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ABSTRACT

This paper describes the actions of one secondary school mathematics teacher in establishing a community of mathematical practice within which students acquire not only knowledge and skills, but also the epistemological values of mathematics. Classroom observations over a period of 18 months, together with interview and questionnaire data, are used to sketch a model of the teacher's interactions with his students as he works toward creating a culture of mathematical sense-making. The results indicate four aspects of the teacher's role that are particularly important in establishing a classroom community of practice: (1) modeling mathematical thinking; (2) cognitive and social scaffolding; (3) encouraging individual reflection, self-monitoring and checking; and (4) introducing tools for mathematical communication. Contains 33 references. (DDR)

Making Sense of Mathematics: The Teacher's Role in Establishing a Classroom Community of Practice

Merrilyn Goos

Recent mathematics curriculum reforms in Australia and the US reflect the growing realisation that traditional classroom practices may have a detrimental effect on students' learning. These changes are consistent with new conceptions of mathematics teaching and learning, which emphasise that *doing mathematics* is both an intellectual and a social activity. Drawing on theories concerning the relationship between social interaction and learning, current research in mathematics education argues that students learn to think mathematically by participating in a community of mathematical practice, within which they acquire not only knowledge and skills, but also the epistemological values of the discipline. This paper describes the actions of one secondary school mathematics teacher in establishing such a community. Classroom observations over a period of eighteen months, together with interview and questionnaire data, are used to sketch a model of the teacher's interactions with his students as he worked towards creating a culture of mathematical sense-making.

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BACKGROUND

The last decade has seen calls for significant changes to the way mathematics is taught in schools. In the United States, the National Council of Teachers of Mathematics (NCTM) has published a series of influential documents which articulates new goals for students' learning, and promises to bring about a radical rethinking of current mathematics teaching practices (NCTM, 1989, 1991). In these documents the traditional emphasis on memorisation and basic skills has given way to arguments that students need to develop reasoning and problem solving skills, to learn to communicate mathematically, and to work collaboratively as well as individually.

It is clear that a similar shift in priorities has also occurred in Australia, with the intent of the NCTM Standards being echoed by the *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991). Like the NCTM Standards, the National Statement proposes a set of reformist goals for school mathematics, for example:

> Students should acquire the mathematical knowledge, ways of thinking and confidence to use mathematics in both familiar and unfamiliar situations.
>
> Students should develop skills in presenting and interpreting mathematical arguments.
>
> Students should develop their capacity to use mathematics in solving problems individually and collaboratively.
>
> Students should learn to communicate mathematically to a range of audiences.
>
> Students should experience the processes through which mathematics develops (e.g. conjecture, generalisation, proof, refutation).

The incorporation of these goals into secondary school mathematics syllabi (e.g. Board of Senior Secondary School Studies, 1992) challenges Australian teachers to re-examine their conceptions of mathematics learning and teaching and to develop new ways of working with students in their classrooms. This paper considers how the challenge may be met.

The paper begins by comparing the goals and practices of "traditional" and "reform" mathematics classrooms, and goes on to offer a set of principles which constitute a theoretical rationale for creating a new culture of school mathematics which is consistent with the agenda for change sponsored by curriculum development authorities. Examples from the research literature are then presented to illustrate how these principles have, to a limited extent, been put into practice. The next part of the paper provides more extensive evidence of the possibilities for change, by reporting on some findings of an Australian research study conducted in the senior secondary school context. A significant outcome of the study has been the specification of the teacher's role in initiating students into mathematical ways of thinking and communicating.

TRADITIONAL MATHEMATICS AND REFORM MATHEMATICS

Traditionally, "learning mathematics" has been defined as mastering a predetermined body of knowledge and procedures. The teacher's job was to present the subject matter in small, easily manageable pieces and to demonstrate the correct procedure or algorithm, after which students worked individually on practice exercises. However reasonable this approach may appear, numerous research studies (e.g. Schoenfeld, 1988) have shown that traditional mathematics instruction can leave students with imperfect understanding and flawed beliefs about mathematics. When students' activity is limited to imitating the technique prescribed by the teacher, they can create the appearance of mathematical competence by simply memorising and reproducing the correct way to manipulate symbols, and may even come to believe that producing the correct form is more important than making sense of what they are doing (Cobb, 1986; Cobb & Bauersfeld, 1995).

Associating competence with symbol manipulation is but one of many undesirable consequences of the traditional approach to teaching mathematics. In addition, reliance on the teacher or text as the source of knowledge reduces students to a passive, accepting role, and leads them to expect that there must be a readily available method or rule for every kind of problem. The term "problem" is itself problematic, as students know that the practice exercises on which they work constrain them to use the algorithm most recently taught, a situation which not only is highly contrived, but also leaves them helpless when faced with genuine problems where the solution method is not immediately obvious (Schoenfeld, 1992). As a result of school experiences such as these students equate mathematics with meaningless practice on routine exercises, and learn that mathematics is not meant to make sense.

The goals and practices of the reform movement stand in contrast to those of traditional instruction (see Figure 1, adapted from Forman, in press). In reform classrooms the goals of mathematical communication, collaborative problem solving, and effective strategy use and explanation are achieved by increasing the variety of participatory roles open to students. Encouraging students to initiate their own inquiries and to debate their ideas with peers also helps them to see themselves and each other as legitimate intellectual resources.

	Traditional Instruction	Reform Movement
Goals	Individual mastery of basic skills (mathematical facts and algorithms). Automaticity and accuracy.	Communication skills. Collaborative problem solving. Effective strategies and explanations.
Practices	Teacher led recitation. Individual seatwork. Teacher and text are sole sources of authority.	Student presentations. Small group work. Students see themselves and peers as intellectual resources.

Figure 1. Goals and practices of traditional mathematics instruction and the reform movement. (Adapted from Forman, in press)

While such a comparison can suggest what a "reform" classroom might look like, changes to classroom practice are unlikely to occur, or to persist, unless they can be justified on theoretical grounds. The next section provides a theoretical rationale for a new pedagogy by drawing together ideas about mathematical thinking, classroom culture, relationships between social processes and individual cognition, and teacher beliefs, to argue that learning mathematics involves entry into a *community of practice* held together by shared mathematical values.

PRINCIPLES FOR A NEW PEDAGOGY

Mathematical thinking is based on cognitive strategies and epistemological values.

Goals for school mathematics are derived from a conception of what mathematics is, and what it means to understand mathematics. There is growing agreement within the mathematical community that learning to think mathematically involves acquiring not only skills, strategies and knowledge, but also habits and dispositions of interpretation and meaning construction (Schoenfeld, 1994), that is, a *mathematical point of view*. When students adopt the epistemological values of the discipline they "come to see mathematics as a vehicle for sense-making" (Schoenfeld, 1989a, p. 81), rather than a collection of arbitrary rules for symbol manipulation.

Learning to think mathematically occurs through a process of enculturation.

If seeing the world in the way that mathematicians do is a fundamental element of mathematical thinking, then mathematics education is as much a socialisation process as an instructional process; most importantly, students' understanding of what the discipline is about is shaped by their participation in the classroom mathematical community. The notion that values are culturally defined raises questions about the kind of classroom environment that is appropriate for developing the habits of mind prized by the discipline (Weissglass, 1992). Some answers are found in the emerging body of literature (e.g. Collins, Brown & Newman, 1987; Lave, Smith & Butler, 1989; Resnick, 1989; Schoenfeld, 1989b) which argues that mathematics is an inherently social and collaborative activity, and that mathematics classrooms should therefore engage students in these authentic practices of the wider mathematical community.

Social processes are the vehicle for internalising strategies and values.
The mechanisms linking social processes and mathematical thinking can be understood in terms of sociocultural theory, first sketched by Vygotsky (1978) in the early part of this century, and now undergoing considerable elaboration and extension (e.g. Bruner, 1985; Forman & Cazden, 1985; Tudge, 1990; Wertsch, 1985). Vygotsky claimed that all higher cognitive functions in a child's development appear first between people, or intermentally, and then within the child, or intramentally, and that these functions are elicited during the child's interaction with expert adults or peers. However, more recent theorising suggests that it is a mistake to limit the analysis of social relations to their cognitive goals or consequences, as social interactions also carry meanings concerned with authority and values (Wertsch & Rupert, 1993). Consequently, when students are socialised into mathematical ways of knowing they internalise both cognitive strategies and the cognitive values implicit in the forms of interaction and communication sanctioned by the teacher.

Teacher beliefs influence classroom practices.
Teacher beliefs about the nature of mathematics and how it is learned determine the features of the classroom environment they create (Fennema & Loef-Franke, 1992; Thompson, 1992). This close connection between epistemology and pedagogy has significant implications for teachers' ability to translate into practice the changing goals of mathematics education. Because their ideas about mathematics were formed as a result of their own school experience, many teachers may not have learned to think mathematically themselves and are thus ill-equipped to model cognitive processes such as conjecture and generalisation (Schifter, 1993). At the same time teachers may have acquired inappropriate mathematical values, which they subsequently communicate to their own students through patterns of classroom social interactions.

The classroom is a community of mathematical practice.
If students are to develop mathematically powerful forms of thinking, as well as appropriate epistemological values, then mathematics classrooms must create a culture of sense-making in which students learn by immersion in the authentic practices of the discipline. Such a culture has the features of a "community of practice" (Lave and Wenger, 1991, p. 98):

> A community of practice is a set of relations among persons, activity, and world, over time and in relation with other tangential and overlapping communities of practice. A community of practice is an intrinsic condition for the existence of knowledge, not least because it provides the interpretive support necessary for making sense of its heritage. Thus, participation in the cultural practice in which any knowledge exists is an epistemological principle of learning. The social structure of this practice, its power relations, and its conditions for legitimacy define possibilities for learning.

Although the "community of practice" metaphor is attracting increasing interest from mathematics researchers, only a few examples of classroom communities fitting this description have yet appeared in the literature. The next section presents some exemplary cases, drawn from primary, secondary and tertiary level classrooms, which bring to life the principles outlined above.

FROM PRINCIPLES TO PRACTICE: EVIDENCE FROM RESEARCH

Lampert (1990) describes a research and development project in a fifth grade mathematics classroom, where the focus was on helping students to construct mathematical meaning while gaining experience with the discourse conventions of the discipline. She explains how she, as the teacher, resolved the inevitable tension between students' individual constructions ("inventions") and the mathematical conventions already accepted within the wider community outside the classroom.

The teacher's role had two components. First, she established norms for social interaction so that the classroom discourse was organised around students' mathematical ideas. This involved: communicating her expectation that students publicly justify their assertions; offering suggestions without imposing her own answers or procedures; creating a safe environment for students to disagree with peers and helping them to clarify their position in an argument; and managing the discussion so that individual student constructions were consolidated into a publicly accepted mathematical procedure. Second, the teacher used her influence as an experienced knower of the discipline to shape students' constructions in a way that initiated them into mathematical conventions. This was accomplished by representing students' ideas (for example, on the blackboard) in a more mathematical form that still retained the intended meaning, thus supplementing students' repertoire of language and symbols.

A similar emphasis on social interactions and mathematical argumentation is found in the university calculus course developed by Alibert and his colleagues (Alibert, 1988). An experimental teaching method was designed to counter first year students' apparent lack of concern for meaning, lack of appreciation of the role of mathematical proof, and failure to recognise the usefulness of mathematics for solving real life problems. The course aimed to establish new customs in the classroom, and was based on four principles:

1. *Uncertainty* in relation to mathematical knowledge was introduced by giving the students responsibility for formulating and verifying conjectures. This created a natural context for requiring argumentation and proof.
2. Students addressed argumentation to their peers rather than the teacher. The purpose was to *convince* others (and themselves) of the truth of a conjecture in order to solve the problem from which it arose.
3. The introduction of *new mathematical tools* became necessary in order to solve a complex, real life problem.
4. Students *reflected* on the nature of their new knowledge, and their own learning process.

As a result of the new teaching method both students and teachers experienced a change in their relationship to mathematical knowledge: students became genuine producers, rather than consumers, of knowledge, and teachers learned to relinquish their position as the sole authority for validating answers. It was also noted that false conjectures provided rich opportunities for learning, because the debate that was generated made students aware of misunderstandings that might otherwise have gone uncontested.

The potential for errors, uncertainty and anomalies to stimulate mathematical thinking is explored more fully by Borasi (1992), who articulates a view of mathematics as humanistic inquiry. Emphasising its human elements acknowledges that mathematics is an ill-structured discipline, full of open questions, ambiguous meanings, and multiple interpretations. The humanistic metaphor arose from her experience in developing a number of innovative teaching approaches in response to the calls for school mathematics reform contained in the NCTM (1989, 1991) Standards. She reports on a ten lesson mini course on mathematical definitions she conducted to help two students who had missed work due to absences from their regular class. Her task was made more challenging by the fact that the students, two sixteen year old females, shared a strong dislike for school mathematics, which they had experienced only as a precise, frustrating and impersonal discipline.

Teaching strategies for promoting and supporting the students' mathematical inquiry included (1) exploiting the complexity of real life problematic situations (2) focussing on traditional mathematics topics where uncertainty and limitations are most evident (3) uncovering humanistic elements within the traditional mathematics curriculum (4) using errors as springboards for inquiry (5) exploiting the surprise elicited by working in new domains (6) creating ambiguity and conflict by proposing alternatives to the status quo (7) using generative reading activities as a means of sustaining inquiry (8) providing occasions for reflecting on the significance of students' inquiry and (9) promoting exchanges among students. At the end of the course the two students revealed that this learning experience had dramatically changed their perceptions of mathematics. Both now recognised that mathematics is the product of human minds, and that mathematical activity offers opportunities for creativity and personal discovery. In addition, the students became more confident in working on unfamiliar and open ended problems, and in initiating their own mathematical explorations.

In each of the case studies described above the teachers worked from the premise that mathematics is both an intellectual and a social activity, and they organised a learning environment in which students were actively engaged in mathematical sense-making. While these positive examples offer a model for teachers who wish to make changes in their own classrooms, the central role of teacher beliefs as the foundation on which such a classroom community is built should not be overlooked. Nor should the practical difficulties of implementing change within a whole school context be underestimated, particularly in secondary schools, where subject specialisation and timetabling divide the students' day between many teachers, each of whom may hold different beliefs about learning.

The issues raised above were investigated in the research study which is the major focus of this paper. The next part of the paper outlines the design and purpose of the study, and then describes the beliefs, goals and practices of one secondary school teacher who was successful in creating a classroom community of mathematical inquiry.

A Classroom Study

The study reported here is part of a two year research project investigating patterns of classroom social interactions that improve senior secondary school students' mathematical understanding, and lead to the communal construction of mathematical knowledge. Four mathematics classes (three Year 11 and one Year 12) and their teachers participated in the first year of the study; three of the teachers, each with a new Year 11 class, are currently continuing their involvement in the project's second year.

Research Design and Purpose

Multiple methods were used to gather data on features of classroom interaction and students' individual thinking. At the beginning of each year questionnaires and associated written tasks were administered to obtain information on students' beliefs about mathematics, perceptions of classroom practices, and metacognitive knowledge. From March until September one mathematics lesson per week was observed for each class to record teacher-student and student-student interactions. At least ten lessons were videotaped in each classroom in the first year of the project. The research plan for the second year included an additional two week period of intensive observation, during which every lesson in a unit of work nominated by the teacher was videotaped. Stimulated recall interviews (Leder, 1990) have been conducted with teachers and students on a number of occasions to seek their interpretations of selected videotape excerpts, and students views' about learning mathematics have been elicited in individual and whole class interviews, and in reflective writing.

One of the purposes of classroom observation was to investigate the teacher's role in a classroom community of mathematical inquiry. Throughout the first year of the project, categories of teacher-student interaction were developed to serve as indicators of such a classroom culture (see Figure 2). Progressive refinement of the categories was guided by continuing observations and the theoretical principles outlined earlier in this paper. Initially, eight categories were identified (numbered one to eight in Figure 2).

Category	*Example*
1. The teacher models mathematical thinking.	*Verbalises and elaborates strategies.*
2. The teacher expects students to clarify, elaborate, critique, and justify their responses and strategies, both to the teacher and to each other during whole class discussion.	*Expects students to take responsibility for validating solutions.*
3. The teacher emphasises sense-making, at both individual and community levels.	*Expects students to make significant contributions to the lesson content by providing, either through individual reflection or peer discussion, intermediate or final steps in solutions or arguments initiated by the teacher.*
4. The teacher makes explicit reference to mathematical language, conventions and symbolism.	*Labels conventions as traditions that permit communication.*
5. The teacher encourages reflection, self-monitoring, self-checking.	*Encourages students to locate and correct their errors.*
6. The teacher uses students' ideas as starting points for discussion.	*Withholds judgment on students' suggestions while inviting comment or critique from other students.*
7. The teacher structures students' thinking.	*Asks questions that elicit strategic steps.*
8. The teacher encourages exploratory discussion.	*Presents "what if?" scenarios.*
9. The teacher structures students' social interactions.	*Asks students to explain ideas and strategies to each other.*

Figure 2. Observation categories developed as indicators of a community of mathematical practice

Although the categories were derived from observations of all four classrooms participating in the first year of the study, it became clear that they were exemplified to varying degrees by the different teachers. Evidence from field notes and videotapes pointed to one teacher who was most successful in sustaining a community of inquiry. However, as the teacher indicated that the values and practices of his Year 12 class had been formed during Year 11, before the research project began, it was decided in the second year of the study to examine his strategies for establishing norms of cognitive and social activity for a new Year 11 class. During this time the previously developed categories of teacher-student interaction were verified, and a ninth category added (number nine in Figure 2). The material is the remainder of the paper is based on data gathered from this teacher and his Years 12 and 11 classes over the two years of the study.

The Teacher's Beliefs about Learning and Teaching Mathematics

Initial inferences about the teacher's beliefs were made by observing the learning environment in which they were played out. Viewing the videotape of a lesson with his Year 12 class later provided the stimulus for the teacher to elaborate on these implicit theories and beliefs (as in Meade & McMeniman, 1992), which are summarised below.

1. Students learn mathematics by making sense of it for themselves, and engagement leads to ownership.

I want to try as much as possible to get them to work it out for themselves.

(Having the students reconstruct a mathematical argument developed in a previous lesson), you're getting them to try to build some sense into it, by getting them to reconstruct it themselves they have to be able to make some sense out of it even if it's only internal consistency with the mathematics ... you hope that way that's building in a more robust cognitive structure they can use later on.

The other important thing about it as well, by doing it this way you've got a degree of ownership involved ... the kids are engaged, and I really think that's because they're owning what's going on, it's not just sitting there, listen to this and away you go.

If you never gave them the opportunity, if you just told them, then they're expecting you—or it's easy enough to wait to be told again. I don't think, long term, that's a great advantage.

2. Teachers should model mathematical thinking and encourage students to make, and evaluate, conjectures.

There's as element of attempting to model the problem solving process in this as well ... at the beginning of Year 11 they do a unit on it and I attempt to keep coming back to these things.

... they won't always offer information and it's important they're encouraged to guess and just have a go. So then other people can criticise it, or they can criticise themselves once they've had a guess.

3. Communication between students should be encouraged so they can learn from each other, sharpen their understanding, and practise using the specialist language of mathematics.

I do think it's important that they're able to communicate with other people and their peers. They will learn at least as much from each other as they will with me. To be able to do that they have to talk to each other. It's also a part, one of the reasons I often force them to say things because they need to be able to use the language because the language itself carries very specific meanings; and unless they have the language they probably don't have the meanings properly either. They need the language to be able to, obviously communicate, but I think it also has something to do with their understanding as well.

Classroom Practices

How these beliefs were manifested in the classroom is illustrated in the annotated observation record of two sequential lessons conducted with the Year 11 class participating in the second year of the research study (see Figures 3 and 4). The annotations refer to the previously developed observation categories shown in Figure 2. In these records the abbreviations T and S refer to the teacher and unidentified students, while other letters of the alphabet are used to identify specific students.

Annotation	Interaction	Blackboard
Structures S's thinking (backward)	T reminds S's of procedure for finding inverse of a 2x2 matrix using simultaneous equations. Asks S's to solve the resulting equations. S's provide equations and solution. T: So the inverse of $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ is $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$.	$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $3a + c = 1$ $5a + 2c = 0$ $3b + d = 0$ $5b + 2d = 1$ $a = 2, b = -1, c = -5, d = 3$
Encourages self-checking	T: Can you check via matrix multiplication that you do get the identity matrix? S's confirm this is so. T: Is it inefficient to do this every time? S's concur.	
Models mathematical thinking *Uses S's ideas* *Sense making & ownership*	T: Could we find a shortcut? L suggests reversing the position of *a* and *d*, and placing minus signs in front of *b* and *c*. T elicits symbolic representation and writes on blackboard.	Inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$?
Models mathematical thinking	T: How could we verify this? S's suggest doing another one. T provides another example; asks students to use "L's conjecture" to write down the hypothetical inverse and check via matrix multiplication. S's do so – they are convinced the method works.	$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{inverse} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$
Structures S's thinking (forward)	T gives another example for S's to try. Gradual increase in S talk as they realise L's conjecture doesn't work for this one (matrix multiplication does not yield the identity matrix).	$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} \xrightarrow{inverse} \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$? $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$
Structures S's thinking (backward)	T reminds S's they can still find the inverse by solving simultaneous equations. S's do so and verify via matrix multiplication.	Inverse is $\begin{pmatrix} \frac{2}{5} & \frac{-1}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix}$
Structures S's thinking (forward)	T: How is this related to L's conjecture? (which is half right). S's reply that the first attempt is too big by a factor of 5, so they need to divide by 5.	
Structures S's thinking (consolidation) *Models mathematical thinking*	T: What did you divide by in the previous example? S's realise they could divide by 1. T: So the new method (dividing by something) works. But how do you know what to divide by?	$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{inverse} \begin{pmatrix} \frac{1}{1} & \frac{-1}{1} \\ \frac{-1}{1} & \frac{2}{1} \end{pmatrix}$
Sense making & ownership	Homework: Find a rule that works for these two cases. Test it on another matrix of your choice.	

Figure 3. Year 11 Maths Lesson #1: Finding the inverse of a 2x2 matrix

Annotation	Interaction	Blackboard
	T asks S's to remind him of the matrix worked on last lesson (homework). The first try gave $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$: you had to adjust by dividing by five. (S's were to find a rule for the divisor).	$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ $\downarrow$ $\begin{pmatrix} \frac{2}{5} & \frac{-1}{5} \\ \frac{-3}{5} & \frac{4}{5} \end{pmatrix}$
	T: What was the divisor?	
	D: *ad - bc*.	
Sense-making and ownership	T: Did you invent your own matrix and test it?	
	S's: Yes, it worked.	
	T names "this thing" (*ad - bc*) as the *determinant*.	
Mathematical conventions and symbolism	T: Let's formalise what you've found. What would I write as the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$?	
	AV volunteers the formula, which T writes on blackboard.	$\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
	L: Would the inverse of a 3x3 matrix be similar?	
	T: Yes, but it's messy—you can use your graphics calculator to do it. You need to be able to find the inverse of a 2x2 matrix longhand.	
	R: What part of that is the determinant?	
	T labels *ad - bc* and writes the symbol and name "del" on blackboard.	$\nabla = ad - bc$ *del*
Models mathematical thinking (test conjecture with another example)	T puts another example on blackboard and asks S's to find the inverse.	Find $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}^{-1}$
	After working for a short time S's begin to murmur "zero". They find that *ad - bc*, the determinant of the matrix, is zero, therefore the inverse cannot be calculated.	
Ownership of ideas	R: Is our method *still* wrong?	
	T: No. Remember, some elements of the real number system have no inverse. So what is the test to find if a matrix is non-invertible?	
	L: The determinant is zero.	
	T: A non-invertible matrix is called a *singular* matrix. What happens if you try to invert this matrix using your graphics calculator?	
	S's try it: see "error" message.	

Figure 4. Year 11 Maths Lesson #2: Inverse and determinant of a 2x2 matrix
(cont. over page)

Annotation	Interaction	Blackboard
Structures S's thinking (consolidation)	T: We can think about this another way. Remember how to use simultaneous equation method to find the inverse ... What happens if the matrix is singular? First find the inverse of this matrix, using simultaneous equations. S's work on solving the simultaneous equations.	$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $2a+c=1$ $2b+d=0$ $a+c=0$ $b+d=1$ $a=1, c=-1, b=-1, d=2$
Structures S's social interaction	T tours the room. Asks AG "Have you done it?" AG: No. T: Then ask AR (sitting beside him) to explain it. S's finish finding solutions.	
Structures S's thinking (backward)	T: What is this related to, from Junior maths? S's: Finding the intersection of two lines. T: These are all linear equations so we could solve them by graphing. S's use graphics calculators to find graphical solutions. T: So one way to find the inverse is to set up simultaneous equations and solve (algebraically or graphically). Now try to find the inverse of $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$ (which we just found is singular) by solving simultaneous equations graphically. S's find parallel lines - no solution.	
Structures S's thinking (forward)	T: Another interesting thing ... you know how to turn a matrix equation into simultaneous equations ... (S's do the conversion and solve the equations) T: Can we do the reverse? What if I gave you the simultaneous equations—how would you make a matrix equation? AR explains how the numbers and the letters are arranged in matrix formation.	$\begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 10 \\ 3 \end{pmatrix}$ $4a+2b=10$ $a+b=3$ $a=2, b=1$
Structures S's thinking (backward)	T: What was the reason we wanted to find matrix inverses in the first place? R: We couldn't divide by a matrix! T reminds S's where they left off previous work on solving a problem that required division of one matrix by another (like the equation on the blackboard). T: Recall the parallel with the real number system ... to solve this algebraic equation you'd multiply both sides by the multiplicative inverse of 3.	$\begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 10 \\ 3 \end{pmatrix}$ $3 \quad x = 6$
Sense-making and ownership	Homework: Solve the matrix equation (by "inventing" matrix algebra).	

Figure 4. Year 11 Maths Lesson #2: Inverse and determinant of a 2x2 matrix (cont.)

Specifying the Teacher's Role

The dominant feature of this teacher's classroom is the manner in which students make sense of, and come to own, the mathematics they construct. In the lessons illustrated above it is the students, not the teacher, who "invent" and test an algorithm for inverting a two by two matrix. The extent of their ownership is made clear by the teacher's labelling of one student's initial suggestion as "L's conjecture" (see Figure 3), and also by another student's question as to whether their discovery of non-invertible matrices makes "our method" wrong (see Figure 4). Contributing to this process of sense-making are the classroom practices instituted by the teacher, which help students to develop the cognitive and metacognitive processes of mathematical thinking (as outlined in Mason, Burton & Stacey, 1985) by participating in the social processes through which mathematical ideas are generated and validated. Four aspects of the teacher's role appear to be particularly important in establishing a classroom community of practice: modelling mathematical thinking, scaffolding students' thinking and social interactions, encouraging individual reflection, and introducing tools for mathematical communication. The features of each are outlined below.

1. Modelling mathematical thinking

Modelling of mathematical thinking occurs on a global, whole-lesson scale. Although the teacher has a specific agenda he does not merely demonstrate how to do the mathematics; instead he involves the students by presenting a problem for them to work on, eliciting students' conjectures and generalisations, withholding judgment to maintain an authentic state of uncertainty as to the validity of conjectures, and asking students to test conjectures and justify them to their peers.

2. Cognitive and social scaffolding

The teacher scaffolds students' mathematical thinking by asking questions which prompt students to clarify, elaborate, justify and critique their own and each other's assertions. These interventions can move students' thinking either *forwards* towards new ideas ("Could we find a shortcut?", "How is this related to L's conjecture?") or *backwards* towards previously developed knowledge or a previously identified goal ("What is this related to from Junior maths?", "What was the reason we wanted to find matrix inverses in the first place?"); or they can serve to *consolidate* students' thinking by drawing together ideas developed during the lesson ("What did you divide by in the previous example?").

The teacher also signals that certain forms of social interaction are valued, for example, by asking students to help their neighbours and discuss ideas with each other.

Both kinds of scaffolding were particularly noticeable during the early weeks of Year 11, when cognitive and social norms were being established. Later in the year, and particularly in Year 12, these forms of argumentation and social interaction appeared spontaneously in both small group and whole class discussion, their appropriation by the students a sign that teacher scaffolding could be withdrawn.

3. Encouraging individual reflection, self-monitoring and checking

Self-directed thinking is initially prompted by teacher questions ("Can you check via matrix multiplication that you do get the identity matrix?"). As the students become accustomed to the teacher's expectations (particularly in Year 12), more subtle interventions are used to promote reflection; for example, allowing time for students to read textbook explanations and examples in order to provide substance for a whole class discussion.

4. Introducing tools for mathematical communication

The teacher avoids using technical terms until students have developed an understanding of the underlying mathematical ideas ("This thing is called the determinant." "Let's formalise what you've found."). The availability of precise language then helps students to make their thinking visible while discussing ideas with their peers.

Student Perceptions

Students' responses to interview questions and reflective writing tasks showed that they were remarkably well-attuned to the teacher's goals, and were aware that their classroom operated differently from others they had experienced. They also felt that there were benefits in the approach practised by their teacher. A sample of their views is presented below.

Stimulated recall interview with three Year 12 students, based on a videotaped lesson segment capturing their discussion about a problem (I is the interviewer, R a student).

I: One of the interesting things is that you don't just accept what each other says.
R: We always assume everyone else is wrong about it!
I: But it's not just saying "No it isn't", "Yes it is".
R: Yeah, we've got to be *proven* beyond all doubt!

Whole class interview with the Year 12 class.

D: So many times I find myself trying to explain something to other people, and you find something you've kind of missed yourself ... Even if they don't really know what they're doing, explaining it to them imprints it to *your* mind.
E: Yeah, and if you can explain it to someone else it means you *know*.

B: In other subjects like (names a non-mathematics subject), the teacher doesn't give you much time to talk to other students. Most of the time, she's (i.e. the teacher) talking. When I talk to D (another student) about something, we get in trouble for talking.
D: It's more like learning parrot fashion.
B: It's mostly pure learning, so what do you discuss? It's already all proven ...

Reflective writing (Year 11 class). Responses to the questions: How did the teacher help you to learn this topic? How did your classmates help you to learn?

AV: The teacher mainly guided us—we learned most things by ourselves. Classmate discussion was very important in this unit, i.e. comparing answers, discussing and explaining things to each other.

DB: We had to work it out by ourselves (with friends and one-on-one with the teacher) which is one of the best learning methods.

Whole class interview with the Year 11 class

L: In other subjects the *teacher* asks the questions; here, *we* do.

DISCUSSION

This paper has been concerned with the changing conceptions of mathematics teaching and learning expressed in both new curriculum documents and the mathematics education research literature. These changes represent a move towards regarding mathematics as a discipline of humanistic inquiry, rather than one of certainty and objective truth, and they pose a significant challenge to teachers to develop classroom practices in keeping with new goals for learning which emphasise reasoning and communication skills, and the social origins of mathematical knowledge and values.

Possibilities for new approaches to mathematics teaching are suggested by the metaphor of the classroom as a community of practice, within which students learn to think mathematically by participating in the intellectual and social practices that characterise the wider mathematical community outside the classroom. This paper has described how one teacher created such a community. Of primary importance were the teacher's beliefs about mathematics, which were the source of both the learning goals he held for his students and the teaching practices he implemented. To achieve his goals of sense-making and ownership the teacher modelled mathematical thinking processes, provided cognitive scaffolding to support students' appropriation of cognitive and metacognitive strategies and the language of mathematics, and through social scaffolding communicated the values of the discipline.

Despite the success of this teacher in socialising his students into mathematical practice, there remain barriers to change which demand consideration. The first, and most obvious, barrier is that raised by teacher beliefs. As beliefs appear to be formed as a consequence of teachers' own experiences of schooling, it is difficult to see how the cycle of (teacher beliefs $\rightarrow$ student beliefs $\rightarrow$ teacher beliefs) can be broken without substantial and long term inservice education.

School structures and philosophies represent a second barrier to change, especially as inquiry mathematics removes teachers from their previously unchallenged position of authority. The problem may be less serious in primary schools, where teachers work with the same class all day and students' learning experiences therefore have some consistency and continuity. Secondary school teachers face greater difficulties in establishing a sense of community in their classrooms, first, because they teach many classes, and second, because their students are also members of many other classroom communities, whose values may not coincide with those of the mathematics teacher. The task of the teacher who participated in the research study described in this paper was made easier by his school's espoused philosophy of encouraging negotiation and collaboration between teachers and students.

Finally, it is important to realise that changes to teaching practices can be resisted by students, whose views about mathematics have been formed through long experience with prescriptive teaching methods (Nickson, 1992). Participation in a community of inquiry makes unfamiliar demands on students as well as teachers, and it is unreasonable to expect students to quickly embrace changes that challenge their ideas about what mathematics is, and how it is best learned. However, the positive responses of the students in the present study suggest that a teacher's patience and persistence will eventually be rewarded.

Although the study described in this paper has identified actions that teachers might take to bring about changes in their classrooms, perhaps the most difficult task confronting the teacher is to learn what *not* to do, that is, to resist the urge to do the mathematics for the students, and to let them grapple directly with ideas in what might appear to be a messy and inefficient fashion. However, it should be clear from the results presented here that such a teacher is far from being an irresponsible or passive participant in the classroom; rather, he or she is the representative of the culture into which students seek entry, and is responsible for structuring the cognitive and social opportunities for students to experience mathematics in a meaningful way.

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