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ABSTRACT

Designed as an avenue of communication for mathematics educators concerned with the views, ideas, and experiences of two-year college students and teachers, this journal contains articles on mathematics exposition and education and regular features presenting book and software reviews, classroom activities, instructor experiences, and math problems. The first of two issues of volume 17 includes the following major articles: "Elementary Probability and Process Control," by J. Susan Milton; "Solutions to X to the Y Power = Y to the X Power and Related Equations," by Yuanqian Chen and Charles Waiveris; "An Out of Math Experience: Quadratic Equations and Polynomial Multiplication as Used in Genetics," by Gregory Fiore; "Foreshadowing as an Assessment Vehicle for Instruction and Problem Design," by Russell Jay Hendel and Dona V. Boccio; "Using Number Theory to Reinforce Elementary Algebra," by Jane D. Covillion; "A Mathematics Educators' Guide to Internet Gophers, " by Jon W. Scott and Elizabeth J. Teles. The second issue of volume 17 contains the following major articles: "Cramer's Rule," by Veena Chadha; "Divisibility Discoveries," by Richard L. Francis; "Tax-Sheltered Annuities," by Harris S. Shultz and Martin V. Bonsangue; "Modeling Data Exhibiting Multi-Constant Rates of Change," by Edward D. Laughbaum; "Strategies for Affecting the Affective Domain: A Math Anxiety Reduction Guide," by Rosemary M. Karr; and "Active Learning in Statistics Supports Students' Understanding," by Mary M. Sullivan. (MAB)

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Joseph Browne, editor.
American Mathematical Association of Two-Year Colleges
Onodaga Community College
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THE AMATYC REVIEW

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Huge quantities of information are available on the Internet.
For help in finding things of interest to mathematics educators, see the "Guide" by Jon Scott and Elizabeth Teles in this issue.

Also in this issue

- Foreshadowing
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The AMATYC Review Fall 1995 Volume 17, Number 1

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About the Cover and Editor's Comments

The Internet has grown from a tool to assist some researchers to a mammoth network with the potential of serving all of mankind. It contains information on virtually all subjects. A problem, however, especially for beginners, is how to find that information. In this issue Jon Scott and Elizabeth Teles provide a guide particularly for mathematics educators. It shows how to use one of the simplest tools, gopher, to locate information on mathematics. Most of the major mathematics organizations have gopher servers, including AMATYC. If you have Internet access, but are only using the e-mail, I urge you to try looking at some of the sources of information discussed in the Scott-Teles guide.

Letter to the Editor:

(The following comments are taken from letters between reader Laurie Golson, a developer of college mathematics textbooks, and our "Mathematical Underground" columnist Alain Schremmer. Some issues are addressed in this issues' column, so we include here some that are not. Ed.)

I agree emphatically with [Prof. Schremmer's] claim that "the way things are presented in the conventional curriculum makes it completely impossible for [students] to see the broader picture, the overall architecture according to which these things hang together." I am continually saddened by textbooks/curriculum that ignore the broad strokes of mathematics, indeed the fundamental starting places and the common threads.

Unfortunately, the fundamental starting places are by no means a settled issue of mathematics, as you certainly are aware. As we dig deeper down to the fundamentals we run into many unresolved questions about the nature of number, the relationship of number to measurement, intuition and logic, etc.

I sympathize completely with your objective. It is especially unnerving, I believe, the way fractions/ratio is taught. There is very little effort made to connect the repeated addition definition of multiplication with the symbolic definition of multiplication of fractions $\left(\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}\right)$. It gets even worse with division. I don't believe your units approach works either, though. Again, what does 2×25 STRAWBERRIES mean if it does *not* mean 25 STRAWBERRIES + 25 STRAWBERRIES? Or, how does your length/width definition of multiplication allow us to conclude that 2×25 STRAWBERRIES = 50 STRAWBERRIES?

I think one thing your piece demonstrates is the importance of teaching unit analysis to beginning students. It is by no means a simple topic, but it is one we mostly sort of assume that students grasp. We ask them to multiply 30 mph by 2 hours and get 30 miles. Quite a lot goes into such a process that we gloss over as "common sense".

Laurie Golson
714 West Washington Avenue
Lake Bluff IL 60044
lsgolson@aol.com

To begin with, there are no *"fundamental starting places"*, never have been, never will be, Bourbaki notwithstanding. It is not a matter of whether or not the issue has been settled. The fundamental starting place is the place where *you* start learning a given piece of mathematics. In other words it depends on your own past mathematical history and on your attitude in life. ... Thus, the starting place cannot be the same for a child who begins to count, for a child who begins to learn arithmetic, for a student who begins to learn algebra, for a student who begins calculus, for a mathematician interested in the foundations, etc. So, when I mentioned "the overall architecture according to which things hang together", I should not have used the definite article. There is no absolute, universal architecture. It isn't that it hasn't been found yet. It is that it is in the eyes of the beholder. I should have said something like, "the overall architecture according to which things can *appear* to hang together *to a given type of students*".

Concerning the "strawberry exchange example", it is indeed the case that "we can make any equation we want to be true as long as we previously claim some equivalency of units." As a child who had just learned about fractions, I once told my mother who had just broken a plate in two neat halves that it did not matters since $\frac{1}{2} + \frac{1}{2} = 1$. This got me a slap for making fun of her at a trying moment but, had I looked upon the denominators as names for units, I would have known that 1 HALF-PLATE + 1 HALF-PLATE is *identical* to 2 HALF-PLATES which, depending on the viewpoint, may or may not be *equivalent* to 1 WHOLE-PLATE (the viewpoint of weight as opposed to the viewpoint of soup for instance). In general, we should distinguish between identity and equivalence.

Alain Schremmer

How is it that little children are so intelligent and men so stupid? It must be education that does it.

Alexandre Dumas

❖ ❖ ❖

Children who are treated as if they are uneducable almost invariably become uneducable.

Kenneth B. Clark

MATHEMATICAL EXPOSITION

Elementary Probability and Process Control

by

J. Susan Milton
Radford University
Radford VA 24142

J. Susan Milton is a professor of statistics at Radford University. She received a B.S. in mathematics and chemistry from Western Carolina University, an M.A. in mathematics from the University of North Carolina at Chapel Hill, and a Ph.D. in statistics from Virginia Polytechnic Institute and State University. She is a Danforth Associate and is a recipient of the Radford University Award for Excellence in Teaching.

Introduction

The Japanese emphasis on quality has forced American industrial leaders to rethink their approach to manufacturing (Imai, 1986). One of the major results of this introspection is a renewed interest in the use of statistical methods in quality control. For students to appreciate the power of mathematics it is imperative that they see the connection between mathematical theory and the solution of "real world" problems early in their lives. In this paper, we present one such problem. In particular we consider the notion of a control chart and show how the geometric distribution is used to study the characteristics of the chart. It is assumed that the reader is familiar with the normal distribution and with the general concept of discrete probability functions. This paper will give students in both calculus and noncalculus based courses in beginning statistics some insight into the use of statistics in industry.

Control Charts for the Mean

A control chart is a statistical device used in industry to monitor a manufacturing process. Control charts can be quite simple or very complex in nature. Here we discuss a simple chart known as an xbar chart. Its purpose is to monitor a process in such a way that on the average the items produced are reasonably close in value to some ideal target average value. For example, consider a process in which "one pound" bags of potato chips are being filled mechanically. We all know that not every bag of chips labeled "one pound" actually contains exactly one pound of chips. Some variability in fill is inevitable. The process is considered to be "in control" relative to the mean if the machinery is performing in such a way that the average amount of fill is close in value to the one pound target; it is thought to be "out of control" if the average amount of fill appears to differ considerably from this target value. Many factors can cause a process to go out of control. Among them are malfunctioning machinery, operator error, use of inferior materials, negligence, or environmental disturbances.

Ideally it is assumed that X, the random variable under study, is normally distributed with mean μ and variance σ^2. Consider a sample, $X_1, X_2, \ldots, X_n$, of observations on X. It can be shown that $\overline{X}$, the sample average, is normally distributed and that the mean of $\overline{X}$ is the same as that of X. The variance of $\overline{X}$ is $\frac{\sigma^2}{n}$, the variance of X divided by the sample size. Its standard deviation is $\sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$. For example, in our potato chip example we might know from past experience that the machines used to fill the bags are such that they dispense an average of one pound (16 ounces) with a variance of .09. In this case the sample mean based on $n = 25$ bags should be normally distributed about a mean of 16 ounces with $\frac{\sigma^2}{n} = \frac{.09}{25}$ and standard deviation $\frac{.3}{5} = .06$ ounce.

One of the well known properties of the normal distribution is the "normal probability or Empirical rule" (Milton & Arnold, 1990). This rule states that a normal random variable will lie within one standard deviation of its mean approximately 68% of the time; within two standard deviations approximately 95% of the time; and within three standard deviations approximately 99% of the time. Figure 1 illustrates this rule as applied to the potato chip example. The rule is

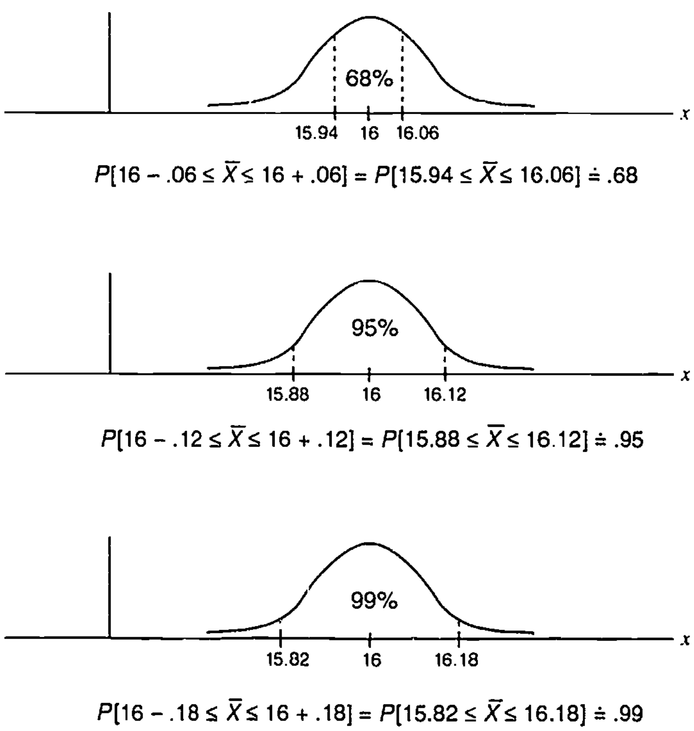

Figure 1. The normal probability rule as applied to the potato chip example.

used to set up an xbar control chart. This is done by drawing a horizontal line to represent the target mean. Parallel lines are then drawn above and below this target value two standard deviations $\left(\frac{2\sigma}{\sqrt{n}}\right)$ away as shown in Figure 2. This creates what is called a two-sigma control chart. If the machinery is operating correctly then 95% of the averages obtained from randomly drawn samples of 25 bags of chips should lie within the band shown. An observed sample mean that falls outside of the band is suspicious. There are two explanations for observing such a value. These are

1. The process is in control and we simply obtained an unusual sample;
2. The process is out of control and something needs to be corrected.

Since the probability that the former explanation is correct is small ($P < .05$), we choose to accept the latter explanation. In this case, the process is examined and possibly shut down to locate the cause of the problem.

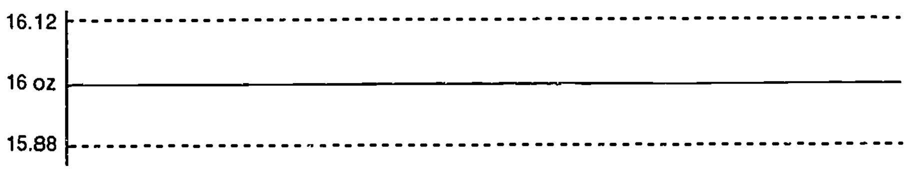

Figure 2. A two-sigma control chart for the potato chip example. In this case $\mu \pm 2\frac{\sigma}{\sqrt{n}}$ is $16 \pm .12$.

Notice that two types of errors are possible when using this approach to process control.

1. We could, by chance, draw an unusual sample whose $\bar{x}$ value falls outside of the control band even though in fact there is no problem. We will make the error of stopping or examining the process to correct a nonexistent problem. We say that a false signal has been received or that a false alarm has been raised. It is known that for a two-sigma chart, $P[\text{false alarm}] \doteq .05$.
2. We could, by chance, draw a sample whose $\bar{x}$ value falls within the band even though the system is out of control. That is, we could fail to detect a problem that does exist. It is hoped that if a problem does exist then eventually an alarm will be raised and the problem will be discovered.

It is common practice to monitor a process at frequent intervals throughout a given time period. Figure 3 shows a series of $\bar{x}$ values relative to the potato chip example taken at set intervals. Notice that the process is considered to be out of control at the fifth observation period since the $\bar{x}$ value at this point lies outside of the control band. Is there really a problem or are we dealing with a false alarm? This question

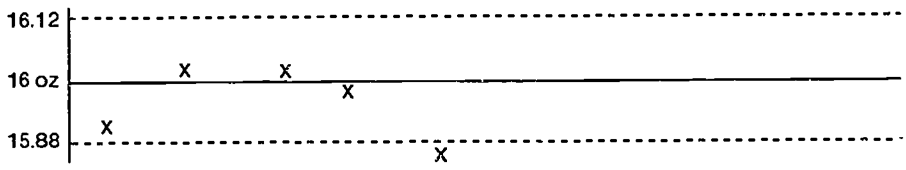

Figure 3. The process of filling bags is deemed out of control at the fifth period. At this time it appears that, on the average, the bags are being underfilled.

can only be answered by examining the process to locate the problem if there is, in fact, a problem present.

The Geometric Distribution and Control Charts

The geometric distribution is a well known discrete probability distribution. It arises when a series of independent and identical trials are observed. It is assumed that each trial results in one of two possible outcomes called "success" or "failure." The probability of success is denoted by p while the failure rate, $1 - p$, is denoted by q. The random variable Y is the number of trials needed to obtain the first success. It is known that the probability function for the geometric distribution is given by

$$P[Y = y] = f(y) = pq^{y-1} \qquad y = 1, 2, 3, \ldots .$$

It can be shown (Milton & Arnold, 1990) that the average number of trials needed to obtain the first success is given by $\frac{1}{p}$.

One property of two-sigma control charts is inherent in the construction of the chart. That is, we know that the chart is constructed in such a way that approximately 5% of observed $\bar{x}$ values will fall outside of the control band by chance even though the process is in control. That is, it is known that roughly 5% of all alarms that are raised will be false alarms.

We now ask, if a process is in control, on the average how many samples will be taken before a false alarm occurs? To answer this question, visualize the taking of a sample as a trial. Since we will sample until we observe an $\bar{x}$ value that falls outside of the two-sigma central band, observing such a value is considered to be a "success." The probability of success is .05. Thus Y, the number of samples needed to obtain the first false signal, follows a geometric distribution with $p = .05$. Its average value is $\frac{1}{p} = 20$. On the average, the first false alarm will be obtained on the twentieth sample. Notice that this means that, on the average, the process will be stopped or carefully examined unnecessarily about once in every 20 samples.

Another interesting question: If the process is out of control in the sense that the true mean has shifted off target, how long will we have to wait before a signal is received? For example, suppose that the machinery used to fill the potato chip bags has been bumped so that its settings are off and it is no longer dispensing an average of 16 ounces of chips per bag. How long will it be before this fact is discovered? This question is harder to answer because the probability of obtaining a signal is dependent upon how far off target the process has become. Intuition tells us that the larger the difference is between the new mean and the old target mean, the sooner a signal will come. For example, suppose that the machine dispensing chips has been disturbed so that it now dispenses an average of 15.925 ounces of chips per bag rather than 16. If the variability in the amount of fill has not changed then $\overline{X}$ is normally distributed with mean 15.925 ounces and standard deviation .06 ounce. A signal will be received whenever an observed $\overline{X}$ value falls outside of the two-sigma control band. In this case $P[\text{signal}] = P[\overline{X} > 16.12$ or $\overline{X} < 15.88]$. Standardization of X yields $P[\overline{X} > 16.12] = P\left[Z > \frac{16.12 - 15.925}{.06}\right]$ $= P[Z > 3.25]$. From the standard normal table, this probability is seen to be .0006. A similar calculation will show that $P[X < 15.88] = P[Z < -.75] = .2266$. Hence if the mean fill has shifted to 15.925, the probability of receiving a signal on a given sample is .2266 + .0006 = .2272. The average number of samples taken in order to receive a signal is $\frac{1}{.2272} = 4.4$. Notice that if sampling is done frequently, a shift of this magnitude should be detected fairly quickly.

A Classroom Simulation

A two-sigma control chart can be simulated easily in the classroom. All that is needed is a bag containing 20 colored marbles, 19 of which are one color (say blue) and 1 of which is another color (say red). To simulate a sampling, draw one marble at random from the bag. If the marble is blue the process is assumed to be in control; the red marble constitutes a signal. Notice that $P[\text{signal}] = \frac{1}{20} = .05$, the same as in a two-sigma control chart. Repeat the draw over and over until a signal is received and record the number of trials needed. Replicate this experiment several times to obtain a sample of observations on Y, the number of trials needed to obtain a signal. Ask the class to compute $\overline{Y}$ and compare the results obtained to the theoretical average value of 20.

Questions for Further Investigation

To assure that students understand the concepts presented here, you might ask that they set up and discuss the properties of a three-sigma control chart. Would such a chart be more or less likely to produce false alarms? If a shift in the true mean occurs, should a three-sigma chart signal the shift quicker or less quickly than a two-sigma chart? How could a random digit generator be used to simulate a three-sigma chart?

It will be instructive to allow students to experiment with various numerical values in the potato chip example. Assume different shifts away from the target of 16 ounces and investigate the effect on the probability of receiving a signal and the average number of trials needed to receive a signal for various shifts. See if a general statement can be developed to describe the relationship between the magnitude of the shift, the probability of detecting an out of control process, and the average time required to detect the problem.

Summary

In this paper we have considered a simple problem in process control. It is one that can be understood with only a minimum amount of background in probability. Notice that to set up a two-sigma chart, good estimates for μ and σ must be available. Methods of obtaining these estimates are given in Milton and Arnold (1990). Readers who are interested in obtaining more information on the subject of quality and process control are referred to any of the texts listed in the bibliography.

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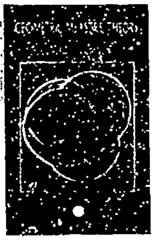

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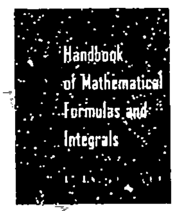

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Solutions to $x^y = y^x$ and Related Equations

by

Yuanqian Chen and Charles Waiveris
Central Connecticut State University
New Britain CT 06050

Yuanqian Chen received her undergraduate degree in mathematics from North China Institute of Technology, and her MA and Ph.D. from University of Kansas. She has been teaching at Central Connecticut State University since 1992, and is currently an assistant professor in mathematics. Her research interest is in lattice-ordered algebraic structures.

Charles Waiveris received his Ph.D. in mathematics from Wesleyan University and has research interests in topology and rings of continuous functions. He has been teaching at Central Connecticut State University since 1990 where he has been working to integrate technology and applications into the introductory mathematics curriculum.

Introduction

Most students' study of logarithms is limited to simplifying complicated computations and solving equations involving exponents like $e^{3.47x} = 19$ or $2^{x+1} = 3^{x-1}$. Logarithms are actually a very powerful tool which can be used to gain insight into more interesting equations involving exponents. One example is $x^y = y^x$. Its solution was given by Goldbach in 1728 by the use of logarithms. Marta Sved (1990) has recently found all the nontrivial rational solutions. We will demonstrate that one can use logarithms, graphing technology, and some calculus to solve equations like $x^{(y^k)} = y^{(x^k)}$, where k is a real number. The methods also serve as a nice introduction to parametric equations.

Example 1. Find all solutions to

$$x^y = y^x, \quad x, y > 0. \tag{1}$$

Students are quick to point out that all non-zero points along the line $y = x$ are obvious solutions. On the other hand, some may also notice that it is not necessary that $x = y$ as $2^4 = 4^2$ illustrates. In fact, the graph in figure 1 shows what appears to be a smooth curve of solutions different from those along the line $y = x$. Obtaining this relation explicitly by solving equation (1) for y in terms of x is not possible. It is very easy to describe, however, if we introduce a parameter t and express x and y as functions of t.

Suppose that (x, y) is a solution to (1). If $x \neq y$ then $y = tx$ for some positive

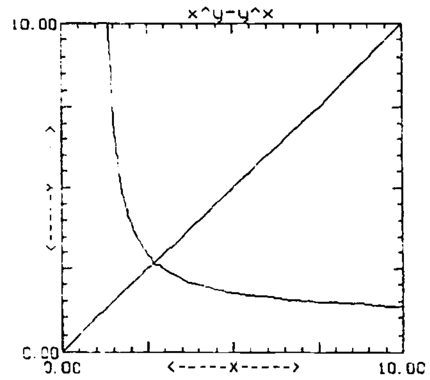

Figure 1

$t \in \mathbb{R}$, $t \neq 1$. Substituting $y = tx$ into equation (1) we have

$$x^{tx} = (tx)^x. \tag{2}$$

Taking natural logs of both sides we obtain

$$tx \ln x = x \ln (tx). \tag{3}$$

Since x is non-zero, we can divide equation (3) by x. After simplifying we obtain

$$\ln x = \frac{1}{t-1} \ln t$$

and therefore

$$x = t^{\frac{1}{t-1}}.$$

Since $y = tx$, we have

$$y = t \cdot t^{\frac{1}{t-1}} = t^{\frac{t}{t-1}}.$$

On the other hand, it is an easy exercise to show that for any positive real number $t \neq 1$, these values of x and y provide solutions to equation (1). Thus all positive solutions to (1) are either on the line $y = x$ or satisfy

$$\begin{aligned} x &= t^{\frac{1}{t-1}}, \qquad t > 0, t \neq 1 \\ y &= t^{\frac{t}{t-1}}. \end{aligned} \tag{4}$$

These parametric equations can easily be plotted by using a computer or graphing calculator. The graph (figure 2) shows quite clearly the function of x defined by all solutions to (1) with $x \neq y$. (This is trivially a function since $y = tx$ for each $t > 0$, hence for each x there is one y.)

Example 2. The equation

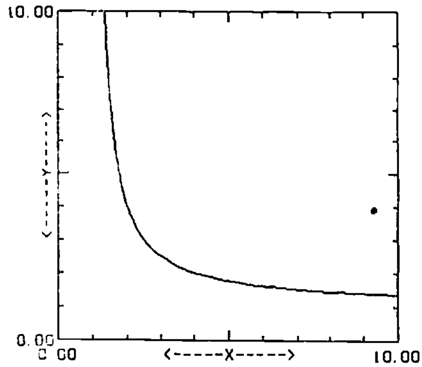

Figure 2

$$x^{(y^k)} = y^{(x^k)}, \qquad x, y > 0 \tag{5}$$

can be solved just like equation (1) yielding

$$\begin{aligned} x(t) &= t^{\frac{1}{t^k - 1}}, \qquad t > 0,\ t \neq 1 \\ y(t) &= t^{\frac{t^k}{t^k - 1}} \end{aligned} \tag{6}$$

for any $k \in \mathbf{R}$, $k \neq 0$. This enables us to investigate the solutions to equations like

$$x^{(y^2)} = y^{(x^2)} \quad \text{or} \quad x^{\frac{1}{y}} = y^{\frac{1}{x}}.$$

Figures 3 and 4 show the pattern of the graphs for $k > 0$, and figures 5 and 6 for $k < 0$. We can see that $y = 1$ and $x = 1$ are horizontal and vertical asymptotes for all of the graphs.

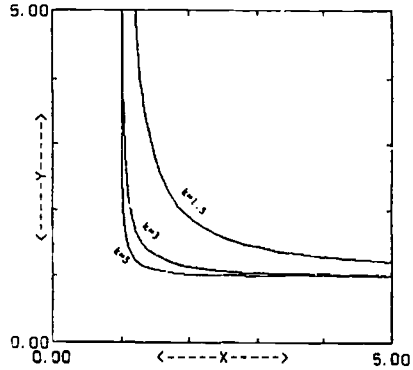

Figure 3

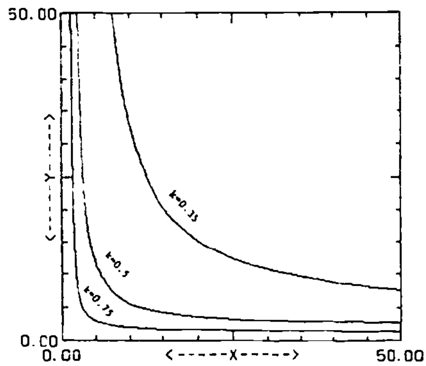

Figure 4

Applying L'Hôpital's rule, we have that for x, y as in (6)

$$\lim_{t\to 0^+} x = \begin{cases} +\infty & \text{if } k>0 \\ 1 & \text{if } k<0 \end{cases}$$

$$\lim_{t\to 0^+} y = \begin{cases} 1 & \text{if } k>0 \\ 0 & \text{if } k<0 \end{cases}$$

$$\lim_{t\to +\infty} y = \begin{cases} 1 & \text{if } k>0 \\ 0 & \text{if } k<0 \end{cases}$$

$$\lim_{t\to +\infty} x = \begin{cases} +\infty & \text{if } k>0 \\ 1 & \text{if } k<0. \end{cases}$$

Letting t approach 1, we obtain

$$\lim_{t\to 1} x = e^{\frac{1}{k}}, \qquad k \neq 0,$$

$$\lim_{t\to 1} y = e^{\frac{1}{k}}, \qquad k \neq 0.$$

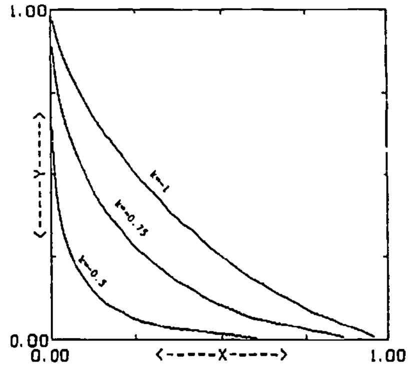

Figure 5

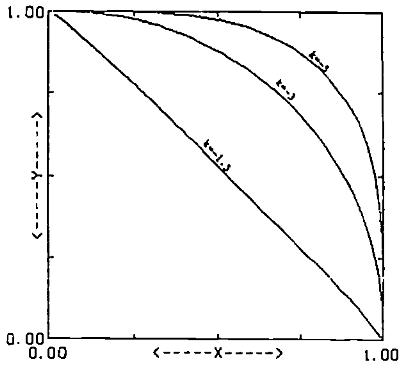

Figure 6

This is consistent with the graphs in figures 3–6 and extends the domain for the functions in (6) to $t > 0$. These limits show that for each non-zero k, the solutions to (5) travel from "right to left" as t increases. There is an interesting symmetry between the graphs for $k > 0$ and $k < 0$. Fixing k and letting $(x(t, k), y(t, k))$ denote a point on a solution to (5), we see that

$$y(t, -k) = \frac{1}{x(t, k)}, \qquad t > 0$$

and

$$x(t, -k) = \frac{1}{y(t, k)}, \qquad t > 0.$$

The graphs for k and $-k$ possess a "reciprocal and coordinate swapping" symmetry through the line $y = 2 - x$. Thus the behavior of the solutions given in (6) for $k < 0$ could have been obtained from those for $k > 0$.

Exercises

The graphing technology with its ability to graph parametric equations quickly is a powerful tool for exploring the solutions in (6). It provides an excellent opportunity for students to learn how to do mathematics with a computer or graphing calculator.

1) Draw graphs of $x(t)$ and $y(t)$ for $t > 0$ and various values of k. Explain why these graphs are consistent with those of figures 3–6.
2) Using the definiton in equation (6), show that $x > 1$ and $y > 1$, for all $k > 0$, and that $x < 1$ and $y < 1$ for all $k < 0$. (Hint: consider the cases for $t > 1$ and $0 < t < 1$.)
3) Aside from (2,4), (4,2) and the points along $y = x$, are there any other positive integer values for x and y satisfying equation (1)?

4) Find the value of $k > 0$ such that the graph of

$$\begin{cases} x(t, -k) \\ y(t, -k) \end{cases}$$

passes through (0.5, 0.5). This curve looks very much like the line passing through (0,1) and (1,0). Are they the same? Use the trace function on graphing technology to compare the values. (Hint: along $y = x$, $t = 1$)

5) The points (–2,–4) and (–4,–2) satisfy equation $x^y = y^x$. Are there any other points with negative coordinates not on the line $y = x$ that satisfy $x^y = y^x$? (Note: if x and y are not both positive, then the value of x^y or y^x need not be a real number. If w and z are complex numbers then

$$w^z = e^{z(\ln|w|+i(Argw+2k\pi))} \qquad k = 0, \pm 1, \pm 2, \ldots\ldots$$

where $Argw$ is the principle argument of w and therefore $Argw = 0$, if w is positive and $Argw = \pi$ when w is negative.)

6) Compute and draw the graphs of dx/dt and dy/dt for $t > 0$ and various values of k. Explain why these graphs are consistent with those of figures 3–6. Note: the differentiability of $x(t)$ and $y(t)$ implies their continuity.

7) The derivative dy/dx is the ratio of dy/dt with dx/dt. Compute and draw the graphs of dy/dx for $t > 0$ and various values of k. Explain why these graphs are consistent with those of figures 3–6.

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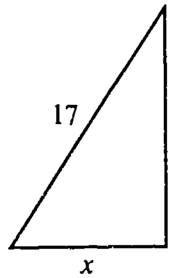
Swokowski, E. (1991). *Calculus.* (5th ed.). Boston: PWS-Kent.

Lucky Larry #18

Larry has an unusual way to use the Pythagorean Theorem.

$$\begin{aligned} x^2 &= 15^2 + 17^2 \\ &= 2 \cdot 15 + 2 \cdot 17 \\ &= 30 + 34 \\ &= 64 \\ x &= \sqrt{64} = 8 \end{aligned}$$



Submitted by Jerry Lieblich
Bronx Community College
Bronx NY 10453

An Out of Math Experience:
Quadratic Equations and Polynomial Multiplication as Used in Genetics

by

Gregory Fiore
Dundalk Community College
Dundalk MD 21222

Gregory Fiore is a professor of mathematics and computer science at Dundalk Community College. He received his MS in mathematics from Purdue University, and his MES in computer science from Loyola College. His interests include adapting applications from other disciplines to the developmental mathematics classroom. He is the author of a series of developmental mathematics textbooks.

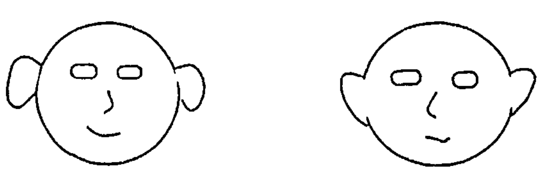
Genes and Earlobes

What I like about bringing applications from other disciplines into the mathematics classroom is the mood change in my class. I introduce my applications this way: "We have studied polynomial multiplication, second degree polynomials in two variables, and quadratic equations. Now let's investigate how these concepts are used in genetics."

A few students look at me as if to say "You've got to be kidding. You mean there is use for this stuff?" The value of using an application from another discipline is that students see direct use for the mathematics they are learning. They also see that knowing it may affect their understanding and performance in another course. These applications make mathematics relevant, and give students a reason for learning it.

Back to genetics. "Let's investigate one of your body parts. What kind of ears do you have? Are they hanging or attached?" I show the class a picture and ask them to feel their earlobes. "Earlobes come in these two varieties. The kind you have is determined by two genes, one from your father, and the other from your mother." The actual set of two genes you have is called your **genotype**. The physical appearance, hanging or attached earlobes, is called your **phenotype**.

Free Hanging Attached



FF or Ff ff

Most biologists believe that two genes, or **alleles**, determine the earlobe trait. Suppose we represent these genes by the letters F and f.

F = the gene for Free hanging earlobes. It is the dominant gene. (Explanation: A gene is a piece of information. A **dominant** earlobe gene contains information that causes some change in the organism. We use F as our symbol for the gene that contains the information to make earlobes that hang free from the side of the head.

f = the gene for attached earlobes. It is the recessive gene. (Explanation: A **recessive** earlobe gene does not contain the information to make earlobes that hang free. Therefore, f is our symbol for the absence of that information.)

Since each person has two genes for his or her earlobe trait, there are three possible earlobe genotypes. They are FF, Ff or fF, and ff. The genotypes Ff and fF are the same.

FF: Hanging. Each parent contributes one dominant gene. So the individual has information to produce hanging earlobes.

Ff or fF: Hanging. Since one of the two genes is F, the individual has the information to produce hanging earlobes.

ff: Attached. Since neither gene is F, the individual has no information to produce hanging earlobes. The result is attached earlobes.

In **population genetics**, we are interested in the percent, or **frequency**, of genes of a certain type in the entire population under study. In other words, take the two earlobe genes from each person in the population and put them in a *bowl*. This *bowl* is called the **gene pool** for the population. Geneticists study the gene pool in order to draw conclusions about the population.

Hardy-Weinberg

Define two variables p and q as follows:

p = percent of genes in the gene pool that are dominant (F)

q = percent of genes in the gene pool that are recessive (f).

Now suppose we have an isolated population in which 75% of the genes in the earlobe gene pool are dominant F, and the other 25% are recessive f. Then $p = 0.75$ and $q = 0.25$. Since p and q give us 100% of all the genes in the gene pool, we have $p + q = 1$.

$$p + q = 1$$
$$(p + q)^2 = 1$$
$$p^2 + 2pq + q^2 = 1 \quad \text{Hardy-Weinberg equation}$$

Fraction of the population with two dominant genes, FF (Hanging)

$$p^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

Fraction with one dominant gene, Ff (Hanging)

$$2pq = 2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) = \frac{6}{16}$$

Fraction with no dominant genes, ff (Attached)

$$q^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

The Hardy-Weinberg equation is an application of quadratic equations in two variables. The three terms p^2, $2pq$, and q^2 on its left side are used to give us the relative frequencies of the three genotypes FF, Ff, and ff in the population. Each of these terms has a unique interpretation. This is an opportunity to point out to our students that individual terms in an equation do have meaning in the real world.

p^2 tells us that the fraction of the population that is dominant FF (**homozygous dominant**) is $\frac{9}{16}$.

$2pq$ tells us that the fraction of the population that is dominant Ff (**heterozygous dominant**) is $\frac{6}{16}$.

q^2 tells us that the fraction of the population that is recessive ff (**homozygous recessive**) is $\frac{1}{16}$.

Note that $\frac{9}{16} + \frac{6}{16} + \frac{1}{16} = 1$.

Question #1: What is the algebraic expression for the fraction of the population with the dominant phenotype?

Answer: $p^2 + 2pq$. We have p^2 for FF, plus $2pq$ (or $pq + qp$) for Ff and f F.

Hardy-Weinberg tells us that with random mating, the frequencies p and q will remain constant in the population from one generation to the next unless disturbed by mutation, selection, or migration. To verify this, suppose the alleles F and f occur with frequencies p and q in the gene pool of the original generation, and $p + q = 1$. The alleles F and f will be passed onto the next (or first) generation with probabilities p and q. The probability of an individual in the first generation getting the genotype FF is p^2, of getting Ff is $pq + qp$ or $2pq$, and of getting ff is q^2. What genes will be passed onto the second generation? The proportion p^2 of the first generation population will contribute an F allele, and q^2 of the population will contribute an f allele. The remaining $2pq$ of the population has an equal probability of contributing either an F or an f allele. Therefore, the F allele is passed onto the second generation with probability

$$p^2 + pq = p(p + q) = p(1) = p.$$

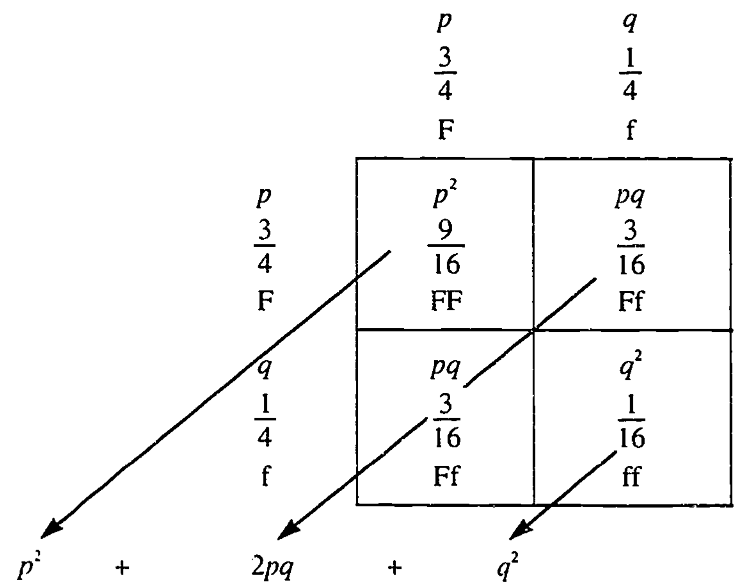
And the f allele is passed onto the second generation with probability

$$q^2 + pq = q(q + p) = q(1) = q.$$

These are the same frequencies as in the previous generation.

The Punnett Square

The Punnett square is sometimes used to calculate and display genotype frequencies. It can also be used to multiply polynomials. The terms of one polynomial are listed horizontally across the top, and of the other vertically down the side. Like terms are combined diagonally. Once they see it, about one-quarter of my students use the Punnett square to multiply polynomials.



Question #2: Suppose a certain population only has two eye color genes. They are B = brown (dominant, $p = 0.6$), and b = blue (recessive, $q = 0.4$). What fraction of the population has each genotype? Each phenotype?

Answer: The fraction of the population with BB is $p^2 = 0.36$, with Bb (or bB) is $2pq = 0.48$, and with bb is $q^2 = 0.16$. The fraction of the population with the dominant brown phenotype is $p^2 + 2pq = 0.36 + 0.48 = 0.84$, and with the recessive blue phenotype is $q^2 = 0.16$.

Determining p and q

Question #3: Describe how you would estimate the values of p and q for the percent of dominant and recessive genes for the earlobe trait in the gene pool for any isolated population.

Answer: If a person has hanging earlobes, his or her genotype can be either FF or Ff. You cannot tell which a person has because you cannot see the genes. But attached earlobes only have the recessive ff genotype. Since we can see ff as an attached earlobe, all we have to do is count the number of people in the population with the recessive phenotype. This is exactly how biologists and geneticists determine p and q.

For example, suppose 900 people are in the population. We count 81 people with attached earlobes. Then 81/900 of the population has attached earlobes. But this fraction is q^2 in the Hardy-Weinberg equation.

$$q^2 = \frac{81}{900}$$

$$q = \frac{9}{30} \text{ or } 0.3.$$

(Note that the negative root is meaningless. Biologists use decimals to represent frequencies, not fractions or roots.) Thus 30% of the genes in the gene pool are f. If $q = 0.3$, then $p = 1 - 0.3 = 0.7$, and 70% of the genes in the gene pool are F. Therefore, the genotypes in our population have the following frequencies:

$p^2 = 0.49$ are FF,

$2pq = 0.42$ are Ff

$q^2 = 0.09$ are ff. Note that $81/900 = 0.09$.

Blood Types

Another application of quadratic polynomials is to blood types. There are three genes in the gene pool for blood, A, B, and O. An individual carries two of these three genes, which makes up his or her blood genotype. There are 6 possible genotypes that can be made from the genes A, B, and O, shown in the table below. These 6 genotypes result in 4 possible blood phenotypes A, B, AB, and O. (For convenience, all persons in this article are assumed to be rh+.)

Geno.	Pheno.
AA	A
AO	A
BB	B
BO	B
AB	AB
OO	O

Blood type A has information to produce a protein called the A antigen. B has information to produce a second, different protein called the B antigen. O does not have the information to produce any proteins.

A and B each dominate O. Therefore AO is type A blood, because though O produces no protein, A produces the A antigen. In like manner, BO is type B. Finally, A and B are codominant. This means the AB genotype produces both the

A antigen and the B antigen, in the same body. Both antigens coexist in harmony.

Protein production can help you understand the rules for blood transfusions. Blood type A cannot be put into an O body. The reason is that A produces a protein, the A antigen. O produces no protein. The O body sees this A antigen as foreign, and produces antibodies to destroy it. Hence the type O body rejects the type A blood transfusion. For the same reason B and AB cannot be put into an O body. Therefore type O blood can only use a transfusion of type O blood.

On the other hand, since O produces no protein, type O blood can be put in a type A, B, or AB body without consequences. Type O is called the **universal donor**. Any blood type can use type O blood.

AB produces both the A antigen and the B antigen. Therefore, the AB body can accept a transfusion of A blood, B blood, or O blood. AB already produces both antigens, and so detects no foreign proteins. For this reason, AB is called the **universal recipient**. It will accept a transfusion of A, B, AB, or O blood.

Question #4: A father is type A, a mother is type O, and their child is type O. Describe the genotype of each parent.

Answer: The child is type O. The child's genotype must be OO, and so has received one O gene from each parent. The mother is type O. Her genotype must be OO. The father is type A. So his genotype is AA or AO. He cannot be AA, or his child would have received an A gene from him. The child would then have genotype AO, and so be type A. Therefore, the father must have the AO genotype.

Define the variables p, q, and r to represent the frequencies of the genes A, B, and O in the blood gene pool. Suppose in an isolated population 30% of the genes in the blood gene pool are A, 20% are B, and 50% are O. Then $p = 0.3$, $q = 0.2$, and $r = 0.5$, and $p + q + r = 1$. Square both sides of this equation. The Punnett square on the right below shows the nine terms that result from squaring $p + q + r$. Combine terms, and we get a second degree polynomial in three variables.

A	B	O
$p = .3$	$q = .2$	$r = .5$

$$p + q + r = 1$$

$$(p + q + r)^2 = 1$$

$$\underset{AA}{p^2} + \underset{AO}{2pr} + \underset{BB}{q^2} + \underset{BO}{2qr} + \underset{AB}{2pq} + \underset{OO}{r^2} = 1$$

	p .3	q .2	r .5
p .3	p^2 .09	pq .06	pr .15
q .2	qp .06	q^2 .04	qr .10
r .5	rp .15	rq .10	r^2 .25

Each of the 6 terms in the squared equation gives the frequency of each of the 6 possible genotypes. With this information you can answer a variety of questions.

Question #5: What percent of the population has the AO genotype?

Answer: $2pr = 2. (0.15) = 0.30$ or 30%.

Question #6: What percent of the population is type A?

Answer: The genotypes AA and AO result in type A blood. The expression representing the type A phenotype is

$$p^2 + 2pr = 0.09 + 0.30 = 0.39 \text{ or } 39\%$$

of the population.

Question #7: What percent of the population is the universal donor type O?

Answer: $r^2 = 0.25$ or 25%. This means 25% of the population is available to make a blood donation to a type O person. If a type O person is in need of blood, this means there is a 1 in 4 chance that a random donor will have the same type. (On the other hand, 100% of the population is available to make a blood donation to a type AB person.)

Question #8: Expert testimony in the preliminary hearing of the O.J. Simpson trial reveals that the percent of the U.S. population having each of the four blood phenotypes is type A, 34%; type B, 16%; type AB, 4%; and type O, 46%. Estimate the frequencies p, q, and r for the U.S. blood gene pool.

Answer: Solve the following system of equations:

$$\begin{aligned} p^2 + 2pr &= 0.34 &\quad& \text{AA and AO} \\ q^2 + 2qr &= 0.16 && \text{BB and BO} \\ 2pq &= 0.04 && \text{AB} \\ r^2 &= 0.46 && \text{OO} \end{aligned}$$

Using a calculator and the quadratic formula, the approximate answers are $p = 0.22$, $q = 0.11$, and $r = 0.68$.

How to Use Applications

There are several guiding principles for introducing an application from another discipline in the classroom.

1. **It should clearly demonstrate how the mathematics being learned is used in another course or in another discipline.** It is important that the application clearly show the student the relevance of the mathematics he or she is investing time, energy, and money to learn.
2. **The application should challenge the students.** It should teach them something new which may include material not normally seen in a mathematics classroom. The amount of detail given depends on the instructor's knowledge base, the ability level of the students, and time. Most of the applications I use take 5 to 15 minutes. The interest and ability level of the students determines

the amount of detail given by the instructor. An application in another discipline may require additional research on your part.

It is essential to be accurate. Converse with colleagues in other disciplines. Request that they review your written materials for accuracy of content outside your discipline.

3. **Follow each application with a question or assignment. Make students responsible for it.** Let them know at the start how they will be responsible for it (for example, as material for a quiz, test, or for an assignment).

Related Readings

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SHORT COMMUNICATIONS

A Note on The Euclidean Algorithm

by

Richard Maruszewski and William Wardlaw
United States Naval Academy
Annapolis MD 21402

Richard F. Maruszewski, Jr. is a member of the Mathematics Department at the United States Naval Academy. He received his MS in mathematics from Marquette University and his PhD in mathematics from the University of Wisconsin in Milwaukee. His research interests include radical theory, the use of the computer in the classroom, and cryptologic mathematics.

William P. Wardlaw received a B.A. in physics from the Rice Institute in 1958. He served two years in the U.S. Navy and worked the next two years for Douglas Aircraft in Santa Monica. He received his Ph.D. in mathematics from the University of California at Los Angeles in 1966. He taught at the University of Georgia from 1966 to 1972. In 1972 he went to the U.S. Naval Academy.

Introduction

The importance of the Euclidean algorithm is well documented. Expositions on abstract algebra, number theory, and associated areas introduce, explain, and use the algorithm [see Gallian (1986), Gilbert (1984), Koblitz (1987), Lang (1987), or McCoy (1987)]. The Euclidean algorithm is applied to find the greatest common divisor, d, of two integers a and b, written as $\gcd(a,b) = d$. The standard presentation of the algorithm finds x and y such that $d = xa + yb$ by backtracking through the process. This backtracking is often very extensive. An alternate form of the algorithm (Blankinship, 1983) computes d, x, and y directly. The purpose of this note is to present and prove this alternate form and to use matrices to simplify the necessary calculations.

The Algorithm

We will begin by giving an example of the algorithm. Let's find the gcd of 42 and 60. We start with the standard presentation.

$$
\begin{array}{ll}
60 = (42)(1) + r_1 & r_1 = 18 \\
42 = (18)(2) + r_2 & r_2 = 6 \\
18 = (6)(3) + r_3 & r_3 = 0 \\
\gcd(60,42) = 6 &
\end{array}
$$

We now redo the example using our algorithm. (Note that the remainders in the two presentations are subscripted differently.)

$$1a + 0b = 60 = r_0$$
$$0a + 1b = 42 = r_1$$
$$1a - 1b = 18 = r_2 = r_0 - q_1 r_1 \; (q_1 = 1)$$
$$-2a + 3b = 6 = r_3 = r_1 - q_2 r_2 \; (q_2 = 2)$$
$$7a - 10b = 0 = r_4 = r_2 - q_3 r_3 \; (q_3 = 3)$$

Our algorithm works as follows. We start by designating the larger of the two integers as a and write this on the first line. We then put the second integer, b, on the second line. In each succeeding line, the number to the right of the first equal sign is the remainder obtained when dividing the number two lines above it by the number one line above it. For example, if we divide 60 by 42 we get a remainder of 18. The process terminates whenever we get a remainder of zero, and the gcd is the number just above the 0. In our case the answer is 6. Also note that whatever we do to the right of this equal sign, we also do to the left. Thus to get our 18, we subtract one times 42 from 60, so we also subtract $0a$ from $1a$ and $1b$ from 0b to get $1a - 1b$. In this way we obtain not only that the gcd is 6 but also that $6 = (-2)(60) + (3)(42)$ ($d = xa + yb$). Furthermore, we find these values without backtracking.

Matrix Implementation

Before we prove that our algorithm works, we will show another way to calculate the result using matrices. Instead of writing the first two lines, we could write the matrix system,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 60 \\ 42 \end{bmatrix}$$

which corresponds to the augmented matrix,

$$\begin{bmatrix} 1 & 0 & 60 \\ 0 & 1 & 42 \end{bmatrix}.$$

We then accomplish the same results as the above by using row operations. In each iteration we choose the smaller of the two entries in the third column and calculate the largest multiple of it which does not exceed the other entry. We then subtract that multiple of the row of the smaller entry from the row of the larger. We repeat until the smaller entry in the third column is zero. In our example:

$$\begin{bmatrix} 1 & 0 & 60 \\ 0 & 1 & 42 \end{bmatrix} \quad (-1)R_2 + R_1 \rightarrow R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 18 \\ 0 & 1 & 42 \end{bmatrix} \quad (-2)R_1 + R_2 \rightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 18 \\ -2 & 3 & 6 \end{bmatrix} \quad (-3)R_2 + R_1 \to R_1$$

$$\sim \begin{bmatrix} 7 & -10 & 0 \\ -2 & 3 & 6 \end{bmatrix}$$

Note that the calculations above are identical to those in the last section.

Proof of the Algorithm

The fact that the algorithm always terminates is well known and can be found in any discussion of the Euclidean algorithm. (See references.) We proceed with the rest of the proof. The matrix form of the algorithm changed

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}$$

into

$$\begin{bmatrix} x & y & d \\ u & v & 0 \end{bmatrix}$$

using only one type of row operation, adding a multiple of one row to another. We claim that d is the gcd. Because the row operation we used does not change the value of the determinant,

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \det \begin{bmatrix} x & y \\ u & v \end{bmatrix} = xv - uy = 1$$

and the matrix is invertible. The inverse of the matrix is

$$\begin{bmatrix} v & -y \\ -u & x \end{bmatrix}.$$

Thus, we have that

$$\begin{bmatrix} x & y \\ u & v \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} d \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} v & -y \\ -u & x \end{bmatrix}\begin{bmatrix} d \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

From the second matrix equation, $vd = a$ and $-ud = b$, so d is a common divisor of a and b. From the first, $ax + by = d$. So if c divides a and c divides b, c also divides d. Hence, d is the $\gcd(a,b)$.

The LCM

We also get a bonus from our computations. In our example above, the last line (and the top row of the last matrix) was $7a - 10b = 0$. Rewriting, we obtain $7a = 10b = 420$ which is the lcm of 42 and 60. We claim that this is always the case. Note that the first matrix multiplication in the last section also yields $ua + vb = 0$. Either ua or vb is less than zero. We will assume without loss of generality that vb is. Let $m = ua = -vb$. We claim that m is the lcm of a and b. By definition, m is a common multiple of a and b, so we must show it is the lowest one. Assume that c is another multiple of a and b. Then $c = pb = qa$ and

$$
\begin{aligned}
c &= c1 \\
&= c(xv - uy) && \text{(the determinant above)} \\
&= pbxv - qauy && \text{(substituting for } c\text{)} \\
&= -pudxv - qauy && \text{(substituting for } b \text{ from the last section)} \\
&= -puax - qauy && \text{(substituting for } vd \text{ from the last section)} \\
&= ua(-px - qy) \\
&= m(-px - qy)
\end{aligned}
$$

Hence c is a multiple of m, and m is the lcm of a and b.

Summary

The matrix implementation of the algorithm proved to be very fruitful. By a series of elementary matrix row operations, we calculate the lcm and gcd of our two integers, and we express the gcd in terms of the original two integers.

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Lucky Larry #19

Larry combined creative use of algebra and calculator to get the right answer to this one.

$$
\begin{aligned}
e^{2x-3} &= 31 \\
\ln e^{2x-3} &= \ln 31 \\
\ln(2x - 3) &= \ln 31 \\
\ln 2x - \ln 3 &= \ln 31 \\
\ln 2x - 1.099 &= 3.434 \\
\ln 2x &= 2.335 \\
\ln x &= \frac{2.335}{2} = 1.168 \\
x &= e^{1.168} = 3.22
\end{aligned}
$$

Submitted by Richard Hartt
Onondaga Community College
Syracuse NY 13215

Foreshadowing as an Assessment Vehicle for Instruction and Problem Design

by

Russell Jay Hendel
University of Louisville
Louisville KY 40292

and

Dona V. Boccio
Queensborough Community College
Bayside NY 11364-1497

Russell Jay Hendel, Ph.D., ASA, currently works as an actuary for the Health Care Finance Administration in Pennsylvania. Dr. Hendel's specialty is number theory. The following article adds to a set of about one dozen publications in pedagogy describing Dr. Hendel's approaches to actual classroom teaching. This article was begun while he was at Dowling College, revised while he was at Morris College and completed while he was a visiting assistant professor of mathematics at the University of Louisville.

Dona V. Boccio is Associate Professor of Mathematics at Queensborough Community College, City University of New York. Dona earned her Ph.D. in mathematics at City University of New York Graduate Center, and holds a M.S. in mathematics from the Courant Institute of Mathematical Sciences at New York University, and a B.A. in Chemistry from New York University. Her research interests include quantitative illiteracy and geometric probability. She is on the Executive Board of NYSMATYC, serving as Legislative Chair.

Introduction

In an educational context, "foreshadowing" refers to previewing or preparing for topics which will be covered in the future. Some examples are presented in Table 1. The major benefit of foreshadowing is that multiple presentations of the same technique should increase ease and familiarity with its implementation by students.

Foreshadowing is, or ought to be, incorporated at several levels. Perhaps most obviously, foreshadowing should be evident in curriculum design. If a course is designated as a prerequisite for another, it should not only contain the prerequisite topics; it should also contain discussion or problems similar in form and complexity to those that will be used in the later course.

Of course, foreshadowing is by no means limited to curriculum design. In most current texts authors use foreshadowing to prepare students for material which comes later. Instructors can select and assign exercises which not only illustrate the current topic, but also familiarize the student with the way(s) the topic will be used in the future.

Table 1: Familiar examples of foreshadowing from precalculus and calculus. As an example of table interpretation consider the next to last column: it suggests that the teaching of the *target* topic—the chain rule, is enhanced through a *preparatory* lesson on function decomposition.

Prepatory Topic	Determinants	Fraction algebra	Rationalizing denominator	Function decomposition	Slope-point formula
Target Topic	Cramer's Rule	Derivative of $1/x$	Derivative of $\sqrt{x}$	The chain rule	The tangent line equation

Thus while foreshadowing is not new, it nevertheless is something that instructors should be aware of and seek to use. In this note we give tips on how to look for and use foreshadowing when examining texts, creating syllabi, selecting homework and preparing instructional units.

Example 1: **Devising instructional language using foreshadowing:** It is well known that the elimination method for solving simultaneous systems foreshadows the row reduction methods of matrix algebra. This foreshadowing of concepts can be reflected in the terminology also. In particular, the elimination method can be formulated in terms of three "equation rules" which parallel the three row rules used in Gauss Jordan elimination. Instructors can employ similar wording for both sets of rules and explicitly inform students that the elementary row operations are a rewording in matrix language of the equation operations that produce equivalent equations.

Example 2: **Supplementing problem sets by using foreshadowing:** Order of operation exercises in (pre)-algebra texts often include problems with multi-nested parenthetical expressions. However business majors are more likely to encounter problems such as

$$\frac{(1.05)^2 - 1}{.05}$$

in future courses. The foreshadowing guideline, which encourages similarity between problems in target and preparatory courses, suggests supplementing and/or revising the algebra course problem sets with exercises such as

$$\frac{(4+5)^2-21}{7-1}.$$

Example 3: **Remediation using foreshadowing:** Calculus texts frequently have a review section on function decomposition. Many required decompositions in chain rule problem sets may only involve powers, radicals, and reciprocals. By contrast, the decomposition section exercise sets may have a richer variety of problem types and thus not provide sufficient practice in power, radical and reciprocal decompositions. As a remedy, the decomposition review section can be supplemented by requesting students to decompose the functions in the chain rule section. Such a foreshadowing of problems can be very beneficial to weaker students.

Example 4: **Enriching problem solving techniques using foreshadowing:** Multi-sign charts facilitate both graphing in calculus and solving quadratic inequalities (by the critical value method) in precalculus. Foreshadowing encourages looking for still other areas where multi-sign charts can be used.

One example of such enrichment is the design of challenging domain and range problems whose solution is facilitated by multi-sign charts. In fact for every real, polynomial graphing problem there is a corresponding domain-range, precalculus problem with similar multi-sign charts. Thus both the calculus problem, "Graph $f(x) = \frac{x^3}{3} - x^2$," and the precalculus problem, "Find the domain of $\sqrt{\frac{x(x-2)}{2(x-1)}}$ use the following multi-sign chart:

Sign of $f'(x) = x(x-2)$	+	−	−	+	Numerator $= x(x-2)$
Sign of $f''(x) = 2(x-1)$	−	−	+	+	Denominator $= 2(x-1)$

---0------1------2---

Example 5: **Reform using foreshadowing:** Foreshadowing can be an important component in the support of course modifications. As an elementary example, foreshadowing justifies introducing problems involving the logistic model into (pre)-calculus courses even though a full treatment requires differential equations.

Conclusion

The preceding examples show how foreshadowing can be advantageously used in a variety of situations. Instructors at all levels are therefore encouraged to seek out new applications of this simple but powerful educational technique.

Using Number Theory to Reinforce Elementary Algebra

by

Jane D. Covillion
Onondaga Community College
Syracuse NY 13215

Jane Covillion is an Associate Professor at Onondaga Community College where she teaches courses in Computer Information Systems and Mathematics. She earned an AB in Mathematics from Cornell University and an MS in Education (Secondary Mathematics) from the State University of New York at Oswego. Jane is currently active in many local, state and national mathematics organizations. She also is the production manager of The AMATYC Review.

One of my primary goals in teaching mathematics is to have students really understand the concepts that are being presented. One way of doing this is to use elementary number theory in algebra classes. This method allows students to use their acquired algebraic skills, as well as to learn something about number theory.

Factoring and the Divisibility Rules

When I teach factoring, students review divisibility rules. They usually know the following divisibility rule: A number is divisible by two if it ends in 0, 2, 4, 6, or 8. But, can beginning algebra students explain why it is true? They probably have never been asked.

At this point, I show the class a place value chart for the year, for example:

1000	100	10	1
1	9	9	5

The students understand that $1995 = 1000 \times 1 + 100 \times 9 + 10 \times 9 + 5$. Now, I ask them what if this is my number:

1000	100	10	1
a	b	c	d

Students see the parallel that

$$abcd = 1000a + 100b + 10c + d.$$

But, with what they have learned about factoring,

$$abcd = 10(100a + 10b + 1c) + d. \tag{1}$$

I now ask the class, "What does this mean?" Since 2 is a factor of $10(100a + 10b + 1c)$

because 2 divides 10, what is left to consider is d. If d is divisible by 2, then the original number $abcd$ is divisible by 2 since it is the sum of two expressions each divisible by 2. This implies that if d is 0, 2, 4, 6 or 8 (an "even" number), then $abcd$ is also divisible by 2. This is the rule that most students learn in elementary school. At this time it should be pointed out that this method can be extended to any number of digits. The factoring process has then reinforced a concept known for years, in addition to reinforcing algebraic skills.

The same procedure can be used to illustrate the divisibility rules for 5 and 10. In fact, the factoring from (1) is still appropriate. Since both 5 and 10 are factors of $10(100a + 10b + 1c)$ because they both divide 10, what is still left to consider is d. If d is divisible by 5, then so is the original number $abcd$ since it is the sum of two expressions divisible by 5. This implies that if d is 0 or 5, then $abcd$ is also divisible by 5. If d is 0, then (1) becomes $10(100a + 10b + 1c)$. This new expression is divisible by 10, which means $abcd$ is also divisible by 10.

Many students are familiar with the divisibility rule for 3: A number is divisible by 3 if the sum of its digits is divisible by three. Is factoring helpful in illustrating why this rule works? The now familiar $abcd = 1000a + 100b + 10c + d$ becomes

$$\begin{aligned} &999a + a + 99b + b + 9c + c + d = \\ &999a + 99b + 9c + a + b + c + d = \\ &9(111a + 11b + c) + a + b + c + d. \end{aligned} \tag{2}$$

Clearly 3 is a factor of $9(111a + 11b + c)$ because 3 divides 9. What is left to consider is $a + b + c + d$. If $a + b + c + d$ is divisible by 3, then so is $abcd$ since it is the sum of two expressions each divisible by 3.

Another divisibility rule, less frequently encountered, is obvious now from (2). A number is divisible by 9 if the sum of its digits is divisible by 9. Why? Since 9 is a factor of $9(111a + 11b + c)$ and $a + b + c + d$ is also divisible by 9, $abcd$ must be divisible by 9.

One should note that the proofs already presented have been for four digit integers. These proofs are appropriate for the level of students in my class. General proofs for examples in this article can be done in the same manner only with more sophisticated notation that would probably obscure the point for elementary algebra students.

At this point, students will wonder if there are other divisibility rules and if factoring can be used to see why the rules work. Presented below are further examples of divisibility rules.

Divisibility by 4:

A number is divisible by 4 if the last two digits are divisible by 4.

$$\begin{aligned} abcd &= 1000a + 100b + 10c + d \\ &= 4(250a + 25b) + 10c + d \end{aligned}$$

Since 4 is a factor of $4(250a + 25b)$, what is left to consider is $10c + d$. If $10c + d$

is divisible by 4, then so is *abcd* since it is the sum of two expressions each divisible by 4. Since $10c + d$ is the representation of the last two digits, the divisibility rule has been illustrated using factoring.

Divisibility by 6:

A number is divisible by 6 if it divisible by both 2 <u>and</u> 3. The divisibility rules for 2 and 3 have already been illustrated using factoring. A number is divisible by 6 if it is even (1) and the sum of its digits is divisible by 3 (2).

Divisibility by 7:

To determine if a number is divisible by 7, do the following: Truncate the last digit, double it, and subtract the result from the resulting number. If the difference is divisible by 7, then so is the original number. This method can be repeated to determine if larger numbers are divisible by 7.

It is first necessary to show the class examples of this rule:

Example 1:

$$\begin{array}{r} 15\cancel{4} \\ \underline{-\ \ 8} \\ 7 \end{array}$$

Since 7 is divisible by 7, so is the original number 154.

Example 2:

$$\begin{array}{r} 2483\cancel{6} \\ \underline{-\ \ \ \ 12} \\ 247\cancel{1} \\ \underline{-\ \ \ \ 2} \\ 24\cancel{5} \\ \underline{-\ 10} \\ 14 \end{array}$$

Since 14 is divisible by 7, so are 245, 2471, and 24836. It should be noticed that in this case, it may be easier just to divide by 7 and see what happens!

Example 3:

$$\begin{array}{r} abc\cancel{d} \\ \underline{-\ \ 2d} \end{array}$$

$$100a + 10b + c - 2d$$

To see why the divisibility rule works, suppose $100a + 10b + c - 2d$ is divisible by 7. Then there exists an integer k, such that

$$7k = 100a + 10b + c - 2d$$

$$7k - 100a - 10b - c = -2d$$

$$-7k + 100a + 10b + c = 2d$$

$$-7/2\,k + 50a + 5b + c/2 = d. \qquad (3)$$

Using (3) for d, $abcd$ becomes, upon expansion,

$$abcd = 1000a + 100b + 10c - 7/2\,k + 50a + 5b + c/2$$
$$= 1050a + 105b + 21/2\,c - 7/2\,k$$
$$= 7(150a + 15b + 3/2\,c - 1/2\,k).$$

Since 7 is a factor of $7(150a + 15b + 3/2\,c - 1/2\,k)$ and $150a + 15b + 3/2\,c - 1/2\,k$ is an integer because k and c are of the same parity, $abcd$ must be divisible by 7.

Divisibility by 8:

A number is divisible by 8 if the last three digits are divisible by 8.

$$abcde = 10000a + 1000b + 100c + 10d + e$$
$$= 8(1250a + 125b) + 100c + 10d + e$$

Since 8 is a factor of $8(1250a + 125b)$, what is left to consider is $100c + 10d + e$. If $100c + 10d + e$ is divisible by 8, then so is $abcd$ since it is the sum of two expressions each divisible by 8. Since $100c + 10d + e$ is the representation of the last three digits, the divisibility rule has been illustrated using factoring.

Divisibility by 11:

A number is divisible by 11 if the difference between the sum of the digits in the odd places and the sum of the digits in the even places is divisible by 11.

Example: 16148

The sum of the digits in the odd places:

$1 + 1 + 8 = 10$

The sum of the digits in the even places:

$6 + 4 = 10$

Since the difference between the two sums is 0 (which is divisible by 11), the original number 16148 is divisible by 11.

Why does this work?

$$abcd = 1000a + 100b + 10c + d$$
$$= 1001a - a + 99b + b + 11c - c + d$$
$$= 1001a + 99b + 11c - a + b - c + d$$
$$= 11(91a + 9b + c) + (b + d) - (a + c)$$

Since 11 is a factor of $11(91a + 9b + c)$, what is left to consider is $(b + d) - (a + c)$. Since $(b + d) - (a + c)$ is the representation of the difference of the sums of the digits in the odd and even places, it follows that if it is divisible by 11, then $abcd$ is also divisible by 11.

Once factoring with divisibility rules has been explored with a class, other related examples can be used as illustrations. For instance, the numbers 252 and 161 have a "special" property. The sum of the first two digits and the sum of the last two digits are seven <u>and</u> both numbers are divisible by seven (Messmer and

Tanner, 1978; Covillion, 1986). Students like these "neat tricks." They should be encouraged to ask themselves if a particular property always works. The answer in this example is yes; the property is just a special case of the divisibility by seven rule. Students at this point should be able to give a short proof using the factoring techniques that have already been presented.

Other possible questions can be posed to students:

1) What is the divisibility rule for 16?
2) The first three digits are the same as the last three digits of a six digit number. This six digit number is always divisible by what number?

Number Patterns

As educators, we all have had "Lucky Larrys," i.e. those students who get right answers from wrong processes, in our classes (Hogg, 1993). I believe that it is our job to use these "creative" solutions to further our classes' insight into the original problem. Many times a lively class discussion can begin by examining the error that was made to see <u>why</u> the correct answer was achieved.

Last summer I was teaching an Intermediate Algebra class. The topic under discussion was simplification of radical expressions. For whatever reason, I had written on the board $12^2 = 144$. Right next to that, by chance, I had $21^2 = 441$. Larry noticed that if he read the first equation "backwards" he got the second equation.

Other Larrys in the class noticed that the observation was true for a few other examples, such as 13 ($13^2 = 169$ and $31^2 = 961$) and even 10. (Note that $01^2 = 001$ is a true statement, even though the zeroes normally are not written.) But, it did not take the class too long to find many examples that did not fit the pattern. The lesson continued, with Larry convinced that he would still use his calculator to help him square numbers.

After class, I followed up on Larry's version of squaring. After a few minutes of analysis, I confirmed that the method would work under certain conditions (the class had already discovered this). If one lets the two digit number be represented by ab, the observation can be stated: If $(ab)^2 = xyz$ then $(ba)^2 = zyx$. The following must be true for Larry's method to work:

1. Both a and b must be less than or equal to 3. If either a or b is greater than 3, at least one of $(ab)^2$ and $(ba)^2$ will have four digits.
2. The sum of a and b must be less than or equal to 4. If $a + b$ is greater than 4, then again $(ab)^2$ and $(ba)^2$ will have more than the desired three digits.

Also, consider

$$\begin{array}{r} ab \\ \times\ ab \\ \hline ab + b^2 \\ a^2 + ab \\ \hline a^2 + 2ab + b^2. \end{array}$$

Since both a and b must be less than or equal to 3, the middle digit of the resulting

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square must be double the product of a and b.

Even though time was short to complete the required material in the course, I thought that it was crucial to reexamine this pattern during the next class. As soon as I represented the two-digit number as ab, the class was able to determine when Larry's method would work. This analysis involved trial-and-error, graphing inequalities, and quadratic equations, all topics previously discussed in class. When I told my students that they had also solved a number theory word problem, they were shocked! After all, word problems are hard to solve!

The class really enjoyed the discussion. Before returning to the "required" material, I left the students with some other thoughts: Will something similar work with squaring numbers with more than two digits and are there any restrictions? Is there also a similar pattern involving cubes or other powers?

Larry's squaring pattern would also be appropriate for interesting questions in other mathematics courses. For example, in a Probability course, the following question could be posed:

> Let ab be a two digit number with a the ten's digit and b the unit's digit. Let xyz be a three digit number with x, y and z also representing digits. Consider all values of a and b such that $(ab)^2 = xyz$ and $(ba)^2 = zyx$. Find the probability that an unknown two digit number satisfies both equations.

A computer instructor could assign a beginning computer class the following problem: Write a short program to generate all numbers that satisfy the following two equations: $(ab)^2 = xyz$ and $(ba)^2 = zyx$ (where a, b, x, y, and z are defined as before). This problem could also be attempted using spreadsheets.

Conclusion

These examples show methods that could be used to attain two particular goals that apply to developmental mathematics, as listed by AMATYC (AMATYC, 1993) in a circulating draft on standards:

1. The developmental mathematics curriculum will emphasize the development of mathematical understandings and relationships.
2. The developmental mathematics curriculum will develop students' confidence in their ability to use mathematics appropriately and efficiently so they will become effective and independent users of mathematics.

These goals were further incorporated into the final draft on standards for introductory college mathematics (AMATYC, 1995).

NCTM has also been investigating standards and has recognized the following "shifts" in their visions (NCTM, 1995):

1. Shift in content *toward* a rich variety of mathematical topics and problem situations *away from* just arithmetic.
2. Shift in learning *toward* investigating problems *away from* memorizing and repeating.
3. Shift in teaching *toward* questioning and listening *away from* telling.

4. Shift in expectations *toward* using concepts and procedures to solve problems *away from* just mastering isolated concepts and procedures.

All of these situations in this article illustrate something that should be happening in the classroom today. Teachers should be reinforcing concepts taught in the classroom. In Larry's case, a teacher should not be too quick to tell him that his observation is incorrect. If by chance, a student's mistake leads to a correct answer, encourage him to investigate why "it worked." (This should only be done after the student understands why he initially made the mistake.) Many times this additional exploring will lead to reinforcing material the student has already learned. Teachers should "challenge students, but at the same time build confidence in their abilities to learn and use mathematics" (AMATYC, 1995).

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Lucky Larry #20

Larry's "methods" are questionable, but his numerical answer is correct, to three significant digits.

$$\begin{aligned} \ln(2x-3) &= 1 \\ \ln 2x - \ln 3 &= 1 \\ \ln 2x - 1.0986 &= 1 \\ \ln 2x &= 2.0986 \\ 2x &= e^{2.0986} \\ x &= e^{\frac{2.0986}{2}} = e^{1.0493} \\ x &= 2.86 \end{aligned}$$

Submitted by Richard Hartt
Onondaga Community College
Syracuse NY 13215

Little Rock
full of surprises

American Mathematical Association of Two-Year Colleges

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STANDARDS:

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- Technology-Based Delivery Systems
- Critical Thinking Skills and Problem Solving
- Professional Development
- Alternative Teaching Strategies
- Curricular Reforms: Calculus, Pre-Calculus, Developmental Math

Arkansas' diverse physical and cultural features, which exist nowhere else, created the "Natural State." Scenic landscapes and outdoor recreational activities abound year-round. From the breathtaking Ozarks, with their wonderful mountain heritage, to the Delta, featuring its traditional southern accent and meandering river, friendly Arkansas will greet you warmly.

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- Museums are within walking distance
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A Mathematics Educators' Guide to Internet Gophers

by

Jon W. Scott and Elizabeth J. Teles

Jon W. Scott
Montgomery College
Takoma Park MD 20912
jscott@umd5.umd.edu

Elizabeth J. Teles
National Science Foundation
Arlington VA 22230
eteles@nsf.gov

Jon Scott is Professor of Mathematics at Montgomery College and currently is spending a sabbatical year as Visiting Mathematician for Professional Development through Technology at the Mathematics Association of America. He received his education at the State University of New York at Albany (BS) and Western Michigan University (MA and SpA).

Elizabeth Teles is Program Director for Mathematics and Lead Program Director for the Advanced Technological Education Program in the Division of Undergraduate Education at the National Science Foundation. Prior to joining the NSF, she was Professor of Mathematics at Montgomery College. She received her education at Winthrop College (BA), Johns Hopkins University (MAT), and the University of Maryland (PhD).

What innovative mathematics curriculum development projects exist for calculus? How can I contact the person(s) involved to learn more?

What precalculus laboratory projects have been funded by the National Science Foundation? Who is involved in these projects?

What Mathematics Faculty Enhancement workshops are being given this summer? How can I register for them?

What is being reported in the news that involves probability and "Chance" events?

Is there public domain software that I can use in my calculus class? Can I access it via the Internet? If so, how?

Who is the Chair of the AMATYC Committee on Education?

What's the address and telephone number for the Executive Secretary of MAA? For a mathematician I met at a conference last week? For the mathematics program directors at NSF?

Until recently, answers to the above questions have been difficult to find. At best, such information is found in many different places, usually in paper copy only, and is often outdated. Now answers to these and many similar questions can be found easily and instantaneously on the Internet through a powerful and easy to use tool known as a *Gopher*.

The Internet has much to offer mathematics educators. For example, one can join several mathematics related e-mail *discussion groups*. Members of the *Calc-TI* list discuss problems and share ideas related to using TI's graphing calculators in the classroom. The *Calc Reform* list is devoted to discussion of all the various issues involved in that subject. One can read, respond to, and initiate messages as one chooses.

Faculty can also receive electronic newsletters and journals. The *Chance Newsletter* consists of short abstracts of "chance" events reported in the news media. Published every other week, it will be automatically sent to the subscriber's Internet address. Likewise, the *Journal of Statistics Education* a refereed journal devoted to improving the teaching of statistics, is published only in electronic format. This article, however, focuses on the use of about a dozen *Gophers* of interest to undergraduate mathematics faculty. Other services, such as the World Wide Web, discussions groups, and list servers, await a subsequent article.

What Is A Gopher?

A Gopher is a tool which allows the user to locate and retrieve information from the huge network of computers around the world. Gophers present the user a menu of choices, some of which point to text documents, some to other menus. Often there are items which allow keyword searches on remote databases, sign-ons to other computers, or downloading of files. How one accesses a Gopher depends on the Internet client software on one's host computer, be it through the World Wide Web (via a browser like Mosaic or Netscape), Windows (WS Gopher) or Macintosh (TurboGopher) software, or a UNIX shell account. As an illustration, on a UNIX shell account or through WS Gopher, one would type the command *gopher* followed by the Internet address of the desired Gopher. To access the MAA Gopher, type *gopher gopher.maa.org*. The first *gopher* is the command and *gopher.maa.org* is the address of the MAA Gopher. At the conclusion of this article, addresses and a short description of several mathematics related Gophers are given.

The article begins with a section on navigating through a Gopher and then describes some common ways to use different mathematical gophers to answer the questions posed at the beginning of the article. These skills include (a) finding, reading, and downloading text-based files, (b) searching a database, and (c) downloading software. Remember, however, that the Internet is a rapidly growing source of information and is changing and expanding daily. For this reason, the Gopher screens the reader sees may vary slightly from what is shown in the article. Rather than discussing specifics, which will in most cases be unique to the particular system at each institution, this article examines resources on different Gophers once the user has access.

Navigating a Gopher

A Gopher is essentially an organized series of menus through which one moves to locate an item of interest. Moving forward in a Gopher is easy. Simply use the arrow keys to move to the desired menu item and press Enter, or, in a Window environment, click on the item. Moving back to the preceding level is just as easy, but will vary with the particular interface. In WS Gopher press the <*Esc*> key, in

UNIX, type "u", and in Mosaic click on the *BackArrow* button on the screen.

Symbols following individual menu item (or icons in Windows) give a clue as to what is underneath that particular item. The following are a few of the symbols that appear as line endings on the gopher menus along with their meanings.

Symbol	Meaning
/	There is another menu underneath. Select (press enter when highlighted) to move to that menu.
. (period) or (blank)	Text file underneath. Select the line to see the file.
<?>	Search a database. Selecting the line will bring up a box in which to enter search criteria.
<CSO>	Computer Search On-Line. A more elaborate phone and address book search.
<Tel>	Telnets to another computer.
<Bin>	A binary file ready for downloading. The user will be asked to designate the destination.

Finding and Using Text Files

Answers to most of the questions asked at the beginning of the article are information items found in text files. Finding such answers involves knowing which Gophers have such information and where that information is located in the Gopher. The American Mathematical Association of Two-Year Colleges (AMATYC), the Mathematical Association of America (MAA), American Mathematical Society (AMS), and the Society for Applied and Industrial Mathematicians (SIAM) all have their own Gophers as does the National Science Foundation (NSF). Special Gophers of mathematical interest include the Mathematics Archives, the Geometry Forum, Educom, Chance, and the Geometry Center. Some university mathematics departments also have their own Gophers. Gophers often have "pointers" to other Gophers allowing the user to easily move from one to another ("surfing the Internet.") Indeed, this is true for most of the Gophers described in this article.

```
                Root gopher server: gopher.maa.org

--> 1. About the MAA organization and its activities/
    2. Celebrating progress in collegiate mathematics/
    3. Committees and governance of the MAA/
    4. Electronic services of other mathematical organizations/
    5. General information of interest to mathematicians/
    6. Meetings calendar/
    7. Publications of the MAA/
    8. Sections of the MAA and their activities/
    9. Student information and activities/
    10. Women and Minorities/
    11. Mathematics Awareness Week --- 23-29 April 1995/
    12. Professional Development Activities/
    13. Suggestion Box
    14. What's new on the MAA Gopher (February 1995)/
```

Figure 1

THE MAA GOPHER

The MAA Gopher is a good place to start. Not only does it contain a lot of useful information in its own right, but it has pointers to many other Gophers described in this article. The opening screen is shown in Figure 1.

Notice that all but one of the items listed ends in a "/" signifying that another menu lies underneath. While most lines are fairly descriptive, it sometimes takes a little investigation to find out what is actually there. (Remember, one can always go back to a previous menu.) Consider the second item on the menu, *Celebrating progress in collegiate mathematics*. Here one can find information about various programs attempting to address important issues in undergraduate mathematics education. Included in these are innovative curriculum development projects.

```
                Celebrating progress in collegiate mathematics

-->  1.  About Celebrating Progress in Collegiate Mathematics
     2.  Alphabetical listing of all programs/
     3.  Assessment/
     4.  Course and curriculum development/
     5.  Professional Development/
     6.  Student issues/
     7.  Teacher Preparation (pre-service and in-service)/
     8.  Teaching of undergraduate mathematics/
     9.  Technology/
```

Figure 2

Selecting *Course and curriculum development* brings up another menu categorizing the projects into area such as *precalculus, calculus, general education* and the like. Choosing precalculus, for example, currently gives the reader more than 25 choices to investigate. The file on each project contains several screens of information including the project name, the institution and contact person (phone, fax, email, mailing address), the intended audience and a description of the project.

```
Press <RETURN> to continue,
   <m> to mail, <D> to download, <s> to save, or <p> to print:
```

Figure 3

After viewing the file several options for obtaining a copy are presented.

Choosing <m> brings up a box in which to enter a valid email address (generally the reader's). The file will be mailed to that address. Selecting <D> (note the upper case) downloads directly to one's own personal computer; <s> to the host computer.

Want to find out about a meeting or a summer workshop? The MAA Gopher is the place. From the opening screen, select *Meetings calendar*. Listed are not only regional and national meetings but also MAA and NSF workshops pertaining to mathematics education. The item *Committees and governance of the MAA* includes names and addresses of MAA Headquarters staff and committee chairs. One of the most useful things about the organization of Gophers is the ability to include pointers to other Gophers. This allows the user to move to another Gopher almost without knowing it. Selecting *Electronic Services of other mathematical organizations* from the MAA Gopher currently allows the user to access to a wide array of mathematics related Gophers including most of the ones mentioned earlier in this article.

THE CHANCE GOPHER

The Chance Gopher deals with current events in probability and statistics as applied to the world around us. Anyone teaching an introductory statistics course ought to take a look at this Gopher and particularly at the menu item *Chance News*. This is an electronic newspaper of sorts, delivered to the Internet every couple of weeks, which contains excerpts, abstracts and discussions of current issues where "chance" has played a role. Much of it is serious, but it generally contains some fun things, too. It is guaranteed to be interesting. What is especially noteworthy is that the material is geared to be used by and with students, and the moderators of Chance often suggest how the material can be used. Here is a short segment from a recent *Chance News* (11 Jan 1995 – 1 Feb 1995.)

> Secondhand smoke: is it a hazard?
> Consumer Reports, January 1995, p27-33.
> January 1995, p27-33.
>
> This article describes the mounting evidence about health risks from exposure to secondhand smoke, as described in recent EPA reports, and the tobacco industry's campaign to discredit the scientific basis for those findings. The conclusion: "The tobacco merchants claim there's still a controversy. We don't buy it." There is some nice non-technical discussion here of the methodology of epidemiological studies and the issues involved in applying meta-analysis to combine the results.
>
> The article responds directly to some of the industry's criticisms of the EPA report on secondhand smoke, some of which were articulated in Jacob Sollum's article in the National Review, (see Chance News, June 10, 1994). For example, the alleged changing of the threshold for significance from 5% to 10% is described here as the result of using one-tailed rather than two-tailed tests. The article argues that the use of a one-sided test is entirely appropriate when there is already independent evidence that a substance is harmful. Also discussed are controversies about possible confounding variables, and criticisms about studies allegedly excluded by the EPA.
>
> DISCUSSION QUESTION:
>
> Do you think that using a one tailed test justifies making the threshold of significance at the 10% level?

THE AMATYC GOPHER

The AMATYC Gopher has recently been established and is currently a part of the Gopher at State Technical Institute at Memphis. To reach it, first gopher to STIM: *gopher stim.tec.tn.us*. At the opening menu, choose *Mathematical Associations* and then *American Mathematical Association of Two-Year Colleges*.

```
        American Mathematical Association of Two-Year Colleges

--> 1.  General Announcements/
    2.  Jobs/
    3.  Leadership/
    4.  about.amatyc
    5.  available_publications.txt
    6.  label_request.form
```

Figure 4

This Gopher is still in its infancy, but one can find information about the organization, its leadership and committees, conference information, publications and how to get them, and job postings. Keep watch here for AMATYC news.

Searching a Database

Many Gophers also allow access to searchable databases. When this is an option, the line usually ends in a <?> or <CSO>. Two examples are given here: Searching the Combined Membership List (CML) on the *AMS e-Math Gopher* to find the name and address of mathematicians (currently all members of AMS, MAA, and SIAM are listed on the CML with AMATYC members to be added in the near future), and searching the NSF database *STIS* for NSF publications and awards.

THE AMS GOPHER: e-MATH

The AMS Gopher contains a rich amount of material of interest to people in the mathematics community. It is well worth exploring. Access to the CML is under *Professional Information for Mathematicians*.

```
           Professional Information for Mathematicians

    1.  About this Node
 >  2.  AMS Combined Membership List (Search) <?>
    3.  AMS Combined Membership List (Instructions)
    4.  Directory of Institutions in the Mathematical Sciences/
    5.  Math Reviews 1991 Subject Classifications/
    6.  Keyword Search of MR Subject Classifications <?>
    7.  Institution Codes & Addresses used in CMP & Math Reviews <?>
    8.  Serials Reviewed in Mathematical Reviews <?>
    9.  Guide for Reviewers (Mathematical Reviews)
    10. Guide for Electronic Submissions (Mathematical Reviews)
    11. MR Author Lookup (1985-current) <TEL>
    12. Professional Opportunities for Mathematicians/
    13. Career Development Resources/
```

Figure 5

Choosing *AMS Combined Membership List (Search)* brings up a search box in which to enter a complete last name. Pressing the enter key and then selecting *Matches for people with last name ...* , invokes the search and returns the selected listing. While searching on the name **Teles** returns one entry, searching on **Scott** returns many. When there is apt to be more than one entry with the same last name, another word can be entered following the last name. This could be any word that might reasonably be expected to be in the listing for that person, such as first name, institution name, or city. Searching on **Scott Montgomery** narrows the scope of the search.

```
              Professional Information for Mathematicians

     1.  About this Node
 --> 2.  AMS Combined Membership List (Search) <?>
     3.  AMS Combined Membership List (Instructions)
     4.  Directory of Institutions in the Mathematical Sciences/
              AMS Combined Membership List (Search)

 Words to search for


 [Help: ^_]  [Cancel: ^G]

     13. Career Development Resources/
```

Figure 6

THE NSF GOPHER: STIS

The NSF database is called STIS for Science and Technology Information System. One can login to the gopher directly - *gopher gopher.nsf.gov* - or through pointers on most of the other mathematics related gophers mentioned in this article. The opening menu consists of two pages.

```
                 Root gopher server: gopher.nsf.gov

 -->  1. About this Gopher
      2. About STIS/
      3. Search NSF Award Abstracts <?>
      4. Instructions for "Search NSF Award Abstracts"
      5. Search NSF Publications <?>
      6. Instructions for "Search NSF Publications"
      7. NSF Phone Directory <CSO>
      8. NSF Publications/
      9. NSB - National Science Board/
      10. Office of the Director/
      11. OIG    Office of the Inspector General
      12. BIO  - Dir. for Biological Sciences/
      13. CISE - Dir. for Computer and Information Science and Engineering
      14. EHR    Dir. for Education and Human Resources
      15. ENG    Dir. for Engineering/
      16. GEO    Dir. for Geosciences/
      17. MPS    Dir. for Math and Physical Sciences/
      18. SBE    Dir. for Social, Behavioral and Economic Sciences
```

Figure 7a

```
                 Root gopher server: gopher.nsf.gov

 -->  19. SRS    Science Resources Studies Division
      20. Cross Directorate Programs/
      21. Other U.S. Government Gopher Servers
```

Figure 7b

Note that choices 3 *(Search to NSF Award Abstracts)*, 5 *(Search NSF Publications)*, and 7 *(NSF Phone Directory)* all lead to searches. Also notice that items 4 and 6 give instructions for searches. In general, STIS gives lots of on-line help in using the system.

One of the questions posed at the beginning of the article concerned NSF funded projects for precalculus laboratories. All the award abstracts for NSF funded projects since 1989 are present on STIS. If the award number is known, simply search on that number prefixed with the letter "a". For example, to find award 9412345, enter **a9412345** in the request box. But, more than likely, the award number will not be known. Searching on the word **precalculus** locates forty-eight awards whose abstract file contains the word "precalculus." Each of these may be viewed or downloaded in the usual manner. Perhaps one wishes to narrow the search some to those involving a laboratory. The Boolean operators AND, OR, and NOT are permissible as are parentheses. Note that these must be entered in uppercase. Entering **precalculus AND laboratory** yields twenty-one awards with both words in the file, while the phrase **(precalculus AND laboratory) NOT algebra** found ten.

One has to be a little careful in claiming that there were twenty-one precalculus laboratory awards. The search is only on the words, and the document will be included if it contains both of these words anywhere in the file regardless of the context.

There is another system which provides even more sophisticated text searches and retrievals from the STIS database. Called On-Line STIS, it can be accessed through the Internet by telneting to *stis.nsf.gov* or through a direct dial-up connection. Details on using this system, as well as other useful information on obtaining NSF publications and award abstracts electronically are provided in document NSF 94-10, *STIS User's Guide.* Like most other NSF publications, this can be obtained through the NSF Gopher by selecting item 5, *Search NSF Publications* and enter **nsf9410** (no spaces!) as the search criteria.

Downloading Software

The Mathematics Archives *(gopher archives.math.utk.edu)* has a number of services and features available ranging from information on faculty workshops and calculus reform to public domain software and shareware for use in undergraduate mathematical instruction. This section discusses accessing software which is ready for downloading and running on the user's own computer.

```
                Root gopher server: archives.math.utk.edu

     1.  About the Mathematics Archives Gopher
     2.  Organization of the Mathematics Archives Gopher
 --> 3.  Software (Packages, Abstracts and Reviews)/
     4.  Teaching Materials and Other Information/
     5.  Other Mathematics Gophers and Anonymous FTP Sites/
     6.  Submitting Materials to the Mathematics Archives
     7.  Other Ways of Accessing the Mathematics Archives
     8.  Other Mathematics Archives Info (newsletters, reports, etc.)/
     9.  What is Gopher? (adapted from Texas A & M's Gopher)
     10. Information About Gopher (from Univ. Minnesota)/
     11. ----------------------------------------
     12. Mathematics Archives Announcement of WWW Server
     13. WWW FAQ - Frequently Asked Questions/
     14. The home page of the Mathematics Archives WWW Server <HTML>
     15. READ ME if you cannot access the above document
     16. ----------------------------------------
     17. Search Mathematics Archives gopher menus using Jughead <?>
     18. About Jughead
```

Figure 8

Software on the Mathematics Archives is arranged by subject content. A program of interest in both a precalculus and calculus class would be listed under both subjects. Each program has been tested and reviewed by the moderators who are available for assistance if users have problems. There are currently more than 300 DOS and 200 Macintosh programs listed on the Archives, including reviews and abstracts of various commercial products.

Instructions are given here to locate Naval Academy's graphing program *MPP: Mathematics Plotting Program*. This excellent DOS program is in the public domain and may be freely used and distributed. Locate *MPP* in the archives by traversing the following path through the menus. From the opening screen, select *Software (Packages Abstracts, and Reviews)* then choose *MSDOS Software arranged by subject*, followed by *Calculus*. Page down and select *Mathematics Plotting Program v.3.80 (mpp.zip)*. The following menu will appear.

```
            Mathematics Plotting Programs v.3.80       (mpp.zip)

--> 1.  Abstract for mpp.zip
    2.  Download mpp.zip <Bin>
    3.  Download the collection of examples for mpp.zip (mppexam.zip) <Bin>
    4.  Download the documentation for mpp.zip (mppdoc.zip) <Bin>
    5.  Readme file from mpp.zip
    6.  Reviews of mpp.zip
```

Figure 9

Select *Download mpp.zip <Bin>*. A prompt will suggest a name for the receiving file which may be changed. There are two things to note here. One is that this particular download copies the file to the user's host computer, not their PC. So it will require another file transfer step to transfer it to the user's PC. Secondly, this is a compressed binary file. This means that before it can be installed, it must be decompressed using a utility like PKUNZIP. The user may also want to download examples or documentation files from the Archives.

Try it!!

This article has discussed most of the useful skill needed to successfully use Internet Gophers and has illustrated some of what is available on a few of the mathematics related ones. But there is much more and the reader is left to explore. As every instructor knows, there is no substitute for active involvement and "doing," so give it a try. If there are difficulties, contact one of the authors by email, for a few hints. "Surfing the Internet" can be great, but, if one is not careful, can also be a real time-sink. Have FUN!!!

> It is our American habit if we find the foundations of our educational structure unsatisfactory to add another story or wing. We find it easier to add a new study or course or kind of school than to reorganize existing conditions so as to meet the need.
>
> John Dewey

A Short List of Gophers of Interest to Mathematics Faculty

<table>
<tr><td>AMATYC</td><td>stim.tec.tn.us</td></tr>
<tr><td colspan="2">American Mathematical Association of Two-Year Colleges. Information about AMATYC leadership, committees, conferences, and publications.</td></tr>
<tr><td>AMS</td><td>e-math.ams.org</td></tr>
<tr><td colspan="2">American Mathematics Society. Serves primarily the concerns of the research mathematics community. Information about AMS publications, conferences and meetings, and links to other mathematical information services. Includes access to the Combined Membership List.</td></tr>
<tr><td>Chance</td><td>via other math Gophers (eg. MAA)</td></tr>
<tr><td colspan="2">Chance Project Database. Materials useful in teaching probability and statistics based on current chance events as reported in newspapers and journals.</td></tr>
<tr><td>Educom</td><td>educom.edu</td></tr>
<tr><td colspan="2">Information relating to information technology and academic computing in higher education.</td></tr>
<tr><td>Geometry Center</td><td>geom.umn.edu</td></tr>
<tr><td colspan="2">The Geometry Center at the University of Minnesota. Information about the activities of the Center to develop, support, and promote computational tools for visualizing geometric structures. Includes an archive of geometric picture files.</td></tr>
<tr><td>Geometry Forum</td><td>forum.swarthmore.edu</td></tr>
<tr><td colspan="2">The Geometry Forum at Swarthmore. Archives of useful and interesting materials related to geometry and its teaching at all levels. Includes software, pictures, articles and project information.</td></tr>
<tr><td>MAA</td><td>gopher.maa.org</td></tr>
<tr><td colspan="2">Mathematics Association of America. Information about the MAA, its meetings, workshops, publications, and activities, as well as issues in collegiate mathematics education. Includes information about recent curricula projects, and links to other mathematics related gophers.</td></tr>
<tr><td>Mathematics Archives</td><td>archives.math.utk.edu</td></tr>
<tr><td colspan="2">The Mathematics Archives. A repository for all kinds of material used in teaching of mathematics. Includes public domain software for downloading and reviews of commercial software.</td></tr>
<tr><td>National Science Foundation</td><td>gopher.nsf.gov</td></tr>
<tr><td colspan="2">NSF Science and Technology Information System. A database of NSF publications and award abstracts.</td></tr>
<tr><td>SIAM</td><td>gopher.siam.org</td></tr>
<tr><td colspan="2">Society for Industrial and Applied Mathematics. Information for and about applied and computational mathematics and those who use mathematics.</td></tr>
</table>

Selected Readings

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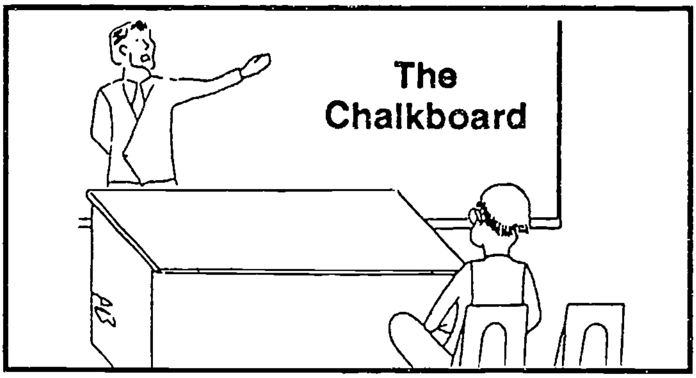


Edited by

Judy Cain and Joseph Browne

Tompkins Cortland Comm. College
Dryden NY 13053
cainj@sunytccc.edu

Onondaga Comm. College
Syracuse NY 13215
brownej@goliath.sunyocc.edu

This column is intended as an idea exchange. We hope to facilitate an open exchange of ideas on classroom management, teaching techniques, tips for helping students get past the usual stumbling blocks, techniques for improving student participation, etc. We know there are lots of good ideas out there, and this is your chance to share them. Please send your contributions to Judy Cain. Our backlog is nearly exhausted, and we would appreciate your participation! Items may be submitted by e-mail or regular mail; please include your e-mail address if available.

A Variation on Integration by Parts

A tabular method of displaying the u and dv in integration by parts is a fairly widely known short cut in spite of the fact that it appears in only a few of the standard calculus texts. A typical example would be $\int x^2e^{2x}dx$, for which the tabular solution would look like this:

$$\begin{array}{ccc} & u & dv \\ \hline + & x^2 & e^{2x} \\ - & 2x & \frac{e^{2x}}{2} \\ + & 2 & \frac{e^{2x}}{4} \\ & 0 & \frac{e^{2x}}{8} \end{array}$$

$$\int x^2e^{2x}dx = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$$

Each term is just the product indicated by the slanted dotted lines, i.e. u times v. The extra + and – signs on the left of the table come from the negative sign in the integration by parts formula; each successive application of the formula inserts another minus, producing the alternating pattern.

Apparently less well known is that the same tabular approach can still be quite helpful even when the u column doesn't eventually differentiate to zero. This can be particularly handy for those problems where it is necessary to use "parts" twice and then solve for the integral. The setup is the same, except that we must add in the integral of the product of terms in the bottom row, i. e. the last $v\,du$. Consider $\int e^{2x} \sin 3x \, dx$.

$$
\begin{array}{lcc}
 & u & dv \\
\hline
+ & e^{2x} & \sin 3x \\
- & 2e^{2x} & -\dfrac{\cos 3x}{3} \\
+\int & 4e^{2x} & -\dfrac{\sin 3x}{9}
\end{array}
$$

$$\int e^{2x} \sin 3x \, dx = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9} \int e^{2x} \sin 3x \, dx$$

$$\frac{13}{9} \int e^{2x} \sin 3x \, dx = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x$$

$$\int e^{2x} \sin 3x \, dx = \frac{9}{13} \left(-\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x \right) + C$$

Submitted by Robert Pumford, Jamestown Community College, Jamestown NY 14701

Testing Strategies for the Lecture Format Class

In lecture mathematics classes, I make every test cumulative. While I do not expect my students to review the text and notes for material related to previous tests, I do expect that they will review all the previous tests. I take questions from the old tests for the current test, changing the numbers to protect the innocent. Without this policy there is no incentive for a student to review a test after it is taken, or to master the skills not achieved for that test. Also, the student often doesn't attempt to master unlearned material until reviewing for the final exam (if at all); and, as we know, if the student didn't understand something on the first test, it may be irretrievable at the end of the semester. Obviously, cumulative tests throughout the semester produce much better performance on a cumulative final exam, as well as enhancing retention for the next course.

I also give a half hour test every two weeks, instead of the one hour exam given by many teachers at the end of a chapter. I believe that a chapter test seems to compartmentalize the material; I can almost hear my students thinking, "We're done with that chapter (finally!), so study, take the test, and we'll move on to something else (which may be easier — keep your fingers crossed)." With chapter tests, a student is tempted to postpone studying until the end of the chapter approaches, which may be four weeks into the course. (Daily or weekly quizzes may also counter this particular behavior effectively.)

Testing every two weeks doesn't give the student much chance to put off learning the material. Since this is the only form of test I give, the biweekly test is very important to the student, and I find that students are more likely to keep up with assignments. They know the test, like death and taxes, is inevitable and is coming soon, regularly, like clockwork.

My tests are one half hour in length, and are given at the end of class. Students don't like that part — they want to cram, then come to class and take the test before they forget. I tell them (and it's true) that I don't give the test at the beginning of class because inevitably someone wants more time, and I don't have the heart to take the test away at that point. Thus, I lecture on new material first, and let the clock do the dirty work. (Of course, like you, I accommodate students with learning disabilities or acute cases of math anxiety with untimed tests at other times.) I also point out to students that you cannot survive an entire semester by cramming. I counsel them to study regularly and know the material when they arrive in class. If they do, they will do fine on the test.

Of course I realize that a few students may be cramming while I'm lecturing on new material before the test. I will refrain from arguing why this may be acceptable, and let the reader ponder that.

Testing every two weeks produces six or seven grades at the end of the semester. Because tests and the final exam are cumulative, I allow the final exam to replace up to two test grades, and I give no make-up tests whatever. In this format, a make-up test is a waste of time. Since the tests are cumulative, the student has the incentive to master the material on a missed test without the incentive of an explicit make-up test. If the student would do well on a make-up, then (s)he will do well on the rest of the tests and on the final. If the student would not do well on a make-up, (s)he will not do well for the rest of the semester anyway.

I believe that the net effect of the strategies outlined above is to promote regular study, encourage mastering of material which a test showed was unmastered, reduce overall anxiety about the grade on any individual test, and produce better results on a cumulative final examination.

Submitted by Philip Mahler, Middlesex Community College, Bedford MA 01730, mahlerp@admin.mcc.mass.edu

> It is ironic that the United States should have been founded by intellectuals, for throughout most of our political history the intellectual has been for most part either an outsider, a servant, or a scapegoat.
>
> Richard Hofstadter

Snapshots of Applications in Mathematics

Dennis Callas
State University College of Technology
Delhi NY 13753

David J. Hildreth
State University College
Oneonta NY 13820

The purpose of this feature is to showcase applications of mathematics designed to demonstrate to students how the topics under study are used in the "real world," or are used to solve simply "charming" problems. Typically one to two pages in length, including exercises, these snapshots are "teasers" rather than complete expositions. In this way they differ from existing examples produced by UMAP and COMAP. The intent of these snapshots is to convince the student of the usefulness of the mathematics. It is hoped that the instructor can cover the applications quickly in class or assign them to students. Snapshots in this column may be adapted from interviews, journal articles, newspaper reports, textbooks, or personal experiences. Contributions from readers are welcome, and should be sent to Professor Callas.

Error-Correcting Codes

(to accompany higher-order polynomials and systems of equations)

by Linda Kurz, SUNY College of Technology, Delhi NY

Question: What do CDs that play your favorite music and satellites like the Voyager have in common?

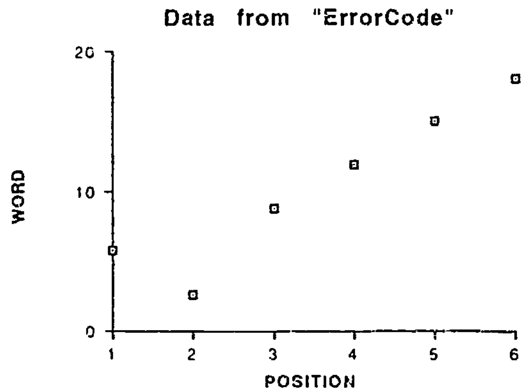
Answer: They both send messages of information over "noisy" channels, and as such need error-correcting techniques built into their systems to ensure that you get the intended information. For example, Voyager signals travel through "noisy" channels interferred by atmospheric storms, and competing frequencies from satellites orbiting the Earth, to name a few phenomena.

Both Voyager and your CDs use mathematics that was proven theoretically possible in 1948 by a mathematician named Shannon, and finally put into practical terms in 1960 by the Reed-Solomon team of engineers at MIT. In simple terms: one collects data, encodes the data using repeated patterns, transmits the data on often noisy channels, de-codes the data by adjusting for errors suggested by the strong patterns transmitted, and finally recovers the corrected data.

Musical CDs obviously "send" information that gets decoded audibly. The Voyager sends back information in the form of 0's and 1's, 00000000 being "white" and 11111111 being "black" with 254 shades of grey in-between being represented by the other 8-tuples you can create with 0's and 1's. Each picture it sends contains 640,000 of these various shades of 0's and 1's, and sends its code in groups of 223 pieces at a time.

The Reed-Solomon technique is based on curve fitting using polynomials as the model for the "fit." So, for Voyager, each coded message is sent using a polynomial of degree 222! Mathematically, this means being able to solve 223 equations in 223 unknowns! Fortunately, the use of computer programs to solve these equations makes this coding technique viable.

Suppose you want to send a 2-word message, coded into numbers. Say, for example, the two coded values are 2.6 and 5.7. The pairs to be sent are: (1, 2.6) and (2, 5.7) where the first part of the pair denotes the position of the word being sent. These two pairs determine a straight line. You can mathematically find the equation of the line, which in this case is: $y = 3.1x - 0.5$. Using the equation, you can create four more pairs that satisfy this equation. Let them be, for example, (3, 8.8), (4, 11.9), (5, 15.0), and (6, 18.1). The message gets sent encoded in the following manner: (2.6, 5.7, 8.8, 11.9, 15.0, 18.1). This establishes a strong pattern that enables one to "recover" data that has incorrectly been received through the noisy channel.



Suppose, for example, this data was received as: (5.7, 2.6, 8.8, 11.9, 15.0, 18.1) as shown on the graph above. The four points that look colinear can be used to determine the equation of the line containing them. Then by substituting the x-coordinate of the points that don't fit, you can determine what their y-values should be.

Exercises

1. A two word message has been sent using the Reed-Solomon encoding process. It was received as: (2.9, 14.1, 8.5, 11.3, 5.7, 16.9).

   Graph these as pairs of points: (1, 2.9), (2, 14.1), (3, 8.5), etc. Determine the equation of the line that fits the most points, and decode the message.

2. If one were to send a three word message using the Reed-Solomon process, one would use a quadratic equation .to create the strong pattern. Here is the message to be coded: (1.2, 2.5, 3.7).

   The ordered pairs for generating the strong pattern would be: (1, 1.2), (2, 2.5), (3, 3.7).

A general quadratic equation has the form: $ax^2 + bx + c = y$.

The system of equations you'll need to solve to create the strong pattern for sending the message is:

$$a(1)^2 + b(1) + c = 1.2$$
$$a(2)^2 + b(2) + c = 2.5$$
$$a(3)^2 + b(3) + c = 3.7.$$

Solve this system for a, b and c either by hand or by using some matrix program on your computer. Write the quadratic equation that creates the strong pattern for the code.

Use your equation to create four more points: $(4, y)$, $(5, y)$, $(6, y)$, $(7, y)$. Then, write the message to be sent.

3. As the number of code words increases, solving the system of equations by hand becomes almost impossible. Use a computer program or one on your calculator to create a strong pattern for sending the following seven word message:

$$(2.3, 1.1, 1.7, 2.4, 2.6, 3.0, 4.0)$$

This snapshot was based on an article by Joseph Malkevitch (1992) and was produced as part of a project sponsored by the State University of New York and the National Science Foundation (Division of Undergraduate Education).

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In teaching you cannot see the fruit of a day's work. It is invisible and remains so, maybe for twenty years.

Jacques Barzun

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The quality of a university is measured more by the kind of student it turns out than the kind it takes in.

Robert J. Kibbee

Notes from the Mathematical Underground

Edited by

Alain Schremmer
Mathematics Department, Community College of Philadelphia
1700 Spring Garden Street
Philadelphia PA 19130

There is a widespread belief that, in order to proceed into mathematics, all that is needed is a certain number of prerequisite skills, collectively going by the name of Basic Algebra. Hence, since those lacking these skills are supposedly barred from mathematics, the need for remediation. What is conspicuously absent from this *Weltanschaung* s summarized by the very first sentence in Bourbaki's *Elements de Mathématique*: There is no prerequisite, only a measure of mathematical maturity.

But then, whether most mathematics remediation is a failure, as (Steen, 1991) maintained, and (Laughbaum, 1992) concurred with and (Rotman, 1993) disagreed with in these pages, must surely be an ill-posed problem. Laughbaum's opening paragraph mentions that "*it is difficult for instructors to spend adequate time on fundamental concepts*" and gives as an example student performance on factoring, one of the "*traditionally taught skills at the remedial level.*" He blames "*symbol manipulation*" and his solution is to use graphics calculator and to bend the curriculum to fit. On the other hand, Rotman's "*own developmental program is quite successful by traditional standards*" and what he advocates is the creation of a task force! Just in case, one supposes.

What are *we* to conclude from all this? A clue is that, if both authors mentioned *topics*, mostly topics in algebra, neither gave any *reason* for learning these particular topics. In reading their "viewpoints," one cannot avoid the impression that God created The Curriculum, probably some time before the Big Bang, even if, deplorably, She didn't have the foresight to create students good enough to learn it from us. Women! Usually, there is a vague, unspoken assumption that it is somehow "useful" for the students. But, surely, as a tool for "applications," basic algebra just won't do. In fact, even "*First Semester Calculus has no applications*" as (Dudley, 1988) pointed out as Conclusion #5 in an article on calculus texts. When confronted with this harsh reality, we usually fall back on something like factoring being good for the students' soul.

The matter must depend on our idea of what mathematics *is* and of what possible *use* it can be to "just plain folk" (Goldstein, 1986). In other words, this raises the question of what *learning* mathematics consists of. Of course, none of the above viewpoints saw fit to disclose any *idea* on that matter. So what *is* mathematics? And what, therefore, should Developmental Courses develop in order for just plain folk to learn mathematics?

As an example, I propose to specify Basic Algebra "equationally" rather than "descriptively" or "prescriptively" namely as the very least needed to deal with the "elementary" initial value problems $f'(x) = f(x)$ and $f''(x) = \pm f(x)$. We argued in (Schremmer & Schremmer, 1989) that the Precalculus and the Differential Calculus could be integrated into a systematic study of functions culminating with the "elementary" functions based on their (Laurent) polynomial approximations. This approach, going back to (Lagrange, 1797) and which we expounded in (Schremmer & Schremmer, 1990), conceptually requires very little beyond familiarity with decimal numbers. Reverse engineering then determines the required Basic Algebra. For instance, we need to divide in ascending as well as in descending powers because while, near ∞, $\frac{x^3 - 1}{(x + 1)^2} = +x - 2 + \frac{3}{x} + (\ldots)$, near 0, $\frac{-1 + h^3}{(+1 + h)^2} = -1 + 2h - 3h^2 + (\ldots)$. (Note that we know when to stop the division: when the quotient has *concavity*.)

Thus, after Basic Algebra, considered as remediation, students can reach First Semester Calculus level in two semesters. Moreover, observe that, if the above definition of differential calculus meets the challenge implicitly posed by Dudley's Conclusion #5, it also has the merit to prepare for an alternative to the traditional Second Semester Calculus better suited for students *not* headed towards Physics, Engineering or Mathematics, namely a course in Dynamical Systems. I will expand on this in a future column and, for now, suffice it to say that reviewers of an—otherwise unsuccessful—NSF proposal were quite taken by the idea.

• • •

Starting from the fact that "*the very nature of number is that number is 'unitless'*", Reader Laurie Golson raises several issues concerning my advocating the use of units. She also indirectly showed how inadequate my presentation was. For instance, I should have pointed out that the unit for the 2 in "2 times 25 strawberries" is "25 strawberries" as, "times" notwithstanding, this is an *additive power* rather than a *multiplication*. Bad language always creates problems. Still, what *is* a number? Since the notion has evolved throughout history, asking what it should be in a given course is certainly a most important question. If readership interest warrants it, we could devote some space to a forum.

• • •

When, a *very* long time ago, I was first presented with the dictum that "minus times a minus is a plus," I of course firmly rejected it as obviously false. Upon consideration however, being a nice middle class child with no particular problem, I decided to *believe* my teacher and to memorize "minus times a minus is a plus" over my own better judgment, knowing full well that this was the only way to the future. You could say that this is when I became schizophrenic or, at the very least, when I learned that to succeed requires being dishonest.

In my previous column, I had held that "*the main problem students have with mathematics* [is that] *the conventional curriculum makes it completely impossible for them to see* [...] *the overall architecture according to which these things hang together.* For instance, the problem in the conventional approach to differential calculus is that limits, continuity, differentiability are introduced in the first few weeks of the course so that, if those concepts are not mastered immediately—and they cannot in such an architecture, it is impossible for the students to function intelligently: All they can do, all they must do, is to believe, memorize and become schizophrenic if not learn being dishonest.

To an extent, four-year schools can get away with it because *they* have enough "good" students, that is students whose social background is such that they have no reason not to trust their instructor. After all, they probably belong to the same social class. By and large however, *our* students are in a different situation as *they* have little ground to trust us or a societal system that is grinding them down. So, even though they have been brainwashed into truly believing that learning equals memorizing, they run into the unfortunate problem that one cannot memorize when in a state of anxiety, mathematical or otherwise.

Thus, it is indeed the very lack of architecture of the conventional approach that is a barrier to *our* students and I would propose that we discuss architectures. For example, if nothing else, the architecture just alluded to above has the merit to leave enough time to "*spend adequate time on fundamental concepts,*" be it in the course of remediation or in that of the differential calculus. But there are, of course, other architectures for which however I would not be a good advocate and advocates of such architectures should use this column.

Speaking of architecture, or rather, the lack thereof, could this be the result of the modern trend to find safety in numbers? It indeed used to be that a large number of reviewers was necessary to ensure the salability of the fat calculus text: Presumably, at least the favorable reviewers would use it. Moreover, and to quote Dudley again: "*If one* [reviewer] *writes that the author has left out the* $\tan(x/2)$ *substitution in the section on techniques of integration, how can he or she do that, we won't be able to integrate* $3/(4 + 5 \sin 6x)$, *how can anyone claim to know calculus who can't do that.*" But now it seems that it is a large number of *authors* that has become necessary in addition to "*the generous support of the National Science Foundation*" mentioned in (Hughes-Hallett et al., 1994), apparently the number one seller and a real critique of which is, I think, vastly overdue. It certainly does not have much of an architecture: It begins with Chapter 1 - A Library of Functions, Chapter 2 - Key Concept: The derivative. This under the name, inter alia, of someone who once wrote that Calculus "*frequently hurries into such questions as differentiation and integration, and often fails to put the proper emphasis on what the subject is all about, namely function of a real variable*" (Gleason, 1967). And then, all of this in 148 pages! I can well understand why the required background is left rather fuzzy: "*We have found that this curriculum to be thought-provoking for well-prepared students while still accessible to students with weak algebra backgrounds.*" Presumably, *their* students "think" about calculus while *our* students can only be expected to "access" it.

As long as we in two-year colleges allow ourselves to be driven by four-year schools, we are doomed to impotence. It is interesting in this respect that even Dudley should have about concluded his article with the statement that "*Calculus is a splendid screen for screening out dummies, but it also screens out perfectly intelligent people who find it difficult to deal with quantities.*" Can there really be such people or is this a convenient way to dispose of them properly?

A tidbit about architecture. On page 967 of his Calculus (Anton, 1988), the famous author of *In Defense of the Fat Calculus Text* (Anton, 1991) essentially defines a differentiable function of *two* variables as a function that can be approximated by an affine function. Of course, on page 150, he had begun by saying that a function of *one* variable is differentiable if it has a derivative.

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Think before you speak. Read before you think. This will give you something to think about that you didn't make up yourself.

Fran Leibowitz

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History teaches us that men and nations behave wisely once they have exhausted all other alternatives.

Abba Eban

Software Reviews

Edited by Shao Mah

Title: Maple V Student Edition Release 3

Author: Waterloo Maple Software

Distributors: Springer-Verlag New York, Inc.
175 Fifth Avenue
New York NY 10010

and

Nelson Canada
1120 Birchmount Road
Scarborough ON M1K 5G4

Hardware Requirements: IBM PC 80486 or 80386 (coprocessor optional but recommended), 4MB extended memory, 16 MB disk space, DOS 5.0 or higher

Macintosh 4MB, 16 MB hard disk space, System 7

Price: Student Edition (with First Leaves: A Tutorial Introduction To Maple V or Maple V Flight Manual plus Getting Started and Release 3 Notes) $99 US or $99 Cdn

Student Edition, Education Version (with Getting Started and Release Notes only) $49.99 Cdn (Nelson Canada only)

Maple V is a Computer Algebra System (CAS), with a set of comprehensive mathematical packages, which does numeric and symbolic computations. Maple V Professional Edition Release 1 has been reviewed by Professor Girvan of Red Deer College in a previous issue of this column (Vol. 13, No. 1, Fall 1991). In this review, the features of Maple V Student Edition Release 3 will be reviewed.

The difference between Maple V's Professional Version and its Student Version is the restrictions placed on the size of problems which the Maple V Student Version can handle. Apart from the size restrictions, the Student Version of Maple V includes all the packages found in the Professional Version of Maple V's library, and it can be as efficient as the Professional Version.

There are several new features in Maple V Release 3. Compared to the previous release version, Release 3 provides more mathematical functions and it has an improved statistics package. It also has an improved user interface with a typeset-quality output, and an enhanced graphics capabilities. The user interface is called a worksheet in Maple V. It can be used to enter input, to display output or to mix text and graphs in a single worksheet for presentations or for print-out purposes. The worksheet also has tool bars and icons to perform common tasks.

It is worthwhile to mention that the Maple V Student Version contains two packages, "student" and "linalg," which are very useful for two-year college students. The "student" package provides the tools for the teaching and learning of calculus and the "linalg" package is for linear algebra. When the command "with(student)" is entered into the worksheet, the package "student" will be loaded into the memory and a list of 33 commands for calculus is shown. Similarly, if "with(linalg)" is entered then a list of 105 commands for linear algebra will appear on-screen. To perform an operation, a user needs to type the command in a specific form. The use of these commands are relatively simple. For example, to learn the concept of slope of tangent line at a given point, a user is given a choice of several different approaches. Maple V can calculate the slope by using the left, right, or two-sided limit of a function at a given point. A user can also find the slope by evaluating the derivative of the function at that point by using a single command. Graphically, one may use the "showtangent" command to show the graph of the function and the tangent line at a given point. Moreover, Maple V can animate a secant line to move towards a point which becomes a tangent line by using the "animate" command.

The reviewer has been using Maple V Student Edition Release 3 for teaching calculus. A class of third-term calculus students was involved in using Maple V Student Edition Release 3 during seminar hours. Students, in general, find it difficult to draw the three-dimensional graphs by hand. Maple V aids the plotting of space curves, animation of three-dimensional graphs and many other related activities, making the work much easier and, thus, stimulating the students' interest in learning calculus. Because of the reasonable price for the Maple V Student Edition Release 3, students are generally willing to purchase their own software. Besides, Maple V is one of the most comprehensive computer algebra systems available on the market today. Students have the advantage of using it for their studies in related fields. Maple V provides a large amount of commands for performing different mathematical tasks. However, its on-screen Help menu has been very useful in explaining the use of the various commands. Therefore, a common feedback from most of the students is that the Maple V Student Edition is one of the best CAS student packages available on the market at present.

Title: Converge Version 4

Author: John Mowbray

Distributor: JEMware
The Kawaiahao Plaza Executive Center
567 South King Street, Suite 178
Honolulu Hawaii 96813-3076
Phone: 808-545-3503 (subtract 5 hours from EDT),
call after 9:00 AM

Computer: Converge requires an IBM PC or a compatible with a hard drive where it needs only 3.57 MB of space, 540K RAM, and a VGA, EGA, MCGA, CGA, or Hercules compatible graphics adapter.

Price: Student price is $39.95 plus $10.00 S/H. LAN license is $545 plus

$15.00 S/H in the U.S.A. Site license for faculty use only for up to 14 faculty is $595.00 plus $15.00 for S/H in the U.S.A. Price for faculty is $139.00 + $10.00 S/H.

Introduction

The writer has compared a number of software packages during the past few years and this software seems to be the one well suited for a classroom as a teaching aid. Other software, the high-end computer aided symbolic software, requires knowledge of the commands and sometimes a bit of programming. Converge, Version 4.0, written and produced by a teacher of mathematics can do most of the numerical and graphical jobs required in a pre-Calculus or Calculus course. Converge has the ability to cut short teachers' work of drawing, creating a table of values of a function, show a limit both graphically and numerically, freeze graph, zoom in and out to watch mathematics dynamically and comprehend the nuances of Calculus. Yes, I said Calculus, as it is Calculus you can teach best with this software although Professor John Mowbray has included Algebra/Trig and other menus in his program. The latest version under review can also import and export functions to and from Derive (Soft Warehouse). This review is not meant to teach Converge, but the manual, which is very well written, provides a very quick tutorial. No programming language is required to work with Converge. Title bar and the context-sensitive help are enough to start using calculus as soon as it is loaded.

There are other technical improvements too. One can import functions from Derive with certain hot keys. Also functions created in Fortran, Basic, and C files can be brought in Converge, thus saving a lot of valuable time. This can also be accomplished by the File menu on the extreme left corner of the title bar.

Summary of New Features in Version 4.0

The program can be learned in about one hour of concentrated learning provided by the manual accompanying Converge. The new features include the following. The program can enhance graph break points. Thus, graphing piece-wise defined functions gives much smoother graphs. A beginning has been made by inducting vectors and their graphs. One expects dot and cross products in the next version. Matrix Calculator can perhaps be used to develop this important area. Mathcad Plus and Maple, among others, do this well.

Paired data can be manipulated now. Curve fitting up to a polynomial of degree 7 is now possible. Polynomial interpolation of (x_r, y_r) for $r = 1, \ldots, 9$ is now possible. After generating these polynomials, one can use these functions elsewhere in Converge to generate tables of values effortlessly. This becomes an excellent teaching tool. Converge can throw data sets back at you if you give it the equation of linear correlation by the submenu option "enter a linear correlation" and specify the number of paired data.

Linear programming problems involving solutions of inequalities resulting in vertical boundary lines are possible now. Also, one can graph up to six inequalities on the same graph. One can also add arrowheads to boundaries.

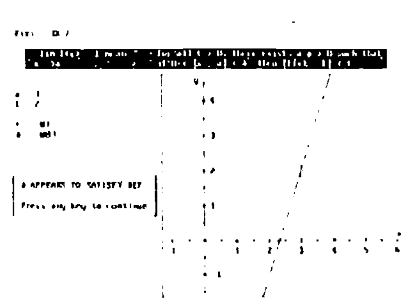
Post Graph

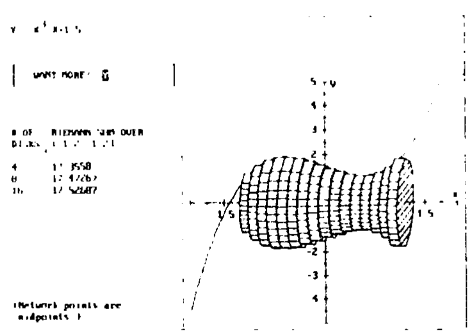
New features are: Intersection of Graphs of type $x = f(y)$. *One at a Time Menu* has been introduced. This helps one to emphasize graph break points on any graph, plot any points and draw tangent lines with the *Option* that the graphs remain intact when you escape to the main menu. This is helpful as one does not have to go through the graphing process again.

Two choices have been added to the *Derivative Extrema Menu* which help one graph the first derivative, table, and point ride (trace) and another set of operations with same features plus the second derivative of the function. This operation overlays these graphs in the same window, hence enhancing understanding of this important aspect of single variable calculus. One can generate tables of values by entering the independent variable values, do tracing on any of the graphs, generate values in the table simultaneously as well as tangent lines.

Calculus

Volumes by disk or washer method can be accomplished by revolution of the region about any horizontal or vertical line (not necessarily the x and y axes). Overlaying the graph of Taylor Polynomials over the graph of the function is a very neat pedagogical device. One can use up to *100th degree polynomial*. Epsilon-delta definition of limit has become a much better demonstration now with the addition of automatic zoom into the region bounded by $L \pm \varepsilon$ and $a \pm \delta$.





Memory Requirements

Converge 4.0 requires only 512,000 bytes of conventional memory compared to 535,000 in version 3.0. It is in contrast to other packages whose appetite for memory is increasing with each upgrade.

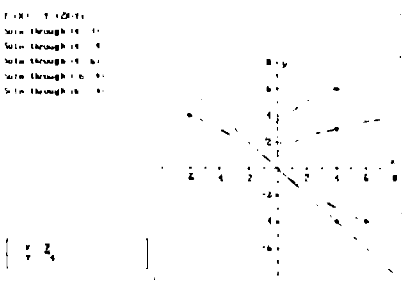
Other Enhancements

The user can print a color screen at the Main Menu even when the text has a blue background. The user can temporarily switch to monochrome mode via the "Options" and then "General Toggle Options" to get the black background for text while importing graphics to word processors or desktop publishing programs. Matrix Calculator is more capable now. Curve-fitting and correlation is sufficient for elementary ideas in Statistics. New factory macro **ALT+I** is a hot-key combination for selecting Enhance graph break points from the One at a Time

Menu. A new Mouse feature allows the user to enter new data by pointing and pressing the left mouse button.

A Graphic from Converge 4.0 to teach concept of Slope Field in Calculus

The figure below shows the direction field of a differential equation of first order. The box to the left bottom of the graph is the prompt to tell Converge where a point can be supplied to obtain a graph through that point.



Three dimensional graphs can have up to sixteen views. It is a quite sufficient list of views. I saved a lot of time in the multivariable class by taking advantage of the intersection of two surfaces.

An advantage of Converge over other CAS at the Calculus level is that a teacher can monitor student work right from her/his machine. Above all, the user does not need the commands of a CAS program.

Wish List

Users might like to see a better editing facility in the text region. Also, long tables cannot be saved in a file; they can only be printed by the hot-key **CTRL-P**. I had to fill in the numbers from these tables from the hard copy into my tables in Framemaker, the desktop publisher. *Graph vectors* is a limited facility which could be enlarged to three dimensional situation. There is a need to calculate double and triple integrals in a multivariable course. This could be added in the next version. The program would need a routine involving double and triple sums. Some software in the market available at this price do that. The inclusion of Cylindrical and Spherical coordinates would make Converge a complete program for a Calculus sequence.

Acknowledgments

Converge 4.0 is a trademark of JEMware. Derive is a trademark of Soft Warehouse. Framemaker is a trademark of Frame Technology. The author thanks the Critical Thinking Steering Committee for a Q-7 (Quality-7) acceleration grant sponsored by St. Cloud State University under the Minnesota State University System's quality initiative.

Reviewed by: R. N. Kalia, Department of Mathematics, St. Cloud State University, 720, 4th Avenue South, St. Cloud, MN, 56301-4498, *ravi@condor.stcloud.msus.edu*

> Everyone who remembers his own educational experience remembers teachers, not methods or techniques. The teacher is the kingpin of the educational situation. He makes or breaks the program.
>
> Sidney Hook

Title: Derive Version 3.01

Distributor: Soft Warehouse, Inc.
3660 Waialae Avenue, Suite 304
Honolulu HI 96816-3236

Price: $125

System Requirements: IBM compatible with MS-DOS 2.1 or later, 512K RAM, and one 3-1/2" disk dr. ve.

Derive is a comprehensive software package for college or high school students and their instructors for learning and teaching mathematics. Derive possesses two-dimensional (2-D) and three-dimensional (3-D) plotting capabilities and it does not require a large computer memory as compared to many other mathematical software packages. However, since pull-down menus or mouse control functions are not provided by Derive, it is, in practice, not as convenient as other mathematical software packages which can be operated under Windows environment.

There are few improvements for two-dimensional and three-dimensional graphs in Derive Version 3.01 as compared to its earlier versions. One of these improvements is the enhancement of 2-D graphs. In Version 3.01, for 2-D plots, one has the option to see the number scales and axis labels. Illustrated in Figure 1 is an example of a 2-D graph with number scales and axis labels generated by Derive. Another new feature is the "Range" option for 2-D plots. By using "Range," a user can specify the size and location of a rectangular area in a graph window. This specific region of the graph can then be enlarged. An example of "Range" option is shown in Figure 2a-2c.

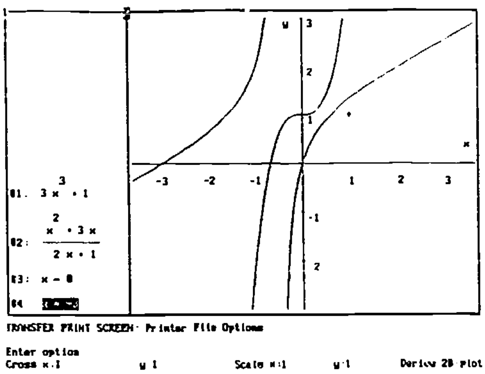


Figure 1

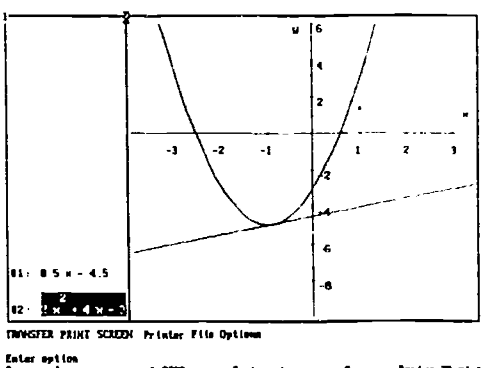


Figure 2a

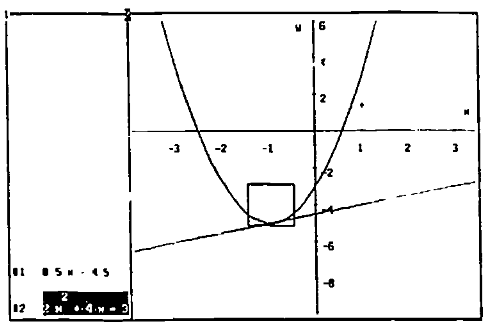


Figure 2b

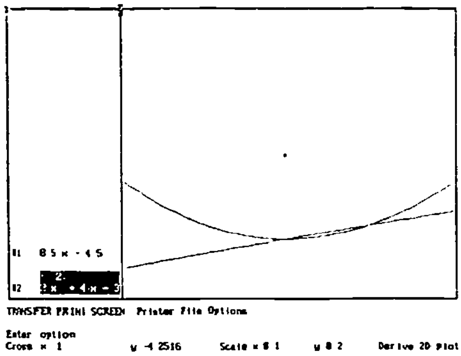


Figure 2c

Figure 2a shows an intersection of two curves. In Figure 2b, a rectangular area is chosen by the "Range" option and is consequently enlarged and becomes the new graphic window as shown in Figure 2c.

As shown in Figure 3, three-dimensional graphs can, also, be plotted by using Derive. Unfortunately, Derive does not have the capability to perform rotations or animations for 3-D graphs as well. It can be time consuming for a user to plot a smooth surface by a 286 or 386 computer.

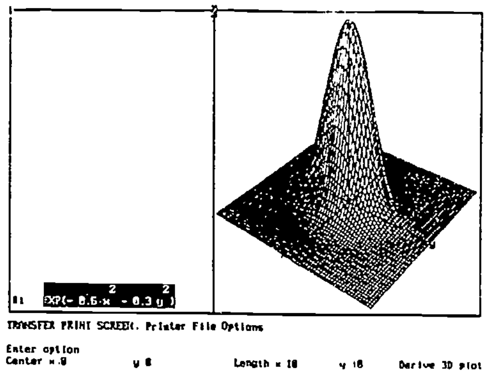


Figure 3

Derive Version 3.01, now, supports color printing. Thus, the users have a choice whether to plot graphs in color or not. It is definitely an improvement to better visualize 2-D and 3-D graphs. However, Derive does not allow a user to put mathematical expressions, text and graphs together in a single document.

Many fundamental mathematical operations can be performed by using Derive. The basic mathematical operations such as trigonometry and calculus can directly be performed by using the command menu. There are also twenty-three utility files in Derive which can perform other mathematical operations such as vector and matrix operations or solving first-order and second-order differential equations. To access the utility files, a user may need a Derive manual or Derive reference book. The on-line help files in Derive provides very limited explanations.

In general, Derive is an inexpensive, easy to operate software package for high school and first-year college level mathematics, especially for the users who do not have computers with large memories. Nevertheless, Derive could be more convenient to users if it provided a capability of superimposing the mathematical expressions and symbols of standard form (Greek letters, subscripts, superscripts, etc.) on the graphs. Furthermore, a Windows version of Derive would definitely be beneficial to the user.

Reviewed by: Liming Dai, Department of Mechanical Engineering, University of Calgary, Calgary, AB, Canada

Send Reviews to: Shao Mah, Editor, Software Reviews
The AMATYC Review, Red Deer College, Red Deer, AB, Canada, T4N 5H5

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Book Reviews

Edited by Gloria S. Dion

THE MOST BEAUTIFUL MATHEMATICAL FORMULAS, Lionel Salem, Frédéric Testard, and Coralie Salem, John Wiley & Sons, Inc., 1992, xiii + 141 pages, ISBN 0-471-55276-3.

THE ENJOYMENT OF MATH, Hans Rademacher and Otto Toeplitz, Princeton University Press, 1957, 205 pages, ISBN 0-691-07958-7 (hardcover) 0-691-02351-4 (paperback).

I have recently read two delightful books that fall at opposite ends of the mathematical spectrum. *The Most Beautiful Mathematical Formulas*, by Lionel Salem, Frédéric Testard, and Coralie Salem, and *The Enjoyment of Math*, by Hans Rademacher and Otto Toeplitz, are similar in that they each contain a collection of short chapters discussing a variety of classical mathematical topics, yet they are completely different in approach, spirit, and intended audience. Both books deserve consideration for a spot on your bookshelf.

Salem, Testard, and Salem's *The Most Beautiful Mathematical Formulas* is a light, whimsical collection of elementary results from geometry, algebra, trigonometry, and statistics. According to the book jacket, *The Most Beautiful Mathematical Formulas* is "An instructive romp through the 49 most interesting, useful, and/or quirky mathematical formulas of all time..." The informal exposition introduces and explains important concepts and results, including geometric and trigonometric formulas, special numbers and series, and results on prime numbers, powers of two, Pascal's Triangle, complex numbers, the binary system, and infinity. There are no proofs in this volume, and the level is appropriate for students in algebra and precalculus. More advanced students already familiar with the topics covered in this book will also enjoy the presentation. *The Most Beautiful Mathematical Formulas* was translated from French and there are a few differences in terminology and notation from American texts, but these are minor and should not cause any difficulty for students.

In *The Most Beautiful Mathematical Formulas*, mathematics is told as a story, which is sometimes fictional and sometimes historically accurate; an "Annex" at the end of the book distinguishes fact from fiction for the interested reader. Pictures are used throughout the book to clarify the mathematics, but they are cartoon-like rather than the traditional mathematical figures we find in textbooks (be sure to look for the contortionists Professor Sine and the scholar Cosine).

To illustrate the approach of *The Most Beautiful Mathematical Formulas*, consider that many of the geometric results are presented in the context of planting a garden. My favorite garden is the one planted by Lord Napier's Scottish gardener in Chapter 19, "The Discovery of Logarithms." The gardener was instructed by Napier to plant gardens x units wide bounded by two parallel lines and various upper boundaries ($y = 1$, $y = x$, $y = x^2$, and $y = 1/x$) with nothing planted in the first meter. The problem was that Napier would not give the gardener any more seeds than necessary for each garden, so the gardener was required to determine the exact area of each garden plot. This was a simple matter for the first three gardens, but the fourth garden required the development of Napierian (natural) logarithms. The chapters that follow explore some of the properties of logarithms and introduce the number *e*. Included in this section of the book is Chapter 25, "Derivatives and Integrals: Areas Viewed from Two Different Perspectives."

In contrast, Rademacher and Toeplitz's *The Enjoyment of Math* presents a detailed history, exposition, and proof (when possible) for a wide variety of interesting mathematical problems, primarily from geometry, number theory, and combinatorics. This book was originally published in German in 1929. It was translated into English in 1957 and reissued in 1994 by the Princeton University Press for the Princeton Science Library.

The 28 chapters of *The Enjoyment of Math* include discussions of prime, perfect and irrational numbers, maximization and minimization problems, results on polygons, circles, and polyhedrons, geometric constructions, special numbers, factorization, and the then unsolved Four Color Theorem and Fermat's Theorem. In general the solutions are not complete, but they contain the flavor of each problem and generally explore one facet of the problem in detail. In the case of the Four Color Theorem and Fermat's Theorem, although these chapters are frozen in time and do not include any indication of the proofs of the theorems, they provide an introduction and historical backdrop for understanding the problems. Readers of *The Enjoyment of Math* should have a level of mathematical sophistication beyond the first two years of undergraduate mathematics. There are ample drawings (of the traditional mathematical figure variety) throughout the book.

The first chapter I turned to when I read *The Enjoyment of Math* was Chapter 25, "Curves of Constant Breadth." After introducing the topic with definitions and the obvious curve of constant breadth – the circle – Rademacher and Toeplitz discuss the construction of the Reuleaux triangle, used in the Wankel rotary engine, the generalization of the Realeaux triangle to figures constructed with *n* circular arcs, and extensions to an infinite class of convex curves of constant breadth in which no part of the curve is a circular arc. They state without proof the result that all curves of constant breadth are, in fact, convex, and they continue with theorems establishing properties of curves of constant breadth. Finally, in the concluding paragraph (again without proof) they mention the remarkable result that all curves of constant breadth with the same breadth have the same perimeter! In thirteen pages the reader comes away with a detailed and complete introduction to the topic, which allows for further investigation for those so motivated. There is much to learn about mathematics and exposition from these expert mathematicians.

The Most Beautiful Mathematical Formulas and *The Enjoyment of Math* are both enjoyable excursions into the beauty and pleasure of mathematics. The stories and drawings of *The Most Beautiful Mathematical Formulas* are perfect for clarifying results in algebra and geometry for students seeking to understand these important results. The exposition of *The Enjoyment of Math* provides enrichment for more advanced students with some of the classic problems and puzzles of mathematics. Both books provide refreshing and enjoyable reading for mathematicians and educators.

Reviewed by Gloria Dion, Penn State Ogontz Campus, Abington, PA 19001

Send Reviews to: Gloria Dion, Penn State Ogontz Campus, Department of Mathematics, 1600 Woodland Rd., Abington, PA 19001-3990

The Problem Section

Dr. Michael W. Ecker
Problem Section Editor
The AMATYC Review
909 Violet Terrace
Clarks Summit PA 18411

Dr. Robert E. Stong
Solution Editor
The AMATYC Review
150 Bennington Road
Charlottesville VA 22901

Greetings, and welcome to still another Problem Section!

The AMATYC Review Problem Section seeks lively and interesting problems and their solutions from all areas of mathematics. Particularly favored are teasers, explorations, and challenges of an elementary or intermediate level that have applicability to the lives of two-year college math faculty and their students. We welcome computer-related submissions, but bear in mind that programs should supplement, not supplant, the mathematical solutions and analyses.

Important Updates! At this time I am fresh out of good problems. If you have been holding on to material, now is the time to send it in, as I have *no* suitable problems left for next issue! To facilitate the process, you may also contact me at DrMichaelE@aol.com via the Internet.

When submitting material for this department, please note that we have separate editors for problems and for solutions. Send **two copies** of your new **problem proposals**, preferably typed or printed neatly with separate items on separate pages, to the Problem Editor. Include **two copies** of a solution, if you have one, and any relevant comments, history, generalizations, special cases, observations, and/or improvements. Please include your name (title optional, no pseudonyms), affiliation, and address of same. Enclose a mailing label or self-addressed envelope if you'd like to be assured a reply. All **solutions** to others' proposals, *except Quickies*, should be sent directly to the Solutions Editor. Send your solutions to **Quickies** to the Problem Editor. These should be sent immediately, as their solutions are published the following issue, leaving at most a few weeks before they are due.

Dr. Michael W. Ecker (Pennsylvania State University, Wilkes-Barre Campus)
Dr. Robert E. Stong (University of Virginia)

Quickies

Quickies are math teasers that typically take just a few minutes to an hour. Solutions usually follow the next issue, listed before the new teasers. All correspondence for this department should go to the Problem Editor.

Comments on Old Quickies

Quickie #19: Proposed by the Problem Editor but similar to what has appeared elsewhere (e.g., Millersville State University math contest circa 1980).

Calculate $\log_{10}(11) * \log_{11}(12) * \log_{12}(13) * \ldots * \log_{98}(99) * \log_{99}(100)$.

Solution: By the usual change-of-base formula, $\log_a(b) * \log_b(c) = \log_a(c)$. Thus, by repeated application of this result, the product is just $\log_{10}(100) = 2$.

The same solution was sent in by Eze N. Nwaogu of York Tech. College, Rock Hill, SC.

Quickie #20: A howler passed on by Michael Andreoli, Miami-Dade Community College.

A 30-ft. ladder is leaning against a wall when the bottom of the ladder is pulled away from the wall horizontally at a constant 1 ft. per sec. Find the height of the top of the ladder (above ground level) when the top of the ladder is falling at twice the speed of light.

Ideas: There are many levels from which to approach this, so I am extending the deadline until we get some definitive answers from readers. But here are my thoughts so far:

a) If you actually solve this, you will get a height so small as to be meaningless in view of the usual tolerances. For example, is the floor perfectly flat?

b) Is the model workable? Can one actually move the ladder as described?

c) As the ladder's vertical component of motion speeds up, doesn't Newtonian mechanics become less and less applicable?

New Quickies

Quickie #21: A classical howler passed on by the Problem Editor.

Consider the integral

$$\int \frac{1}{x}\,dx$$

(pretend we don't know it) and apply integration by parts ($u = \frac{1}{x}$ and $dv = dx$). Obtain

$$\int \frac{1}{x}\,dx = 1 + \int \frac{1}{x}\,dx.$$

Evidently, then, we have proven that $0 = 1$! (You might also consider definite integrals.)

Quickie #22: Proposed by Frank Flanigan, San Jose State University.

The power series $1 + 2x - 3x^2 + x^3 + 2x^4 - 3x^5 + x^6 + 2x^7 - 3x^8 + x^9 + \ldots$ does not converge at $x = 1$, but it does represent on $(-1, 1)$ a function $f(x)$ that is analytic on $(-\infty, \infty)$. Calculate $f(1)$.

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New Problems

Set AC Problems are due for ordinary consideration April 1, 1996. Of course, regardless of deadline, no problem is ever closed permanently, and new insights to old problems – even Quickies – are always welcome. However, our Solutions Editor requests that you please not wait until the last minute if you wish to be listed or considered on a timely basis.

An asterisk * on a problem indicates that the proposer did not supply a solution with the proposal.

Please note again that we more desperately need good new problem proposals than we have at any point in my 14 years as your Problem Editor!

Problem AC-1. Passed on by Harry J. Smith (Saratoga, CA) and the Problem Editor (Michael W. Ecker, Pennsylvania State University, Wilkes-Barre Campus).

There are numerous ways to evaluate the expression

$$\sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2}\,.$$

Evaluate and prove mathematically that your answer is correct.

Comment: This expression appeared as a challenge in a math user group last year on the Internet. Though I used it subsequently in my publication, *Recreational & Educational Computing,* I felt that it deserved a wider audience. It is particularly appropriate here.

Problem AC-2. Proposed by David Shukan, Los Angeles, CA (passed on by Problem Editor).

Consider any natural number n. Write n in bases 2, 3, 4, and 5. Add the digits in these representations and call the resulting natural number $f(n)$. Iterate to calculate $f^2(n) = f(f(n))$, $f^3(n) = f(f(f(n)))$, etc. Prove or disprove: For each n there exists a natural number k (which may depend on n) such that $f^m(n) = 10$ for all $m \geq k$.

Problem AC-3*. Proposed by the Problem Editor.

Let $S = \{1, 2, \dots, n\}$ for a natural number n, and suppose $f\colon S \to S$ hereafter. The number of such functions f is n^n, and of such permutations is $n!$ (n-factorial). The permutations all satisfy

$$\sum_{i=1}^{n} f(i) = \sum_{i=1}^{n} i.$$

However, there are surely other functions f that satisfy this condition. How many are there?

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Problem AC-4. Proposed by Kenneth G. Boback, Pennsylvania State University, Berks Campus, Reading, PA 19610.

Let q be a prescribed positive constant. Find all values of the constant c that satisfy

$$\lim_{x \to \infty}\left(\frac{x + c}{x - c}\right)^x = q.$$

Set AA Solutions

Depth Perception

Problem AA-1. Proposed by Philip Mahler, Middlesex Community College, Bedford, MA 01730.

A heavy rock is dropped into a deep well. Three seconds later a splash is heard. How far down is the surface of the water?

Solutions by Charles Ashbacher, DecisionMark, Cedar Rapids, IA; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Kenneth G. Boback, Pennsylvania State University - Berks Campus, Reading, PA; Matt Foss, North Hennepin Community College, Brooklyn Center, MN; Donald Fuller, Gainesville College, Gainesville, GA; Gulf Coast Community College Math Solvers Group, Panama City, FL; Steve Kahn, Anne Arundel Community College, Arnold, MD; Carl O. Riggs, Jr., Largo, FL; and the proposer.

Taking the acceleration of gravity to be 32 feet per second per second and the speed of sound to be 1100 feet per second, we find the distance down to the water is $16t^2 = 1100(3 - t)$, where t is the time until the rock hits the water. Solving the quadratic equation gives $t = 2.88$ seconds, and the distance down to the water is approximately 133 feet.

Base Switch

Problem AA-2. Proposed by Kenneth G. Boback, Pennsylvania State University, Berks Campus, Reading, PA 19610.

Convert 9.3 (base 10) into its base 7 equivalent representation.

Solutions by Charles Ashbacher, DecisionMark, Cedar Rapids, IA; Nicholas G. Belloit, Florida Community College at Jacksonville, Jacksonville, FL; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Jane D. Covillion, Onondaga Community College, Syracuse, NY; Mike Dellens and Tony Vance, Austin Community College, Austin, TX; Matt Foss, North Hennepin Community College, Brooklyn Center, MN; Bill Fox, Moberly Area Community College, Moberly, MO; Donald Fuller, Gainesville College, Gainesville, FL; Gulf Coast Community College Math Solvers Group, Panama City, FL; Stephen Plett, Fullerton College, Fullerton, CA; Carl O. Riggs, Jr., Largo, FL; Michael Sawyer, Houston Community College, Houston, TX; Grant Stallard, Manatee Community College, Bradenton, FL; and the proposer.

Clearly 9 = 2 + 7 (base 10) is 12 (base 7). Also 3/10 (base 10) is 3/13 (base 7) and long division gives

$$\begin{array}{r} .2046 \\ 13\,\overline{)\,3.0000} \\ \underline{26} \\ 100 \\ \underline{55} \\ 120 \\ \underline{114} \\ 3 \end{array}$$

so 9.3 (base 10) is the repeating decimal 12.20462046... (base 7).

Bill Fox says he was taught to change base by taking remainders of successive divisions for the integer part and to change the fractional part by taking the integer parts of successive multiplications. Thus

$$7(.3) = 2.1,\ 7(.1) = 0.7,\ 7(.7) = 4.9, \text{ and } 7(.9) = 6.3$$

so that .3 (base 10) is .20462046... (base 7).

Fibonacci Foolishness

Problem AA-3. Proposed by Juan-Bosco Romero Marquez, Universidad de Valladoled, Valladoled, Spain.

Evaluate $F_nF_{n+p+3} - F_{n+1}F_{n+p+2}$, where the $<F_n>$ are the usual Fibonacci numbers defined here by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. (Addendum by Problem and Solution Editors: What if we use $F_0 = 1 = F_1$ instead?)

Solutions by Charles Ashbacher, DecisionMark, Cedar Rapids, IA; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Matt Foss, North Hennepin Community College, Brooklyn Center, MN; Donald Fuller; Gainesville College, Gainesville, GA; Gulf Coast Community College Math Solvers Group, Panama City, FL; Michael Sawyer, Houston Community College, Houston, TX; Grant Stallard, Manatee Community College, Bradenton, FL; and the proposer.

$$\begin{aligned} F_{n+1}F_{n+1+p+3} - F_{n+2}F_{n+p+3} &= F_{n+1}(F_{n+p+3} + F_{n+p+2}) - (F_{n+1} + F_n)F_{n+p+3} \\ &= -(F_nF_{n+p+3} - F_{n+1}F_{n+p+2}) \end{aligned}$$

and inductively

$$F_nF_{n+p+3} - F_{n+1}F_{n+p+2} = (-1)^n(F_0F_{p+3} - F_1F_{p+2}) = (-1)^{n-1}F_{p+2}$$

For the Addendum, only the indexing changes and

$$F_nF_{n+p+3} - F_{n+1}F_{n+p+2} = (-1)^nF_{p+1}.$$

Circumscription Description

Problem AA-4. Proposed by J. Sriskandarajah, University of Wisconsin, Richland Center, WI 53581.

Around any equilateral triangle circumscribe a rectangle so that each side of the original triangle cuts off a right triangle from the rectangle. (Note: We assume that one vertex of the triangle coincides with one of the rectangle.) Prove that the sum of the areas of the two smaller right triangles equals the area of the largest one thus formed.

Solutions by Charles Ashbacher, DecisionMark, Cedar Rapids, IA; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Kenneth G. Boback, Pennsylvania State University - Berks Campus, Reading, PA; Matt Foss, North Hennepin Community College, Brooklyn Center, MN; Bill Fox, Moberly Area Community College, Moberly, MO; Donald Fuller, Gainesville College, Gainesville, GA; Gulf Coast Community College Math Solvers Group, Panama City, FL; Stephen Plett, Fullerton College, Fullerton, CA; Grant Stallard, Manatee Community College, Bradenton, FL; Bella Wiener, University of Texas - Pan American, Edinburg, TX; and the proposer.

Letting A be the acute angle in one of the right triangles at the vertex in common with the rectangle, the acute angle in the other right triangle at that vertex is 30-A. The acute angles in the third right triangle are then $30 + A$ and $60 - A$. For a right triangle with hypotenuse s and acute angle A the area is

$$\frac{1}{2} s^2 \sin A \cos A = \frac{1}{4} s^2 \sin 2A.$$

The identity $\sin x + \sin y = 2\sin \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$ then gives

$$\begin{aligned}\frac{1}{4} s^2 \sin(60 - 2A) + \frac{1}{4} s^2 \sin 2A &= \frac{1}{2} s^2 \sin 30 \cos(30 - 2A) \\ &= \frac{1}{4} s^2 \cos(30 - 2A) \\ &= \frac{1}{4} s^2 \sin(60 + 2A)\end{aligned}$$

which is the desired equality of areas.

Harmonic Means

Problem AA-5. Proposed by Frank Flanigan, San Jose State University, San Jose, CA 95192.

Given the positive reals $a_1, a_2, \ldots, a_n$, define the products

$$A_i = a_1 \ldots a_{i-1} a_{i+1} \ldots a_n \text{ for } i = 1, 2, \ldots, n.$$

Solve for x:

$$A_1(x - a_1) + A_2(x - a_2) + \ldots + A_n(x - a_n) = 0.$$

Solutions by Jim Africh, College of DuPage, Glen Ellyn, IL; Charles Ashbacher, DecisionMark, Cedar Rapids, IA; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Kenneth G. Boback, Pennsylvania State University - Berks Campus, Reading, PA; David L. Farnsworth, Rochester Institute of Technology, Rochester, NY; Matt Foss, North Hennepin Community College, Brooklyn Center, MN; Donald Fuller, Gainesville College, Gainesville, GA; Gulf Coast Community College Math Solvers Group, Panama City, FL; Steve Kahn, Anne Arundel Community College, Arnold, MD; Stephen Plett, Fullerton College, Fullerton, CA; Michael Sawyer, Houston Community College, Houston, TX; Bella Wiener, University of Texas - Pan American, Edinburg, TX; and the proposer.

The given equation is

$$a_1 \dots a_n \left(\frac{x - a_1}{a_1} + \frac{x - a_2}{a_2} + \dots + \frac{x - a_n}{a_n} \right) = 0$$

so

$$\left(\frac{1}{a_1} + \dots + \frac{1}{a_n} \right) x = n$$

or

$$x = \frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}} = \frac{n a_1 \dots a_n}{A_1 + .. + A_n} .$$

Odd Odd Polynomials (Corrected)

Problem Z-1. Proposed by the Problem Editor, Pennsylvania State University, Wilkes-Barre Campus, Lehman, PA.

Characterize all invertible, odd, fifth-degree polynomials. That is, determine necessary and sufficient conditions on the real numbers a and b to make $p(x) = x^5 + ax^3 + bx$ invertible.

No additional solutions to the corrected problem have been received.

In order that p be invertible, it must be monotone (increasing because of the coefficient of x) and thus one must have $p'(x) = 5x^4 + 3ax^2 + b \geq 0$ for all x. By considering the value at $x = 0$, one sees that $b \geq 0$. Then for $a \geq 0$, p' is the sum of three nonnegative terms, so is nonnegative. If $a < 0$, then

$$p'(x) = 5 \left[\left(x^2 + \frac{3a}{10} \right)^2 + \frac{b}{5} - \frac{9a^2}{100} \right]$$

in which the minimum value of the first term is zero, so p' is nonnegative if and only if $b \geq 9a^2/20$. Thus the conditions are:

1) both a and b are nonnegative,

or

2) $a < 0$ and $b \geq 9a^2/20$.

Correction. Grant Stallard, Manatee Community College, Bradenton, FL was omitted from the list of solvers of Problem Z-2.

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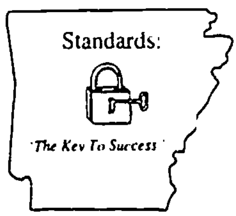

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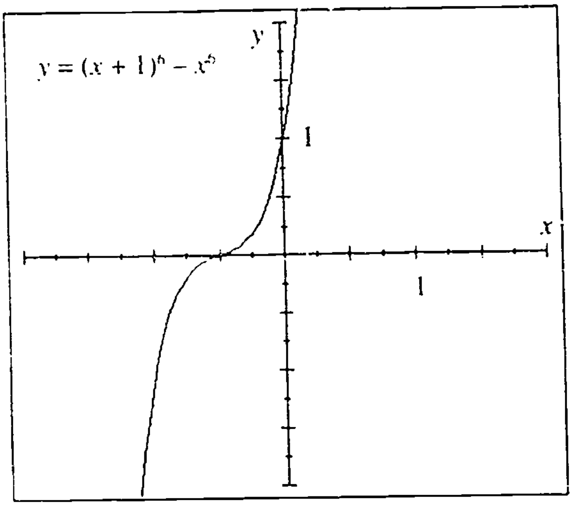

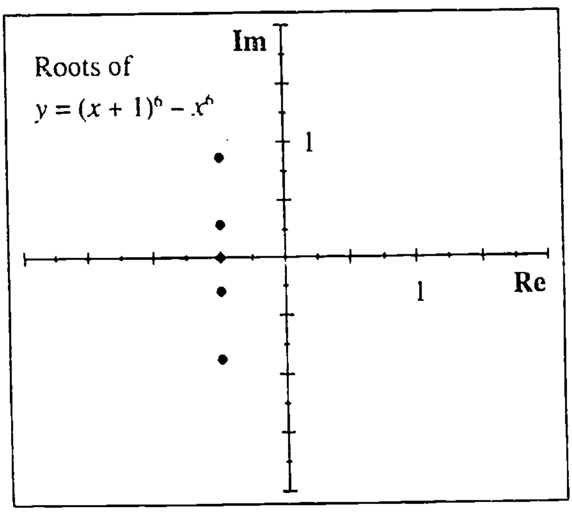

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The Official Journal of the

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- to lead the development and implementation of curricular, pedagogical, assessment and professional standards for mathematics in the first two years of college;
- to assist in the preparation and continuing professional development of a quality mathematics faculty that is diverse with respect to ethnicity and gender;
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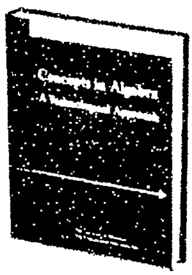

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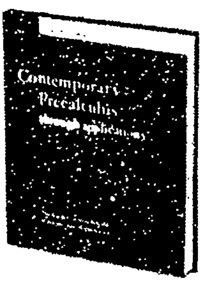

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About the Cover and Editor's Comments

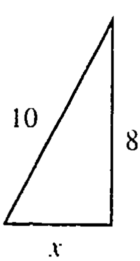
The illustration on our cover shows a graph of a polynomial in the xy-plane and a graph of its roots in the complex plane. The (real and complex) roots of a polynomial often make an interesting pattern in the complex plane. One well known example is the n-th roots of a number, r, (i.e. roots of $x^n - r$) which are n evenly spaced points on a circle of radius $\sqrt[n]{|r|}$ centered at the origin of the complex plane. In this issue, Russell Euler looks at polynomials of the form $(x+1)^k - x^k$ and gives an elementary proof that the real part of all roots must be $\frac{-1}{2}$.

The quotations used as fillers in this issue are "famous last words" of some notable people on the prospects of success for some ideas in technology. Let us not be guilty of the same closed-mindedness in our dealings with students. These came from the Internet; unfortunately, I can't give proper credit to the source since I received them n-th hand with no references.

Letter to the Editor:

Lucky Larry #18 (Fall, 1995) depicted an unusual way to use the Pythagorean Theorem. What would happen if Larry had been given this problem?

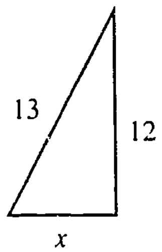
Find the missing side.



Following the pattern in #18, Larry would proceed as follows:

$$\begin{aligned} x^2 &= 8^2 + 10^2 \\ &= 2 \cdot 8 + 2 \cdot 10 \\ &= 16 + 20 \\ &= 36 \\ x &= \sqrt{36} = 6 \end{aligned}$$

which, again, happens to be the correct answer. Larry would not believe that he is doing something incorrectly until he is given a problem such as



Larry's method would yield $x = \sqrt{50}$, whereas the correct answer is 5. When does Larry's method work (i.e. give correct answers)?

Using a right triangle with legs x and y and hypotenuse z, the Pythagorean Theorem yields $x^2 + y^2 = z^2$, or $x^2 = z^2 - y^2$. But Lucky Larry thinks $x^2 = y^2 + z^2 = 2y + 2z$. These will be equal when

$$z^2 - y^2 = 2y + 2z$$

$$(z + y)(z - y) = 2(y + z)$$

$$z - y = 2.$$

Thus, we see that Larry's method will produce correct answers if and only if the hypotenuse is two units longer than the given side.

Jane Covillion
Onondaga Community College
Syracuse NY 13215

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AMATYC
State Technical Institute at Memphis
5983 Macon Cove
Memphis, TN 38134

Phone: (901) 383-4643 • Fax: (901) 383-4503

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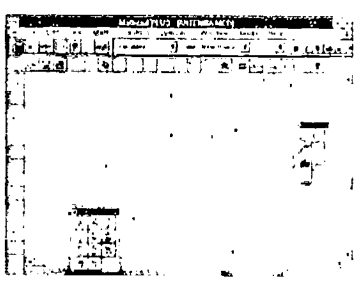

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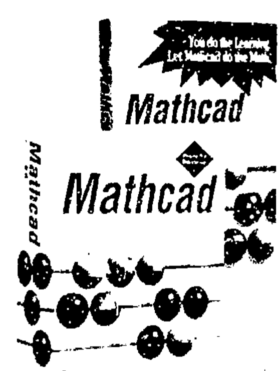

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MATHEMATICAL EXPOSITION

Cramer's Rule

by

Veena Chadha
University of Wisconsin
Eau Claire WI 54702

Veena Chadha received her Ph.D. from G.K. University of India in fixed point theory and stability problems. Her other areas of interest include optimization, discrete and finite topics, computer algorithm development and analysis. She has been enjoying undergraduate teaching for the last 17 years.

Introduction

Cramer's Rule, named after Gabriel Cramer (1704-1752), uses determinants to solve a system of n linear equations in n variables. In this note, I shall derive both Cramer's rule and, as a consequence, an interesting property of square matrices with constant non-zero determinants.

Notation

If $A = (a_{ij})$, $1 \le i \le n$; $1 \le j \le n$, is an $n \times n$ matrix, then A'_{ij} will denote the co-factor of a_{ij}, $|A|$ will denote the determinant of A, and A' will denote the transpose of A.

The system of linear equations,

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, 2, \dots, n$$

can be represented in matrix form as $AX = b$, where

$$A = (a_{ij}), X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ x_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ . \\ b_n \end{pmatrix}.$$

The matrix obtained by replacing the jth column of A with B will be a square matrix and its determinant will be denoted $|A_{Bj}|$. We have the following well-known theorem,

CRAMER'S RULE Let $AX = b$ be a system of n linear equations in n variables and $|A| \neq 0$. If X is the unique solution to $AX = b$, then

$$x_j = \frac{|A_{Bj}|}{|A|} \quad \text{for } j = 1, 2, \ldots, n.$$

Proof of Cramer's Rule

The determinant of the matrix A of coefficients of the system $AX = B$, is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{vmatrix}.$$

Multiply the elements of column 1 of $|A|$ by x_1. Then,

$$x_1|A| = \begin{vmatrix} x_1a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ x_1a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ x_1a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{vmatrix}.$$

Add x_2 times the elements of column 2, x_3 times the elements of column 3, ... , and x_n times the elements of column n to the elements of column 1. This gives,

$$x_1|A| = \begin{vmatrix} x_1a_{11} + x_2a_{12} + \ldots + x_na_{1n} & a_{12} & \cdots\cdots & a_{1n} \\ x_1a_{21} + x_2a_{22} + \ldots + x_na_{2n} & a_{22} & \cdots\cdots & a_{2n} \\ \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ x_1a_{n1} + x_2a_{n2} + \ldots + x_na_{nn} & a_{n2} & \cdots\cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} b_1 & a_{12} & \cdots\cdots & a_{1n} \\ b_2 & a_{22} & \cdots\cdots & a_{2n} \\ \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdots\cdots & \cdot\cdot \\ b_n & a_{n2} & \cdots\cdots & a_{nn} \end{vmatrix}$$

$$= |A_{B1}|.$$

It follows that $x_1 = \frac{|A_{B^1}|}{|A|}$ if $|A| \neq 0$. Similarly, $x_2 = \frac{|A_{B^2}|}{|A|}$, $x_3 = \frac{|A_{B^3}|}{|A|}$, ... , $x_n = \frac{|A_{B^n}|}{|A|}$ if $|A| \neq 0$.

We now state and prove the following theorem.

THEOREM For B a fixed $(n \times 1)$ matrix

$$A \begin{pmatrix} |A_{B^1}| \\ |A_{B^2}| \\ \cdot \\ \cdot \\ |A_{B^n}| \end{pmatrix}$$

is the same $(n \times 1)$ matrix for all $(n \times n)$ matrices A with the same non-zero determinant $|A|$.

PROOF Let X be an $(n \times 1)$ matrix such that $AX = B$. Then, since A is non-singular

$$X = A^{-1}B. \tag{1}$$

By Cramer's rule we have, since $|A| \neq 0$,

$$X = \frac{1}{|A|} \begin{pmatrix} |A_{B^1}| \\ |A_{B^2}| \\ \cdot \\ \cdot \\ |A_{B^n}| \end{pmatrix}. \tag{2}$$

From (1) and (2),

$$A \begin{pmatrix} |A_{B^1}| \\ |A_{B^2}| \\ \cdot \\ \cdot \\ |A_{B^n}| \end{pmatrix} = |A| \cdot B. \tag{3}$$

Since $|A|$ and B are fixed so is the left hand side of (3). This proves the theorem.

Special Case

Consider $a = (a_{ij})$, $1 \leq i \leq n$; $1 \leq j \leq n$; choose $\hat{A}$ and B such that,

$$\hat{A} = diag(|A|, 1, 1, \ldots, 1), \text{ and } B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

Then the following are true:

a) $|A| = |\hat{A}|$

b) $|A_{Bj}| = A^c_{ij}$ in A

c) $|\hat{A}_{Bj}| = \begin{cases} 1 \text{ if } j = 1 \\ 0 \text{ if } 2 \le j \le n. \end{cases}$

Applying the above theorem to A, we have

$$A \begin{pmatrix} |A^c_{11}| \\ |A^c_{12}| \\ \cdot \\ \cdot \\ \cdot \\ |A^c_{1n}| \end{pmatrix} = |A| \begin{pmatrix} A_{B1} \\ A_{B2} \\ \cdot \\ \cdot \\ \cdot \\ A_{Bn} \end{pmatrix} = \hat{A} \begin{pmatrix} |\hat{A}_{B1}| \\ |\hat{A}_{B2}| \\ \cdot \\ \cdot \\ \cdot \\ |\hat{A}_{Bn}| \end{pmatrix} = |\hat{A}| \cdot B = |A| \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = \begin{pmatrix} |A| \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

Hence,

$$\Sigma a_{ij} A^c_{1j} = \begin{cases} |A| \text{ if } i = 1 \\ 0 \text{ if } i \neq 1. \end{cases}$$

We now deduce the following elegant property of non-singular square matrices.

$$\Sigma a_{ij} A^c_{kj} = \alpha_{ik} |A| \text{ for } \begin{matrix} i = k, \alpha_{ik} = 1 \\ i \neq k, \alpha_{ik} = 0. \end{matrix}$$

Acknowledgement

I would like to thank an anonymous referee for his valuable suggestions in light of which this paper has been improved.

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> "I have traveled the length and breadth of this country and talked with the best people, and I can assure you that data processing is a fad that won't last out the year."
>
> —The editor in charge of business books for Prentice Hall, 1957.

Divisibility Discoveries

by

Richard L. Francis
Southeast Missouri State University
Cape Giradeau MO 63701

Richard L. Francis is a professor of mathematics at Southeast Missouri State University, where he has taught since 1965. He received a B.S. degree from Southeast Missouri State University and master's and doctorate degrees (including post-doctoral work) from the University of Missouri (Columbia). His major scholarly interests are number theory and the history of mathematics.

Do you occasionally feel compelled to "look-up" a divisibility test? Perhaps it is the test for dividing by 7. Do you also find such a test one that is easily forgotten? The problem is not as overwhelming as may seem likely at first glance. Actually, such tests are readily derived, no matter how large the divisor, and appealingly supplement one of the most basic operations of arithmetic.

The overall matter of divisibility test derivation is essentially an exploration of the integers. Evidently, the set of integers is closed with respect to addition, subtraction, and multiplication. However, in the case for division, the integers are semi-closed (Francis, 1975). That is, the process of dividing an integer by an integer sometimes yields an integer and sometimes not. It is within this latter setting of semi-closure that some of the most challenging problems of number theory arise. These challenges include the topics of primality, abundant and deficient numbers, amicable numbers, perfect numbers, and other divisibility outgrowths.

Ideally, divisibility tests shift the focus from the number being divided to that of another more easily managed number. Ideally also, this change of focus is accomplished in but a few simply described steps. Some divisibility tests are highly intuitive (as for 2, 5, 10, or 100) whereas others are more subtle (as in the case for 3, 9, or 11). Beyond the simpler cases, the question of generalization emerges. It also must be acknowledged that stated divisibility tests can be as complicated if not more so than long-division itself (i.e., the most practical test for dividing by 97 may be to divide by 97). Even if impractical, a certain insight accompanies the various test developments and their application. Such extended tests build on elementary notions, are thought-provoking, and prove rich in generalizing appeal.

Highly Useful Notation of Congruence

Significantly, divisibility tests, even for a single divisor, exist in great abundance (infinitely so) and are easily derived. Moreover, it proves unnecessary, as noted above, to "look-up" such criteria as those for dividing by 7 or 13 or 19 (or whatever the divisor choice). The key is that of congruence (Francis, 1992). A critical page in the history of mathematical notation, written by Carl Friedrich Gauss (1777-1855), concerns this equivalence relation on the integers. It appears within chapter 1 of his 1801 treatise, the *Disquisitiones Arithmeticae* (Gauss, 1966). More precisely, "a is congruent to b modulo m" if and only if "$a - b$ is divisible by m." That is,

$$a \equiv b \pmod{m}.$$

For example, $25 \equiv 11 \pmod 7$ or $1997 \equiv 1945 \pmod{13}$ or $6 \equiv -13 \pmod{19}$. Such congruence forms have manipulational properties comparable to equations. For example, the congruence remains valid if the same number is added to, subtracted from, or multiplied by each member. It is important also to note that if c and m are relatively prime, then $ac \equiv bc \pmod m$ implies $a \equiv b \pmod m$. Should $P(x)$ be a polynomial of integral coefficients, then $a \equiv b \pmod m$ implies $P(a) \equiv P(b) \pmod m$.

The Test for 7: An Illustration

Consider some positive integer expressed in the form $10t + u$ (as in the writing say of 1997 as $199(10) + 7$). If $10t + u \equiv 0 \pmod 7$, then the adding or subtracting of a multiple of 7 to or from the left member will not destroy the validity of the congruence. Accordingly, one may for example subtract $7t$ from $10t$ and $7u$ from u. Then $3t - 6u \equiv 0 \pmod 7$. Upon dividing both members by 3, it follows that $t - 2u \equiv 0 \pmod 7$. This is probably the best known of the tests for divisibility by 7. It sometimes reads "a number is divisible by 7 if and only if the result of subtracting the double of the last digit from the number formed by the preceding digits is divisible by 7." Note that 259 is divisible by 7 as 25 – 18 is divisible by 7. Likewise for 322 as 32 – 4 is 28, a multiple of 7.

But other variations apply to $10t + u \equiv 0 \pmod 7$. For example, simply add $7u$ to u so as to form $10t + 8u \equiv 0 \pmod 7$ and then divide by 2. That is, $5t + 4u \equiv 0 \pmod 7$. This means a number is divisible by 7 if and only if the sum of 4 times the units digit and 5 times the number formed by the preceding digits is divisible by 7. Note that 91 is divisible by 7 as $5(9) + 4(1)$ or 49 is divisible by 7. The process of altering the congruence $10t + u \equiv 0 \pmod 7$ is endless (obviously so by the random addition of multiples of 7 to $10t$ or u) in which case the mathematician may write as many divisibility tests for 7 as desired.

Parallels to the Test for 7

Further examples with larger divisors reinforce the claim of test variety. Suppose a test for divisibility by 13 is required. Then $10t + u \equiv 0 \pmod{13}$. One approach is that of subtracting $13t$ from $10t$ and subtracting $13u$ from u. This gives $-3t - 12u \equiv 0 \pmod{13}$, or in dividing both members by -3, $t + 4u \equiv 0 \pmod{13}$. That is, a number is divisible by 13 if and only if the sum of 4 times the units digit and the number formed by the preceding digits is divisible by 13. Illustratively, 52 is divisible by 13 as $4(2) + 5$ or 13 is divisible by 13.

Randomly generated tests for larger divisors (e.g., 17, 19, 97, etc.) are easily designed. Consider a test for 97. Building on $10t + u \equiv 0 \pmod{97}$, simply add $97u$

to u and divide both members by 2. That is, $5t + 49u \equiv 0 \pmod{97}$. Or add $7(97u)$ to u so as to yield $10t + 680u \equiv 0 \pmod{97}$ and divide both members by 10. The congruence $t + 68u \equiv 0 \pmod{97}$ basically describes the test. Rephrased, a number is divisible by 97 if and only if the sum of 68 times the units digit with the number formed by the preceding digits is divisible by 97. For example, 291 is classified quickly as a multiple of 97 as 29 + 1(68) is 97. Or, 1261 yields 126 + 1(68) which is 194 (the double of 97). Admittedly, the test is not very appealing in practice (it may actually yield a larger number) but nevertheless provides considerable insight as to the wide scope of divisibility test construction and subtleties inherent in the Hindu-Arabic (place-value) system of numeration.

The reader may wish to verify the following: a number is divisible by 23 if and only if the difference of the double of the units digit and 3 times the number formed by the preceding digits is divisible by 23. It is easily applied to such numbers as 46 or 529. How might a divisibility test for 29 be constructed?

Caution must be exercised in the division of both members of the congruence by the same number. The outcome of a valid congruence is guaranteed only if the divisor and the modulus are relatively prime.

Derived divisibility tests are not restricted to primes. Should the divisor be composite (for example 12), be fully aware of the word of caution above. Suppose $10t + u \equiv 0 \pmod{12}$. One approach is to add $12t$ to $10t$ and subtract $12u$ from u. Then $22t - 11u \equiv 0 \pmod{12}$. Dividing both members by 11 yields $2t - u \equiv 0 \pmod{12}$. A number is thus divisible by 12 if and only if the result of subtracting the units digit from the double of the number formed by the preceding digits is divisible by 12. For example, 24 is divisible by 12 as $2(2) - 4 = 0$ and the difference 0 is clearly a multiple of 12.

Other Groupings

Instead of isolating the units digit, one may also isolate terminal digits by groups. For example, an integer may be written in the form $100h + k$ as in writing 1997 as 19(100) + 97. Or an integer may be written in the form $1000x + y$ as in expressing 1997 as 1(1000) + 997. Such representations are easily generalized.

Consider the development of a test for divisibility by 7 by use of the form $100h + k$. Beginning with $100h + k \equiv 0 \pmod 7$, subtract $98h$ (a multiple of 7) from $100h$ and add $7k$ to k. Then $2h + 8k \equiv 0 \pmod 7$ or $h + 4k \equiv 0 \pmod 7$. Accordingly, a number is divisible by 7 if and only if 4 times the number formed by the last two digits added to the number formed by the preceding digits is divisible by 7. For example, 2366 is divisible by 7 as 4(66) + 23 or 287 is divisible by 7.

Consider still another grouping of terminal digits, say $1000x + y \equiv 0 \pmod 7$. Subtract $1001x$ (a multiple of 7) from $1000x$. Then $-x + y \equiv 0 \pmod 7$. That is, a number is divisible by 7 if and only if the difference of the number formed by the last three digits and the number formed by the preceding digits is divisible by 7. To illustrate, note that 15736 is divisible by 7 as 736 − 15 or 721 is divisible by 7.

A Polynomial Approach

The technique above provides a quick and direct method for generating tests of divisibility. It involves an addition or subtraction of modulus multiples to or from the basic form $10t + u$ or its generalization. Clearly, such a technique, here called the ADDITION OR SUBTRACTION MODULUS METHOD, permits constructing as many divisibility tests as desired for any divisor of one's choosing. It is built on the foundation of congruences.

Other forms besides $10t + u$ could be used in test derivation and are mentioned here briefly. The polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$$

can be used to represent any positive integer by the simple restriction of the coefficients to the set of digits and letting x equal ten. Hence, 1997 becomes

$$1(10)^3 + 9(10)^2 + 9(10)^1 + 7.$$

Subject to the digital coefficient restriction, $P(10)$ becomes the number itself, $P(1)$ becomes the sum of the digits, and $P(-1)$ is the sum of the digits with alternating sign.

As $a \equiv b \pmod{m}$ implies $P(a) \equiv P(b) \pmod{m}$, a tremendous variety of divisibility tests comes to light. For example, $10 \equiv 1 \pmod{9}$. Hence, $P(10) \equiv P(1) \pmod{9}$. Any integer is thus congruent modulo 9 to the sum of its digits. This is a well-known test and is the basis for the famous technique referred to as "casting out nines" (Francis, 1976). Yet other divisors may be utilized. For example, consider 7 again. Note that $10 \equiv 3 \pmod{7}$ in which case $P(10) \equiv P(3) \pmod{7}$. This implies that 1981 is divisible by 7 if and only if $1(3)^3 + 9(3)^2 + 8(3)^1 + 1$ or 133 is divisible by 7. The conclusion of divisibility by 7 immediately follows. A similar test stems from $P(10) \equiv P(-4) \pmod{7}$.

Of course, digits may be considered in pairs or triples or n-tuples. Note, for example, that 22,431,216,157 can be written as

$$22(1000)^3 + 431(1000)^2 + 216(1000)^1 + 157.$$

In this approach, $P(1000)$ represents the number itself, $P(1)$ represents the sum of digital triples, and $P(-1)$ is the sum of digital triples with alternating sign. As $1000 \equiv -1 \pmod{7}$, then

$$P(1000) \equiv P(-1) \pmod{7}.$$

But $P(-1)$ is $157 - 216 + 431 - 22$ or 350. As 350 is a multiple of 7, so is the given number 22,431,216,457. As before, this test admits great diversity in approach. Note: the illustrated test applies to 11 and 13 also. That is, $1000 \equiv -1 \pmod{11}$ and $1000 \equiv -1 \pmod{13}$. It likewise provides a test, by multiplication of relatively prime divisors, for 77 and 91 and 143.

The POLYNOMIAL METHOD lends itself nicely to the answering of divisibility questions in the context of non-decimal arithmetic. Obviously, a modification in the listing of coefficients is needed. In base 6 arithmetic, the allowable coefficients become 0, 1, 2, 3, 4 and 5. Interestingly, since $b \equiv 1[\bmod (b-1)]$, then $P(b) \equiv P(1)[\bmod (b-1)]$. That is, in base b arithmetic, any integer is congruent modulo $(b - 1)$ to the sum of its digits. To test for divisibility by $(b - 1)$, simply express the number in base b notation and examine the sum of the digits for divisibility by $(b - 1)$. For example, one may show that 100 is divisible by 5 by first writing 100 as 244_{six}. The sum of the digits is 14_{six}. Repeating the process, the sum of the digits becomes 5 in which case divisibility by 5 is established. Such a test is excellent, theoretically speaking, but lacking in terms of practicality. Should the base exceed ten, symbols beyond the familiar digits will be required. However, the scheme permits a simple word description of a divisibility test for any positive integral divisor whatever.

Divisibility is vast. It constitutes one of the cornerstones of number theory. Too, its forerunners appear glaringly in such places as Books VII, VIII, and IX of Euclid's *Elements* (Eves, 1990). With the advances of place-value notation and modern day symbolism, great strides were taken. Hard-to-express relationships became within reach and number theory (the higher arithmetic) accordingly brought to an impressive level of elegance.

The topic of divisibility is thus an old one but yet ever new in the sense of discovered tests. It brings varied long-ago quests into the classroom of today and adds zest and excitement to an arithmetic activity that is far from routine.

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Lucky Larry #21

$$\lim_{x \to 0} \frac{\sin 7x}{x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot 7$$
$$= 1 \cdot 7$$
$$= 7$$

Submitted by Steven J. Rottman (student)
Catonsville Community College
Catonsville MD

Tax-Sheltered Annuities

by

Harris S. Shultz and Martin V. Bonsangue
California State University, Fullerton
Fullerton CA 92634

Harris Shultz has served as principal investigator of numerous NSF- and state-funded projects for secondary mathematics teachers in Orange County, CA; in The Northern Mariana Islands; and in American Samoa. In 1992 he received the Southern California Section of the Mathematical Association of America's Award for Distinguished College or University Teaching.

Marty Bonsangue received his PhD in Mathematics Education from The Claremont Graduate School in 1992, where he was awarded the Peter Lincoln Spenser Memorial Award for Outstanding Dissertation of the Year. He has taught mathematics to middle school, high school, and college students for twenty years, during which time he has faithfully contributed to a Tax-Shelter Annuity.

Introduction

Many educators qualify to invest part of their annual salary in a *tax-sheltered annuity*. By doing so, no income tax is paid on the amount invested until withdrawals are made later in life. The income tax on interest earned is similarly deferred. A common argument made in favor of participation in such a plan is that when the funds are withdrawn you are likely to be in a lower tax bracket. Indeed, Julian Block's *Guide to Year-Round Tax Savings* states that the money is free from taxes until retirement, "when your tax bracket is likely to be substantially below what it was during your working years." (1985, p. 29).

However, your retirement income might well be high enough so that your income tax bracket is no lower than it was when you were employed. With such being a reasonable possibility, it certainly is worthwhile to investigate the tax implications of participation in a tax-sheltered annuity under the assumption that there is no change in your tax bracket upon retirement.

A Simple Example

Let us begin by looking at an example in which a teacher is in a 50% tax bracket, has $100 discretionary salary, and is one year from retirement. Suppose she can obtain a 10% annual rate of interest either in a tax-sheltered annuity or in an ordinary non-sheltered investment. If she puts the entire $100 in a tax-sheltered annuity, she will have $110 after one year, $55 of which she will retain after paying income tax. On the other hand, if she does not shelter her $100, she must pay $50 in taxes, leaving $50 to invest. After one year, that investment will grow by 10% to \$55. With half of the \$5 earned in interest payable as income tax, she will be left with \$52.50 after the one year. This compares unfavorably with the \$55 she would have had by sheltering the original \$100.

Thus, even if there is no change in tax bracket, investment in a tax-sheltered annuity will work to a financial advantage. In this simple example, our teacher has one year free use of \$50 ultimately payable in taxes. That \$50 generates \$5 in interest, half of which she will keep. That \$2.50 after-tax interest is precisely the difference between the amounts \$55 and \$52.50 computed above. Thus, the investor in a tax-sheltered annuity gains additional income through the interest earned on the temporary use of funds which otherwise would have been paid in taxes a year earlier.

A Long Term Example

Consider now the case of a teacher with \$100 discretionary salary to use for retirement 15 years from the present. Let us assume, more realistically, that she is and will be in a 28% income tax bracket and that she can earn 8% interest on her money. If she shelters the \$100, she will have

$$100(1.08)^{15} = 317.22$$

at the end of 15 years. Since she will get to keep 100% – 28% = 72% of this amount, after 15 years she will have

$$(.72)(100)(1.08)^{15} = 72(1.08)^{15} = 228.40 \qquad (1)$$

after taxes. Notice from (1) that sheltering money earned in the present year is mathematically equivalent to paying the 28% income tax in the present year and then earning and keeping interest *tax free* for the entire period of sheltered investment.

If she does not shelter her \$100, she will be able to invest 72 after-tax dollars for 15 years. Since she must pay income tax on the interest earned each year, the after-tax annual interest rate will be $.72 \times .08 = 5.76\%$. Therefore, after 15 years she will have

$$72(1.0576)^{15} = 166.78.$$

The amount \$228.40 is about 37% greater than \$166.78.

Let us compare these results with a comparable investment in an individual retirement account (IRA). In this case, the \$100 is immediately subject to income tax, but the interest can be deferred until the funds are withdrawn in 15 years. Thus, she will invest \$72 at 8% and in 15 years the account will be worth

$$72(1.08)^{15} = 228.40.$$

At that time, the interest of \$228.40 – \$72 = \$156.40 will be subject to a 28% income tax. Thus, after 15 years she will have

$$72(1.08)^{15} - .28(72(1.08)^{15} - 72) = 184.61.$$

Thus, the IRA provides a more lucrative investment than no shelter at all, but a less lucrative investment than the tax-sheltered annuity.

Annual Contributions to a Tax-Sheltered Annuity

We now suppose that our teacher is able to set aside \$100 this year and each of the next 14 years. If the present is taken to be year 0, then the analysis summarized in (1) tells us that \$100 sheltered in year k will result in an after-tax net of $72(1.08)^{15-k}$ in the retirement year 15 years from the present. Therefore, in 15 years, the total \$1500 investment will be worth

$$72(1.08)^{15} + 72(1.08)^{14} + 72(1.08)^{13} + \ldots + 72(1.08)$$

$$= \frac{(72)(1.08)(1.08^{15} - 1)}{.08} = 2111.35$$

after taxes.

If she does not shelter her money, her annual \$72 investment will grow to

$$72(1.0576)^{15} + 72(1.0576)^{14} + 72(1.0576)^{13} + \ldots + 72(1.0576) = 1740.34$$

after taxes.

In practice, one would not withdraw the entire annuity in the first year of retirement. Rather, much of the account would remain for several years, earning even more tax-deferred interest. Therefore, in real life the difference between sheltering and not sheltering is even greater than this example illustrates.

Conclusion

This discussion underscores the extensive role mathematics plays in tax and investment considerations. While many tax books such as *Guide to Year-Round Tax Savings* (1985) and *Shelter What You Make, Minimize the Take* (1982) discuss the advantages of sheltering, the advantage is usually assumed to be because of the lower income tax bracket. However, a careful analysis shows that sheltering is advantageous even if the tax bracket remains unchanged.

Standard 4 of *The Curriculum and Evaluation Standards for School Mathematics* states that

> the mathematics curriculum should include investigation of the connections and interplay among various mathematical topics and their applications so that all students can...use and value the connections between mathematics and other disciplines. (1989, p. 146)

This article demonstrates the key role that a knowledge of exponential growth and geometric series plays in understanding annuities, and, perhaps more importantly, the powerful link between mathematics and events that affect people's lives.

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SHORT COMMUNICATIONS

The Chain Rule: Multiply Slopes for the Slope of the Composite Function $f \circ g$

by

David L. Farnsworth
Rochester Institute of Technology
Rochester NY 14623-5603

R·I·T

Often, David Farnsworth teaches calculus, and he likes to draw pictures to illustrate ideas for his classes. He also takes great pleasure in using the zoom key on his graphing calculator to linearize differentiable functions.

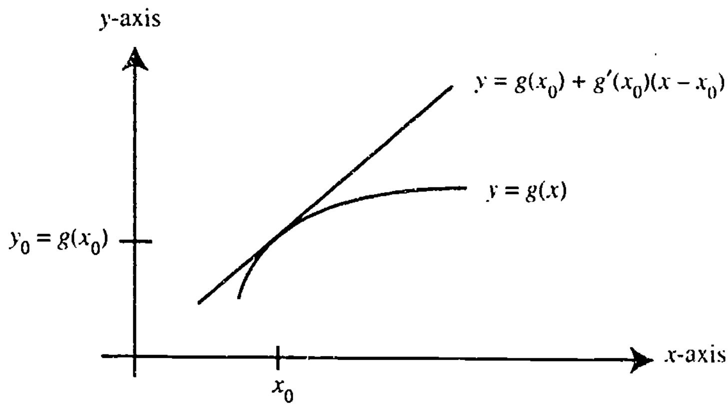

Figure 1. Near x_0, $g(x) \approx g(x_0) + g'(x_0)(x - x_0)$

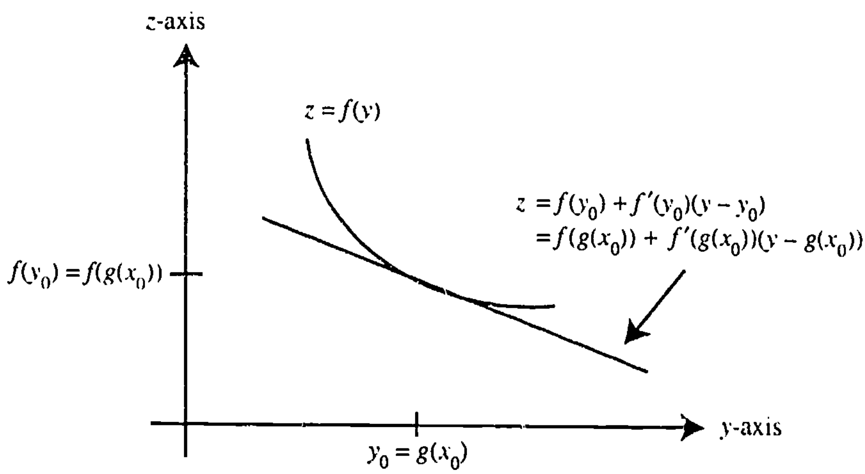

Figure 2. Near y_0, $f(y) \approx f(y_0) + f'(y_0)(y - y_0)$

$$f(g(x)) \approx f(g(x_0)) \;\; + f'(g(x_0))(g(x) - g(x_0))$$
$$f(g(x)) \approx f(g(x_0)) \;\; + f'(g(x_0))(g(x_0) + g'(x_0)(x - x_0) - g(x_0))$$
$$f(g(x)) \approx f(g(x_0)) \;\; + f'(g(x_0))g'(x_0)(x - x_0)$$
$$(f(g(x_0)))'$$

Figure 3. Locally, the differentiable functions g, f, and $f \circ g$ are approximately linear.

"Computers in the future may weigh no more than 1.5 tons."

—Popular Mechanics, forecasting the relentless march of science, 1949.

❖ ❖ ❖ ❖

"I think there is a world market for maybe five computers."

—Thomas Watson, chairman of IBM, 1943.

The Distribution of Roots of an Equation

by

Russell Euler
Northwest Missouri State University
Maryville MO 64468

Russell Euler received his Ph.D. degree from the University of Missouri-Kansas City. His Ph.D. research was directed by the late Y.L. Luke. He is a professor in the department of mathematics and statistics at Northwest Missouri State University. Russell is the author of several analytical and pedagogical papers.

The purpose of this paper is to find the distribution of the roots of the polynomial equation

$$\sum_{k=0}^{n-1} \binom{n}{k} z^k = 0 \tag{1}$$

where n is an integer such that $n > 1$.

Roots for several values of n were generated and are shown in Table 1. In the table, the ordered pair (a, b) represents the complex number $a + bi$ where $i^2 = -1$. The results suggest that if z is any solution, then $\mathbf{Re}\, z = -\frac{1}{2}$.

n	$\sum_{k=0}^{n-1} \binom{n}{k} z^k$	Roots
2	$1 + 2z$	$-.5$
3	$1 + 3z + 3z^2$	$(-.5, \pm .288675134595)$
4	$1 + 4z + 6z^2 + 4z^3$	$-.5, (-.5, \pm .5)$
5	$1 + 5z + 10z^2 + 10z^3 + 5z^4$	$(-.5, \pm .688190960236)$, $(-.5, \pm .162459848116)$
6	$1 + 6z + 15z^2 + 20z^3 + 15z^4 + 6z^5$	$-.5, (-.5, \pm .866025403784)$, $(-.5, \pm .288675134595)$

Table 1

In order to prove the above conjecture, notice that (1) is equivalent to

$$(z+1)^n = z^n. \tag{2}$$

Since $z \neq 0$, divide equation (2) by z^n to get

$$\left(\frac{z+1}{z}\right)^n = 1$$

$$= e^{2k\pi i}$$

where k is an integer. Then

$$1 + \frac{1}{z} = e^{\frac{2k\pi i}{n}}$$

for $k = 0, 1, 2, \ldots, n-1$. Hence,

$$z = \frac{1}{-1 + e^{\frac{2k\pi i}{n}}}.$$

In order to express this in standard form, multiply the numerator and denominator of the latter fraction by the conjugate of the denominator to obtain

$$z = \frac{-1 + e^{\frac{2k\pi i}{n}}}{2 - e^{\frac{2k\pi i}{n}} - e^{\frac{2k\pi i}{n}}}.$$

However, since $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ and $e^{i\theta} = \cos\theta + i\sin\theta$, we have

$$z = \frac{-1 + \cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n}}{2 - 2\cos\frac{2k\pi}{n}}$$

$$= -\frac{1}{2} - i\left(\frac{\sin\frac{2k\pi}{n}}{2 - 2\cos\frac{2k\pi}{n}}\right).$$

Hence, **Re** $z = -\dfrac{1}{2}$.

Suppose the restrictions on n are relaxed and n is also allowed to be a negative integer less than -1. Let $n = -m$ where $m > 1$. Then equation (2) becomes $(z+1)^{-m} = z^{-m}$ and so $(z+1)^m = z^m$. As a result, if n is any integer other than ± 1 and z is a root of (2), then **Re** $z = -\dfrac{1}{2}$.

Using similar techniques, the interested reader may wish to prove that if n is a positive integer and z is a root of

$$2z^n + \sum_{k=0}^{n-1} \binom{n}{k} z^k = 0,$$

then **Re** $z = -\dfrac{1}{2}$. The result can be extended to negative integer values of n also.

MATHEMATICS EDUCATION

Modeling Data Exhibiting Multi-Constant Rates of Change

by

Edward D. Laughbaum
The Ohio State University
Columbus OH 43210

Ed Laughbaum is Professor Emeritus of Mathematics from Columbus State Community College and is currently the Associate Director of the Demana/Waits technology-based Ohio State University Short Course Program.

The *Standards for Introductory College Mathematics Before Calculus* (Revised Final Draft), published by the American Mathematical Association of Two-Year Colleges in February of 1995, recommends that mathematical modeling be taught in all college mathematics courses. Does this mean that we must only use the regression models on calculators like the TI-83 or TI-92 to model data? No. There is an entire group of relationships that do not lend themselves to be modeled with the built-in regression models on calculators like the TI-83 or TI-92. Below are four examples of data relationships found in health, business, aviation, and government that are appropriate from beginning algebra to college algebra. 1) The level of the drug Digoxin or Imipramine in the blood of a patient rises at a constant rate until the patient is at the prescribed level and then the rate of change remains at 0% until the patient is taken off the drug at a constant rate. 2) The relationship between the medical charges filed by a subscriber and the medical charges paid by an insurance company that pays at a rate of 0% for the first \$200 in medical charges filed and pays at the rate 80% for charges over \$200. 3) The relationship between time and height of an airplane that ascends at a constant rate for 30 minutes, levels off at a 0% rate of change for 60 minutes and then descends at a constant rate for the last 30 minutes of the flight. 4) The 1994 federal income tax rate 1040 schedule (for single filers) where the rate of taxation is 15% on the first \$22,700 of taxable income, 28% on the next \$32,350, 31% on the next \$59,900 of taxable income, etc. (There were two more brackets that will be ignored for sake of simplicity.)

All of these relationships have a common theme: they have different constant rates of change for selected subsets of the problem domain of the relationship. While it is possible to model this data with piecewise defined linear functions, experience with college algebra students shows that some have difficulty with piecewise defined functions. This further suggests developmental students may have the same difficulties. To develop models for these situations without piecewise defined functions, an analysis of the sum of absolute value functions of the form $y = d|x + e| + f$ needs to be undertaken. Consider the graphical or numerical representations of the functions $y = |x + e_1| + |x + e_2| + f$, where e_1 and e_2

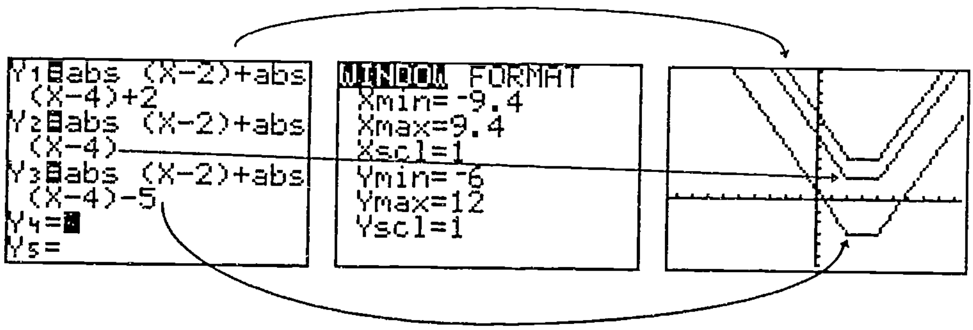

Figure 1

have values of −2 and −4, and f has values of 2, 0 and −5. A visual or numeric investigation of these functions will show there are corners of the graph when x is 2 or 4. Further investigation with the calculator shows corners when x is $-e_1$ or $-e_2$. Investigation of the function $y = d(|x + e_1| + |x + e_2|) + f$ will also show corners when x is $-e_1$ or $-e_2$. Symbolic analysis of these functions will lead to the same conclusion.

A graphical investigation of the function $y = d(|x + e_1| + |x + e_2|) + f$ will lead to the conclusion that the maximum or minimum (depending on the sign of d) value of the function is $d(||e_1| - |e_2||) + f$ when e_1 and e_2 are the same sign or $d(|e_1| + |e_2|) + f$ when e_1 and e_2 are opposite in sign. The rate at which the function changes on each branch can easily be determined from a numerical perspective. Students will typically use this method to determine the rate of change on each branch. This discovery by students many times leads to an in-class confirmation by symbolic methods. The rates of change (slopes) can be found analytically by recognizing that the expression $d|x + e_1| + d|x + e_2|$ simplifies to

$-d(x + e_1) - d(x + e_2)$, which becomes $-2dx - e_1 - e_2$, when $x \in (-\infty, e_1]$;

$d(x + e_1) - d(x + e_2)$, which becomes $0x + e_1 - e_2$, when $x \in [e_1, e_2]$; $\quad (e_1 < e_2)$

$d(x + e_1) + d(x + e_2)$, which becomes $2dx + e_1 + e_2$, when $x \in [e_2, \infty)$.

Thus, the slopes of each branch of the sum of two absolute value functions with equal coefficients are $-2d$, 0, and $2d$.

What happens when three, four, or more absolute value functions are added? The graph of $y = 2|x + 1| + |x - 3| + 4|x - 5| - 20$ on the window [−10, 10] by [−15, 30] in Figure 2 demonstrates still another kind of behavior. The function has different constant rates of change on the intervals $(-\infty, -1]$, $[-1, 3]$, $[3, 5]$, and $[5, \infty)$, and

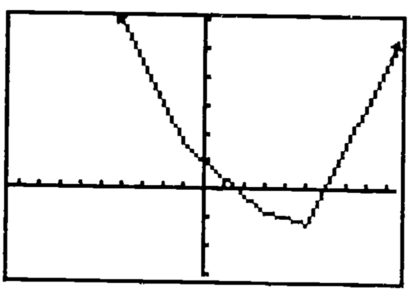

Figure 2

there are corners at −1, 3, and 5. The actual rates of change on each interval will be discussed later.

Once students know when there are corners, what the rates of change are for each branch of the function, the maximum or minimum value of the function, and how to make the graph open up or down, they are ready to model data.

Imipramine Example

Table 1 shows the level of Imipramine in the blood of a patient as he is given the drug over a 26 week period. An interesting mathematical feature of Imipramine is that it must be phased in and out at a constant rate as shown in the data in Table 1. The drug level I is measured in nanograms per milliliter of blood. Figure 3 shows a

t	0	1	2	3	5	10	15	20	23	24	25	26
I	0	58	121	178	180	181	180	179	180	123	61	0

Table 1

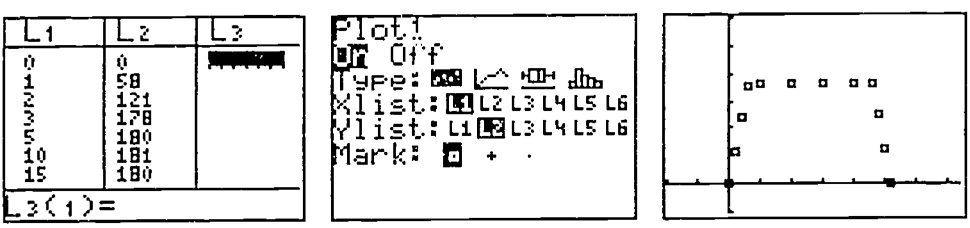

Figure 3

scatter plot of the data. The shape suggests the sum of two absolute value functions. Thus, the form of the model is $I = d(|t + e_1| + |t + e_2|) + f$. From the exploration described above, students know e_1 and e_2 immediately to be −3 and −23. Further, they know that d must be negative and a calculation of the phase-in/phase-out rate shows it to be approximately $\frac{178}{3} = 59.333 \approx 60$. Since there are two absolute functions in the model, $d = -30$. It appears that the desired maximum is 180 nanograms per milliliter. Rather than finding f analytically, experience shows that students prefer to use the calculator. For example, graphing $I = -30(|t - 3| + |t - 23|)$ shows the maximum to be −600; thus, if 780 is added to the function, it will have a maximum value of 180. The model for the Imipramine is $I = -30(|t - 3| + |t - 23|) + 780$. Figure 4 shows the data and the model. While the domain of

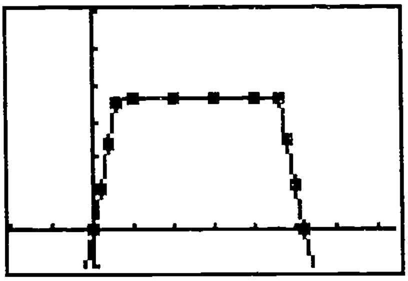

Figure 4

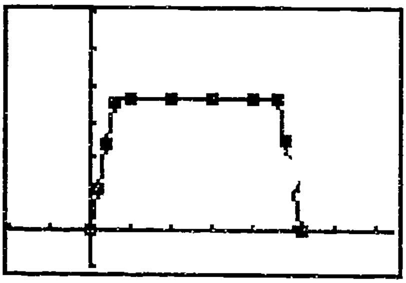

Figure 5

the model is $(-\infty, \infty)$, it can be restricted to the problem domain by adding the change of domain function $0\sqrt{-t(t-26)}$ to the model as shown in Figure 5.

Knowing the model of the data encourages an entire new level of mathematical inquiry that can take place in a group or lecture setting. Sample questions are:

- What should the Imipramine level be after $3^1/_2$ days?
- What should it be at $24^1/_2$ weeks?
- What function parameter(s) should be changed if the doctor wants the maximum level to be 200 nanograms per ml?
- What function parameter(s) should be changed to reflect a longer phase-in phase-out period?
- When is the mathematical model at a constant rate of change?
- When is the model decreasing, increasing, or constant?
- What are the limitations to the model?
- Does the model apply if the phase-in rate is different than the phase-out rate?

Health Insurance Example

Many health insurance companies have a deductible amount that each subscriber must pay before the insurance company starts to reimburse. A typical deductible may be \$200. After a subscriber has accumulated \$200, the insurance company might pay 80% of all medical charges above \$200. The data in Table 2 shows the relationship between the medical charges (C) of each subscriber and the

C	0	50	100	180	200	225	275	305	380	450	600
P	0	0	0	0	0	20	60	84	144	200	320

Table 2

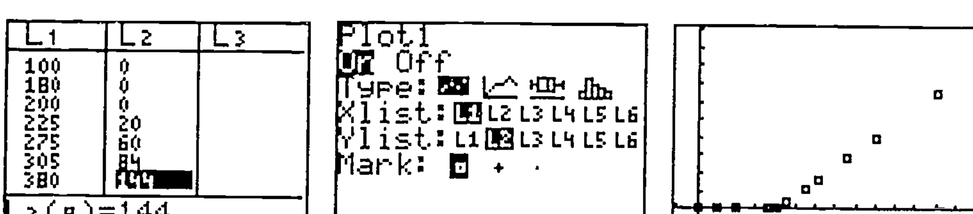

Figure 6

amount paid (P) by the insurance company. Can this data be modeled with the sum of two absolute value functions? The general model type can be determined by making a scatter plot as shown in Figure 6. The scatter plot suggests that the data looks like two branches of the three-branched shape of the sum of two absolute value functions. With the domain-controlling function $0\sqrt{C}$, one branch can be deleted. Using the standard form $P = d(|C + e_1| + |C + e_2|) + f$, the corners are known as 0 and 200. The value of the parameter d is known because the insurance company pays at a rate of 0.8 and the graph opens up; thus, d is positive 0.4. The model $P = 0.4(|C| + |C - 200|)$ has a minimum value of $(0.4) \times (200)$, or 80; therefore, to cause the model to have a minimum of zero for the first \$200 in claims, make f equal to −80. The final model $P = 0.4(|C| + |C - 200|) - 80 + 0\sqrt{C}$ is shown with the data in Figure 7. Like the previous example, now that students know the symbolic form of the model, they should be expected to answer typical mathematics questions like: If a subscriber has a total of \$452 in medical bills for the year, what reimbursement will the company send to the subscriber? In programming the insurance company's computer, what parameter(s) should be changed if the company changes to a \$300 deductible? What parameter(s) should be changed if the company changes to a 30% co-pay? Does the model give an exact answer to question 1?

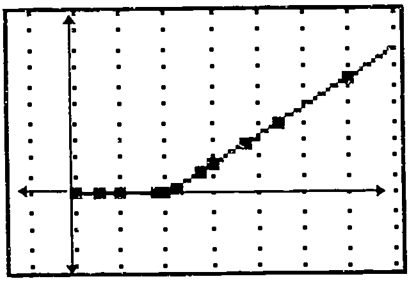

Figure 7

Income Tax Example

The final example demonstrates how to model the 1994 1040 tax rate schedule for a single taxpayer, with a sum of absolute value functions. The tax form gives the following information:

\$0	\$22,750	pay 15%
\$22,750	\$55,100	pay \$3,412.50 + 28% of excess over \$22,750
\$55,100	\$115,000	pay \$12,470.5 + 31% of excess over \$55,100

The tax schedule suggests the corners of the graphical representation of the model are at 0, −22750, −55100, and −115000; thus, the model is the sum of four absolute value functions $T = a|I - 0| + b|I - 22750| + c|I - 55100| + d|I - 115000| + f$ where T is the tax for income I. Many students try to find the remaining parameters by guess and check; however, on a problem this difficult they may spend several hours with no solution — others will succeed. Many try to use the numbers 0.15, 0.28, and 0.31 for some of the parameters. Now is a good time for the use of algebra.

On each tax bracket, the individual absolute value functions simplify to:

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	$a\lvert I\rvert$	$b\lvert I-22750\rvert$	$c\lvert I-55100\rvert$	$d\lvert I-115000\rvert$
$I \in (-\infty, 0]$	$-aI$	$-b(I-22750)$	$-c(I-55100)$	$-d(I-115000)$
$I \in [0, 22750]$	aI	$-b(I-22750)$	$-c(I-55100)$	$-d(I-115000)$
$I \in [22750, 55100]$	aI	$b(I-22750)$	$-c(I-55100)$	$-d(I-115000)$
$I \in [55100, 115000]$	aI	$b(I-22750)$	$c(I-55100)$	$-d(I-115000)$

Since the tax bracket constants have no effect on the slope for each interval, this table can be simplified to show only the slopes of each bracket.

	$a\lvert I\rvert$	$b\lvert I-22750\rvert$	$c\lvert I-55100\rvert$	$d\lvert I-115000\rvert$
$I \in (-\infty, 0]$	$-aI$	$-bI$	$-cI$	$-dI$
$I \in [0, 22750]$	aI	$-bI$	$-cI$	$-dI$
$I \in [22750, 55100]$	aI	bI	$-cI$	$-dI$
$I \in [55100, 115000]$	aI	bI	cI	$-dI$

Finally, form 1040 gives the tax rate for each interval, and each row above is the rate of change (slope) for each interval. Set them equal and solve the system for a, b, c, and d. Figure 8 shows

$$-aI - bI - cI - dI = 0$$
$$aI - bI - cI - dI = 0.15$$
$$aI + bI - cI - dI = 0.28$$

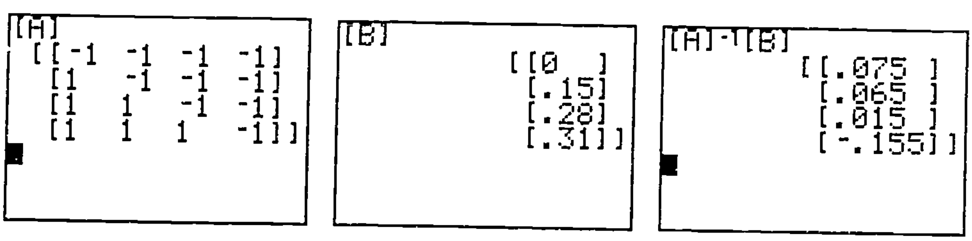

Figure 8

$$aI + bI + cI - dI = 0.31$$

the solution to the system is $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0.075 \\ 0.065 \\ 0.015 \\ -0.155 \end{bmatrix}$. The model for federal income taxes

owed (T) with a taxable income of I is $T = 0.075\lvert I\rvert + 0.065\lvert I - 22750\rvert + 0.015\lvert I - 55100\rvert - 0.155\lvert I - 115000\rvert$, where the value of f and the problem domain are yet to be determined. When the model is graphed, the branch with a 0% tax rate has a value of −15519.75; thus, if 15519.75 is added to the model, the vertical shift will make the taxes start at \$0 for \$0 in taxable income. A reasonable problem domain should be [0, 115000] and the domain-controlling function $0\sqrt{-I(I-115000)}$ can be added to restrict the domain of the model. The final model is

$$T = 0.075|I| + 0.065|I - 22750| + 0.015|I - 55100| - 0.155|I - 115000| + 15519.75 + 0\sqrt{-I(I-115000)}.$$

The graph of the model is in Figure 9 shown in two windows, [-5000, 56000] by [-5000, 10000] and [-25000, 120000] by [-5000, 35000] so that each branch can be clearly seen.

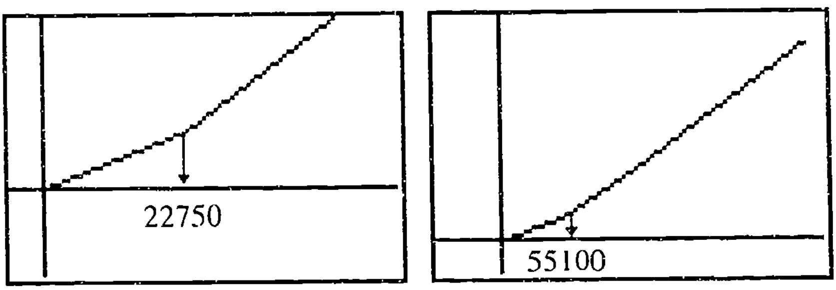

Figure 9

Typical questions that follow after the students have developed the model are:

- If your taxable income was \$84,320 in 1994, what tax did you pay?
- If a friend said that her taxes paid in 1994 were \$8,975, what was her taxable income?
- What tax bracket is your friend in?
- If your taxable income is \$80,000, is your tax rate 31%?
- The 1040 tax rate schedule for taxable income on [22750, 55100] is "\$3,412.50 plus 28% of income in excess of \$22,750." Explain why \$3,412.50 is used in the tax algorithm.
- What function parameters will change if a fourth tax bracket is added?
- If the fourth tax bracket is added, do you expect parameters *a*, *b*, and *c* to increase or decrease? Why?

Conclusion

It is difficult (or maybe meaningless) to study the sum of absolute value functions without using multiple representations. It is difficult to "see" that the sum of absolute value functions can be used to model anything in the real world when only studying them in symbolic form. There is no visualization to suggest rate changes or numeric representation to clearly show the constant rate changes from one interval to another. It is exploration with a graphing calculator that leads to discoveries.

The three examples used in this article do not exhaust the real-world situations where absolute value sums can be used. Many other multi-constant rate relationships are within the grasp of most developmental/college students.

> "But what ... is it good for?"
>
> —Engineer at the Advanced Computing Systems Division of IBM, 1968, commenting on the microchip.

❖ ❖ ❖ ❖

Collaborative Computer Calculus Workshop

The Borough of Manhattan Community College's Mathematics Department will host two NSF sponsored workshops on how new learning strategies and technologies can improve calculus and pre-calculus instruction. The workshop will emphasize combining collaborative learning with the latest software to create a new learning experience. Special interest will be given to creating Calculus Movies. We will use IBM P.C. and Macintosh computers as well as the TI-81, 82 graphing calculators, and particularly the TI-92 with *Derive*. Participants will both learn to use software and create new student collaborative computer calculus projects.

Two workshops will take place. The workshop for faculty inexperienced with computers will be from the Tuesday, May 28 through Friday, May 31, 1996. The workshop for faculty experienced with computers will be from Monday, June 3 through Friday, June 7, 1996. Both will be at the Borough of Manhattan Community College, (212) 346-8530.

While individuals may apply, two-person teams consisting either of two professors or one professor and one lab technician will receive preference. Two-person teams should submit a single application. The application should consist of a one page typed essay describing:

1. Your previous computer experience in education
2. Available computer equipment at your college
3. Plans for using collaborative computer experiences in your calculus and pre-calculus classes during the next academic year.

In addition, fill in the coupon below, and return with a letter of support from your chairperson.

Name(1)______________________________
Name (2)______________________________
College______________________________
Address______________________________
Phone (Office) 1(   )__________ 2(   )__________
(Home) 1(   )__________ 2(   )__________

Send to: Drs. L. Sher and P. Wilkinson, Mathematics Department
Manhattan Community College, 199 Chambers St., New York, NY 10007
Applications should be received no later than May 15, 1996. Fax: (212) 346-8550

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Strategies for Affecting The Affective Domain: A Math Anxiety Reduction Guide

by

Rosemary M. Karr
Collin County Community College
Plano TX 75074

Rosemary M. Karr is a professor and coordinator of Developmental Mathematics at Collin County Community College in Plano, Texas. She earned a bachelor's and master's degree from Eastern Kentucky University and is currently writing her dissertation at the University of North Texas.

Smith (1991) indicates that many students in developmental mathematics curricula suffer from more than a cognitive inadequacy in their preparation for advanced courses in mathematics. Often, student's self-esteem has become negatively impacted by their past difficulty with mathematics. Consequently, students enter a course hampered by their own perception of an impending crisis, i.e. *Math Anxiety*.

To ameliorate this prevalent condition, a seminar was designed to address the *affective* components of learning inhibitors. This student-centered experience enabled the participant to explore the emotional impediments to learning that are often present in a mathematics environment. After conducting the seminar several times, certain strategies were effective in reducing anxiety. The methods and techniques used during the seminar presentation were then compiled into a Seminar Guide which could be utilized by teachers who wish to implement anxiety-reduction concepts in the mathematics classroom, or (with appropriate modifications) in other subjects.

The Math Anxiety seminars addressed the following areas of concern:

- General Attitudes – background material on the attitudinal manifestations of math anxiety;
- Anxiety Evaluation – assessment instruments for determination of student levels of math anxiety;
- Anxiety Reduction Strategies – guidance in reducing and eliminating the effects of math-induced stress;
- Classroom/Study Strategies – useful tips and techniques for active involvement of the student, both in and outside the classroom setting, during the instructional phases of learning;
- Test Taking Strategies – hints and resolution tips to the student for active stress reduction during the evaluative phases of learning.

Front of Card

Name S.S. Number

Phone Number

Address

Hours employment, per week, outside of class

Course Hours attempted this semester

College Major

Math Background

Back of Card

Grade expected in this class

hours you expect to study, per week

Why are you taking this course?

What can **I do** to help you achieve your goal of receiving course credit?

Figure 1. Classroom Card Introduction

Several of the effective strategies used during the seminar are discussed in detail. For example, the first day of class instructors may ask students to fill out biographic information. A first-day card is illustrated in Figure 1. The purpose of the card is two-fold: to obtain the biographic information on the front, and to use the information on the back in an advisory role if the student seems to be misled in any way as to the expectations of the semester. A colleague uses a similar card technique. She tells this story:

> The first day of Beginning Algebra, students were asked to either write a few sentences about how they feel about math or draw a picture of their feelings. A particular student drew a picture of a brick wall. After successfully completing the course, this same student enrolled in Intermediate Algebra. On the first day of class she drew a brick wall again. The instructor, surprised by the same picture, asked her about the drawing. The student replied, "Don't you see the cracks in the brick, now?"

Study strategies are a key factor in reducing anxiety. Keeping a journal of study habits, as illustrated in Figure 2, increases student awareness of consistent interruptions. Successful homework strategies must be presented and reinforced by the instructor. Students are encouraged *not* to do all of their homework in one sitting. The spiraling technique for homework reinforces the material just as it does in the classroom. One-third of the problems in an assigned set could be done at the first attempt, one-third after a significant break, and the last third on the next day.

Time studying began: ________________

Where are you studying? ________________

How is the lighting? ________________

How is your posture? ________________

Do you have any constant distractions such as: telephone, television, radio, children, etc.? ________________

Do you have any specific interruptions such as: phone calls, dinner, doorbell, pets, etc.? ________________

If you were interrupted, how long did it take you to regain your concentration?

____

Time studying was completed: ________________

How much time was actually spent studying? ________________

Do you feel this was a successful study session? ________________

If not, why not? ________________

Did you attend class regularly? ________________

Did you actively participate in class? ________________

Figure 2. Journal Entries (recommended time is two weeks)

Many students comment, "I have always been told to complete the entire set in one day. Now, when I do my homework in split sessions, I realize where I need help."

A successful study strategy has been the use of cards, punched and placed on a ring. Cards are not used as a memorization technique, but rather as a method of organization for test preparation. Sample exercise problems and the directions are written on the front of the card. A separate card would be made for each "type" of problem in an exercise set. The back of each card contains a step-by-step solution, as well as a page-number reference. Students study from the cards by working the problem out on a sheet of paper. They may check their answer on the back of the card. There are three clear advantages to using cards: they are easily portable, there is no preassigned order, and the solution to the problem is not readily visible. Many times students comment, "If I knew which section this came from, I know I could do it." With cards, no section ordering exists. By placing the solution on the back of the card, the eye is unable to trick a student into believing they "know" the first step, as may occur when studying directly from homework papers.

Practicing at home, under simulated test conditions, reduces the anxiety of taking the test. In other words, the test will be given with problems in no specific order and with no examples to follow. During the exam, all that is present is the student, the problem to be worked on the test, and the student's knowledge. Thus, studying at home must eventually reach the level of competency which requires nothing but the instructions and problem to be worked.

A common teaching technique for developmental mathematics teachers is a two-color writing approach where steps in problem solving are given emphasis by the instructor through the use of colored annotations. Students may duplicate this same technique while they do their homework in order to achieve increased readability of the steps involved in their work. Figure 3 contains a simple example to illustrate this technique. Compare the clarity of the step when using color. Students taking the time to master the effective changing of color have increased their understanding of the material.

SUBTRACT:

$$\frac{5x}{(x+3)(x-3)} - \frac{1}{(x+3)(x+4)}$$

METHOD 1.

$$\frac{x+4}{x+4} \cdot \frac{5x}{(x+3)(x-3)} - \frac{x-3}{x-3} \cdot \frac{1}{(x+3)(x+4)}$$

$$\frac{5x\,(x+4)}{(x+3)(x-3)(x+4)} - \frac{1\,(x-3)}{(x+3)(x-3)(x+4)}$$

METHOD 2.

$$\frac{\boldsymbol{x+4}}{\boldsymbol{x+4}} \cdot \frac{5x}{(x+3)(x-3)} - \frac{\boldsymbol{x-3}}{\boldsymbol{x-3}} \cdot \frac{1}{(x+3)(x+4)}$$

$$\frac{5x\,(x+4)}{(x+3)(x-3)(x+4)} - \frac{1\,(x-3)}{(x+3)(x-3)(x+4)}$$

Figure 3. Color Coding your Notes.
(Note: Colored type is shown in bold here)

A direct result of increased understanding is reduced anxiety. It is often useful to explain procedures in words as well as mathematical symbols. This technique emphasizes to the student the thought processes necessary to work the problem. For example, when solving the problem $2x^2 - 10 = 16x$ by completing the square, it can be helpful to write the steps in words that correspond to the mathematical step taken. Refer to Table 1. This English/Math steps approach incorporates the "writing across the curriculum" theme.

Knowledge of test-taking strategies before, during, and after an exam help to reduce math anxiety. Many such strategies should be included in a seminar. An in-house video tape of your math anxiety seminar is also suggested. Many times a student cannot or will not go to the seminar but would take advantage of the

Given	$2x^2 - 10 = 16x$
Put in standard form, $ax^2 + bx + c = 0$	$2x^2 - 16x - 10 = 0$
Divide by 2 to get a coefficient of 1 on the x^2 term	$x^2 - 8x - 5 = 0$
Move the constant to the right side of the equation	$x^2 - 8x = 5$
Divide the coefficient of x by two, square the result, add this new number to both sides of the equation	$x^2 - 8x + \mathbf{16} = 5 + \mathbf{16}$
Simplify the right side of the equation	$x^2 - 8x + 16 = 21$
Factor the left side of the equation	$(x - 4)(x - 4) = 21$
Rewrite the left side of the equation as a binomial squared	$(x - 4)^2 = 21$
Apply the Square Root Property	$x - 4 = \pm\sqrt{21}$
Solve for x	$x = 4 \pm \sqrt{21}$
Write the solution set	$\{4 - \sqrt{21}, 4 + \sqrt{21}\}$

Table 1. English/Math Steps

opportunity to obtain the information by watching a video tape.

During the seminar a bibliography of all library holdings, complete with call numbers on topics such as math anxiety reduction, stress reduction, study skills, test taking skills, and time management, is available to the students for "outside" pursuit of information.

Many strategies to increase the quality of student performance exist. What must be emphasized in any seminar or classroom is the value of **practice**. An analogy is typically made between performing mathematics and playing a musical instrument. You cannot play well if you do not practice. Nor can you excel at mathematics without practicing in the form of homework. This must be reiterated many times during the semester.

Much of what an instructor can accomplish intellectually is dependent upon the anxiety levels of their students. Reduction of these levels can lessen the impact of the emotional barriers to learning and significantly increase the effectiveness of classroom instruction. Instructors cognizant and skilled in the application of these strategies will indeed lead the way toward a more productive and meaningful educational experience for their students.

The bibliography which follows contains references useful in the preparation of math anxiety seminars.

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Lucky Larry #22

$$\lim_{x \to 0} \frac{\sin^2 x}{x + x\cos x} = \lim_{x \to 0} \frac{1 - \cos^{\cancel{2}} x}{x + x\cancel{\cos x}}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{2x}$$

$$= 0$$

Submitted by Steven J. Rottman (student)
Catonsville Community College
Catonsville MD

Active Learning in Statistics Supports Students' Understanding

by

Mary M. Sullivan
Curry College
Milton MA 02186

Mary M. Sullivan is Associate Professor of Mathematics at Curry College, where she has taught since 1979. She holds degrees in mathematics from Boston College and a doctorate in Mathematics and Science Education from the University of Massachusetts-Lowell. She teaches mathematics and statistics and her research interests include investigating students' conceptual understanding in those areas.

Statistics is an active discipline. Statisticians gather data; they study it to discover obvious patterns and anomalies; they seek solutions to data problems such as missing data; and when they are satisfied that the data is the best available, they analyze it using tools that have developed over time. Statisticians get a "feel" for the data before they "crunch numbers." Why should students learning statistics be different from statisticians doing statistics?

Statistics educators advocate for active learning experiences in the elementary course (Moore, 1993; Snee, 1993). They realize that students build their understanding and believe that activities which include connections to the theory both enhance statistical understanding and provide a context from which students can think about the concepts (Scheaffer, et al., 1996). Several learning activities that I have found to be successful in facilitating students' understanding of elementary statistics concepts are presented.

Which exam should Prof. Dee Viation scale?

For statistical understanding it is necessary but not sufficient that students can compute summary statistics and create visual representations of the data. Students need opportunities to make decisions concerning data and to provide justifications for their decisions. An illustration appears in Figure 1.

In my class, students analyze the data individually and prepare results for a class discussion. Rarely do they agree on the choice of exam to be scaled. In the explanations for their choice, they reveal personal beliefs about scaling that sometimes conflict with their analysis. As class members debate which exam to scale, they increasingly support their position with more solid reasoning until consensus is reached. Their confidence increases as they justify their decision. When students agree, I thank them for their assistance in resolving the dilemma on behalf of Prof. Dee Viation, and initiate a discussion on bias and subjective interpretations in statistics.

Below are the scores for 22 students in three exams. The teacher has promised to scale ONE of the exams. Analyze the data to decide which one should it be, and justify your choice. Explanations usually include a discussion of shape, center, and spread.

Exam 1		Exam 2		Exam 3	
27	72	30	85	25	89
33	74	51	85	28	89
45	75	55	93	30	89
57	75	56	93	34	89
62	75	72	94	35	89
62	75	72	94	36	89
62	78	73	95	36	92
64	83	74	97	37	92
65	85	75	100	71	92
66	96	75	100	72	92
67	96	85	100	74	98

Figure 1. Prof. Dee Viation's exam scores.

Probability distributions from Mars — M & M, that is.

Students find any venture into probability beyond simple dice, card, and coin illustrations very confusing. Statistics educators argue that the introductory course should deemphasize its importance (Moore, 1993; Scheaffer, 1992; Snee, 1993). I use an activity with M & Ms to discuss basic ideas about probability and probability distributions that includes a review of variability and prepares for a later treatment of chi-square goodness of fit.

While students organize the contents of a small size bag of M & Ms by color, I ask whether they think the Mars candy company has a set color distribution for M & Ms. Students report the frequency of each color and total piece count, which I record on an overhead. I ask questions that require students to consider the data and make conjectures:

- What is the probability of red, $P(R)$, in the sample?
- What would you expect for the $P(R)$ in the next bag, if the only information available is your sample?

Students realize that empirical probability from small samples varies considerably, which is useful later when I discuss drawing inferences from samples.

I prompt students to observe data patterns, such as the number of identical samples and the colors that appear with the greatest or least frequency. Students calculate the probability for each color in their sample and sum the individual probabilities. We discuss how we could discover the theoretical distribution, whose existence surprises them (Corwin & Friel, 1990; Moore & McCabe, 1989). They sum the sample frequencies for each color and find the total number of pieces among all samples in order to calculate the class' probability distribution of M & Ms. I ask students whether they think our class sample is a good illustration of the theoretical distribution and let them know that we will study statistical methods that test goodness of fit.

At this point, we define a discrete random variable, which we illustrate with the total number of M & Ms in the sample packages. We create a chart of the probability distribution for the total number of M & Ms, verify that $\sum P(x) = 1$, and work through properties and visual representations. In my experience, students grasp ideas more quickly when we start with concrete materials, create measurable variables, and use the numerical quantities in our own analysis.

Descriptive statistical summaries—what about the data?

Faculty routinely give students experience in calculating statistics from raw data. Many texts provide data from actual studies to capture student interest and remove the sterility of the numbers. How many instructors ask students to process the statistical information in the opposite direction? I like to give students a median value and a sample size and ask them to construct a possible distribution. A repeat of the task with a mean provides insight into degrees of freedom. Since multiple representations of the same concept enhance understanding, I present students with visual representations of a distribution, such as a boxplot or histogram, and ask them to construct a possible distribution for a particular sample size.

Other activities that include visual representations for data work well with small groups. Students gain understanding of visual data representations when they match variables and summary data to them, match different forms of visual representation (Scheaffer et al., 1996), and discuss reasons for choices.

Telephones: a necessary expense, but is the expense necessary? A computer activity.

Data relevant to students' lives captures their attention. Opportunities to verify numerical conjectures sharpens estimation skills. I record students' estimates of the number of minutes they use the phone for long distance calls in a two-week period, and ask them to bring in a copy of their long distance phone bill that includes the specific two-week period and to mark off the calls that they made.

A computer file with the following variables suffices for this activity: student name; long distance carrier; total number of long distance minutes; total number of long distance calls; number of call minutes in the day, evening, and night rate periods, respectively; total cost; and original estimate. After data entry is complete, I give each student a hard copy of the data file to inspect, to plan analyses for, and to form hypotheses for future data collections.

Students have many variables with which to practice creating graphical data representations and preparing descriptive analyses. A consideration of level of measurement for the included variables and the comparison of estimates with actual values lead to issues that underlie data analysis, including most appropriate measures of center and spread, measurement error, and sampling types.

This data set provides opportunities for students' writing assignments. If students agree to consider this data as a population, they can write a response to the question, "How would you construct simple random, stratified, cluster, or systematic samples using these data?" The data set also offers opportunities for students to generate testable hypotheses, such as: "Should one believe the claims by MCI that it is cheaper than AT&T?" They can also direct reports of their analysis to a specific person, such as the president of MCI or AT&T, and emphasize aspects which would interest the recipient.

Conclusion

Activities that are question-based or require students to collect and use their own data provide active, concrete learning experiences and facilitate the development of students' conceptual understanding. The process of working in small collaborative groups and participating in full-class, consensus-building discussions empowers students to believe they can do mathematics. Spoken and written communications, aside from calculations, support their mathematical and statistical thinking. Familiar situations facilitate a grasp of new ideas because students already have a mental framework to which they can attach new material. Hands-on activities provide a vehicle with which students can be active learners of statistics.

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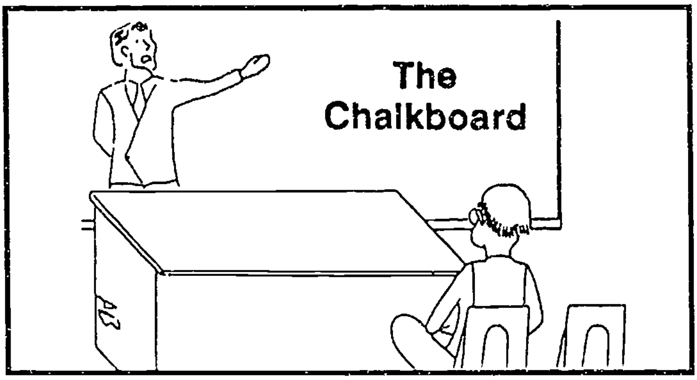
"640K ought to be enough for anybody."

—Bill Gates, 1981.

REGULAR FEATURES



Edited by

Judy Cain
Tompkins Cortland Comm. College
Dryden NY 13053
cainj@sunytccc.edu

and

Joseph Browne
Onondaga Comm. College
Syracuse NY 13215
brownej@goliath.sunyocc.edu

This column is intended as an idea exchange. We hope to facilitate an open exchange of ideas on classroom management, teaching techniques, tips for helping students get past the usual stumbling blocks, techniques for improving student participation, etc. We know there are lots of good ideas out there, and this is your chance to share them. Please send your contributions to Judy Cain. Our backlog is again nearly exhausted, and we would appreciate your participation! Items may be submitted by e-mail or regular mail; please include your e-mail address if available.

Conflict Resolution in Math

The introduction of team projects into mathematics classes may result in the problem of a team member not contributing equally, frustrating others on the same team. Here are two strategies to deal with the problem:

1. Don't deal with it at all, but make several test questions similar to or dependent on project questions. The deadbeats will learn their lesson.

2. With each project, require the team to turn in an assessment of the percentage of the grade each team member has earned. The rules are that if there are n members on the team, the percentages must add up to n*100 percent; and all team members must agree on the assessment. This has the advantage that each student can decide just how much work (s)he wants to do on a project; if (s)he has, for example, a biology exam the same week a math project is due, (s)he can elect to do 10% of the work for the math project. Then on a later project, (s)he can do 200% of the work, or whatever portion (s)he chooses.

In determining individual assessments for this strategy, the students practice a form of conflict resolution, a useful tool in the "real world." And the instructor does not have to deal with a student problem.

Submitted by Rosemary Hirschfelder, University of Puget Sound, 1500 North Warner, Tacoma WA 98416, hirsch@ups.edu

The Importance of Mathematical Vocabulary

I have been surprised to learn how many mathematical terms there are, and how many hundreds of those terms teachers use in class. Because many students pay little attention to mathematical vocabulary, their understanding of mathematics and their resulting grades often suffer. These realizations have led me to set aside a corner of the blackboard in my classroom for vocabulary. Every time I use a new mathematical term, or a term from a previous course that hasn't yet come up in the current course, I write it in the designated place. Then I go over its meaning, its pronunciation, its spelling, and sometimes its etymology.

As an example, let's take the word asymptote that comes up in precalculus courses. I explain that an asymptote is, in non-technical terms, a line that a curve "gets closer and closer to" as the curve becomes infinitely long. I mention that asymptotes may be horizontal, vertical, or oblique. I say the word several times and point out that the *p* isn't really pronounced. In order to head off the common mispronunciation asymptoPe, I stress that the final consonant is a *t* and not a *p*, and I joke that asymptoTes aren't radioactive the way that isotoPes are. I then go around the room and have each student say the word out loud to prove that it isn't so terrible.

As for the etymology of asymptote, I explain that the word is made up of three Greek roots. The first component is a-, meaning "not," as found in words like atypical, apolitical, and atheist. The second component is sym-, meaning "together," as found in words like symphony, synthesis, and sympathy. The third and main component is from the Greek root pt-, meaning "rush, fly, feather." It is also found in words like pterodactyl (a winged dinosaur) and helicopter (a vehicle that has helical "wings"). An asymptote is literally a line that "doesn't rush together with" (i.e., doesn't make contact with) a curve as the curve extends to infinity. I am also careful to point out that in spite of the etymology, a curve may intersect its horizontal or oblique asymptote in a place where that line doesn't function as an asymptote. Because the histories of so many technical terms are interesting, at least some students find etymological explanations fascinating.

On each test I include a vocabulary section typically containing ten items worth one point apiece. Students who spend just a few minutes a night reviewing their vocabulary list are assured of earning an easy ten points on the next test. Usually I put definitions or descriptions on each test and ask students to come up with the mathematical terms that fit those definitions or descriptions. There are no surprises: the class knows all the words that can appear on a given test. Because of that, I take half a point off for misspelling; it's no harder to learn the right spelling than to learn a wrong one. Occasionally I alter the format by providing a mathematical term and asking for a definition. For example, if I ask what $\log_b c$

means, I expect an answer like "$\log_b c$ is the exponent that b has to be raised to in order to produce c."

When language is not stressed, students do more poorly than they would otherwise. For instance, few mathematics students seem to understand the difference between an expression and an equation, as evidenced by their attempts to "solve" an expression, as well as their mistakenly multiplying a fractional expression by its denominator to get rid of the fraction. When language is emphasized, students not only do better in mathematics classes, they also get better at explaining other things articulately. Articulate expression has been sadly lacking among many students in recent years, but it is an ability that can be cultivated again through regular inclusion of vocabulary in our classes and on our tests.

Submitted by Steven Schwartzman, Austin Community College, Austin TX

> "The concept is interesting and well-formed, but in order to earn better than a 'C,' the idea must be feasible."
>
> —A Yale University management professor
> in response to Fred Smith's paper proposing
> reliable overnight delivery service.
> (Smith went on to found Federal Express Corp.)

Snapshots of Applications in Mathematics

Dennis Callas
State University College of Technology
Delhi NY 13753

David J. Hildreth
State University College
Oneonta NY 13820

The purpose of this feature is to showcase applications of mathematics designed to demonstrate to students how the topics under study are used in the "real world," or are used to solve simply "charming" problems. Typically one to two pages in length, including exercises, these snapshots are "teasers" rather than complete expositions. In this way they differ f om existing examples produced by UMAP and COMAP. The intent of these snapshots is to convince the student of the usefulness of the mathematics. It is hoped that the instructor can cover the applications quickly in class or assign them to students. Snapshots in this column may be adapted from interviews, journal articles, newspaper reports, textbooks, or personal experiences. Contributions from readers are welcome, and should be sent to Professor Callas.

The AIDS Epidemic: Making Predictions

(to accompany the study of rates of change, curve fitting and the normal distribution)

by Dennis Higgins, SUNY College at Oneonta, Oneonta, NY

Has the AIDS epidemic peaked? Will the number of new cases reported continue to increase forever? In 1840, without the benefit of Louis Pasteur's discovery of bacteriology, and certainly without any concept of viral infection, William Farr studied various diseases and developed a theory of epidemiology which can still be applied to modern epidemics. Among the illnesses he studied was a bovine epidemic, epizootic. In particular, he examined monthly data on new cases reported by a royal commission:

Date	Total cases reported	Number of new cases
Oct. 7	11,300	....
Nov. 4	20,897	9,597
Dec. 2	39,714	18,817
Dec. 30	73,549	33,835
Jan. 27	120,740	47,191

Table 1. Bovine Epidemic

The paper made dire predictions and certainly, the number of new cases seemed to grow and grow. But Farr noted that although in the first month the number of new cases nearly doubled, this rate of increase did not continue. If that trend had continued there would have been over 76,000 new cases in the January 27 record. But there were only 47,191. In fact, the percent rates of increases for the first three periods are 96.07, 79.81, 39.47. (Dividing current new cases by previous new cases, note that 96.07% = 18,817/9,597.) The rate of increase of reported new cases was declining. Farr was able to predict to within two weeks the date at which the bovine epidemic would crest.

In epidemic theory, the epidemic crests when the number of new cases in a given period is the same as the number of cases in the preceding period; thereafter, the new case reported should decline. Epidemics are seen to follow a bell shaped or normal curve. Farr's theory has been applied by some epidemiologists to the AIDS epidemic.

Years of Diagnosis	No. of new Cases	First Ratio	Second Ratio
1982	920	...	...
1983	2,573	2.7967	...
1984	5,237	2.0354	0.7278
1985	9,328	1.7812	0.8751
1986	14,705	1.5764	0.8850
1987	19,333	1.3147	0.8340
	Projections by Farr's Law		
1988	21,978	1.1368	0.8647
1989	21,604	0.9830	0.8647
.			
.			
.			

Table 2. AIDS Epidemic

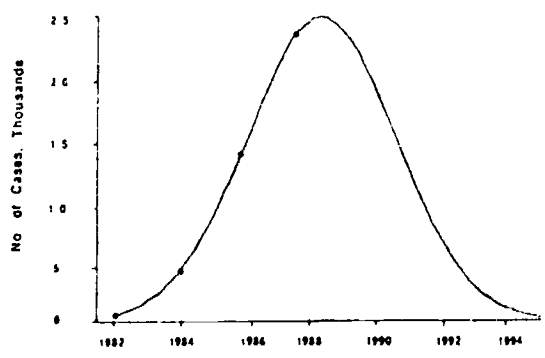


Figure 1

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In the data for AIDS in Table 2, the second ratios have been calculated. In a normal curve, the second ratio is constant. (See Exercise 6.) Averaging the second ratios from 1985 to 1987 we get 0.8647. If we draw a normal curve with this second ratio value we can make predictions about the AIDS epidemic. (See Figure 1.) Notice, if the second ratio is known, and constant, we can calculate all future first ratios and thus fill in more of the table than the data that's been collected. For example, in 1987 the number of cases is 19,333 and the first ratio is 1.3147. If the second ratio for 1988 is 0.8647 we can solve for the 1988 first ratio: $0.8647 = \frac{fr}{1.3147}$, or $fr = 0.8647 * 1.3147$. So $fr = 1.1368$. Now we can solve for the new cases in 1988 since $1.1368 = \frac{\text{new cases}}{19{,}333}$. Similarly, we can fill in predicted cases for 1989 through 1995. (See Table 2 and Figure 1.)

This method of projecting new cases in the AIDS epidemic can be criticized for a number of reasons. For one thing, cases of AIDS for a given year continue to be reported in later years. Data from IV drug users and infants may be hard to get or late in coming. Worse still is the fact that, in 1987, the Center for Disease Control started using a new method to classify AIDS victims. According to the new method previous numbers of cases reported were too low. But this doesn't mean that Farr's method will not give good predictions, only that, like many mathematical modeling techniques, it is dependent on accurate data.

Exercises

1. Verify the values of the ratios given in columns 3 and 4 of Table 2.
2. For Table 2 (AIDS Epidemic) calculate the number of new cases for 1989 through 1995, i.e., fill in column 2.
3. Check that the value 0.8647 is the average of the second ratios in Table 2 for 1985, 1986, and 1987. What is the average of the second ratios from 1984-1987? Would using this value change our predictions? How?
4. Get new data on number of AIDS cases reported for a series of years and compute the first and second ratios. Now average your second ratios. Calculate the number of AIDS cases that will be reported for the next few years. Has AIDS crested?
5. Plot the bovine epidemic curve and predict when it crested.
6. The Normal distribution function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

The symbols σ and μ are real numbers representing the standard deviation and mean of the distribution. Their particular values are not important for this exercise. For arbitrary values x_1, x_2, x_3 find the first ratio and second ratio, i.e. find $fr_1 = \frac{f(x_2)}{f(x_1)}$ and $fr_2 = \frac{f(x_3)}{f(x_2)}$ The second ratio is $\frac{fr_2}{fr_1}$. If the x values are separated by a uniform interval h, can you show that the second ratios for a Normal distribution are constant?

This snapshot was based on an article by Dennis Bregman and Alexander Langmuir (1990) and on an historical review by Robert Serfling (1952) and was produced as part of a project sponsored by SUNY Delhi and the National Science Foundation (Division of Undergraduate Education).

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Notes from the Mathematical Underground

Edited by

Alain Schremmer
Mathematics Department, Community College of Philadelphia
1700 Spring Garden Street
Philadelphia PA 19130

Conversing with a couple of friends at the AMATYC conference in Little Rock during a break, I was arguing that, by now, most so-called calculus books, a good example being the "Harvard Calculus" as it is now known, were really texts about what should more properly be called Data Analysis and that mathematics appeared to be going the way of Greek and Latin. My friends countered of course with the great successes that they were having with the "Harvard Calculus", how the students liked it, how great the problems were, etc. Yeah, but do they learn some mathematics? I kept asking. They thought I was joking. Then, a Precalculus Editor joined us. I immediately kidded her about how could they have missed what was clearly going to be a bestseller, what with the Harvard imprint and a two and a quarter million dollars NSF grant to get going. (By the way, is this what they talk about when they talk about welfare?). She claimed that she hadn't been with the publisher at the time but that, anyway, "*people were now dropping it left and right.*" When I asked why, she said "*because they think there is too little mathematics in it.*" Of course, we all laughed but it did not occur to me then to ask her *who* was dropping it.

What is important, what we should be discussing in our Department meetings, at the AMATYC meeting, is what is really happening in our classes. Specifically, what are our students learning? I contend that, for the most part, they do not acquire any "mathematical maturity" and are learning nothing substantial about mathematics. But then, why is it so difficult to have a discussion on what it is that we are trying to achieve; why is it that we just go from one fad about what are essentially "delivery methods" to the next but never examine what it is that we want delivered to our students and why we should want to do so?

A few weeks after the Little Rock meeting, I was bemoaning the very low attendance at precisely these presentations that had the most mathematical substance while the most popular presentations seemed to consist mostly of anecdotes about how to teach this and that—preferably calculator in hand. A friend mentioned that this was quite consistent with what McGrath & Spear (1991) call the "community college practitioners' culture" in that this culture

> accords only very weak status to theory, analysis, and debate. Inevitably, precedence is granted to anecdote over theory and to informal classroom experience over rigorous method as sources of knowledge. Within a practitioners' culture the conscious link between theory and practice is broken. If and when theory is proposed, it is treated as something to be mined for practical suggestions, for what to do on Monday.

And I'll bet that those who are dropping the "Harvard Calculus" aren't in two-year colleges either.

To be a bit more precise, I attended a presentation in Little Rock about teaching some predictable table of contents to future teachers. The topic, as ever, was sharing ways to make said contents more palatable to the future teachers. But, at some point, I couldn't take it anymore and I tried to suggest that the mathematical nature of said contents did not entirely go without saying or at least was not entirely obvious to me, and that this might be a place to look for the source of some of the difficulties the students were having. The immediate reaction was interesting: No one seemed to believe that I was serious when I asked what they meant by multiplication or by fractions and it seemed to me that I had elicited "an extremely strong negative response from the audience" in trying "to argue, to debate, to make intellectual progress together on some of the most critical issues." The reason McGrath and Spear give for this type of response is that

> among community college faculties, unlike university faculties, social organization does not follow the academic/intellectual organization. The elevation of teaching over research or scholarship may have turned faculty into "generic teachers," but it also stripped away any intellectual norms that might bind them together. And so they search for commonalities, or in any event don't raise matters they see no rational way to resolve. They come to undervalue intellectual exchange and mutual criticism, and to overvalue "sharing" as sources of professional and organizational development.

This is fairly convincing, at least as far as it goes. In fact, it also explains why I should find the atmosphere at AMATYC meetings so much more pleasant than that at AMS-MAA meetings. However, it does not explain why all calculus texts should be cloned from Thomas as well as from each other given that their authors abide in four-year institutions. Nor does it explain the above mentioned trend away from mathematics towards data analysis. The problem has to be recast in a larger category, namely our profound dislike of logic.

"Governor Ridge proposed a new state budget yesterday that would again cut benefits for the poor and give tax breaks to business in hopes of creating jobs." (*Philadelphia Inquirer*, February 7, 1996). If the lack of logic is mind boggling we should keep in mind that the Governor of Pennsylvania went to school and most probably took some mathematics courses. But of course, non-sequiturs are not the privilege of Governors alone. Consider the following: "Teaching students how to visualize and explore mathematical concepts helps ensure that they have a solid educational foundation for their personal, academic and professional growth — a major AMATYC goal. Thus, the appropriate use of technology plays a key role in both conceptual instruction and in the teaching of problem solving skills" (Policy Statement of the AMATYC on Instructional Use of Technology in Mathematics). Now, with that "*thus*", we are not talking simple fuzzy logic, we are talking mind boggling lack of logic of gubernatorial proportions. The Governor must have taken these math courses in a two-year college. Am I being unfair to two-year colleges? See McGrath and Spear.

Also directly relevant to the issue is an article by Colin McGinn, Homage to Education, in the August 16, 1990 issue of the *London Review of Books*, which I once discussed in the PSMATYC Newsletter. The article is a review of a book of, and of a book about, R. G. Collingwood. The relevant part is where McGinn spells in his own way what he thinks Collingwood is getting at here.

> Democratic States are constitutively committed to ensuring and furthering the intellectual health of the citizens who compose them: indeed, they are only possible at all if people reach a certain cognitive level. Democracy and education (in the widest sense) are thus as conceptually inseparable as individual rational action and knowledge of the world.

But what is education?

> Plainly, it involves the transmission of knowledge from teacher to taught. But what exactly is knowledge? [It] is true justified belief that has been arrived at by rational means. Thus the norms governing political action incorporate or embed norms appropriate to rational belief formation. The educational system of schools and universities is one central element in this cognitive health service.

The quasi-mathematical language in which this is stated should have a special resonance for mathematicians.

> It would be a mistake to suppose that the educational duties of democratic state extended only to political education, leaving other kinds to their own devices. How do we bring about the cognitive health required by democratic government? A basic requirement is to cultivate in the populace a respect for intellectual values, an intolerance of intellectual vices or shortcomings. The forces of cretinisation are, and have always been, the biggest threat to the success of democracy as a way of allocating political power: this is the fundamental conceptual truth, as well as a lamentable fact of history.

However,

> people do not really like the truth; they feel coerced by reason, bullied by fact. In a certain sense, this is not irrational, since a commitment to believe only what is true implies a willingness to detach your beliefs from your desires. Truth limits your freedom, in a way, because it reduces your belief-options; it is quite capable of forcing your mind to go against its natural inclination. This, I suspect, is the root psychological cause of the relativistic view of truth, for that view gives me license to believe whatever it pleases me to believe. One of the central aims of education, as a preparation for political democracy, should be to enable people to get on better terms with reason – to learn to live with the truth.

Finally, I will adduce the following taken from an article by Umberto Eco, Ur-Fascism, in the June 22, 1995 issue of the *New York Review of Books*, in which Eco is trying to define, or at least circumscribe, what characterizes fascism.

1. The first feature of Ur-fascism is the cult of tradition. This new culture had to be syncretistic. Syncretism is not only, as the dictionary says, "the combination of different forms of belief or practice"; such a combination must tolerate contradiction. As a consequence, there can be no advancement of learning. Truth has been already spelled out once and for all, and we can

only keep interpreting its obscure message.

2. Traditionalism implies the *rejection of modernism.* The Enlightenment, the Age of Reason, is seen as the beginning of modern depravity. It this sense Ur-fascism can be defined as *irrationalism.*

3. No syncretistic faith can withstand analytical criticism. The critical spirit makes distinctions, and to distinguish is a sign of modernism.

4. Besides, disagreement is a sign of diversity. Ur-fascism grows up and seeks for consensus by exploiting and exacerbating the natural *fear of difference.*

Could it be that teachers in two-year colleges in fact seek a consensus on a combination of different pedagogical beliefs? That the subject has already been spelled out once and for all (in the textbook) and that we can only keep...?

References

McGrath, D., & Spear, M. B. (1991). *The academic crisis of the community college.* Albany: State University of New York Press.

Software Reviews

Edited by Shao Mah

Title: Resampling Stats Version 3.16

Authors: Julian Simon, Dan Weidenfeld, Peter Bruce and Carlos Puig

Distributors: Resampling Stats Inc.
612 North Jackson Street
Arlington VA 22201

Computer: IBM PC compatible (with DOS 3.0 or later) or MacIntosh computers

Price: $225

Resampling statistics, first introduced by Julian Simon in 1966, is a form of computer-intensive methods. The term "resampling" refers to the use of sampled data which has been randomly generated by computer simulations. These computed results can then be analyzed. The resampling method resembles the Monte Carlo and nonparametric statistical techniques. The advantages of the resampling statistical method are formula-free and free of the assumptions from theoretical statistics such as samples being normally distributed, and homogeneity of variances in statistical testing. In principle, the resampling statistical method can be used to solve any problems treated by probabilistic and statistical analysis.

"Resampling Stats" is software created by Julian Simon and three others to perform the resampling method. This software requires a user to use a set of keywords to make up a very simple simulation program which will perform the required statistical calculations; therefore, programming knowledge is not needed.

The following examples illustrate the programs of "Resampling Stats" and compare their results with the answers from theoretical approaches:

1. A group of five people is selected at random. What is the probability that two or more of them have the same birthday?

   a. By Resampling Stats

```
repeat 1000
generate 5 1,365 a
multiples a >= 2 j
score j z
end
count z >= 1 k
divide k 1000 pr
print pr
```

(Result: pr = 0.02)

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b. By mathematical calculation

Pr (two or more of the 5 persons have the same birthday) =

$$1 - \frac{365 \cdot 364 \cdot 363 \cdot 362 \cdot 361}{365^5} \approx 0.0271$$

2. Six married couples went to a party. The host pairs them at random for the first dance. What is the chance that exactly two couples will get the partners they came with?

a. By Resampling Stats

```
copy 1,6 m
copy 1,6 w
repeat 1000
shuffle w f
subtract m f r
count r=0 t
score t z
end
count z=2 k
divide k 1000 pr
print pr
```

(Result: pr = 0.18)

b. By mathematical calculation

$$\text{Pr (exact two couples)} = \frac{1}{2}\left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right) = 0.1875$$

3. A farmer wished to determine which of the two brands of pig foods A or B is more effective in pig weight gains by his 20 pigs. The weight gains in pounds in a period of one week for pigs fed on brand A or B, with 10 in each group, are as follows:

A: 31, 29, 30, 30, 32, 28, 29, 24, 27, 31

B: 24, 31, 27, 28, 32, 28, 31, 23, 27, 31

a. By Resampling Stats

```
copy(31 29 30 30 32 28 29 24 27 31) a
copy(24 31 27 28 32 28 31 23 27 31) b
concat a b c
repeat 1000
sample 10 c aa
sample 10 c bb
mean aa ma
mean bb mb
subtract ma mb d
score d z
end
count z >= 0.9 k
```

```
divide k 1000 kk
subtract .5 kk pr
print pr
```

(Result: pr = 0.475)

b. By t-test of two sample means

$t = 0.74 \quad P = 0.47 \quad DF = 18$

Comparing the results of "Resampling Stats" with the theoretical approaches, the three sets of answers from the examples above were very similar to one another. For each problem, "Resampling Stats" can possibly create different answer sets due to the different set of random numbers generated by the computer.

The reviewer has used the software "Resampling Stats" for a finite mathematics class to experiment with problems in probability. Since the resampling method requires no mathematical formulas, the students needed to clearly understand each question involved. They then employed their critical thinking skills to break down the probabilistic question into required steps, and wrote their "Resampling Stats" program for computer processing. The students have found it quite enjoyable to solve the mathematical problems using this method. It is really an innovative approach in statistical problem solving.

Reviewed by: Jane Mah, Springbank High School, Calgary, AB, Canada

Send Reviews to: Shao Mah, Editor, Software Reviews
The AMATYC Review, Red Deer College, Red Deer, AB, Canada, T4N 5H5

Advertiser's Index

Book Reviews

Edited by Gloria S. Dion

THE MATHEMATICAL UNIVERSE, William Dunham, John Wiley & Sons, Inc., 1994, vi+314 pages, ISBN 0-471-53656-3.

In *The Mathematical Universe*, William Dunham presents "an alphabetical journey through the great proofs, problems, and personalities" of mathematics. He begins in "Chapter A: Arithmetic," with a look at number theory and Euclid's proof that there are infinitely many primes, and takes us through the alphabet to "Chapter Z: Z," where he gives an overview of the complex number system. Dunham presents clear and intuitive explanations of important mathematical results, interspersed with an anecdotal history of mathematics and famous mathematicians. Dunham does all this with wit and humor.

Referring to the alphabetic structure of the book, Dunham notes in the preface that "Such a format imposes severe restrictions upon a book meant to be read from cover to cover. Mathematical topics, after all, do not align themselves in a logical progression so as to mirror the Latin alphabet." I set out to read *The Mathematical Universe* from beginning to end (so as not to upset my compulsive need for order), only to find that I could not keep myself from jumping around, either to a chapter cross-referenced by Dunham or just turning to a chapter on a topic that sparked my interest. For example, "Chapter C: Circles" discusses Archimedes' approximation to π, and it is hard to resist following this with "Chapter S: Spherical Surface," where we work through Archimedes' insightful derivation of the surface area of a sphere.

In "Chapter F: Fermat" we learn about Pierre de Fermat's contributions to probability, analytic geometry, and differential calculus, as well as number theory. The exposition of Fermat's Last Theorem is current as of the date of publication, including mention that Andrew Wiles' proof of Fermat's Last Theorem is awaiting verification from the mathematical community.

In "Chapter U: Utility" we find that mathematics has practical uses. Examples include using mathematics to measure trees, mountains, and solar systems. In this chapter Dunham wonders

> Does nature, as is often said, obey mathematical rules? Such an obedience suggests that the outside world is somehow constrained by mathematical principles. Or do nature and mathematics exhibit parallel but essentially unrelated behavior? Is it just serendipitous that mathematics, with its orderly character, is the perfect language to describe the intrinsic order of the world? Perhaps the rhythms and structures of intangible mathematics simply mimic the rhythms and structures of tangible reality with neither obeying the other.

"Chapter W: Where Are the Women?" addresses the question of the title in case "the reader is keeping score." In Dunham's words,

> ... it should be apparent that more men than women have appeared on the pages of this book. This imbalance reflects the historical male dominance of

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the mathematical sciences. But does it mean women have never contributed to the subject, that women are not currently contributing, or that women will not contribute in the future? The answers to these questions are, respectively, "no," "of course not," and "get serious." Women appear in the history of mathematics as far back as classical times and are far more active today than ever before. Their presence comes in spite of obstacles that male mathematicians can hardly imagine, among which are not only a lack of encouragement but an active *discouragement* of women's participation.

Dunham lists three major impediments to the participation of women in mathematics: a pervasive negative attitude toward women who participate in mathematics combined with the attitude that women should not do mathematics, the denial of formal education to women, and the fact that even when women have overcome the obstacles they face a lack of support to pursue their work apart from the demands of everyday life. In spite of this, many women have distinguished themselves in mathematics, and Dunham discusses their lives and mathematical accomplishments in this chapter.

It is always refreshing to step back and laugh at yourself, and Dunham does this for mathematicians collectively in "Chapter M: Mathematical Personality." I had the misfortune of reading this chapter while I was sitting in a hospital waiting room, and I found myself very embarrassed that I could not keep from laughing out loud. Dunham discusses the notorious eccentricity and absentmindedness of mathematicians ("Upon his death, Dirichlet's brain was removed for later study, surely an instance of carrying absentmindedness to an extreme."), mathematicians' proclivity for humor based on distortions of logic (".. . consider Stephen Bock's description of a sheltered man and his dreams: 'Reading was something Jay knew about only from books, yet he was quite anxious to experience it for himself.'"), and mathematicians' attire and physical appearance ("It seems clear that mathematicians select apparel with an eye toward comfort rather than style... There is little doubt that male mathematicians are disproportionately bearded. Full facial hair is the unofficial uniform of the profession, perhaps because shaving is illogical (if men were meant to have smooth faces, why would little hairs grow out of their chins?). . . .The only place one is likely to encounter more beards is at a Santa Claus convention..."). Dunham summarizes: "If attire and humor, eccentricity and absentmindedness set mathematicians apart, their shared identity may be viewed as something of a defense mechanism. They truly find strength in numbers."

Dunham's style achieves a balance that is interesting to professionals and accessible to laymen. *The Mathematical Universe* is not a comprehensive dictionary or history of mathematics. It is a well-written, easy-to-read collection of facts, anecdotes, and biographical sketches that makes the history of mathematics very much alive. Dunham reminds us in the preface that "In the end, this book is the response of a single individual to the immense mathematical universe. It represents one of countless journeys that could have been undertaken by countless authors, and I make no claim to having followed the comprehensive or definitive route from A to Z."

Reviewed by Gloria S. Dion, Educational Testing Service, Princeton, NJ 08541

Send Reviews to: Sandra Coleman, 4531 Parkview Lane, Niceville, FL 32578

The Problem Section

Dr. Michael W. Ecker
Problem Section Editor
The AMATYC Review
909 Violet Terrace
Clarks Summit PA 18411

Dr. Robert E. Stong
Solution Editor
The AMATYC Review
150 Bennington Road
Charlottesville VA 22901

Greetings, and welcome to still another Problem Section!

The AMATYC Review Problem Section seeks lively and interesting problems and their solutions from all areas of mathematics. Particularly favored are teasers, explorations, and challenges of an elementary or intermediate level that have applicability to the lives of two-year college math faculty and their students. We welcome computer-related submissions, but bear in mind that programs should supplement, not supplant, the mathematical solutions and analyses.

At this time I am extremely low on good problems. In particular, I seek diverse material from as many different problem-posers as possible. Contact me by one of these means:

E-mail (the Internet): mwe1@psu.edu – via which I will acknowledge your problem, comment, suggestion, or whatever.

Regular mail: Send **two copies** of new **problem proposals** to the Problem Editor. Please submit separate items on separate pages each bearing your name (title optional; no pseudonyms), affiliation, and an address. If you want an acknowledgement or reply, please include a mailing label or self-addressed envelope.

In either case, if you have a solution to your proposal, please include same (two copies would be appreciated if sent by traditional mail) along with any relevant comments, history, generalizations, special cases, observations, and/or improvements.

Please also send your solutions to *Quickies* to the Problem Editor. These should be sent immediately, as their solutions are published the following issue, leaving at most a few weeks before they are due.

All solutions to others' proposals (except *Quickies*) should be sent directly to the Solutions Editor.

Dr. Michael W. Ecker (Pennsylvania State University, Wilkes-Barre Campus)
Dr. Robert E. Stong (University of Virginia)

Quickies

Quickies are math teasers that typically take just a few minutes to an hour or two. Solutions usually follow the next issue, listed before the new teasers. All correspondence for this department should go to the Problem Editor.

Comments on Old Quickies

Quickie #21: A classical howler passed on by the Problem Editor.

Consider the integral

$$\int \frac{1}{x}\, dx$$

(pretend we don't know it) and apply integration by parts ($u = \frac{1}{x}$ and $dv = dx$). Obtain

$$\int \frac{1}{x}\, dx = 1 + \int \frac{1}{x}\, dx.$$

Evidently, then, we have proven that $0 = 1$! (You might also consider definite integrals.)

Solution: There is no contradiction, as each side represents a *class* of functions. (The constants of integration may be thought of as differing by 1 on each side.) In the case of definite integrals, don't forget that the product uv needs to be evaluated between limits of integration.

Quickie #22: Proposed by Frank Flanigan, San Jose State University, CA.

The power series $1 + 2x - 3x^2 + x^3 + 2x^4 - 3x^5 + x^6 + 2x^7 - 3x^8 + x^9 + \dots$ does not converge at $x = 1$, but it does represent on $(-1, 1)$ a function $f(x)$ that is analytic on $(-\infty, \infty)$. Calculate $f(1)$.

I have a solution. To allow more readers to have a chance to send in their solutions, I will postpone this until next issue in the hope that readers still send me their own solutions.

New Quickies

Quickie #23: Proposed by Florentin Smarandache, Tuscon, AZ.

For each positive integer n, define the Smarandache function S by $S(n)$ = the least positive integer k such that n divides $k!$. Characterize $\{n \in Z^+ : S(n) = n\}$; i.e., when is $S(n) = n$?

Quickie #24: Proposed by the Problem Editor.

What is the average length of a chord in a circle of radius r?

Quickie #25: Proposed by the Problem Editor.

Given any modulus m, prove that every additive sequence of integers mod m is eventually periodic. (This result can be easily extended to other kinds of sequences. Note that an additive sequence f is one satisfying $f(n) = f(n-1) + f(n-2)$.)

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New Problems

Set AD Problems are due for ordinary consideration October 1, 1996. Of course, regardless of deadline, no problem is ever closed permanently, and new insights to old problems – even *Quickies* – are always welcome. However, our Solutions Editor requests that you please not wait until the last minute if you wish to be listed or considered on a timely basis.

Problem AD-1. Proposed by Michael Andreoli, Miami-Dade Community College, FL.

A well-shuffled deck of 52 cards is to be turned over one card at a time. For each card that is not an ace, the player wins \$1. The player may elect to stop at any time and keep his accumulated winnings. However, if an ace is turned over, the player loses everything and the game ends. How many cards should the player turn over to maximize his expected winnings?

Problem AD-2. Proposed by the Problem Editor.

The assumption of no air resistance or friction in short-distance, falling-body problems leads to the conclusion that heavier bodies fall no faster than lighter bodies. Suppose we more realistically include a force due to air resistance, one that is proportional to velocity at each moment. Prove (or disprove) that heavier bodies subject only to constant gravitation and such air resistance *do* fall faster when dropped than lighter ones.

Problem AD-3. Proposed by the Problem Editor.

For each integer $n > 1$, an *aliquot* divisor is any divisor of n, including 1, other than n itself. Let $s(n)$ = the sum of the aliquot divisors of n, and similarly $p(n)$ = the product of these divisors.

a) Characterize the integers $n > 1$ according to whether $s(n) < p(n)$, $s(n) = p(n)$, or $s(n) > p(n)$.

b) Find all $n > 1$ for which $s(n) = p(n) = n$.

Problem AD-4. Proposed by Michael Andreoli, Miami-Dade Community College, FL.

Balls numbered 1 through n are placed in an urn and drawn out randomly without replacement. Before each draw a player is allowed to guess the number of the ball that is about to be drawn. The player is told only whether his guess was right or wrong. The player decides to adopt the following strategy: Keep guessing 1 until correct. Then switch to 2 until correct, and so on. Find the expected number of correct guesses. What happens as n increases without bound?

Problem AD-5. Proposed by the Problem Editor.

Consider all linear functions $L(x) = ax + b$. For which pairs (a, b) is it true that there exists a unique real number h such that, for any real x, the sequence of function iterates $\langle L^n(x) \rangle$ converges to h? For such functions, find h.

Set AB Solutions

Circulation Problem

Problem AB-1. Proposed by the Problem Editor (Michael W. Ecker, Pennsylvania State University, Wilkes-Barre Campus).

Let a be a suitably small positive constant and define:

$$x_n = \cos a + \cos 2a + \ldots + \cos na$$

$$y_n = \sin a + \sin 2a + \ldots + \sin na.$$

Prove that the points (x_n, y_n) lie on a circle, give the center and radius in terms of a, and find any restrictions on the parameter a.

Solutions by Carl O. Riggs, Jr., Largo, FL; and the proposer.

Let $z_n = x_n + iy_n = u + u^2 + \ldots + u^n$, where $u = e^{ia}$. If a is not an integral multiple of 2π, then $z_n = \dfrac{u(1 - u^n)}{1 - u}$ and, because $|u| = 1$,

$$\left| z_n - \frac{u}{1-u} \right| = \left| \frac{u^{n+1}}{1-u} \right| = \frac{1}{|1-u|}.$$

Thus z_n lies on the circle with center $\dfrac{u}{1-u}$ and radius $\dfrac{1}{|1-u|}$.

Quartic Quandry

Problem AB-2. Proposed by Stanley Rabinowitz, Westford, MA.

Let a, b, and c be real numbers with $a > 0$ and $b^2 - 4ac < 0$. Express the quartic polynomial $ax^4 + bx^2 + c$ explicitly as the product of two quadratic polynomials with real coefficients. (The square root of a negative number or a complex number must not appear anywhere as a sub-expression within your answer.)

Solutions by Rick Armstrong, Florissant Valley Community College, St. Louis, MO; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Kenneth G. Boback, Pennsylvania State University, Wilkes-Barre Campus, Lehman, PA; Donald Fuller, Gainesville College, Gainesville, GA; Michael J. Keller, St. Johns River Community College, Orange Park, FL; Shirley A. Murray, Cuyahoga Community College, Parma, OH; Stephen Plett, Fullerton College, Fullerton, CA; Robert A. Powers, Front Range Community College/Larimer Campus, Fort Collins, CO; Carl O. Riggs, Jr., Largo, FL; Grant Stallard, Manatee Community College, Bradenton, FL; and the proposer.

Completing the square gives

$$ax^4 + bx^2 + c = (\sqrt{a}x^2 + \sqrt{c})^2 - (\sqrt{4ac} - b)x^2$$
$$= (\sqrt{a}x^2 + \sqrt{(\sqrt{4ac} - b)}x + \sqrt{c})(\sqrt{a}x^2 - \sqrt{(\sqrt{4ac} - b)}x + \sqrt{c})$$

and all coefficients have the desired form.

The Long Wait

Problem AB-3. Proposed by Michael H. Andreoli, Miami-Dade Community College.

A fair coin is tossed repeatedly and you observe heads (H) or tails (T). Which pattern has a longer expected time to occur: HH or HT?

Solutions by Rick Armstrong, Florissant Valley Community College, St. Louis, MO; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Mike Dellens, Austin Community College, Austin, TX; Donald Fuller, Gainesville College, Gainesville, GA; William T. Long, Broward Community College, Coral Springs, FL; Frank Soler, De Anza College, Cupertino, CA; and the proposer.

Let E be the expected number of tosses to get HH. When the coins are tossed, you get either a T first, or HT, or HH. Thus

$$E = \frac{1}{2}(1 + E) + \frac{1}{4}(2 + E) + \frac{1}{4}(2)$$

so $E = 6$. In order to get HT, you must first get a head (2 tosses expected) and then continue tossing until you get a tail (2 more tosses expected), so 4 tosses are expected to get HT. Thus HH has a longer expected time to occur.

(Mike Andreoli notes that a more elegant and general solution can be found on pages 231–233 of *Stochastic Processes* by Sheldon Ross (Wiley, 1983).)

Triangle Trisection

Problem AB-4. Proposed by J. Sriskandarajah, University of Wisconsin-Richland.

The adjacent pairs of angle-trisectors of equilateral triangle ABC meet in a triangle $A'B'C'$. Find the ratio of the area of $A'B'C'$ to the area of ABC.

Solutions by Stephen Plett, Fullerton College, Fullerton, CA; and the proposer.

Suppose the triangle ABC has edge 1. In the right triangle formed by B, A', and the midpoint of side BC, one sees that $|BA'|\cos 20° = \frac{1}{2}$. In the isosceles triangle $A'BC'$, one sees that the edge e of $A'B'C'$ satisfies $\frac{e}{2} = |BA'|\sin 10°$. Thus $e = \frac{\sin 10°}{\cos 20°}$ and the ratio of the areas is $e^2 = \frac{\sin^2 10°}{\cos^2 20°}$.

Fermat Polynomials

Problem AB-5. Proposed by Frank Flanigan, San Jose State University.

Given distinct primes $p < q < \ldots < r$, choose a rational number b and form the following polynomial of degree r over the rationals:

$$P(x) = bx + \frac{x^p}{p} + \frac{x^q}{q} + \ldots + \frac{x^r}{r}.$$

Are there choices of b with the property that $P(n)$ is integral whenever n is? (Give necessary and sufficient conditions.)

Solutions by Robert Bernstein, Mohawk Valley Community College, Utica, NY; and the proposer.

The necessary and sufficient condition is that $P(1)$ be integral; i.e.,

$$b + \frac{1}{p} + \frac{1}{q} + \ldots + \frac{1}{r}$$

must be integral. It is clear that if $P(n)$ is integral for all integers n, then $P(1)$ is integral. On the other hand,

$$P(n) = nP(1) + \frac{n^p - n}{p} + \ldots + \frac{n^r - n}{r}$$

and if $P(1)$ is integral, each of these summands is integral.

That's the Breaks

Problem AB-6. Proposed by Michael H. Andreoli, Miami-Dade Community College.

From the interval [0, 1] choose two numbers at random. What is the probability that the resulting segments can be used to form a triangle?

Solutions by Rick Armstrong, Florissant Valley Community College, St. Louis, MO; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Donald Fuller, Gainesville College, Gainesville, GA; Stephen Plett, Fullerton College, Fullerton, CA; Carl O. Riggs, Jr., Largo, FL; Frank Soler, De Anza College, Cupertino, CA; and the proposer.

The three segments a, b, c with sum 1 can be made to correspond to a point in the interior of an equilateral triangle with altitude 1 where the distances to sides A, B, and C are a, b, and c. The three segments form the sides of a triangle when the point lies in the subtriangle with vertices at the midpoints of the sides (i.e., none of a, b, c exceeds $\frac{1}{2}$). This middle triangle has area $\frac{1}{4}$ of the total area, so the probability is $\frac{1}{4}$.

A Smooth Wiggle

Problem AB-7. Proposed by Stephen Plett, Fullerton College, CA.

Produce a smooth function $P(x)$ defined on $\{x: x \geq 2\}$ that is polynomial on each interval $[k, k + 1]$ with k zeros equally spaced, including the endpoints. (For instance, on [3, 4], zeros must occur at 3, 3.5, and 4.)

Solutions by Robert Bernstein, Mohawk Valley Community College, Utica, NY; Donald Fuller, Gainesville College, Gainesville, GA; and the proposer.

Let $P_k(x) = \prod_{j=0}^{k-1} \left(x - \left(k + \frac{j}{k-1}\right)\right)$. The function $P(x)$, which is equal to

$$(-1)^{1+\left[\!\left[\frac{k}{2}\right]\!\right]} \frac{(k-1)^{k-2}}{(k-2)!}$$

times $P_k(x)$ on the interval $[k, k+1]$ for each k, has simple zeros and is smooth. (Note: These multipliers make the derivatives agree at the endpoints.) The function $Q(x)$, which is the square of $P_k(x)$ on the interval $[k, k+1]$, has multiple zeros at the desired points and is smooth.

"There is no reason anyone would want a computer in their home."

—Ken Olson, president, chairman and founder of Digital Equipment Corp., 1977.

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Conference Committee Chairpersons

Marilyn McBride
Skyline College
3300 College Drive
San Bruno, CA 94066
(415) 738-4354
mcbride@smcccd.cc.ca.us

Melanie Branca
Southwestern College
900 Otay Lakes Road
Chula Vista, CA 92010
(619) 421-6700 Ext. 5519
mbranca@sunstoke.sdsu.edu

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See page 35 for more details

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