, they are also working to develop their understanding of numbers, patterns, measurement, data analysis, and mathematical connections. Additional ideas for activities relating to this standard can be found in the chapters discussing these other standards.

Standard 15 — Conceptual Building Blocks of Calculus — Grades 3-4

Indicators and Activities

The cumulative progress indicators for grade 4 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 3 and 4.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 3-4 will be such that all students:

1. Investigate and describe patterns that continue indefinitely.

- Using the constant multiplier feature of a calculator, students see how many times 1 must be doubled before one million is reached. They might first guess the number of steps to one million, and to half a million.
- Students start with a long piece of string. They fold it in half and cut it in two, setting aside one piece. Then they take the remaining half, fold it in half, cut it apart, and set aside half. They continue this process. They discuss how the length of the string keeps getting smaller, half as much each time, so that after about ten cuts, there is essentially no string left. Some students may understand that the process could keep going for several more steps, if we could only cut more carefully, and some may realize that in theory the process could continue forever.
- Students count out 1, 2, 3, 4, 5, 6, ... , and recognize that this pattern could continue forever. They also count out other patterns, like the even numbers, or the square numbers, or skip-counting by 3s starting with 2, and recognize that these patterns also could be continued indefinitely.
- Students investigate the growth patterns of sunflowers, pinecones, pineapples, or snails to study the natural occurrence of spirals.

2. Investigate and describe how certain quantities change over time.

- Students begin with a number like 5, add 3 to it, add 3 to it again, and repeat this five times. They record the results in a table and make a bar graph which represents the numbers that they have generated. They draw a straight line connecting the tops of the bars. They experiment with numbers other than 5 or 3 to see if the same thing occurs.
- Students begin with a number less than 10, double it, and repeat the doubling at least five times. They record the results of each doubling in a table and make a bar graph which represents the numbers they have generated. They discuss whether they can connect the tops of the bars with a straight line.
- Students measure the temperature of a cup of water with ice cubes in it every fifteen minutes over the course of a day. They record their results (time passed and temperature) in a table and plot this information on a coordinate grid to make a line graph. They discuss how the temperature changes over time and why. Initially the temperature will increase rapidly, but as the water warms up, its temperature will increase more slowly

until it essentially reaches room temperature.

- Students are given several examples of bar graphs with straight lines connecting the tops of the bars. They are asked to describe a motion scenario which reflects the data. For example, they might indicate that a graph reflects their running to an after-school activity, staying there for an hour, and then slowly walking home to do their chores.
- Students keep a monthly record of their height and record the data collected on a bulletin board. At the end of the school year, they describe what happened over time. They also find each month the average height of all the students in the class, and discuss how the change in average height over the year is similar to, and how it is different from, the change in height over the year of the individual students in the class.
- Students plant some seeds in vermiculite and some in soil. They observe the plants as they grow, measuring their height each week and recording their data in tables. They examine not only how the height of each plant changes as time passes but also whether the seeds in vermiculite or soil grow faster.

3. Experiment with approximating length, area, and volume, using informal measurement instruments.

- Students measure the length of their classroom using their paces and compare their results. They discuss what would happen if the teacher measured the room with her pace.
- Students use pattern blocks to cover a drawing of a dinosaur with as few blocks as they can. They record the number of blocks of each type used in a table and then discuss their results, making a frequency chart or bar graph of the total number of blocks used by each pair of students. Then they try to cover the same drawing with as many blocks as they can. They again record and discuss their results and make a graph. They look for connections between the numbers and types of blocks used each time. Some students simply trade blocks (e.g., a hexagon is traded for six triangles), while other students try to use all tan parallelograms since that seems to be the smallest block. (It actually has the same area as the triangle, however.)
- Students compare the volumes of a half-gallon milk carton, a quart milk carton, a pint milk carton, and a half-pint milk carton. They also measure the length of the side of the square base of each carton and its height. They make a table of their results and look for patterns. The students notice that the difference between the height measurements is not the same as the difference between the volume. The differences in volume grow more quickly than the differences in the heights. They see how many small cubes or marbles fill up each of their containers, and they try to explain why more than twice as many fit into a quart container than a pint container.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

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Standard 15 — Conceptual Building Blocks of Calculus — Grades 5-6

Overview

Students in grades 5 and 6 extend and clarify their understanding of patterns, measurement, data analysis, number sense, and algebra as they further develop the conceptual building blocks of calculus. Many of the basic ideas of calculus can be examined in a very concrete and intuitive way in the middle grades.

Students in grades 5 and 6 should begin to distinguish between patterns involving **linear growth** (where a constant is added or subtracted to each number to get the next one) and **exponential growth** (where each term is multiplied or divided by the same number each time to get the next number). Students should recognize that linear growth patterns change at a constant rate. For example, a plant may grow one inch every day. They should also begin to see that if these patterns are graphed, then the graph looks like a straight line. They may model this line by using a piece of spaghetti and use their graph to make predictions and answer questions about points that are not included in their data tables. In contrast, exponential growth patterns change at an increasingly rapid rate; if you start with one penny and double that amount each day, you receive more and more pennies each day as time goes on. Students should note that the graphs of these situations are not straight lines. At this grade level, students should also begin to imagine processes that could in theory continue forever even though they could not be carried out in practice; for example, although in practice a cake can be repeatedly divided in half only about ten times, nevertheless it is possible to imagine continuing to divide it into smaller and smaller pieces.

Many of the examples used should come from other subject areas, such as science and social studies. Students might look at such linear relationships as profit as a function of selling price, but they should also consider nonlinear relationships such as the amount of rainfall over time. Students should look at functions which have "holes" or jumps in their graphs. For example, if students make a table of the parking fees paid for various amounts of time and then plot the results, they will find that they cannot just connect the points; instead there are jumps in the graph where the parking fee goes up. A similar situation exists for graphs of the price of a postage stamp or the minimum wage over the course of the years. Many of the situations investigated by students should involve such **changes over time**. Students might, for example, consider the speed of a fly on a spinning disk; as the fly moves away from the center of the disk, he spins faster and faster. Students might be asked to write a short narrative about the fly on the disk and draw a graph of the fly's speed over time that matches their story.

As students begin to explore the decimal equivalents for fractions, they encounter non-terminating decimals for the first time. Students should recognize that calculators often use approximations for fractions such as .33 for 1/3. They should look for patterns involving decimal representations of fractions, such as recognizing which fractions have terminating decimal equivalents and which do not. Students should take care to note that π is a nonterminating, nonrepeating decimal; it is not exactly equal to 22/7 or 3.14, but these approximations are fairly close to the actual value of π and can usually be used for computational purposes. The examination of decimals extends students' understanding of **infinity** to very small numbers.

Students in grades 5 and 6 continue to develop a better understanding of the approximate nature of **measurement**. Students are able to measure objects with increasing degrees of accuracy and begin to consider significant digits by looking at the range of possible values that might result from computations with approximate measures. For example, if the length of a rectangle to the nearest centimeter is 10 cm and its width to the nearest centimeter is 5 cm, then the area is about 50 square centimeters. However, the rectangle might really be as small as 9.5 cm x 4.5 cm, in which case the area would only be 42.75 square centimeters, or it might be almost as large as 10.5 cm x 5.5 cm, with an area of 57.75 square centimeters. Students should continue to explore how to determine the surface area of irregular figures; they might, for example, be asked to develop a strategy for finding the area of their hand or foot. They should do similar activities involving volume, perhaps looking for the volume of air in a car. Most of their work in this area in fifth and sixth grade will involve using squares or cubes to approximate these areas or volumes.

Standard 15 — Conceptual Building Blocks of Calculus — Grades 5-6

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 5 and 6.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 5-6 will be such that all students:

4. Recognize and express the difference between linear and exponential growth.

- Students develop a table showing the sales tax paid on different amounts of purchases, graph their results, note that the graph is a straight line, and recognize that this situation represents a constant rate of change, or linear growth.
- Students make a table showing how much money they would have at the end of each of eight years if $100 was invested at the beginning and the investment grew by 10% each year. They note that the graph of their data is not a straight line; this graph represents exponential growth.
- Students make a table showing the value of a car as it depreciates over time. They note that the graph of their data is not a straight line; this graph represents "exponential decay."
- Students are presented stories which represent real life occurrences of linear and exponential growth and decay over time, and are asked to construct graphs which represent the situation and indicate whether the change is linear, exponential, or neither.

5. Develop an understanding of infinite sequences that arise in natural situations.

- Students make equilateral triangles of different sizes out of small equilateral triangles and record the number of small triangles used for each larger triangle. These numbers are called *triangular numbers*. If the following triangular pattern is continued indefinitely, then the number of 1s in the first row, the number of 1s in the first two rows, the number of 1s in the first three rows, etc. form the sequence of triangular numbers. The triangular numbers also emerge from the handshake problem: *If each two people in a room shake hands exactly once, how many handshakes take place altogether?* If the answers are listed for 2, 3, 4, 5, 6, 7, ... people, the numbers are again the triangular numbers 1, 3, 6, 10, 15, 21, ...

```
          1
        1   1
      1   1   1
    1   1   1   1
  1   1   1   1   1
1   1   1   1   1   1
```

- Students imagine cutting a sheet of paper into half, cutting the two pieces into half, cutting the four pieces into half, and continuing this over and over again, for about 25 times. Then they imagine taking all of the little pieces of paper and stacking them on top of one another. Finally, they estimate how tall that stack would be.
- Students describe, analyze, and extend the Fibonacci sequence (1, 1, 2, 3, 5, 8, ...). They research occurrences of this sequence in nature, such as sunflower seeds, the fruit of the pineapple, and the rabbit problem. They create their own Fibonacci-like sequences, using different starting numbers.

6. Investigate, represent, and use non-terminating decimals.

- Students use their calculators to find the decimal equivalent for 2/3 by dividing 2 by 3. Some of the students get an answer of 0.66667, while others get 0.6666667. They do the problem by hand to try to understand what is happening. They decide that different calculators round off the answer after different numbers of decimal places. The teacher explains that the decimal for 2/3 can be written exactly as $.\overline{6}$.
- Students have been looking for the number of different squares that can be made on a 5 x 5 geoboard and have come up with 1x1, 2x2, 3x3, 4x4, and 5x5 squares. One student finds a different square, however, whose area is 2 square units. The students wonder how long the side of the square is. Since they know that the area is the length of the side times itself, they try out different numbers, multiplying *1.4 x 1.4* on their calculators to get 1.89 and *1.5 x 1.5* to get *2.25*. They keep adding decimal places, trying to get the exact answer of 2, but find that they cannot, no matter how many places they try!

7. Represent, analyze, and predict relations between quantities, especially quantities changing over time.

- Students study which is the better way to cool down a soda, adding lots of ice at the beginning or adding one cube at a time at one minute intervals. Each student first makes a prediction and the class summarizes the predictions. Then the class collects the data, using probes and graphing calculators or computers and displays the results in table and graph form on the overhead. The students compare the graphs and write their conclusions in their math notebooks. They discuss the reasons for their results in science class.
- Students make a graph that shows the price of mailing a letter from 1850 through 1995. Some of the students begin by simply plotting points and connecting them but soon realize that the price of a stamp is constant for a period of time and then abruptly jumps up. They decide that parts of this graph are like horizontal lines. The teacher tells them that mathematicians call this a "step function"; another name for this kind of graph is a piecewise linear graph because the graph consists of linear pieces.
- Students review Mark's trip home from school on his bike. Mark spent the first few minutes after school getting his books and talking with friends, and left the school grounds about five minutes after school was over. He raced with Ted to Ted's house and stopped for ten minutes to talk about their math project. Then he went straight home. The students draw a graph showing the distance covered by Mark with respect to time. Then, with the teacher's help, the class constructs a graph showing the speed at which Mark traveled with respect to time. The students then write their own stories and generate graphs of distance vs. time and graphs of speed vs. time.

8. **Approximate quantities with increasing degrees of accuracy.**

- Students find the volume of a cookie jar by first using Multilink cubes (which are 2 cm on a side) and then by using centimeter cubes. They realize that the second measurement is more accurate than the first.
- Students measure the circumference and diameter of a paper plate to the nearest inch and then divide the circumference by the diameter. They repeat this process, using more accurate measures each time (to the nearest half-inch, to the nearest quarter-inch, etc.). They see that the quotients get closer and closer to π.
- Using a ruler, students draw an irregularly shaped pentagon on square-grid paper, taking care to locate the vertices of the pentagon at grid points. They estimate the area of the pentagon by counting the number of squares completely inside the pentagon and adding to it an estimate of the number of full square that the partial squares inside the pentagon would add up to. Then they divide the pentagon into triangles and rectangles and find the area of the pentagon as a sum of the areas of the triangles and rectangles. They compare the results and explain any difference.

9. **Understand and use the concept of significant digits.**

- Students measure the length and width of a rectangle in centimeters and find its area. Then they measure its length and width in millimeters and find the area. They note the difference between these two results and discuss the reasons for such a difference. Some of the students think that, since the original measurements were correct to the nearest centimeter, then the result would be correct to the nearest square centimeter, while the second measurements would be correct to the nearest square millimeter. However, when they experiment with different rectangles, for example, one whose dimensions are 3.2 by 5.2 centimeters, they find that the area of 15 square centimeters is not correct to the nearest centimeter.
- Students find the area of a "blob" using a square grid. First, they count the number of squares that fit entirely within the blob (no parts hanging outside). They say that this is the least that the area could be. Then they count the number of squares that have any part of the blob in them. They say that this is the most that the area could be. They note that the actual area is somewhere between these two numbers.

10. **Develop informal ways of approximating the surface area and volume of familiar objects, and discuss whether the approximations make sense.**

- Students trace around their hand on graph paper and count squares to find an approximate value of the area of their hand. They use graph paper with smaller squares to find a better approximation.
- Students work in groups to find the surface area of a leaf. They describe the different methods they have used to accomplish this task. Some groups are asked to go back and reexamine their results. When the class is convinced that all of the results are reasonably accurate, they consider how the surface area of the leaf might be related to the growth of the tree and its needs for carbon dioxide, sunshine, and water.
- Each group of students is given a mixing bowl and asked to find its volume. One group decides to fill the bowl with centimeter cubes, packing them as tightly as they can and then

to add a little. Another group decides to turn the bowl upside down and try to build the same shape next to it by making layers of centimeter cubes. Still another group decides to fill the hollow 1000-centimeter cube with water and empty it into the bowl as many times as they can to fill it; they find that doing this three times almost fills the bowl and add 24 centimeter cubes to bring the water level up to the top of the bowl.

11. Express mathematically and explain the impact of the change of an object's linear dimensions on its surface area and volume.

- While learning about area, the students became curious about how many square inches there are in a square foot. Some students thought it would be 12, while others thought it might be more. They explore this question using square-inch tiles to make a square that is one foot on each side. They decide that there are 144 square inches in a square foot; they make the connection with multiplication, noticing that 144 is 12 x 12 and that there are 12 inches in a foot. They realize that the square numbers have that name because they are the areas of squares whose sides are the whole numbers.
- Having measured the length, width, and height of the classroom in feet, the students now must find how many cubic yards of air there are. Some of the students convert their measurements to yards and then multiply to find the volume. Others multiply first, but find that dividing by 3 does not give a reasonable answer. They make a model using cubes that shows that there are 27 cubic feet in a cubic yard and divide their answer by 27, getting the same result as the other students.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Standard 15 — Conceptual Building Blocks of Calculus— Grade 7-8

Overview

In grades 7 and 8, students begin to develop a more detailed and formal notion of the concepts of approximation, rates of change for various quantities, infinitely repeating processes, and limits. Activities should continue to emerge from concrete, physical situations, often involving the collection of data.

Students in grades 7 and 8 continue to develop their understanding of **linear growth, exponential growth, infinity,** and **change over time.** By collecting data in many different situations, they come to see the commonalities and differences in these types of situations. They should recognize that, in linear situations, the rate of change is constant and the graph is a straight line, as in plotting distance vs. time at constant speeds or plotting the height of a candle vs. time as it burns. In situations involving exponential growth, the graph is not a straight line and the rate of change increases or decreases over time. For example, in a situation in which a population of fish triples every year, the number of fish added each year is more than it was in the previous year. Students should also have some experience with graphs with holes or jumps (discontinuities) in them. For example, students may look at how the price of mailing a letter has changed over the last hundred years, first making a table and then generating a graph. They should recognize that plotting points and then connecting them with straight lines is inappropriate, since the cost of mailing a letter stayed constant over a period of several years and then abruptly increased. They should be aware of other examples of "step functions" whose graphs look like a sequence of steps.

Students in these grades should approximate irrational numbers, such as square roots, by using decimals; they should recognize the size of the error when they use these approximations. Students should take care to note that π is a nonterminating, nonrepeating decimal; it is not exactly equal to 22/7 or 3.14, but lies between these values. These approximations are fairly close to the actual value of π and can usually be used for computational purposes. Students may also consider sequences involving rational numbers such as 1/2, 2/3, 3/4, 4/5, ... They should recognize that this sequence goes on forever, getting very close to a limit of one. Students should also consider sequences in the context of learning about fractals. (See Standard 7 or 14.)

Seventh and eighth graders continue to benefit from activities that physically model the process of approximating **measurement** results with increasing accuracy. Students should develop a clearer understanding of the concept of significant digits as they begin to use scientific notation. They should be able to use these ideas as they develop and apply the formulas for finding the areas of such figures as parallelograms and trapezoids. Students should understand, for example, that if they are measuring the height and diameter of a cylinder in order to find its volume, then some error is introduced from each of these measurements. If they measure the height as 12.2 cm and the diameter as 8.3 cm, then they will get a volume of $\pi(8.3/2)^2(12.2)$, which their calculator may compute as being $660.09417\ cm^3$. They need to understand that this answer should be rounded off to $660.09\ cm^3$ (five significant digits). They also should understand that the true volume might be as low as $\pi(8.25/2)^2(12.15) \approx 649.49\ cm^3$ or almost as high as $\pi(8.35/2)^2(12.25) \approx 670.81\ cm^3$.

Students in these grades should continue to build a repertoire of strategies for finding the surface area and volume of irregularly shaped objects. For example, they might find volume not only by approximating irregular shapes with familiar solids but also by submerging objects in water and finding the amount of water displaced by the object. They might find surface areas by first laying out patterns of the objects called "nets"; for example, the net of a cube consists of six squares connected in the shape of a cross — when creased along the edges of the squares, this "net" can be folded to form a cube. Then they would place a grid on the net and count the small squares, noting that the finer the grid the more accurate the estimate of the area.

Explorations developing the conceptual underpinnings of calculus in grades 7 and 8 should continue to take advantage of students' intrinsic interest in infinite, iterative patterns. They should also build connections between number sense, estimation, measurement, patterns, data analysis, and algebra. More information about activities related to these areas can be found in the chapters discussing those standards.

Standard 15 — Conceptual Building Blocks of Calculus— Grades 7-8

Indicators and Activities

The cumulative progress indicators for grade 8 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 7 and 8.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 7-8 will be such that all students:

4. **Recognize and express the difference between linear and exponential growth.**

   - Students measure the height of water in a beaker at five second intervals as it is being filled, being careful to leave the faucet on so that the water runs at a constant rate. They make a table of their results and generate a graph. They note that this is a linear function.
   - Students investigate patterns of exponential growth with the calculator, such as compound interest or bacterial growth. They make a table showing how much money is in a savings account after one quarter, two quarters and so on for ten years, if $1000 is deposited at 5% interest and there are no further deposits or withdrawals. They represent their findings graphically, noting that this is not a linear relationship, although in the case of simple interest, where the interest does not earn interest, the graph is linear.
   - Students obtain a table showing the depreciated value of a car over time. They graph the data in the table and observe that it is not a straight line. The value of the car exhibits "exponential decay."
   - Students compare different pay scales, deciding which is a better deal. *For example, is it better to be paid a salary of $250 per week or to be paid $6 per hour?* They realize that the answer to this question depends on the number of hours worked, so they create a table comparing the pay for different numbers of hours worked. They make a graph and decide at what point the hourly rate becomes a better deal.
   - Students predict how many times they will be able to fold a piece of paper in half. Then they fold a paper in half repeatedly, recording the number of sections formed each time in a table. Students find that the number of folds physically possible is surprisingly small (about 7). The students try different kinds of paper: tissue paper, foil, etc. They describe in writing any patterns they discover and try to find a rule for the number of sections after 10, 20, or n folds. They also graph the data on a rectangular coordinate plane using integral values. They extend this problem to a new situation by finding the number of ancestors each person had perhaps ten generations ago and also to the situation of telling a secret to 2 people who each tell two people, etc.

5. **Develop an understanding of infinite sequences that arise in natural situations.**

   - Students discuss how the Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, ...) is related to the following problem: begin with two rabbits (one male and one female), each adult pair of rabbits produces two babies (one male and one female) each month, the babies themselves become adults (and start having their own babies) after one month, and none of the rabbits ever die. The students decide that the Fibonacci sequence shows how many pairs of rabbits there are each month. The students explore other patterns in this sequence, noting that each term is the sum of the two preceding terms.

- Students look for infinite sequences in Pascal's triangle. Starting at the top 1 and moving diagonally to the left, there is a constant infinite sequence 1, 1, 1, 1, ... Starting at the next 1 and moving diagonally to the left there is the sequence 1, 2, 3, 4, 5, ... of whole numbers. Starting at the next 1 and moving diagonally to the left, there is the sequence 1, 3, 6, 10, 15, ... of triangular numbers, which records the solutions to all handshake problems. Also the sum of the numbers in each row yield the exponential sequence 1, 2, 4, 8, 16, ...

```
        1
      1   1
    1   2   1
  1   3   3   1
1   4   6   4   1
```

6. **Investigate, represent, and use non-terminating decimals.**

- Students investigate using simple equations to iterate patterns. For example, they use the equation $y = x + 1$ and start with any x-value, say *0*. The resulting y-value is *1*. Using this as the new x yields a *2* for y. Using this as the next x gives a *3*, and so on, resulting in the sequence 1, 2, 3, 4, ... Then students use a slightly different equation, $y = .1x + .6$, starting with an x-value of .6 and finding the resulting y-value. Repeating this process yields the sequence of y-values *.6, .66, .666, .6666, ...,* which approximates the decimal value of 2/3.

- Students explore the question of which fractions have terminating decimal equivalents and which have repeating decimal equivalents. They discover that the only fractions in lowest terms which correspond to terminating decimals are those whose denominators have only 2 and 5 as prime factors.

- Students explore the question of which fractions have decimal equivalents where one digit repeats and learn that these are the fractions 1/9, 2/9, 3/9, ... They generalize this to find the fractions whose decimal equivalents have two digits repeating like .171717 ... .

7. **Represent, analyze, and predict relations between quantities, especially quantities changing over time.**

- Students describe what happens when a ball is tossed into the air, experimenting with a ball as needed. They make a graph that shows the height of the ball at different times and discuss what makes the ball come back down. They also consider the speed of the ball: *when is it going fastest? slowest?* With some help from the teacher, they make a graph showing the speed of the ball over time.

- Students use probes and graphing calculators or computers to collect data involving two variables for several different science experiments (such as measuring the time and distance that a toy car rolls down an inclined plane or measuring the brightness of a light bulb as the distance from the light bulb increases or measuring the temperature of a beaker of water when ice cubes are added). They look at the data that has been collected in tabular form and as a graph on a coordinate grid. They classify the graphs as straight or curved lines and as increasing (direct variation), decreasing (inverse variation), or mixed. For those graphs that are straight lines, the students try to match the graph by entering and graphing a suitable equation.

- Students measure the temperature of boiling water as it cools in a cup. They make a table showing the temperature at five-minute intervals for an hour. Then they graph the results and make observations about the shape of the graph, such as "the temperature went down the most in the first few minutes," "it cooled more slowly after more time had passed," or "it's not a linear relationship." The students also predict what the graph would look like if they continued to collect data for another twelve hours.
- Students make Ferris wheel models from paper plates (with notches cut to represent the cars). They use the models to make a table showing the height above the ground (desk) of a person on a Ferris wheel at specified time intervals (time needed for next chair to move to loading position). After collecting data through two or three complete turns of the wheel, they make a graph of time versus height. In their math notebooks, they respond to questions about their graphs: *Why doesn't the graph start at zero? What is the maximum height? Why does the shape of the graph repeat?* The students learn that this graph represents a periodic function.
- Students compare two ways of cooling a glass of soda, adding lots of ice at the beginning or adding one cube at a time at one minute intervals. Each student first makes a prediction about which cools the soda faster, and the class summarizes the predictions. Then the teacher collects the data, using probes and graphing calculators or computers and displays the results in table and graph form on the overhead. The students compare the graphs and write their conclusions in their math notebooks. They discuss the reasons for any difference between these two methods with their science teacher.
- Students compute the average speed of a toy car as it travels down a ramp by dividing the length of the ramp by the time the car takes to travel the ramp. They try different angles for the ramp, recording their results. They make a graph of average speed vs. angle and discuss whether this graph is linear.
- Students make a graph that shows the minimum wage from the time it was first instituted until the present day. Some of the students begin by simply plotting points and connecting them but soon realize that the minimum wage was constant for a time and then abruptly jumped up. They decide that parts of this graph are like horizontal lines. They look for other examples of "step functions."

8. Approximate quantities with increasing degrees of accuracy.

- Students measure the speed of cars using different strategies and instruments and compare the accuracy of each. For example, they first determine the speed of a car by using a stopwatch to find out how long it takes to travel a specific distance. They note that the speed of the car actually changes over the time interval, however. They decide that they can get a better idea of how fast the car is moving at a specific time by shortening the distance. They collect data for shorter and shorter distances. Finally, they ask a police officer to bring a radar gun to their class to help them collect data about the speed of the cars going past the school.

- Students find the area of a "blob" using a square grid. First, they count the number of squares that fit entirely within the blob (no parts hanging outside). They say that this is the least that the area could be. Then they count the number of squares that have any part of the blob in them. They say that this is the most that the area could be. They note that the actual area is somewhere between these two numbers. Finally, the students put together parts of squares to try to get a more accurate estimate of the area of the blob.

9. Understand and use the concept of significant digits.

- Students measure the radius of a circle in centimeters and find its area. Then they measure its radius in millimeters and find the area. They note the difference between these two results and discuss the reasons for such a difference. Some of the students think that, since the original measurements were correct to the nearest centimeter, then the result would be correct to the nearest square centimeter, while using the second measurements would give a value for the area which is correct to the nearest square millimeter. However, after experimenting with circles of different sizes, they find that if the radius is measured to the nearest centimeter.
- Students explore the different answers that they get by using different values for π when finding the area of a circle. They discuss why these answers vary and how to decide what value to use.
- Students estimate the amount of wallpaper, paint, or carpet needed for a room, recognizing that measurements that are accurate to several decimal places are unnecessary for this purpose.

10. Develop informal ways of approximating the surface area and volume of familiar objects, and discuss whether the approximations make sense.

- In conjunction with a science project, students need to find the surface area of their bodies. Some of the students decide to approximate their bodies with geometric solids; for example, their head is approximately a sphere, and their neck, arms, and legs are approximately cylinders. They then take the needed measurements and compute the surface areas of the relevant solids. Other students decide to use newspaper to wrap their bodies and then measure the dimensions of the sheets of newspaper used.
- Students estimate the volume of air in a balloon as a way of looking at lung capacity. Some of the students decide that the balloon is approximately the shape of a cylinder, measure its length and diameter, and compute the volume. Other students think the balloon is shaped more like a cylinder with cones at the ends; they measure the diameter of the balloon at its widest part, the length of the cylinder part, and the height of each cone and then compute the volume of each shape. Some other students decide that they would like to check their work another way; they place a large graduated cylinder in the sink, and fill it with water. They submerge the balloon, and read off how much water is left after the balloon is taken out. Since they know that 1 ml of water is 1 cm^3, they know that the volume of the water that was displaced is the same as that of the balloon.
- Students develop different strategies for finding the volume of water in a puddle.

11. Express mathematically and explain the impact of the change of an object's linear dimensions on its surface area and volume.

- Students analyze cardboard milk containers to determine how the dimensions of the container affect the volume of milk contained in the carton and how the amount of cardboard used varies. In addition to measuring actual cartons, students make their own cartons of different sizes by varying the length, width, and height one at a time. They write up their results and share them with the class.
- Placing a number of identical cereal boxes next to or on top of one another, students learn that doubling one of an object's length, width, or height doubles its volume, that doubling two of these dimensions increases the volume by a factor of 4, and that doubling all three dimensions increases the volume by a factor of 8.
- Students sketch a 3-dimensional object such as a box or a cylindrical trash can. They then make a sketch twice as large in all dimensions. *How much larger is the volume of the larger object? How much larger would it be if the dimensions all increased by a factor of 3?* Square grid paper might be helpful for this exercise.

On-Line Resources

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Standard 15 —Conceptual Building Blocks of Calculus — Grades 9-12

Overview

This standard does not advocate the formal study of calculus in high school for all students or even for all college-intending students. Rather, it calls for providing opportunities for all students to informally investigate the central ideas of calculus: limit, the rate of change, the area under a curve, and the slope of a tangent line. Considering these concepts will contribute to a deeper understanding of the function concept and its usefulness in representing and answering questions about real-world situations.

Instruction should be highly exploratory, based on numerical and geometric experiences that capitalize on both calculator and computer technology. Activities should be aimed at providing students with an understanding of the underlying concepts of calculus rather that at developing manipulative techniques.

The development of calculus is one of the great intellectual achievements in history, especially with respect to its use in physics. Calculus is also increasingly being used in the social and biological sciences and in business. As students explore this area, they should develop an awareness of and appreciation for the historical origins and cultural contributions of calculus.

Students' earlier study of patterns is extended in high school to the study of finite and **infinite processes**. Students continue to look at **linear growth** patterns as they develop procedures for finding the sums of arithmetic series (e.g., the sum of the numbers from 1 to 100). They may consider this sum in many different ways, building different types of models. Some students may look at $1 + 2 + 3 + \ldots + 100$ geometrically by putting together two "staircases" to form a rectangle that is 100 by 101. Other students may look at the sum arithmetically by adding $1 + 2 + 3 + \ldots + 100$ *to* $100 + 99 + 98 + \ldots + 1$ and getting 100 pairs of numbers that add up to 101. Still others may look at the sum by finding the limit of the sequence of partial sums. Students also look at **exponential growth** as they develop procedures for finding the sum of finite and infinite geometric series (e.g., $2 + 4 + 8 + 16 + 32$ or $6 + 3 + 3/2 + \ldots$ or finding the total distance traveled by a bouncing ball). Students' work with patterns and infinity also includes elaborating on the intuitive notion of limit that has been addressed in the earlier grades.

High school students further develop their understanding of **change over time** through informal activities that focus on the understanding of interrelationships. Students should collect data, generate graphs, and analyze the results for real-world situations that can be described by linear, quadratic, trigonometric, and exponential models. Some of the types of situations that should be analyzed include motion, epidemics, carbon dating, pendulums, and biological and economic growth. They should use Calculator Based Labs (CBLs) in conjunction with graphing calculators to gather and analyze data. Students should recognize the equations of the basic models ($y = mx + b$, $y = ax^2 + bx + c$, $y = \sin x$, and $y = 2^x$) and be able to relate geometric transformations to the equations of these models. Students should develop a thorough understanding of the idea of slope; for example, they need to be able compare the steepness of two graphs at various points on the graph. They also need to be able explain what the slope means in terms of the real-world situation described by a graph. *For example, what information does the slope give for a graph of the levels of medicine in the bloodstream over time?* Students also extend their understanding of the behavior of functions to include the concept of the continuity of a function, considering features such as removable discontinuities (holes or jumps), asymptotes, and corners.

Students in high school apply their understanding of approximation techniques not only with respect to numbers in the context of using initial portions of nonrepeating, nonterminating decimals but also with respect to **measurement** situations. Students further develop their understanding of significant digits and the arithmetic of approximate values. They also use repeated approximations to find the areas of irregular figures, including experimenting with situations in which they need to find the area under a curve.

Looking at the conceptual underpinnings of calculus provides an opportunity for high school students to pull together their experiences with data analysis, patterns, algebra, measurement, number sense, and numerical operations. It also provides the opportunity to apply technology to real-world situations and to gain experience with mathematics as a dynamic human endeavor.

Standard 15 — Conceptual Underpinnings of Calculus — Grades 9-12

Indicators and Activities

The cumulative progress indicators for grade 12 appear below in boldface type. Each indicator is followed by activities which illustrate how it can be addressed in the classroom in grades 9, 10, 11, and 12.

Building upon knowledge and skills gained in the preceding grades, experiences in grades 9-12 will be such that all students:

12. Develop and use models based on sequences and series.

- Students work in groups to collect data about a bouncing ball. They first decide how to measure the height of each bounce and then record their data in a table. They notice the pattern of the heights and make two graphs, one of height vs. bounce and the other of total distance traveled vs. bounce. They note that the distance traveled involves adding together the heights of each of the bounces, and so is represented by a series. They describe the general behavior of each graph and have their graphing calculators compute various regression lines. In their report, they describe what they did, their results, and why they think that the type of function they used to describe each graph is reasonable.
- Students use M&Ms to model exponential decay. They spill a package of M&Ms on a paper plate and remove those with the M showing, and record the number of M&Ms removed. They put the remaining M&Ms in a cup, shake, and repeat the process until all of the M&Ms are gone. They plot the trial number versus the number of M&Ms removed and note that the graph represents an exponential function. Some of the students try out different exponential functions until they find one that they think fits the data pretty well.

13. Develop and apply procedures for finding the sum of finite arithmetic series and finite and infinite geometric series.

- Students investigate a situation in which a contractor is fined $400 if he is one day late completing a project, $475 more if he is two days late, $550 more if he is three days late, and so on. They want to find out how much he will lose if he is two weeks late finishing the job. They recognize that this is an arithmetic series where the first term is $400 and each term is obtained from the preceding one by adding $75. They draw upon several techniques they have learned to add up the terms of this series. One method that they have discussed involves reversing the order of the terms of the series and adding the two series. Some of the students thus solve the problem by writing the fourteen terms of the series and underneath writing the same fourteen terms backwards, a technique sometimes called Gauss' method because, according to legend, he discovered it as a child while walking to the back of his class to perform his punishment of adding together the first hundred numbers. They obtain the following format:

$$\begin{array}{l} 400 + 475 + 550 + \ldots + 1300 + 1375 \\ 1375 + 1300 + 1225 + \ldots + 475 + 400 \\ \hline 1775 + 1775 + 1775 + \ldots + 1775 + 1775 \end{array}$$

They recognize that they have 14 pairs of numbers, each of which adds up to 1775. This gives them a total of $24,850 which they divide in half (since they added together both sequences) to find the answer, $12,425. Another group decides that they can separate out the 14 charges of $400, for a total of *14 x 400 = $5600,* and then deal with the remainders $0 + 75 + 150 + 225 + ... + 975, or $75(0 + 1 + 2 + 3 + ... + 13); this series they recognize as (13x14)/2, so the total fine is $5600 + $75x91 or $5600 + $6825, for a grand total of $12,425. Still another group of students uses a formula for the sum of a finite arithmetic series.

- Students are asked to find a method similar to Gauss' method to find the sum of the series *9 + 3 + 1 + 1/3 + 1/9 + 1/27.* The students notice that this series is not an arithmetic series since different amounts have to be added in order to get the next term. They discover, however, that each term is 1/3 of the previous term, and they write down 1/3 of the series and arrive at:

$$\begin{array}{l} 9 + 3 + 1 + 1/3 + 1/9 \\ \quad\;\; 3 + 1 + 1/3 + 1/9 + 1/27 \\ \hline \end{array}$$

  They subtract to get 9 - 1/27 or 242/27 . Since they subtracted 1/3 of the series from itself, this total is 2/3 of the sum of the series, so the sum is 3x242/2x27 or 121/9. The teacher uses this technique to motivate the standard formula for the sum of a finite geometric series, where a is the first term of the series and r is the common multiple:

$$S_n = a\,(1 - r^n)/(1 - r).$$

- After investigating how to find the sum of a finite geometric sequence, students begin looking at infinite geometric sequences. They realize that the same technique they used for the finite geometric series works for the infinite one as well. Thus for example, if we added the first 100 terms of the series by the method above, the sum would be $9 - 1/3^{97}$, which is very close to 9. Since this sum is again 2/3 of the sum of the original series, the actual total is 27/2. For those students who are likely to use a formula, the teacher generalizes this discussion and tells them that the sum gets closer to a/(1 - r) as the number of terms expand. They confirm this conclusion by checking out the partial sums of some sequences.

14. Develop an informal notion of limit.

- After a class discussion of the repeating decimal .9999 ... , the students are asked to write in their journals an "explanation to the skeptic" on why .9999 ... is equal to 1. Among their explanations: There is no room between .9999 ... and 1; .9999 ... is 3 times .3333 ... which everyone agrees is 1/3 ; if you take 10 times .9999 ... and subtract .9999 ... , you get 9 and 9 times .9999 ... — so 1 must be .9999 ... ; if you sum a geometric series whose first time is .9 and whose common multiple is .1 you get a/(1-r) which amounts to .9/(1-.9), or 9. Given all these convincing reasons, the class decides that the limit of the sequence is 1.

- Students consider the sequence 1/2, 1/4, 1/8, ... in different contexts. First, they look at it as representing a situation in which someone eats half of a pizza, then half of what is left, then half of what is left, etc. They decide that, while theoretically there will always be some of pizza left, in the end it would be all gone. However, in practice, by the end of ten stages or so the entire pizza would in effect have disappeared. Similarly, if a sheet of paper is repeatedly torn in half, then in theory some part is always left; however, in practice, after about ten tearings the paper will have disappeared.

15. **Use linear, quadratic, trigonometric, and exponential models to explain growth and change in the natural world.**

   - Students use a graphing calculator, together with a light probe, to examine the relationship between brightness of a light and distance from it. They do this by collecting data with the probe on the brightness of a light bulb at increasing distances and then analyzing the graph generated on the calculator to see what kind of graph it is. They use other CBL probes to investigate the kinds of functions used to model a variety of real-world situations.

   - Students learn about the Richter Scale for measuring earthquakes, focusing on its relation to logarithmic and exponential functions, and why this kind of scale is used.

   - Students use recursive definitions of functions in both geometry and algebra. For example, they define n! recursively as $n! = n\,(n-1)!$ They use recursion to generate fractals in studying geometry. They may use patterns such as spirolaterals, the Koch snowflake, the *monkey's tree curve*, the *chaos game*, or the Sierpinski triangle. They may use Logo or other computer programs to iterate patterns, or they may use the graphing calculator. In studying algebra, students consider the equation $y = .1x + .6$, start with an x-value of .6, and find the resulting y-value. Using this y-value as the new x-value, they then calculate its corresponding y-value, and so on. (The resulting values are .6, .66, .666, .6666, etc. — an approximation to the decimal value of 2/3!) Students investigate using other starting values for the same function; the results are surprising! They use other equations and repeat the procedure. They graph the results and investigate the behavior of the resulting functions, using a calculator to reduce the computational burden.

   - Students work through the *Breaking the Mold* lesson described in the Introduction to this *Framework*. They grow mold and collect data on the area of a pie plate covered by the mold. They make a graph showing the percent of increase in the area vs. the days. The students graph their data and find an equation that fits the data to their satisfaction.

16. **Recognize fundamental mathematical models (such as polynomial, exponential, and trigonometric functions) and apply basic translations, reflections, and dilations to their graphs.**

   - Students work in groups to investigate what size square to cut from each corner of a rectangular piece of cardboard in order to make the largest possible open-top box. They make models, record the size of the square and the volume for each model, and plot the points on a graph. They note that the relationship seems to be a polynomial function and make a conjecture about the maximum volume, based on the graph. The students also generate a symbolic expression describing this situation and check to see if it matches their data by using a graphing calculator.

   - Students look at the effects of changing the coefficients of a trigonometric equation on the graph. *For example, how is the graph of* $y = 4 \sin x$ *different from that of* $y = \sin x$? *How is* $y = .2 \sin x$ *different from* $y = \sin x$? *How are* $y = \sin x + 4$, $y = \sin x - 4$, $y = \sin (x - 4)$, *and* $y = \sin (x + 4)$ *each different from* $y = \sin x$? Students use graphing calculators to look at the graphs and summarize their conjectures in writing.

- Students study the behavior of functions of the form $y = ax^n$. They investigate the effect of "a" on the curve and the characteristics of the graph when n is even or odd. They use the graphing calculator to assist them and write a sentence summarizing their discoveries.
- Students begin with the graph of $y = 2^x$. They shift the graph up one unit and try to find the equation of the resulting curve. They shift the original graph one unit to the right and try to find the equation of that curve. They reflect the original graph across the x-axis and try to find the equation of that curve. Finally, they reflect the original graph across the y-axis and try to find the equation of the resulting curve. They describe what they have learned in their journals.

17. Develop the concept of the slope of a curve, apply slopes to measure the steepness of curves, interpret the meaning of the slope of a curve for a given graph, and use the slope to discuss the information contained in the graph.

- Students collect data about the height of a ball that is thrown in the air and make a scatterplot of their data. They note that the points lie on a quadratic function and use their graphing calculators to find the curve of best fit. Then they make some conjectures about the speed at which the ball is traveling. They think that the ball is slowing down as it rises, stopping at the maximum point, and speeding up again as it falls.
- Students take on the role of "forensic mathematicians," trying to determine how tall a person would be whose femur is 17 inches long. They measure their own femurs and their heights, entering this data into a graphing calculator or computer and creating a scatterplot. They note that the data are approximately linear, so they find the y-intercept and slope from the graph and generate an equation that they think will fit the data. They graph their equation and check its fit. They also use the built-in linear regression procedure to find the line of best fit and compare that equation to the one they generated. (An instructional unit addressing this activity can be found in the Keys to Success in the Classroom chapter of this *Framework*.)
- Students plot the data from a table that gives the amount of alcohol in the bloodstream at various intervals of time after a person drinks two glasses of beer. Different groups use different techniques to generate an equation for the graph; after some discussion, the class decides which equation they think is best. The students consider the following questions: *What information does the slope give for this situation? Would that be important to know? Why or why not?*
- Students investigate the effect of changing the radius of a circle upon its circumference by measuring the radius and the circumference of circular objects. They graph the values they have generated, notice that it is close to a straight line, and use the slope to develop an equation that describes that relationship. Then they discuss the meaning of the slope in this situation.

18. Develop an understanding of the concept of continuity of a function.

- Students work through the *On the Boardwalk* lesson found in the Introduction to this *Framework*. A quarter is thrown onto a grid made up of squares, and you win if the quarter does not touch a line. A grid is drawn on the floor using masking tape, and a circular paper plate is thrown onto the grid several hundred times to simulate the game.

The activity is repeated several times, varying each time the size of the squares in the grid. The students collect data and make a graph of their results (size of squares vs. number of wins out of 100 tosses). The graph looks like a straight line, suggesting that as the size of the squares increases without bound, so does the percentage of "hits". But, of course, the percentage of hits cannot exceed 100%, so the line is actually curved, with an asymptote at $y=100$.

- The school store sells pencils for 15¢ each, but it has some bulk pricing available if you need more pencils. Ten pencils sell for $1, and twenty-five pencils sell for $2. The students make a table showing the cost of different numbers of pencils and then generate a graph of number of pencils vs. cost. The students note that the graph has discontinuities at ten and twenty-five, since these are the jump points for pricing. They also note that if you need at least seven pencils, it is better to buy the package of ten and if you need 17 or more, you should get the package of 25.
- Students make a table, plot a graph (number of people vs. cost), and look for a function to describe a situation in which the Student Council is sponsoring a Valentine's Day dance and must pay $300 to the band, no matter how many people come. They also must pay $4 per person for refreshments, with a minimum of 50 people. The students note that the cost will be $500 for anywhere from 0-50 people and then increase at a rate of $4 per person. They decide that this is a function with a corner and needs to be defined in pieces:

$$f(x) = 500 \quad \text{for } x \leq 50$$
$$f(x) = 500 + 4x \quad \text{for } x > 50$$

19. Understand and apply approximation techniques to situations involving initial portions of infinite decimals and measurement.

- Students investigate finding the area under the curve $y = x^2 + 1$ between -1 and 1. They approximate the area geometrically by dividing it into rectangles 0.5 units wide. They find the height of each rectangle that fits under the curve and use it to find the areas. Then they find the height of each rectangle that contains the curve and use these measurements to find the areas. They realize that this gives them a range of values for the area under the curve. They refine this approximation by using narrower rectangles, such as 0.1.
- After some experience with collecting data about balls thrown into the air, students are given a table of data about a model rocket and its height at different times. They plot the data, find an equation that fits the data, and use the trace functions on their graphing calculators to find the maximum height.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

STANDARD 16
EXCELLENCE AND EQUITY FOR ALL STUDENTS

Introduction

This overview discusses excellence and equity and how they are described in the *Mathematics Standards* of the New Jersey State Department of Education's *Core Curriculum Content Standards*. It is followed by five sections entitled ***High Expectations in Mathematics for All Students, The Significance of Mathematics***, ***Overcoming the Barriers to Equity***, ***Challenging All Students to Maximize Their Achievement,*** and ***Identifying Equity Concerns in Districts and Schools***.

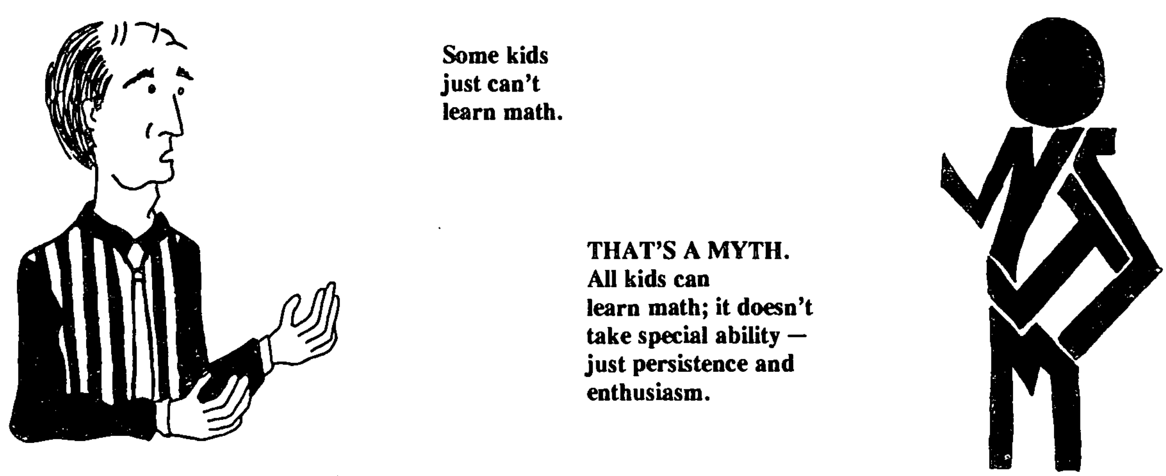
> All students will demonstrate high levels of mathematical thought through experiences which extend beyond traditional computation, algebra, and geometry.

Descriptive Statement

High expectations for all students form a critical part of the learning environment. The belief of teachers, administrators, and parents that a student can and will succeed in mathematics often makes it possible for that student to succeed. Beyond that, this standard calls for a commitment that all students will be continuously challenged and enabled to go as far mathematically as they can.

High Expectations for All Students

The first fifteen standards set out high expectations for all students, and this final standard insists that all students **need to**, **can**, and will meet those standards.



Math doesn't run in my family.

THAT'S A MYTH. Children don't inherit their parents' difficulties with math.

As discussed in the Introduction to this *Framework*, our students **need to** meet these standards in order for them to be well prepared for careers in the 21st century, and in order for our state and country to have suitable employees in the 21st century.

But **can** all students meet high standards? Our answer is unequivocally *Yes*. New Jersey's *Mathematics Standards* sets high standards for all students, and at the same time insists that those standards are indeed attainable by all students. There will undoubtedly be exceptions, but those exceptions should be exceptional.

Achieving these standards for all students will not be easy, but it must start from the assumption that every individual student can achieve these standards. There are many barriers that have to be dealt with, and these are discussed later in this chapter, but none of these barriers are more powerful than the pervasive sense that many of our children cannot achieve these standards.

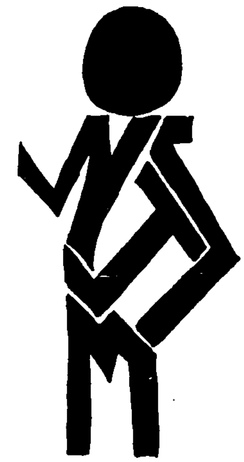
We must take seriously the goal of preparing ALL students for twenty-first century careers. In order to do this, we must overcome the all too common perception among people that many students simply lack mathematical ability. *Everybody Counts*, a report prepared by the Mathematical Sciences Education Board (MSEB) of the National Academy of Sciences (1989), notes the following:

> Only in the United States do people believe that learning mathematics depends on special ability. In other countries, students, parents, and teachers all expect that most students can master mathematics if only they work hard enough. The record of accomplishment in these countries — and in some intervention programs in the United States — shows that most students can learn much more mathematics than is commonly assumed in this country (MSEB, 1989, 10).

Curricula that assume student failure are bound to fail; we need to develop curricula which assume student success. We need to develop attitudes in students, in parents, and in school personnel which assume student success. And we need to translate those positive attitudes and high expectations into programs which ensure that students **will** meet the standards. This chapter is intended to provide information and guidance to districts on how to make that translation.

Girls are just not good at math.

THAT'S A MYTH.
All children, girls and boys, are equally good at math.[1]



Opportunities to Exceed the Standards

By insisting that all students can meet high expectations, the *Mathematics Standards* does not imply that all students will become professional mathematicians, scientists, or engineers. Certainly, as with other areas of human endeavor, some students have more interest in and talent for mathematics than others.

The *Mathematics Standards* does not describe expectations for those students who might be going on to careers which require higher levels of mathematics. Its focus is on the high expectations which are appropriate for all students. However, the *Mathematics Standards* insists that raised expectations for all students should not result in lowered expectations for our high achieving students.

Indeed, to increase the number and success of high achieving students it is necessary to provide all students with opportunities to learn more mathematics than is encompassed in the *Mathematics Standards*. This is discussed further in the section entitled ***Challenging All Students to Maximize Their Achievement.***

What is Equity?

All students need to achieve, and can achieve the high expectations of New Jersey's *Mathematics Standards*. Ensuring that all students have every opportunity to achieve these high expectations is the focus of our concerns about equity.

The United States has traditionally promoted education as the most effective vehicle for access to intellectual development and economic independence. The *Curriculum and Evaluation Standards for School Mathematics* of the National Council of Teachers of Mathematics (NCTM, 1989) cites data to support this premise. For example, jobs requiring mathematical knowledge and skills in such areas as data analysis, problem solving, and statistics are growing at nearly twice the rate of overall employment.[2] In addition, the strongest predictor of earnings nine years after graduation from high school is the number of mathematics courses taken (after having taken into account demographic factors) (NCTM, 1992, 3). In

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[1] Drawings by Mira Rosenstein.

[2] *Making Points for Mathematics Education Reform*, the Mathematical Sciences Education Board, p. 3.

order to compete effectively in today's global, information-based economy, and in today's increasingly high-tech work environment, students must be able to reason logically, solve problems and communicate effectively. Clearly they must be educated at a more sophisticated level of mathematical and scientific reasoning than ever before. Mathematics and science, therefore, have been called the "critical filters" for determining future success.

Achievement in mathematics, science, and technology, however, has not been equally accessible to all students. Certain ethnic and racial minorities in the United States, including populations of students from economically disadvantaged backgrounds, are substantially *underrepresented* among top achievers, and are included in disproportionate numbers among those whose achievement is unsatisfactory. Young women are still underrepresented in these disciplines, as compared to young men.

According to a statement developed by the Equity Focus Group for the Statewide Systemic Initiative supported by the National Science Foundation (1994) "equity means equitable access to high-quality science, mathematics, and technology education and equitable treatment in the classrooms, schools, and post-secondary education institutions for every student. The goal is to eliminate the academic performance gap between mainstream groups and underrepresented groups, and to raise the level of knowledge and skills in mathematics and science of all students."[3] This goal is reflected in Standard 16 of the New Jersey's *Mathematics Standards*.

A recent study by the Rand Corporation[4] confirmed just such improvement in the last fifteen years: standardized test scores for African-American and Latino teenagers improved significantly between the mid-1970's and 1990, narrowing the gap with Caucasian students, who also made gains. The Rand study said that "the gains suggested that desegregation and increased spending, especially for programs designed for minority students, had paid off."

Each district should commit itself to reducing the gap between its own mainstream and underrepresented groups, without diminishing the performance of the mainstream groups. The final section of this chapter, ***Identifying Equity Concerns in Districts and Schools,*** provides suggestions for how a district or school might evaluate its own situation with regard to this goal.

[3] National Science Foundation Statewide Systemic Initiative. (1994). *Equity Framework in Mathematics, Science, and Technology Education*. Washington, DC.

[4] Grissmer, D. W. (1994). *Student achievement and the changing American family*. Santa Monica, CA: Rand Corporation.

Cumulative Progress Indicators

New Jersey's *Mathematics Standards* provides the following eleven cumulative progress indicators for Standard 16. They are arranged in four groups, and their implications are discussed in the next four sections of this chapter.

The first two indicators address our commitment to high expectations in mathematics for all students, that all children learn mathematics in accordance with the vision of the *Mathematics Standards*.

By the end of grade 12, all students:

1. **Study a core curriculum containing challenging ideas and tasks, rather than one limited to repetitive, low-level cognitive activities.**
2. **Work at rich, open-ended problems which require them to use mathematics in meaningful ways, and which provide them with exciting and interesting mathematical experiences.**

In order for children to value mathematics, they need to understand the importance of mathematics in their own culture and other cultures, they need to understand that the quantity and quality of their own mathematical achievements will affect their futures, and they need to know that members of their community use mathematics in their own occupations. The following three indicators address these issues:

By the end of grade 12, all students:

3. **Recognize mathematics as integral to the development of all cultures and civilizations, and in particular to that of our own society.**
4. **Understand the important role that mathematics plays in their own success, regardless of career.**
5. **Interact with parents and other members of their communities, including men and women from a variety of cultural backgrounds, who use mathematics in their daily lives and occupations.**

Children need to hear the message that all students can learn mathematics and that their schools are making a commitment to their achieving the high expectations in the New Jersey's *Mathematics Standards* through successful completion of the core curriculum. The following three indicators address these issues:

By the end of grade 12, all students:

6. **Receive services that help them understand the mathematical skills and concepts necessary to assure success in the core curriculum.**
7. **Receive equitable treatment without regard to gender, ethnicity, or predetermined expectations for success.**
8. **Learn mathematics in classes which reflect the diversity of the school's total student population.**

Finally, all students should be provided with encouragement and opportunities to go beyond the expectations of the *Mathematics Standards*. The following three indicators address this issue:

By the end of grade 12, all students:

9. **Are provided with opportunities at all grade levels for further study of mathematics, especially including topics beyond traditional computation, geometry, and algebra.**
10. **Are challenged to maximize their mathematical achievements at all grade levels.**
11. **Experience a full program of meaningful mathematics so that they can pursue post-secondary education.**

High Expectations in Mathematics for All Students

Engaging and Challenging All Students

The first two indicators specify that students should "study a core curriculum containing challenging ideas and tasks, rather than one limited to repetitive, low-level cognitive activities" and that they "should work at problems which require them to use mathematics in meaningful ways." The vision of the *Mathematics Standards* (see pp. 8-9 of this *Framework*) speaks of "students who are excited by and interested in their activities", "students who are learning important mathematical concepts," and "students who are posing and solving meaningful problems."

Students need to be engaged and challenged. To accomplish this, we need to involve them in hands-on activities, to provide them with settings where they can participate in mathematical discovery, to decrease the focus on repetitive tasks, to make available alternate ways of learning concepts, and to offer them activities which they recognize as meaningful.

Above all, we need to challenge them with meaningful problems. Indeed, problem solving should be the central focus of all mathematics instruction. Students benefit from a classroom environment in which they are working together to find solutions for meaningful problems. Levels of engagement, communication, and achievement are all enhanced when students accept these kinds of challenges. Moreover, such an instructional approach also tends to work more effectively than any other with heterogeneously grouped students. Students tend to find ways to contribute to the overall group progress by sharing their own skills and understandings with their classmates.

Traditionally, problem-solving oriented approaches to mathematics instruction have been used with only special populations of students. Remedial teachers frequently relied on such an approach to recapture the students' lost interest or to rekindle their motivation to learn the subject. At the other end of the spectrum, problem-solving also frequently form the basis for an elementary school program for gifted-and talented students. Some of the best available curriculum units in elementary mathematics were written to be used with a gifted population. It was thought that these units were necessary to fully engage bright students and to illustrate for them the power and pervasiveness of mathematics.

But we must now focus on making this curricular focus the focus for all students. For just the same reasons that teachers thought this approach to be useful for special populations, it turns out to be useful for all. Motivation, engagement, and appreciation for the usefulness and power of mathematics are dispositions that enable all students to be effective learners.

Core Curriculum

As noted on p. 12 of this *Framework* and reflected in Indicators 1 and 6, implicit in the vision of the *Mathematics Standards* is the notion that there should be a core curriculum. What do we mean by a "core curriculum"? We mean that every student will be involved in experiences addressing all of the expectations of each of the other fifteen content standards. We anticipate that over time each district will review the *Mathematics Standards* and, using this *Framework* as a guide, will develop its own core curriculum. All courses of study in the district should then have a common goal of completing this core curriculum, no matter how students are grouped or separated by needs and/or interests.

A core curriculum does not mean that all students will be thrown together into one program. There may be different programs with different goals, but completing the core curriculum should be a goal that is common to all of the programs. Students have different aptitudes, interests, educational and professional plans, and learning styles. Different groups of students may address the core curriculum at different levels of depth, and may complete the core curriculum according to different timetables. Nevertheless, all students should complete all elements of the core curriculum recommended in the *Mathematics Standards*.

For example, it is anticipated that those students who normally go on to take calculus in the 12th grade will complete the core curriculum, and go substantially beyond it, by the end of 10th or 11th grade. Indeed, at the present time, with curricula currently in place, those students are likely to complete most of what would be in the core curriculum, and more, by the 11th grade, with the major exception of the content described in Standards 12 (Probability and Statistics) and 14 (Discrete Mathematics). On the other hand, a substantial percentage of the students in most districts might well be involved with the core curriculum through the end of the 12th grade.

At the elementary school level, each district already has in effect a core curriculum which all students are expected to complete, but it may be focused primarily and even exclusively on arithmetic; however, the *Mathematics Standards* recommends a core curriculum with more substantial expectations than the current curricula. In addition, the recommendation of a core curriculum at the high school level has major implications for the elementary school level, since every student needs to enter the upper grade levels with both the knowledge and the confidence to achieve the expectations of the *Mathematics Standards* during her or his remaining years in school.

The core curriculum recommended in the *Mathematics Standards* is appropriate also for students in vocational education. According to the Secretary's Commission on Achieving Necessary Skills (SCANS), a study of competency in the workplace across the *entire* spectrum of the economy revealed a clear pattern of requirements. Workers need a solid foundation in retrieving, analyzing, and evaluating data, applying technology to specific tasks, and the ability to reason, think creatively, and solve problems.[5] Clearly this is the case for students involved in vocational education programs as well as college preparatory programs, and fulfilling the core curriculum of the *Mathematics Standards* would address the SCANS recommendations.

[5] *Learning a Living*, SCANS (April 1992, 5-6).

High Expectations in the Elementary and Middle School — The Issue of Grouping

In the abstract, it seems to make sense to group students according to ability; higher ability students can proceed more rapidly, and lower ability students can receive additional instruction. In practice, however, homogeneous grouping of students (i.e., where students of similar abilities are grouped together), and its logical extension of tracking students into entire programs based on selected abilities, has limited the achievement of a substantial percentage of children. The research shows that once placed in a track, there is little chance of moving to a higher track; that tracking in mathematics is often based on reading scores or poor behavior rather than on mathematical ability; and that a disproportionate percentage of low-income and minority students are placed in low tracks. Students get the message that less is expected of them.

Moreover, the practice of homogeneous grouping and tracking of students is based on the premise that the abilities of students are substantially different. In fact, exceptional children are indeed the exceptions; all students have the ability to succeed in mathematics and to be empowered to use it successfully. "Individual differences in ability are not great enough to warrant differences in curriculum, except in unusual circumstances such as major learning disabilities or extraordinary talent," said leading mathematics educator Zalman Usiskin.[6] The reason some students appear to have high ability is often because they have been better prepared, they are building on a foundation of greater knowledge, and they have greater interest and willingness to work.

Heterogeneous grouping (i.e., where students who appear to have different abilities are grouped together) reinforces the message that all students can succeed and can meet high standards. The research shows that heterogeneous grouping does have the desired effect: all students' self-image and self expectations rise, as does their performance (SCDOE, 1992, 26).[7]

What happens in the first few years of school is crucial. If all children in the elementary grades develop mathematical competency and positive attitudes toward mathematics, then equity problems will be less significant later on. If we want all students to complete the elementary levels with the knowledge and confidence that are needed for success in the upper grades, then heterogeneous grouping must be a key component of our strategic plan.

Heterogeneous grouping by itself, however, will not ensure success. It is a strategy, not a solution. The focus must be on providing all children with opportunities to learn, with encouragement to succeed, and with continued mathematical growth. In some settings, the strategy of heterogeneous groups may be counterproductive — resulting in some children experiencing only the frustration of failure, and others the boredom of stagnation. This may often be the case because teachers have not been prepared to implement this strategy. Training and support for teachers are critical; they must be in place before and during implementation of heterogeneous grouping, so that teachers can respond flexibly to the diverse needs of the students in the classroom. Teachers must be prepared to function as problem-solvers, ready to use a variety of strategies to ensure that all children learn; they must be familiar with these strategies, and understand when they should be used, and when they should not be used. We do our children a disservice if we abandon the old dogma of homogeneous grouping only to adopt a new dogma of heterogeneous

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[6] *Individual Differences in the Teaching and Learning of Mathematics*, presentation at the Ninth Annual UCSMP Secondary Conference, November 1993. See also *If Everybody Counts, Why Do So Few Survive?* in *Reaching All Students with Mathematics*, NCTM, 1993, Reston, VA.

[7] South Carolina Department of Education. (1992). *South Carolina Mathematics Framework*. Columbia, SC.

grouping; we do our children a greater disservice if we adopt heterogeneous grouping without assisting teachers to implement it properly. Administrators must ensure that this does not happen; they must encourage teachers to learn how to teach children who are grouped heterogeneously, and how to strike an appropriate balance between that and homogeneous grouping. The rationale for any form of grouping must be reflected in the structure of the classroom, in the activities that take place there, and in the instructional strategies used by the teacher. We suggest that:

✓	*SUGGESTIONS*

✓ districts use heterogeneous grouping as a strategy, at least through 6th grade, and possibly in 7th and 8th grades depending on local circumstances.

✓ districts provide opportunities for continuous and rigorous professional development to its teachers to help them develop the variety of instructional strategies needed for effective student learning, with a focus on heterogeneous grouping and how and when it can best be used.

✓ districts use heterogeneous grouping as an opportunity to provide an important advantage for the better prepared and highly motivated students — the opportunity to strengthen their own understanding through sharing it with others.

✓ districts provide assistance to those students who require it, enrichment for all students, and additional opportunities for those who are better prepared and highly motivated.

✓ since students progress at different rates, schools use flexible strategies (including enrichment activities) that ensure that all students have the opportunity to learn when they are ready.

✓ teachers closely monitor student performance and provide assistance to those students who need it. Priority should be placed on preventing students from falling behind through:

* additional instructional sessions;
* individual assistance in class;
* tutoring outside of class;
* cross-age tutoring in after-school programs;
* after-school access to technology;
* activities between school sessions;
* summer sessions; and
* enlisting support and assistance from adults in parental or community support roles.

High Expectations in the Secondary School

Each district is expected to develop a core curriculum based on the *Mathematics Standards* which embodies high achievable expectations for all students. In developing its core curriculum, districts should consider the following suggestions:

✓	*SUGGESTIONS*

✓ districts may institute different courses and programs for different groups of students, but all ofthese should have a common goal — completing the core curriculum.

✓ the core curriculum should not be simply a rearrangement of traditional mathematics topics taught traditionally, which in the past has served to filter out substantial numbers of students, but should focus on making mathematics, including both traditional topics (such as algebra and geometry) and new topics (such as probability and discrete mathematics) relevant, exciting, challenging, and accessible to all students.

✓ all general mathematics, consumer mathematics, and other courses which focus primarily on mastery of lower-level basic skills should be transformed into courses which address the core curriculum as well as basic skills.

✓ districts should maintain support systems and enrichment opportunities that appropriately address the needs of their students.

✓ students should be permitted and enabled to cross over from one sequence of courses to another by demonstrating mastery of the material.

✓ all students, regardless of their backgrounds and abilities, should have the opportunity to study this core curriculum in an environment which allows them to develop to their fullest potential. Such an environment is one where[8]:

* students are encouraged to communicate freely and to take risks in asking mathematical questions and proposing solutions to mathematical problems, and feel safe in doing so;
* teachers recognize that students have the potential to learn mathematics at a high level;
* instructional strategies accommodate the cultural diversity, varied learning styles, and different amounts of time needed by different students;
* placement decisions are made in the interest of elevating students to the most challenging course they can be successful in;
* placement decisions are based on multiple assessment measures and recommendations of classroom teachers;
* classroom assessment includes continuous evaluation of students' understanding and performance using a variety of assessment strategies;
* all students have the opportunity to study all of the core mathematics curriculum;
* students are expected to do complete work by thinking, drawing on mathematical ideas, and using appropriate tools and techniques;
* students are expected to communicate their reasoning and problem-solving strategies and are able to hear and discuss different strategies used by other students;
* students are encouraged to work harder, and are provided with the necessary support if they are having difficulties;
* students with a special interest in mathematics pursue issues with similarly interested peers, while still participating in the common core curriculum;
* students have sufficient time to process questions, formulate their answers, and present them to the class;

[8] This list is adapted and extended from a similar list in the Field Review Draft (1992) of the *South Carolina Mathematics Framework*, which in turn based its list on the draft (1991) *Mathematics Framework for California Public Schools*, 51.

* students have a chance to revise and resubmit their work until it meets quality standards;
* students have access to the same quality of technology;
* the dynamic of students teaching students is harnessed through grouping strategies to leverage the teacher's instruction; and
* services are available to help students who have gaps in their skills and understandings.

The Important Role of Mathematics

In order to achieve Indicators 3-5, students need to understand the importance of mathematics in their own culture and in other cultures, they need to understand that their futures will be affected by their mathematics achievement, and that mathematics is indeed used by adults in their communities.

Mathematics in All Cultures

Mathematics plays an integral role in art, music, games, explorations, inventions, and commerce within virtually any culture. People in all societies have devised their own ways of doing mathematics, and an inclusive study of cultures and their various contributions to mathematics is an effective way to demonstrate its relevance to all students[9] [10]. We suggest that teachers:

✓	*SUGGESTIONS*

✓ ask groups to research the role of mathematics in various fields of human endeavor, and to report to the entire class on their projects.

✓ design instructional units that help students experience how decisions involving mathematics are made in relation to the needs and practices of various cultures. Such discussions can be used to connect mathematics with other content areas such as history, literature, sociology and art. For example, when designing rugs or quilts (see vignette), students are using the mathematics of geometry and measurement. A broader topic is the use of patterns in various cultures — for example, in art or in clothing. Another topic might be the types of architecture which characterize various times and places. For instance, a study of housing might revolve around the concepts of size, shape, perimeter, and area, and may be used to develop skills of estimation and approximation. Types of housing might include an African round house, a tipi, an urban apartment, and a suburban ranch house. (NCTM, 1993, 54).

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[9] For information about mathematics and culture, contact the International Study Group on Ethnomathematics, Math-Tech, 9155 N. 70th Street, Milwaukee, WI 53223.

[10] Some helpful sources are *Multicultural Mathematics Materials*, Marina Krause, NCTM, Reston, VA; *Ethnomathematics: A Multicultural View of Mathematical Ideas*, Marcia Ascher, Brooks/Cole Publishing Company, Pacific Grove, CA; and *Africa Counts: Numbers and Patterns in African Culture*, Claudia Zaslavsky, Warren Mill Books.

Mathematics, Folk Art and Literature: Making Connections

In a third grade classroom, students planned and then created a quilt as a culminating experience. As a class, the children used measurement to divide the quilt into separate squares. Decisions about symmetry, alignment, design, and effective use of space were addressed by the whole class. Then each child designed one quilt square which represented an important piece of learning they valued during the year. Next, students, working in cooperative groups, organized the separate quilt squares into thematic groupings. Children looked for patterns among the separate quilt squares, proposed several ways to organize the squares and reached consensus. Earlier, the children read several fiction and nonfiction texts about quilting as a folk art. The finished quilt was displayed prominently at the Board of Education for the entire school community to view.

✓ review the counting words in various languages and analyze the different schemes for counting; in some cultures, for example, grouping is done by twenties, rather than by tens (NCTM, 1993, 54).

✓ use materials that reflect the diversity of cultural backgrounds of the students in the classroom or school.

✓ include history of mathematics in daily lessons, including examples of male and female mathematicians and scientists from a variety of cultures.

✓ be alert to the possibility that cultural perspectives may have the effect of discouraging some students from succeeding in mathematics.

Mathematics in Their Future

Students need to be aware that the mathematics that they learn, the problem-solving that they do, will affect their futures. One of the problems described in the New Jersey State Department of Education's *Directory of Test Specifications and Items*[11] asks students to determine a schedule for employees at a fast food place so that each employee has a reasonable schedule, so that sufficient staff is available at peak times, and so that labor costs are minimized. Having the problem-solving skills to tackle this kind of problem will differentiate between those who are eligible for managerial positions and those who be unable to advance beyond minimum-wage positions. This point is also made by the National Action Council for Minorities in Engineering's (NACME) "Math Is Power" video public service announcement and print materials[12].

Students need to be aware that mathematics is used by artists and musicians, by scientists who are involved in space travel, and by designers of skateboards. This point is stressed in the following materials:

- *Mathematics: Making a Living, Making a Life*, a booklet demonstrating how mathematics is all around us, and *Mathematics: Making the Connection*, a video which shows viewers how a jazz musician,

[11]New Jersey State Department of Education. *Directory of Test Specifications and Items — 1991 Grade 11 High School Proficiency Test.* Trenton, NJ, 1993.

[12]The National Action Council for Minorities in Engineering (NACME) is dedicated to expanding minority participation in engineering and the sciences. NACME can be contacted at 3 West 35th Street, New York, NY 10001, by phone at (212) 279-2626, or via their World Wide Web site http://www.nacme.org.

architect, and newspaper publisher connect mathematics to their daily lives and professions; available from the National Council of Teachers of Mathematics (NCTM)[13].

- *SETQuest Career Discovery* video, CD-ROM, and print materials that explore careers that use science and mathematics with candid career profiles of "on-the-job" educational requirements for a variety of careers; available from The Consortium for Mathematics and its Applications (COMAP)[14].
- *101 Careers in Mathematics*, biographical essays of individuals with careers that require a solid background in mathematics, and *She Does Math! Real Life Problems from Women on the Job*; available from the Mathematical Association of America (MAA)[15].

A very important way in which teachers can reinforce this message is to have community members come to the school and share their biographies and their experiences. Visitors should include both those whose professions are obviously mathematical or scientific and those who professions are not. In the first instance, the students will be impressed by their activities, and, in the second instance, they will be surprised at the problem-solving activities in which "ordinary" people are involved.

Overcoming Barriers to Equity

The third group of indicators focus on ensuring that all students are provided the opportunity to succeed in mathematics — that they receive the appropriate services and are treated equitably. In this section we discuss the variety of barriers to achieving equity in our schools and how those barriers can be overcome. These include the following types of barriers, each of which is discussed in a separate section below:

- attitudinal barriers — attitudes and beliefs of students, parents, and educators;
- gender barriers;
- economic, language, and disability barriers; and
- geographical barriers — special problems faced in urban and rural settings.

Overcoming Attitudinal Barriers to Equity

An important set of barriers to equity are the attitudes and beliefs of students, parents, and educators which influence student outcomes in mathematics.

Students' **attitudes toward mathematics** are an important factor in their learning of mathematics; those who enjoy mathematics and have confidence in their mathematical abilities are more likely to succeed. Studies have shown that minority students have positive attitudes toward mathematics in the primary

[13]The National Council of Teachers of Mathematics (NCTM) catalog of materials can be requested by calling (703) 620-9840 or by e-mail orders@nctm.org.

[14]The Consortium for Mathematics and its Applications (COMAP) catalog of materials can be requested by calling 1-800-772-6627 or by e-mail order@comap.com.

[15]The Mathematical Association of America (MAA) catalog can be requested by calling 1-800-221-1622.

grades. Like other students in the United States, however, they become less positive about mathematics as they proceed through school; both confidence and enjoyment of mathematics decline as students move from elementary through secondary school (NCTM, 1993, 22)[16]. Similarly, although girls do better in mathematics than boys in the early grades, a major drop in their self-confidence, and a concomitant decline in their performance, occurs in the fifth and sixth grades.

Students' **beliefs about mathematics** are also an important factor. Those for example who believe that all mathematics problems should be solved quickly will curtail their efforts if they encounter problems which take more than a few minutes, or if they find that a problem takes them more time than other students. Similarly, we have to be aware that all students experience both positive and negative emotions as they learn mathematics, and that students will develop negative beliefs about their mathematical abilities if their negative emotions are not balanced with positive emotions about mathematics. Learning mathematics is not always easy, and will at times generate frustration, but we must ensure that our students' experience of frustration is not converted into a sense of failure.

Another important factor is students' **beliefs about the utility of mathematics**. Majority students, in general, rate mathematics higher in utility than do minority students (The Mid-Atlantic Equity Center, 1992, 9)[17]. Males in general perceive mathematics as more useful and valuable to their futures than do females. One significant consequence of these differences in attitudes is that males are therefore more likely to take more mathematics courses and, as a result, have more desire and more opportunity to pursue mathematics-related careers.

Also important is **how students account for their success or failure in mathematics**. As noted earlier, in other countries success or failure in mathematics is usually attributed to hard work or its absence, whereas in the United States success or failure in mathematics is usually attributed to ability. *This is particularly true of female students, who tend to attribute their success to extra effort and their failures to lack of ability; males on the other hand, are more likely to attribute their success to ability and their failures to lack of effort.* Research on confidence in learning mathematics indicates that males tend to be more confident even when females have more reason (based on their achievement) to be confident.

Research indicates that **attitude and achievement interact with each other in subtle and often unpredictable ways**. For example, while students in Japan express a greater dislike for mathematics than students in other countries, they exhibit a very high level of proficiency in mathematics. More common here, however, is that identification as a low achiever, or placement in a lower track, has a negative impact on a student's self-confidence and belief in his or her own ability to learn mathematics.[18]

Finally, **peer pressure** is often a barrier to student achievement in mathematics. Students from underrepresented groups sometimes hold attitudes which are counter-productive and discourage their peers from achieving. Peer pressure to avoid academic excellence can be particularly difficult to combat among minority adolescents because they sometimes link it to majority cultural values or to disloyalty to their group (USDOE, 1993, 13).

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[16] National Council of Teachers of Mathematics. 1993. *Reaching All Students with Mathematics*. Reston, VA.

[17] Mid-Atlantic Equity Center. (1992). *Opening up the Mathematics and Science Filters*. Washington, DC: The American University.

[18] *Mathematics Framework for California Public Schools*, California Department of Education, 62.

Attitudes and beliefs which discourage achievement are not limited to students; parents, educators, and policy makers also exhibit reduced expectations for some students.

Parents and teachers of high achieving girls sometimes reduce their expectations when girls reach adolescence out of concern that high achievement may jeopardize social approval. One mother refused to allow her seventh grade daughter to enroll in an algebra program for which she had been selected. She explained, "The stress of such a high powered class is likely to make her acne worse. She doesn't need that at her age." In class, teachers call on girls less frequently than boys, and tend to have shorter interactions with them. Well-meaning teachers also sometimes limit gifted girls to protect their popularity. In one eighth grade class, the two gifted boys were sent to the high school for math, while the parents of the two girls, who had similar abilities, were told that their daughters would be better served by remaining in the classroom because "there's a stigma for girls who go to the high school for math."

Parents and teachers of minority students also frequently believe that their children cannot be successful in mathematics and try to protect them by placing them in lower level courses; policy makers try to protect them by arguing against challenging expectations. While it is very difficult for students to overcome inadequate elementary and middle school experiences with mathematics, there is no reason to believe that minority students with positive early experiences with mathematics will not be successful in high school.

Parents who themselves had negative experiences with mathematics tend to transmit their apprehensions to their children and communicate that lower expectations in mathematics are acceptable.

It's a myth that some kids can do math and others just can't. It's a myth that girls are not as good at math as boys. It's a myth that children inherit trouble with math from their parents. We need to encourage *all* children to succeed in mathematics.

We suggest that teachers, schools, and districts:

✓	*SUGGESTIONS*

✓ use instructional strategies (see Chapter 17) that provide positive experiences for students and engender in them enjoyment in mathematics and confidence in their abilities to do mathematics.

✓ convey to students that it is normal to find mathematics frustrating at times and, through examples, that it sometimes takes a considerable amount of time to solve mathematical problems. (Students need to understand the difference between an exercise, which usually involves applying a simple specified procedure, and a problem, part of which involves determining what methods are appropriate.)

✓ maintain and convey high expectations for all children, and reinforce the message that all students can meet those high expectations.

✓ encourage students to strive for excellence in mathematics by providing opportunities for them to participate in math clubs, math teams, and other math activities[19].

✓ reflect on your own attitudes toward students and your expectations for them by answering the

[19] See footnotes 52-56 for information about these kinds of activities.

following self-reflective questions[20]:

* How do I interact with girls, minority students, and "low achievers" in class?
* Do my interactions reflect lesser expectations for some groups than for others?
* Do I believe that all students can learn mathematics?
* Do I interact more with the "high achievers" and give less attention to "low achievers?"
* Do I ask more questions and questions requiring higher levels of thinking to the "high achievers?"
* Do I provide sufficient wait time for all students to formulate responses to questions?
* Do I seat "high achievers" more closely to myself than "low achievers?"
* Do I praise "high achievers" more often than "low achievers?"
* Do I provide detailed feedback to "high achievers" and provide less precise feedback to "low achievers?"
* Do I demand more work and effort from "high achievers" and accept less work and work of a lower standard from "low achievers?"
* Do I provide a positive classroom environment in which students are willing to share their questions and their answers without fear of being blamed or shamed?

✓ provide effective professional development for teachers who lack sufficient content knowledge, or who have a limited instructional repertoire.

✓ use positive peer influence to help shape the attitudes and beliefs of students. Peer tutoring, especially between younger students and older peer tutors, can increase students' interest and motivation.

✓ seek actively to dispel parental myths their children must inherit their anxieties and difficulties with mathematics.

✓ communicate with parents and encourage them to encourage their children to learn mathematics, even if their own experiences were not positive.

✓ utilize adult members of the community to help shape students' attitudes about mathematics and help them recognize the usefulness of mathematics in their future careers. Female and minority members of the professional community who use mathematics in their daily jobs and lives can be enlisted to serve as role models for female and minority students and to help others recognize the contributions and possibilities of women and minorities.

✓ obtain information about how other schools and districts are addressing issues of equity[21].

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[20] Adapted from *Opening Up the Math and Science Filters*, MidAtlantic Equity Center, 1992, 1-7.

[21] A good source is districts involved in the New Jersey Statewide Systemic Initiative (NJ SSI), "Achieving Excellence in Mathematics, Science and Technology Education" funded by the National Science Foundation and the New Jersey Department of Education. The NJ SSI is committed to the advancement of equity, and districts involved in NJ SSI can provide assistance in planning and implementation. For further information, call Roberta Schorr at (908) 445-2342.

✓ review and make use of the literature[22] [23] [24] on equity in mathematics education.

Overcoming Gender Barriers to Equity

An equitable learning environment is one that encourages the mathematics development of male and female students from every cultural background. We suggest that teachers:

✓	*SUGGESTIONS*

✓ self-assess your individual beliefs and habits in regard to gender-role stereotyping. Ask the following self-assessment questions[25] [26]:

* do I believe females are less capable in mathematics than males?
* do I allow male students to control the discourse?
* do I interact more with male students than female students?
* do I encourage risk-taking and autonomous behavior in female students?
* do I believe it is more important for the boys to "get it?"

Check your responses by inviting a friend to tape your class and then analyzing your interactions on the tape with male and female students.

✓ avoid using the technique of increasing attention to and interaction with boys as a method for controlling the class. There is an apparent benefit to paying more attention to male students or allowing them to call out the answers, since it keeps the class on task and discourages disruptive

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[22] The Consortium for Educational Equity at Rutgers University, New Brunswick has the nation's most extensive library collection on equity in mathematics and science education. Subject bibliographies are created and updated on a regular basis to highlight the collection and to assist borrowers in reviewing materials they wish to use. Bibliographies on the following topics are particularly relevant: Cooperative Learning, Educational Equity: Training and Technical Assistance, Equity in Science and Mathematics, Futures Unlimited: Career Information, Futures Unlimited in Mathematics and Science (grades 7-12), Guidance and Counseling, Science-Related Storybooks for the Classroom, Science Stuff for Girls and Boys (grades K-6), Sources for Resources (for Librarians and School Library Media Specialists). In addition, subject bibliographies which support the understanding of the major cultural/ethnic groups in New Jersey have been developed for educators. Contact the Consortium at (908) 445-2071 to gain access to its collection.

[23] The Mid-Atlantic Eisenhower Consortium for Mathematics and Science Education, located at Research for Better Schools, Philadelphia, has published a useful resource guide, *Equity Materials in Mathematics, Science, & Technology*, by Marylin A. Hulme. This guide includes print materials, resources for career information, and audiovisual materials for use with students and also in staff development programs. All the materials listed in this guide are included in the collection in the Resource Center at the Rutgers Consortium for Educational Equity, and are available for borrowing. For a copy, please call Research for Better Schools (215) 574-9300.

[24] The Mid-Atlantic Eisenhower Consortium for Mathematics and Science Education maintains a World Wide Web Page dedicated to Equity Issues in Mathematics and Science. The address of this page is http://www.rbs.org/eisenhower/res_equity.html/. From this page you can link to a wide variety of equity sites.

[25] List adapted from *Teaching Mathematics Effectively and Equitably to Females*, Katherine Hanson, ERIC Clearinghouse on Urban Education: New York, NY, 1992, 31.

[26] See also the list referenced in footnote 20 which is adapted from *Opening Up the Math and Science Filters*.

activity. On the other hand, an important consequence is that it permits many female students to remain passive in their dealings with mathematics and invisible in the mathematics classroom.

✓ use language and materials that are free from gender-role stereotyping[27].

✓ provide career examples early in life to female students and encourage them to develop as mathematical thinkers. Research shows that girls do not think about advancing in mathematics because they have no idea of how they could use it in life (Hanson, 33).

✓ consider the following recommendations of the Institute for Urban and Minority Education[28]:

* use situations that introduce girls and boys to a variety of mathematical-related career options;
* provide role models of men and women working in mathematics, technology, and the sciences;
* encourage female students as well as male students to participate in extracurricular math and science activities;
* be sensitive to the meanings of words — children do not translate "man" to mean both "man" and "woman";
* encourage female students as well as male students to explore;
* introduce female and male students to action toys and activities such as team sports which increase their spatial visualization skills;
* make mathematics fun and appealing to both male and female students, using word problems that relate to the interests of both and that emphasize non-stereotyped roles;
* devise comfortable ways for students of both genders to play and interact;
* teach mathematics to young children through play; and
* discover early and correct promptly gaps in previous mathematical education so as to encourage both female and male students to continue in mathematics.

Overcoming Economic, Language, and Disability Barriers to Equity

Overcoming economic barriers to equity. More than 20% of the school children in the United States come from families in poverty (USDOE, 1991,1)[29]. In New Jersey, more than 11.5% of school children are below the poverty line[30]; of those who are in families with three or more children, over 20% are below

[27] Marylin A. Hulme of the Rutgers Consortium for Educational Equity has developed a series of questions for teachers and curriculum specialists to use when evaluating mathematics books for bias and stereotyping. These "guidelines" are intended to make the users aware of how the texts and illustrations depict girls and boys, women and men, the language used, and the roles that are assigned to both females and males. *The Guidelines for Evaluating Mathematics for Bias* can easily be used in conjunction with an evaluative chart/checklist. For more information, call (908) 445-2071.

[28] List adapted from *Teaching Mathematics Effectively and Equitably to Females*, Katherine Hanson, ERIC Clearinghouse on Urban Education: New York, NY, 1992, 31.

[29] United States Department of Education. (1991). *The Search for Effective Instruction of Children of Poverty.* Washington, DC.

[30] United States Department of Education. Digest of Education Statistics (1994), Table 20: Household Income and Poverty Rates. Based on information from United States Department of Commerce, Bureau of the Census.

the poverty line[31]. A greater percentage of these children experience failure at every level when compared to children from the middle class. Research points to three characteristics of the instruction these children typically experience that exacerbate problems with learning: *low expectations for what they can accomplish, misdiagnosis of their learning difficulties, and a failure of the schools to reexamine and restructure programs for these students* (USDOE, 1990, 6)[32]. What steps can we take to ensure equity for disadvantaged students? We suggest that teachers:

✓	*SUGGESTIONS*

✓ provide instruction that utilizes students' prior knowledge, especially from real-work experience. Research indicates that minority students, for example, perform best in mathematics classes when the content is related to their previous experience (Mid-Atlantic Equity Center, 1992, 1-3). Thus, while recognizing the gaps in the students' formal knowledge, teachers need to build on students' experiential knowledge while at the same time expanding their knowledge base.

✓ provide mathematics instruction that is rich with problems and activities which connect mathematics to the everyday experiences of students. Activity-based programs have been demonstrated to significantly improve minority student performance in mathematics and science process skills (Mid-Atlantic Equity Center, 1992, 1-4). Supplement instruction with field trips to laboratories, college campuses, various worksites, and other similar places where it is apparent that mathematics is relevant and useful to the tasks at hand.

Overcoming language barriers to equity. Students who are not native English speakers are at a disadvantage in English-speaking mathematics classrooms and when taking tests constructed for English-speaking students. We suggest that teachers:

✓	*SUGGESTIONS*

✓ use tools other than those that require English language proficiency in assessing the students' mathematical understanding and in making decisions about which mathematics courses should be taken by students who are not native English speakers. As their language skills improve, so will their performance in mathematics class.

✓ facilitate both learning of mathematics and proficiency in English[33] by

* designing and implementing activity-based programs in mathematics with built-in linguistic objectives;
* teaching mathematics as a component of bilingual programs;
* having students participate in purposely structured cooperative learning groups which will provide development in oral and written communication skills and enhance the student's academic self-image;

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[31] New Jersey State Data Center, New Jersey Department of Labor. Table S6: Family Poverty Status in New Jersey. (November 1994). Based on information from United States Department of Commerce, Bureau of the Census.

[32] United States Department of Education. (1990). *The Search for Effective Instruction of Children of Poverty.* Washington, DC.

[33] Recommendations of the MidAtlantic Equity Center, 1992, I-4.

* instructing students in problem-solving strategies which include tools for decoding words and phrases; and
* presenting mathematics content in simplified or "sheltered" English. This has been shown to increase language competencies.

✓ make use of the literature on ESL instruction[34].

Overcoming disability barriers to equity. Research indicates that students with disabilities often have low academic self-images (The Mid-Atlantic Equity Center, 1992, 1-5). Neither special programs nor mainstreaming have been shown to significantly reduce these negative beliefs about the self. Poor self-image often persists from elementary through secondary school. Students with disabilities can master the curriculum content requirements for a high school diploma (NYDOE, 1994, 91)[35]. Many can attend regular classes when provided with supplemental instruction and services. It is important, therefore, that such students receive instruction, from the elementary years on, in the core curriculum in mathematics recommended by the Standards. Following are some suggestions[36]:

✓	*SUGGESTIONS*

✓ enrich the mathematics program and classrooms by using technology in order to enhance learning experiences of students with disabilities[37].

✓ use a partner or assistant to work with the students with disabilities in the classroom as an additional resource to clarify classroom discussions and activities.

✓ encourage students to express their understanding, both with a partner and with the teacher.

✓ provide a suggested timeline and benchmarks for extended tasks and projects to students who have difficulty with organizational skills.

✓ use alternative testing strategies by modifying testing procedures and formats.

✓ align the mathematics and special education programs by providing teachers with collaborative planning time.

✓ provide professional development opportunities for special education teachers that further enhance their own mathematical backgrounds.

✓ provide special education teachers and students with video and audio tapes of classroom activities and discussions for further review.

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[34] Some helpful sources are *Children and ESL: Integrating Perspectives* (TESOL, Washington, DC, 1986) and *When They All Don't Speak English: Integrating the ESL Student into the Regular Classroom.* (NCTE: Urbana, IL, 1989).

[35] New York Department of Education. (1994). *Curriculum, Instruction, and Assessment Framework for Mathematics, Science, and Technology.* New York, NY.

[36] List adapted from *Curriculum, Instruction, and Assessment Framework for Mathematics, Science, and Technology*, New York Department of Education (NYDOE), 1994, 91-92.

[37] Suggested resources are: *Directory of Resources of Technology for Special Education* and the May/June 1992 issue of *Technology and Learning.*

Overcoming Geographical Barriers to Equity

Overcoming barriers to equity in urban schools. Students in urban schools present a special challenge for mathematics educators. Among the issues are:

More time on task: Many urban students are behind, and getting farther behind every year. Thus one key element of an equitable curriculum is providing students with the classroom time required to catch up.

Engaging students: Additional instructional time alone is not sufficient. It must also be used well — just doing more of the same will not spark students' interest or inspire achievement. We need to get beyond basic skills and present students with tasks, challenges and perspectives that are varied, interesting, and appropriate for employment and citizenship in the 21st century.

Contextual learning: Because urban students often have a very different experience base, standard classroom approaches that have been developed for students with middle-class backgrounds may seem foreign or contrived. We need to embed mathematical principles and learning within a range of projects and activities which students can identify as authentic.

Encouraging positive interactions: To overcome the negative peer pressure which devalues achievement in mathematics, we need to encourage students to have more positive and productive interactions in a mathematical context.

✓	*SUGGESTIONS*

✓ provide opportunities for additional time on mathematics though additional instructional sessions, tutoring by teachers and students outside of class, tutoring by adults in the community, after-school access to technology, and summer sessions.

✓ select and design learning activities and projects which incorporate the use of mathematics in familiar contexts of interest to students. These could include locally important employment markets, the natural environment, educational puzzles and games, and sports or entertainment. Also appropriate are:

- team-taught lessons or units with science teachers;
- reports and projects jointly assigned and assessed with English teachers;
- integrated lessons or projects with vocational teachers;
- field trips to science museums and work sites;
- use of technology to support both individual learning and group projects;
- team-taught lessons with bilingual and ESL teachers, who sometimes are not aware of potential opportunities to enrich their lessons with mathematical perspectives; and
- curricula specifically designed to incorporate hands-on mathematics laboratory activities such as *Applied Mathematics*[38].

✓ accommodate students with varying levels of proficiency in mathematics and monitor progress

[38] *Applied Mathematics* is a set of 36 modular learning materials prepared to help high school students and others develop and refine their job-related math skills. It was developed by a consortium of 41 states, in cooperation with the Center for Occupational Research and Development (CORD) of Waco, Texas.

daily; the following practices[39] will help meet the needs of individual students:

* start from where students are;
* do not needlessly review;
* do not undermine students' self-confidence by placing them in courses for which they are unprepared;
* allow students of different ages to do the same mathematics together;
* provide remediation immediately and powerfully;
* use technology; and
* incorporate applications and real problem solving into the curriculum.

✓ determine the different learning style preferences of students, and design activities to accommodate a wide variety of learning styles.

✓ use cooperative learning techniques to encourage students to interact positively and productively with each other.

✓ use a variety of instructional styles and learning modes to capture and keep the attention of the students.

✓ provide professional development to teachers about learning styles and instructional strategies and help them to implement their new knowledge in their own classrooms.

✓ inform students about opportunities in higher education.

✓ review Chapters 17, 18, and 19 for further information about instructional strategies, assessment, and technology.

Overcoming barriers to equity in rural schools. Because of their relatively small size and less adequate resources, many rural school districts cannot afford to have as broad a curriculum or as many specialized teachers as do wealthier districts. Moreover, they often offer a narrower range of courses and require individual teachers to cover more subjects than their larger counterparts. Rural students often are shortchanged when it comes to more advanced courses. The Children's Defense Fund (Sherman, 1992)[40] has reported, for example that (1) calculus was offered by approximately 33% of rural schools in the early 1980s, 50% of urban schools, and 67% of suburban schools, and (2) advanced placement classes — which offer talented students an important head start on earning college credit — were available in only 20% of the nation's rural schools in the early 1980s compared to nearly 50% of the suburban schools.

With regard to achievement, average rural achievement scores in most subjects are reported to be slightly below those in metropolitan areas and far below those in suburban areas. The Children's Defense Fund report cites a National Assessment of Educational Progress (NAEP) study conducted in 1981-1982 that provided a picture of students' mathematical skills based on the population size of the community where they attended school. Consistently students in the smallest communities and largest cities performed slightly below average in math. Students in medium-sized cities and suburbs performed somewhat better

[39] Zalman Usiskin (Director, University of Chicago School Mathematics Project), *If Everybody Counts, Why Do So Few Survive?* in *Reaching All Students with Mathematics*, NCTM, 1993, Reston, VA.

[40] Sherman, A. (1992). *Falling by the Wayside, Children in Rural America.* Washington, DC: Children's Defense Fund.

than average. These results suggest that communities at both extremes of population size face special problems in education.

In a more recent analysis of the assessment of student performance in rural schools[41], the following data is reported: On the National Assessment of Educational Progress mathematics assessments in 1978 and 1982, the mean proficiency scores for rural students were below the national average. By 1986 and again in 1990, however, rural mean scores essentially matched the national average. The study also reported data on the National Education Longitudinal Study of 1988, indicating that rural 8th graders scored at about the national average on measures in mathematics, but that they scored significantly lower than their suburban counterparts.

✓	*SUGGESTIONS*

The Children's Defense Fund recommended that states and school districts take a variety of steps to improve rural students' access to a broad range of programs and courses, including the following four points:

✓ all school districts should eliminate tracking and ensure that all students are taught a rigorous core curriculum.

✓ school districts should aggressively explore distance learning technologies that bring interactive classes to rural schools via satellite and interactive video; federal and state governments should promote distance learning, for example by expanding the Federal STAR schools program.

✓ states or consortia of schools should fund regional alternative schools or specialized magnet schools to increase the range of programs available to students.

✓ rural districts should not become distracted by issues such as consolidation of school districts with small populations. Consolidation is neither a magic answer, nor always a disaster. Communities must focus on the real need — good teachers in good facilities with good support.

Challenging All Students to Maximize Their Achievement

All students should be challenged to reach their maximum potential. For many students, the core curriculum recommended here will indeed be challenging. But if we do not provide this challenge, we will be doing our students a great disservice — leaving them unprepared for the technological, communication, and information age of the 21st century.

For other students, this core curriculum itself will not be a challenge. We have to make sure that we provide these students with appropriate mathematical challenges. We have to make sure that the raised expectations for all students do not result in lowered expectations for our high achieving students. A core curriculum does not exclude a program which challenges students beyond the expectations set in the Standards. Indeed, the *New Jersey Mathematics Standards* calls for all schools to provide opportunities to

[41] Stern, J.D., ed. (1994). *The Condition of Education in Rural Schools*. Washington, DC: US Department of Education, Office of Educational Research and Improvement.

their students to learn more mathematics than is contained in the core curriculum.

Students who learn quickly, who have a high level of interest in mathematics, who are industrious and who are bored with repetition, are often under-challenged, and therefore, may not achieve their full potential. Most top students in the United States are offered a less rigorous curriculum, read fewer demanding books, complete less homework, and enter the work force or post-secondary education less well prepared than top students in many industrialized countries (USDOE, 1993, 5).[42]

Poor preparation in elementary and secondary school translates into student performance far below their potential. Only one-half of America's high-ability high school seniors from the class of 1980 (the top 25 percent as indicated by achievement tests) were estimated to have received a bachelor's degree by 1987, and only one in eight had entered a graduate program in any field by that date. In addition, U.S. students are not aspiring to, or are not qualifying for, our graduate programs in mathematics. In 1990, 57 percent of doctorates granted in the United States in mathematics went to students from other countries (USDOE, 1993, 11).

Most elementary and secondary school programs for students termed "gifted and talented" are often modest in scope. "The vast majority of talented students spend most of the school day in a regular classroom where little is done to adapt the curriculum to their special needs." The exceptional programs — specialized schools, magnet schools, and intensive summer programs — serve only a fraction of the secondary students who might benefit. Moreover, dual enrollment (where secondary school students also enroll in college) is uncommon (USDOE, 1993, 21-22).

Effective programs do, however, exist around the country; programs such as residential schools, summer activities, and enriched curricula are often aimed at developing analytical thinking skills in students, and often offer innovative approaches to scheduling and other organizational aspects of the mathematics program. In the past, these approaches generally "have not been implemented in regular education because educators did not realize their potential for improving all of American education. Now, however, many educators believe that the knowledge gained from these and other outstanding programs can be used to upgrade all of education."[43] (USDOE, 1993, 23).

Accordingly, the USDOE makes the following recommendations:

* expand effective education programs and incorporate more advanced material into the regular school program;
* provide all students with the opportunities to solve problems, analyze materials and situations, and learn from real-life experiences;
* identify students who need individual or special opportunities, using test data only as appropriate;
* serve especially talented students in many places: the regular classroom, special class, the community, at a university or a museum, in front of a computer, or anywhere the opportunity

[42] United States Department of Education. (1993). *National Excellence: The Case for Developing America's Talent.* Washington, DC.

[43] The Accelerated Schools Project at Stanford University is a process based on whole school change which has shown remarkable results. When at-risk students are provided with the rich and varied approaches usually reserved for the gifted, their learning is accelerated. For further information, contact Dr. Henry M. Levin at (415) 723-0840 or at the Accelerated Schools Project, Stanford University, Stanford, California 94305.

meets the need; and

* create flexible schools that enable all students, including the most able, to be grouped and regrouped according to their needs and interests (USDOE, 1993, 24).

Assessment and Learning

A statewide assessment which assesses student understanding of the *Mathematics Standards* will be a major step in ensuring that all students indeed receive and learn the core curriculum. The New Jersey Department of Education is developing a fourth-grade statewide assessment aligned with the *Mathematics Standards*, called the Elementary School Proficiency Assessment (ESPA), and over the next few years will adapt the Eleventh-Grade High School Proficiency Test (HSPT) and the Eighth-Grade Early Warning Test (EWT) so that will continue to evolve to reflect the *Mathematics Standards*. The expectation is that these statewide assessments will reinforce the message of high achievable standards for all students and will support the vision of mathematics education reflected in the *Mathematics Standards*.

At the local level, we must move away from the notion that the major purpose of assessment is to filter out students, and move toward the notion that the major purpose of assessment is to improve learning. In order for assessments to serve this purpose, they must enable each student to demonstrate what he or she understands and is able to do. This is what equity implies in the context of assessment. This cannot be accomplished by any single instrument, and certainly not by one whose grading is only in terms of correct and incorrect. Assessments which are timed favor those who work best under those conditions, and assessments which are interactive favor those for whom interaction is an important component of learning. If the focus of assessment is improving learning, and not simply ordering students against a linear standard or preparing grades for a report card, then multiple options become available to the teacher.

Regular use of a variety of assessments, such as those discussed in Chapter 18, can provide the teacher and each student with specific information about his or her progress, so that efforts can be initiated, with both individual students and with entire classes, to address promptly any problems that students are having with the mathematical topics under discussion.

In discussing criteria for choosing an equitable set of assessments, one should consider how the assessments[44]:

* support the vision of New Jersey's *Mathematics Standards*;
* enable students with diversity in experience and mathematical sophistication to respond to the assessments in ways that demonstrate what the students know and can do;
* individually and collectively provide a composite picture of each students' understanding and skills;
* communicate effectively to the student the mathematics problems being posed (using appropriate and multiple representations, including verbal descriptions, graphs, and other visuals);
* support the learning of students who are bilingual or are developing their use of the English language;

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[44] Adapted from the *Assessment Standards for School Mathematics*, National Council of Teachers of Mathematics (1993, 86-87).

* encourage self-assessment and reflection leading to self-improvement, and motivate students to take responsibility for their learning;
* occur under conditions (i.e., space, tools, and time) that enable each student to exhibit best work; and
* seem reasonable for all students to complete in terms of outside school requirements, including access to libraries, technology, and other resources, affordability, and family and time constraints and responsibilities.

Community Involvement

Support from the community is essential for achieving equity and excellence. We must communicate to parents our message that all children must and can succeed in mathematics, and encourage them to communicate that message to their children. On the other hand, the community at large is a resource that can be tapped for services in extending students' mathematical experience beyond the school day and beyond the school curriculum. How then do we involve community members and parents? We suggest that teachers, schools, and districts:

✓	*SUGGESTIONS*

✓ convey to parents the message that all children must and can succeed in mathematics[45], and encourage them to communicate that message to their children.

✓ convey to parents that even though they may have had negative experiences with mathematics when they were in school, their children need to be encouraged to succeed in mathematics.

✓ enlist parents, retired persons, and local business people to serve as mentors, tutors, translators, and encouragers.

✓ enlist guests from local business and industry to share their personal and professional biographies in the fields of mathematics and science.

✓ make arrangements for students to have adequate study facilities (in schools and in the community) and access to technology during after-school hours.

✓ arrange after-school study groups and encourage parents to exchange phone numbers so that students can review classwork and discuss homework.

✓ establish a homework hotline[46] [47].

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[45] The New Jersey Mathematics Coalition has produced a guide for parents, *Mathematics to Prepare our Children for the 21st Century: A Guide for New Jersey Parents*. This document can serve as support for your message, and as an outline for a group discussion. Call (908) 445-2894 to purchase single or multiple copies of the revised Parents' Guide.

[46] The Richard C. Crockett Middle School (Hamilton Township, Mercer County) has had a homework hotline since 1990. Call (609) 890-3800 for information.

[47] The Extra Help Homework Hotline is a live call-in program for Newark students, which airs each Tuesday on Channel 3 (Cablevision) from 3:30 to 5:30 p.m. (with taped repeat on Wednesday from 8 to 10 p.m.). For information call Kenneth Herskovits, Central High School, at (201) 733-8656 or, in the evenings, at (212) 733-8656.

✓ sponsor activities for parents and students each April during Math, Science, and Technology Month[48] (and at other times of the year) which involve parents actively in mathematics and which inform them about the kinds of mathematical activities that are taking place in the school, and how they reflect national and state recommendations.

✓ provide opportunities for ongoing mathematics education for parents and other adults. Programs such as Family Math[49] can be initiated to increase parents' awareness of the need for mathematics and its role in shaping the future successes of their children. Family Math offers activities for parents and their children which heighten children's interest and encourage parental support for mathematics.

✓ reach out to parents who appear nonsupportive and who do not participate by conducting activities where parents normally congregate, such as churches or community rooms in housing projects, and by offering a wide variety of opportunities for involvement which require a low level of commitment, such as establishing friendly home to school communications.

✓ encourage parents to engage their children in extra-curricular mathematics projects in the home[50]

EQUITY 2000[51], a project of the College Board, has as its goal to close the gap in the college-going and success rates between minority and non-minority students, and advantaged and disadvantaged students by proposing academic excellence for all students. This is accomplished through:

* creation of district-wide policy changes to end tracking and raise standards for all students, beginning with the requirement that all students complete algebra by the ninth grade and geometry by the tenth grade, and including reform of the curriculum to reflect standards set by the National Council of Teachers of Mathematics and other discipline-based organizations;
* establishment of ongoing professional development for teachers, counselors, and principals to increase their professional knowledge and skills and to raise their expectations for students;
* improvement in schools' involvement with students' parents and families to create a consistent climate for learning as well as to empower parents to be advocates for their children's education;
* development of a "safety net" for students through academic enrichment programs that provide

[48] For additional information about Math, Science, and Technology month contact: The New Jersey Mathematics Coalition, P.O. Box 10867, New Brunswick, NJ 08906 or call (908) 445-2894.

[49] Family Math is an innovative parental involvement program which provides parents and children in grades K-8 with opportunities to build understanding of mathematical concepts using inexpensive hands-on materials such as beans, toothpicks, and coins. The teaching emphasis is on doing mathematics; time is not spent on worksheets or in a lecture format. Children and adults come together once a week for six weeks to work cooperatively, learn to reason and think logically, and talk about the ways they are solving problems. Teachers act as facilitators, encouraging cooperation, motivating students in future careers and everyday life. All Family Math activities underscore the NCTM Standards and complement the school mathematics curriculum. Developed by EQUALS at Lawrence Hall of Science, University of California, Berkeley, Family Math was introduced to New Jersey schools by the Rutgers University Consortium for Educational Equity in 1985. Three-day workshops are offered by the Consortium for teachers and parents who want to conduct Family Math courses in their school. Other parent involvement programs include Family Science and Family Tools & Technology. For further information call: (908) 445-2071.

[50] Partners for Reform in Science and Math (PRISM) videos and outreach sheets show parents how to become involved in their children's mathematics and science education. PRISM is a collaboration between the National Urban League and Thirteen/WNET. To order materials call the Annenberg/CPB Math and Science Collection at 1-800-965-7373.

[51] For further information about EQUITY 2000, call (212) 713-8268.

extra academic support;

* formation of school-community partnerships that include links with colleges and universities, the business community, and community-based organizations; and
* use of student course enrollment and achievement data broken down by ethnic group and gender to monitor progress toward reform goals.

Extracurricular activities such as math clubs[52] [53], math teams[54] [55], and other math activities[56] can also be used to encourage students to strive for excellence in mathematics.

Identifying Equity Concerns in Districts and Schools

In order to address the issues raised in this chapter, each district, each school, and each individual educator needs to reflect on the issues in conjunction with a realistic appraisal of their situation. Most of the suggestions in this chapter are addressed to teachers; this section, however, is addressed to school and district administrators.

Identifying Equity Concerns in Districts

In order to assure that we are providing an equitable learning environment for all students, we must review all district policies and practices which have an effect on equity. Each district should review the following questions, and consider the consequences, intended or unintended, of its policies and practices in each of these areas:

Administrative Policies and Practices

* What are district policies on equity issues in general, and mathematics education in particular,

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[52] A Math Club in your school provides an opportunity for students to pursue their interests in mathematics beyond the curriculum. Even some elementary schools offer after-school activities for interested students. One way to start is to offer a challenging problem to students in your grade or post it in all the math classes. Students may be encouraged to submit solutions and the solutions can be presented and discussed, with prizes for correct solutions, at the first meeting of the Math Club! For information about math clubs and the over 50 math contests that are available each year to stimulate your "mathletes", contact David Marain, Sparta High School, 70 West Mountain Avenue, Sparta, NJ 07871 or email dmarain@ix.netcom.com or call (201) 791-3118 (PM).

[53] MATHCOUNTS is a nationwide program that promotes math excellence among junior high school students and helps them and their parents become aware of career opportunities in math. Funded by the National Society of Professional Engineers and business and industry, MATHCOUNTS also sponsors a contest for "mathletes". For information, call (703) 684-2828.

[54] Enter your high school in the New Jersey High School Math Contest and encourage your students to participate. Sponsored by the Association of Mathematics Teachers of New Jersey, the contest is held each November. For information, call Mary E. Froustet at 908/686-2767.

[55] Enter your middle school in **Solve It**, a Middle Grades Mathematics League. For information, write to the League Director, William B. Moody, Solve It, University of Delaware, Dover, DE 19716-2901 or call (302) 831-1658.

[56] Students in grades 8-12 can participate in the Gelfand Outreach Program in Mathematics at Rutgers University, and work on monthly problem sets which are reviewed by Rutgers faculty and graduate students. For information, call Harriet Schweitzer at (908) 445-0669.

including such areas as employment, school and classroom practices, student treatment, etc.?

* Has the district discussed and adopted an explicit statement on equity?
* What are district policies on tracking and grouping students in mathematics?
* What are district referral and classification practices for special education?
* What are district practices and expectations regarding various groups of children — gifted, basic skills, compensatory education, language-minority, etc. children?
* What are the very highest expectations of the school district, for those students who are most successful in mathematics, science, and technology? To what extent are such high expectations encouraged widely, and to what extent might opportunities to achieve them be artificially or unnecessarily limited?
* How do individual school's student populations reflect the racial/ethnic composition of the district as a whole?
* What happens to students after they complete school in the district?
* Are funds specifically allocated to advance equity concerns?
* Do teaching assignments in mathematics and other teaching and administrative areas reflect the racial/ethnic diversity of the larger community and provide role models for all students?
* What kinds of professional development activities are available to teachers, administrators, and guidance counselors to sensitize them toward the integration of equity and diversity in mathematics, science, and technology education?
* What kinds of professional development activities are available to teachers which reflect the mathematics and science standards?
* What kinds of professional development activities are provided to teachers regarding instructional/learning styles, expectations, teacher/student interaction?
* Are professional development activities available to guidance counselors which focus on expectations, equity, and career development activities to encourage students to consider careers in mathematics and science?

Curriculum, Instruction, and Assessment

* Are curriculum materials for mathematics and other subject areas free of bias? Do they represent all groups and encourage the participation of all students?
* Are curriculum materials and instructional practices aligned with the recommendations of the *New Jersey Mathematics Standards*? Are equity issues addressed in implementation?
* Are a variety of teaching strategies in mathematics used?
* To what degree are cooperative learning groups in mathematics used to encourage all students to be actively engaged?
* What kinds of alternative assessment strategies are being utilized to assess student achievement in mathematics, science, and technology education?
* What are the requirements in mathematics, science, and technology for students? Do the requirements differ for different populations of students?
* What is the level of participation of underrepresented students in advanced and honors mathematics courses? What are the percentages for the traditionally-achieving students?
* How are students selected to participate in advanced and honors mathematics and science courses?

What criteria are used to make this selection?

* Are guidance counselors involved in conveying to students the importance of mathematics and science education? In what specific ways do they reach out to underrepresented groups of students to encourage them to pursue courses and careers in mathematics and science?
* How are students encouraged to see mathematics and science as integral to the development of all cultures and civilizations?
* How is the relevance of mathematics to all careers demonstrated?

Community Outreach

* Have parents been involved in the district's mathematics and science reform efforts?
* Does the district offer opportunities for parents to learn about the importance of mathematics and science education, and to enable them to assist their children in learning these subjects?
* What kinds of outreach strategies have been targeted to parents to educate them about the kinds of mathematics, science, and technology courses and programs that are available to students in the district?
* In what specific ways have local business, industry, and community organizations been involved in district mathematics and science reform efforts?

Identifying Equity Concerns in Schools

Teachers and supervisors do not have much control over the areas enumerated above, especially policy areas where the responsibility belongs to the district. But those of us who are teachers and supervisors do have a lot of control over what happens in our classrooms and in our school buildings.

In order to identify equity concerns in our schools, a first step might be to take a good hard look at what is happening in the school. We should start by collecting data to ascertain how many students from each group are enrolled in the various courses offered by the school.

Such data, disaggregated by relevant groups, might include:

* enrollment data for high school mathematics and science classes;
* numbers of students in various tracks at various grade levels, including gifted and talented programs and special education; and
* achievement data including grades and test scores in mathematics and science.

For instance, you might want to review enrollments in 12th grade mathematics courses; you should go back a year or two in collecting data and use a table like the one below (with additional rows and columns as appropriate) which presents information by gender and ethnicity.

Such tables will provide information to answer questions like the following:

* How many students are not taking any math courses in their Senior year?
* To what extent are various groups underrepresented in advanced math courses?
* Are some groups underrepresented among students who take math in their senior year?
* Are some groups overrepresented among students who enroll in senior year math courses but fail

to complete them satisfactorily? (To respond to this question, you would need two tables, one recording numbers of those who register and the other recording numbers of those who complete the various courses.)

* What is the selection process for admission to advanced math classes? Does it allow too few students in? Can students elect to take these courses or do they have to be selected? Can students take multiple advanced courses or must they choose between mathematics and another discipline?
* What are the grades of students from different groups?
* How many girls and boys move "up a level" (e.g., from average to above average) in math and science courses each year? How many move "down a level" (e.g., from honors to above average)?

Number of Students	African American		Asian		Hispanic		White	
	Girls	Boys	Girls	Boys	Girls	Boys	Girls	Boys
AP-Calculus								
Calculus (non AP)								
Pre-calculus								
Consumer Math								
Basic Skills Math								
General Math								
Discrete Math								
Prob./Stat.								
No math course								
TOTAL STUDENTS								

In other areas, the data to review would be suggested by some of the questions in the previous section on district responsibility. For instance are some groups more likely than others to be classified as learning disabled? Are factors unrelated to educational need encouraging these differential assignments? Are "special education" children getting the levels of mathematics they are capable of?

Once problems are identified, the next step should be to develop a plan for corrective action which will include a review of policies and practices which may have had the effect of discouraging the achievement of some groups. Suggestions for overcoming the barriers to equity in our schools are provided in the preceding section of this chapter.

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On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

http://www.rbs.org/eisenhower/res_equity.html/

The Mid-Atlantic Eisenhower Consortium for Mathematics and Science Education maintains a Web page on Equity Issues in Mathematics and Science.

LEARNING ENVIRONMENT STANDARD 17
KEYS TO SUCCESS IN THE CLASSROOM

> All students' mathematical learning will embody the concept that engagement in mathematics is essential, and that decision-making, risk-taking, cooperative work, perseverance, self-assessment, and self-confidence are frequently keys to success.

(This "learning environment standard" was developed and approved by the task force that prepared the *Mathematics Standards* and appears in the Introduction to the *Mathematics Standards* chapter of the New Jersey State Department of Education's *Core Curriculum Content Standards*; however, since it was not considered a "content standard", it was not presented to the New Jersey State Board of Education for adoption.)

Descriptive Statement

Engagement in mathematics should be expected of all students, and the learning environment should be one where students are actively involved in doing mathematics. Challenging problems should be posed and students should be expected to work on them individually and in groups, sometimes for extended periods of time, and sometimes on unfamiliar topics. They should be encouraged to develop traits and strategies — such as perseverance, cooperative work skills, self-assessment, self-confidence, decision-making, and risk-taking — which will be key to their success in mathematics.

Meaning and Importance

New Jersey's *Mathematics Standards* describe what a high-quality mathematics education should comprise for all of New Jersey's students, from kindergarten to 12th grade. Central to these standards is the development of mathematical power for all students, including the ability to explore, conjecture, and reason logically; to solve nonroutine problems; to communicate about and through mathematics; and to connect ideas within mathematics and between mathematics and other intellectual activity. Mathematical power also includes the development of personal self-confidence and a disposition to seek, evaluate, and use quantitative and spatial information in solving problems and in making decisions. Students' flexibility, perseverance, interest, curiosity, and inventiveness also affect the realization of mathematical power.

Keys to Success

Standard 17 discusses the "keys to success" in mathematics, and what teachers can do to make it more likely that each student will acquire these keys.

The keys to success are enumerated in this chapter as cumulative progress indicators, as with other standards, and for each key to success, specific suggestions are provided for how it may be achieved. The first three of these indicators address the student's orientation toward mathematics, and the remaining five indicators address the nature of the student's engagement in mathematical activity.

The other sections of this chapter discuss the learning environment that will facilitate students' achieving this standard. The main sections of this chapter, following this section on **Keys to Success**, are entitled **Organizing the Classroom, Organizing Instruction,** and **Organizing the Content.**

Experiences will be such that all students:

> **1. Demonstrate confidence as mathematical thinkers, believing that they can learn mathematics and can achieve high standards in mathematics, and accepting responsibility for their own learning of mathematics.**

New Jersey's *Mathematics Standards* are based on the assumption, discussed in detail in Standard 16, that **all** students can succeed in mathematics. Students need to develop confidence in their own mathematical abilities and performance; they need to recognize that they can understand and do mathematics, and believe that they will succeed in achieving the high standards now expected of them. They need to recognize that success in mathematics, as in other areas, is the result of persistence and hard work.

To help all students achieve mathematical self-confidence, teachers should;

- believe themselves that *all* students can learn mathematics, even when their students are experiencing difficulties.
- set high expectations for *all* students, expect students to take responsibility for their own learning, and provide the increased support necessary to help students meet these higher standards.
- build self-confidence in each student by making mathematics understandable, using effective and meaningful teaching methods, relating past learning to concepts currently being taught, establishing short-term, obtainable goals, and providing mathematical experiences at which students can succeed.
- help students make appropriate attributions about effort, persistence, and strategy; student "failures" are often due to the choice of an incorrect strategy or to a lack of effort or persistence, rather than to a lack of ability.
- convey to students that *their* effort and persistence are critical to their success, and will lead to their success. And *our* efforts to do this must also be persistent and creative. While New

Jersey's *Mathematics Standards* asserts that all students can succeed in mathematics, all the recommended changes in curriculum, instruction, and assessment will have little impact on students who are unwilling to make the effort.

2. Recognize the power that comes from understanding and doing mathematics.

Students need to realize that by developing their ability to do mathematics, they will be able to understand, formulate, and solve problems in a wide variety of situations. This is because mathematical tools and perspectives are often the key to understanding and solving a problem, even if at first glance mathematics doesn't even appear to be involved. The better they understand and can do mathematics, the greater will be the variety of problems they can solve. As a result, a greater variety of school and career options will be open to them.

To help students recognize the power of mathematics, teachers should;

- provide students with *worthwhile mathematical tasks*. Such tasks are inherently interesting to students, helping them to develop mathematical concepts and skills, and drawing on their own experiences. They stimulate students to connect mathematics to itself and to other subjects, they encourage students to communicate about mathematics, and they help to develop the students' dispositions to do mathematics. Worthwhile mathematical tasks help students develop a coherent framework for mathematical ideas and present mathematics as an ongoing human activity (NCTM *Professional Standards for Teaching Mathematics*).
- introduce a variety of activities since different students make mathematics their own in different ways; some students learn best by constructing actual models, others learn best by hearing someone else talk about a mathematical idea, still others learn best by writing down the mathematical concepts for themselves or explaining them to others.
- arrange the journey for the students through the world of mathematics, filling the journey with rich, important experiences which will allow the students to achieve mathematical power.

3. Develop and maintain a positive disposition to mathematics and to mathematical activity.

Students begin school with great interest in solving puzzles and problems, and an enthusiasm for measuring and counting. Over the next few years, many come to believe that mathematics consists of rules which are unintelligible to them, and that math class means rote problems which are not connected to their reality. It is important to convey to students that mathematics is a dynamic human activity to which they can relate.

To help students develop a positive disposition to mathematics and to mathematical activity, teachers should;

- convey their own enthusiasm for and enjoyment of mathematics and solving mathematical problems.
- model persistence, confidence, self-reliance, flexibility, curiosity, inventiveness, and enthusiasm during problem solving, and many students will imitate those traits.

- structure their lessons so that students have fun doing mathematics; of course, this should not diminish the message that learning mathematics is often hard work and is not always exciting.
- make mathematics meaningful through the use of concrete models and discovery lessons.
- demonstrate the usefulness of mathematics in everyday life and relate mathematics frequently to careers. This is extremely important in the middle grades and high school, as students begin to make choices about their futures.
- consider students' interests when planning instruction; field trips, guest speakers, and videos may help spark interest and communicate the ever-present nature of mathematics.
- promote students' intrinsic motivation to learn mathematics by stimulating their curiosity, and provide students with choices and some control over the learning environment, especially in the middle and secondary grades.
- recognize that students have different learning styles and address those different learning styles regularly to ensure that all students remain engaged with mathematics.
- recognize that some students will take more time than others to develop their understanding of a specific topic. Teachers should ensure that students who are proceeding more slowly than their classmates receive the additional support they need and do not become discouraged.

4. Participate actively in mathematical activity and discussion, freely exchanging ideas and problem-solving strategies with their classmates and teachers, and taking intellectual risks and defending positions without fear of being incorrect.

Students should be actively involved in the learning of mathematics. Although some students absorb mathematics through teacher presentations, all students learn better when they are actively engaged in the learning process. Students need to be active participants in their mathematics classes, discussing mathematics with the teacher and with each other, engaged in activities which enhance their learning. They need to be prepared to propose strategies for solving problems, to provide explanations for why things work as they do, and to make conjectures for the consideration of their classmates. In order for them to do this, they must have a supportive classroom environment which encourages diversity of thought.

Teachers should establish this supportive classroom environment by;

- making mathematical discussions a daily activity.
- encouraging students to make suggestions and conjectures, and to propose strategies and explanations.
- conveying to all students that they must all listen to their classmates respectfully and respond to their suggestions as members of a learning community. Students are often reluctant to speak about mathematics unless they are sure of their answers. They need to feel that their incorrect answers will be respected and are part of the learning process; they need to be sure that their answers will not simply be rejected and that they will not be humiliated. The teacher

sets the tone of the classroom by ensuring that all who speak will be treated respectfully and their suggestions will be taken seriously.

- encourage students to take intellectual risks, and convey that everyone can learn from their mistakes.

- lead students from a partial to a complete solution through asking a question or through posing a new problem that offers an opportunity to think about evidence not previously considered. It is crucial that teachers go beyond simply rejecting a response or pointing out the student's error.

5. Work cooperatively with other students on mathematical activities, actively sharing, listening, and reflecting during group discussions, and giving and receiving constructive criticism.

Students should see themselves as participants in a learning community, where students learn from each other as well as from the teacher and print and electronic materials. While it is sometimes difficult for each student to participate in a whole-class discussion, when students work in groups on mathematical tasks, they can all be active participants, each sharing in the discussion. They also need to listen to and reflect on what other students are saying, so that they can learn from each other's approaches and insights. They need to complete their objective, whether it is a common understanding of a situation or a common solution to a problem, without diminishing each other's unsuccessful attempts along the way.

To help develop students' abilities to work cooperatively with other students, teachers should;

- create regular opportunities for students to work in groups on mathematical tasks, which may include quiet reflective time, so that all students have an opportunity to articulate their understanding of a concept and their strategy for solving a problem.

- structure the group activity so that all students participate appropriately — sharing in the discussion, listening to each other, responding to each other's suggestions with respect, and reflecting on each other's suggestions for completing the task.

- ensure that all students are provided with ample time to think through a problem situation before moving on to its solution.

- teach mathematical ideas, whenever possible, through posing a problem, setting up a situation, or asking a question. Resist the impulse to give an answer or an explanation when a student is confused; try to provide the student with an opportunity to think in a different way about the situation, for example, by asking a related question or posing a different problem which may help them find a solution to the original problem.

- spend more time on a few rich activities rather than a little time on a lot of different ones. Similarly, cover a smaller number of topics well rather than many different topics superficially.

6. Make conjectures, pose their own problems, and devise their own approaches to problem solving.

As noted in the First Four Standards, posing and solving problems is an important cornerstone of New Jersey's *Mathematics Standards*. Students should be encouraged to become mathematically active, looking for mathematics in the world around them, formulating mathematical questions, and engaged in answering those questions. They need to understand that problem solving means being able to solve problems to which routine methods may not apply, so that they may need to be creative in choosing problem-solving strategies.

To encourage their students to be mathematically active, teachers should;

- encourage their students to explore the world of mathematics and mathematics in the world by asking questions themselves and by encouraging their students to ask questions.
- present the students with problems that they don't already know how to solve, and encourage them to make conjectures about what the solution might be and how it might be obtained.
- encourage students to generate other problems suggested by a problem that has just been solved.
- recognize that there are often many paths to a solution, and should encourage students to develop their own solutions when appropriate.
- model that errors happen in the course of solving problems, and that what is important is how one gains information and recovers from one's errors.
- understand that posing non-routine problems provokes emotional issues from students who are experiencing cognitive difficulties with the subject matter, and find ways of discussing and dealing with affective issues that arise in the context of problem solving (DeBellis, 1996).

7. Assess their work to determine the effectiveness of their strategies, make decisions about alternate strategies to pursue, and persevere in developing and applying strategies for solving a problem in situations where the method and path to the solution are not at first apparent.

Students need to be able to reflect regularly on their mathematical activity and determine whether they are making appropriate progress toward solving a problem. Just as a carpenter makes decisions about which tools to use, the problem solver needs to reflect regularly on whether the right tool is being used. If a problem-solving strategy is not successful, the student may need to try another strategy, although it is possible that later she will discover that the original strategy was after all appropriate.

To encourage students to assess their problem-solving efforts, teachers should;

- encourage their students to step back regularly from their problem solving activity, articulate the strategy that they are currently pursuing, reflect on whether their efforts seem fruitful, and, if appropriate, formulate other options and make the decision to alter their strategy. As in other areas, teachers need to model this behavior, occasionally stepping back and reflecting on whether the whole class activity is moving toward a solution of the problem, and if not what alternate strategy should be adopted.

- encourage students to persevere if they are moving in the right direction, since they tend to give up too easily, particularly if the problem is one which can't be solved in a short time.

8. Assess their work to determine the correctness of their results, based on their own reasoning, rather than relying solely on external authorities.

Students need to develop self-confidence not only in their mathematical abilities, but also in their mathematical performance. If they understand a problem, develop a strategy for solving it, and then obtain an answer, they need to be able to conclude, with some measure of certainty, whether or not they have done the problem correctly. An answer supplied by the teacher, or by an answer key, should provide confirmation of what they already have shown to be the case.

To help students become more willing to assess their own work, teachers should;

- encourage students regularly to validate their own work rather than depending on someone else to tell them whether they are right or wrong; when students request the answer to a problem, teachers should ask them first to explain how they arrived at their answer, whether they believe it to be correct or incorrect, and why. Through this process, if they are correct, they should be able to accept the validity of their reasoning. If they are incorrect, they may be able to correct their errors; alternatively, the teacher will pose a question which will enable them to reconsider and revise their solution.

- encourage students to use each other as sounding boards; students can often find their errors or validate their claims through such discussions.

- take note that in some cases students who exhibit inappropriate self-confidence in their mathematical understandings; in such cases, teachers must ensure that their self-confidence does not interfere with their critical analysis (DeBellis, 1996).

The *Professional Standards for Teaching Mathematics* (NCTM), suggests that we shift:

Toward:	**Away from:**
classrooms as mathematical learning communities	collections of individuals under the control of an adult
mathematical tasks that engage students' interests and intellect	repetitive drill and practice
logical and mathematical evidence as verification	the teacher as the sole authority for right answers
mathematical reasoning	merely memorizing procedures
providing opportunities for students to deepen their understanding of the mathematics being studied and its applications	trying to "cover" too many topics in too little time at a superficial level
promoting the investigation and growth of mathematical ideas through classroom discourse	passive absorption of information by students as the teacher lectures
conjecturing, inventing, and problem solving	stressing mechanistic answer-finding
using technology and other tools to pursue mathematical investigations	using only paper-and-pencil to do mathematics
connecting mathematics, its ideas, and its applications and helping students seek connections to previous and developing knowledge	presenting mathematics as a body of isolated concepts and procedures
students working individually, in small groups, and as a whole class	students working individually at desks lined up in neat rows

Adapted from NCTM *Professional Standard for Teaching Mathematics*, pp. 1-3

Students will develop positive attitudes toward mathematics when they are taught good mathematics in a supportive, enabling environment, when *all students' mathematical learning embodies the notion that engagement in mathematics is essential and that where decision-making, risk-taking, cooperative work, perseverance, self-assessment, and self-confidence are frequently keys to success.*

I Love Math!

On Valentine's Day (or as close to that day as the calendar allows), the math department at Lower Cape May Regional School District has "I Love Math" Day. This is a day when students express their love of mathematics in many different ways: writing songs, making posters, making buttons or other "jewelry" to wear, baking cakes and decorating them with math terms and symbols. The students earn extra-credit points and must display their love of math all day. One teacher did not have his students participate and waited until the following day and had his own "I Love Math Even More" Day.

Organizing the Classroom

The Physical Environment

The physical environment of the classroom is an important consideration in planning for effective instruction. The vision for mathematics in our schools calls for students who are excited by and interested in their activities, who work together to find solutions to real problems, who use technology and other tools as an integral part of the process, and whose learning is conceptually-based, meaningful, and connected to previous mathematics learning and to the real world. The standards call for students who accept responsibility for their own learning, using time effectively and efficiently in order to further their understanding of mathematics.

The physical environment of the classroom must be congruent with the vision we have for our schools. The organization of the classroom should foster a spirit of discovery. Technology and other tools, such as manipulatives, need to be readily and easily accessible to all students. Teachers need collections of supplementary materials to serve as resources. All students should have calculators appropriate to their grade level, and each classroom should have at least one computer and software available at all times for both teacher demonstration and student use. Additional computers should be available for use by the class as a whole. An overhead projector and screen should be available at all times, and teachers should have easy access to other audiovisual equipment as needed.

The classroom itself should be a pleasant place. It should be cheerful, bright, well-lit, and attractive. It should be furnished with flat-top desks or tables and chairs; these can be easily rearranged to accommodate different sized groups and work well with manipulatives, calculators, and printed materials. While each teacher will want to develop a semipermanent room arrangement that conforms to his/her dominant form of instruction, the room should provide sufficient flexibility to rearrange the furniture for different types of activities. Ample space for movement is essential, as is convenient storage space for supplementary resources, manipulatives, supplies, calculators, and computers. Classrooms must also be equipped with adequate electrical power and communications capabilities. Classrooms for younger children should have "messy" areas where children can use water, paint, or sand.

In developing an arrangement of the classroom, the teacher must consider several factors. "How will an arrangement affect my ability to move around the room, checking on student progress?" "Will this arrangement help my students to feel involved in the class activities?" "Will I be able to use small groups as well as have individual work, pairs, and whole class activities?" Some teachers like to organize student desks into groups of four (e.g., Marilyn Burns), while others like to have student desks grouped in pairs, facing the front of the room. Still other teachers prefer a U-shaped arrangement two rows deep (e.g., David Johnson).

The classroom envisioned in New Jersey's *Mathematics Standards* is not a quiet place. It is a place in which students are excited about learning mathematics, in which they work together purposefully to accomplish a task, in which they talk and make noise and sometimes even jump for joy!

Classroom Routines

Planning effective routines for beginning class, checking and discussing homework, taking roll, collecting and distributing papers, and handling make-up work can save valuable class time and minimize disruptions.

It is important to teach students classroom rules and procedures as they are needed, with special emphasis on this area in the first weeks of school. Many teachers have found that it is helpful to have students rehearse procedures. Others use incentive systems to help shape student behavior. Still others coach students to respond to specific signals, such as the teacher's raising her hand for silence or turning down the lights. Students also need to be taught how to follow directions, how to copy assignments from the board, how to find pages in their text, how to take notes, and how to prepare for tests and quizzes.

Beginning the Class. The *first five minutes of a class* are the most important. They set the tone for the class and the pace for the entire class period. It is important for students to spend this time productively, rather than in waiting for the teacher to complete clerical tasks. Some possible ways to start a class include the following.

✓	*SUGGESTIONS*

✔ Give students a do-now or warm-up problem. It may be a review problem, a puzzle, a quiz, a question relating to the new lesson, or a problem similar to the ones done for homework. At the elementary level, this is sometimes referred to as *five-minute math.*

✔ Have some students do problems related to their homework at the board, while the others do the same problems at their seats. Spend no more than five minutes in this activity.

✔ Start each class with mental computation activities or a drill on mathematical facts.

Calendar Math

Mr. Haynes starts his second-grade class each day by asking students to name the day of the week and then write a sentence that states the day of the week and the date. "Today is Monday, September 8th, 19--." He asks them questions such as "What day of the week will it be two days from now?" or "What will the date be in two weeks?" He also has the students keep track of the number of days they have been in school in two different ways: by writing the number on a long roll of adding machine tape and by adding one craft stick to bins marked for ones, tens, and hundreds. The weather symbols that are put on the calendar are used at the end of the month for data analysis activities. Temperature is read from a thermometer and posted on the calendar. Special events are posted on a sheet near the calendar, with the date at the top of the page. These sheets are saved and used later for sequencing, calendar, and data analysis activities.

✔ There are frequently other spots during the class period or, in the elementary grades, during the day, when a short span of time is available for students to complete a brief task. Teachers need to plan *sponge activities* to absorb productively these short gaps: questions to think about, mental math, estimation, or problems to solve. Such activities are sometimes called *minute math*.

Organizing Distribution of Materials. In order to minimize the time spent setting up, distributing, or gathering equipment, supplies, or materials, it pays to plan ahead. Teachers should have everything set up in advance, if at all possible. Much time is wasted if teachers have to move from one room to another between periods.

✓	*SUGGESTIONS*

✔ Take roll by using a seating chart and noting which seats are empty or while checking to see who completed their homework. Remember that, while this should be done early in the period, it does not have to be the very first thing. Take roll while students are working.

✔ Designate one person from each group (if students are working in groups) to distribute materials, another to collect materials, and a third to distribute and collect papers.

✔ To help students who have been absent, teachers have devised a number of strategies which *minimize class disruptions*. Some teachers post assignments on the bulletin board in a central location. Others keep a notebook of assignments throughout the year. Still others give carbon paper to a student in the class, asking them to make a copy of their notes for absent students.

✔ Organize manipulatives in order to reduce time lost in distributing and collecting them. Some teachers, especially those in the primary grades, prefer to have manipulatives set up in learning centers around the classroom, easily accessible to students working in small groups. Other teachers prefer to use manipulatives with the whole class at one time. These teachers find that it is helpful to package manipulatives in quantities appropriate for the group that will be using them (individuals, pairs, groups of four or six), using baggies, boxes, or buckets. Teachers who move from room to room may find see-through boxes or labeled tote bags, especially those with zippers, useful for storage of materials.

✔ Organize calculators for use in the classroom. In some schools, students are issued calculators individually, like textbooks. In other schools, students keep calculators in their desks. In still other schools, teachers keep classroom sets of calculators, which they distribute for use in class. In any case, it is useful to number the calculators and assign each student a number, which they will use throughout the year. Calculator caddies, looking something like hanging shoe bags, may be purchased to hold the calculators. The numbered pockets for each calculator make it easy to see which calculators have been returned.

Organizing Homework Review and Collections. Teachers should plan specifically how they will accomplish such tasks as collecting or checking homework.

✓	*SUGGESTIONS*

✔ Try an alternative to calling on students for answers when homework questions have short answers. Some teachers write the answers to the homework problems on a transparency, while others write answers on the board. Some teachers simply read the answers aloud from an answer key or ask a student to do so. In some classes, it may be appropriate for students to have answer keys themselves, which they may use in checking their work as they do it at home.

✔ Emphasize how students did their homework problems as well as whether or not they got the right answer. Regardless of the method chosen to check homework, some time must be spent discussing the homework. Discussing all of the homework problems, however, is likely to be a waste of time for many of the students (and will discourage them from completing it prior to class time!) One technique that some teachers use is to have students indicate which problems they would like to discuss on the chalkboard as they enter the room by placing tally-marks next to the question number. The process of "going over the homework" can be speeded up by having several students at the board at one time, each doing a different problem. Students at their seats may be given a different practice problem to do while others are writing at the board. Another technique which may be useful is giving students an opportunity to correct a problem that they missed prior to discussing the homework.

✔ Motivate students to do their homework. Since homework is intended to be a learning experience for students, it is generally not advisable to collect it and grade it for correctness. More beneficial is checking to see that students have done their homework, either by moving around the room as students are working or by giving homework quizzes, such as the one shown below (Johnson, 1982, p. 27).

Homework Quiz

Directions: Do not open your textbook. Open your notebook to your assignment section. In your homework, find each problem given below. Copy your work for that problem, including all steps and solutions. No credit will be given for just an answer.

1) Feb. 13 - Page 196, problem 6
2) Feb. 18 - Page 203, problem 7
3) Feb. 23 - Page 215, problem 15
4) Feb. 26 - Page 220, problem 14

✔ Have students keep mathematics notebooks to record homework assignments, vocabulary words, examples of problems with explanations, and records of group work. Open-notes quizzes allow students to use their notebooks to find the needed information and to develop strategies for checking their own work.

Summary

Clearly teachers need substantial amounts of time, materials, and equipment to create an effective learning environment. More time and access to resources are needed for planning and developing the types of lessons envisioned in the *Standards*. Schools may need to consider the possibility of having double mathematics periods, extended mathematics classes to provide time for the use of manipulatives and technology, or combined mathematics/science classes. Students also need sufficient time, free of distractions and interruptions, to learn. Scheduling must be guided by the needs of students and what they are expected to learn.

Organizing the classroom appropriately can help teachers to find the time to focus on students' understanding, to build the bridges from the concrete to the abstract, to make the connections between mathematics and the real world, to develop students' problem solving and communication skills, and to help students develop their ability to reason mathematically.

Organizing Instruction

> "In reality, no one can *teach* mathematics. Effective teachers are those who can stimulate students to *learn* mathematics. Educational research offers compelling evidence that students learn mathematics well only when they *construct* their own mathematical understanding. To understand what they learn, they must enact for themselves verbs that permeate the mathematics curriculum: 'examine,' 'represent', 'transform,' 'solve,' 'apply,' 'prove,' 'communicate.' This happens most readily when students work in groups, engage in discussion, make presentations, and in other ways take charge of their own learning." (*Everybody Counts*, 1989, pp. 58-59)

Although it is important that instruction be problem-based, no one instructional strategy can develop the mathematical power that students need. Research findings on learning, on learning styles, and on attitudes all suggest that use of a variety of instructional strategies is most appropriate.

In order to accomplish these expectations, teachers need to use different instructional methods for different purposes. Some of these instructional strategies may involve whole-class, teacher-directed activities, while others may involve students in working in cooperative learning groups. Some instructional strategies may involve discovery learning, with students working in a variety of settings.

Whole-Class, Teacher-Directed Instructional Strategies

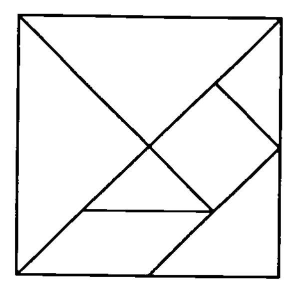
Whole-class, teacher-directed instructional strategies may involve the teacher as a "sage on the stage," as in a lecture or demonstration, or as a "guide on the side," as in a discussion or question and answer session. The variety of strategies in this category are illustrated in the vignette on the following page.

Vignette — Tangrams

Mr. Hudson began class by asking the students about some of the games they enjoyed playing on rainy days. He listed each game on the board and then asked the students to describe the mathematics that they used in playing that game. After some discussion, he explained that they would be making an ancient Chinese game in class. He distributed square pieces of construction paper to each student. He held up a square and asked the students, "What can you tell me about this shape?"

"It's a square."
"It has four sides."
"It has four corners."
"All the sides are the same length."
"It has four right angles."

Mr. Hudson continued to solicit student responses until he was satisfied that all students had an opportunity to reflect upon the shape and tell him everything they knew about that shape. He continued, "Today we are going to use this shape to make a puzzle by folding and tearing apart the square. First, fold the square along the diagonal and tear it along that fold. What can you tell me about the shapes of the pieces you get?"



As the students folded the squares, Mr. Hudson continued to question them. They told him that the shapes were two isosceles right triangles, that the two triangles were congruent, and that each triangle had one right angle and two 45° angles.

Then Mr. Hudson asked the students to put their pieces back together into a square. After each student had completed this task, Mr. Hudson continued to the next fold, again questioning the students about the resulting shapes and asking them to put the pieces back together into the original square. This continued until the students had cut their squares into the seven pieces of the tangram.

"Let me tell you about this puzzle," said Mr. Hudson. "Legend has it that this puzzle was invented by a poor Chinese boy as a present for the boy emperor, who loved puzzles. In 1813, a Chinese book about tangrams stated that, 'the origin of the seven-piece puzzle is not known,' suggesting that, even then, the tangram was thought to be very old. However, references prior to 1800 have not been found. There was a lot of interest in China in tangrams around 1800; the puzzle was considered primarily a game for children and women at that time. The puzzle became known about this time in the Western countries. At first, most books on tangrams were translations of Chinese works. Later books were more original. Most noteworthy is *The 8th Book of Tan* by Sam Lloyd, published in 1913. Lloyd provides much historical documentation regarding tangrams, all of it completely made up, claiming that the puzzle originated with a monumental work in seven volumes, which was compiled in China 4000 years ago. However, at that time, the Chinese were still at the stage of using crude pictographs scratched on turtle shells for writing.

The name of the puzzle, tangram, most likely comes from the South Chinese word for 'Chinese,' 'tang,' combined with the common ending 'gram.'

"The tangram can be used to make more than 300 different designs, including the numerals and the letters of the alphabet. Let me show you how to make an *A*. Mr. Hudson used tangram pieces on the overhead projector to demonstrate how the pieces could be used to make an *A* Then he used the pieces to show some different designs, asking for each, "What does this look like to you?" The first design was a boat, the next a candle, and the last a bell. "Tangrams can be used to depict many things, even the story of Cinderella. Today, however, we are going to use them to make some geometric shapes." Mr. Hudson distributed a sheet with several geometric shapes and asked the students to work in pairs to make these shapes out of the tangram pieces.

In the vignette, Mr. Hudson uses a variety of whole-class, teacher-directed instructional strategies before asking the students to work in pairs on a lab activity. First, he involves the students in a *discussion* about games that they had played. During discussions, both teacher and student are talking. Generally, discussions are initiated by questions posed by the teacher, although it is also possible to generate a discussion from student-initiated questions. Discussions are useful mechanisms for forcing students to develop their own thoughts about mathematics and to communicate their ideas to others. Some teachers have found that the quality of discussions is enhanced by asking students first to respond briefly in writing to a question before beginning a discussion. Discussions will be more effective if listening is not a passive activity; students should listen to one another, incorporate new ideas into their own thinking, and evaluate their own understanding. Some teachers follow up discussions by asking students to write about what they learned, what they think are the key ideas, or what they do not understand.

Following the discussion, Mr. Hudson involves the students in a *question-and-answer session*. In this, the teacher poses questions for students. Question-and-answer may be used quite effectively to provide a quick check on student understanding of a concept by asking all students to respond on chalkboards, flip books which they all show the teacher at the same time, or by showing their answers with manipilatives. Although this strategy can be used as a quick check on factual information, not all questions should be single-answer, low-level responses. Questions should also encourage students to explain different ways to find an answer. Some questions should challenge students' thinking, asking them to clarify their answers and justify their thinking. (A practical reference on questioning is David Johnson's book, *Every Minute Counts: Making Your Math Class Work.*)

Next, Mr. Hudson provided some background information on tangrams by lecturing briefly. *Lectures*, or teacher talk, are presently the dominant form of instruction in many mathematics classrooms. Lectures are most appropriate for providing factual information for motivated students. They are most effective when they are directly related to the objectives of the lesson, when the teacher uses relevant examples, and when the teacher avoids the use of vague terms. Most lectures are based on the philosophy of filling students' empty minds with factual information; such an approach runs contrary to research findings indicating the importance of involving students actively in processing information and of relating new learning to past learning. A major disadvantage of the lecture strategy is the low level of retention that can generally be expected.

The last whole-class, teacher-directed instructional strategy that Mr. Hudson uses is the *teacher demonstration*. Mr. Hudson uses the overhead to show the students how tangrams can be used to make several different designs. Another teacher might use a geoboard on the overhead projector to demonstrate the relationship between the area of a right triangle and that of a rectangle. Still another teacher might use models of a cube and a square pyramid with the same base and height to demonstrate the relationship between the volume of a prism and that of a pyramid. Demonstrations are often used in combination with other instructional strategies. Effective classroom demonstrations require clarity, enthusiasm, good questioning techniques, and student involvement (see Prichard & Bingaman, 1993).

Discovery Learning

Discovery learning may take place in a whole-class setting, in small groups, or on an individual basis. In a whole-class setting, it is generally guided by the teacher. In small groups, discovery learning usually involves some type of mathematics lab. Individual discovery learning generally involves student projects. Discovery activities should involve important mathematics, make connections among mathematical topics, link mathematics to other subject areas and to the real world, and include assessment as an integral part of the instructional activity.

Projects

Mr. Arnold has assigned individual projects to students in his seventh grade class. One such project is to determine how much postage is wasted on the "junk mail" that arrives at that student's home during a typical month. Barbara has been carefully examining the mail each day when she arrives home, and has been taking careful notes on how much postage appears each day on this junk mail. She then brings the data into class, where she enters it on a graph she has been keeping in her portfolio. She also adds the amount spent on the postage to a running total which she also keeps. When Mr. Arnold asked Barbara what the graph was being used to describe, she informed him that she had noticed that more junk mail seemed to arrive on Tuesdays than on any other day. The graph was being used to determine on which day the most junk mail actually arrived. Although not technically a part of her assigned project, it was something that had occurred to Barbara, she had discussed it with Mr. Arnold, and they had agreed that she should keep a record on a day-by-day basis.

In this vignette, the students have been working by themselves as much as possible. This technique has permitted the students to determine what techniques they will use to gather the data they need for their project, and to decide what the best method of keeping and presenting this data will be. The projects have been assigned individually, and Mr. Arnold has given them one month to complete their study. Some time has been allotted in class periodically, but most of the work is done at home or during free time.

Mr. Arnold has found that the use of individual work puts the student into the role of the living mathematician. The projects are "real-world" in nature, and show the students some areas in which their mathematics is used outside of class. This independent work provides an opportunity for some statistics, as the students decide what data they want, how they will best collect it, and finally, how they will write up and present their findings to the entire class. Mr. Arnold has been using the projects technique for several years now, and the students seem to enjoy being "living mathematicians," as one student put it.

Mathematics Labs

> Ms. Horatio has students working in small groups in her first-year algebra class. The students have been given a laboratory sheet that leads them through an activity designed to help them discover trinomial factoring. Using a series of large squares (representing x^2), rectangles (dimensions x by 1, representing an area of x), and unit squares (representing units), the students are building a series of rectangles whose areas are expressions such as $x^2 + 4x + 3$ (length = $x + 3$, width = $x + 1$); $x^2 + 5x + 6$ (length = $x + 3$, width = $x + 2$); etc. After experiencing five or six of the rectangles, the students begin to see the quadratic expressions as areas, and the factors as the length and the width.
>
> While the students are working in their groups, Ms. Horatio moves about the classroom helping the students build their rectangles. She occasionally asks a leading question, designed to keep the students "on-track." Once they have demonstrated their ability to "build" the rectangles, Ms. Horatio gathers them into a large group and helps them make the transition to the abstract, in which students are given a quadratic trinomial to factor without using the manipulatives.

The type of teaching strategy used in the vignette at the left permits the students to make use of a laboratory setting to discover how to factor. The discovery technique permits them to make use of manipulatives until they gain the concept at hand. The discovery technique permits the students to enjoy creating their own mathematics. Using a laboratory setting at this level is somewhat different for her students. One disadvantage which Ms. Horatio faces, however, is the distinct possibility that some students may discover some concepts which differ from those she is attempting to have them learn, or may even be incorrect concepts. She controls these possibilities to some extent by moving about the classroom during the discovery period and "helping" the students with their work. Her lab sheet is somewhat directed, so that students will not move too far from the desired discovery.

Lab activities involving data collection are also becoming more common in the mathematics classroom. Such activities involve doing an experiment, often with some type of equipment, recording data in a suitable form, analyzing the data (looking for patterns and making generalizations), and reporting back the results. Usually these labs are completed by small groups of students working together. Sometimes the lab report is generated as a group; at other times, it may be prepared by each individual in the group.

Whole Class Activities

Ms. Lovelace's second-graders are investigating addition by using base ten blocks. Ms. Lovelace begins by pairing up the students. She asks one student in each pair to represent the number 25, using as few blocks as possible, while the other represents the number 28, again using as few blocks as possible. Then she asks the students to combine the blocks and tell her how many there are in all. She asks several pairs to explain how they found the answer and then introduces a method for recording what the students have done with the blocks.

Mrs. White wants her high school class to see how changes in an equation affect the graph of that equation. She believes that the overhead graphing calculator will be the best tool to use to accomplish this. (It's easier to keep everyone looking at the same thing with the overhead than it is with individual graphing calculators, and it's easier to set up the graphing calculator than it is to use the computer.) She begins by graphing the equation $y = x^2$. She asks students to predict how the graph will change if it is changed to $y = x^2 + 3$. She enters this on the graphing calculator and asks the students to describe what happened. She does several more problems involving adding a constant and then asks the students to write down the pattern they have found. She continues the lesson by examining the effects of other constants.

Both of the examples above illustrate discovery learning in a whole-class, teacher-guided situation. This instructional strategy is frequently used in conjunction with a demonstration and discussion. Keeping all of the students in a single group provides a more consistent experience for all of them but also generally provides fewer opportunities to articulate patterns and describe generalizations than does working in small groups. (Useful references on discovery learning are the books by J. G. Brooks and M. G. Brooks, M. Burns, C. K. Kamii, and D. Chazan and R. Houde listed in the References.)

Cooperative Learning

In this vignette, we see a classroom of youngsters working in cooperative or collaborative groups. In this case, the teacher has placed the students into the groups, in order to ensure that each group has a wide distribution of abilities and that personal problems are kept to a minimum. All students in the group are encouraged to contribute, answering questions within their own group. Each student is expected to be able to explain what the group did. Mrs. Exton changes the groups every four or five weeks. Her role is to establish the objectives of the lesson, provide the stimulus for discussion, monitor the groups, and to intervene only when absolutely necessary.

> In Mrs. Exton's sixth-grade class, the students have been presented with a task to perform. The following sequence of numbers has been placed on the board:
>
> 9, 16, 23, 30, 37, 44, 51, ...
>
> Mrs. Exton has placed the students into working groups of four, and each group has been asked to examine the sequence of numbers and tell all they can about the sequence. Mrs. Exton is circulating among the groups, observing the students as they work, and from time to time, making a comment to some of the groups as she passes by. As the groups work, there is a chattering noise throughout the classroom. "The difference is always 7." "The first two numbers are squares. Are there any other squares?" "Take 2 from each number and you get the 7's table." "If we add 5 to each number, we get the 7's table." "Two of the numbers are primes. Are there any more if we go on?" "Look at this pattern that you get if you add up the digits in each number!" In most of the cases, other students within the group answer the questions as the group continues to explore. After about 10 minutes, Mrs. Exton calls the groups to order and has one person from each group present their group's findings.

Other teachers may organize cooperative learning groups in different ways. Some teachers prefer to let students select their own groups, at least on occasion. Some teachers prefer to change groups more frequently, depending on the activity planned for a particular day. Some teachers prefer to occasionally have students of like abilities and backgrounds working together. Some teachers like to assign roles to individuals in the group: facilitator (keeps the group on task), recorder (writes down ideas), reporter (reports to class), reader (reads problem and checks accuracy of facts), materials handler (gets needed materials). Still another cooperative learning strategy is the Jigsaw method, in which each member of a group is assigned an area to research, all those having the same topic work together to learn about that topic, and then they disperse back to their original groups to teach the rest of their group about that topic. (Useful references on cooperative learning are the books by A. Artzt and C. Newman, and by N. Davidson in the References.)

Math Buddies

In Janet Pike's sixth-grade math class at Reed Avenue School in Millville, students work in pairs called Math Buddies. Buddies work together in checking homework, correcting homework mistakes, working at the chalkboard, writing in math notebooks, using manipulatives, and calculator activities. They also work on projects and reports as a team. The students look forward to specific days when Math Buddies are required. Not all lessons or daily objectives lend themselves to the "buddy system," but on most days the arrangement allows for maximum involvement, understanding, and exploration.

It is said that we really learn a concept when we teach it to someone else. Cooperative learning allows for peer teaching-tutoring, actively engaging students in a process which facilitates long-term retention of mathematical concepts. Cooperative learning has also been shown to have a positive effect on student achievement and interpersonal relationships (Slavin, 1990). It improves student attitudes toward racial and ethnic groups. The group often helps to keep low-achieving students on-task. Above all, the group helps keep students more actively involved in the learning process. However, using cooperative groups to the exclusion of any other teaching strategy is as inappropriate as using only lectures.

Summary

A variety of teaching strategies have been described in this section. Individual lessons lend themselves to different styles of teaching. It is the job of the teacher to match the particular strategy of teaching with the topic of the lessons and to utilize as wide a variety of strategies as possible. Using a variety of strategies helps to make the mathematics classroom more interesting to students and to make the subject of mathematics come alive as they learn. The mathematics teacher must plan the most effective strategies to engage students in learning, retaining, and applying important mathematics.

"The most useful metaphor for describing the modern teacher is that of an intellectual coach. At various times this will require that the teacher be

- A *role model* who demonstrates not just multiple paths to a solution but also the false starts and higher-order thinking skills that lead to the solutions of problems.
- A *consultant* who helps individuals, small groups, or the whole class to decide if their work is keeping 'on track' and making reasonable progress.
- A *moderator* who poses questions to consider but leaves much of the decision making to the class.
- An *interlocutor* who supports students during class presentations, encouraging them to reflect on their activities and to explore mathematics on their own.
- A *questioner* who challenges students to make sure that what they are doing is reasonable and purposeful and who ensures that students can defend their conclusions."

(*Counting on You*, 1991, pp. 13-14)

Organizing the Content

Identifying the Problem

The Mathematics Curriculum Committee for the Busytown School District began their discussions by reviewing how the present curriculum seemed to be working.

> "The students in fourth grade seem to do fine on their tests, but they don't do nearly as well on the standardized tests. It's as if they've forgotten everything they studied!" said Ms. Brown.
>
> "The first-graders are doing pretty well on the standardized tests and on the assigned work, but they don't really seem to understand a lot of the content. I'm concerned about what they can build on in second grade," said Mr. White.
>
> "I think we really need to think about how we approach the different topics in math — maybe there's a better way," suggested Mrs. Hope.

As they worked, the teachers discovered that their curriculum tended to break mathematics content down into very small, isolated bits of content that were often taught in isolation. Students just were not putting those bits together to form broader understandings of concepts.

The approach taken by the teachers in this district is not unfamiliar. For many years, the mathematics curriculum has been viewed as a fragmented collection of discrete, isolated topics. Each topic has typically been broken down into its component tasks, with lessons focussing on each of these tasks. Students often fail to see the connections among the component tasks, working instead with each task in isolation.

Students often learn topics through rote manipulation of symbols rather than developing an understanding of a concept for themselves; this leads to rapid "forgetting" and a need for continual reteaching of concepts. The "spiral curriculum" (see Bruner, 1977) seems to have degenerated into "going around in circles," teaching and reteaching the same content year after year after year.

What Bruner's spiral curriculum actually suggests is that there is a set of simple but powerful basic ideas in mathematics (like functions and variables) that are central to the curriculum and that can begin to be addressed by students even at very young ages. These ideas are first addressed concretely in very basic forms. As students grow older, they revisit these ideas in more depth and learn to apply them in more complex and abstract situations. The ultimate goal is to develop a complete understanding of the basic ideas. Perhaps it is more useful to describe such a curriculum as a concentric one (see McKnight, et al., 1987) in which each encounter with an idea significantly moves learning to a deeper level.

The concentric/spiral approach originally suggested by Bruner can assist teachers and administrators in organizing the curriculum. The descriptions in this document of the content that New Jersey's students are expected to learn illustrates how students at each grade level can study the same topics in greater and greater depth. In reading through the illustrations at each grade level, it is clear how understanding is expected to develop over the years.

Studying each of the content and process strands in isolation, however, is neither efficient nor effective. Students need to recognize related ideas within mathematics as well as those in other subject areas. They need to understand and appreciate the history of mathematics. They need to see that people are always developing new mathematics and new ways of learning mathematics. Students should learn mathematics through exploration, discovery, and problem solving. Thus, it is desirable to strive for integration of content across strands.

Units of Instruction

Most teachers think of mathematical content as being organized into instructional units. Frequently, these units correspond to the chapters in a textbook. They may also be organized in terms of mathematical or interdisciplinary themes. A unit usually includes investigations, problems, and other learning activities, integrated with assessment, that develop students' understanding of specific concepts. Units can be as short as a week or as long as six weeks, depending on the concepts and tasks involved. In units, it is important to include mathematics that makes sense all by itself, with a clear purpose. All the work in a unit should be related to a primary goal, exploring and eventually consolidating a set of related ideas that will be useful later. Furthermore, a unit should be interesting and engaging to the students, connecting to what they've learned before and helping them see mathematics as an integrated whole.

In planning for an instructional unit, teachers should first consider what concepts are central to the mathematics program at their particular grade level. The mathematical ideas in the unit should all fit together in a cohesive way, connecting to each other and to other areas of mathematics. Some important connections to consider in planning include what students have already learned and what students will be learning in the future, either in that grade or in future grades. Teachers may also want to consider what connections can be made to other subject areas. Activities included in the unit should focus on worthwhile mathematical tasks. The First Four Standards should be included in all instructional units, but not all units will deal with all content strands. Over the course of the year, however, students should have experiences with all of the content strands. Finally, teachers should consider what instructional and assessment strategies and activities are most likely to help them accomplish the goals they have established for that particular unit. By considering all of these components, teachers can design effective instructional units that will help to meet New Jersey's *Mathematics Standards*.

Components of an Instructional Unit

- Important mathematics
- Mathematical connections
- Bridge between past and future experiences
- Connections to other content areas and to the real world
- Worthwhile mathematical tasks
- The First Four Standards
- Variety of appropriate instructional strategies
- Assessment embedded in instruction

Not all mathematics need be done in units. It is also appropriate to include briefer activities unrelated to units. Such activities may be favorites from previous years, problems of the week, short exercises, and so on. Students also benefit from looking back across several units to identify related ideas.

The Appendix to this chapter contains three examples of instructional units. The first example describes the mathematics portions of an interdisciplinary unit in a primary classroom; this unit uses playgrounds as a theme for studying geometry and machines. The second example describes a teacher's thinking as he develops an introductory unit on common fractions for students in the middle grades. The third example shows high-school students using data analysis to generate mathematical models and to describe them with algebraic equations.

Daily Decisions

Organizing the curriculum at the classroom level requires different kinds of decisions — decisions such as those Ms. Kee is facing: homework, review, assessment, lessons and emphasis.

Ms. Kee was beginning a new school year in a new school district. She had talked with the mathematics supervisor and principal about the mathematics program in the school and had picked up copies of the textbook adopted for each course she would be teaching, as well as the district curriculum. She also had copies of materials she had used in her previous school, some of which she would like to continue using. She began working on eighth grade math by looking through the textbook and reading over the district curriculum. One of the first things she noticed was that the district curriculum specified thematic units to be taught over the course of the year, with references to specific pages in the text as well as some other materials.

"This district curriculum is a big help! I'm glad I won't be just following the text this year. I need to talk to Mrs. Hughes again, though, to find out about all these other resources."

"Let's see -- what else do I need to figure out? I want to look at different ways to handle homework and review. I've also been reading a lot about alternative assessment; I'd like to try some different techniques this year. And ..."

While teachers must teach mathematics every day and must plan for a daily mathematics lesson, it is not necessary to teach a different objective each day. Such an approach reinforces a view of mathematics as isolated concepts. *More desirable is a structure in which most lessons involve more than one objective addressed over several days in a more unified way. This approach provides the time that is needed for concepts to build, allowing daily lessons to be more connected. Students may explore a concept one day, look for patterns the next, generalize their results on a third day, and apply those generalizations on a fourth day. Review is provided for in each lesson, building on previously-learned concepts and making explicit connections to other mathematical ideas.*

Traditionally, students are given daily homework and/or classwork assignments from the textbook. These are to be done individually and often involve "drill and practice." A steady diet of such assignments is not desirable. The purpose of homework is not only to provide practice but also to help students identify what they do and do not understand by raising questions to be answered the next day in class. *Thus it is desirable to have a variety of homework assignments, including exploration, projects, discovery, and writing in journals. Some assignments may extend over several days and may require considerable time outside of class. These assignments usually are stated in the form of a broad, complex problem to be investigated by a group of students, with findings presented in the form of an individual or group report, a presentation to the class, or some other type of visual display.*

Testing has always been considered extremely important in the mathematics classroom. Traditionally, tests closely match assignments, with a narrow focus on skills. Mastery is expected. *More desirable is use of a range of forms of assessment, including checking for understanding throughout a lesson, providing guided practice for students in class, quizzes, observation, interviews, and group work. Tests should* *include questions that ask students to explain their thinking. Students may be expected to keep portfolios of work throughout the year, revising their work to meet the expected standards.*

In the vignette below, Mr. Fry begins to make some decisions about pacing, assignments, and review.

> Mr. Fry has been teaching at New Jersey High School for the past fifteen years. He is very familiar with the district curriculum and with the textbooks but has become increasingly unhappy with his students' performance in geometry over the past few years. Last spring he was able to attend a mathematics conference and got some new ideas for ways to organize his geometry course. He also got some good ideas about using writing in the mathematics classroom and about integrating review activities into instruction on a regular basis. Now he is planning for next week's lessons on angle relationships.
>
> First, Mr. Fry looks over the text and reviews what he did last year. "Not too much student involvement — I need to get them thinking about what will happen before I state the theorems. Maybe I can find or develop some investigations that will get them started on vertical angles and linear pairs and ..." Mr. Fry develops some brief activities that will help the students discover the general relationships for themselves.
>
> "Now, I'd better think about how much time I want to spend on angle relationships. These are pretty important, so I don't want to rush through. On the other hand, the students will see these ideas again and again throughout the course. I think one week ought to be about right." Mr. Fry balances the competing needs of all the topics in the course with the need to develop understanding on the part of the students.
>
> Mr. Fry looks at the assignments he gave last year. "These were all pretty computational. I'll use some of these, but I think I'll take out some and have students write something each night. Then we can begin reviewing the homework by having them exchange what they wrote and comment on it."
>
> "What about review? I could add a few review problems to each homework assignment ... that would work well on Monday and Wednesday. On Tuesday and Thursday, I could begin class with a few review questions. And Friday, we'll do this project; it involves all of the ideas from this week plus some others they've already studied." Mr. Fry selects different strategies for incorporating review into each day's lesson.
>
> As for assessment, Mr. Fry uses a system that includes homework, classwork, tests, quizzes, journals, and group work. "This week, I'll give a homework quiz on Wednesday, and I'll grade their journals on Friday. I think I'll focus on how well the groups monitor their progress in working on Friday's project, too. The students should be ready for a written test next Friday."

✓	*SUGGESTIONS*

✔ Plan most lessons so that they involve more than one objective and take place over more than one day, giving students enough time to explore a situation thoroughly and solidify learning.

✔ Provide for ongoing review in each lesson.

✔ Vary the types of homework assignments given to students, including some projects and group assignments.

✔ Use several different types of assessments and include "explain" questions on written tests.

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Appendix

A. Primary Grades Sample Unit — Geometry

This second-grade unit focuses on the playground as a context for studying geometric figures and properties as well as simple machines. In addition to the connections between mathematics and science, the teacher wanted to try to incorporate some language arts activities into the unit. She began her planning by reviewing New Jersey's *Mathematics Standards* and identifying the important mathematical concepts that would be addressed in the unit. She adapted a number of the cumulative progress indicators, primarily relating to Standard 7 (Geometry and Spatial Sense), and developed the following list of expectations for this unit.

- Students will explore and understand properties of three-dimensional and two-dimensional geometric shapes.
- Students will identify and use geometric shapes in various orientations: circle, square, rectangle, triangle, sphere, cone, cylinder, prism, pyramid.
- Students will explore properties of geometric shapes in conjuction with the concepts of area and perimeter.
- Students will develop tessellations.
- Students will explore the various ways geometric shapes and objects can be measured.
- Students will identify examples of congruent objects, similar objects, and self-similar objects.
- Students will identify the types of symmetry found on the playground, including both line symmetry and rotational symmetry.
- Students will use shadows (projections), points-of-view (perspective), and maps to describe the relationships among objects on the playground.

She next considered what connections students should make to other areas of mathematics; she decided that she would like to try to incorporate some measurement and some data collection and analysis. She also thought about how this unit could build on what students already knew. The second-graders had studied a little bit of geometry last year and they had worked with geometric figures in creating and extending patterns, but she knew that there was a wide range of understanding about some of the specific properties of different shapes. She decided to use several activities that would have students describe what they already knew as a springboard for further discussion. She thought about the unit later in the year that would develop linear measurement, perimeter, and area more formally; she decided to include measurement of shadows and an activity with designing a tile patio to begin to develop some of these ideas more informally. She also thought that, by having students build models of playground equipment, some size issues might arise naturally. Then the teacher considered the First Four Standards. She decided that she wanted to incorporate all four into the unit, but that she would emphasize connections and communication.

Next the teacher considered whether her initial ideas for activities represented worthwhile mathematical tasks. She thought that collecting examples of shapes, building playground models, planning a tile patio, and collecting data about shadows would be interesting to the students, that the activities would draw on students' previous experiences, and that the activities would stimulate connections within mathematics and across subject areas. She planned to build sufficient time into the unit for students to reflect on and discuss each of the activities. She also considered whether the instructional strategies she planned to use were appropriate and addressed different learning styles. She planned to include some work in cooperative groups, some work in pairs, and some individual work. She planned to have some oral work and some written work. She planned for projects, work with concrete materials, discovery learning, and whole-class discussions. She thought these would be sufficiently varied for her students.

Finally, the teacher thought about assessment. She knew she had already built into the instructional activities a number of assessment tasks, ranging from diagnostic activities, like writing about what they already know about triangles, to a summary assessment project, in which individual students would collect examples of different three-dimensional shapes. She wanted to be sure to provide for multiple measures of students' achievement. Student work in groups and comments in student journals would provide evaluative information on a daily basis. In addition to the independent project, she planned to randomly select students to report on group projects and, at the end of the unit, to give a quiz and ask students to evaluate their own learning in their journals.

Day 1: Shapes on the Playground. In math, the class goes on a geometry walk on the playground in order to check the list from science class and to find out what students already know about geometric shapes, congruence, similarity, self-similarity, and symmetry. Students look for examples of geometric ideas in nature and in objects found on the playground. Students take notes in their journals. Back in the classroom, during language arts, students write about and draw the geometric shapes they saw on the geometry walk. In science class, students make a list of types of equipment found on the playground. They discuss how the different objects found on the playground are alike and different. They generate a list of questions to investigate.

Day 2: Polygon Models. In math class, pairs of students use straws (cut to different lengths) and string to build models of figures composed of line segments (triangles, squares, rectangles, and possible others). Students discuss how all of the triangles (squares, rectangles, etc.) are alike and how they are different. The terms congruent, symmetric, similar, and self-similar are used. In science class, students study about the seesaw (lever), building models (what measurements do we need and why?), and investigating what happens when weights are placed at different locations on the lever.

Day 3: Polygon Properties. In math class, students continue their discussion of polygons. Groups of three or four students work together to write down what they know about triangles, squares, and rectangles and to list examples of these shapes found on the playground. They identify congruent and similar figures, as well as examples of self-similar shapes. They also describe lines of symmetry. They summarize what they have learned in their journals. In science class, students begin learning how each six of the simple machines — lever, inclined plane, wedge, pulley, wheel and axle, and gear wheel — work; they do tasks at learning stations set up for each machine.

Day 4: Investigating Shadows. In math class, students look at the shadows cast by simple objects placed on the overhead projector and make predictions about the shapes of the shadows cast by objects on the playground. They also discuss how the shapes of the shadows change as the sun moves through the sky and make predictions about how the shapes will change. They develop a plan for collecting data about the shadows made by the seesaw, the slide, the merry-go-round, and the swings throughout the next day. A group of students will be assigned to each object and will sketch the shape of the shadow and measure its length in links at three points during the day: 9 a.m., 11:30 a.m., and 2:00 p.m. In science class, students finish the learning station activity about simple machines.

Day 5: Collecting Shadow Data. At three points during the day, students go outside to sketch the shape of the shadows cast by each of the playground objects and to measure their size. On these outdoor excursions, they also look for examples of simple machines. At the end of the day, they list the different simple machines that they found, summarize the shadow data, identifying patterns in how the shadows changed, and write about how well their predictions from the previous day matched the actual data collected.

Day 6: Three-Dimensional Objects. The teacher begins the math lesson by showing the students the possible shadows of some mystery objects (ball, ice cream cone, soda can, a Unifix cube, and a pyramid from the kindergarten block corner) and asking the students to identify the objects. The students then discuss the mathematical terms for these three-dimensional objects and list objects on the playground (and elsewhere) that have those shapes. The teacher assigns a homework project to assess students' understanding and ability to identify shapes; the students must find five pictures in magazines of each of the shapes studied and write about why they chose those particular pictures. The project is due in one week. In science class, the students begin an investigation of swings (pendulums), looking at what factors affect how high the swing goes.

Day 7: Playground Models. Today the students begin working on building a model of an "ideal" playground. Since this is a primary class, the teacher has decided not to address the issue of scale directly, but rather to encourage students to compare the size of their models to those of other groups to see if they "match." Each group is assigned a piece of playground equipment to build. They are encouraged to build a model that "works" but are not required to do so. Students write in their journals about what they are doing in their groups.

Day 8: Finishing Models. The students finish their models and place them on a large board, creating their own class playground model. Each group explains their model. In language arts, they read a story that takes place on a playground.

Day 9: Playground Maps. Working in groups, the students sketch maps of their class playground model. In language arts, they begin writing their own playground stories.

Day 10: Orientation. Students place teddy bear counters on the playground maps according to clues given by task cards. They also sketch the orientation of the pieces of playground equipment that a specific teddy bear would see and check out their answers by using the actual model.

Day 11: Transformations. The students investigate transformations by making predictions about the positions of a teddy bear counter at various points on the slide, on the swing, and on the merry-go-round. They use their model to test their predictions. They also discuss the relationship between two teddy bears on the seesaw or opposite each other on the merry-go-round. The teacher then introduces the terms slide, flip, and turn, relating them to the students' findings. Students make sketches of each transformation in their journals.

Day 12: Tile Designs. Students investigate tessellations by designing a tile patio for a playground. Working in groups, they use pattern blocks to design a tessellation. They are given some constraints: (1) the distance around the patio must be at least 20 units, where the length of one side of the triangle is one unit; (2) the patio must be smaller than 40 squares; and (3) no more than $100 may be spent (costs for each shape are given).

Day 13: Tile Design Reports. Each group of students reports on their tile designs. In order to assess each students' contribution to the project, the teacher randomly selects a student from the group to answer some standard questions: *How did you find the distance around your patio? Is it smaller than 40 squares? How do you know? How did you determine the cost of the patio?*

Day 14: Quiz & Self-Assessment. Students take a short paper-and-pencil quiz and describe in their journals what they learned about geometry in this unit.

B. Middle Grades Unit — Common Fractions

Mr. Lopez is planning an introductory unit on common fractions for his sixth-grade class. Since this is a critical concept for sixth-graders to understand, he decides to focus almost exclusively on mathematics and deal more with applications in later units. From conversations with the fifth-grade teachers, he knows that the students worked at some length on fractions the previous year, using circle models to investigate fractions concretely. This year, he wants to extend the intuitive investigations to strips and rectangles and begin to develop the algorithms for addition, subtraction, multiplication, and division. His class has already studied decimals to some extent in connection with place value and the algorithms for addition and subtraction of whole numbers and decimals. He plans to use fractions extensively in the next two units, involving measurement and data analysis, and decides to postpone making connections to other subjects until that time. He also plans to introduce students to fraction calculators in the first of those units.

Mr. Lopez begins his planning by reviewing New Jersey*'s Mathematics Standards*, and indentifying the important mathematical concepts to be addressed in this unit. He adapts a number of the cumulative progress indicators related to Standard 6 (Number Sense) and Standard 8 (Numerical Operations) and he develops the following list of expectations for this unit:

- Students will extend their number sense by constructing meanings for rational numbers.
- Students will expand the sense of magnitudes of different number types to include rational numbers.
- Students will develop and use order relations for rational numbers.
- Students will investigate the relationships among fractions, decimals, and percents and use all of them appropriately.
- Students will extend their understanding and use of basic arithmetic operations to fractions.
- Students will develop, analyze, apply, and explain procedures for computation and estimation with fractions.

From the First Four Standards, Mr. Lopez adapts the following expectations:

- Students will use mathematical language and symbols to represent problem situations and explain what they have done.
- Students will link conceptual and procedural knowledge.
- Students will understand and use various representations of concepts, and connect them to one another.
- Students will analyze mathematical situations by recognizing and using patterns and relationships.

From Standard 17 (Keys to Success), he selects the following for emphasis:

- Students will make conjectures, pose their own problems, devise their own approaches to problem solving and use their results for informed decision-making.
- Students will spend the time and use the tools needed for mathematical exploration and discovery.

- Students will gain confidence through successful experiences in mathematics and thereby develop a positive disposition toward doing mathematics.
- Students will regularly self-assess to determine the effectiveness of their strategies, the correctness of their results, and to monitor their attitudes toward learning mathematics.

Mr. Lopez makes a list of these expectations to keep in mind as the students proceed through the activities of the unit. Mr. Lopez next begins to outline a sequence of activities for the students. He first works in content blocks rather than days:

A. Different ways of writing numbers

B. Using strips to represent fractions

C. Equivalent fractions

D. Comparing fractions

E. Adding and subtracting fractions

F. Exploring multiplying fractions

G. Exploring dividing fractions

After outlining the activities, he goes back and tries to determine how many days should be in each block and what the specific activities should be for each day. While he is doing this, he begins thinking about how students will be assessed. He decides that he would like to use both quizzes and a project of some type. He plans to give a short paper-and-pencil quiz each week, including at least one question which requires the students to explain their thinking. In addition, he decides to have each student develop a book, *The Story of Fractions*. Students will complete individual pages for the book throughout the unit and assemble them at the end into a book to share with others. Mr. Lopez makes notes about some ideas for pages in conjunction with the activities he has outlined. As he makes his notes, he tries to make sure that he uses worthwhile mathematical tasks and a variety of instructional strategies.

A. Different ways of writing numbers

Day 1:

- Warmup: Write as many "incredible equations" as you can that have an answer of 123.
- Discuss warmup: 123 = 100 + 20 + 3 = 120 + 3 = 110 + 13 etc.
  (relate this idea to addition/subtraction algorithms)
  also 123 = 6 x 20 + 3 (when would writing 123 this way be useful?)
- Use base ten blocks, with 1 flat = 1 whole. How could we describe 5 rods?
  5/10 = 1/2 = 50/100 = 50% = .5 = .50
- When would writing the number in each way be useful? Example: 5 out of 10 games were won, 1 out of 2 apples were damaged, 50 out of 100 pennies or 50 cents, a 50% off sale, 5 dimes as a decimal is .5 as parts of a dollar, which is $.50, and so on.
- Have students work in small groups on several similar problems, and come up with as many

different ways to write each number as they can. Remember to provide calculators! Post their results on the bulletin board. For homework, ask them to find examples from newspapers and magazines of the various notations that they used.

Day 2:

- Warmup: What are some different ways of writing 6/10? Discuss.
- Discuss the examples from newspapers and magazines that students found and have them put them on the bulletin board.
- Have students do more examples, using hundred grids instead of base ten blocks. This time, have students work individually and then compare their results in pairs before discussing them as a class.
- Make the point that we would like to be able to use whatever form of a number is easiest for computation.
- Remind students about work done with decimals and their connection to our curent focus on fractions. Homework assignment will be a decimal review.
- Note: Remember to come back to the idea of different ways of writing numbers again and again.

B. Using strips to represent fractions

Day 3:

- Warmup: Mental math with whole numbers and decimals.
- Have students use construction paper to make sets of fraction strips by folding and cutting. Students also keep one fraction worksheet without cutting it to refer to it later. Use different colors for each denominator. Make one set in class and have students make two more for homework. (Write up a handout with instructions.)
- Ask the students questions about the strips, such as *how many sixths make a whole?* or *which is bigger, a half or a third?* Ask how we would write two thirds, developing the terms numerator and denominator.
- Use the strips to do problems, such as *Suzanne ate a fourth of a candy bar and Luis ate a fifth of the candy bar. Who ate more candy?* Be sure to point out the importance of making sure the candy bars were the same size, since a fourth of a small candy bar may be smaller than a fifth of a larger one. Bring in examples of candy bars to illustrate this concept. Also do some problems like *Carmen ate half of a pizza and her sister ate one-fourth. How much was left?* (This will begin to lay a foundation for later work with operations.)
- Have students create their first page for their fraction book by picking a number from a hat to give them the denominator for their fraction and completing a page entitled *How Many Make a Whole?*
- For homework, ask students to explain why 1/4 is bigger than 1/5.

C. Equivalent fractions

Day 4:

- Warmup: Mental math with decimals and whole numbers.
- Have students share what they wrote for homework with another student.

- Discuss homework as a group.
- Explain that today we will be working on different ways of naming the same fraction, or equivalent fractions. Relate this to a pizza that may be cut into halves or fourths or twelfths. Have students work in pairs to find different equivalent fractions for 1/2, 2/3, etc., using their fraction strips. Record their results on newsprint sheets. Introduce the idea of simplest form (using as few pieces as possible.)
- Use pattern blocks and one of the hexagons. Try to cover the hexagon with blocks of a new shape which uses the fewest pieces (2 trapezoids). Each represents ½ of the whole hexagon. Now cover it with the most pieces (6 triangles). Each is 1/6 of the hexagon. *How many triangles cover one trapezoid?* (3/6 = 1/2) Students use other pattern blocks to form equivalent names for fractions.
- Give students a challenge problem that cannot be done with their strips: Find as many equivalent fractions for 75/100 as you can. Are there other ways to write this number?
- Discuss the patterns that have emerged in looking at equivalent fractions: How can you tell if two fractions are equivalent?
- For homework, ask students to write about equivalent fractions: what are they, what have you learned, and what don't you understand?

Day 5:

- Warmup: Find simplest form for 2/4, 6/8, 8/10. Discuss.
- Have students work in groups of three on equivalent fraction problems, some where they fill in the missing numerator or denominator and some where they must put the fraction in simplest form. Include problems that can be done with the strips as well as some that cannot. While they are working, read over their homework and note the extent to which students are using the strips.
- Discuss the problems, having students explain how they solved each problem. (If students have a lot of difficulty with these problems, divide into two days and repeat the activity.)
- Have students apply the methods discussed to several new problems as guided practice.
- Homework will be doing problems similar to those in class.

Day 6:

- Discuss homework. Quiz.
- Have students do another page for their book on *What Are Equivalent Fractions?*

D. Comparing fractions

Day 7:

- Warmup: Equivalent fractions.
- Return quizzes and discuss as needed.
- Use concrete real-life situation to pose question, *How can we tell which of two fractions is larger?*
- Write responses from class discussion on the board, then have students work in pairs, using concrete materials as needed, to compare pairs of fractions. Discuss.
- Homework will be similar to classwork.

Day 8:

- Warmup: Comparing fractions.
- Discuss homework. Compare some of the fractions found in newspapers and magazines that are posted on the bulletin board. Ask students when you might have to compare fractions in real life.
- Have students make up a word problem that requires comparing fractions.
- Have students complete another page for their fraction book on *Which is bigger?*
- Homework will be preparation for addition and subtraction of fractions: *How can we express 1/4 + 1/4 as a single fraction?* involving fractions with like denominators, followed by some word problems that require adding and subtracting fractions with like denominators.

E. Adding and subtracting fractions

Day 9:

- Warmup: Estimation with fractions — e.g., *Which is closer to 1, 5/6 or 7/8?*
- Discuss homework. Generalize procedure used to add and subtract fractions with like denominators.
- Pose problems involving fractions with unlike denominators: *Mary ate 1/3 of a candy bar and Susie ate 1/6. How much did they eat all together? How can we express 1/3 + 1/6 as a single fraction?*
- Have students explore this question in pairs, using their strips, then discuss what they did. Record what students did, using symbols.
- Have students do more problems involving unlike denominators (some word problems), using their strips and recording their results symbolically.
- Have students also do a subtraction problem with their strips: *There was half of a candy bar left. Joe ate 1/3 of the candy bar. What fraction of the candy bar was left?* Discuss results and record symbolically. *How is this like the addition problems?*
- For homework, have students do a few problems, including word problems and one that cannot be done with strips.
- Note: This may be too much for one day.

Day 10:

- Warmup: Estimation with fraction addition and subtraction. Discuss.
- Discuss homework. Lead students to idea that you need to express fractions as equivalent fractions which have a common denominator.
- Have students do a few addition and subtraction problems (some that can be checked with strips and some not, some word problems and some not) and discuss.
- Homework: Problems involving addition and subtraction of unlike and like fractions, including word problems.

Day 11:

- Warmup: Estimation with fraction addition and subtraction. Discuss.
- Discuss homework. Quiz. Have students do two more pages for their fraction books on *How to*

Add Fractions and *How to Subtract Fractions*. Have them make up one word problem to include on each page.

- Homework: Finish fraction book pages.

F. Exploring multiplying fractions

Day 12:

- Warmup: Mental math with fraction addition and subtraction. Discuss.
- Introduce each of the following expressions with a word problem, discuss what the expression means and ask students how to represent them concretely:
  - 3 x 1/4 (3 groups of 1/4, or 3/4)
  - 1/3 x 12 (1/3 of a group of 12, or 4)
  - ½ x 1/3 (1/2 of 1/3, or 1/6 — fold a 1/3 strip in half vertically or horizontally)
- Have students work in groups to do similar problems concretely, recording their results symbolically and looking for patterns.
- Homework: Explain what each problem means and solve it: 2 x 1/6, 2/3 x 9, 2/3 x 1/2.

Day 13:

- Warmup: Equivalent fractions, comparing fractions
- Discuss homework. Have different students explain different ways they solved the problems.
- For class, have students do more multiplication problems and then do another page for their fraction book on *Multiplying Fractions*. Have them include at least one word problem on each page.

G. Exploring dividing fractions

Day 14:

- Warmup: Estimation with fraction multiplication
- Discuss homework.
- Focus on developing meaning for division by asking students to solve some word problems involving the following expressions:
  - 6 ÷ 2 *How many groups of 2 are there in 6?* or *What do you get when you divide 6 into 2 equal parts? Is this like another problem we have done recently (*1/2 x 6*?)*
  - 1/2 ÷ 3 *What do you get when you divide one-half into three equal parts?* Ask if this reminds them of another problem they have done recently, like 1/3 x 1/2.
  - 6 ÷ 1/2 *How many halves are there in 6?*
  - 1/2 ÷ 1/6 *How many sixths are there in a half?*
- Give pairs of students a set of problems to explain the meaning of, represent concretely, and solve.
- Ask them to look for patterns.
- For homework, ask students to write about what they have found out about dividing with fractions.

Day 15:

- Warmup: Mental math with fractions
- Have students trade homework papers and comment: *Do they agree? Is it clear?*

- Have students do more division problems, including word problems and one or two that cannot be done with strips. Discuss their procedures.
- Homework: Division of fractions problems, including word problems.

Day 16:

- Warmup: Estimation with fractions.
- Discuss homework. Quiz.
- Have students complete the last page of their fraction book on *Dividing Fractions* and the cover.
- Homework: finish fraction book.

H. Unit summary

Day 17:

- Review of unit, including binding *Fraction Book*.

Day 18:

- Review of unit — make up some kind of game?

Day 19:

- Unit test — be sure to include some questions which ask students to explain their thinking.

C. High School Sample Unit — What's My Line?

Ms. Albert's students have been working with solving equations and graphing lines, including learning about slopes and intercepts. They just finished a statistics unit in which they learned how to make stem-and-leaf plots and boxplots. In both these units, they have been using graphing calculators to graph equations and to generate one-variable statistical information and graphs. Now Ms. Albert would like to have them work with some real data involving applications of linear equations. She has been talking with the science teacher about some possible links between mathematics and science. She thinks that this work will provide an important foundation as they begin to look at systems of equations and quadratics. She hopes to include another unit like this one, using a greater variety of types of equations, later in the year.

She begins her planning by reviewing New Jersey's *Mathematics Standards*. She is somewhat surprised to find that while the unit focuses primarily on Standard 13 (Algebra) and Standard 12 (Probability and Statistics), it can also easily include some elements of Standard 15 (Building Blocks of Calculus). She also decides that the unit should include work on Standard 1 (Problem Solving), Standard 2 (Communicating Mathematics), and Standard 5 (Tools and Technology). She adapted a number of cumulative progress indicators and developed the following list of expectations for this unit.

- Students will model and solve problems that involve varying quantities with variables, expressions, and equations.
- Students will interpret algebraic equations geometrically and describe geometric objects algebraically.
- Students will represent, analyze, and predict changes of quantities over time using linear and non-linear models.
- Students will use linear and non-linear models to explain growth and change in the natural world, such as situations involving motion, periodic phenomena, and biological and economic growth.
- Students will use curve fitting to interpolate and predict from data.
- Students will construct and use graphical and symbolic models to represent problem situations.
- Students will determine, collect, organize, and analyze data needed to solve problems.
- Students will effectively apply processes of mathematical modeling in mathematics and other areas.
- Students will use mathematical language and symbols to represent problem situations and explain what they have done.
- Students will explain their conclusions, thought processes, and strategies in problem-solving situations.
- Students will use technology to gather, analyze, and display mathematical data.

Ms. Albert continues her planning by preparing an outline of the unit. As she works, she keeps in mind the components of an effective instructional unit, especially using a variety of instructional strategies and making sure that tasks are worthwhile mathematically. She also considers assessment as she plans, incorporating a paper-and-pencil test at the end of the unit, written projects done in pairs, and homework assignments done individually most nights.

Day 1: Posing a Problem. The class begins with a problem: *Archaeologists have found a human thigh bone that is 38 cm long in a dig in New Jersey. They would like to use this bone to estimate the height of the person from which it came. How might they do this?* After some discussion, the class decides to collect data on the length of their thigh bones (as accurately as they can measure them) and their height. For homework, they each collect data from five other individuals of varying ages.

Day 2: Finding a Line Freehand. The students enter their data individually into their own graphing calculators at a prescribed location. The teacher then connects all of the calculators to assemble a complete data set. (If this cannot be done, then the class data is collected and entered into the calculator for the overhead projector.) The class decides to look at a scatterplot of the data. They decide that the data seem to fall more or less in a straight line. Using trial and error and working in pairs, the students suggest some potential equations for lines to use to describe the relationship between the length of the thigh bone (x) and the height of a person (y). For homework, each student uses a given set of data to make a scatterplot and determine a line that fits by using trial and error.

Day 3: Median-Median Line. After discussing the homework, the students learn how to find a median-median line, using their thigh bone vs. height data set. They compare the equation generated in this way with the equations they generated freehand. The teacher introduces the concept of residuals, the vertical distances between the line of best fit and each point in the data set. The students discuss the advantages and disadvantages of the freehand method and the median-median line method. For homework, they use the same data set as the previous night, finding the median-median line and the residuals for each point in the data set. (The median-median line, available on many calculators which have statistics capabilities, is found by dividing the data points on the x-y plane into three equal sets, grouped by x-value, finding a single point for each set whose coordinates are the medians of the respective coordinates of the points in the set, connecting the first and third points by a straight line, and shifting this line 1/3 of the way toward the second point. The median-median line is discussed in Contemporary Precalculus through Applications by the North Carolina School of Science and Mathematics, Janson Publications, 1991.)

Day 4: Regression Line. The teacher develops the idea of using the principle of least squares to find a line of best fit (regression line) and shows the class how to use their calculators to do this with their thigh bone vs. height data set. The class compares this equation to those generated using the freehand and median-median line methods. They discuss the advantages and disadvantages of each method. For homework, the students find the least squares regression line for the data set they have been using previously for homework.

Day 5: Unit Projects. Today the students begin working on their unit projects. Each pair of students has two problems to investigate. One problem requires them to collect data, either on the circumference and diameter of round objects, the length of a person's foot versus her/his height, the height of a candle as it burns, or a simulation of decay in which the students monitor the number of M&Ms that remain after each trial, involving spilling the M&Ms from a cup, when those that land M–side up are removed. The other problem provides a data set for the students: temperature and pressure data collected by chemistry students, cricket chirps per minute and temperature, the relationship between fouls committed in basketball and points scored, or the relationship between winning times in the Olympics for the 100–yard dash and the year. For each problem, the students must prepare a scatterplot and determine whether or not a linear relationship exists. If so, then they must use each of the three methods (freehand, median–median line, and linear regression) to find the equation of a line of best fit. They must then prepare a report explaining what they have done and why and discussing which line they feel best describes the data set and why.

Days 6-7: Unit Projects. Students continue working on their projects.

Day 8: Discussion and Review. Students hand in their completed projects and discuss their results. They also review for a test the following day.

Day 9: Test. The students take a paper-and-pencil test covering the concepts studied in this unit as well as the computational techniques used.

LEARNING ENVIRONMENT STANDARD 18
ASSESSMENT

> All students will be evaluated using a diversity of assessment tools and strategies to provide multiple indicators of the quality of every student's mathematical learning and of overall program effectiveness.

(This "learning environment standard" was developed and approved by the task force that prepared the *Mathematics Standards* and appears in the Introduction to the *Mathematics Standards* chapter of the New Jersey State Department of Education's *Core Curriculum Content Standards*; however, since it was not considered a "content standard," it was not presented to the New Jersey Board of Education for adoption.)

Descriptive Statement

A variety of assessment instruments should be used to enable the teacher to monitor students' progress in understanding mathematical concepts and in developing mathematical skills. Assessment of mathematical learning should not be confined to intermittent standardized tests. The learning environment should embody the perspective that the primary function of assessment is to improve learning.

Overview

An important goal of this chapter is to broaden understanding of both the purposes and the tools of assessment. The popular conception of assessment is restricted to evaluating individual student performance by tests designed to determine, at the end of a unit of time or instruction, what the student has already learned. But assessment should also be used during the learning process to enable teachers to monitor students' understanding and to modify curriculum and instruction, as well as to assess the effectiveness of school programs. Assessment of individual student performance should be a continuous process that involves many types of assessment activity. Students should play active roles in assessment so that each assessment experience is also an educational experience.

This chapter therefore has three main sections, dealing with, **Alternative Assessment Strategies**, **The Student's Role in Assessment**, and **Educational Purposes of Assessment.**

The *Assessment Standards for School Mathematics* of the National Council of Teachers of Mathematics describes assessment as "the process of gathering evidence about a student's knowledge of, ability to use, and disposition toward mathematics, and of making inferences from that evidence for a variety of purposes." The kinds of inferences that can be drawn from that evidence are discussed in the section on **Educational Purposes of Assessment**.

However, it is important to establish at the outset the perspective that the major purpose of assessment is to promote learning. The assessment is not the goal, but a means to achieve a goal. *Measuring What Counts*, a 1993 Policy Brief of the Mathematical Science Education Board, begins as follows:

"You can't fatten a hog by weighing it." So said a farmer to a governor at a public hearing in order to explain in plain language the dilemma of educational assessment. To be useful to society, assessment must advance education, not merely record its status.

Measuring What Counts lists "three fundamental educational principles which form the foundation of all assessment that supports effective education":

The Content Principle — Assessment should reflect the mathematics that is most important for students to learn.

The Learning Principle — Assessment should enhance mathematics learning and support good instructional practice.

The Equity Principle — Assessment should support every student's opportunity to learn important mathematics.

These three principles are reflected in the first three cumulative progress indicators for this standard.

Experiences will be such that all students:

1. **Are engaged in assessment activities that function primarily to improve learning.**
2. **Are engaged in assessment activities based upon rich, challenging problems from mathematics and other disciplines.**
3. **Are engaged in assessments activities that address the content described in all New Jersey's *Mathematics Standards*.**

The Content Principle, the Learning Principle, and the Equity Principle were incorporated into the first three of the six assessment standards in the NCTM *Assessment Standards for School Mathematics*:

- Assessment should reflect the **mathematics** that all students need to know and be able to do.

  New Jersey's *Mathematics Standards* provide a vision of the mathematics that all students should know and be able to do. Assessment should match this vision.

- Assessment should enhance mathematics **learning**.

  Assessments should be learning opportunities as well as opportunities for students to demonstrate what they know and can do. Although assessment is done for a variety of reasons, its main goal is to improve students' learning and inform teachers as they make instructional decisions. As such, it should be a routine part of ongoing classroom activity rather than an interruption.

- Assessment should promote **equity**.

  Assessment should be a means of fostering growth toward high expectations rather than a filter used to deny students the opportunity to learn important mathematics. In an equitable assessment, each student has an opportunity to demonstrate her or his mathematical power; this can only be accomplished by providing multiple approaches to assessment, adaptations for bilingual and special education students, and other adaptations for students with special needs. Assessment is equitable when students have access to the same accommodations and modifications that they receive in instruction.

- Assessment should be an **open** process.

  Three aspects of assessment are involved here. First, information about the assessment process should be available to those affected by it, the students. Second, teachers should be active participants in all phases of the assessment process. Finally, the assessment process should be open to scrutiny and modification.

- Assessment should promote valid **inferences** about mathematics learning.

  A valid inference is based on evidence that is adequate and relevant. The amount and type of evidence that is needed depends upon the consequences of the inference. For example, a teacher may judge students' progress in understanding place value through informal interviews and use this information to plan future classroom activities. However, a large-scale, high-stakes assessment such as the HSPT11 requires much more evidence and a more formal analysis of that evidence.

- Assessment should be a **coherent** process.

  Three types of coherence are involved in assessment. First, the phases of assessment must fit together. Second, the assessment must match the purpose for which it is being conducted. Finally, the assessment must be aligned with the curriculum and with instruction.

These principles should be kept in mind as changes in assessment strategies are contemplated, developed, tested, and implemented. They should be kept in mind by classroom teachers and all others involved in assessment — for example, district committees selecting a standardized norm-referenced test, district supervisors or department chairs analyzing data from a collection of student portfolios, and state mathematics content development committees reviewing proposed test items for the statewide tests.

Alternative Assessment Strategies

The next cumulative progress indicator for this standard refers to a wide variety of assessment techniques that are now available to help make informed judgments and to assure continued progress.

Activities will be such that all students:

4. **Demonstrate competency through varied assessment methods including, but not limited to, individual and group tests, authentic performance tasks, portfolios, journals, interviews, seminars, and extended projects.**

Making use of a variety of assessment methods provides a more complete picture of students' learning. Some types of assessment tasks provide information about students' abilities to perform mathematical procedures. Others involve higher-level thinking and problem-solving skills, represent meaningful instructional activities, and/or invoke real-world applications. Stenmark (1991) describes some of the changes in mathematics learning that result from using these alternative assessment strategies:

Students:

- think more deeply about problems;
- feel free to do their best thinking because their ideas are valued;
- ask deeper and more frequent questions of themselves, their classmates, and their teachers;

- improve their listening skills and gain an appreciation for the role of listening in cooperative work;
- feel responsibility for their thoughts and ownership of their methods;
- observe that there are many right ways to solve a problem;
- experience the value of verbalization as a means of clarifying one's thinking;
- form new insights into mathematical concepts;
- learn ways to identify the places they need help;
- increase their self-confidence and self-esteem as a result of genuine interest shown by a teacher or classmate;
- feel more tolerance and respect for other people's ideas;
- focus their energy on exploring and communicating ideas about mathematical relationships rather than simply finding answers;
- develop strategies for conducting self-interviews while solving problems in other settings;
- find satisfaction and confidence in their ability to solve problems; and
- look less to the teacher for clues about the correctness of their methods and focus less on imitating the "right" way.

Teachers:

- gain access to student thinking;
- enhance their ability to use non-threatening questions that elicit explanations and reveal misconceptions;
- strengthen their listening skills;
- show respect for students by being non-judgmental;
- use interview results as a source of questions to pose on written assignments for the whole class;
- encourage respect for diversity by modeling appreciation of varied approaches;
- pose questions that encourage students to construct and share their own understandings;
- feel reinforcement for letting go of "teaching as telling."

A good source of samples of different types of assessment tasks is *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions* (Stenmark, 1991). Many of the examples and definitions in this section come from that source.

Individual and Group Tests

Traditionally, the dominant mode of assessment has been paper-and-pencil testing of individual students. This testing often includes both selected-response items — such as matching, multiple-choice, and true/false questions — and constructed-response items — such as problems to solve, or short-answer, fill in the blank, or "show your work" questions. Large-scale testing often uses primarily selected-response items, since these are easy to score. However, constructing good selected-response test items is quite difficult, so many teachers rely more on constructed-response items for classroom assessments. Some teachers check only answers, while others ask students to show their work and provide partial credit to varying degrees. More recently, individual tests have also begun to include open-ended questions ("solve and explain your solution"), such as those found on New Jersey's Eighth-Grade Early Warning Test (EWT) and Eleventh-Grade High School Proficiency Test (HSPT), since these provide more insight into student thinking. The following are some suggestions for creating and/or selecting open-ended questions, as well as for lightening the burden created by having students write.

✓	*SUGGESTIONS*

✓ The questions should be constructed so that they cannot be answered by simple multiple-choice responses. They should address the ability of the student to form and communicate mathematical ideas and arguments, see and make connections between the various content strands of mathematics, make conjectures, justify results, organize and analyze data, and make estimates and predictions based on incomplete data or patterns of events.

✓ When students have been using an arithmetic operation or algebraic procedure, ask them to explain, in writing, or with a diagram, what that operation or procedure means and how it works.

✓ The New Jersey Department of Education's *Mathematics Instructional Guide: Linking Classroom Experiences to Current Statewide Assessments* provides many examples of open-ended questions for the 7-12 grade levels. Problems used in other states' assessment programs can be found in the National Council of Supervisors of Mathematics' *Great Tasks and More!!, A Source Book of Camera-Ready Resources on Mathematics Assessment.*

✓ Ask students to explain how they got their answers. This works quite well not only for textbook word problems but also for mental math and estimation problems.

✓ Sample student papers randomly. Select a few papers or a few questions on each paper each day to review. Be selective about what is commented on; choose one or two aspects for evaluation or scoring and detailed feedback.

✓ Teach students to assess each others' work. Review sample student answers with the class, asking students to suggest improvements. Share scoring rubrics with the students.

✓ Evaluation of open-ended questions can be done in several ways. For classroom tasks, the "piles" method is often appropriate and efficient; put each paper as it is read in one of three (or more) piles so that comparable papers are in the same pile. Labels or points can be assigned to the piles after all of the papers are sorted. For assessments with more substantial consequences, it is important to ensure the reliability of the scoring (two different graders will give the same paper the same score); thus, scoring rubrics have been developed. Some of these are quite general, such as the one on the following page that is used in New Jersey's statewide assessments; this scoring guide can be found in the New Jersey State Department of Education's *Grade 11 High School Proficiency Test: Directory of Test Specification and Items*. Other scoring rubrics can be found in the National Council of Supervisors of Mathematics' *Great Tasks and More!!, A Source Book of Camera-Ready Resources on Mathematics Assessment.*

The General Scoring Guide

Student Demonstrates Proficiency — Score Point = 3

The student provides a satisfactory response with explanations which are plausible, reasonably clear, and reasonably correct, e.g., includes appropriate diagram(s), uses appropriate symbols or language to communicate effectively, exhibits an understanding of the "mathematics" of the problem, uses appropriate processes and/or descriptions to answer the question, and presents sensible supporting arguments. Any flaws in the response are minor.

Student Demonstrates Minimal Proficiency — Score Point = 2

The student provides a "nearly satisfactory" response which contains some flaws, e.g., begins to answer the question correctly but fails to answer all of its parts or omits appropriate explanation, draws diagram(s) with minor flaws, makes some errors in computation, misuses mathematical language, or uses inappropriate strategies to answer the question under consideration.

Student Demonstrates a Lack of Proficiency — Score Point = 1

The student provides a less than satisfactory response that only begins to answer the question, but fails to answer it completely, e.g., provides little or no appropriate explanation, draws diagram(s) which are unclear, exhibits little or no understanding of the question being asked, or makes major computational errors.

Student Demonstrates No Proficiency — Score Point = 0

The student provides an unsatisfactory response that answers the question inappropriately, e.g., uses algorithms which do not reflect any understanding of the question, makes drawings which are inappropriate to the question, provides a copy of the question without an appropriate answer, fails to provide any information which is appropriate to the question, or fails to attempt to answer the question. (New Jersey State Department of Education)

✓ Develop a specific scoring rubric for the item that you are using. This initial analysis of your item will help you to consider exactly what a "good" response should include. Remember, however, that students may come up with an approach that you have not previously considered, so be ready to adapt your rubric as necessary. Sample scoring rubrics for individual problems can be found in the publications cited above from the New Jersey State Department of Education and the National Council of Supervisors of Mathematics.

✓ Use an analytic scoring rubric for problem-solving tasks that rates students on three phases of problem-solving: understanding the problem, developing a solution plan, and finding the answer. The rubric in the following table assigns three scores for a problem, one for each phase. Each of the three can vary from 0 to 2. The advantage of this type of rubric is that it provides more specific information about where students are having difficulty than does a general scoring rubric.

Analytic Scoring Guide	
Understanding the Problem	0: Complete misunderstanding of the problem 1: Part of the problem misunderstood or misinterpreted 2: Complete understanding of the problem
Planning the Solution	0: No attempt, or totally inappropriate plan 1: Partially correct plan based on part of the problem being interpreted correctly 2: Plan could have led to a correct solution if implemented properly
Getting an Answer	0: No answer, or wrong answer based on an inappropriate plan 1: Copying error; computational error; partial answer for a problem with multiple answers 2: Correct answer and correct label for the answer

Open-ended questions and scoring rubrics can be used for group as well as individual tests. Such tests are particularly appropriate for assessing problem solving, reasoning, communication, and use of tools and technology. These tests may involve open-ended questions that must be answered with a single written response from the group (a paragraph, a letter, a graph, or a poster), or they may involve observing the group as it works for indications of appropriate understanding and use of the mathematical concepts and techniques. Some teachers like to combine both methods, observing students working together in a group and then having them write up independent responses based on their group work. Students may be assigned to groups randomly (e.g., by drawing a card) or they may be tested in the same groups in which they have been working.

Authentic Performance Tasks

An authentic performance assessment starts with asking a student or group of students to engage in some mathematical task or investigation. Observations of the students as they work, questioning of the processes they use to solve the task, and examination of their final results are all standard activities that create useful data about what the students know and are able to do. Great care must be taken to be sure that the task is rich and motivating enough to produce sustained effort on the part of the students. Tasks which are boring or too simple will not be very good performance assessment tasks.

How Many Buttons?

Ms. Allen's fourth-graders had just finished a unit on numeration that emphasized estimation. As a review, she had them estimate the number of buttons *in their class* on that day, collect and analyze data, and share their results with the class. As a performance task, she had them work in pairs the next day to estimate the number of buttons *in the school* on that day.

Ms. Allen has developed what she calls a protorubric — the beginning of a scoring rubric — which she will refine as she reviews the students' papers.

Good response	Considers work from day before, estimated number of children in each class, number of classes in school, age variations in number of buttons worn, and adults' buttons. Gives reasonable justifications for numbers used. Uses appropriate arithmetic processes. Shows steps used to solve problem.
Average response	Considers work from day before, estimated number of children in each class, number of classes in school, but misses age variations and/or adults' buttons. Gives reasonable justifications for numbers used. Uses appropriate arithmetic processes. Shows some but not all steps used to solve problem.
Poor response	Refers to previous day's work. Some other values chosen reasonably, but some chosen inappropriately or arithmetic processes used are incorrect. Explanation of steps is incomplete or unclear.

(Adapted from *Measuring Up*, 1993, pp. 95-100)

The above task is adapted from *Measuring Up: Prototypes for Mathematic Assessment* which provides a number of assessment activities for the middle grades. The following sample performance tasks are adapted from Stenmark (1991, pp. 14-22):

- Ask a group of third graders who are learning about fractions to show you with manipulatives how they would divide different items, such as 5 candy bars, 10 pencils, or 11 comic books, among 4 students.
- Have students explain how they would teach a younger sibling to understand the meaning of tens and ones in place value.
- Give a group of high school students the task of finding the function that "best fits" certain data by using computer software and a printout of their results.
- Ask students, "How many bicycles are there within two miles of this school?" Have them make a plan for investigating the question and prepare an oral report, with overheads or other displays, for the class.
- Ask a group of seventh graders, "If you measured an object with five different rulers and got five different answers, how would you decide which answer was correct?"

Sample Performance Task— South Brunswick

Teachers in South Brunswick have been working on a plan to use at least one common performance assessment task across all sections of each grade level in the district. Some of the tasks they've settled on for the first year of the program are:

- kindergarten students sort a group of fifteen stickers into smaller groups and make up a rule that explains the way they have sorted them.
- second graders use toothpicks to make 3 different geometric shapes that each have a perimeter of 8 toothpicks and then discuss some of the properties of the shapes they have made.
- fifth graders plan a class party, paying attention to the creation of a budget, the number of people attending the party, and real-world costs of entertainment, food, and drinks, and then write a letter to their teacher explaining the choices they've made.

Designing appropriate performance tasks is not a simple procedure. Many guidelines must be considered and the tasks must be tested with real students before they can be used on a wide scale. A useful set of criteria for the tasks is presented in the following table, attributed in NCTM's *Mathematics Assessment* to Leinwand and Wiggins:

Criteria For Performance Tasks

A good task is:	rather than:	meaning that:
Essential	Tangential	- The task fits into the core of the curriculum. - It represents a "big idea."
Authentic	Contrived	- It uses processes appropriate to the discipline. - Students value the outcome of the task.
Rich	Superficial	- The task leads to other problems. - It raises other questions. - It has many possibilities.
Engaging	Uninteresting	- The task is thought-provoking. - It fosters persistence.
Active	Passive	- The student is the worker and decision-maker. - Students interact with other students. - They construct meaning and deepen understanding.
Feasible	Infeasible	- The task can be done within the allotted time. - It is developmentally appropriate for students.
Equitable	Inequitable	- The task develops thinking in a variety of styles.
Open	Closed	- It contributes to positive attitudes. - The task has more than one right answer. - It has multiple avenues of approach, making it accessible to all students.

Portfolios

A portfolio is a showcase of student work; it is a place where students can demonstrate their mathematical power in specific and general ways. Student thinking, growth over time, mathematical connections, students' views of themselves as mathematicians, and the problem-solving process are each emphasized in creating, maintaining, revising, and assessing student portfolios. In addition, teachers should be working with students to regularly review their portfolios in order to establish short-term and long-term goals.

What's in a portfolio? Many of the following kinds of items may be included in portfolios. Of course, no one student would be asked to present all of these:

- table of contents
- introductory and self-assessment letters
- long-term projects
- daily notes
- journal entries
- excerpts from dialogue notebooks
- test problems
- physical models of mathematical concepts
- mathematical models of real world phenomena
- interviews between student and peer
- interviews between student and teacher
- art work done by the student
- a mathematical autobiography
- audio and video tapes of work in progress and/or finished products
- scale drawings
- photographs
- homework
- peer critiques and evaluations
- commentary from parents about portfolio contents
- self-generated problems and solutions
- papers showing student's corrections of errors and misconceptions
- teacher observations of student
- group projects
- excerpts from team notebooks

Students can be informed about their specific portfolio assignment through discussion of the following sample hand-out:

Middle School Portfolio Assignment

In this unit, you will be building a math portfolio that demonstrates your understanding of the content. Your portfolio will include several different kinds of work.

1. Cover	Include an illustration of one or more of the concepts studied in this unit.
2. Table of Contents	List what is in the portfolio.
2. Self-evaluation	What did you learn? What do you understand well? What do you still need to work on?
3. Key ideas	Select 3-5 key concepts from this unit, write about what they are and why they are important, and include two examples for each.
4. Work that needs improvement	Explain what you did not understand originally and revise the work.
5. Best work	Explain what you learned and why you are proud of it.
6. Favorite activity or assignment	Explain what you learned and why it is your favorite.
7. Creative piece	Create a story, poem, or picture that illustrates one or more concepts from the unit.

Evaluation of portfolios may be accomplished in much the same way as performance tasks or open-ended tasks. Vermont uses a portfolio rating system for grades 4 and 8 in which seven pieces of work are rated on a four-point scale for understanding of the task, quality of approaches/procedures, decisions along the way, outcomes of activities, use of mathematical language, use of mathematical representations (graphs, tables, diagrams, manipulatives, etc.), and clarity of presentation. Portfolios are also used to provide a picture of the instructional opportunities, the content areas of programs, and anecdotal indicators of disposition towards mathematics. Most classroom teachers do not use such an elaborate rubric, preferring to use a more holistic approach, such as the following:

Level 4 (top level)	Exciting! Includes a variety of work, with evidence of use of many different resources. Papers show understanding of content, organization and analysis of information, clarity of communication, enthusiasm for math. Includes self-assessment.
Level 3	Variety of work. Fairly good explanations, with some use of resources. Good understanding of basic mathematics processes. Missing indications of enthusiasm, self-assessment, extensive investigations, and/or student analysis of information.
Level 2	Little evidence of original thinking. Minimal student explanations. Over-concentration on low-level tasks, such as computation.
Level 1	Almost no creative work. Mostly ditto sheets or textbook problems. Almost no evidence of student thinking. No evidence of discussion of mathematical ideas. No explanations.

(Adapted from Stenmark, 1991, p. 44)

Managing the development and review of portfolios is a time-consuming process. Following are some suggestions to help streamline that process.

✓	*SUGGESTIONS*

✓ To demonstrate students' growth over time, pose similar questions periodically throughout the year. These questions should explore the same content in different situations.

✓ Include observation checklists, interview notes, and parent communications regularly.

✓ Keep two portfolios for each student, a work portfolio and an assessment portfolio. Have younger students keep work portfolios in a plastic crate or box to pick up as they enter the room; older students may use a notebook. Include journals in portfolios and put copies of group work in the folders of each group member. Every two to six weeks, have students review their work portfolios. They should make a list of all the work they have completed, make revisions if they wish, review their journals, and circle or label pieces that they feel reflect growth or understanding. Ask them to select work to put in their assessment portfolio, writing a paragraph about each item selected.

Journals

Another assessment strategy being used by New Jersey teachers is the *Math Journal*, a small booklet or notebook section in which students write on a regular basis. A journal entry can be a regular daily part of the math class or something just done occasionally. It can be very structured, with the students responding to a *writing prompt* from the teacher, or very free, with the students choosing their own math topic and question to write about. The journal is not graded, since teachers want students to write freely in it. The consistent aspect of the journal is that it provides a means of communication between the student and the teacher. It is a vehicle by which the student can express understanding, attitudes, delight, creativity, and a myriad of other responses. Students frequently have a hard time creating journal entries when they first begin to write them, especially if they are already well-advanced in school. Some sample prompts, from a collection being created in South Brunswick, are shown below. They can help children feel more comfortable and more focused in constructing an entry.

Sample Journal Prompts

- How would you teach someone to add two-digit numbers?
- What is one area of math you feel confident in? Why?
- Judy was absent today. What will you say to her if she calls you tonight and asks what we did in class?
- Write a story problem using this number sentence: 65 - 32 = 33.
- Write everything you know about the fraction 1/2.
- Make up a rhyme or a funny sentence to help you remember the algebraic order of operations.
- The most important thing I learned in math this week was ... .
- Make up a number pattern and tell about it.
- Describe what is going on in your mind mathematically right now by completing one of these statements: I think ..., I feel ..., I know ..., I wonder ..., I guess ... .

Teachers who have students write in their journals are, of course, obligated to read and respond frequently to the entries. But they will find many of the students' responses refreshing and original and the process will be one of the most pleasant of their tasks. To reduce the amount of time spent reading journals, some teachers have students box or underline portions which they would like the teacher to read. Opinions vary on whether or not to give comments, especially ones that suggest corrections or a need for improvement; some teachers like to put comments on post-it notes so as to leave students work unmarked. Journals frequently are good sources of information about student's attitudes and misconceptions as well as emerging classroom problems. Teachers can often find out more about the students' thinking and degree of understanding from journals than from most other methods of assessment.

Observations, Interviews, and Conferences

Observations, interviews, conferences, and questions provide teachers and students with numerous opportunities to assess progress. At the heart of these types of assessment is the student and teacher working in collaboration in order to ascertain the progress the student has made. A student can analyze her or his own mathematical work by focusing on the quality of the product, the need for revision, changes or additions to be made, and different problem-solving strategies to employ. Performing an effective, critical analysis of one's work is a process that students can adapt for use later in life in their careers.

✓	*SUGGESTIONS*

✓ Write observations on index cards or on a sheet of address labels that can be transferred to student portfolios later. Some teachers print up labels with names corresponding to the seating chart so that it is easy to locate students.

✓ Give the same problem to the whole class but choose one or two groups to observe closely.

✓ Use a checklist of observation details for each group of students. Rate each group (or simply check if you see the behavior) on such items as whether all members are participating, whether they are estimating before computing answers, whether they are using more than one process, whether they are recording their work, whether they are reporting their work, and whether they evaluate themselves.

Seminars, Labs, and Extended Projects

Assessment may also include student presentations (seminars), labs, and extended projects, particularly in the higher grades. These methods assure the connection between instruction and assessment as well as provide opportunities for relating mathematics to other subject areas. These methods are especially appropriate for assessing students' ability to identify and define a problem and what they already know; to make a plan; to create and modify problem-solving strategies; to collect, organize, and analyze needed information; to discuss, review, revise, and explain results; to persist in work on a problem; and to produce a quality product or report. Possible topics for these types of assessment might include maps, gears and ratios, sound waves and music, traffic patterns, collecting and analyzing litter, studying local water use, census studies, lunch preferences, sports statistics, plant growth, measurement of parts of the body, or nutrition.

The following are a sample of projects assigned to high school students at Glassboro High School. All projects involve a final presentation to the class, either in person or on videotape. Sophomores must complete three of the following:

Plan a Day Trip to an Amusement Park
Redecorate This Classroom
Origami/Tangrams
Logic Puzzles
Conduct and Evaluate a Survey
Menu Planning
Chores
Consumer Reports

More specific detail is provided for each choice. For example, for *Consumer Reports*, students are told:

> Research an item you are considering purchasing in the near future by reviewing the ratings given all makes of this product by a consumer periodical such as *Consumer Reports*. Demonstrate to the class how to read the charts and what each symbol means. Research the actual cost locally of the three highest-rated makes. Choose one and justify your choice.

Seniors must complete one of the following projects, providing status reports at two points.

- Start Your Own Business (cost projections, loan application, graph of costs and income)
- Buy a Home (financing options, loan application, taxes)
- Design Your Dream Home (scale model, costs)
- Develop a Financial Plan for College or Other Post-Secondary Education (compare costs at four institutions, financial aid application)
- Plan a Community Service Project (survey, cost projections, grant application)
- Simulate Applying for and Getting Your First Post-HS Job and Plan Your Budget for the First Year at this Job
- Plan a Vacation for a Group of Five People for Two Weeks

Math In The News

Susan Simon of Morris Knolls High School gives students in a high school discrete math course an article from a newspaper or magazine on a mathematical topic (such as Andrew Wiles' proof of Fermat's Theorem, DNA, or the Traveling Salesman Problem), and asks them to find two other sources on that topic, write a 3-page paper, and present the information to the class.

Creating opportunities on a regular basis to observe, interview, confer, and question students while at work on their products is an essential component. Strategies such as think alouds, planned conferring periods, dialogue notebooks, and team notebooks afford both teachers and students with the opportunity to investigate student thinking.

The Student's Role in Assessment

The remaining cumulative progress indicators addresses an area that gets far too little attention – the students' role in assessment. The conventional perspective is that the student plays a passive role, simply answering questions or performing activities so that the instructor can evaluate his or her work. This standard calls for the student to play a more active role in assessment – to engage in self-assessment, to better communicate mathematical understanding, knowledge, and attitudes, and to use assessments as opportunities for reflection and growth.

5. **Engage in ongoing assessment of their work to determine the effectiveness of their strategies and the correctness of their results.**

6. **Understand and accept that the criteria used to evaluate their performance will be based on high expectations.**

7. **Recognize errors as part of the learning process and use them as opportunities for mathematical growth.**

8. **Select and use appropriate tools effectively during assessment activities.**

9. **Reflect upon and communicate their mathematical understanding, knowledge, and attitudes.**

"If students are to function as independent learners, they must reflect on their progress, understand what they know and can do, be confident in their learning, and ascertain what they have yet to learn" (NCTM *Assessment Standards for School Mathematics*, p. 14). Clearly, the student who wrote the following journal entry is an independent learner who can reflect on work in progress, knows what she is able to do, has confidence in what she does, and has the ability and practice to determine what she still needs to learn.

From a Sophomore Student's Mathematics Journal Entry

In the past whenever we were evaluated in class, it was a day where no learning seemed to take place. I'd spend the night before looking over the problems we had been solving in class since the last test. I'd prepare as best I could by looking to see if there were some patterns to the problems. I'd come to class not really sure of what I knew and just hoping I'd pass.

This year, I have trouble telling when I'm being "tested!" It all seems so connected. I've never been asked to create ways to test what I know and how I know. By keeping a dialogue notebook with my partner Sarah and my teacher, Mr. Cray, where I discuss what and how I'm learning, I've made some important discoveries. For example, one thing I've discovered about how I learn is that it helps if I can see what the written problem looks like. At first I was kind of intimidated by the graphing calculator. For example, I really had a lot of trouble understanding what Mr. Cray meant by the y-intercept. It wasn't until he had us graph $y=x$, $y=x+5$, and $y=x-2$, that I could see it. It was so clear! By using the calculator I can construct a lot more graphs to observe than I can by hand. As Mr. Cray always says, "When you find something that works for you, put it in your tool box." Making sense through pictures works for me. I think that the more I use pictures to help me learn new concepts, the less I will need to use pictures in the future. I can already see that I don't need to see certain graphs anymore, because I can picture them accurately in my head.

I also appreciate being able to demonstrate my knowledge, even when I'm really unsure of what I know. I find that by talking with Sarah, or other members of my team, I can begin to discover what I'm learning.

Seamless, connected instruction and assessment help to ensure learning and to make it more authentic, more real. In this journal entry, the student is responsible for self-assessment. She is able to be reflective because a system is in place to afford her the opportunity to practice regular inquiry. Through the use of a dialogue notebook, she is able to "discuss" with a partner and the teacher what she is learning and how she is learning. Important discoveries are made when students are afforded both time and strategies to delve into big questions such as: *How do I make sense? What do I do when I don't know or understand? What tools do I have and use to help me make sense? How can I repeat success?* Although not specific to mathematics, such questions form the basis of all learning. Knowing how one thinks and what one does when challenged with unfamiliar tasks or an unfamiliar method of inquiry is critical for all mathematics students.

✓	*SUGGESTIONS*

✓ For problem-solving activities, have students complete a self-assessment form such as the following one from Stenmark (1991, p. 58).

Problem-Solving Strategy Inventory

Think about your use of strategies when solving the problem and check the following that apply.

1. ____ I didn't think about using strategies at all.
2. ____ The idea of using strategies came to my mind, but I didn't think about it much more.
3. ____ I looked at a strategy list, but didn't try a strategy.
4. ____ I looked at a strategy list and picked a strategy, which I tried.
5. ____ I didn't look at a list, but just thought of a strategy to try.
6. ____ I used at least one strategy and it helped me find a solution.
7. I tried the following strategies:

_____ guess and check _____ solve a simpler problem _____ make a table
_____ work backward _____ look for a pattern _____ draw a picture
_____ write an equation _____ make an organized list
_____ other ____

✓ Use bookmarks for a weekly self-grading system. Have students give themselves a grade each day for classwork and for homework on one side of the card. On the other side, have them fill in responses to questions such as the following:

The thing I enjoyed most in math this week was ____________.

I am most proud of ______________________.

One thing I didn't like was ________________.

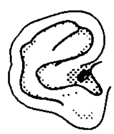
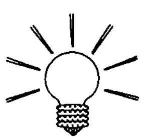
I wish ______________________________.

Collect the bookmarks on Fridays and distribute new ones on Mondays. Monitor responses, looking for patterns of learning for students. Give students feedback individually, perhaps through interviews.

✓ Have students assess their work as a group during cooperative learning activities. You may wish to develop a form similar to the one shown on the next page.

Group Assessment

Names: ____

Did your group:	☺	😐	☹
1. listen? 			
2. talk about the task? 			
3. cooperate? 			
4. suggest good ideas? 			
5. finish the task? THE END			
What went well?			
What do you wish you had done in a different way?			

Educational Purposes of Assessment

The NCTM *Assessment Standards for School Mathematics* presents four categories of educational purposes for assessment: making instructional decisions, monitoring students' progress, evaluating individual students' achievement, and evaluating programs. These four categories are discussed in separate sections below.

A particular assessment task or instrument may well serve more than one of these purposes, but would be unlikely to be able to serve them all. An individual, teacher-administered, student interview designed to probe a third grader's understanding of place value and number base ideas, for example, could certainly help to make instructional decisions and to evaluate individual student achievement, but would be unlikely to be able to help evaluate the mathematics program as a whole. A nationally-normed, standardized test, on the other hand, **if consistent with these standards and vision of mathematics education**, can certainly help to evaluate the effectiveness of the educational program, but would be of limited value in making instructional decisions about individual students.

Making Instructional Decisions

Teachers gather assessment data continuously in order to ascertain how effective their teaching has been and to make changes in what and how they teach. Through observation, questioning, and student products, teachers are able to judge the effectiveness of their instruction. The judgments then form the basis for decisions about immediate and future instructional activity. Long-range, short-range, and moment-to-moment planning are all interconnected. The teacher who creates a long-range plan that spans a year includes specific benchmarks that can be used at intervals to mark students' progress. In addition, the moment-to-moment instructional decisions that are made each day in the classroom are influenced by knowledge of the long-range and short-range plans. More formal types of assessments are usually used for longer-range decisions, however, and more informal strategies for shorter-range decisions.

Long-range planning involves using assessment to help make decisions about a full year or semester of instruction. A critical component in these assessments is the determination of the most important mathematics to be learned during the period. Assessments are then tailored to create the data that best reveal students' understanding of this mathematics. Many teachers are beginning to use portfolios of student work to inform their long-range planning decisions. Others are using performance tasks that tend to require student integration of the content covered during a particular period of time.

Short-range planning involves using assessments to influence the students' educational activity throughout a unit. Teachers review their long-range plans and the previous assessments in order to create a unit of study that will call on students' background knowledge to help them construct new knowledge. In an effort to gauge how well students are acquiring this new knowledge, teachers establish benchmarks within the unit of study. These benchmarks are assessments that help the teacher monitor student progress. Often-used strategies for making these kinds of instructional decisions are think-alouds, journal entries, student interviews, and short pencil-and-paper instruments.

Using assessments to influence moment-to-moment decisions about instruction involves observing and listening to students while they are engaged in ordinary classroom activity. Such assessments happen continuously in most classrooms. Often teachers need to probe in order to truly understand what sense their students are creating. As a result, they frequently revise their plans based on the new information they have gleaned. Central to moment-to-moment planning is the recognition that a long-range plan and specific benchmarks are clearly in place. Knowledge of both influences the daily decision making. The following vignette describes a teacher who finds think-alouds useful for daily decision-making in a second-grade class.

Think-Alouds

Mrs. Seeliger, a second grade teacher, uses the technique of a think-aloud with her students in order to learn how they problem solve. "In order to see how my students are understanding new concepts, constant assessment is required. At this age, my students are changing rapidly and how they understood something yesterday is no guarantee as to how they'll respond today. As a result, I use think-alouds frequently. The children love to be verbal, so talking aloud while they think through a problem is something they enjoy. This technique helps me to hear what the children are thinking and to hear how they are problem solving. Such insights help me to see what changes I need to make in order to teach better."

These moment-to-moment decisions come most effectively as a result of rich, fruitful classroom discussion. The more students are actively engaged in the exploration of a particular topic, the more evident will be their level of understanding and the more natural it will be for the teacher to slightly alter course. The NCTM *Professional Standards for Teaching Mathematics* (1991, p. 35) provide direction in how to promote this kind of productive discussion in the classroom. In discussing the teacher's role in discourse, the following suggestions are made:

✓	*SUGGESTIONS*

✓ Pose questions and tasks that elicit, engage, and challenge each student's thinking.

✓ Listen carefully to students' ideas.

✓ Ask students to clarify and justify their ideas orally and in writing.

✓ Decide what to pursue in depth from among the ideas that students bring up during a discussion.

✓ Decide when and how to attach mathematical notation and language to students' ideas.

✓ Decide when to provide information, when to clarify an issue, when to model, when to lead, and when to let a student struggle with a difficulty.

✓ Monitor students' participation in discussions and decide when and how to encourage each student to participate.

Monitoring Student Progress

The process of using assessments to monitor student progress is similar to that for making instructional decisions in that long-range, short-range, and daily components need to be considered. However, the emphases of the two processes are different. When using assessments to plan instruction, the emphasis is on examining data in order to create units of study. When monitoring student performance, the focus is on understanding what sense students are making of the mathematics. Monitoring student progress successfully requires good communication between the students and the teacher. The following vignette demonstrates one strategy for monitoring student progress throughout a unit.

Well, What Do You Know?

At the beginning of a new unit of study, I have students list in their notebooks everything they know and want to know about the general topic we will be studying. Next, the children share aloud what they know and want to know, as I record their findings on an overhead transparency. As a class we then take an inventory of what we know collectively. I ask students who volunteered pieces of information to share their knowledge with the class by elaborating on what they said. This process helps to build knowledge for those students who did not know the particular pieces of information. In addition, the children then clarify those items they want to know about. I include their self-generated list of information and ideas they wish to explore in my unit plan. This process of naming what we know and what we want to know better prepares us to learn any new information. Throughout the unit, I invite the students to return to their list of what they know and want to know in order to add additional information about what they now know and to clarify what they still want to know. This process helps me to make changes in the content I teach and especially in the sequence of activities I design.

Evaluating Student Achievement

Teachers use a variety of assessment methods to gather evidence about students' learning and then provide feedback to the students about their progress. Students need to understand clearly not only what is expected but also whether their work is of acceptable quality. "For assessment to be equitable and valid, each student must received feedback over time on multiple occasions and in multiple formats on tasks that address the breadth of important mathematical content." (NCTM *Assessment Standards for School Mathematics*, p. 34). While feedback may be oral or written, formal or informal, private or public, geared toward an individual or a group, it should be always be descriptive, specific, relevant, timely, and encouraging.

Reports of student achievement indicate a student's demonstrated mathematical accomplishments at a specific moment in time. These reports may be grades based on a teacher's judgments about the student's understanding, scores from exams, checklists of competencies mastered by the student, or narrative reports of student progress. In each case, the purpose is to compare the student's progress with the goals he or she was expected to have achieved.

Evaluations of student achievement have three important characteristics, regardless of their form. First, they are summative and cannot be adjusted or changed. Second, they are designed to inform audiences beyond the classroom walls about the performance of individual students or groups of students. Finally, these evaluations are often used to make important decisions for students, such as admission, placement, or certification.

Grades should reflect each student's level of mathematical understanding rather than reflecting their comparative performance. They should be based on evidence that accurately reflects the progress of each student toward the attainment of mathematical power. They should indicate understanding of important mathematical content, and they should be derived from learning situations in which students are actively engaged. When assessment is based on multiple sources of evidence, a single letter or number grade cannot adequately represent the breadth and depth of information about what students know, what they can do, and their disposition toward mathematics. Thus, these grades must be used with caution and with an awareness that much information is lost during the process.

The NCTM *Assessment Standards for School Mathematics* describes a comprehensive performance assessment system designed to document student achievement by using checklists, portfolios, and summary reports. The checklist provides information about student progress in mastering specific content and is completed based on a variety of assessment tasks, including observation. The portfolio is compiled by the student to demonstrate their understanding. The Summary Report provides information to parents and others outside the classroom; in it, the teacher summarizes student achievement by providing ratings for the checklist, portfolio, and progress.

SUMMARY REPORT OF MATHEMATICAL THINKING						
	Checklist		Portfolio		Progress	
	Developing as expected	Needs develop-ment	Developing as expected	Needs develop-ment	Developing as expected	Needs develop-ment
Approach to mathematical thinking						
Patterns & relationships						
Number concepts & operations						
Geometry & spatial relations						
Measurement						
Probability & statistics						

Evaluating Programs

A program evaluation uses information about student performance, along with other evidence, to judge the quality and success of the instructional program. Data on student achievement may be used for making modifications to a program or for making decisions about continuing a program. However, in addition to student achievement data, program evaluations should take into consideration information about goals, curriculum materials, instructional methods, students' opportunity to learn, and the responsibilities of teachers and administrators. This section of this chapter focuses only on the use of student achievement data in program evaluation.

A major distinction between the use of assessment data for program evaluation and for other purposes is that decisions can be made using results from groups of students. Statistical methods such as matrix sampling make it possible to expand the scope and types of tasks that can be administered to students, since not all tasks need be completed by all students. Short-answer or multiple-choice tests can elicit some information on skills if the balance among topics is appropriate for the intended purpose. Performance assessment tasks can provide other types of information, especially concerning problem solving, reasoning, communication, mathematical connections, and use of tools and technology.

It is important to recognize that the relevant information an assessment can provide is limited by the choice of question formats. For example, multiple-choice questions are poorly suited to furnishing information about problem solving and mathematical communication. Similarly, using a performance task to ascertain whether students know their addition facts is inefficient. Any single form of assessment often limits the scope of what is being tested to what fits that format. It is important to incorporate a variety of assessment methods into any program evaluation to ensure that the full spectrum of mathematical content is addressed.

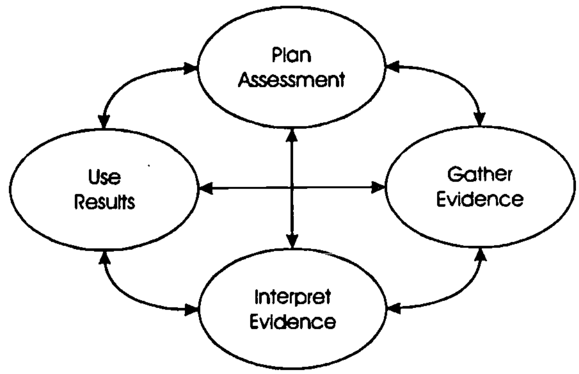
Many large-scale tests unfortunately report too little information to be useful in evaluating programs. For example, at the district level, only the average percentile ranking for students in mathematics may be available. This information does not indicate the range or distribution of student scores – for example, are scores low because of a few very low-achieving students or because most students are achieving at slightly below-average levels? Nor does it provide information about the specific content areas in which the instructional program may have strengths or weaknesses. It may be necessary to provide more detailed information in order to determine, for example, whether an instructional program is adequately serving both the gifted and the low-achieving student.

New Jersey's *Mathematics Standards* can be used as yardsticks, as measurement tools, against which school mathematics programs can and should be measured. Data should be collected to enable reviewers to determine the extent to which each of the standards, each of the recommendations of the *Framework*, and each other local objective, is being met.

The Assessment Process

The material in this section is taken from the NCTM *Assessment Standards for School Mathematics*.

"The assessment process can be thought of as four interrelated phases that highlight points at which critical decisions need to be made. The figure below shows the four phases — plan the assessment, gather evidence, interpret the evidence, and use the results. The division is arbitrary however, and makes the process seem more orderly that it actually is. In practice the phases are interactive, and the distinctions between them are blurred. Assessment does not proceed through them in a neat, linear fashion.



"Each phase of the assessment process can be characterized by the decisions and actions that occur with that phase as follows:

Plan Assessment

- What purpose does the assessment serve?
- What framework is used to give focus and balance to the activities?
- What methods are used for gathering and interpreting evidence?
- What criteria are used for judging performance on activities?
- What formats are used for summarizing judgments and reporting results?

Gather Evidence

- How are activities and tasks created or selected?
- How are procedures selected for engaging students in the activities?
- How are methods for creating and preserving evidence of the performances to be judged?

Interpret Evidence

- How is the quality of the evidence determined?
- How is an understanding of the performances to be inferred from the evidence?
- What specific criteria are applied to judge the performance?
- Have the criteria been applied appropriately?
- How will the judgments be summarized as results?

Use Results

- How will the results be reported?
- How should inferences from the results be made?
- What action will be taken based on the inferences?
- How can it be ensured that these results will be incorporated in subsequent instruction and assessment?

"The phrase *assessment process* ... emphasizes the complex process that underlies the purposes for which assessments are done, the decisions made by the assessors, and the standards to which assessments are held. This vision of assessment should apply to any assessment purpose."

SUMMARY

As New Jersey*'s Mathematics Standards* are implemented in classrooms throughout the state, assessment changes will also be implemented, corresponding to the curricular and instructional choices that are made. Students will experience a wide variety of assessment methods and will become more informed and active participants in the assessment process. The goals and purposes of assessment will be considered as assessments are constructed, implemented, and revised, and as the results of assessment are used in making instructional decisions, monitoring student progress, evaluating student achievement, and evaluating programs. Each of the changes outlined in this chapter represents a shift along a continuum toward improved practice. These shifts are summarized in the following table, highlighting important ideas involving assessment.

Major Shifts in Assessment Practice

Toward ...	Away From ...
Assessing students' full mathematical power	Assessing only students' knowledge of specific facts and isolated skills
Comparing students' performance with specific performance criteria	Comparing students' performance with that of other students
Giving support to teachers and credence to their informed judgment	Designing "teacher-proof" assessment systems
Making the assessment process public, participatory, and dynamic	Making the assessment process secret, exclusive, and fixed
Developing a shared vision of what to assess and how to do it	Developing assessment by oneself
Using assessment results to ensure that all students have the opportunity to achieve their potential	Using assessment to filter and select students out of the opportunities to learn mathematics
Aligning assessment with curriculum and instruction	Treating assessment as independent of curriculum or instruction
Basing inferences on multiple sources of evidence	Basing inferences on restricted or single sources of evidence
Viewing students as active participants in the assessment process	Viewing students as the objects of assessment
Regarding assessment as continual and recursive	Regarding assessment as sporadic and conclusive
Holding all concerned with mathematics learning accountable for assessment results	Holding only a few accountable for assessment results
Communicating with students about their performance in a continuous, comprehensive manner	Simply indicating whether or not answers are correct

Using multiple and complex assessment tools (such as performance tasks, projects, writing assignments, oral demonstrations, and portfolios)	Sole reliance on answers to brief questions on quizzes and chapter tests
Using evidence of every student's progress toward long-range goals in instructional planning	Planning for content coverage with little regard for students' progress
Making program decisions based on high-quality evidence from multiple sources	Relying on over-simplified evidence from a single test or test format

Adapted from NCTM *Assessment Standards for School Mathematics*, p. 83.

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On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

CHAPTER 19 – IMPLEMENTING A TECHNOLOGY PLAN

Introduction

This chapter has two major sections, each focusing on a specific aspect of technology. The first, *Technology: A Bridge Between Mathematics Education and Thought*, opens with a brief vision of a technology infused classroom. It is followed by a discussion of the connections between mathematics education and technology. Since technology influences how and what we teach, the types of changes we might need to make concerning content, instructional practices, and assessment are also discussed.

In the second section, *Incorporating Technology into an Existing Mathematics Program*, a sample game plan is provided. The steps necessary to add technology to a program are explored and discussed in the order in which they should take place in a well-conceived curriculum revision. Special attention is paid to taking an inventory of current technology use, creating a vision of future use, drafting a plan for the district, providing for professional development, selecting appropriate hardware and software, and creating a budget and locating funding.

Technology: A Bridge Between Mathematics Education and Thought

Overview

The mathematics education community is in general agreement that learning is enhanced by doing mathematical assignments and projects that have meaning, by employing multi-sensory stimuli that improve retention, and by working in cooperative learning groups that foster group decision making and interpersonal skills. The use of technology in schools provides a vehicle through which all of these modes of learning can be realized. In order for New Jersey's *Mathematics Standards* to be fully implemented, technology must be infused into the daily lives of mathematics students in all New Jersey classrooms.

What might a mathematics classroom that uses technology to enhance all learning look like?

A Look at the Future

On the way to school, Ms. Gomez thinks about yesterday's excitement in her sixth grade class as students completed the tasks of collecting data about the circumference and diameter of a variety of 20 cans and jars they had brought to school. She wonders if the homework assignment in which students were to draw a conclusion by writing about the relationship between the circumference and diameter of the cans was too difficult for some of the students.

Shortly after arriving at the school, Ms. Gomez checks a variety of types of messages that she has received about the assignment: e-mail, faxes, and voice messages left by students and their parents as the students were working on the assignment at home.

She thinks about the messages from parents and students and then opens up a file on her portable computer to review the day's lesson plan to see if she needs to make further adjustments.

Ms. Gomez decides to present alternative strategies that may appeal to the various types of learners who are having trouble with the assignment. Her strategies include: using a calculator to guess and check, using a computer with a geometric sketchpad to draw diagrams of circles, and using a spreadsheet or a database to extend the problem to a general solution through investigation.

In order to assess whether these new strategies have provided students with a better understanding of the problem and solution process, Ms. Gomez circulates around the room with her hand held scanner recording individual student responses as her students work in cooperative groups. She will then be able to study and analyze each student's level of concept mastery.

After students discover the relationship, Ms. Gomez uses the video, *The Story of π*, to help students visualize and connect historically with how the concept of π was developed. Students who want additional information use several History of Mathematics sites on the Internet and retrieve more information about the development of π and the formula for the circumference of a circle.

As a culminating activity, students will create presentations about π using the multi-media workstation in their classroom.

While this vignette may seem a bit futuristic, Ms. Gomez and her students use only equipment and technology that people in business and industry use in their jobs every day and which are becoming ever more present in New Jersey classrooms! Central to this vision of teachers and students using technology to learn about π is the realization that technology can help to enhance, deepen, and extend learning. Technology is not simply a collection of faddish products that are fun to use. Technology does not lead us away from inquiry, but rather enables us to think anew about how we make sense of our world. As technology shapes our reality, we, in turn, through selective and specific use, shape it.

Curricular Implications

With the infusion of technology into the schools, the educational community will need to reevaluate the content, instructional strategies, and assessment devices that are current practices in our schools. What we teach, how we teach, and the means by which we evaluate the relative success of that teaching and learning are inextricably influenced by technology. As a result, some of what we have been teaching needs to be eliminated or revised. In addition, skills and strategies we previously didn't teach now need to be stressed.

This section addresses some curricular implications that arise from the incorporation of technology into the math classroom. While the following lists are not intended to be comprehensive, the examples provided are intended to illustrate the kinds of changes that should take place in the related areas of content, instructional practices, and assessment at both the K-8 and 9-12 levels.

Suggested K-8 Curriculum Changes and Revisions

Content:

- Greatly reduce the amount of cumbersome paper-and-pencil computations such as: unreasonable column addition, large multi-digit operations with whole numbers and decimals, long division, and fractions with unrealistic denominators. Students must not, however, become dependent on calculators and computers to perform computations that can more efficiently be completed without the use of technology.

- Move away from teaching and learning which are focused on disconnected skills.
- Add content that emphasizes real-world settings and problems which call for the intelligent use of estimation, mental math, and number sense.
- Add projects and investigations that use technology to help students make connections among the different strands of mathematics as well as to content from other disciplines.

Instruction:

- Move away from rule-based mathematics, rote drill and practice, and teaching strategies that confine the teacher's roles to providing information and confirming solutions.
- Use group explorations with calculators, data bases, spreadsheets, and computer environments such as Logo. The explorations should frequently revolve around open-ended problems or projects for which there are many possible solutions.
- Use multi-media technology to address the diverse learning styles of students. Multi-media workstations can offer all students dynamic presentations of real-world applications that cannot easily be replicated in the traditional classroom setting.
- Use technological research tools, like the Internet, to enable students to collect real-world, up-to-the-minute data about mathematical or scientific issues of concern to them, and then to share their findings and conclusions with other students around the world.

Assessment:

- Avoid assessments that stress only or primarily computation.
- Use assessments that reflect the changes made in content and instruction. Be sure to provide for the use of appropriate technology when assigning group and individual projects, open-ended questions, and journal writing.

Suggested 9-12 Curriculum Changes and Revisions

Content:

- Revise courses that do not allow or encourage the use of calculators and other technology. Reduce the emphasis on procedural and symbolic manipulation skills.
- Eliminate segmented, discrete subjects and courses that treat various strands of mathematics as separate entities rather than parts of the whole. The United States is one of the few industrialized countries of the world that does not organize and teach mathematics as an integrated discipline.
- Add topics, subjects, and courses to the curriculum that reflect an integrated approach to mathematics so that computers and calculators can be utilized to enable students to see the connections between data analysis and algebra and between algebra and geometry.
- Add technology-enriched topics such as matrices, self-similarity, the iterative process, dynamic systems (chaos), probability and statistics to the curriculum for all students. These topics are both

easier to teach with the new technologies and more important for an understanding of today's mathematical world.

- Use the investigative power of computers and calculators to build intuitions about mathematics. Such topics as key sequence, scaling, zoom-in, zoom-out, domain, range and cell definition take on new and deeper meanings in a technological context.
- Use calculators and computers to help students explore and develop conceptual understanding. Graphing utilities and instructional software such as the *Geometric Supposer* or the *Geometers' Sketch Pad* enable students to visualize relationships and test ideas quickly.

Instruction:

- Greatly reduce the occurrence of *watch and do* mathematics. Instructional practices which emphasize teachers lecturing and students sitting quietly, practicing procedures and memorizing rules, fail to take advantage of the variety of tools available to help students build solid mathematical understanding.
- Emphasize estimation and visualization so that conjectures can be confirmed through computer and calculator use.
- Use technology to diversify instruction to take better advantage of whole class, small group, and one-to-one opportunities.
- Use multi-media technology to address the diverse learning styles of students. Multi-media workstations can offer all students dynamic presentations of real-world applications that cannot easily be replicated in the traditional classroom setting.
- Use technological research tools, like the Internet, to enable students to collect real-world, up-to-the-minute data about mathematical or scientific issues of concern to them, and then to share their findings and conclusions with other students around the world.

Assessment:

- Reduce the use of assessments and questions that only require recall of knowledge and note manipulation of rules and fail to assess conceptual understanding.
- Increase the use of assessments that require calculator and computer use.
- Use assessments that encourage student investigation through the use of technology.
- Allow students to construct answers to open-ended, essay-type questions with the use of a word processor that has the ability to import graphs and charts to encourage the connections between language and mathematical representations.

Incorporating Technology into an Existing Mathematics Program

Developing a Technology Plan

Any technology plan must recognize that the velocity of change and growth in technology will make last year's innovation archaic by next year or the year after. In order to make recommendations for curricular improvement, a technological infusion plan needs to be developed. The plan for increases in technology in the mathematics classroom needs to be clearly outlined with distinct educational goals. Additionally, the plan needs to be flexible enough to allow the professional staff to modify their expenditures in response to the available hardware and software and the changing technological environment.

In the box on the next page, a game plan is provided which describes the steps that need to be taken to assure appropriate integration of technology in the mathematics program. While the list is organized chronologically, and the steps are intended to be completed in the order listed, no real-life process will ever be as smooth and easy as this theoretical model. Given the different technologies to be integrated, the realities of merging the instructional needs of mathematics with those of other content areas, and the myriad of issues that surround funding possibilities for the infusion, many of the items on the list will most likely be addressed at more than one time and quite possibly out of the ideal sequence. The list does serve, however, to remind planners of the many issues that need to be addressed in this kind of innovation and of their interrelationships. In the box, items marked with a * are discussed further in separate sections, while the remaining items are relatively self-explanatory.

Making an Inventory

One of the first steps in ascertaining what is required to create technology-oriented mathematics classrooms is to inventory current practices concerning technology use. A sample questionnaire to aid in the conduct of such an inventory is provided in the Appendix. There are several points to remember in the survey:

✓	*SUGGESTIONS*

✓ Use separate survey instruments for elementary and secondary levels. The needs and objectives at the two levels are very different and a single instrument will be incapable of probing both.

✓ Focus on instructional uses of technology only. Do not try to address administrative and school record-keeping or scheduling needs at the same time or with the same equipment.

✓ Create questionnaires that ask probing questions and require the respondents to provide detailed answers. Some of the best ideas and suggestions will come from staff members who have been prompted to share additional ideas.

Creating a Vision

Understanding the relationships between technology and mathematics education is critical, and an important starting point is a strong statement that expresses those relationships as they will be embodied in your district. Steven Willoughby wrote, "Rapid developments in technology are changing (and ought to be changing) the way we teach mathematics both because they modify our goals for the mathematics education of people and because they provide new tools with which we can better achieve our goals" (*Willoughby,* 1990, p. 60). It is evident that what is stressed in instruction and assessment will be altered as a result of the use of technology. "Template exercises and mimicry mathematics — the staple diet of today's texts — will diminish under the assault of machines that specialize in mimicry. Instructors will be forced to change their approach and their assignments. It will no longer do for teachers to teach as they were taught in the paper-and-pencil era" (*Everybody Counts*, 1989, p. 63). We have the opportunity and the responsibility to use and be influenced by technology in order to help all of our students become mathematically powerful.

Technology Infusion Game Plan

1. Establish a District Technology Steering Committee which includes at least one mathematics teacher.

2. Establish a K-12 mathematics technology committee to develop plans which clearly identify issues, develop time lines, and focus on determined goals. This committee should function as a math-specific subcommittee of the District Technology Steering Committee and should report its findings and recommendations to that group. It should also propose a professional development plan in mathematics for the district staff.

3. * Inventory the current mathematics program with regard to technology infusion and use.

4. * Develop a vision and then goals (long term, short term, and immediate) that will enable the district to achieve technology-oriented mathematics classrooms.

5. * Write a draft of a district technology plan using the information gained through the inventory, as well as input received from all of the content-specific committees and other stakeholders. Special attention should be paid to the gap between the results of the inventory about present technology use and the vision and goals expressed for future use. Widely disseminate the draft, asking all stakeholders to respond, react, suggest, and question the technology plan. Revise the draft and again disseminate the plan. Continue this process until consensus as to the vision, goals, means, and assessment is reached.

6. * Begin a strong staff development program even before hardware and software purchases are made and provide additional training whenever new software or hardware is purchased.

7. Evaluate available software. Decide on the appropriate balance between software that promotes higher order thinking skills and software that provides drill and practice.

8. Begin to plan courses and instructional programs that truly integrate the technology.

9. Be sure that your district plan addresses compatibility of equipment throughout the district, buildings, and departments. While it is sometimes acceptable to use incompatible hardware in different buildings or even for different uses within the same building, such decisions should be made with great care and for compelling reasons.

10. Decide on a maintenance program before equipment is purchased. Decide on a person at each site who will be responsible for keeping the equipment in running order.

11. * Determine the hardware that is best suited to run the selected software.

12. * Prepare a budget and seek internal and external sources of funding. Technology should become an annual budgetary item and a board of education policy should be passed to support the use of technology for all students.

13. Buy enough equipment so *all* students have equal access.

14. Begin to offer the technology-enriched program.

15. Make computers available to staff and students before, during, and after school.

16. Plan for the continuous upgrading of software and hardware and for the regular evaluation of the instructional programs which use the technology.

To that end we need to consider the role of technology in connection with how we educate our students and prepare them to be life-long mathematical learners. Generating a mission statement, goals, and curricular objectives, and reviewing educational guidelines established by state, national, and professional organizations represents our beginning point. It is important to note, however, that we in New Jersey are not starting at ground zero. There is much work that has been done in this area on a statewide level that represents both valuable resource material and even mandates. The 1993 report *Educational Technology in New Jersey: A Plan for Action*, prepared by the New Jersey Department of Education, outlined a bold plan for the entire state's progress. Current documents with information about educational technology, such as the Department's Strategic Plan, are posted on the Department of Education's home page at http://www.state.nj.us/education. Districts should be aware that the New Jersey State Board of Education adopted a resolution in August, 1992 that requires that the Early Warning Test (EWT) and the High School Proficiency Test-11 (HSPT-11) "be constructed on the assumption that all students will be using calculators as they take those tests." Given this resolution, districts must start their technology planning by establishing student proficiency with a variety of calculators as an absolute minimum level of expectation.

Professional Development

Even before hardware and software purchases are made, and for a considerable period of time afterwards, professional development of mathematics teachers in the district is essential. Studies have shown that the amount of technology-related teacher education in a district can be a significant factor affecting student achievement.

✓	*SUGGESTIONS*

✓ The National Council of Teachers of Mathematics (NCTM) recommends that teachers model the use of calculators in computation, problem solving, concept development, pattern recognition, and graphing. The NCTM Position Statement on *Calculators and the Education of Youth* suggests that "school districts conduct staff development programs that enhance teachers' understanding of the use of appropriate state-of-the-art calculators in the classroom." (*NCTM,* 1991)

✓ NCTM also calls for teachers to be "educated on the use of computers in the teaching of mathematics and in examining curricula for technology modifications ... Teachers should be able to select and use electronic courseware for a variety of activities, such as simulations, generation and analysis of data, problem solving, graphical analysis, and practice." The NCTM Position Statement on computer use goes on to state that teachers should be able to use various programming languages and spreadsheets and to keep up with advances in technology. *(NCTM, 1994)*

✓ The Association of Mathematics Teachers of New Jersey, in an effort to support the professional development efforts of the state's teachers in this area, published *The New Jersey Calculator Handbook* in 1993. It contains numerous sample professional development plans and workshop outlines and includes many suggestions for the use of calculators with K-12 students.

✓ Teachers' professional improvement plans may include objectives for implementing technology into the classroom. In order to assist the teacher in reaching the objectives, staff development needs to be extensive and ongoing. Districts must provide time for extensive exposure regarding the use of technology in the mathematics classroom via conferences, expert instruction, individual exploration, refinement, experimentation, observation of other teachers, review and discussion of print and electronic materials, and teamed instruction.

✓ Attendance at professional conferences is essential for motivation, development, and maintenance of technological awareness. Conferences such as the annual meeting of the Association of Mathematics Teachers of New Jersey and regional NCTM conferences provide a broad overview of the many technological products available for classroom use. In addition, they also provide a fertile environment for proposing and exchanging ideas.

✓ Programs such as TRANSIT-NJ promote systemic change in mathematics education through the use of technology. In-service workshops and follow-up meetings help teachers become proficient at enhancing the instructional environment through the use of calculators and computers. Schedules of programs for teachers can be obtained by calling (201) 655-5353 or via email at wolffk@alpha.montclair.edu.

✓ Professional development resources to facilitate the use of technology in promoting inquiry-oriented instruction in K-12 mathematics and science are available from the Center for Improved Engineering and Science Education (CIESE) at Stevens Institute of Technology. Resources include workshops, video series and related print support material, Internet-based lesson activities, administrator conferences, and turnkey training programs. Use of the Internet for collaboration, consultation with experts, and access to "real-time" data on scientific and natural phenomena, as well as instructional mathematics software are emphasized. For more information, contact CIESE at (201) 216-5375 or via email at pdonnell@stevens-tech.edu. Web site: http://k12science.stevens-tech.edu.

✓ Provide professional and support staff inservice activities focusing on specific aspects of technology in the mathematics classroom. Consultants (within and outside the district), curriculum coordinators, and telecommunications networks are some of the sources who can provide services.

✓ Provide teachers with work areas furnished with the same technology equipment and supplies found in their classrooms or laboratories. These work areas need to be available before, during, and after school.

✓ Professional development must focus not only on instruction in the operation of particular pieces of hardware and software, but also on the instructional strategies that are most effective for the successful integration of computers, calculators, and other technologies into the curriculum.

✓ Professional development opportunities should also include specific sessions on the advantages and on the special problems associated with the use of technology in assessment.

The Department of Education, through a competitive grant program, is establishing Educational Technology Training Centers, one per county, by the summer of 1997. Grant awards will be made to those local education agencies with demonstrated experience in providing effective professional development for the implementation of instructional technology practices. For an overview of the program, see the request for proposals (RFP) on the Department's home page at: http://www.state.nj.us/education/ (under grants) or directly through: http://www.pingsite.com/njded/grants/eetc/toc.htm/.

Types of Hardware and Software Needed in The Technology-Oriented Mathematics Classroom

Technology helps to contextualize mathematics by building bridges between theory and life. Imagine, if you will, a student who is learning about the mathematics involved with navigation. Through Virtual Reality she/he is living on Columbus' ship, learning first hand about Columbus' difficulties with navigation. Or perhaps that same student through Virtual Reality is exploring space and planning how to adjust her/his own orbital changes. Later that student might use a Verbal Computer Communications (VCC) system to report her/his learning. Through VCC, this oral text would be transcribed into written text, and if desired, the written text could be further transcribed into another language. VCC is a reality today, and will be in the classroom tomorrow. Perhaps that same student is learning about the development of the Arabic numeral systems by "visiting" the Middle East thousands of years ago and experiencing the sights, smells, sounds, and tastes of the environment, forever imprinting the experience and the information in her/his mind. Early versions of this type of multi-media instruction are currently being explored by Disney World, and will be available in the classroom in the near future. Technology can help to extend our students' learning by building bridges across disciplines, thus connecting prior knowledge with new knowledge.

But what about today? Students now can access information through global information retrieval, using either commercial enterprises or the Internet. As more locations hook up with satellite transceivers, additional two-way and multiple hook-up interactions will become available throughout the world. Students will be able to statistically analyze the information received and compare their knowledge country by country, solving real problems on a real time basis. By using this technology to interact with professionals in their fields of interest, students will be entering the workforce with skills that were unheard of ten years ago. The use of technology in the mathematics classroom is exciting, challenging, thought provoking and necessary.

Calculators

Whereas Virtual Reality, Verbal Computer Communications systems, and multi-media instruction will transform instruction in the classroom in the near future, the premier and universal technological tool for today's mathematics classroom is the calculator. John Kemeny, comparing how they performed the calculations used to develop the atomic bomb fifty years ago with the power of today's pocket calculators, said, "It took twenty of us working twenty hours a day for an entire year, to accomplish what one student now can do in an afternoon." Teaching students how and when to use calculators is critical.

A primary question that teachers and curriculum planners must deal with when incorporating calculators into mathematics classrooms is *What functions are most useful for this grade level? How sophisticated (and costly) do these calculators have to be?* The New Jersey Department of Education provides at least a floor for this discussion in their *Guidelines for Acceptable Calculators for Use on the HSPT11 and EWT from 1993-1994 through 1995-1996.* This document establishes these functions as the minimum set acceptable for use on the tests:

- algebraic logic (i.e., automatically follows the standard order of operations);
- exponent key to do powers and roots of any degree;
- at least one memory; and
- a reset button or other simple, straightforward way to clear all of memory and programs.

In addition, the benefits of many other functions are described in great detail in the mathematics education literature and instructional approaches incorporating them are appearing in many commercially available programs. Algebraic logic is now available in very simple, four-function calculators for young children. Extensive capabilities to change the form of and operate on fractions provide for the creation of a class of calculators which are used widely in the middle grades. Statistical functions and even graphs and plots are available in inexpensive calculators which are finding their way into all grades from four through college. However, probably the most revolutionary effect of all is being provided by two classes of calculators that are capable of graphing functions and doing symbolic manipulations for algebra and calculus. Graphing calculators, used in conjunction with Calculator Based Laboratories (CBL), provide vivid examples of the applications of mathematics and provide excellent opportunities for collaboration between mathematics and science teachers.

Computers

In spite of the growth of calculator use in the classroom, with the many new features and the tremendous flexibility that they provide, the use of the computer is still essential. Software utility programs afford students opportunities to use spreadsheets to analyze data, to graphically represent mathematical formulas, to manipulate pictures, to make and test conjectures, to run multiple simulations, to write about mathematics, and to perform a myriad of other functions. They enable students to make connections between mathematics and the real world, to expand their own sense of reality, and to participate in generative and reflective learning.

The picture of mathematics classrooms where technology is fully integrated in intelligent, meaningful, and necessary ways is attractive. What types of hardware are needed to make this paper sketch a reality? The suggestions below may help develop a response to this question.

✓	*SUGGESTIONS*

✓ Introduce students to calculators at the earliest levels of schooling and progress to scientific calculators with algebraic logic, graphing, and programmable calculators as students advance through the mathematics curriculum. Overhead versions for all of the calculator models mentioned are available and should be used by the teacher for demonstration purposes.

✓ Provide mathematics teachers with access to computers with color monitors and liquid crystal displays for demonstration purposes. These stations should be equipped with high-speed CD-ROM and laser disk drives.

✓ Make a fully-equipped computer lab available to all mathematics classes. The lab should include modern computers with substantial internal memory, color displays, printer access, and appropriate software to accommodate teacher- and student-directed activities. The stations must have hard drives, and if possible should be networked.

✓ Choose a computer only after deciding on the software to be incorporated into the mathematics program, since programs that are well suited to specific needs may run on certain computers and not others.

✓ Make VCR's and video disc players available to every mathematics teacher, so that they can take advantage of the increasing number of quality mathematics programs being produced in these formats.

✓ Provide for interactive television, whether using fiber optics or ordinary telephone lines, as a practical means of visual communication between multiple sites. Mathematics courses which are otherwise too small or specialized can be offered with this tool. Specialized experiments and complex demonstrations can be conducted, and instructional resources can be accessed from remote sites, through this technology.

✓ Equip each classroom in the school with full telephone capability to support Local Area Networks as well as on-line search capabilities through electronic networks and data bases. Fiber optic cable is preferred for new installations. The state maintains a list of approved service providers (ISPs). The list is available online via the homepage for state contracts (http://www.state.nj.us/infobank/noa/t1572a.htm) or through a link on the NIE home page (http://k12science.ati.stevens-tech.edu/connect/connect.htm).

✓ Equip schools to receive externally produced programming via antenna, cable, or satellite. Advanced math classes are now available through the Satellite Education Resources Consortium (SERC). In addition, a school should have the capacity to disseminate the programming to the individual classrooms.

Making Connections

Miss Johnson spent part of her summer watching the 26 half-hour episodes of *Algebra: In Simplest Terms*, hosted by Saul Garfunkel. She made notes describing the application portion of each episode, since she planned to introduce these topics in her classroom.

During the school year, Miss Johnson showed her students specific sections from the series that highlighted direct application of algebra to the real world. For example, one episode the students enjoyed was about the use of ellipsoids in removing kidney stones. After the students had viewed the segment, Miss Johnson had her students conduct a laboratory investigation about ellipses with their graphing calculators. Wanting to stress connections between algebra and the real world, Miss Johnson provided her students with many tasks that year that involved investigating practical applications of the algebraic concepts they had just seen portrayed in a video segment. During her twenty-eight year career as a teacher, she had never seen such enthusiasm from students.

Miss Johnson is waiting for the laser disc version of the series so that retrieval and multi-media presentation can be more easily accomplished. To obtain information about this particular video series phone: 1-800-LEARNER.

Budget and Funding

Each district would like to provide the kinds of services and experiences that have been described in this chapter for their students. The most serious deterrent is obviously their cost. Schools have well-established funding mechanisms and sources to cover the basic day-to-day services they need to offer, but are ill-equipped to deal with necessary purchases of expensive equipment such as that needed to fully integrate technology into an instructional program. Many schools and districts have been successful, however, in accomplishing a major portion of the vision. They have used a variety of strategies including fund raising, some reliance on the local tax base, special bonding, corporate-sponsored grant and award programs, government-sponsored research and demonstration programs, and many more creative approaches. This section provides some information about activity at the state level and a sample of suggestions made by successful districts.

The State Plan

The state plan, *Educational Technology in New Jersey: A Plan for Action* (*NJDOE*, 1993), contains action plans to engage interest, guide collaboration, promote funding, and monitor policy implementation in order to ensure widespread integration of technology in all areas and at all levels across the school. Another document, *Giving New Jersey's Students Power to Perform*, developed by the 1993 Commissioner's Ad Hoc Council for Technology, suggests that it is imperative that key groups in the state work together to attract funding support for technology. The Commissioner's Ad Hoc Council for Technology made the following recommendations for funding the state plan activities:

Funding Recommendations

1. State legislature to appropriate funds to provide an annual entitlement of $50 per pupil for New Jersey's 1.2 million public school students. Fund to be renewed annually to assure that up-to-date resources for learning are available to all students.
2. State legislature to direct appropriations to fund a one time capital investment project to develop a statewide fiber optic telecommunication highway for education which will have the capacity to carry voice, video, and data communications throughout the state.
3. State legislature to provide financial incentives for districts to engage in new school construction and technology retrofit projects.
4. State legislature to appropriate funds to implement a megasystem for data management for the State Department of Education.
5. State legislature to appropriate funds for technology modeling incentives to support the planning and implementation of exemplary uses of educational technology; schools would demonstrate need and a commitment to become state-of-the-art centers for excellence.

Progress has been made in addressing these recommendations. The New Jersey Department of Education's Comprehensive Plan for Educational Improvement and Financing (May 1996), recommends that $50 million be included in the FY 1997/98 state budget (and for the four following years) for a distance learning network aid. Funds will be distributed on a flat, per pupil rate to all districts which amounts to $43 per student. The network for delivery of voice, video, and data offers all districts (including those that are poor and have large numbers of disadvantaged students) an opportunity to obtain quality programs for their students to effectively implement the standards. The third recommendation is addressed in part by pending funding legislation (S40 and A20). (Legislation is available on the NJ Legislative home page at http://www/njleg.state.nj.us/.) Data management is being addressed through the Department's Office of Technology, established in 1995. The fifth recommendation is addressed through the FY 95 and FY 96 grant programs for Classrooms: Connections to the Future and Educational Technology Consortia, which provided $1.3 million in funding for technology modeling incentives. For details on these and other Department of Education initiatives, see the Department's home page at http://www.state.nj.us/education/.

Local Suggestions

School districts that have already made considerable progress in educational technology recommend the following strategies.

✓	*SUGGESTIONS*

✓ Allocate a standard fixed percentage of the local school district budget, perhaps 1 or 2 percent, for technology equipment and maintenance.

✓ Establish a consortium of school districts for the purpose of negotiating cost effective benefits for hardware and software through group purchase, site licensing, etc.

✓ Develop a procedure for evaluating aging and obsolete hardware in light of local instructional needs, normal service lifetimes, and the need for systematic upgrades and replacements.

✓ Use special state or federal funds such as Chapter I and Chapter II funds. Consistently designating these funds for technology can provide technology for special need groups and for specially identified projects and programs.

✓ Seek funding from the National Science Foundation (NSF), the United States Department of Education, and other federal agencies. Work with colleges, state agencies, or district consortiums which have been awarded grants to promote the use of technology.

✓ Lease purchase current technology over a two or three year period. This plan makes large district-wide implementation easier to accomplish over a shorter period of time.

✓ Use Eisenhower funds. Each year create training models for mathematics and science teachers that target appropriate use of technology in the classroom.

✓ Hold a budget referendum. Present the public with a technology plan that is designed to create district-wide state-of-the-art technology-oriented schools.

✓ Establish at least one magnet school. Create a school specially focused on state-of-the-art technology and preparation for high-tech careers.

✓ Establish a business partnership. Many large corporations have competitive grants available in varying amounts. Technology products may also be obtained through the Computer Learning Foundation's non-profit program "Technology for Education" which sponsors multiple corporate partner label collection activities; call (415) 327-3347 for information.

Summary

Technology has changed and will continue to change what and how mathematics is taught. When we think about technology and the myriad of reasons why we need to infuse technology in the mathematics classroom, we should not lose sight of the following:

- In 1950, 40 percent of jobs in the United States were for unskilled workers. By the year 2000, only 15 percent will be.
- In 1900, 85 percent of all agricultural jobs were filled by unskilled workers. Today, only 3 percent of agricultural jobs require unskilled workers.

- 20 out of the 21 largest industrialized nations require one year of applied physics for all students. Only the United States does not require physics.
- In 1943, U.S. education was 5.5 hours a day for 180 days. In Japan it was 3.25 hours a day for 120 days. Today, in the U.S. it is still 5.5 hours a day for 180 days. In Japan, it is now 8.5 hours per day for 243 days.
- The percentage of college students graduating with degrees in mathematics, science, or engineering in these countries is: Japan, 20%; Great Britain, 14%; Germany, 13%; and the United States, 8%.

We are preparing our children for a different world from the one in which their parents grew up. In order to succeed in this increasingly technological world, we must provide them with the best possible education; an education that includes the most advanced tools, techniques, and methodologies available. Our nation must have the opportunity to compete in the global economy on an even playing field. Anything less will reduce our children's prospects and weaken our nation's future.

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Software

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Video

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On-line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/
The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Appendix — Sample Inventory Checklist

This checklist includes questions that address each of the categories of planning, staff preparation, curriculum, methodology, setting, availability of technology, and budget. For each question, there are three statements which are designed to help districts assess whether they are at a Minimal (M), Intermediate (I), or Advanced (A) stage of implementation.

1. **Planning. Is there an organized effort to address technology issues in mathematics instruction?**

    M Curriculum planning and documentation do not normally address technology.

    I The mathematics program is part of a district-wide technology plan that has just begun to be implemented.

    A Studies of the appropriate uses of technology in mathematics are ongoing at the department level.

2. **Staff Preparation. (a) Is support provided for staff development to include technology?**

M Staff is encouraged and expected to attend conferences, workshops, and courses with regard to technology.

I On the job support is used to encourage the use of technology in instruction.

A There is a staff development program with specific goals for all teachers.

(b) For immediate instruction — Does the staff have technology available outside of class so that opportunities to include technology in the classroom are realistic?

M Equipment and software are available to staff without denying student use.

I Staff have office and personal devices for preparation of lessons.

A Technology is available for clerical and communication tasks.

3. **Curriculum. (a) Documents — Is there a formal, written commitment to including technology in mathematics instruction?**

M Courses of study include technology.

I New topics/approaches are being added because of technology.

A Topics/approaches are being de-emphasized because of technology.

(b) Source of Content — What is the source of decisions made on content and how are those decisions translated into instruction?

M Local staff creates instructional sequences which use technology.

I Resource materials for creating technology-supported lessons are in use.

A Materials purchased for student use are purchased with the expectation of routine use of technology to solve problems and conduct investigations.

(c) Technology included in student materials — Are instructional materials selected and acquired to assure effective use of technology?

M Texts have technology addenda.

I Texts integrate technology into content.

A Texts are written based on technology as a way of learning.

4. **Methodology. (a) Regular Instruction — Is technology used in the actual instructional process?**

M Technology is used by the teacher during lectures and demonstrations.

I Students use technology in a class-laboratory setting.

A Inquiry methods replace some lecture sessions and technology is regularly used by all.

(b) Assessment — Is technology used by students during evaluations?

M Students are permitted to use technology while taking some tests.

I Students are permitted to use technology during assessments whenever appropriate.

A Assessments are designed to capitalize on technology.

5. Setting. (a) Demonstration equipment — Is instruction supported by technical presentations which can be seen and heard by all?

M Mobile units are available in a classroom with advance notice.

I Mobile units are available in classrooms on demand.

A Fixed units are installed in every mathematics classroom.

(b) Communications — Is there provision for exchanging ideas with others facilitated by technology?

M Technology is used to encourage students to write in mathematics classes.

I Access to a modem and a telephone line is provided to the mathematics department or program.

A Students regularly communicate about mathematics via computer.

6. Availability of Technology. (a) Technology is current— How are devices kept current?

M The mathematics program competes with other programs for new technology.

I A technology committee continuously reviews district needs.

A The mathematics program contains a plan for continual update.

(b) Individual Device — Are devices such as graphing calculators available to students in numbers that facilitate instruction with technology?

M There are sufficient numbers for mandated testing.

I There are sufficient numbers for classroom use.

A There are sufficient numbers to allow each student to use one at school and at home.

(c) Community Devices — Are devices such as computers available in numbers that facilitate instruction with technology?

M Demonstration devices are in classrooms.

I A laboratory setting is available for whole class access.

A Library-like availability exists for individual access.

7. Budget. (a) Are provisions made for continuing the purchase of technology?

M Yearly or special requests for the purchase of equipment/software is the vehicle used to obtain material.

I There is an established budget for software and minor equipment updating.

A There is an established budget to ensure continual modernization.

(b) Supplies — Are supplies (ribbons, paper, diskettes, batteries, etc.) that are required for the use of technology available in appropriate quantity and accessible to facilitate instruction?

M Supplies are available in limited quantity.

I Supplies can be obtained through advanced requests.

A Supply purchases are planned on an annual basis and controlled locally.

(c) Repair and Back-up — Is there a plan in place to continue instructional activities in the face of technical failure?

M Facilities are serviced when down.

I Back-up facilities are available to replace down items.

A Support services are in place to assure continual access to technology.

CHAPTER 20 — PLANNING FOR CHANGE

Introduction

Implementing the *Mathematics Standards* of the New Jersey State Department of Education's *Core Curriculum Content Standards* will require **consistent, long-term and widespread commitment to systemic change** on the part of districts and school communities.

Changing a district's approach to staff development, staffing, curriculum, and assessment is necessary but insufficient. Involving the representatives of various stakeholders in the community — parents, students, staff, board members, business and industry — is vital, but also insufficient. Restructuring the decision making practices of individual schools and across whole districts is crucial, but not enough. Improving the instructional practices of a few individual teachers, or even a school's entire instructional staff is only a beginning, not an end. Systemic change means changing every aspect of every school in every school system. It is not simply doing a better job at what we have always done; it requires that we redefine the nature of the system and institute whatever practices are deemed necessary to carry out that change.

This task may seem daunting, but it can be done. Recognizing that the teaching and learning of mathematics cannot be accomplished in a vacuum, that mathematics must be connected to other disciplines and to other aspects of the overall school systems holds the potential to make the present reform do what past efforts have failed to do: germinate, take root, flower, and seed again. Before us is an important opportunity we need to seize. Developing and then nurturing a new culture in all facets of the educational community and encouraging continuous self-reflective growth and renewal for all stakeholders is the very nexus of change.

There are several assumptions that have guided the development of this chapter. They are:

- Change is a process, not an event. It takes time and is on-going.
- Change is not linear. Change in one area can affect change in another, often as a catalyst and/or a model.
- Change is accomplished by individuals who react at different rates and in different ways and intensities to new and continuous challenges.
- Although the change process requires an initiator, leadership is provided by a variety of people within an educational community. The process however, cannot be person-dependent or it is bound for failure. There needs to be a respect for the existing organization and a plan to invite the appropriate stakeholders into the process at the onset.
- A culture must be created that values continuous learning, problem solving, reflection, and sharing of knowledge among staff, parents, and students.

A model representing systemic change as occurring along a continuum was designed by Beverly Anderson (1993). This model was adapted and then condensed into the following four stages:

1. **Awareness and exploration**
2. **Transition**
3. **Emergence of a new infrastructure**
4. **Predominance of the new system**

One way in which educators can become involved in systemic change is by employing Anderson's model as a framework to guide and develop an understanding of this complex process. In the four sections that follow, each stage is described. Anecdotal information and resources pertaining to each stage as well as suggestions for implementation are provided for each section. Although each of these stages is presented separately, it is important to keep in mind that change is a recursive process.

Following the discussion of the four stages in the change process, two additional important areas related to change are discussed. The first, professional development, is a primary vehicle for addressing change throughout all of the stages. In this section, the need for professional development is addressed, followed by a discussion of various formats appropriate for professional development activities and of the resources needed to provide these activities. The second section discusses some of the organizational issues that should be addressed by schools undergoing systemic change and provides examples of different types of school organization.

Stage 1 — Awareness and Exploration

Stakeholders become aware that the current system is "out of sync" at the beginning of the change process. As a result of the National Council of Teachers of Mathematics' *Curriculum and Evaluation Standards for School Mathematics,* the National Education Goals for the Year 2000, economic concerns, workforce requirements, and general public discussion, there is widespread recognition that current mathematics curricula and instruction are inadequate for the world of today and tomorrow. Thus there is an opportunity to promote change in the light of existing awareness by building consensus among the various stakeholders of the vision of what mathematics curriculum and instruction should be and what changes are needed to achieve that vision.

Creating a Vision

When any group contemplates organizational change, it is critical to have a collective understanding of what their present circumstances look like (inventory), as well as what they would like them to look like (vision). This comparative process helps name what their strengths, weaknesses, and needs are. The *New Jersey Mathematics Curriculum Framework* provides pertinent information that supports this process. The importance of reaching out to all persons comprising the school and district community cannot be minimized. Teachers, support staff, administrators, students, parents, and other neighborhood members must be involved. The diversity of experience represented by the whole group will make possible a broader vision, thus creating a richer base from which to work. This initiating process takes time and energy; it develops leadership perspective. When carried out with care and attention, it will form a comprehensive, informed basis for decision making and later evaluation. Theodore Sizer noted: "To pretend that serious restructuring can be done without honest confrontation is a cruel illusion" (1991, p. 34).

So how do we get input from stakeholders? Involving as many people as possible, in as many ways as possible, is an important credo for those involved with change. The final section of this chapter describes the roles of various stakeholders (parents, teachers, supervisors, school board members, administrators) in creating a climate for systemic change.

✓	*SUGGESTIONS*

✔ Schedule a variety of times when people can meet to share thoughts and feelings about what they want for their school in the area of mathematics education. These "town meetings" happen when cross sections of population convene to discuss their aspirations for the best possible educational outcomes for all students. A guest facilitator might help organize the discussion. Leaders of the meeting should listen without responding and should encourage deeper thinking. Careful notes should be made, preferably on chart paper so everyone can see what is written. Use the speaker's words rather than your personal understanding or clarification of what you think the speaker said. Perhaps you will want to videotape and then transcribe sections of the meeting. Share the collected information concerning the meetings.

✔ Invite adults and children to describe what their "dream" mathematics classroom would look like and what their role within it would be. Perhaps some will chose to draw their vision. Mount these drawings on school corridor walls creating a collage of visions. Invite additional sketches from all stakeholders in the process (artists' names are not required). Perhaps others might choose to represent their vision through other media. Form is not important. Rather, the collected "vision" becomes a focal point that turns us inward to be reflective and, in doing so, helps to extend our own vision.

✔ Attempt to involve parents through the Parent Teachers Association (PTA) or Organization (PTO). Create multiple opportunities for these important stakeholders to provide input concerning the vision. Brochures published by the New Jersey Mathematics Coalition or the National Council of Teachers of Mathematics[1] (NCTM) can be used to share what's happening in mathematics with parents. At a back-to-school night in one New Jersey school district, parents received a brochure designed by teachers to explain their mathematics program and how parents could become involved. The following vignette illustrates how involving parents might be effective.

Opening Doors: Parents and Teachers

A committee of middle school teachers prepared a presentation for parents which advocated the use of calculators in the classroom. The committee wanted to share an aspect of their vision and receive vital parental support concerning calculator use. The teachers used materials obtained at an institute on the High School Proficiency Test (HSPT). As a group, they had already developed a shared recognition about the power of the calculator as a tool in support of mathematical thinking. Many parents had never thought of calculators being appropriate for the support of problem-solving strategies and everyday mathematics. Through the use of hands-on activities and a subsequent debriefing session, a deeper understanding and a lively discussion between the parents and the teachers about the use of calculators ensued. A groundswell of parental support began and is still flourishing. On that night, a door was opened.

✔ Help to build and extend knowledge by circulating journal and newspaper articles that address the changes and are appropriate for non-educators. Outside sources do help to validate the perception that the vision is appropriate.

[1]For additional information about CBAM, contact the Association for Supervision and Curriculum Development (ASCD), Alexandria, VA 22314.

✔ A volunteer ad hoc committee, with representation from all sectors involved, should begin to draft a vision. This draft should be circulated, translated into the languages spoken by the parents when necessary, and reviewed by everyone. The committee should review all comments, make amendments and/or other editorial adjustments, and circulate another draft. This process should continue until everyone finds the document acceptable. A sense of ownership and personal involvement in the process ensures commitment to the efforts necessary to bringing these dreams to reality. The draft, a representation of the learning community's vision, should be finalized and posted with the understanding that it will be reviewed periodically and changed as necessary.

Making an Educational Inventory

The educational inventory should be conducted by as many people as possible. It should include not only the relevant topic areas to be considered but also some sort of scale on which to rate the degree to which the respondent feels the element is functioning. In creating an inventory it is critical to ascertain strengths, weaknesses, and needs. The results of the inventory should be communicated widely.

✓	*SUGGESTIONS*

✔ Generate questions for the inventory from general areas of interest as defined by the school community. These questions can be generated from brainstorming and then condensing the suggestions into a smaller list. Perhaps your community will choose to create clusters of ideas and inquiries on large sheets of paper. Whatever strategy you use to elicit response, it is important to first encourage and honor diverse responses that catalyze discussion, followed by organizing those ideas and inquiries into similar categories.

✔ Help the learning community define the present reality by having small groups generate survey and/or interview questions/statements about each topic area (e.g., mathematics curriculum, instructional techniques, assessment, parent attitudes and involvement, school climate, student attitudes towards mathematics, staff expectations).

✔ Compile and share responses to all questions/statements with the entire school community. The results of any data collection needs to be organized and interpreted. This work is best done in small groups and then shared upon completion with all participants in the process.

Identifying Gaps

Once an inventory of the present school status has been completed, the next step is to identify gaps between what is (the inventory) and what is desired (the vision). Categories need to be clearly identified which all stakeholders agree to be areas of concern and need. In addition, it is also critical to identify those areas that represent strengths. Being able to name what is done well enables the school to build on success.

✓	*SUGGESTIONS*

✔ Compile a list through brainstorming or clustering; then pare it down to a few clearly defined areas that should be addressed first. Keep the number of identified needs relatively small, however. If too many needs are identified, the task of addressing them may seem overwhelming, thus frustrating those involved and dooming the reform effort to failure. It may be decided that some areas will be addressed only at certain grade levels or grade spans initially, with plans to broaden the effort in subsequent years. The following vignette shows the results of one school's discussion.

Identifying Needs: Seeing the Big Picture

Members of one New Jersey school spent half a day reviewing information gathered through a variety of sources. They proceeded to reorganize twenty brainstormed items into five general statements that articulated their needs.

We need:

1. Reading experiences in mathematical contexts.
2. Assessments which don't rely on multiple choice questions or single, exact answers.
3. Flexible daily time commitments to allow for intense mathematical experiences.
4. Opportunities for collaborative, collegial planning.
5. Regular parental involvement in support of children's understanding mathematical concepts.

Exploring Paths

Although it is tempting to settle quickly on an easily available solution, extensive, open-minded exploration is necessary because change is a complex process. The way to attain the vision is rarely by traveling the easiest path.

✓ *SUGGESTIONS*

✔ Arrange for small teams of people to begin to research innovations, practices, and programs pertinent to their topics. This may involve literature searches, conversations among colleagues, or requests to publishers. Each team should circulate key articles or other sources of information they have identified with other stakeholders.

✔ Arrange for visits to classrooms within the school and district to observe other professionals implementing new instructional techniques. Some teachers choose to forge ahead, experimenting with techniques and strategies that others might find threatening. Support and input from peers can encourage and sustain change models as they emerge. In addition, these teachers' experiences serve as a resource for the school and district.

✔ Arrange for visits to exemplary classrooms outside the district to allow for on-site access to quality program implementation. Such visits create opportunities for questioning other teachers or educators as to how their programs evolved, what reactions from others they have experienced, how they perceive children have benefited from the program, what activities are in place to support and sustain the effort, what might be done differently, and what real changes have taken root.

✔ Assign teams to select the most favorable means of addressing needs and to share their recommendations with the larger group. The paths sought should represent a consensus of the involved staff and other stakeholders. If a team learns of serious concerns from other stakeholders relative to its findings, additional input should be sought and a resolution forged. Consensus is a decision-making model, which when applied properly results in a win-win situation for all parties. Unlike majority rule (when one group is the absolute victor and the other group submits until the next opportunity to block the majority presents itself) or compromise (when both groups relinquish parts of their chosen positions), consensus allows everyone to support, or at least be able to live with, an outcome. Much discussion is required, and everyone's concerns must be addressed. No one may abstain from participation. Each member of the group is required to share his or her understanding in confirmation or challenge and present real issues to which all can respond. These interactions should bring about modifications to which all can subscribe. All decisions need not be made through consensus, but for those decisions where there needs to be long-lasting support, consensus is the preferred way.

✔ Write a plan that articulates the strategies, activities, individual or group responsibilities, timelines, evaluation or assessment methodologies, and monies required. Teams should be aware that this plan represents the best thought of the moment; as the vision is implemented, the plan will continue to be monitored and revised.

✔ Secure the approval of the plan by the school, district, or board of education. Commitment of resources sends a clear message that there is commitment to implementing the plan.

Developing Shared Responsibility

Understanding change in mathematics education is predicated upon developing a common vocabulary and a clear picture of what effective mathematics instruction looks like in the classroom. Developing a sense of shared responsibility in every person involved in the change process is an important and necessary part of the process. By creating opportunities for people to meet and hear about all of the activities that are taking place, everyone comes to understand what the change process looks like. As a result, all become an integral part of the process.

✓	*SUGGESTIONS*

✔ Provide opportunities for leaders to convey information about effective mathematics instruction through modeling, discussion, both informal and formal, and research sharing.

✔ Develop and support an atmosphere of consensus, trust, and care in order to build and extend leadership.

✔ Provide professional development experiences in mathematics, team building, collaboration, and the process of decision-making.

Stage 2 — Transition

As the change process continues, resources are committed and the staff attempts new approaches toward consensus. The explorations of key stakeholders become the basis for decision making, and a willingness to risk change emerges. Success depends on building support for these changes.

Building Support for Change

James Joyce wrote, "A man's errors are the portals of discovery." So it is with the change process. Those involved with change experience the dynamic flux between knowing and coming to know. Along that continuum are doorways that sometimes are stumbled upon and at other times are sought with certainty. The paths to and from such portals can be taxing, exhilarating, calming, and/or frightening. Building support for the changes being created and for the change agents becomes essential. Publicizing the vision through a variety of media is important. However, at the heart of building support for change is understanding the effect the change process has on individuals so that a program of support can be created.

The Concerns-based Adoption Model (CBAM — Hall, Wallace, Dosset, 1973)[2] is an empirically based conceptual framework that describes seven stages individuals pass through as they implement a new innovation. In CBAM, the stages of concern are identified as: awareness, informational, personal, management, consequence, collaboration, and refocusing. The seven stages are defined and suggestions are provided in the chart below.

Teachers will move from a focus on self to a focus on the task at hand and finally to the impact of the changes. It is important to support staff in transition with patience and reassurance while maintaining an awareness of what works well at each stage.

Stages of Concern from the Concerns-Based Adoption Model

Stages of Concern and Definitions	Suggestions
Awareness: Some concern about or involvement with the innovation is indicated.	Share enough information to spark interest but not so much as to overwhelm. People need to know that the results of the change are not known and that it is acceptable, appropriate, and necessary to question.
Informational: A general awareness of the innovation and interest in learning more details about it is indictated. Teachers are interested in substantive aspects of the innovation such as: general characteristics, effects, and requirements for use.	Provide clear, concise and accurate descriptions of the planned change. Communicate with large and small groups orally and in writing. Encourage visitations to other classrooms and schools. Compare and contrast what is presently being done with the proposed innovation. Most of all, be enthusiastic and publicize the enthusiasm of others.

[2]For additional information, contact the Association for Supervision and Curriculum Development, Alexandria, VA 22314.

Personal: The focus is on teachers' concerns about the demands of the innovation affecting them personally or professionally.	Reinforce and encourage one-to-one exchange. Encourage even small increments of change. Suggest that those at this stage talk to others whose concerns have diminished and who can be supportive.
Management: The focus is on mechanics of using the innovation. Issues related to efficiency, organizing, managing, scheduling, and time demands are critical.	Further clarify the change and address the small "how to" issues. Suggest concrete solutions to the everyday problems. Attend to the here-and-now.
Consequence: The focus is on the impact of the plan on students. The plan should be re-evaluated for relevancy and success in meeting predicted student outcomes. Any needed adjustments should be made.	Provide opportunities for those at this stage to observe students in other settings using the changes successfully.
Collaboration: The focus here is on coordination of efforts in cooperation with other staff regarding the uses of the innovation.	Bring together those at this stage so that they may develop skills and strategies collaboratively; encourage those further along to interact with beginners to foster mentor relationships.
Refocusing: The focus is on the exploration of benefits from further fine-tuning of the plan by means of substitutions, deletions, or new alternatives which may be introduced in order to achieve greater results.	Encourage initiatives to find better ways. Help those at this stage to access resources that can help them refine their ideas and questions. Be ready to accept suggestions that continue the change process.

Vignette — Supporting Change by Encouraging New Voices: The Circular Process

The following vignette shows what the CBAM model looks like in real life.

At a New Jersey elementary school, the principal and representative teachers from grades K-5 met regularly as the curriculum committee. As a result of reviewing and discussing the *Mathematics Standards* and the *New Jersey Mathematics Curriculum Framework*, this group of educators had been sharing research about how children learn and instructional practices that foster learning. They were excited about what they had discovered and wished to bring about change in the way teachers taught mathematics at their school. They were convinced that past practices needed to change in order to enable all of the students to meet the challenges that the new century would bring. At the same time, they were faced with planning how to best share their enthusiasm and expertise in order to convince other members of the faculty and community to contribute and be part of the change. Having read the *New Jersey Mathematics Curriculum Framework*, they decided to use CBAM as a guide to help them create appropriate activities.

Awareness: Within their respective grade levels, the members of the committee talked informally with colleagues about the *Mathematics Standards* and the *Framework*. They summarized what they had discovered in the literature and sought to discover the levels of understanding commonly shared by the teachers and parents. No push was made to "convert" anyone to their way of thinking. The intent was to make teachers and parents aware of the existence of the *Mathematics Standards* and the *Framework* and the fact that they promoted alternative methods of instruction.

Informational: Having gotten the attention of many teachers, they held meetings with small groups across grade levels and provided more substantive examples of why children learn and retain more by constructing their own understanding. They showed videos of teachers conducting classes where small cooperative groups were at work with interesting problems and a variety of manipulative materials. They invited peers to observe their classrooms and offered to conduct model lessons. They encouraged questioning and were candid in their responses. Similar activities took place at parent/teacher meetings.

Personal: The curriculum committee actively sought out comments from teachers and parents. They held brainstorming sessions of 8-10 people at a time to get their ideas about the strengths and shortcomings of the mathematics program. At the same time, they conducted a survey of parents, teachers, and students to ascertain their beliefs, feelings, values, and conceptions of mathematics. The committee members analyzed the responses and prepared materials and workshops to address the fear, lack of confidence, and misconceptions that had emerged, while continuing the informational campaign.

Management: As teachers continued to observe and request demonstration lessons, they became more attuned to the day-to-day benefits and challenges of this new mode of instruction. Professional development increased as the need for outside assistance emerged. The principal participated in most of these activities. The participants engaged in heated discussions about different aspects of this approach and learned to share and respect their collective expertise. The curriculum committee invited all staff members to bring their concerns about implementation to the meetings. No one was overlooked.

Consequence: Teachers began to implement change at varying levels with different instructional practices in their classrooms. They used a variety of assessment instruments and strategies to evaluate the impact of these new approaches on their students. They maintained daily logs that reflected not only their activities and strategies but also their feelings about the relative success of their efforts. They were becoming reflective and analytical about themselves and their students' progress.

Collaboration: Teachers began to observe each others' classrooms on a regular basis and to confer about their progress, as well as the progress of their students. Regular planning periods were now built into daily schedules, and teachers often found that they met outside of this provided time as well. A library of activities and resources had been established for teachers in the building. In addition, a special section was created that included materials for parents to use at home. Teachers regularly sent home notices inviting parents to extend activities that were begun but could not be completed in class. A newsletter, initiated by the curriculum committee, had specialty columns addressing research, ideas for parents, cross-curricular explorations, and challenges for students. Teachers and the principal worked to write grants. As a result of their efforts, enriched opportunity for conference attendance, post-graduate work, curriculum writing, and the integration of math and science occurred; the art teacher shared ideas which were included in the math curriculum, as did the music and physical education teachers.

Refocusing: New voices were heard from teachers and parents expressing interest in participating actively as leaders in the curriculum process. The innovative teaching experiences and positive student outcomes caused them to think about the process. As a result, they desired greater input. Now they have taken leadership roles in the change process. The curriculum process has come full circle.

Fostering Communication and Administrative Support

Long-term support of change is crucial to success. Careful planning is needed so that adequate resources, professional development, planning time, and expert help are available. In addition, during this transition stage, it may be necessary to modify or suspend some district policies that might be inconsistent with the desired changes.

✓	*SUGGESTIONS*

✔ *Arrange time so that collaboration among teachers is possible.* Organize the schedules so that some teachers have the same preparation period or find some other common time within the schedule. Organize departmental sharing sessions where various members of the department present what they are doing and lead discussions with their peers. Consider the stages of concern (CBAM) in planning appropriate activities. Logistical problems in scheduling such as those illustrated in the following vignette can be overcome if staff and administrators are committed to making change. (See the section on school organization later in this chapter.)

Scheduling Sharing Sessions

In one high school, departments rotate out-of-classroom duties so that each department has a block of time during parts of the year to meet. Another high school includes staff from other departments in their sharing sessions to promote interdisciplinary work. In another district, the school day was expanded through contract negotiations to provide time for teacher communication about teaching ideas.

✔ *Utilize peer support or coaching, as in the following vignette.* By encouraging a mechanism for teachers to help each other, administrators express respect and support for the practitioner, and thereby help create a climate that values innovation and collaboration.

The Consulting Teacher

One district identified consulting teachers to provide peer support. The consulting teacher (CT) is not an administrator, supervisor, or department head, and has no evaluative responsibilities. The CT teaches a reduced load and uses the gained time to work with colleagues in a non-threatening manner on a variety of professional development topics. The job description defines the responsibilities of the CT as follows: "To work collaboratively with all other consulting teachers and district directors to facilitate the use of varied instructional strategies in subject area classrooms; act as a consultant to department administrators in the areas of budget, curriculum, textbooks, and in final examinations."

✔ *Create opportunities for administrators and supervisors to become knowledgeable about the changes.* This means that they need to attend and actively participate in staff development activities. Changing behaviors is difficult, yet when attempted by administrators will communicate their commitment. For example, an administrator who used to exclusively lecture during staff meetings, now applies what she has learned and uses a variety of other methods to convey and explore information, ideas, and questions, thereby demonstrating a commitment through action.

✔ *Create opportunities where necessary.* Coach administrators, particularly at the school level, to encourage risk taking and be open to suggestions. Implementation of the *Mathematics Standards* requires empowered professionals. This means that principals need to be ready to share their decision-making responsibility. Specific staff development activities should focus on the changing role of the supervisor and principal, including skills such as consensus building and conflict resolution.

✔ *Organize cross-grade meetings to encourage and support continuous communication between grade levels and schools.*

Providing Appropriate Staff Development

As the transition stage continues, staff development becomes more intensive and focused. At this stage, when changes are being piloted by selected schools, teachers and administrators are the primary groups that should be involved in initiating, planning, and participating in staff development activities. Some mechanism should be developed to communicate information about the activities to non-participating teachers and administrators. (See the section later in this chapter on professional development.)

✓	*SUGGESTIONS*

✔ *Provide a professional growth activity at each departmental and grade level meeting.* The following vignette illustrates this strategy.

Making Connections with the *Standards*

One department chairperson developed a list of examples of connections within mathematics, including:

fractions .... probability
functions .... geometric transformations
multiplication of mixed numbers .... areas of rectangles
midpoint in coordinate geometry .... averages
division by zero .... slope

The list became the discussion point in a department meeting, with teachers sharing some examples of what they had already done. The chairperson began by sharing a lesson she had taught. For several subsequent meetings, the first fifteen minutes was devoted to someone sharing a lesson illustrating connections.

✔ *Schedule workshops so that adequate time and support are available for the sessions and for follow-up activities.* Brief after-school or part-day sessions are least effective (Browyer, Ponzio, Lundholm, 1987). Multiday sessions are best to introduce new teaching strategies and curriculum; follow-up sessions can be shorter.

✔ *Encourage participants in staff development activities to keep a log where they write about their experiences.* People write in order to make discoveries (Murray, 1992), and by using this tool they can make new sense and understand more fully what they have experienced. This reflection is as important to the adult learner as it is to the child.

✔ *Provide opportunities to attend out-of-district conferences, seminars, and meetings.* The vignette shows how one district used this strategy. Those who attend should have the responsibility to communicate to the rest of the staff what they have learned.

✔ *Support alternative mentorship opportunities.* Teachers benefit from observing the craft of others in the classroom. Learning to work cooperatively as adults supports similar efforts with children as well as efforts in site-based planning and decision making. Efforts should be made to arrange observations and classroom demonstrations to include parents and administrators whenever possible.

Developing Support Groups

One district encouraged teachers to sign up in pairs for workshops. They completed homework assignments together, provided each other cognitive coaching, observed each other's lessons, and videotaped their lessons for collaborative review. This laid the groundwork for continued professional peer support in a non-threatening and honest environment. Ideas could be shared while both developed and enhanced their teaching strategies and classroom environments.

Stage 3 — Emergence of the New Infrastructure

By the time a school and district have reached this stage in its change cycle, many critical issues have already been addressed. Stakeholders are aware of strengths and weaknesses; a vision has been forged and a set of shared values has emerged; priorities have been set and solution strategies have been pilot-tested. However, it is still important to remember that change takes time and that it progresses at an uneven level of development for the participants. Keys to bridging the next step in this process are found in the ability of people to decide upon the structures necessary to reach their goals; an environment that nurtures a willingness to acquire new skills and that provides needed resources; and the courage to shed those behaviors and beliefs that are inconsistent with the changes being introduced. Collaboration, respect for the contributions of all stakeholders in the educational community, commitment, and an atmosphere that allows shared risk-taking must all exist.

At this stage, stakeholders will no longer be asking *What Mathematics Standards?* Instead, they will be discussing how they have used the standards in their lessons, and sharing the ways in which they have invited children to solve *real* problems about *real* issues. Enthusiastic conversation about children's discoveries and understanding of mathematics concepts will be the talk in the staff room. Mathematics will be perceived more as an integral part of all subject areas and all teachers will expand their own understandings through collaboration and team teaching. The question then is how do we support and extend these efforts?

✓	*SUGGESTIONS*

✔ *Provide multiple opportunities for intensive staff development.* Instructional change in schools ultimately rests with classroom teachers. New mathematics programs should be enhanced by comprehensive, ongoing professional development. This education should be hands-on, experience-based, and conducted in cooperative groups. It should be supported by on-site coaching, available on a daily basis. Its focus should be both on content and pedagogy. Teachers should form study groups where discussions range from collaborative lesson planning and delivery of instruction to reflective inquiry of the learning and teaching processes. Through a commitment to action research, these study groups would serve to continuously inform the learning community.

✔ *Support alternative approaches to instruction.* Just as the traditional role of the teacher working alone, isolated in a classroom, must be challenged, so too must the traditional role of the student. We need to reconsider the model of the student working alone and competitively. Information and understanding are the collective responsibility and property of all who come to school to learn. We need to understand and employ a variety of approaches to help all students build a conceptual understanding of mathematics. We need to develop and extend our instructional repertoire to include: cooperative groups, student-centered classrooms, interdisciplinary studies, use of appropriate manipulative materials, ongoing opportunities for written and oral communication, challenging problems to be solved over extended periods of time, and encouragement of student questions. Additionally, teachers should, on occasion, work in pairs in the classroom, thus enriching the background brought to students and modeling the collaboration which is so critical to most learning situations.

✔ *Establish mentorships.* In addition to sharing strategies, knowledge and experience, a mentoring relationship among professional staff encourages heightened self-esteem and increased personal expectations. Formal, positive, collaborative peer support systems help to create meaningful and enduring professional growth. Although informal relationships evolve, because they are self-selective the benefits are limited. A more encompassing, well-designed network for both new and tenured teachers results in additional support for staff development and new instructional approaches. What does mentorship look like? It can be as simple as teachers visiting other teachers' classrooms, sharing their professional observations, and collaborating on ways to improve, extend, and enrich the art and craft of their teaching. Moreover, this collaborative spirit induces self-reflection and evaluation, as well as adaptation. The result: Better outcomes for students!

✔ *Encourage and organize visits from others outside the school or district.* Having established and embarked upon a well thought-out path, it is important to share your model with others who are just starting or who not yet involved in the change process. Think of how visits helped (or might have helped) you when first starting. Moreover, perspectives from those not involved in your innovation can be helpful in making evaluations of your progress.

✔ *Continue to learn with the community.* There is a continuous need to inform and engender support from the community outside of the school building. Parents in particular should be informed and involved in the changes that are taking place. They can and should provide encouragement to their children. Also, those who are implementing change need to continue to learn from parents. They can be resources and sounding boards regarding the progress, attitudes, values, and needs of their children.

✔ *Continue to communicate.* Providing updates of activities and reporting successful and less successful efforts allows participants in the change process to learn and to revise their plans in an iterative manner. It is essential to understand that no procedure is fail-safe; it can be expected that errors will occur. But progress and improvement will also be taking place. Communicating in an ongoing fashion and through various media is essential. When interested stakeholders feel left out of the "information pipeline," misconceptions and negative feelings about the change may develop. Public relations need to be attended to throughout all four stages of change.

✔ *Reduce stress.* Stress, anxiety, and strain are some of the by-products of innovation. In seeking to establish and understand our roles in the change process, stress is a natural outgrowth of the process. As we relinquish long held practices and beliefs, our credibility, trust, and professionalism may feel challenged, resulting in stress. Another source of stress is prolonged misunderstanding. Building a common vocabulary, assuring everyone's active participation, involving all in the decision-making process through committee representation and/or review helps to reduce the possibility of misunderstanding. Stress blocks the ability to think creatively, act positively, and behave rationally, which in turn inhibits change.

✔ *Seek regular responses.* Periodically secure information that answers a few important questions. Students, staff, and parents should all be asked from time to time *How are we doing?*

✔ *Use ongoing assessment practices.* Through the use of traditional (tests, quizzes) and non-traditional (performance, authentic, anecdotal records, logs) assessments a picture of how the change has affected outcomes for students can be discerned. Consider asking children about their performance and how responsible they feel for their success or failure. Through anecdotal records and student and teacher logs, additional information can be added to refine the picture. The more ways you invest in seeing what effect the change has had on students' learning the better the analysis and revision you will be able to do.

Stage 4 — Predominance of the New System

After the district has made the commitment to a new way of operating, the next step is to solidify ownership of the change among all stakeholders: the parents, the school board, the teachers, other staff, and when appropriate, the students. Ownership implies more than acceptance and compliance. Rather, ownership is the result of each person making the new system his or her own.

In the last stage of the model, change becomes institutionalized. The system is predominant. Because continuous learning and change have become integral parts of the school and district culture, rethinking, innovation, and willingness to take risks are second nature to the staff.

Many of the same strategies that have been used at other stages are appropriate here. Particularly important are efforts that encourage the staff to reexamine parts of the system in light of the changes that have been made. In particular, the new system needs to match the vision. This happens naturally since perspectives have changed; incongruous elements of the system are immediately obvious. What strategies can we use to help staff reexamine parts of the system and to continue shaping reality to match the vision?

✓	*SUGGESTIONS*

✔ *Reexamine staff supervision.* In small groups, the staff should examine the processes and procedures for staff supervision to ensure consonance with changes. It is easy to slip back into old behaviors if support mechanisms are not maintained. For example, if cooperative learning has become the norm, the supervision process should focus on strengthening teacher skills in collaboration and group processes.

✔ *Reexamine assessment practices.* The assessment changes which began earlier should now be led by the teachers, as in the following vignette. Students and parents should also be actively involved.

Reversals

At a grade-level meeting in one school, teachers agreed to use reversals of traditional problems each day as a way of better assessing student understanding. The following are some examples generated by the teachers:

- Instead of asking *What is the probability of a four when one die is tossed?*, students were asked to describe an event for which the probability would be 1/6.
- Instead of simplifying 22/33, students were asked to write four fractions that simplify to 2/3.
- Instead of asking for the average of three numbers, students were asked to write four sets of numbers, three in each set, with a given average.

✔ *Reexamine incentive systems.* A system of incentives should be designed to reward desired behaviors and change without penalizing undesired behaviors.

✔ *Reexamine staff development.* Teachers should meet to design staff development activities they need. The principal should provide less direction and more support to teacher initiatives.

✔ *Reexamine ways to rekindle enthusiasm.* In order to maintain, nurture, and extend the change, ways of sharing and recognizing success need to continue and be varied. Activities like the contest described below can be used to spark enthusiasm.

Staff Problem Solving Contest

In one K-8 district, the staff was given a challenging problem to solve and then asked to explain their solution. Teachers worked in teams to discuss the problem and how they would present their solution. Each member of a team that presented their solution received a prize.

✔ *Reexamine board of education involvement.* Members of the board of education should meet regularly with staff to analyze and improve components of the system. They should visit schools to see innovations in action. Their involvement should result in budget decisions that reflect new understanding.

✔ *Reexamine facilities.* Changes in facilities through renovation and construction should occur in concert with needs that emerge during the change process.

✔ *Reexamine hiring criteria for professional employment.* Criteria for hiring administrators and staff should be redesigned in light of the new system. Content specialists should be involved in hiring new teachers. New staff members need to be acculturated by existing staff members using a teacher-designed process.

Personnel Choices

In one district, the interview questions asked of candidates changed in light of the *Mathematics Standards*. Now candidates are asked questions like:

- What are some connections between mathematics topics that you might emphasize in your classroom?
- How is mathematical knowledge assessed?
- In your classroom, would a student be more like a sponge or a construction worker? Illustrate your answer.
- How would students participate in your class? Orally? In writing? In groups?
- As a teacher, do you consider yourself more of a dispenser of knowledge or a facilitator? Explain.

The expectation is that by this last stage the staff has reached a state of mind that provides self-generated and mutually supported change:

> "We now understand that the only way we can ensure our own growth is by helping others to grow; the only way to maximize our own potential is by helping others to improve little by little, day by day" (Bonstingl, 1992, p. 5).

Professional Development

In the preceding sections of this chapter, a process for systemic change has been illustrated, focusing on improving mathematics teaching and learning. It is evident from this discussion that, in order for mathematics learning to improve, mathematics teaching must change. These changes require substantial investments of time, energy, and support. Professional development, affecting the beliefs, attitudes, knowledge, and practices of teachers in the school, is central to achieving this change. In order for the vision described in this *Framework* to become a reality, it is critical that professional development activities focus on mathematics specifically. Generic staff development does not provide the understanding of content, of instructional techniques, and of critical issues in mathematics education that is needed by classroom teachers.

Throughout the earlier sections of this chapter, many professional development activities have been illustrated. In this section, after examining the characteristics of effective professional development programs, some of the different formats for professional development activities will be analyzed. This section will conclude with discussions of resource issues and responsibilities related to professional development.

Characteristics of Effective Professional Development Programs

Without carefully planned professional development programs, it is unlikely that the vision described in this *Framework* can be implemented. For example, integrated approaches to teaching mathematics require planning, curriculum development, design of appropriate assessment activities, ongoing planning, and revision during implementation; in addition, staff members need to have a knowledge base of the subject areas to be integrated with mathematics. Each of these activities involves professional development for the participants in them.

How can we ensure that this professional development is effective? Ten key principles have been identified from the research as being critical for the success of professional development programs in mathematics (Clarke, 1994).

✓	*SUGGESTIONS*

✔ *Address issues of concern and interest, largely (but not exclusively) identified by the teachers themselves, and involve a degree of choice for participants.* Teachers are more likely to respond positively to staff development if they are involved in determining its format and content. They should also have the opportunity to select among a variety of alternatives, since not all teachers have the same needs and interests.

✔ *Involve groups of teachers rather than individuals from a number of schools and enlist the support of the school and district administration, students, parents, and the broader school community.* When individual teachers are the only ones involved in a staff development activity, the likelihood that activity will have any real impact on the classroom is slight. Teams or groups of teachers can provide each other with the ongoing support needed to implement change in the classroom. The support and encouragement of the administration, students, and parents are vital for implementing and maintaining the innovations learned through professional development activities.

✔ *Recognize and address the many impediments to the professional growth of teachers, which exist at the individual, school, and district levels.* Unless professional development activities take into account and specifically address some of the barriers to change, they are unlikely to have any lasting effect. Some of the more important areas to consider include providing for administrative support, funding, and follow-up; taking into consideration site-specific differences among schools; ensuring that student assessment and teacher evaluation are consistent with the proposed changes; ensuring a common vision among all constituencies; and assuring that teachers see the proposed changes as practical and feasible. In addition, it is important that the activities emphasize and encourage professional growth rather than focus on correcting deficits. Professional development activities must also reflect the philosophy of teaching and learning that will be used in the classroom.

✔ *Use teachers as participants in classroom activities or show students in real situations to model desired classroom approaches.* Teachers prefer and learn more from active, hands-on sessions than from lecture-type presentations. Such sessions allow teachers to construct their own meanings for instructional approaches, incorporating them into their own view of teaching. Activities that show real students in real classrooms demonstrate the feasibility and effectiveness of an innovation.

✔ *Solicit teachers' conscious commitment to participate actively in the sessions and to undertake required readings and classroom tasks, appropriately adapted for their own classroom.* Two types of commitment are necessary: commitment to active participation and to the philosophy and approaches underlying the professional development activities. Teachers who are unwilling to make the commitments necessary for a particular professional development activity are not yet ready for that activity and are unlikely to benefit from it; they would be better served by a different activity, perhaps one more appropriate for an earlier stage of concern.

✔ *Recognize that changes in teachers' beliefs about teaching and learning are derived largely from* *classroom practice.* It is most likely that changes in instructional practice will take place only after teachers are able to validate these changes by observing them in practice in classrooms. It is thus critical that professional development provide for trying out new approaches and then discussing the results.

✔ *Allow time for planning, reflection, and feedback.* Teachers need to have opportunities to report successes and failures, to share their own experiences as they try out new approaches, and to discuss problems and solutions regarding individual students and new teaching approaches.

✔ *Enable participating teachers to gain a substantial degree of ownership by involving them in decision-making and by respecting their roles as true partners in the change process.* Having frequent meetings, allowing for individual choice, providing appropriate assistance from consultants, and establishing an ongoing program for professional development in mathematics are some of the ways in which ownership can be enhanced.

✔ *Recognize that change is a gradual, difficult, and often painful process, and provide opportunities for ongoing support from colleagues and critical friends.* Ongoing support is vital for systemic change. Some support is affective, such as recognition of changes in mathematics instruction as one of the school goals or encouragement from parents that they concur with the goals of the mathematics program. Other support is cognitive, such as help in working with cooperative groups or attending a workshop given by a consultant on discrete mathematics or fractal geometry.

✔ *Encourage participants to set further goals for their own professional growth.* Professional development should be regarded as a continuous process in which teachers constantly reflect on their own teaching and seek to improve it.

In planning for ongoing professional development, each of the preceding principles should be considered. Addressing every one of these areas increases the likelihood that professional development activities will result in the desired outcomes.

Types of Professional Development Activities

In the previous sections of this chapter, many different types of professional development activities have already been mentioned. These can be categorized in several different ways: individual, with a partner, or as part of a group; at the school, in the district, or outside the district; scheduled or not; with a presenter or with a facilitator. Each type of professional development activity has advantages and disadvantages and may or may not be appropriate for a given individual at a given stage in the change process.

One general type of professional development activity is the *inservice session.* These are scheduled times at which a group of teachers meet with a presenter to learn about a specific topic or approach. This may be a one-time workshop (relatively ineffective except to build awareness), a series of hands-on workshops focusing on a specific grade level (especially appropriate for those beginning to use an innovation), a conference (effective for novice and experienced users of an innovation), or a course (especially appropriate for those integrating an innovation into their everyday practices).

Similar to the *inservice session* is the *work group* or *discussion group.* These also are meetings held at scheduled times, perhaps with a facilitator. Some examples of these types of sessions include a town meeting held to discuss the school's vision of what a mathematics program should look like (Stage 1: Awareness), discussions of school needs (also Stage 1), groups that meet to discuss articles that they have read (Stage 1) or visits made to other schools (Stage 1 or 3), committees charged with developing a school vision (Stage 1), grade-level or department meetings at which instructional approaches are discussed (Stage 2: Transition), and informal study groups such as the one described in the following vignette.

Starting a Study Group

In an elementary school, six K-3 teachers have met for the last school year on Wednesdays from 3:30 to 5:00 p.m. The teachers meet to discuss new trends in early education research, whole language, New Jersey's *Mathematics Standards*, and initiatives to create opportunities for their students to "do" mathematics and science.

"I found that making the change was a bit intimidating. I wasn't sure that the kids would learn," said Mr. Halloran. "So Carla and I decided to work together." After a bit of time, the two teachers were joined by two more of their colleagues, who later invited two additional teachers.

"It really helped to have Larry working with me. This year we were able to create thematic units of study that blended science and mathematics," reported Mrs. Garcia. "The students are really excited about learning how to investigate their world."

Some *work groups* take place outside the school or district. Some good examples of this are curriculum consortia such as the one in Cumberland County, math alliances which involve K-12 teachers and university or college faculty, committees of the New Jersey Mathematics Coalition or of the Association of Mathematics Teachers of New Jersey, or groups planning for special activities, such as Math, Science, and Technology Month. Still another type of *work group* is the professional development network being established by the New Jersey Statewide Systemic Initiative (NJ SSI).

Still another type of professional development activity provides individualized support for teachers as they implement new instructional approaches. Peer coaching and mentoring usually involve pairs or small groups of teachers working together in order to improve and/or refine their skills, understandings, and performance. Specifically, in peer coaching, teachers work with a colleague, whom they have selected, to achieve specific instructional goals through a process of regular observations and feedback. In a mentoring relationship, the teachers do not choose one another. Rather, a more experienced teacher is teamed with a lesser experienced teacher for the purpose of providing, improving, and/or refining specific instructional techniques and/or specific content. (See the earlier vignette about the consulting teacher.) In order for peer coaching and mentoring to succeed, teachers should receive training in the use of peer coaching and mentoring models, and time during the school day should be allotted for conferences and in-class observation of one another. The amount of time required for the initial seminar is from one to three days. Further, in order for mentoring to succeed, the roles of mentor and student need to be clearly developed, understood, and accepted.

Mentoring and Teacher Portfolios

To introduce the concept of portfolio assessment, a supervisor initiated the idea of an unofficial teacher portfolio. The teacher portfolio generated a positive attitude towards portfolio assessment and provided an opportunity for sharing instructional creativity. Three teachers were each asked individually to prepare a portfolio by selecting three examples of assessments they had used with students in the past quarter. They were encouraged to select assessments that typified life in their classrooms. They each wrote a paragraph describing each assessment and then explaining why they had selected it. The supervisor met with each of the teachers individually to give them feedback on their portfolio and then suggested that they might find it helpful to meet together to discuss their assessment strategies. The supervisor also sent a memo to other math teachers highlighting outstanding assessment examples. Some of the types of assessments that the teachers included were:

Making up functions which exemplified properties learned
Lab summary sheets that drew together concepts learned in various labs
Creation of a test and answer key on a given topic
Designing a physical model to illustrate the wrapping function in trig
Lots of "explain how you would find ..." questions
Student journals

Resources Needed for Professional Development

Many types of resources are needed to support professional development. Some of these are not difficult to provide, such as encouraging teachers to try new approaches or providing them with support to ease the stress often engendered by change. Others, such as providing a time and place for teachers to meet collaboratively or providing computers for teachers to use, require reallocating or expanding resources.

Providing time for reform is probably one of the most critical issues facing school districts. The desire to revise the mathematics program may be present, yet without adequate time for planning, staff development, and implementation, the impetus to actually make the reforms may not be generated. When districts are examining the use of time in order to advance reform of the mathematics program, they need to decide:

- Which staff members should work together
- How often key participants should be away from their classrooms
- The duration of additional time demands for staff members

Researchers Susanna Purnell and Paul Hill (*Finding Time for Reform*, 1990) studied schools and businesses which had successfully implemented reform and identified six strategies used to provide time for reform. "An integral part of any attempt to restructure school is the need to create time for the school staff to help design, endorse, and enact reform." How can we organize time in order to make reform a reality? The six strategies Purnell and Hill found present in successful efforts at reform are:

✓	*SUGGESTIONS*

✔ *Refocus the purpose of existing time commitments.* For example, faculty and department meetings and district-wide staff development days can be used to support reform efforts. Thus, part or all of regularly-scheduled faculty meetings might be devoted to planning for an integrated mathematics and science program.

✓ *Increase the amount of time available.* Common strategies that have been used include issuing supplemental contracts, providing stipends for additional time spent in professional development activities, and increasing the number of contract days for faculty.

✓ *Promote volunteer time.* Some schools have increased teacher participation in reform efforts by providing incentives for teachers to volunteer their own time. This may be done by providing services such as child care after school or on Saturdays. It may also be done by providing for credits on the salary scale on the basis of time spent in professional development activities.

✓ *Promote more efficient time use.* Time can often be reallocated by conducting more effective meetings and using telecommunication and computer technology to better manage communications and time. For example, procedural matters can often be handled better in writing than in a meeting.

✓ *Reschedule the school day.* Adjustments can be made in the master schedule or hours can be banked towards early dismissal or late arrival of students. This time can be used for professional development and can be built into the school's weekly and monthly schedule.

✓ *Provide time outside the classroom during the school day.* The most frequently cited time-creation strategies include the use of substitutes, cooperative arrangements with universities in which faculty and students take over classrooms, the use of outside resources (e.g., outside speakers, parent volunteers), and the use of school personnel (e.g., school administrators serving as substitutes, combining classes for joint presentations). Some schools provide double planning time for mathematics and science faculty involved in developing interdisciplinary courses, others relieve staff of "duty" assignments, and still others schedule concurrent planning periods for faculty involved jointly in curriculum efforts. Some schools designate a "team" leader, who is provided with assigned time for coordinating plans among members of the team. Still others build a "conference" period into the day that can be used for teacher planning, student-teacher conferences, or parent-teacher conferences. The following vignette shows how release time was used to provide for ongoing staff development.

Managing Time in Order to Promote Collaborative Exchanges

At one county vocational school system, the basic skills teachers of mathematics from four high schools met monthly for a half-day throughout the entire school year as a product team. The purposes of the product team meetings were for the teachers and their supervisor to work collaboratively to develop appropriate instructional practices; to reorganize and revise the course content; to investigate new software, manipulatives, and teaching practices; and to share innovations, products, and information learned through workshops or professional reading to ensure a quality mathematics program. Since all of the basic skills teachers were teamed with secondary mathematics teachers, the basic skills teachers were able to leave their schools for one-half day per month without disrupting instruction. In turn, since each team of teachers had a common planning time and a team planning period built into their work schedules, they were easily able to share their insights and new learning with their partner throughout the school year. Further, teams of teachers participated together in staff development opportunities outside of the school district. During the two years the teachers met in product teams, major shifts in the course content, instructional practices, and pedagogy occurred.

In addition to the previous strategies for increasing the amount of time available for work on reform efforts, Purnell and Hill discovered that the following strategies were also employed in order to reduce barriers to time:

✓	SUGGESTIONS

✔ Waive policies and contract provisions that limit time.

✔ Minimize administrative requirements associated with reform.

✔ Refrain from introducing programs that compete for time; trying to change everything at once generally leads to chaos.

✔ Incorporate non-classroom time for teachers as a routine part of the school day and calendar.

✔ Provide or enlist adequate resources for reform. Write grants to fund summer staff development activities, set up an education foundation that provides mini grants for teachers to integrate technology into their classrooms, work with the Parent Teacher Association (PTA) or Organization (PTO) to raise funds for calculators or computers, or solicit corporate contributions for equipment and materials. (One school had a fund-raising night at the local fast food restaurant, with a certain percentage of the gross sales donated to the school.)

School Organization

If genuine reform is to take place, there need to be changes in how schools are organized (Canady, 1988). Schools must be organized in such a way that they facilitate rather than inhibit collaborative work among faculty and students, active learning of mathematics, integration of mathematics with other subject areas, and use of technology in the mathematics classroom. Specifically, issues concerning time, staffing, and space need to be revisited, analyzed, and evaluated.

Time

The National Education Commission on Time and Learning in its report, *Prisoners of Time*, states that learning is a prisoner of time (1994, p. 7). In summarizing data regarding time use in American schools, they reported that the typical secondary school:

- opens and closes its doors at fixed times in the morning and early afternoon
- is in session for nine months
- offers a six period day with approximately 5.6 hours of classroom time
- provides students with 51 minute instructional periods
- is in session for 180 days per year
- bases a graduation requirement on seat time

Further, the Commission wrote that four false premises support how schools are organized (1994, 8). These false premises are as follows:

- All students arrive at school ready to learn in the same way.
- Academic time can be used for non-academic purposes with no effect on learning.
- Because yesterday's calendar was good enough, it is good enough now.
- It is reasonable to expect world-class academic performance from students within the current time constraints.

In *Prisoners of Time*, the Commission reported that America may be ready to accept a change of plan for its schools. More than 52 percent of Americans favor students spending more time in school, with a plurality favoring an increase in the number of school days, but only 33 percent favoring a longer school day. At the elementary, middle, and secondary levels, we need to analyze how we organize time in light of our plans for mathematics program reform.

Some of the issues regarding time that should be addressed include whether periods are to be flexible (determined by the teachers) or rigid (following an unchanging bell schedule), whether students attend the same classes all year or only part of it, whether students attend the same classes daily, and how much time is needed by students for mathematics learning.

Flexibility

With cross-curricular integration as an important component of effective mathematics programs, it is important for teachers to be able to reorganize time during the school day in order to address specific content needs. This is most easily accomplished in the self-contained elementary school classroom, as can be seen in the following vignette.

Science, Mathematics, Reading, and Writing in a First Grade Classroom

In my first-grade class each fall, the students and I investigate leaves. This is always a favorite unit for students. They learn about science, reading, writing, and mathematics during this unit. For example, after collecting and identifying their chosen leaf, the students do leaf rubbings in their notebook and then work with their partner, taking turns telling what is similar and different about each of their leaves. Students' responses as to similarities or differences between the leaves are recorded using a Venn diagram. Children note size and the absence or presence of certain shapes. I use the students' responses to introduce the terms *pinnate* and *palmate*. Next, using a two-column chart marked at the top with characteristics of a pinnate or palmate pattern, the children place their leaf in the appropriate column. The children then record in their notebooks their understanding of the chart through pictures and, in some cases, text. Concepts of similar/different and more/less are examined by the students. This unit culminates with the children hearing stories and reciting poems about the fall, about leaf collecting, and about ways to investigate our world. Because I organize instruction using large blocks of time, I can shift and reorganize time according to instructional needs. This strategy helps me to deliver the best possible instruction to my children.

Classrooms in which math is integrated with other subjects are also beneficial in the middle school. Some middle schools accomplish this by assigning teams of 100-125 students to a designated group of teachers.

In this situation, teachers from two or more disciplines (e.g., mathematics, science, social studies, and English) meet regularly to coordinate instruction for a particular group of students. In the meetings, the teachers plan to capitalize on what is being presented in one another's classes. Monitoring student progress, designing assessment projects, and discussing discipline concerns are all a part of the dialogue. By providing for a flexibly scheduled day, teachers can rearrange and allot different amounts of time based on specific content and student needs. For example, teachers might combine science and mathematics instruction time twice a week in order to address the needs of a specific project. An example of a collaborative effort involving 7th grade mathematics and language arts teachers is provided in the vignette below:

Geometry and Reading: Perfect Together

One of the concerns I had with my seventh-graders was finding out what they already knew about geometry. I could see by the expressions on their faces that they were confused by many of the terms I was using and the by the physical shapes I was showing them. Since I work in an instructional team with a language arts teacher, I decided to ask Ms. Carl for some suggestions. She told me about two techniques and volunteered to teach them to my students.

I began the next class by using the first technique, K-W-L (know, want to know, learned). I asked the students to briefly write in their notebooks everything they knew about geometry and everything they wanted to know. The students shared their responses while I recorded them on the overhead. My students' responses showed me that they knew much more than I had suspected. However, my hunch that they had difficulty connecting terms with the actual objects was also confirmed. During the next few classes, we worked primarily with solid geometric shapes, geoboards, and several computer programs (e.g., *Geometer's Sketchpad*) in an effort to solidify (no pun intended) their understanding.

Ms. Carl then taught a vocabulary lesson using a semantic feature analysis chart. The students had used this technique before in reading class but were surprised to see how well it worked with geometry. The students and Ms. Carl listed the names of polygons and then they listed a few features such as plane figure, straight sides, and all sides have equal lengths. The students then made a chart like the one below, labeling each column as sometimes (S), always (A), or never (N).

Shape	Plane figure?	Straight sides?	Four sides?	4 right angles?	Closed figure?	Sides equal?
Triangle	A	A	N	N	A	S
Rectangle	A	A	A	A	A	S
Quadrilateral	A	A	A	S	A	S
Square	A	A	A	A	A	A
Parallelogram	A	A	A	S	A	S

Later we added additional features: 2 pairs of opposite sides have equal length, 2 pairs of opposite sides are parallel, 3 sides and 3 vertices, and more. On the next day, the students reviewed the items that they wanted to know (W) and marked an L next to those they now had learned.

Another technique sometimes used in the middle school involves scheduling intact groups. A group of students moves to classes throughout the school day as a unit. Students are kept together for instruction in from two to five different disciplines per day. Large blocks of time are provided to teachers and they are asked to schedule appropriate amounts of time for instruction in areas assigned to them. Common planning time is provided, and activities are usually coordinated by a unit leader.

A majority of secondary schools in the United States uses a *traditional scheduling* model based upon ability grouping, subject matter, and grade-level divisions (Wrangell, 1990). This mass-production model should be studied carefully, because, all too frequently, students' needs become secondary to the efficacy of the model, rather than the model being adjusted to meet the needs of students. The flexibility that is desirable at the lower grade levels is often more difficult to arrange at the high school level, but it is equally desirable.

One way of providing flexibility is to keep the structure of the traditional schedule but offer class periods at varying times during the week. For example, although a seven-period schedule is in existence, only six periods might meet each day, thereby allowing each class to meet for a double period over a seven-day cycle. This model also allows for a daily rearrangement of periods during the school day and for organizing the school year in cycles. A similar schedule in which each period is lengthen but one class is skipped each day allows for slightly longer class periods for all subjects.

Variations in choice of time patterns for class periods, instructional practices, and number of students in group settings are also served by a *modular schedule*. In this type of situation, the day is divided into twenty-minute modules, and students are scheduled into classes that may have differing lengths. For example, in one school, students at the lower achievement levels were assigned to mathematics classes that were 60 minutes long (three modules), while higher-achieving students had mathematics classes that were only 40 minutes long (two modules). The number of modules that a course meets can vary from day to day or can remain stable.

A *vertical* or *flexible scheduling* technique, generally used with individualized pacing, also provides flexibility. In this model, courses are designed around themes. Regardless of age, students schedule themselves for their courses from a master schedule and negotiate long-term contracts for each term. The amount of time spent on a particular course may vary from student to student or from course to course.

Frequency and Duration

Other scheduling considerations also arise in connection with improving the mathematics program. How much time should students spend on mathematics? How frequently should classes meet? Should the class meet all year? Walter Borg, in a review of research about time and school learning, found that of the three areas typically measured by standardized tests (reading, language arts, and math), "... mathematics is the most strongly influenced by potential quantity of schooling." (1980, p. 49). Rosenshine (1980) reports that second-graders spend an average of 35 minutes a day on math and fifth-graders spend an average of 45 minutes a day on math. This finding was based on actual observations of classes and including time spent doing math-related activities in other subjects. New Jersey's students typically spend about 45 minutes a day on math in the secondary grades; students in other parts of the country (e.g., Texas and Oklahoma) spend about 55 minutes a day on mathematics.

✓	*SUGGESTIONS*

✔ Increase the amount of time elementary school students spend on mathematics to a minimum of five hours a week (*Agenda for Action*, NCTM, 1980, 20).

✔ Increase the amount of time middle grades students spend in mathematics classes to a minimum of seven hours a week (*Agenda for Action*, NCTM, 1980, 20).

✔ Require high school students to take mathematics each year they are in school and increase the amount of time spent each week in mathematics classes.

✔ Consider different possibilities for scheduling mathematics classes. For example, some schools now use *block scheduling* in which students take four 85-minute classes each day in the fall and then take four different classes in the spring. One difficulty with this kind of schedule is that students are not studying mathematics for an extended period. A variation of this which provides for greater continuity is one where students take four 85-minute classes one day and four different ones the following day. Another variation is to have two 15-week sessions in the fall and spring during which students follow typical schedules, and have a six-week winter schedule in which alternative schedules are used.

Staffing

The major issue with respect to the staffing of mathematics classes is whether classes should be taught by a generalist or a specialist. In most elementary schools, mathematics is taught by the general elementary teacher in a self-contained class. Supplementary instruction may be provided by a basic skills teacher or a gifted and talented teacher. In most high schools, mathematics is taught by a specialist who teaches mathematics all day to different groups of students of varying achievement levels. Other organizations are also possible. In selecting an organization appropriate for a particular situation, it is important to consider both the needs of the students and the talents and interests of the faculty.

Many elementary schools prefer self-contained classes, especially for the primary grades. It is generally felt that younger students respond better to the more consistent environment provided by a single teacher. In some elementary schools, however, older students may be taught by two or more teachers. In one model, teachers are paired so that one teacher teaches mathematics and science to two different classes, while the other teaches language arts and social studies. This model allows the teachers to develop a specialty area and to integrate content reasonably well while working with a relatively limited number of students.

Another possible model for elementary school organization relies upon a mathematics specialist who teaches only mathematics to students in five different classes. In this case, students spend about half the day with a homeroom teacher who teaches reading, language arts, health, and social studies to two different classes. The students spend the remainder of the day in "specials," including both math and science. Such a model provides for greater expertise in mathematics by the teacher while maintaining a stable base for the students.

One concern at the elementary level is the common practice of pulling out students for special instruction or support service. This practice may result in a loss of instructional time for students, thus frustrating them. An alternative is to provide in-class support for these students. The two teachers (classroom teacher and teacher providing supplemental instruction) work with two groups of children within an instructional area at the same time. While one group is receiving instruction, the second group relocates to another area within the classroom to receive supplemental or special instruction. This in-class delivery of supplemental instruction is favored by such federal programs as the Basic Skills Instructional Program and is also effective for gifted students. It offers schools flexibility in staff assignments and fosters teacher creativity in choice of instructional design. Care must be taken, however, to ensure that the teachers are equal partners in the planning and the implementing of instruction and assessment.

Research findings have suggested that educators should carefully consider the physical, emotional, and social needs of twelve to fourteen year-olds when designing how time will be utilized at the middle school level. While departmentalization may produce more high-quality instruction in a particular content area, positive student-teacher relationships may suffer. Similarly, self-contained classrooms at this level generate positive student-teacher relationships but often the quality of instruction is lessened (Hollifield, 1988).

Our Roles In Improving Mathematics Education

I'm a Teacher. What are my roles in implementing change?

- I am the direct link to children in the classroom. If changes do not happen in the classroom, the community loses faith in the ability of the educational system to reflect the real world. My success as a professional educator impacts the future opinions of my students and reflects my own capacity to learn and grow along with my students.
- Parents expect me to develop the skills and focus the creativity of their children, making them productive learners. They want me to enhance the natural curiosity of their children in fields such as mathematics and science. Parents will support a school system that delivers on its promises.
- I have a professional responsibility to work with my colleagues in mathematics, including those who are elementary generalists, to make mathematics an exciting and productive experience for the children we teach. I need to find ways to cross disciplinary lines.

I'm a Supervisor. What are my roles in implementing change?

- As a leader in mathematics, I am convinced that mathematics reform is needed. I am accountable for my understanding, interpreting, and implementing the recommended changes. I am an advocate for improvements in mathematics education. I will take an active role in mathematics reform because students need a world class education in mathematics.
- The only thing constant in progress is change. A person's attitude can accelerate or block the change process. I need strategies for working effectively with people.
- My role as a leader involves communicating, and often convincing teachers, administrators, parents, policy makers, the community at large, and students of the need for change. I need techniques in order to be effective.

I'm an Elementary School Principal. What are my roles in implementing change?

- I am more than a manager. I am the instructional leader with the responsibility of helping teachers understand the "why" and "how" of mathematics reform. I have an important role to play in the creation of staff development that helps my teachers keep abreast of change and responsive to new knowledge of how children learn.
- I am the first person whom parents and the community hold accountable for providing a current education that looks to the future. I am the educational link between the school, the parents, and the community.
- The first exposure to structured learning occurs in my school. In mathematics, I need to be sure this experience builds on young children's natural curiosity and love of puzzles, games, and riddles that encourage mathematical thinking.
- My school's mathematics program has to be productive and exciting so children will see mathematics as more than just arithmetic and far more than boring drill. They need challenging opportunities that will keep them enthusiastic and prepare them for the next level of learning.

I'm a Secondary School Principal. What are my roles in implementing change?

- My school is the last step in student transition to work or higher education. The community and the administration hold me responsible for providing students the means by which either choice will open challenging opportunities.
- I am the instructional leader in my school to whom teachers look for guidance and direction. I set the tone for progress or I become a barrier for creative change by my teachers.
- I need to work with my teachers to plan staff development that helps them understand the "why" and "how" of change in mathematics education. The changes in mathematics education can be a blueprint for the educational change that will be occurring in other disciplines as additional standards are implemented.

I'm a Superintendent. What are my roles in implementing change?

- The school board and the community hold me accountable to provide an educational program that does not lag behind the progress made in this field as more schools implement the *Mathematical Standards* adopted by the New Jersey State Board of Education. My school board needs to be prepared, and I must assist them to be responsive to community expectations.
- Principals and teachers look to me for guidance and support of their educational needs. I need to provide professional growth opportunities and resources that keep them abreast of the changes in mathematics education.
- I need to assist my school board with current information about mathematics reform to guide their review of needed policy changes.
- I need to ensure my board's ability to respond to questions which compare our district's mathematics programs and achievements with those in similar districts.

I'm a School Board Member. What are my roles in implementing change?

- My community holds me accountable for the success or failure of the school program. Taxpayers support the schools and expect the school board to provide students with an education which prepares them for the future workforce and gives them the ability to apply what they learn to solving real work problems.
- I help set the policies that direct the actions of the superintendent. In mathematics education, this direction must assist the timely improvement of our curriculum.
- I am the communication link between the school system and the larger community. I have the obligation to explain the need for changes that are coming in our schools and the specifics of the steps that our board is taking to ensure that the reform is successful. I can build supporters of our school system by showing other adults who do not have children in the schools the advantages of this reform effort, and why mathematics, with a curriculum framework now in place, is the logical starting point.

I'm a Parent. What are my roles in implementing change?

- A parent is a child's first teacher and role model. Children believe what they see and hear at home. If they hear that mathematics is too hard or not important to their future, they will carry that myth with them to school, impairing their chances of success.
- All parents want their children to succeed in life. As parents, we need to show our children early on that school mathematics is not only necessary for survival, but that it is the door to most careers they will later choose. The right skills and the right choices will open the doors to opportunities in the world outside of school.
- Children have a natural curiosity. Helping them find answers to the constant question "why" is a natural bridge into mathematics and science learning. Early exposure at home to simple puzzles and games helps us build the bridge from learning at home to success at school.
- Throughout their school experience, children need support and encouragement. We need to stay involved in the mathematics options that are available to our children, enabling them to keep moving ahead rather than relegating them to lower level jobs.

Who can help me?

- Teachers addressing changes in their own classrooms will welcome the opportunity to give information and guidance to parents and colleagues.
- The National Council of Teachers of Mathematics offers many publications. Write to NCTM, 1906 Association Drive, Reston, VA 22901.
- The Association of Mathematics Teachers of New Jersey provides many conferences. Write c/o Nancy Schultz, 20 Aberdeen Avenue, Wayne, NJ 07470
- The Web Site is http://dimacs.rutgers.edu/nj_math_coalition/framework.html/ for on-line resources related to the *Mathematics Standards*.
- The mathematics specialist(s) at the New Jersey State Department of Education can provide information about initiatives at the state level, including the statewide assessments. See the Department's home page at http://www.state.nj.us/education/.
- The New Jersey Mathematics Coalition, at 908/445-2894 for assistance in obtaining workshop leaders. The home page of the Coalition, http://dimacs.rutgers.edu/nj_math_coalition/, contains the Parents' Guide to the Mathematics Standards, the Coalition newsletter, and other timely information.

Summary

In order to achieve the world-class mathematics classroom that students need and deserve, and in order to realize our vision, systemic changes must take place. After recognizing, and then developing an awareness of other possibilities through exploration, we move to a transition period where we make commitments to try new teaching and learning approaches. We begin to see what needs to be re-envisioned and rearticulated. We begin to examine professional development and school organization in light of our goals. We institutionalize the things that we have introduced.

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Software

Geometer's Sketchpad. Key Curriculum Press.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

Framework

New Jersey Mathematics Curriculum Framework

The New Jersey Mathematics Curriculum Framework is designed to provide guidance and assistance to teachers and schools in implementing New Jersey's Mathematics Standards.

It was developed by the New Jersey Mathematics Coalition, in collaboration with the New Jersey Department of Education, with support from an Eisenhower grant from the United States Department of Education.

The New Jersey Mathematics Coalition, organized in 1991, draws together all sectors of the community — education, public policy, business and industry, and the public — in a sustained effort to improve mathematics education throughout the state, and to increase the public's awareness of the importance of mathematics to the future of New Jersey's children.

Further information about the Coalition, the text of this framework and any subsequent extensions, and a Guide for Parents on New Jersey's Mathematics Standards can be obtained at the Coalition's Home Page, http://dimacs.rutgers.edu/nj_math_coalition, at (908)445-2894, or P.O. Box 10867, New Brunswick, NJ 08906.

The New Jersey State Department of Education, through its Strategic Plan, has prioritized the effort to define New Jersey's expectations for student learning. The resulting New Jersey Core Curriculum Content Standards, including standards in seven content areas and cross-content workplace readiness standards, were adopted by the New Jersey State Board of Education in May, 1996.

Inquiries concerning the New Jersey Core Curriculum Content Standards, and requests for copies of this framework should be addressed to the Office of Standards and Assessment, New Jersey State Department of Education, CN 500, Trenton, New Jersey 08625-0500. The New Jersey Standards, in all content areas, can be found at http://www.state.nj.us/education/, the Department of Education's Home Page.

Curriculum

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