DOCUMENT RESUME

ED 390 706 SE 057 558

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TITLE Coemergence: Four Enactive Portraits of Mathematical Activity.

SPONS AGENCY Social Sciences and Humanities Research Council of Canada, Ottawa (Ontario).

PUB DATE Apr 95

CONTRACT 410-93-0239; 752-93-3268

NOTE 37p.; Paper presented at the Annual Meeting of the American Educational Research Association (April 18-22, 1995).

PUB TYPE Reports - Research/Technical (143) -- Speeches/Conference Papers (150) -- Collected Works - General (020)

EDRS PRICE MF01/PC02 Plus Postage.

DESCRIPTORS Algebra; *Cognitive Processes; *College Students; Educational Research; Foreign Countries; Higher Education; *Number Concepts; Observation; *Problem Solving; Research Methodology

ABSTRACT

Team research is important in studying cognition as enactive. This paper contains four different pieces of research directed toward the evidences and artifacts of two students in Canada engaging in a sustained mathematical activity. These four portraits of mathematical cognition in action consider the conversation in which the activity occurs; the structures manifested in the beliefs of these students about mathematics; the patterns of reasoning in action; and the dynamical growth or changes in the mathematical understanding of this pair of students. These approaches and pieces of research can be observed as coemergent. An introductory paper, "Enactivism and Education, Especially Mathematics Education" is followed by four research papers: "A Portrait of Mathematical Conversation" (Lynn Gordon Calvert); "A Portrait of the Coemergence of Reasoning" (David A. Reid); "A Portrait of Beliefs in Action" (Elaine Simmt); and "A Pathway Portrait of Mathematical Understanding in Inter-Action" (Thomas E. Kieren). Appendixes contain verbal snapshots and writing samples of the two students. Contains 52 references. (MKR)

COEMERGENCE:

FOUR ENACTIVE PORTRAITS OF MATHEMATICAL ACTIVITY

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* Research for this work has been supported in part by grants 410-93-0239 and 752-93-3268 of the Social Science and Humanities Research Council of Canada and by grants from the J. Gordon Kaplan Fund of the University of Alberta.

ENACTIVISM AND EDUCATION, ESPECIALLY MATHEMATICS EDUCATION

Knowledge is not in a book or in the library,
Knowledge is not in our heads.
<u>Knowledge is in the inter-action!</u>

Paraphrase of Karl Tomm, Jan., 1989

It is common sense to think about knowing and understanding as entailing specific responses to pre-given "problems" or conditions in the environment. And it is easy to think of a product of that activity, especially in mathematics, as an "answer" which matches pre-given conditions. While such a view is popular with and useful to those who develop standardized tests as well as to some cognitive scientists, this view is problematic on several counts. In using a "mind as (digital) computer" metaphor, it reduces cognitive activity to the making and processing relatively accurate symbolic or quasi-symbolic representations of a pre-given world. Such a status for the world suggests that knowledge indeed exists "out there" (as Karl Tomm says "in the book or library") and that, as such, the environment is the privileged (and for some indeed the only) source of knowledge. One might say that in this view, the individual cognition is contained in or implied by the world in which it exists. Further, this position, which could be called representationism, is based on the dichotomy of mind and body, and the knower and known, positions which are being called into questions both for philosophical reasons (e.g., Merleau-Ponty, 1962; Varela, Thompson and Rosch, 1991) and reasons derived from contemporary views of human functioning (e.g., Damasio, 1994). Such a view divorces cognitive activity from the broader personal histories of the persons engaged in it and reduces the complexities of the situations in which this cognition takes place. This is a prescriptive view of cognition – any product of cognitive activity which does not match pre-given criteria or which is not right, is wrong. While such a view might seem relevant if one is looking at very narrow tasks, it essentially ignores the broad range of responses, many unpredictable, which "work" for people in most situations as they continue to live them. Further, such a view appears to involve an omniscient evaluator checking a person's actions against "the library" of knowledge as it ought to be.

One response to the problems identified above is to see knowing and knowledge as located "in the head" (or at least the body and brain) of the person. Under this view of cognition the person constructs her or his knowable world. Put another way, the world, at least as knowable to the person, is contained in or implied by the person's cognitive acts. While this construction is seen as based on reflections on actions in a space, it is the person who is the privileged (or even the sole) source of knowledge. The environment fades into the background. This position seems more consistent with contemporary work on brain functioning and cognition (e.g., Domasio,1994); in fact, under this view the cognitive activity of a person as seen by an observer is an emergent phenomenon, arising out underlying neural events (sometimes termed connectionism) (Minsky, 1986; Varela, Thompson and Rosch, 1991). Such a point of view is useful in mathematics education research where such privileging of the constructing individual's knowledge both provokes and promotes the search for and identification of mechanisms that individuals appear to use in building up mathematical knowledge (Steffe and Cobb, 1988; Steffe and Weigel, 1994; Steffe and Kieren, 1994). While a person's action is seen as a significant site for the genesis of cognition, the site of the action and the related drift which allows for the environmental possibilities for and the bounding of that action is ignored or allowed to fade into the background. Thus, while such a position, focusing on knowledge as "in our heads" offers rich ideas with respect to the study of person's learning, in its extreme case at least, it is silent or at most proscriptive with respect to teaching.

The research described here - in terms of its content, its process itself and its "results" - calls for us to think otherwise about cognition in general and mathematical cognition in particular. Through this alternate view cognition is not best seen in terms of its products nor its mental structures, but in terms of action in, perhaps better yet, as living in a world of significance with others. One name given to this position is enactivism (Maturana and Varela, 1980,1992; Varela,Thompson and Rosch, 1991; Davis, 1994; Sumara, 1994; Davis, Kieren and Sumara, forthcoming).

Some Tenets of an Enactive View of Cognition

- Cognition is viewed as an embodied interactive process **coemergent** with the environment in which the person acts. Cognition does not entail a reactive representation of a pregiven world, the goal and success of which is measured by its match with that world. Nor is cognition sufficiently described as emergent from more primary brain functioning or even bodily functioning.

- In fact cognition is viewed as **doubly embodied** (Merleau-Ponty,1962; Varela, Thompson and Rosch, 1991) on-going action in an environment. The persons' plastic, in fact, ever changing structures - arising from their own biological organizations, but also from a history of previous action in an environment - fully determine their current actions (in-person embodiment). But the environment is seen as a significant part of the sphere of behavioral possibilities for the persons and as providing the occasion and the (bounded) space for actions (person-in embodiment). Put another way the person's body then serves two roles. Ones body is the site of those elements which can be viewed as the person's structure and allows the observer to see the person as separate from the surroundings. But at the same time it is the body that allows the person to be viewed as fully connected to or part of an environment. Through such a doubly-embodied view both the person and his or her history and the environment and its history are coimplicated in any cognitive act.

- In such a view of cognition, neither the person nor the environment is privileged; the **knowing is seen to occur in the inter-action,** or as suggested by Maturana and Varela (1992) is the structural coupling between the structure of the person and the structure of the environment (including others, such as the teacher, in it). From the point of view of the structure determined action of the individual, the environment does appear to fade into the background or is seen as part of that person's world of significance which they are bringing forth. But similarly, if viewed from the point of view of the environment, the individual actor and cognition fades into the background or is simply one feature of the environment. On this view one might say that personal cognition is like the topologists' "Klein bottle". the world is contained in the person's cognition which is contained in the world.

- To study cognition in such terms then is to consider **at once the structural dynamics of the individuals but also the inter-actional dynamics of actions in a situation. Cognition and knowing are seen as non-linear, recursive, self-organizing processes** (e.g., Pirie and Kieren, 1989; Kieren, Reid and Pirie, 1994) through which one builds and acts in a world of significance with others, while understanding is observed in like terms as a sequence of events through which ones world stands forth (Johnson,1987).

- Local situated knowing is not seen as activity to be eclipsed by or its products shunted aside by more sophisticated or formal knowledge, but it is viewed as **the heart out of which all cognitive activity unfolds and is projected** (Johnson, 1987; Varela,Thompson and Rosch,1991; Kieren, 1994; Pirie and Kieren, 1994).

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- The teacher or teacher/researcher is seen to be situated in the middle of students' cognitive activity and is observed as a key part of the environment as one who provides the occasions for student actions and who acts to bound or proscribe such actions and their outcomes. The teacher is not seen as a front and centre source of knowledge to transmit nor as a checker of the "truth" of student cognitive activity; the teacher is not simply in the cognitive background as a facilitator of student action. The teacher is essentially seen as engaged with students in building a world, albeit coming to that task with very different perceptions and structures.

Such an enactive view of cognition can be compared with two other contemporary views. Like radical or non-representationalist constructivism (vonGlasersfeld, 1987), the role of action in cognition is central and the students' structures are seen as determining their actions in bringing forth a world. But unlike such constructivism, the knowledge in the head is not privileged. The environment provides the sphere of possibilities for action, and if viewed more actively, occasions such action. The rightness of a person's actions in a situation under radical constructivism is solely judged by the person themselves according to their felt constraints (which are themselves personal constructions). Under an enactive view, the environment can be seen to proscribe or bound actions in many ways and at minimum, the environment is a source for "sedimented" internalized history of action which manifests itself in the felt constraints for the individual. Under radical constructivism, action and personal reflection on that action are central; enactivism adds the environment as the occasioner of and codeterminer of cognition, adding the "inter" – to the constructivist "action".

As mentioned above, an enactive view of cognition considers action in a situation. Thus enactivism is related to social constructivism and particularly to ideas from theories of situated cognition (Lave, 1988). My reading of those interested in situated cognition – if they do not take a fully realistic view of cognition, as does Gibson in his theory of environmental affordances (in Varela, Thompson & Rosch, 1991) – is that the weight of privilege in the inter-action is with the environment. Like Vygotsky (1986) they would likely say that while, of course, the individual constructs her or his knowledge, such knowledge has an already constructed existence at a social level. Such a view is a basis for "SITUATED cognition". Such societal level constructions are seen as a necessary "cloud" of knowledge hanging over the individuals or in another sense surrounding them. Neither enactivism nor radical constructivism sees need for such a "cloud" of knowledge. Recent manifestations of the latter either push the situation into the background, or at least feature mechanisms of individual cognition in action in a specified environment (Steffe and Weigel, 1994) and the dynamical change process in individual cognition in interaction with an environment (Pirie and Kieren, 1994). In such cases, it might be said that the cognition is considered as "situated COGNITION". An enactive view with its notions of coemergence and coimplication suggest that it is a theory of more balanced "SITUATED COGNITION" (for example Sumara, 1994).

Such a point of view of cognition has its consequences for research into the phenomenon of cognition itself. Research cannot simply consider disembodied tests, instruments or measurements and their results however complex or reliable these might be. While observing the mechanisms of a person's structure which are used to act in a world of significance, or belief structures which manifest themselves in a certain way in action are important elements of enactivist research[1], these do not make up the end nor the goal of such research. Such research must trace patterns of actions

[1] In a related sphere of research, there is considerable on-going effort to identify the neural substrata which might impy such cognition or the neural sites for it (e.g. Crick, 1994; Calvin and Ojemann, 1994; Damasio, 1994 to name three sources which have received more popular attention.) While it is beyond the scope of this report to elaborate on this, we see an enactivist positon of doubly embodied action as having important implications for such neurologically based research.

and inter-actions in the environment as well as try to observe and account for ways in which the environment occasions cognitive actions of various sorts. That is, not only does this research seek to understand the features of the world brought forth by structure determined actions, but also observes how that world and its features – which are themselves implied by the actions – are fully implicated in the cognition of the individuals.

Such a view of cognition and such a research posture in mathematics education is especially interesting. It could be argued that of all phenomenon and all cognition mathematics shows itself in discrete, well defined problems or questions and that mathematical cognition is seen as a process aimed at providing answers or at least progressively better attempts at answers to such questions. Thus applying an enactivist theory and an enactivist approach to research in mathematics education is a strong test of the fallibility of the theory in Popperian terms. There is a significant amount of evidence from recent research that suggests that there is much more to even school mathematics than single or at least convergent "answers" matched against preset questions. Such a view does not account for the inter-actions, the mathematical reasoning and the diversity of adequate actions of students of all ages as they engage in mathematical activities in various environments (e.g., Cobb, 1994; Cobb, Yackel and Wood 1992; Confrey, 1991, 1995; Pirie and Kieren, 1994; Kieren,forthcoming; Pothier and Sawada, 1983; Steffe and Wiegel, 1994; Steier, 1995; vonGlasersfeld, 1987). Thus there is support for the continued testing of enactivist theory in mathematics education. The "results" of student actions reported here substantiate the point of view that cognition can be seen as the bringing froth of a mathematical world rather than as simply solving discrete problems. But more than that, the research described here seeks to observe mathematical cognition as such a recursive, self-organizing process coemerging with a community of significance and within an environment which both occasions and bounds it in significant ways.

Beyond that, the research approaches described here are themselves enactive in nature. They are **bricological** (Reid, 1995). They entail the flexibility and pragmatics of a bricolage using a wide variety of techniques which respond to the "materials" at hand - actions. inter-actions, transcripts, tapes, artifacts, conversations about any of the above, in fact research reporting such as this. In this way the research methodology itself responds to the fact that cognitive activity (including this research itself) can, in part, be observed as form of evolutionary drift (Maturana, 1987; Varela, 1987; Tomm, 1989) . But cognitive activity also is governed (should one say "is structurally determined by") the logics of inquiry of the researchers. Thus bricological research entails two key features of enactivist theory.

The work described below contains four different pieces of research directed toward the evidences and artifacts of two students engaging in a sustained mathematical activity. These four portraits of mathematical cognition in action consider the conversation in which the activity occurs; the structures manifested in the beliefs of these students about mathematics; the patterns of reasoning in action; and the dynamical growth or changes in the mathematical understanding of this pair of students. What is important to the four of us at least is that these approaches and pieces of research themselves can be observed as coemergent. While each of the researchers is bringing forth a world, that world is occasioned and conditioned by the other portraits being constructed. Each of the "portraits" is an entity unto itself, but each is better understood in its inter-action with the others. Indeed, we have seen this coemergence feature in the sequence of work over the past several years and see it continuing into this presentation. We see team research as important in studying cognition as enactive, not so that the work can be broken down or reduced to a set of component studies, but so that cognition about cognition occurs in a way that provokes and provides the occasion for coemergence.

A Portrait of Mathematical Conversation

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Conversation "*is nothing but the mutual stimulation of thought, a kind of artistic creation in the reciprocation of communication.*"

(Gadamer, 1975, 188)

Each mathematical conversation may be thought of as an improvisation.

Improvisation in theatre is occasioned by an initial dramatic situation. The plot that unfolds is a creation of spontaneous action, interaction and communication that is unique not only to the audience who watches, but to the actors themselves. Although the improvisation is spontaneous and unpredictable it is by no means random. Each actor brings into the situation a history of knowing and experience and as the drama unfolds, the actions they take and the roles they play are a reflection of their individual histories in inter-action.

The improvisation is simultaneously occasioned and constrained by the initial situation and the actors own knowing and experience, but it is also occasioned and constrained by the environment they are in, which includes the play-space, the props available and the actors' awareness of the audience. The audience may be regarded as "a many-headed monster sitting in judgment", or "a looker-in" that needs to be tolerated, or as "a group with whom [they] are sharing an experience" (Spolin, 1963, 13).[2] In all cases, the actors are not only trying to make the play believable to the audience, perhaps more importantly, they are trying to make it believable and authentic for themselves. Some of the most appealing improvisations are when both the actors and the audience have lost themselves in the play.

I am only a single member of the audience (or potential audience). It would be ridiculous to assume that there is a hidden script that is available to all of us in the audience, but not to the actors themselves. It would also be foolish for me to think that the inter-actions that take place are really insignificant and to think instead, that what is important is that it ends as I expect it to. If either situation were true, I would naturally be compelled to make judgments of good or bad, and right or wrong according to these preset conditions.

But neither of these situations is appropriate to the improvisation. Instead I sit back, sometimes having no idea of what they are about to do from moment to moment. But just as the actors bring *their* histories of experience to the improvisation, I bring my own. I have seen a large number of improvisations in the past, sometimes with identical or similar starting points, and I am also aware of many techniques that can be used. I may be able to make some inferences about where the play is going and whether it is more or less appropriate given the initial and evolving situation, but I am often surprised. Many improvisations that I thought were going down some unusual path led me to places I have never been or even before considered. At these times I often turn to the people beside me to see if they understand what the actors are doing or where they are going. Sometimes someone recognizes it as a parody of another play, or someone else explains the technique being used, but many times none of us knows, so we move closer to the edge of our seats to see what happens next.

[2] I should also mention that when the actors are willing to share the experience with the audience, they often do so in literal ways. Sometimes the actors request help from members of the audience for getting and arranging props throughout the improvisation; sometimes the audience is invited directly on stage as partial or full participants in the improvisation; and sometimes some audience members get so caught up in the drama that they cannot help but jump on-stage themselves.

The improvisation metaphor suggests that the unfolding of a mathematical conversation is occasioned and constrained by the initial problem situation, the available resources, the researchers' and/or teacher's presence, and the physical context in general. Within the sphere of possibilities that the environment provides, the improvised conversation, which includes the action, inter-action and the verbal and non-verbal communication, is determined by the participants themselves.

In this project, the arithmagon was given as the initial "dramatic" situation. The resulting conversation between Stacey and Kerry is unique and is "a kind of artistic creation" in action. What should not be surprising, if the improvisation metaphor is seen as analogous to mathematical conversations, is that *all* mathematical conversations are unique. The conversations can neither be predetermined nor do they converge to a predictable ending. Each group in conversation brings forth a different, sometimes a very different world of mathematics through their interaction. The conversation then needs to be thought of, not as a problem solving event, but as persons entering into a shared world of experience. The mathematical conversation becomes an occasion for all participants (students, teacher, researcher) to lose themselves in the *play*.

Just as an actor brings his or her history of experience into the improvisation, Stacey and Kerry brought their individual histories of experience into the conversation. Although the conversation was spontaneous and unpredictable it was by no means random.[3] As this mathematical conversation unfolded – as they layed down the path of their exploration – who they are, what they believe, and their mathematical skills, and interests are recognizable in what they do, what they say, and in the roles that they play.

Their histories become braided together in the conversation; they remain distinct as individuals but they become structurally coupled in the interaction. What emerges is an improvisation or conversation that is choreographed in the moment through the co(n)-sensual coordination of action. The "consensual coordination of action" is less a consensus of action and more an agreement to maintain the relationship, or structural coupling, through interaction (Tomm, 1989). My alteration of the word to co(n)-sensual implies that conversations are inter**text**ual and inter**textur**al; they are co-sensually guided by rational-emotional intuitions and through an awareness and acceptance of the other.

The conversation develops through the explicit coordination of action – conjectures made, ideas generated, calculations performed; however, the actions are implicitly occasioned and conditioned by the participants' beliefs and purposes; the construction and negotiation of meaning; and the roles, expectations and patterns of interaction that are formed over time with each other (and with all others). As David Reid (in this paper) explains, the reasoning that is employed by Stacey and Kerry in doing the arithmagon is the co-emergence of the situation with Stacey and Kerry's interacting structures; and as Elaine Simmt elaborates in her portion of this paper, Stacey and Kerry's structures include their unformulated philosophies and beliefs about mathematics: Kerry sees mathematics as a system with specific rules while Stacey sees mathematics as a game of exploration. In the conversation, we see these two beliefs about mathematics, and the different types of reasoning that they employ meshed together in conversation. The mathematics that is brought forth is unique to Stacey-and-Kerry (together). The path taken would have been quite different if walked alone, and it would have been quite different if Stacey or Kerry were in a conversation with a different person.

The following is an elaboration of the conversation as it occurred in episode 5, lines 22 to 44. This portion of the transcript comes shortly after Stacey and Kerry have exhausted the mechanical means they can think of for solving the original arithmagon – first using algebra to solve the system of equations and then by using a matrix. Not surprisingly, these formalized approaches were suggested by Kerry who sees mathematics as a system of rules and procedures.

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[3] "Improvisation always has its rules, even if they are not *a priori* rules. When we are totally faithful to our own individuality, we are actually following a very intricate design. This kind of freedom is the opposite of "just anything." We carry around the rules inherent in our organism" (Nachmanovitch, 1990, 26).

Episode 5, begins with Stacey suggesting an exploration, "What can we do with three other numbers? We can extend lines –" Such a suggestion seems consistent with Stacey's structure which includes a belief that mathematics is itself a game of exploration.

S: You keep going.
K Where you going?
S: You keep going. We could find numbers for this.
K: For that?
S: Yeah.
K: Okay, so you want to solve that then?
S: Umhmm. Okay —
S: It will go right to zero.
K: Are you saying–
S: This I don't, I don't know.
K: Are you saying the numbers would keep getting smaller?
S: Yeah, these would have to be –
K: Yeah I guess they will be getting smaller –
S: Like decimals. 'Cause you got 1 on one side –
S: Okay mister.
K: You, want, you want to try to solve it then?
S: Yeah sure. Away you go.
K: Okay then, let's label it.
S: I like drawing it.
K: Okay, we'll label these anyway. So we've gone. So go. D, E –
S: and F.

The transcript above is bare, containing only what was heard on video (and is therefore, slightly different than the one in Appendix A). My purpose for leaving it bare is to illustrate that a mathematical conversation does not exist merely as spoken words, and its interpretation cannot be lifted from a transcript. Conversation is a dynamic process of communication (both verbal and non-verbal), action and inter-action that is improvised in the moment, but that is brought forth from a history of communication, action and interaction with other people, places and situations. The following interpretation is itself occasioned and constrained by my own history of communication, action and interaction with Stacey and Kerry, the researchers co-authoring this paper, other individuals with whom I have discussed this research, as well as by other pieces of data such as videotapes of Stacey and Kerry in additional problem solving sessions and in an interview.[4]

After drawing the original arithmagon with an outer, inverted triangle on a new sheet of paper, Stacey suggests a course of action: "You keep going" (line 22) (see diagram in Appendix B).

Kerry moves in closer to Stacey, who has the new problem directly in front of her.

"Where you going?"

"You keep going. We could find numbers for this." Stacey points to the vertices of the new outer triangle.

"For that?"

"Yeah."

Stacey's choice of language, "We could", (as opposed to, for example, "I will") not only invites Kerry into the exploration but it also leaves the suggestion as hypothetical and open to other possibilities or revisions.

[4] I invite any readers to participate in this bricological process by e-mailing their comments and interpretations of the conversational aspects to me. E-mail addresses are included in the back of this paper.

Kerry actively tries to understand what Stacey is suggesting, and he at once accepts the exploration and contributes to it by suggesting a course of action, *"Okay, so you want to solve that then?"* In "solving", Kerry is implicitly suggesting that they use a system of equations to find answers for the new triangle's vertices. They could solve it using trial and error or they could try to find a formula, but it not surprising that they use a technique that was used earlier (in episode 1); the enacted pattern of solving by using a system of equations is carried through to the end of the problem solving session. The difference, however, between the initial episode and subsequent episodes is this technique is now used as a tool to further the exploration rather than as a means to find the final solution to the problem.[5]

Kerry's suggestion to solve it is temporarily put aside as Stacey makes a prediction. She tries here and at several points later on in the conversation to find a connection between the inner and outer triangle(s). Her prediction is at first stated firmly, *"It will go right to zero,"* but when Kerry tries to interpret her by starting with *"Are you saying–"* she interrupts and says, *"This I don't, I don't know"* (line 32). Similar hesitant responses are made by Stacey in other parts of the conversation. After suggesting the original exploration, *"We can extend lines"* (also said hypothetically rather than assertively), Kerry asks what she is doing. Stacey responds, *"I don't know what I'm doing yet"* (line 19). Later on in the transcript she prefaces another conjecture with, *"Well I don't know about this. This is just the one little step"* (Line 88). This suggests a pattern of interaction that Stacey has brought to this mathematical conversation. Stacey frequently makes spontaneous predictions and conjectures that are intuitive and unformulated. When questioned, her response may honestly be that she is not certain of the conjecture's direction or importance. Another possibility is that she is following a more historical and gendered pattern of structural coupling in which she undermines her intuition.

Kerry continues after being interrupted, *"Are you saying the numbers would keep getting smaller?"* Kerry is again actively trying to understand Stacey's utterances and is surprisingly able to make an interpretation of Stacey's cryptic comment. However, it should not be assumed that her words "transmitted" the meaning to him directly; instead, according to the enactivist theory of cognition, her words, which are embedded in the situation, act as a trigger to which Kerry responds or does not respond according to his structure (Maturana & Varela, 1992). In this instance, Kerry responds to the conjecture – *"It will go right to zero"* – by describing the conjecture as he sees it, *"the numbers would keep getting smaller"*. This relationship between the inner and outer triangle(s) is probably not an unexpected one to Kerry; which may be why he is immediately able to interpret Stacey's unformulated conjecture. As Bruner (1986) remarks, "The more expected an event, the more easily it is seen or heard" (p. 46).

Together, Stacey and Kerry agree that the numbers in the outer triangle must be getting smaller (lines 34-36). However, we cannot say if they have "equivalent interpretations" or "parallel interpretations" (Cobb, Yackel, Wood, 1992). What "getting smaller" means may be different for both of them. As Kieren and Reid both elaborate in their portions of this paper, Stacey may be suggesting that the numbers have a limit of zero. We cannot speculate on Kerry's meaning.

At this point Kerry again asks, *"You want to try to solve it then?"* (line 39).

Stacey says, *"Yeah sure. Away you go."* She then pushes the paper that has been in front of her over to Kerry. Stacey agrees that solving the new arithmagon is not only the appropriate action to take at this time, but it is part of Kerry's role in the conversation to do the calculations. This functional role can be attributed to the pattern of interaction that emerged in the initial episode (and possibly from previous problem solving interactions); in episode 1, Kerry did all of the mechanical calculations on paper (with the paper directly in front of him), while Stacey leaned over onto him and followed along verbally. Because Stacey quite easily follows the calculations Kerry is making, we cannot assume that this is Kerry's role because he is more capable at solving

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[5] "To do anything artistically you have to acquire technique, but you create *through* your technique and not *with* it" (Nachmanovitch, 1990, 21).

systems of linear equations, but Stacey may perceive him as being more capable or at least more interested in doing the calculations than she is.

At the end of this section we hear statements again indicative of their individual preferences. Kerry says, *"Let's label it."* Stacey says, *"I like drawing it"* (lines 41-42).

But Kerry needs the labels for his calculations and says, *"Okay, we'll label these anyway. So we've gone. So go D, E – "*[6]

Stacey now leans over onto Kerry (as she did in the first session) and actively participates in solving the system of equations, *"And F."*

All conversations, including mathematical conversations are spontaneous improvisations and even artistic creations when a particular path or an end-goal is not externally pre-determined. The path chosen is visible through the explicit interaction and communication between the participants. However, less visible are the occasions and the constraints provided by the environment and by the participants' inter-acting histories of experience. The negotiated actions, purposes, meanings, expectations and roles that emerge in the present are fully entangled in the biological and phenomenological past that each person brings forth in the conversational moment.

[6] This is perhaps another pattern of interaction that has been established between the two of them and through their histories of interaction with others. While Stacey often makes suggestions and conjectures that are tentative and hypothetical, Kerry makes relatively few conjectures and his suggestions are often assertive and correspond to his purposes for engaging in the activity. As a result he occasionally rejects her suggestions. The most obvious example comes when they are initially faced with the problem; in lines 4-6 in episode 1, Stacey tentatively suggests, "Trial and error?" While Kerry says "No ... Let's assign variables to these then..." which is a more systematic and thus more mathematical approach according to his beliefs. This course of action is fully accepted by Stacey who says, "Go for it." (See Simmt in this paper.) We can question whether this is an issue of power (male domination), and/or an issue of purpose (finding a formal solution and maintaining the relationship).

A Portrait of the Coemergence of Reasoning

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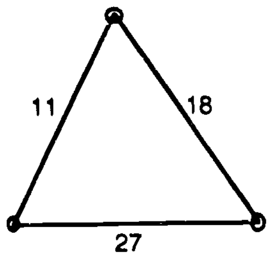
Mathematical reasoning is sometimes portrayed as a characteristic of a person. We might say "Pat is great at seeing patterns but never proves anything" meaning that Pat can reason inductively, but not so well deductively. On the other hand, researchers interested in problem solving in mathematics education occasionally makes claims that "This problem requires the solver to deduce the solution from the given information," implying that the reasoning used by someone who is successful in solving that problem is a characteristic of the problem. Enactivism shows a middle way between these two interpretations of mathematical thinking. An enactivist view would be that the person and the situation coemerge through their interaction, and so the reasoning employed is both determined by the structure of the person, and occasioned by the sphere of possibilities implicit in the situation.

The situation of a person engaged in mathematical activity includes more than just the problem prompt. The resources available, the social context, and the physical context, all play a part in making the situation what it is. As well, the person's own structure, through its coemergence with the situation, makes the situation what it is. The following portrait describes two undergraduate students, Stacey and Kerry, investigating the Arithmagon problem (see Figure 1). They and the problem prompt play a part in defining a situation, which also includes the presence of two observers, the social context of their relationship and the research project they are a part of, and the physical context (including video cameras, microphones, computers, windows, walls, etc.). Within this situation they reason in ways which are characteristic of each of them, but also of all the other aspects of the situation.

The numbers on the sides of this triangle are the sums of the numbers at the corners. Find the secret numbers.



Generalize the problem and its solution.

Figure 1: The Arithmagon prompt (adapted from Mason, Burton, and Stacey, 1985)

The language used to describe reasoning has been developed in the course of a long term research program on deductive reasoning in problem solving and proving situations. This language distinguishes needs which motivate reasoning, kinds of reasoning, kinds of deductive reasoning (proving) and forms of proofs. A full description of this language is given in Reid (1995). In the example given below, Stacey and Kerry are motivated by a need to explore a situation that has coemerged with them, in the course of their investigation of it. They reason inductively and deductively, and their deductive reasoning is further distinguishable as what I call "unformulated proving" and "mechanical deduction".

Mechanical deduction is an example of meaning-free reasoning employing a tool, such as algebra or a computer, to facilitate mathematical activity. It is deductive in that the underlying structure of the activity, if it were analyzed, would be deductive. The rules of algebra and the logic of computer circuits are both based on deduction. Mechanical deduction is meaning-free in the sense that it requires that the connection between the meaning of concepts and their formal representations be suspended. In adding two equations an algebraist is not concerned with the meaning of those equations. In fact, in some contexts the meaning might be such that adding the equations would be nonsense. By suspending the meaning of the equations methods of manipulating them usefully become available, which would not be available if the situation encoded in those equations were thought about meaningfully.

Unformulated proving is deductive reasoning in which the meanings of concepts are not suspended; in fact, they drive the reasoning process. At the same time, however, the prover is not aware of the overall form of the reasoning process. The prover draws conclusions according to a deductive logic, but does not consciously choose the path of the proving. Instead the connections between meanings inspire deductive connections, leading to conclusions which have resulted from proving, without the proving being explicit. In unformulated proving there are often implicit assumptions, and a certain lack of articulation of steps. This makes such proving difficult to follow for others, though it may seem to the prover that the conclusion is obvious. Social pressures to make the proving more explicit can occasion the formulation of it, and proving can also be formulated from the beginning.

In terms of both meaning and form, mechanical deduction and unformulated proving are opposites, but both play important roles in students' reasoning in problem solving situations. In episode 5 there occur several examples of mechanical deduction and one of unformulated proving.

Mechanical Deduction

"Do my – Oh! How are we doing here?" asked Stacey.
"Okay, we'll subtract, we'll take uh –"
"1 minus 2."
"1 minus 2. So we've got –"
"F minus E is 9," they say in unison.
Kerry continues, "And then we'll go –"
"That plus 3," Stacey suggested.
"Hmm?"
"Plus 3."
"Plus 3?"
"Yeah."
"Okay, E plus F equals 17 and then cancel, cancel. 2 F equals 26."
"F equals 13."
"13."
"So it's 13 here."
"13 there. And, want another one?"
"Yeah."
"Okay then – Oh! I think we're going to get a nice negative number."
"Oooh!"
"D equals minus 3 –"
"Negative 3 and E is then –"
"E is –"
"4," Stacey whispers.

Stacey and Kerry's use of systems of equations to solve particular triangles is an example of mechanical deduction. They are manipulating equations, without attaching meaning to the equations themselves. In lines 52, 53, 56, 58, and 59, the numbers 1, 2, and 3, refer not to quantities, but to the equations they are adding. The suspension of the meaning of these equations is such that there is no need to refer to what they are about. The process of solving the system depends entirely on the form of the equations.

The separation of their mechanical deduction from the original situation is also visible at the transition point when the mechanical deduction has produced an answer (line 65). They don't need to continue solving the system at this point in order to find the other two secret numbers, but Kerry does so anyway. The numbers he is generating are still meaning free for him, so continuing the mechanical deduction makes sense for him. Stacey, on the other hand, has been relating the answers back to the triangle, and in line 72 she gives Kerry the answer, having seen what it must be in the triangle. For both of them the situation was one of meaningless equations, as they engaged in mechanical deduction. Their structures, which enable them to reason in this way, contributed to the transformation of the situation from one about numbers on the sides of a triangle to one about meaningless equations. At the same time, it was the nature of the relationships in the problem which allowed them to be represented as equations, and particularly as a system of equations, occasioning Stacey and Kerry's mechanical deduction.

<u>Unformulated proving</u>

"Okay, so you want to solve that then?"
"Umhmm," Stacey responds but is deep in thought.
"Okay."
"It will go right to zero."
"Are you saying–"
"This I don't, I don't know."
"Are you saying the numbers would keep getting smaller?" Kerry was beginning to catch on to what Stacey was doing.
"Yeah, these would have to be-"
"Yeah I guess they will be getting smaller –"
"Like decimals. 'Cause you got 1 on one side." Stacey paused and then challenged Kerry, "Okay mister."

Because unformulated proving involves a lack of articulation and includes implicit assumptions, it is sometimes difficult to see what reasoning is actually taking place. In this section of transcript Stacey is proving, in an unformulated way, two conjectures about the Arithmagon situation. The first is that the numbers on the sides of the triangles, as she adds new ones to the outside, will approach a limit of zero. The second is that this will require that the numbers will include decimals. She expresses these conclusions in her cryptic comments "It will go right to zero. ... These would have to be like decimals."

In some unformulated proving she did earlier in the session she saw that the total of the known values (11+18+27=56) must be twice the total of the secret numbers (1+10+17=28). So the total for her new 1-10-17 triangle, must be half of the total for the original 11-18-27 triangle. This halving process is repeated for each new triangle added to the outside so that the limit "will go right to zero." But it is not only the limit of the sum which is going to zero. She is also saying that each of the numbers will approach a limit of zero. This follows from an implicit assumption that the numbers must be positive, as they have been in all the cases she has seen so far.

For the numbers to decrease to zero, decimal values must be used as some stage. The "1 on one side" suggests that stage must occur soon. Knowing that at some stage it "would have to be decimals," and suspecting that stage to be soon, she then turns the reasoning over to Kerry, as she needs some mechanical deduction done to find the actual values.

Stacey's unformulated proving put her in a certain relationship to the situation, and to Kerry. The presence of a limit in the problem situation was within the sphere of possibilities for it, but it was Stacey's reasoning which created the limit as a feature of the situation. Similarly, decimal values were not a part of the situation Stacey and Kerry were investigating, until Stacey's proving brought them into it. The unformulated nature of Stacey's proving also changed the way Kerry was involved in the situation, and with Stacey. He is in the position of accepting Stacey's conclusions, without having access to the reasoning which led to them, which limits the use he can make of them. Stacey also shifts the responsibility for solving systems of equations onto Kerry, leaving herself free to maintain contact with the meanings which are so important to her unformulated proving. This specialization of roles illustrates an important aspect of coemergence: that coemergence is not a process of structures becoming more alike. In fact, as in this case, coemergence can lead to divergence of structures, which nonetheless contributes to the ongoing relationship which permits the coemergence of the participants.

Final comments

In developing a language to talk about student' reasoning it has been an important part of my work to remain aware of the role reasoning plays in the coemergence of students and their situations. It is the structure of a student which makes their reasoning inductively, deductive, or in some other way, possible. At the same time it is the structure of the situation in which they find themselves which occasions the reasoning they do. Both the student's structure, and the structure of the situations, are changed by the reasoning which takes place, so that the student and situation coemerge through reasoning, at the same time the reasoning is a product of that coemergence.

A Portrait of Beliefs in Action

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Students who engage in mathematical problem settings do so not as blank slates but as persons with unique biological and phenomenological histories. In order to understand students' actions and interactions in these settings, it is important to consider not only what they do in these settings, or how they do something, but also, what it is of themselves that they bring to these settings based on their individual histories as mathematics students and as doers of mathematics.

From an enactivist point of view, one's history of interaction and one's *structure* determines, and at the same time is determined by, how one acts in a given setting and under various perturbations. Enactivism is based on this premise of *structural determinism*. The environment does not instruct or specify which particular changes will occur, rather, the person's interactions with the environment act as perturbations to trigger potential changes that are then determined by the living being's structure. Potentially, many paths of change are possible since whatever is not forbidden by the structure may be allowed; the path selected is ultimately an expression of the particular structural coherence of the living being. Therefore, as Varela (1987) suggests, in order to understand a unity's behavior (thus, a person's acts of cognition) an observer should consider the unity's structure.

A group of mathematics education researchers have been engaged in a project to study mathematical cognition from an enactivist perspective. This project is centered around a problem solving situation in which two undergraduate students (Stacey and Kerry) worked together on an open-ended mathematical problem (see Reid in this collection). This paper is only one discussion of the research and focuses on specific aspects of the students' structures as they were manifested in interactions between the students while they worked on the problem. This paper begins with a short discussion on structure and then introduces Stacey and Kerry by sharing parts of an interview which focused on their philosophies of mathematics. Finally, the paper attempts to demonstrate how the students' philosophies are not like those we find in books in the libraries and neither are they something in stored their heads; rather, their philosophies are lived-philosophies of mathematics and co-emerge with the students' understandings of mathematics when the students engage in mathematical activity.

Structure

The word structure is sometimes difficult to accept as suitable for talking about people in human science research. For those who are not familiar with this language, the word structure "feels" quite mechanical and static. However, for the biologist it is quite full of implicit meanings about living entities (Maturana and Varela, from whom the word is borrowed are biologists). In the biological sciences it is a common word and one that carries a significant amount of meaning. A living being's structure refers to the components and relations that constitute that particular living being (Maturana and Varela, 1987). These components are physical as well as relational. Further, this structure is plastic; that is, changeable. Thus, a living being's structure allows for ongoing structural changes to occur "either as a change triggered by interactions coming from the environment in which it exists or as a result of its internal dynamics" (pg. 74). The life history of the organism then, is a social and biological history of structural changes of that particular living being.

Human beings have complex structures and their social and biological histories are lived in every moment of their lives. Yet these histories are difficult to grasp or see in their totality, even for or in a given instant. Enactivist research acknowledges that structure is complex and elusive and recognizes that structure can only be understood in-action. Therefore, in enactivist research observations and analyses are made from many perspectives so that from the interaction of these many perspectives understandings of what it means to teach and learn mathematics might co-emerge.

A Philosophy of Mathematics as a Tenet of Structure

Recent research in mathematics education demonstrates the pervasiveness of beliefs, attitudes, and conceptions about the nature of mathematics and mathematics education in both teacher and student work in mathematics (Thompson, 1984; Lerman, 1989; Ruthven and Coe, 1994; Williams, 1993; Simmt, 1993). It is not surprising that beliefs, attitudes and conceptions about mathematics are so pervasive in teaching and learning mathematics. Enactivism suggests that this should be the case since human cognition is structure determined.

Mathematics students have beliefs, attitudes and conceptions about the nature of mathematics. These views might include beliefs about the nature of mathematical objects, the nature of mathematical truth, and/or the nature of mathematical processes. They also have conceptions about what it means to do mathematics and what constitutes appropriate mathematical behaviour. Each student's beliefs, attitudes, and conceptions about the nature of mathematics and doing mathematics comprise a personal philosophy of mathematics. This philosophy of mathematics is determined by the student's experiences (history) with mathematics, mathematics teachers, and fellow students and it co-emerges with the student's understanding of mathematics as the student brings forth his or her mathematical world. That is, not only does an experience reflect a student's philosophy but the experience informs and contributes to the philosophy as well. Students' philosophies of mathematics are unformulated and dynamic and exist in context. As students engage in mathematical activity they live their philosophies of mathematics and these philosophies can be observed in their actions.

Stacey and Kerry on Mathematics

Stacey and Kerry, the two students who participated in this project, were interviewed and asked about their beliefs, attitudes, and conceptions about mathematics and doing mathematics two months after working on the Arithmagon problem. This interview provided the opportunity to think about their interactions with the problem solving situation from the point of view of their personal beliefs about mathematics. In particular, two discussions came up in the interview which enrich the understanding of their actions in the problem situation. The first concerns how they see themselves and each other as mathematical problem-solvers and the second is their conceptions about the nature of mathematics. In order to give the reader a better sense of these two people other portions of the interview will also be reported.

Both Stacey and Kerry recognize that they do not approach mathematics the same way the other person does. Kerry commented about Stacey's approach to mathematics. "She guesses more than I do." And Stacey said about Kerry, "[He] always starts out with finding the formula." But Kerry has a reason for beginning in this fashion. He suggests, that in mathematics "there are systems you have to go through, step by step. There are rules." It makes sense to Kerry that when he does mathematics he should work in an organized way. After all, in mathematics there are processes to solve given problems and once these processes are determined then it is simply a matter of carrying them out.

Although they favor different approaches to work through mathematical problem situations, they do seem to appreciate the ways in which the other person works at given problems. Kerry recognized the contributions Stacey made to the problem situation. He gave her credit for sparking the session when it was slowing down. "Oh, we wouldn't be talking about that today even, if Stacey hadn't sat down and said, 'Well what happens if you extend the lines?' And that's what happened. She said, 'Let's look outside of this one triangle and just make a triangle around it.' And then all of a sudden that's what sparked us and got us going.... And then I figured out the mathematical relationship." Notice how he recognizes that his role in the interaction also helped keep the session going. (See Gordon Calvert for a discussion of the conversation.) Kerry appreciated Stacey's contributions to the problem solving activity, even though he did not do mathematics the way she did, nor did he always understand what she was doing. Kerry's contributions to the session were usually in the form of doing the mechanical work. (This aspect is discussed further in Reid.)

Stacey and Kerry were asked if they agreed with the statement, "Once a mathematical structure has been developed and a theorem formulated its proof is a technical detail." Kerry responded that this is the sort of thing that happened to them when they did the triangles (see transcript). "We were looking at those triangles and we figured out there was a relationship there. Then we figured out how we could formulate what the next side was going to be just by the relationship we developed. And after we came back the next day, David came up to us and showed us the proof. That was very impressive. But it is true it is a technical detail." As Kerry related, when Stacey asked, "what happens if ... that's what got us sparked and going. And once we got that, and she figured out that there. She got creative and figured out that there was something past that and then I figured out the mathematical relationship and then Dave came back and showed us the technical proof after." Both Stacey and Kerry agreed that the Arithmagon was not a geometry problem, yet they both acknowledge that Stacey's 'geometric' interpretation of this problem led them in a very interesting direction and that this diversion was dependent on the "geometric" form in which the original problem was presented.

They were asked to comment on the statement, "The general proof is a highly creative part of mathematics since it can lead to new structures, reformulated hypotheses ..." Kerry agreed that it could lead to new structures but, "reformulated hypotheses? That means to revise things that you previously thought? Well if you are revising them then obviously they were wrong anyway. And if they are wrong then they weren't real math anyway." Stacey said, "It seems to me there is only one right answer. I'm pretty sure that is how it is supposed to be – that's how math should be – just one answer." Kerry pointed out, "It's in the interpretation of mathematical results where you can get different answers."

Kerry talked about math this way. "Math is a big system out there. It is always – If you do it right you always come out with the same – the right answer. There is one right answer and it's a system and you use it." Stacey agreed with Kerry but she describe it differently. "I really see math as different games you can play – card games or whatever. And either you – and there are different rules to all of them and the answers are different you don't – I mean you don't question it unless you understand the rules. The rules are straight forward and that's just the way it works. But they are all different. I don't know if I am talking in circles and getting anywhere, but if there's a different system its just a different game."

One of the most interesting aspects of the interview came up in a discussion about the nature of mathematics and whether or not they felt mathematics was influenced by culture. Kerry responded, "Calculus was there before it was discovered. Even if it wasn't discovered it would still be there." But Stacey did not agree with him. "No. I don't think so," she replied. To try and sort this out they were asked if they thought mathematics was discovered or invented[7].

"It's discovered," asserted Kerry.
"I'd say it's invented," Stacey countered.
"You think its invented?"
"It's just try this and try that – so that could be discovery too."

[7]Davis and Hersh in The Mathematical Experience point to this distinction as "the philosophical plight of the working mathematician." "[T]he typical working mathematician is a Platonist on weekdays and a formalist on Sundays." (1980/1981, pg. 321)

"They don't, make it up they just discover it," Kerry insisted.

"They make it up and they add the rules."

"The rules are already there. Calculus already works. If you go before calculus was invented and you try to take the area underneath the graph you can still use calculus – before it was invented. It works. It's there already. It's not invented. It's already there."

These pieces of the interview provide different kinds of insights about Stacey and Kerry's beliefs, attitudes and conceptions about mathematics – insights that are not readily available by only listening and watching them work in the mathematical context. Yet the watching and listening seem to reveal these general beliefs. Their beliefs are not stored in their heads. They are, however, manifested as beliefs in action. First when they worked on the problem and again when they talked about their views of mathematics in the interview. To understand that representations (of beliefs, attitudes or conceptions) which correspond to some external reality are stored in a person's head in some computer like way is a misconception about cognition. There is a reality – but, it is the reality that is brought forth in action. Kerry and Stacey probably never even thought about mathematics as discovered or invented before this interview. Yet they believed those things as is evidenced in their actions on a number of occasions: in the interview and while working on the mathematical problem.

<u>Stacey and Kerry Beliefs in Action</u>

When students participate in mathematical settings, their prior experiences as students and doers of mathematics co-emerge with the context and inform their actions. It is in context that their lived philosophies (in part) are expressed. That is, their beliefs, attitudes, and conceptions about mathematics are manifested in action.

The following are pieces from the transcript found in the appendix

Episode 1:

Stacey and Kerry, two university students, were given the Arithmagon problem and they examined it silently for a few moments.
Finally Kerry began to speak, "I guess this is uh, — uh,—"
"Trial and error?" interrupted Stacey.
"No. We'll go — Let's assign variables to these then. Do you want —" . . .
"Go for it," encouraged Stacey.

Episode 5:

After having solved the original puzzle they then began to explore.
"Okay, this is what we have. 18, 11, and 27 and we're given three other numbers. Right?" Stacey paused, "What can we do with three other numbers? We can, extend lines—"
"What ya doing? Making another big triangle?" Kerry asked.
"Yeah. — I don't know what I'm doing yet." . . . "And we'll call— What's 11, 27? Right. What was this? 10?"
. . . Stacey and Kerry continue to transfer the numbers onto the extended drawing of the Arithmagon (see below) . . . Once the new problem is labeled, Stacey says, "You keep going."
"Where you going?"
"You keep going. We could find numbers for this," pointing to the vertices of the new triangle.
"For that?"
"Yeah."
"Okay, so you want to solve that then?"

Episode 6:

"Do you know what?" Stacey offered another idea.
"What?"
She paused and then said. "Okay, you want a prediction?
"Okay, sure."
"Well I don't know about this. This is just the one little step. This is decreased by 14. This is decreased by 14. And this is decreased by 14."
There was a short pause before Kerry asked, "What's that mean?"

In Episode 1 notice how Stacey immediately wants to do trial and error but Kerry wants to set this up as a system of equations. One possibility here is that Stacey does not immediately see that a system of equations will solve this problem whereas Kerry sees that it will. Although that may be true, it does not seem to be the only explanation. Stacey by her own admission likes to do trial and error as a means of problem solving. Further, when we look later in the vignette (Episodes 5 and 6) it is Stacey who pushes them on to attempt different things with the Arithmagon. She does not view this as simply a problem for which there is a single correct solution. Rather she views this context as one in which there are a variety of resolutions. Each time they come to some resolution she poses another question. She indicated she thinks of mathematics as a game - or rather many games. Clearly she is playing here. Although any particular game might come to an end that does not mean the playing is over. There is always the possibility of different games. Throughout this problem solving setting she is willing to explore and to invent new structures to play with.

Kerry, on the other hand, views mathematics as a single large system of algorithms, rules and answers. Consistent with his Platonistic view that mathematics is discovered, he looks for *the* appropriate way of solving the problem. Once he finds a method for solving the problem then he carries out the mechanical processes necessary to find the one, and only, solution. In this case, upon completing the system he acts as though the problem is finished. He does not ask what can they do next. Instead, he sits back and lets Stacey make conjectures and suggest new directions in which to explore. When some mechanical skills are required to push the exploration along, he obliges. He does listen to Stacey and make some attempt to understand what she is saying, but he does not seem to understand her motivation for asking the questions she does.

Conclusion

Each student has, as an aspect of his or her structure, a philosophy of mathematics. This philosophy of mathematics is a compilation of beliefs, attitudes and conceptions about mathematics and is significant to the student's mathematical cognition. However, the philosophy is not a well formulated nor fixed entity stored somewhere in the student's mind. Rather, it is unformulated and dynamic and exists in context. That is, it co-emerges with the student's understanding of mathematics as the student's mathematical world is brought forth. As the student engages in mathematical activity then, she or he lives her or his philosophy of mathematics and it can be witnessed in action.

The discussion in this paper leads to the suggestion that it is informative for researchers to focus on persons' beliefs about the nature of mathematics and what it means to do mathematics when trying to understand mathematical cognition; because, those beliefs are an aspect of a person's structure and because cognition is structure determined. Stacey's and Kerry's philosophies of mathematics are not simply a collection of justified beliefs or propositional statements. They are not theoretical arguments about the nature of mathematics. Rather, they are the underpinnings of Stacey and Kerry's actions when they live a mathematical experience.

A Pathway Portrait of Mathematical Understanding[8] in Inter-Action

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"...one can know a proof thoroughly and follow it step by step, and yet not <u>understand</u> what it was that was proved."
L. Wittgenstein, 1956 (1978, p. 282)

" 'understanding a mathematical proposition' - that is a very vague concept"
L. Wittgenstein, 1956(1978, p. 298)

"Wanderer..., you lay down a path in walking.
In walking you lay down a path."
A. Machado (quoted in Varela, 1987)

<u>Abstract</u>: This essay represents an attempt to portray the mathematical problem solving activity of Stacey and Kerry in terms of the changing understanding manifested in and by that activity. In so doing the Pirie/Kieren Dynamical Theory for the growth of mathematical understanding is invoked as a tool for such a portrayal. The paper reports part of a sequence of work through which we are attempting to consider mathematical understanding from an ontological perspective. Here both the personal structural dynamics and the interactional dynamics are featured. In particular, the changing understanding of Kerry and Stacey is revealed as a pathway in the modes, formal and informal , of understanding activity. There are several important features of this portrait: It reveals that if these students had stopped their activities at a "normal" terminus point - after two parallel formal processes of solving the problem, then they would have not laid down a path which led to many hours of novel mathematical activity for them and a "new" extension of the given problem. The "folding back", to which the previously described extension could be attributed, is a clear example of the theoretical construct of folding back in that while the folded back to activity of these students is clearly informal in its intent; it just as clearly uses the more formal understandings and actions which they had developed previously. Finally, this pathway portrait of mathematical understanding does not stand alone. It is informed by portraits of beliefs, reasoning patterns and inter-actional patterns of these students. This finding supports the contention that mathematical understanding in action and inter-action is a complex phenomenon.

<u>Background Remarks</u>

In studying the mathematical understanding of persons in action one can consider the general concern of Warren McCullough for what it is about mathematics which allows for its understanding by humans. In considering transitions in mathematical understanding of a person in terms of the over-coming of a sequence of (usually ever-more sophisticated) epistemological obstacles, Sierpinska (1991, 1995) has developed the grounds for one approach to this concern in an eloquent and elegant manner. She has provided a variety of examples of the kinds of changes in

[8] As noted below, this paper is a re-presentation based on one of a sequence of efforts to extend elaborate as well as test the Dynamical Theory for the Growth of Mathematical Understanding which Susan Pirie and I have been working on for many years. The theoretical ideas presented stem from the many interactions we have had and from the many writings we have done together. My interpretations of the data, although my responsibility, reflect much thinking that she and I have done together. Although she is not responsible for what is said in this paper, I am indebted to her for the many and continuing contributions she makes to the general program of work of which this paper is a part.

[9] Research underlying this paper has been supported in part by Grant 930239 of the Social Sciences and Humanities Research Council of Canada.

mathematical thinking and in one's mathematical knowledge (related to what Davis (1994) would term the body of mathematical knowledge) which would be necessary in facing the various obstacles which she describes. Because understanding involves overcoming obstacles, the rather simple step-by-step following of or producing of a mathematical product is, as suggested by Wittgenstein above, no grounds for observing a person's mathematical understanding.

If one can characterize Sierpinska's work as delineating the epistemological obstacles at the heart of mathematical understanding (and this is indeed an over-simplification), this paper represents research and interpretation which focuses on the on-going personal action of mathematical understanding (part of which is the "over-coming"). In so doing, this work can be thought of as a study related to McCullough's second concern – what in the nature of human beings allows them to understand mathematics?

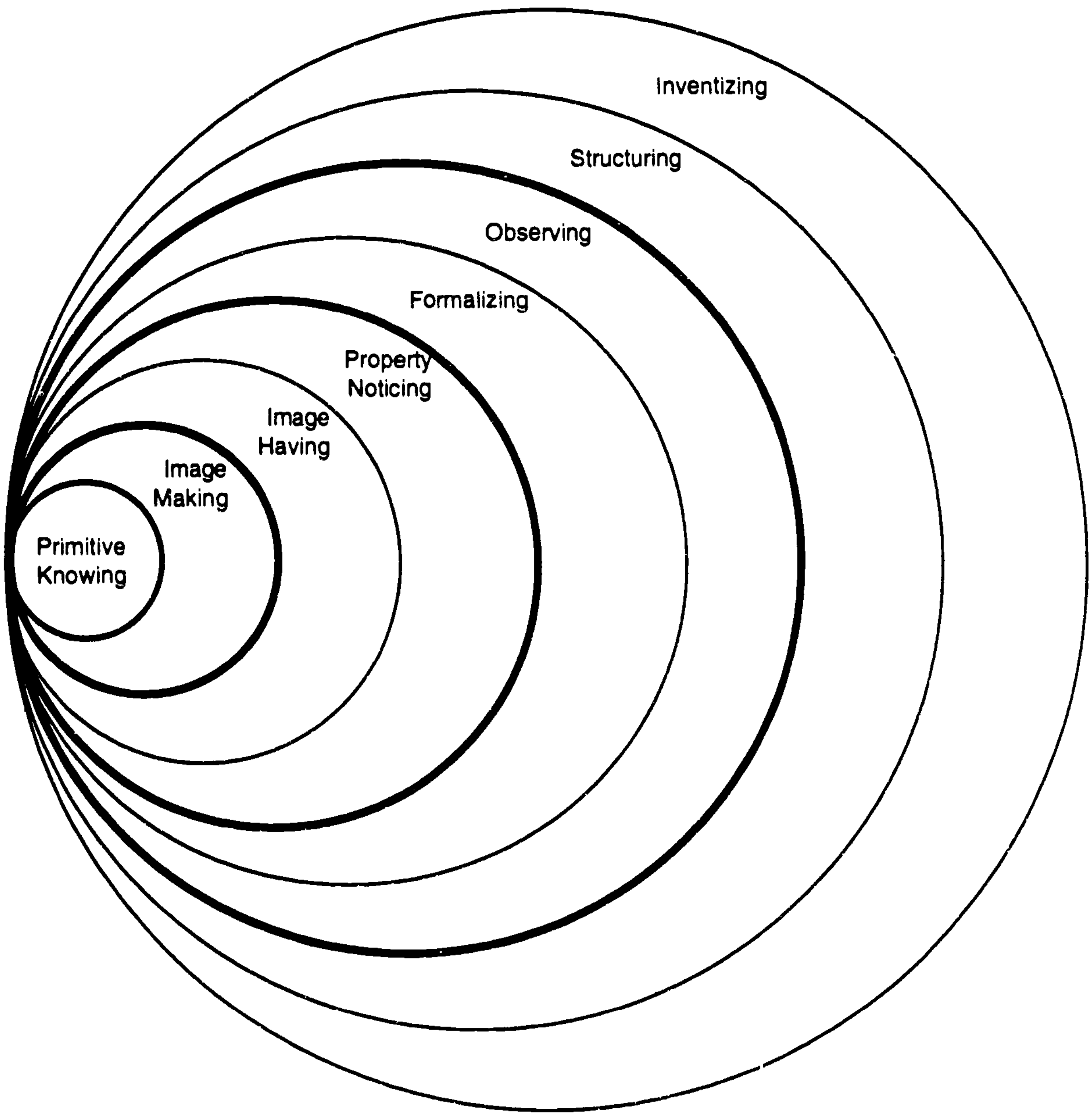

Figure 1

This paper, which follows and uses the interpretations of mathematical understanding found in Kieren, Pirie and Reid (1994), explicitly invokes the Dynamical Theory for the Growth of Mathematical Understanding which has been undergoing development by Pirie and Kieren (e.g., 1989, 1994) over the past seven years. There are numerous approaches to understanding as categories of traits and states useful in considering mathematical understanding as a personal acquisition (e.g., Skemp, 1987; Herscovics, 1992; Miller, Malone and Kandl, 1992). The Dynamical Theory, in keeping with the interest of this paper, is a tool for observing the on-going actions of students as a key source for knowledge of those students' mathematical understandings. Under its use understanding is viewed not as a state but as a dynamical, non-linear, non-monotonic process involving eight unfolding and enfolded modes of mathematical activity (see Figure 1).

The more formal, sophisticated and abstract modes of understanding of a person of a topic – Formalizing, Observing and Structuring – under the ideal, enfold less formal, more local modes of mathematical activity – Image Making, Image Having and Property Noticing. The growth of mathematical understanding of a person(s) with respect to a topic is portrayed by a non-linear pathway involving activity in the various modes as observed (see Figure 2 for the key pathway used in this paper).

It is explicit in this pathway oriented model that understanding is best observed in action; it is dynamic. This fits with the view of cognition – espoused by Maturana and Varela (1987); or Varela, Thompson and Rosch (1991) – as structurally determined action in a medium. In particular, the pathways of the dynamical model could be interpreted as a trace of part of the structural dynamics of the understanding person(s).

It is beyond the scope of this paper to even outline all of the features of the Dynamical Theory, which has been done in detail elsewhere (e.g., Pirie and Kieren, 1994). But three other features which are salient to the interpretations offered in this paper bear mentioning. One key feature of the growth of understanding as we observe it is that its pathway involves frequent **folding back** from more formal or general activity to inner, more local, image-oriented activity. But such folding back is not the same as naive, informal activity. A person folding back, say from formalizing to image making or image having, brings with her or him both the methods and results from the more formal level which they now use for different less formal intents.

A second feature ot the Dynamical theory is the "don't need" boundaries indicated by the bold rings interior to image having, formalizing and structuring on the model. Such boundaries indicate that the student whose understanding activity is observed to be "outside" a boundary may not feel the need for activities of an inner nature. For example, students who are acting to understand mathematics through formalizing see no need for inner image-oriented activity.

Finally, it is useful to think of the influence of actions of others in the environment on the understanding activity of a person. For example, a teacher (or another student, or even the person themselves) can act intending to provoke the understanding activity of the other, that is pushing them to outer level action. Similarly, a teacher can act with invocative intent, trying to get the other to "fold back". Of course, the teacher can simply seek validation of understanding as well. But as we have shown in other writing, it is the action of the understanding person, not the intent of the teacher, which determines the actual nature of the intervention As will be seen all three of these feature in the interpretation of mathematical activity offered below.

This brief discussion of the nature of interventions suggests that there are two dynamics which are important to understanding the growth of mathematical understanding. Maturana and Varela (1987) describe these as the structural dynamics, reflecting the changing nature of the structure of the person which guides action in a changing medium; and the inter-actional dynamics which suggests that understanding occurs in a coupling by the individual with an environment which can include other human actors such as teachers and fellow students. Because of concern for both of these dynamics of cognition, the pathway portrait developed below is best considered in its relationship to other portraits (of reasoning patterns, belief patterns, and conversational features) which serve to help us understand further the changing understanding portrayed by the pathway.

A note on the research itself

The partial transcript found in Appendix A, was developed as part of a research project called "Portraits of Mathematical Understanding". As one component of this project, some 30 pairs of university level students have been observed while they work on a particular problem or mathematical activity. In the particular activity discussed here there were two research observers present who would directly answer any question asked by the students, but who offered no other prompting. The two students, Stacey and Kerry, whose transcript is the object of study here, engaged in four such activities and were interviewed several times over the course of 18 months. Both students have studied a number of university level mathematics courses including a course in linear algebra.

While the current "standard" orientation of the project is to have each pair engage in three problem settings for as long as they wished and then participate in one interview session, in keeping with the bricological methodology outlined in one of the other papers here (Kieren, 1995) and by Reid (1995) these students were asked to and participated in a number of other task sessions and interviews as well. In addition, they worked on the problem discussed here on their own for many hours extending and elaborating the mathematics in both unexpected and unobserved ways. The research makes use of video records of all activities where possible, all written and graphic artifacts produced by the students, of transcripts of these videos, of observer field notes, of observer and research team conversations, of notes on tape viewings and of conversations with extended audiences viewing the research videos. Because each of the involved members of the research team brings a particular logic or logics of enquiry to this work, other interpretive activities arise as well. For example, the pathway portrait offered here is based on the Dynamical Theory discussed above. To facilitate the portrayal of the activity using pathways on our model, another bricological research tool, the mathematical activity trace, MAT, was developed. This trace (see Kieren, Pirie and Reid, 1994; Reid, 1995) is arrived at by breaking the transcript down into salient episodes (two of which are given here) and then creating capsule descriptions of the essential activities in each episode. The portrait below is then a product of the conceptual structure, the Dynamical Theory, brought to the task, the actual mathematical actions and inter-actions of the two students, and the interpretive acts of creating and ordering the mathematical activity trace.

Features of a pathway portrait

The attached partial transcript present two (and part of a third) of the ten episodes from Stacey and Kerry's work on the arithmagon problem. They are useful episodes for this discussion of a pathway portrait in that they reveal many aspects which can be observed using the Dynamical Theory and are reflective of key aspects of the growth of mathematical understanding observed over the course of the 80 minute problem solving session. (And likely reflective of the growing understanding of this problem setting which occurred as these two students worked, unprovoked or invoked, and unobserved on their own for many hours on other "arithmagons".) Figure 2 below is an adaptation from Kieren, Pirie and Reid (1994) where the whole pathway is discussed.

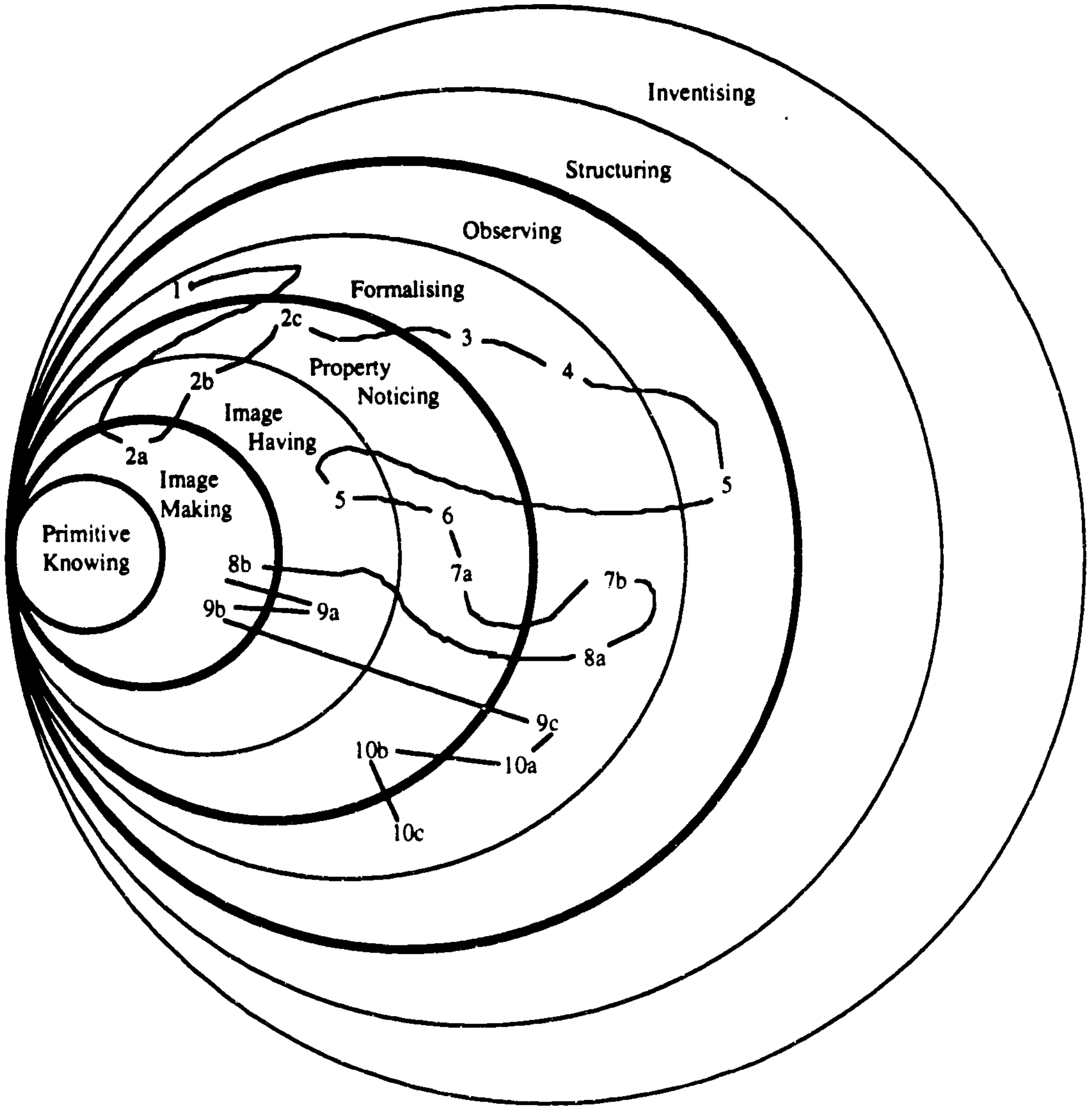

Figure 2

In general, this pathway portrait shows the non-linear, non-monotonic nature of the Stacey and Kerry's changing understanding of the arithmagon problem. From looking at episodes 1,5 and 6 (1, 5 and 6 on Figure 2) we can observed instances of several of the modes from the Dynamical Theory. Take for instance the following interchange:

Finally Kerry began to speak, "I guess this is uh,---uh,-"
"Trial and error", interrupted Stacey.
"No. We'll go- Let's assign variables to these then. Do you want –"..
"Go for it!" encouraged Stacey.
"We'll start with A, B and C"..."And I guess we'll get a system of equations out of that. And we'll try and– And we'll subtract one equation from another and try to deduce one variable and plug it back in. OK, so that's the sum of A and B.
"Yup." Stacey helped as Kerry started writing equations.

There are two features evident in this snippet of dialogue. First we have an obvious example of both the method applying and the method justifying activities of **Formalizing** (lines 7-11 and following). In the brief encounter in lines 3 - 6 we can see Kerry **provoking** them to engage in more formal activity. While anyone who tries the arithmagon problem can see, Stacey's informal local trial and error **Image Making invocation** actually does give an answer to the problem (usually in short order). But Kerry, perhaps applying earlier formalizing understanding of linear systems, provokes them to engaging in **formalizing.** It is clear that Kerry, while answer seeking in his intent, also sees this problem as one of a larger class of problems – linear systems – and he asks them to use that methodology to solve it.[10]

In episodes 3 and 4 in which Kerry pushes them to check their earlier "answer" by now representing the problem using matrices, and then using row reduction formalizing both in the particular problem and a more general arithmagon. At this point under a conventional view of mathematical problem solving many students (and teachers) would consider themselves "done" – they had solved the problem, checked it another way, and observed that it was only one of a class of problems which would yield to similar methods. But now Stacey acts in a way which proves to be **invocative**:

After having solved the original puzzle they then began to explore.
"Okay, this is what we have. 18, 11, and 27 and we re given three other numbers. Right?" Stacey paused, "What can we do with three other numbers? We can, extend lines—"
"What'ya doing? Making another big triangle?" Kerry asked.
"Yeah. — I don't know what I'm doing yet." . . . "And we'll call— What's 11, 27? Right. This one. (I)
What was this? 10?"
. . . Stacey and Kerry continue to transfer the numbers onto the extended drawing of the arithmagon (see below) . . . Once the new problem is labeled, Stacey says, **"You keep going." (II)**
<u>"Where you going?"</u> (KI)
"You keep going. We could find numbers for this," pointing to the vertices of the new triangle.
"For that?"
"Yeah."
"Okay, so you want to solve that then?"
"Umhmm," Stacey responds but is deep in thought.
"Okay."
"It will go right to zero."
"Are you saying-"
"This I don't, I don't know." (III)
<u>"Are you saying the numbers would keep getting smaller?" Kerry was beginning to catch on to what</u>
<u>Stacey was doing</u>. (KII)
"Yeah, these would have to be-"
<u>"Yeah I guess they will be getting smaller –"</u> (KII)
"Like decimals. 'Cause you got 1 on one side." Stacey paused and then challenged Kerry, "Okay mister."
"You, want, you want to try to solve it then?"

---

[10] As Wittgenstein argued such even justified answer seeking is not the hallmark of understanding. Although not found in the attached transcript, episode 2 allows us to observe Stacey invoking folding back to action on the specific triangle and getting an image in terms of its features - relationships between the unknown vertices and given side numbers.

"Yeah sure. Away you go."
"Okay then, let's label it.
"I like drawing it"
"Okay, we'll label these anyway. So we've gone. So go. D, E –"
"and F." (IV)

In lines 15-19 (I above) Stacey is proposing that they fold back to trying to create a new or add to the image of arithmagons. That such image making and image extending is almost literal in nature can be seen in the drawing which Stacey makes. She is making (and continues to make throughout their many hours of work on the problem) use of the "geometric" nature of the problem. Just as we saw that both Kerry and Stacey saw the problem as linear algebraic in nature before, Stacey now tries to capitalize on the triangular aspect of the problem. Of course, Pirie and I have argued in other places that such mental/verbal play does not mean that Stacey either has a new image or has extended her old one. She senses this as well in line 16 and later in lines 28-32 (II) as she now notes without expressing it that if a side number of her new triangle is 1 then the vertex numbers enclosing it (an in succeeding triangular extensions) must be smaller than one (and indeed in the limit approaching zero). The fact that this idea is in flux is signaled by "This I don't, I don't know."(line 32)

What is interesting is that Stacey is generating an informal image of the arithmagon problem which is at once very powerful but "wrong". This rather simple invocative sequence (Episode 5 in Figure 2) in total, leads to very extensive continuing mathematical activity of the part of this pair, when it might be expected that they would have stopped after their elaborated solution to the original problem. It could be argued that this folding back resulted in greatly extended understanding by Kerry and Stacey. But the specific initial new feature of the problem image that Stacey comes up with does not prove to be "true" in the long run. Thus it could be said that while folding back led to new understanding, there is no guarantee that such new informal understanding will prove to be true.

A second feature of this folded back to understanding activity is that the informal activity now carries with it the formal linear algebraic methods and results. This is seen in lines 40-85 of the appended transcript and indicated by (IV) above. But just as the folded back to informal activity shown in Episode 5 has contained within it, formal action, the formal action here takes on a different character. While in Episode 1 (or 3 or 4 in the full transcript) the formalizing was done for its own sake or the sake of "answer" and even a general answer, the activity in Episode 5 (and in 7-10) including the formal algebraic activity is done to support more local explorational activity which led to this pair establishing an extended image for arithmagons, indicated by Kerry's remark "We **did** get a **negative** number."

This transcript piece also contains a nice example of understanding activity aimed at developing a new property of arithmagons or of the relationship among nested arithmagons. This is seen in the conversation snippet from Episode 6 illustrated below.

She paused and then said. "Okay, you want a prediction?
"Okay, sure."
"Well I don't know about this. This is just the one little step. This is decreased by 14. This is decreased by 14. And this is decreased by 14."
There was a short pause before Kerry asked, "What's that mean?"
"From here to here." Stacey said pointing to the diagram.
"I see that. That's –"
"Yeah."
"– pretty neat."

This **property noticing** activity is announced by Stacey in line 86. Even though this first prediction did not stand up it led to a very long path of property noticing and expressing in which both students engaged actively and which greatly extended Stacey and Kerry's pathway of growth.

It has been the purpose of this section of the essay to do two things. First, the pathway portrait was presented and modes of understanding drawn from the Dynamical theory were illustrated (formalizing, image making, image having and property noticing activities). Second the concept of, internal features of, and consequences of folding back were illustrated as well. From these one can see that folding back is indeed one way by which students can and do extend their mathematical understanding in the ontological sense of coming to act differently in a medium. But this folding back activity also extended the mathematics of the arithmagon problem, at least for these students. From an enactive view, Stacey and Kerry were bringing forth a piece of a mathematical world together but at the same time their own mathematical thinking structures were being modified as well.

Whose understanding anyway?

The pathway portrait whose features are discussed above is taken to be our re-presentation of our interpretation of the growth of Kerry and Stacey's mathematical understanding. The Dynamical Theory, which was developed around the idea of a single person's changing understanding, was used as a tool in developing this portrait. Two questions could be asked: "Should there really be two independent paths of growth? What is obscured by presenting a single path in the portrait?"

It is beyond the scope of this paper to answer such questions in detail. But first let me say that thinking of two person growing in mathematical understanding or building a mathematical world together is congruent with the notion of Maturana and Varela (1987) of cognition as coming to live in a world with others. At a very minimum the transcript above suggests a coordination of actions between Stacey and Kerry. While Kerry takes the lead in most of the formalizing and formal activities, it is clear from the text that Stacey is a knowledgeable and active partner. But there is more to it than that. One of the key ideas of the pathway portrait is folding back. Because she appears to invoke this behaviour, it might be thought that while Stacey's growing understanding involves this folding back to extensive informal understanding activity, Kerry's does not. But the transcript suggests otherwise. Initially Kerry was not involved in informal thinking as witnessed by his questions, "Where **you** going?" in line 23 (KI) for example. But clearly by line 34-36 (KII) Kerry is now involved in the image-oriented activity and continues his active co-involvement with Stacey over the next several hours in such informal understanding acts. It is clear that Episodes 5 and 6 as illustrated in the pathway portrait can re-present Kerry's changing understanding as well as Stacey's. Even if Kerry is the lead actor in the formalizing events - e.g. episodes 1,3 and 4 in Figure 2, Stacey is an active participant in them. Similarly, if Stacey's actions provide the occasion for her informal understanding acts and Kerry's as well, Kerry is acting in his own structure determined way to also engage in these local image-oriented understanding activities.

The relationship of the Pathway Portrait to the other portraits

It has been the central purpose of this paper to develop a portrait of the changing understanding of Stacey and Kerry as they worked on the Arithmagon problem and to try to suggest some consequences of various features of this dynamical process. Because this is a portrait of cognition in action, principles of enactivism prompt us to ask two questions: What are features of Stacey's and Kerry's structures that relate to aspects of the Pathway Portrait? Because this portrait purports to re-present an interpretation of a dynamical process, are there other aspects of inter-actional dynamics which support it?

It is beyond the scope of this paper and even the research as conducted to date to provide a very complete response to such questions. As Sumara (1994) has suggested, an enactive view of cognition is necessarily a complex one; there are a wide variety of factors which effect the cognition and these factors effect it "allatonce". Thus it may, in theory, be impossible to provide a complete response to the above questions. But the three other portraits discussed in this paper set can and do offer both ideas and further questions which relate to the questions above.

Stacey and Kerry appeared to have well developed and articulated beliefs about the nature of mathematics (Simmt, this collection)[11]. Kerry believed that mathematics and its methods were "out there" to be known and that there were defined ways to go about the activity ("rules"). On the other hand Stacey was more ambivalent, but thought of mathematical activity as being inventive - mathematics was a human construction. While we did not test this, such beliefs are likely a product of each of their past histories of mathematical activity. Even a cursory look at the transcript reveals that they have very different approaches to mathematical action. But where their beliefs, part of their lived historic structures, most affect the pathway portrait lies in the kinds of mathematical actions that each of them pushes for and occasions. This is most evident in lines 5 and 17-37. In the former, Kerry stops an attempt at informal activity and sets the stage for formal work to find an answer that he knows is "out there". In other words Kerry connects this problem to already known formal mathematics and sees their mathematical activity as being about furthering that connection. He provokes Formalizing. On the other hand in lines 17-37, Stacey's notion that mathematics is invented ("they just try things", see Simmt) is very evident. She does not know where her exploration is going or even its exact character. But her persistent following of her beliefs acts to invoke both of them to very extensive inner mode understanding activity – she invokes image-oriented local exploration.

The Pathway Portrait and these beliefs appear to be inextricably connected to reasoning patterns, capabilities and felt needs of the person, that is other elements of a person's structure through which are manifested in their actions in environments (Reid, this collection). As described by Reid, the mechanical deduction, led by Kerry, plays a key role in this work especially in Episode 1 (but is part of more exploratory work). In terms of understanding, mechanical deduction might be one key reasoning pattern associated with formalizing, the definition of which suggests that it entails a definable and justifiable method. Thus considering the patterns discussed in Reid's reasoning portrait can help inform the development of a portrait of understanding using the Dynamical Theory. As we have argued elsewhere (Pirie and Kieren, 1994) any mode of understanding activity will be associated with appropriate ways of justifying ones actions. Thus it is not surprising to associate informal deductions and proving with informal modes of understanding both theoretically and in the activity such as is shown in Episode 5 in the attached transcript and identified on the model in Figure 2 with "5". But the inter-portrait implications go both directions. The pathway portrait itself suggests that there is an interaction between the needs for "proving" actions in Reid's portrait and the character of proving activities. Thus it might be argued that the mechanical deduction, aimed at finding "the answer" (a kind of verifying by matching) in Episode 1 (or 3 and 4) is different from the mechanical deduction for answer-producing aspects of Episode 5. It is clear that the quest of mechanical deduction in Episode 5 is not for "the" answer but for useful mathematical information to aid in the exploration. In Episode 1 the two students proceed in a very mechanical way, whereas even the "mechanical" activities in Episode 5 are interrupted and added to by less formal deductive work (e.g. Stacey's input in lines 70-75).

Finally, it is important to note that the growth in understanding portrayed here took place in a conversation between peers and friends. Gordon Calvert (this collection) identifies coordination

[11] It is important to note that Simmt's interviews which form the basis of her interpretations of Stacey and Kerry's beliefs occurred several months after their work illustrated in the transcript. It seems unlikely that either of them had discussed their philosophies with other before their work on the Arithmagon.

of actions, expression of rational-emotional responses, expressions which propel the conversation onto unknown paths, and conjecture making/following-accepting as features of this mathematical conversation which have implications for the pathway portrait. It is clear that the last two of these features are important both to folding back and to provoking outward movements in the model (lines 16-17, 82,97). That is, elements of conversation appear to be important in providing occasions or spaces for growth in mathematical understanding.

Closing remarks

The pathway portrait developed here is one way in which to look particularly at the structural dynamics which underlie mathematical activity. Such portraits illustrate ways in which teacher/researchers can "listen" to on-going mathematical activity of students and through such listening better understand that activity and perhaps be informed by it as to how better they might provide the next occasions for learning. The collection of portraits and their "inter-actions" remind us of two considerations in observing mathematics in action. The first is that this activity is best viewed as a complex phenomenon but one in which patterns or order can be discerned. Finally these portraits and inter-actions suggest that personal and interpersonal mathematical cognition need not be seen as simply discrete problem solving, but as a way or part of living - an entering into a world of shared significance.

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Appendix A: Stacey and Kerry: A Snapshot

Transcripts from episodes 1, 5 and 6

Episode 1:

Stacey and Kerry, two university students, were given the Arithmagon problem and they examined it silently for a few moments.

Finally Kerry began to speak, "I guess this is uh, — uh,—"

"Trial and error?" interrupted Stacey.

"No. We'll go — Let's assign variables to these then. Do you want —" . . .

"Go for it," encouraged Stacey.

"We'll start with A, B, and C." . . . "And I guess we'll probably get a system of equations out of that. And we'll try and — And we'll subtract one equation from the other. And try and deduce one variable, plug it back in. Okay, so that's the sum of A and B."

"Yup." Stacey helped as Kerry started writing equations. "A and B is 11. B and C is 18. And A and C is 27 —"

"Okay, these are our three formulas. Our three equations."

Kerry did the writing but consulted with Stacey as he solved the system.

Episode 5:

After finding a solution to the original puzzle they then began to explore.

"Okay, this is what we have. 18, 11, and 27 and we're given three other numbers. Right?"

Stacey paused, "What can we do with three other numbers? We can, extend lines—"

"What'ya doing? Making another big triangle?" Kerry asked.

"Yeah—I don't know what I'm doing yet ... And we'll call—What's 11, 27? Right. This one."

"What was this? 10?"

. . . Stacey and Kerry continue to transfer the numbers onto the extended drawing of the arithmagon . . . Once the new problem is labelled, Stacey says, "You keep going."

"Where you going?"

"You keep going. We could find numbers for this," pointing to the vertices of the new triangle.

"For that?"

"Yeah."

"Okay, so you want to solve that then?"

"Umhmm," Stacey responds but is deep in thought.

"Okay."

"It will go right to zero."

"Are you saying–"

"This I don't, I don't know."

"Are you saying the numbers would keep getting smaller?" Kerry was beginning to catch on to what Stacey was doing.

"Yeah, these would have to be-"

"Yeah I guess they will be getting smaller –"

"Like decimals. 'Cause you got 1 on one side." Stacey paused and then challenged Kerry, "Okay mister."

"You, want, you want to try to solve it then?"

"Yeah sure. Away you go."

"Okay then, let's label it.

"I like drawing it"

"Okay, we'll label these anyway. So we've gone. So go. D, E –"

"and F."

"F. Okay. D plus F equals 10. D plus E equals 1. E -"

"E plus F."

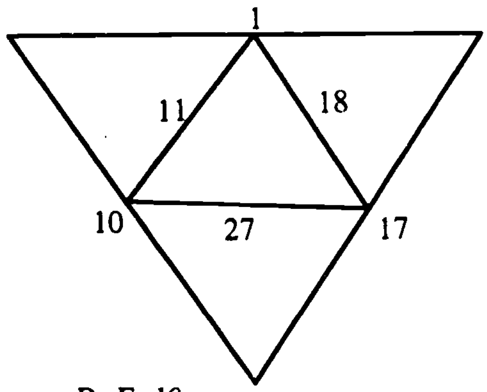
"Plus F."
"17."
"Equals 17 — Okay, so uh –"
"Do my – Oh! How are we doing here?"
"Okay, we'll subtract, we'll take uh –"
"1 minus 2."
"1 minus 2. So we've got –"
"F minus E is 9," they say in unison.
Kerry continues, "And then we'll go –"
"That plus 3, " Stacey corrected Kerry.
"Hmm?"
"Plus 3."
"Plus 3?"
"Yeah."
"Okay, E plus F equals 17 and then cancel, cancel. 2 F equals 26."
"F equals 13."
"13."
"So it's 13 here."
"13 there. And, want another one?"
"Yeah."
"Okay then – Oh! I think we're going to get a nice negative number."
"Oooh!"
"D equals minus 3 –"
"Negative 3 and E is then –"
"E is –"
"4," Stacey whispers.
"4?" Kerry whispers back to her.
"Sorry. 4," laughed Stacey.
"Okay, so we've got – We **did** get a negative number."
"Oooh! Let's keep going!"
"Okay –"
"This is negative 3. This is 4. And they join each other."
"You didn't drop this low enough. Soon you'll need a huge sheet of paper."
"Okay." Stacey hesitated. The figure had grown bigger than the paper.
"Pretend the corner's there," Kerry suggested to her.
"That was 4. And this is – 13."
"Okay, G, H, and I. – So I plus G. This equals negative 3. G plus H equals 4. H plus I equals –"

Episode 6:
"Do you know what?" Stacey offered another idea.
"What?"
She paused and then said. "Okay you want a prediction?
"Okay, sure."
"Well I don't know about this. This is just the one little step. This is decreased by 14. This is decreased by 14. And this is decreased by 14."
There was a short pause before Kerry asked, "What's that mean?"
"From here to here." Stacey said pointing to the diagram.
"I see that. That's –"
"Yeah."
"– pretty neat."
"Yeah."
"Yes."

Stacey told Kerry, "Keep going. Does that mean this will decrease by 14? For this line here?"
"Okay, for H – well."
"Is that a prediction that H is 3? – Go for it."
Kerry then continued to solve for H, and found it to equal 10, thus falsifying Stacey's prediction. Although the prediction was not correct it did act as a trigger for more predictions.

Appendix B: Stacey and Kerry's writing in episodes 5 and 6.

"Okay, this is what we have. 18, 11, and 27 and we're given three other numbers. Right?" Stacey paused, "What can we do with three other numbers? We can, extend lines—"



45 "F. Okay. D plus F equals 10. D plus E equals 1. E -"

D+F=10
D+E=1
E+F=17

"F minus E is 9," they say in unison.

F-E=9

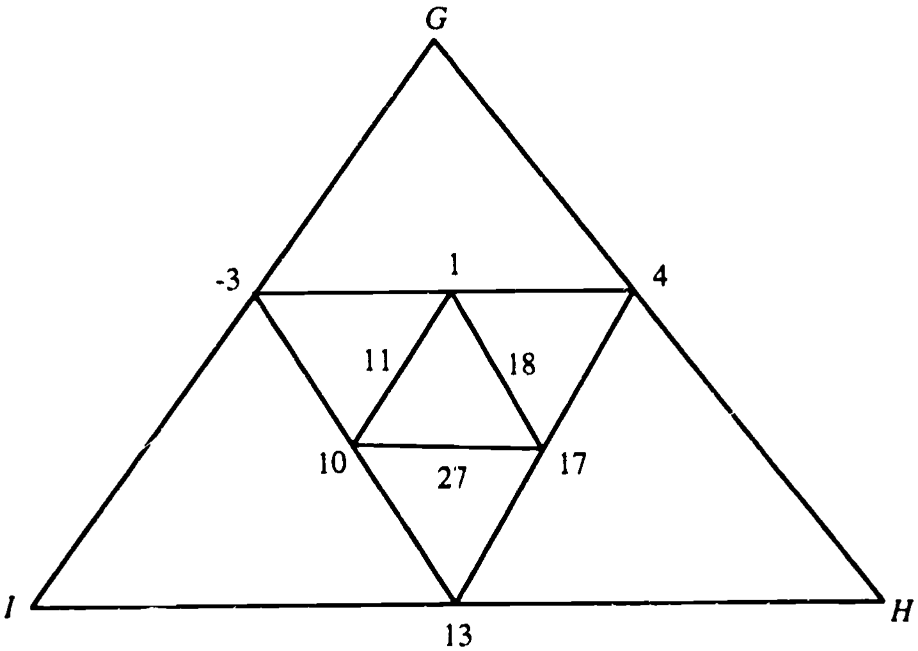
"Okay, E plus F equals 17 and then cancel, cancel. 2 F equals 26."
"F equals 13."

E+F=17
2F=26
F=13

"D equals minus 3 –"

D=-3

80 "Okay." Stacey hesitated. The figure had grown bigger than the paper.
"Pretend the corner's there," Kerry suggested to her.
"That was 4. And this is – 13."
"Okay, G, H, and I. – So I plus G. This equals negative 3. G plus H equals 4. H plus I equals –"



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