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ABSTRACT

This paper describes two uses of the software package TEMATH (Tools for Exploring Mathematics) with calculus students: (1) as a demonstration tool in the classroom to visually explore with students the many mathematical models introduced in a first year calculus course; and (2) as a part of a lab where students use a set of laboratory explorations to experiment on their own and write down their findings, interpretations, and conclusions. Guidelines for presenting a successful classroom computer demonstration are given, as well as examples of assigned explorations and activities. These activities include sequences of secant lines, parametric equations, using real data, modeling the men's summer olympics gold medal long jump distances, numerical integration, the derivative as a function, and Taylor polynomial approximations to the circle. (MKR)

Our Experiences with Using Visualization Tools In Teaching Calculus

by Robert E. Kowalczyk
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OUR EXPERIENCES WITH USING VISUALIZATION TOOLS IN TEACHING CALCULUS

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TEMATH (Tools for Exploring Mathematics) is a mathematics software package containing a set of easy-to-use computational and visual tools. It's design takes full advantage of the Apple Macintosh® interface and it encourages the user to freely explore and visualize the important concepts of calculus. We use *TEMATH* in two different ways with our calculus students: (1) as a demonstration tool in the classroom to visually explore with the students the many mathematical models introduced in a first year calculus course, and (2) as part of a lab where the students use a set of laboratory explorations to explore and experiment on their own and to write down their findings, interpretations, and conclusions.

Computer classroom demonstrations can be a great success or they can be a disaster. If your typical computer demonstration consists of: setting-up the equipment, turning down the lights, typing in long expressions or commands, not knowing all the intricacies of the software you are using, not working from a well prepared script, talking the entire period with little or no student input, then your students will be asleep within 15 minutes. From our experiences over the past years, we have come up with the following guidelines for presenting a successful classroom computer demonstration:

- Have all your computer equipment set up before the class begins. Nobody enjoys watching you fumble with equipment.
- Have a well rehearsed script prepared. Don't try to "wing" the presentation. Technology is always full of unexpected surprises. But do deviate from your script to answer students' "What if" questions.
- Know the software that you are using. There is nothing worse than trying to figure out how to do something on the computer while a classroom full of students anxiously look on.
- Have files prepared ahead of time. Students get bored watching you enter one long expression after another.
- Select software that allows you to increase the font size so that it is visible and readable by all.
- Design your demonstration to include lots of questions for students to answer. Make them an active part of your demonstration.
- Use lots of visual tools and graphics. Screen after screen of text and symbolic expressions is tedious.

Outside the classroom, we assign our students computer lab explorations which they complete on their own. We usually assign 5-6 labs per semester and we allow 1-2 weeks to complete each lab. The major objective of a laboratory exploration is to have the students use the computer to perform a visual experiment which leads them through the development of a mathematical concept. This can be especially important since many students do not read through the concept development given in their text books. We have designed these explorations so that students will:

- take a more active role in their own learning
- take a more worldly approach to the learning of mathematics by using math to model real life phenomena
- become active experimenters by performing experiments to generate or gather data that can be modeled mathematically
- write about their discoveries
- make mathematical conjectures based on their findings

As an example of the pedagogical philosophy that we use in these explorations, we will briefly discuss one of the explorations that we assign to our first year calculus students. This particular exploration has the students use a sequence of secant lines to develop a definition for the tangent line. In the first part of the exploration, the students graph the quadratic function $f(x)=5-x^2$ and then overlay on this graph the sequence of secant lines passing through the points $(x_0, f(x_0))$ and $(x_0+h, f(x_0)+h)$ where $x_0 = 1$ and $h = 1, 0.1, 0.01$, and 0.001 as is shown in Figure 1.

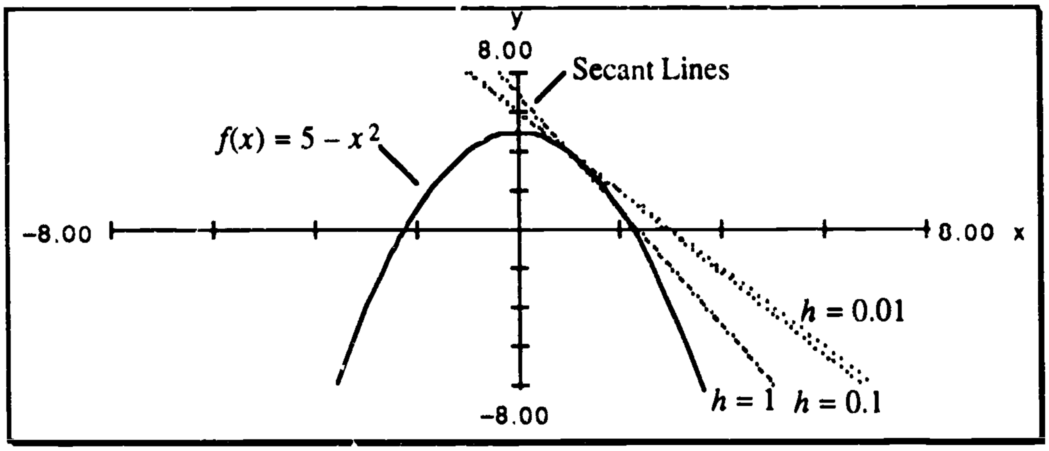


Figure 1 Secant Lines Approaching the Tangent Line

The students then repeat this process for the negative values of $h = -1, -0.1, -0.01$, and -0.001. At this point in the exploration, we tell the students to call the line that these secant lines are approaching the *tangent line* and we ask them to use words to give an accurate definition for the *tangent line* to the curve $y = f(x)$ at the point $(x_0, f(x_0))$.)ne of our student's answer to this question was

> *"The tangent line only intersects the curve at the point $(x_0, f(x_0))$ and nowhere else. It represents the slope at that point on the graph. A rate of change of the curve, you could say, at that point."*

We next have the students test their definition on some other examples and at the same time ask if they want to change their definition of the tangent line. The first example test function we use is $f(x)=|1-x^2|$. Again, we ask the students to graph $f(x)$ and to overlay the two sequences of secant lines as is shown in Figure 2.

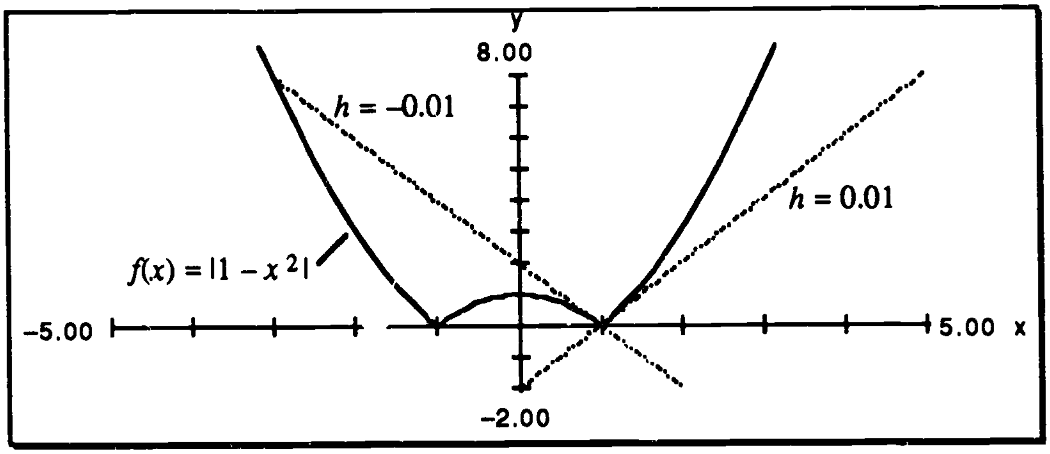


Figure 2 Testing the Definition of the Tangent Line

The students are now asked if there is a tangent line at the point (1, 0). The same student quoted above answered *no* and when asked if his definition for the tangent line held for this example, he also answered *no*. When asked to give a new definition, he gave the following:

> *"The slope of the tangent line has to be the limit of the slope function as h approaches 0. The line still must only touch the graph at one point. If the limit does not exist, you cannot find the tangent line so it does not exist."*

The slope function this student refers to is the function entered into *TEMATH* that finds the slope of the secant lines. Notice how this student has rethought his definition of the tangent and has finally used the idea of a limiting process.

The last example we give the students is the function $f(x) = (x-1)^{1/3}$. Again, we ask the students to graph $f(x)$ and to overlay the sequences of secant lines as is shown in Figure 3.

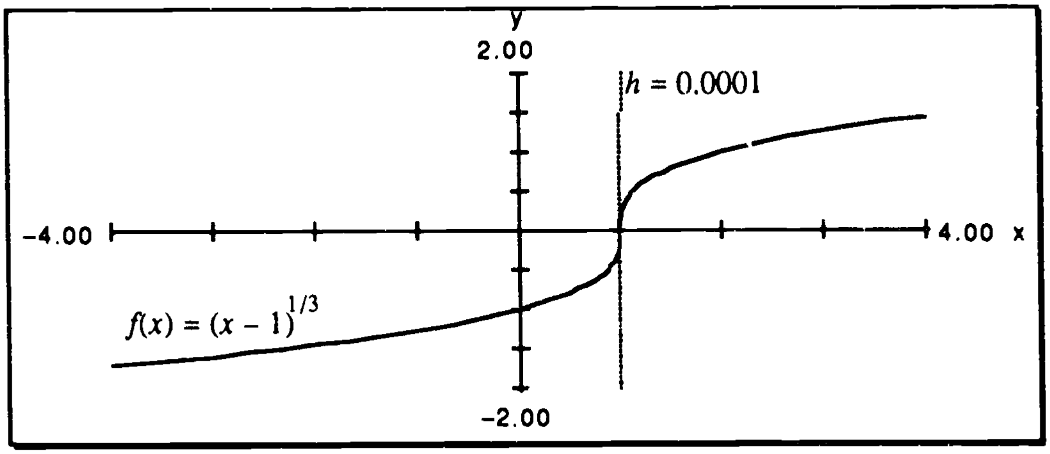


Figure 3 A Vertical Tangent Line

The same student quoted above wrote that this example function does not have a tangent line at the point (1, 0) because

> *"It appears to have a tangent line but the slope would be +∞ and the tangent line would be vertical. Also, the tangent would seem to pass through the graph."*

and he revised his definition of the tangent line by adding

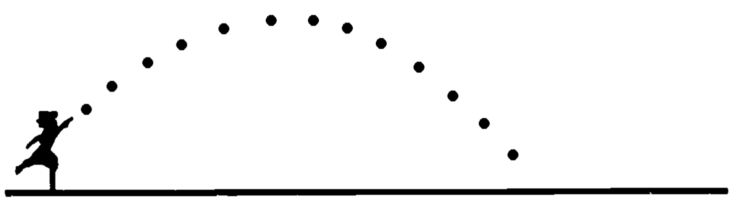
> *"The slope of the tangent line must exist as a real number. That means it must have a two sided limit and cannot be ±∞."*

As you can see from this particular student's responses, we successfully got him to seriously think about the definition of a tangent line. Our goal in this lab was to have the student perform a visual experiment, describe in writing what he or she had observed, make a conclusion, test the conclusion with other examples, and refine his or her conclusion with the information obtained from the additional examples. We are trying to get our students to take an active part in the development of the mathematical concepts in calculus and to be able to describe these concepts in writing.

We now give some examples of classroom computer activities that we have used successfully during the past years.

Parametric Equations

When we teach our students about parametric equations, we try to use examples that can be easily visualized by the student. One such example is that of throwing a ball.



The first part of this demonstration consists of having our students derive the equations of motion for the horizontal and vertical distance the ball has traveled. Using the parameters h_0 = height of the ball when released, v_0 = the initial velocity of the ball, θ = the angle the ball is thrown as measured from the horizontal, and g = the acceleration of gravity, the equations of motion are given as

$$x(t) = v_0 \cos(\theta)t \quad \text{and} \quad y(t) = h_0 + v_0 \sin(\theta)t - \left(\frac{g}{2}\right)t^2$$

We now pose such questions as: "At what angle should you throw the ball to obtain the maximum horizontal distance?", "At what time does the ball attain its maximum height?", and "What is the maximum height the ball achieves?". To get approximate answers to these questions, we use *TEMATH*'s Parametric Tracker tool to visualize the ball throwing experiment for various values of the parameters. For example, we can experiment to see how the maximum horizontal distance the ball is thrown varies with the angle the ball is thrown. If we let $\theta = \pi/6, \pi/4,$ and $\pi/3$, *TEMATH* can track the trajectories for the flight of the ball for all three angles as is shown in Figure 4.

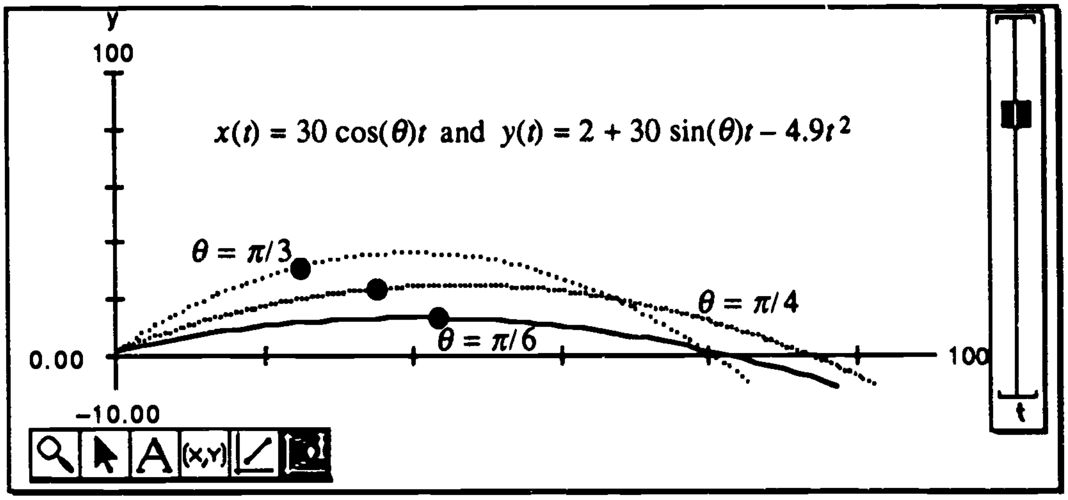


Figure 4 Parametric Equations for Throwing a Ball

As you drag the slide box at the right, the balls move along their trajectories and the values of t, x, and y for a selected trajectory are shown in the Domain and Range window (see Figure 5).

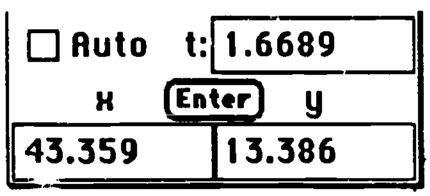


Figure 5 Coordinate Values for a Point on the Trajectory

With this type of dynamic visualization of the ball throwing experiment, the students can gather data and intuition about the experiment and, thus, better understand the mathematics involved.

Using Real Data

We are constantly looking for rich data sets that we can use to stimulate our students' interest in using mathematics to model data. One data set that we have used is the distances of the summer Olympics gold medal performances in the men's and women's long jump event. Most almanacs contain this type of data. We begin this computer activity by first plotting the men's distances and then asking the students to describe the data. For example, we ask them to comment on the shape of the data (linear, quadratic, ...), we ask them to discuss reasons for the jumps in the data, and we ask them about the trend of the data. Next, we use *TEMATH* to find the least squares line fit to the data. For the men's data, the line fit is given by

$$\text{distance} = -1064.7 + 0.70676(\text{year}).$$

We now ask the students to interpret this model, in particular, what is the meaning of the slope (the men's gold medal long jump distance tends to increase on average by 2.8 inches between each of the Olympics). See Figure 6.

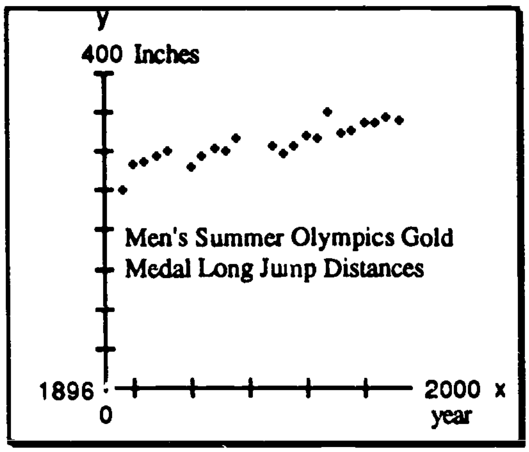


Figure 6 Modeling the Men's Summer Olympics Gold Medal Long Jump Distances

After analyzing the men's distances, we overlay the plot of the women's distances and we find the least squares line fit for this data. The women's line fit is given by the equation

$$\text{distance} = -2041.9 + 1.1702(\text{year}).$$

Note that the rate of increase in long jump distances for the women is about 4.7 inches per Olympics while the men's is 2.8 inches per Olympics. Does this suggest anything? At this point we discuss the consequences of using these models to extrapolate the data into future years. Will these trends continue? Will the athletes' performance continue to improve? If after all this discussion we dare to extrapolate the data, we can use *TEMATH* to find the point of intersection of the two trend lines and estimate that in the 2108 summer Olympics, the women will catch the men in the long jump with a gold medal performance of 425.3 inches. See Figure 7.

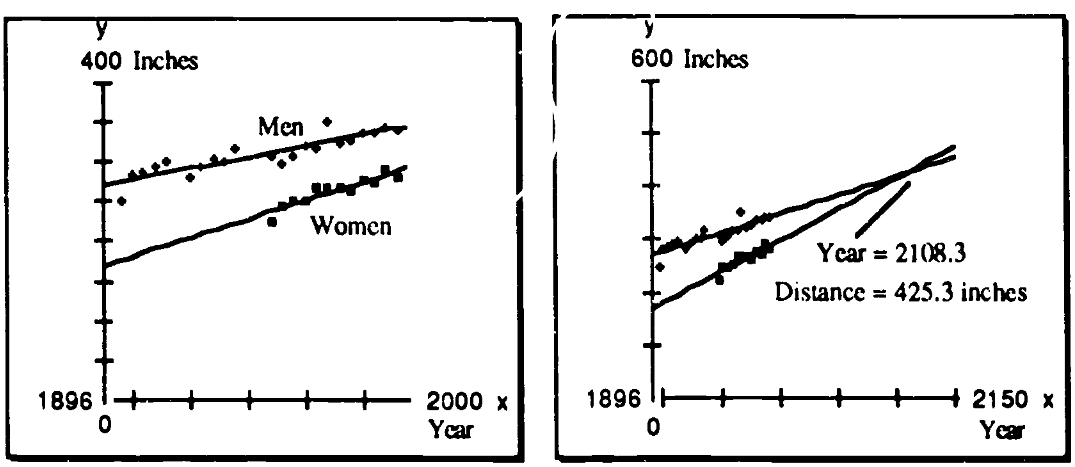


Figure 7 Comparing the Men and Women

Integration

We are presently class testing a visual integration tool that we have written for *TEMATH*. With this tool, you can use the Left Endpoint Rule, the Right Endpoint Rule, the Midpoint Rule, the Random Point Rule, the Trapezoidal Rule, Simpson's Rule, or Monte Carlo Integration to approximate the definite integral of a function. To use this tool, you enter the function $f(x)$, the approximate integration rule you want to use, the value of n (for example, n might represent the number of rectangles), the interval of integration, and then you click the button. A visual representation of the approximate integration rule is drawn to the Graph window (see Figure 8) and the approximate value of the integral is written into *TEMATH*'s Report window.

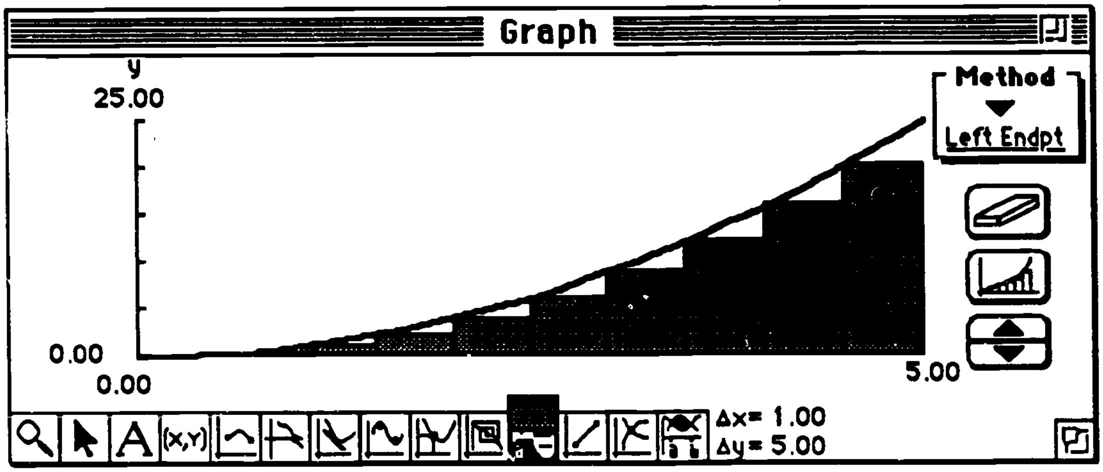


Figure 8 The Definite Integral and Riemann Sums

Each time you click on the up-arrow button, the number of rectangles (trapezoids, parabolas, ...) doubles. As you keep clicking on this button, you can actually watch the sum of the areas of the rectangles (trapezoids, parabolas, ...) converge to the area under the curve. This gives the students a visual representation of the convergence of Riemann sums and an intuitive interpretation of the definite integral as an area under the curve.

We have also used this integration tool in our calculus classes to visually compare the convergence of the Rectangular rules to the Trapezoidal rule and to Simpson's rule. An example of how this can be done is presented in the sequence of snapshots given in Figure 9.

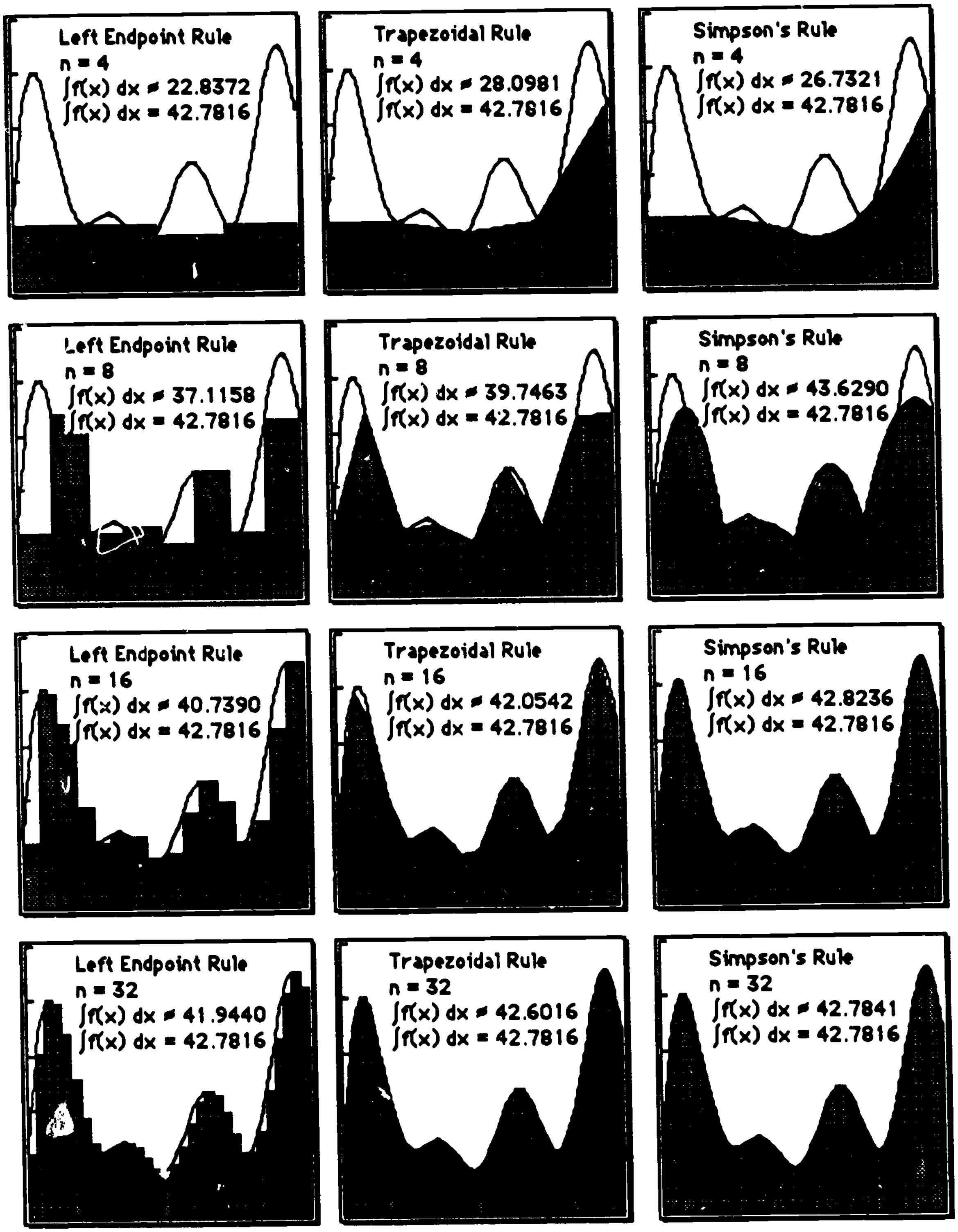


Figure 9 Comparing Numerical Integration Techniques

The Derivative as a function

Many of our beginning calculus students have a difficult time grasping the input-output process of a function. They have an even harder time understanding the concept of the derivative as a function which outputs the value of the slope of a tangent. To help our students through this difficulty, we try to get them more actively and visually involved in the development of the derivative. For example, we have them plot a simple function such as $f(x)=x^2$. Then using *TEMATH*'s Tangent tool, they enter an x-value and *TEMATH* draws a tangent to the curve at that point and *TEMATH* writes the value of the slope of the tangent into the Report window. Next, the student plots the point whose x-coordinate is the x-value they entered and whose y-coordinate is the slope of the tangent. They repeat this for a number of points. Next, we ask them if the set of points they plotted could represent a function and, if so, what is the input and output of the function. Since it appears that all the plotted points lie on a straight line, we have the students use *TEMATH*'s Line tool to draw a line through the points. *TEMATH* calculates the slope of this line and writes its value into the Report window. The student can now write down the equation of the line and, hence, the function representing the derivative of $f(x)=x^2$. This process is shown in Figure 10.

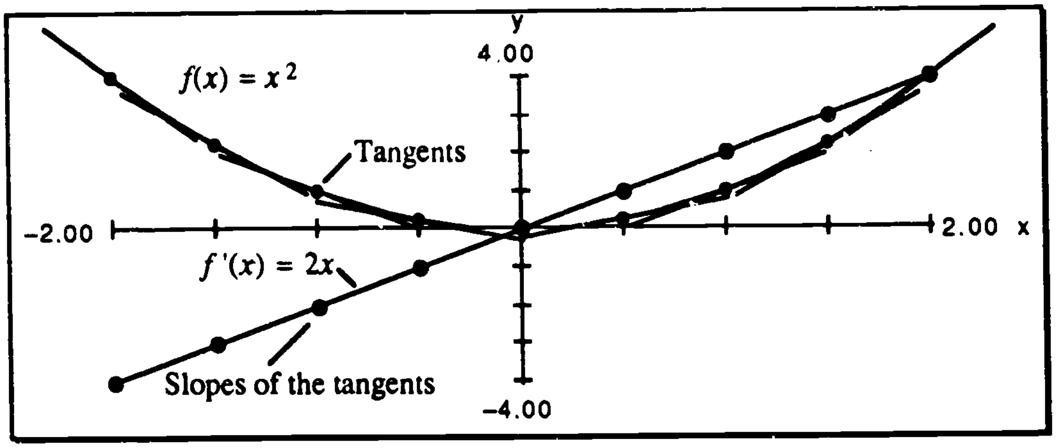


Figure 10 Visualizing the Derivative as a Function

Taylor Polynomial Approximations to the Circle

A standard visual exercise in a first year calculus course is to graph the function $f(x)=\sin(x)$ along with its Taylor approximating polynomials of various degrees and to observe the convergence of these polynomials to $f(x)=\sin(x)$. A good follow-up activity to this exercise (and another opportunity to use polar coordinates) is to use Taylor polynomials to approximate the circle given by the polar equation $r=\sin(t)$. The graphs of the Taylor approximating polynomials up through degree seven are shown in Figure 11.

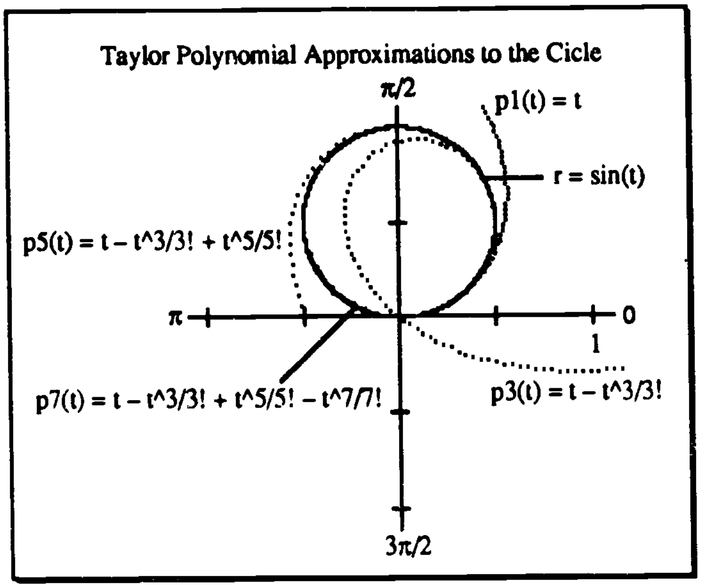


Figure 11 Taylor Polynomial Approximations to the Circle

As an additional example, the Taylor approximations to the cardiod $r = 1 + \sin(t)$ are shown in Figure 12.

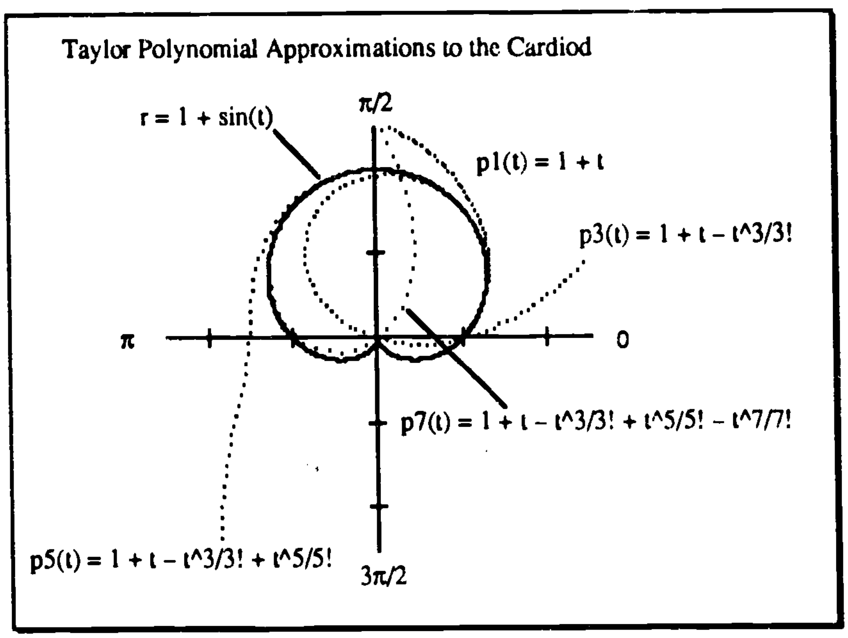


Figure 12 Taylor Polynomial Approximations to the Cardiod

For further information on Taylor polynomial approximation in polar coordinates, see Sheldon Gordon's article *Taylor Polynomial Approximation in Polar Coordinates* in the September, 1993 issue of The College Mathematics Journal.