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ABSTRACT

This paper is a report on the results of a 3-year teaching experiment conducted in Ithaca, New York, introducing students to the concepts of multiplication, division, and ratio as a trio, and to ratio and proportion in a project-based curriculum with heterogeneous grouping. Fractions were introduced as a subset of ratio and proportion. The paper outlines curricular changes in grades 3-5 and focuses on the major representational forms used by the students including: Venn diagrams, daisy chains, contingency tables, tables of values, dot drawings, two-dimensional graphs, and ratio boxes. Also discussed is the role these tools play in the development of students' understandings of the multiplicative world. Results show that these 10- and 11-year-olds exceeded the comparative performance of 14- and 15-year-olds on ratio and proportion test items. Contains 14 references. (Author/MKR)

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SPLITTING REEXAMINED: RESULTS FROM A THREE-YEAR LONGITUDINAL STUDY OF CHILDREN IN GRADES THREE TO FIVE

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A report is made on the results of a three-year teaching experiment introducing students to the concepts of multiplication, division and ratio as a trio, and to ratio and proportion in a project-based curriculum with heterogeneous grouping. Fractions were introduced as a subset of ratio and proportion. The paper outlines curricular changes in the third through fifth grades and focuses on the major representational forms used by the students including: Venn diagrams, daisy chains, contingency tables, tables of values, dot drawings, two-dimensional graphs and ratio boxes, and discusses the role these tools played in the development of students' understandings of the multiplicative world. Results of the study are presented showing that these 10 and 11 year olds exceeded the comparative performance of 14 and 15 year olds on ratio and proportion test items.

Ratio and proportion is arguably the most critical concept to learn in the elementary curriculum in order to make a successful transition into advanced mathematics. Its centrality is secured by both its conceptual and practical characteristics. Proportional thinking represents increased cognitive complexity in comparison to other arithmetic procedures of the elementary curriculum and demands considerable mental flexibility. It underlies such notions as scale, rate of change, acceleration, algebraic fractions, etc. Proportional thinking is involved in all kinds of applications of mathematics, from gears to weights, from motion to conversion tables. The learning of ratio and proportion has garnered significant attention from researchers around the world (Hart, 1988; Lamon, 1994). Its relationship to fractions has been hotly debated (Behr, Harel, Post & Lesh, 1992), its placement in multiplicative conceptual fields explored (Harel & Confrey, 1994; Vergnaud, 1994), and its developmental sequences articulated multiple times (Karplus, Pulos & Stage, 1983; Noelting, 1980; Piaget, Berthoud-Papandropoulou & Kilcher, 1987).

One of the most compelling and startling analyses of personal knowledge of rational numbers is offered by Kieren (1988). He proposes that this "complex and textured" (Kieren, 1988, p. 162) knowledge is comprised of multiple constructs including partitioning, equivalencing, measure, quotient, ratio number, and others. More recently extending and simplifying these constructs, Confrey proposed the *splitting conjecture* (Confrey, 1988). This conjecture posits that counting and splitting are two of the primitives that spawn our number system. Confrey argued that just as the act of partitioning is a primitive that cannot be reduced to repeated subtraction, a complementary construct, the inverse of partitioning, exists that is the precursor to multiplication and cannot be reduced to repeated addition. These partitioning acts which are precursors to multiplication and division evolve from a primitive she called "splitting" that involves the activities of sharing and folding, and geometric constructs which create a fundamental relationship to similarity.

Furthermore, Confrey argues that splitting is a basic cognitive structure that parallels, but differs from, counting.

This conjecture implies profound alterations in the scope and sequence of the typical course, particularly from third through fifth grade (and before and beyond). To examine these changes, a three-year longitudinal study at Belle Sherman Elementary School in Ithaca, NY was undertaken starting with a group in the third grade who would remain together until fifth grade and entry to middle school. The curriculum used by the experimental class incorporated significant changes which are described below.

In third grade, 1) multiplication, division and ratio were introduced as a trio. The order of introduction followed a splitting sequence, starting with twos, fours, fives, tens, eights, threes, sixes, nines and then sevens. 2) Extensive exploration of partitive and quotitive division and their interrelationship through the use of arrays was investigated.

In fourth grade, 1) least common multiple (LCM) and greatest common factor (GCF) were introduced early in the curriculum using prime factoring as students were encouraged to increase their mental flexibility in multiplicative conceptual space. 2) Ratio and proportion were introduced prior to the development of any operations on fractions, except simple recognition and naming of fractional parts. 3) The operations of multiplication and division within rational numbers were developed as extensions of ratio relations. 4) Explorations of ratio involved the two-dimensional plane and similarity relations on geometric figures. 5) Fractions were developed as a subset of ratios which share a common unit, and addition and subtraction of fractions as therefore requiring the identification of a common measurement unit.

In fifth grade, further extensions of ratio thinking, especially as regards multiplication and division, were developed. 1) Transitions to decimals were facilitated using the notion of ratio conversions between smaller and larger units. Mixed systems (such as weight measured in ounces, pounds, and tons) were contrasted with the "pure" system (where one n:1 ratio serves as the conversion factor between adjacent sized units) of decimal notation which utilizes the 10:1 ratio (Lachance, 1995 this volume). 2) Percent was treated in relation to decimal as ratio is in relation to fraction. 3) A transition to the use of algebraic symbolism was undertaken (Luthuli & Confrey, in progress).

Most units were taught using a project-based approach. Students were presented with project challenges, and materials and tools were provided for explorations (Preyer, in progress). For example, during the fourth grade year, the children designed handicap ramps. They were given a child's wheelchair and went outside to find a slope they could go both up and down while remaining in control. Students used a plumb line, measuring tape, and level to figure out how to describe their slope. Each group of students used their slope to create scale drawings and a model ramp for a given height of stairs. They also predicted the cost of materials given a certain set of conditions. Children were heterogeneously grouped with the assumption that all students would complete a performance assessment and an individual open-ended written assessment on all topics.

During the three-year teaching experiment, an exploratory methodology was used. Curriculum units were developed that were aligned with the splitting conjecture. This meant that ratio and proportion were assumed to be intimately connected to multiplication and division, that addition and subtraction of fractions were assumed secondary to multiplication and division, and that connections to geometry were given priority over additive relations. These subconjectures were modified as the experiment evolved in light of student work. All classes were videotaped and when the children worked in small groups, a single group was selected for videotaping for the duration of the project.

We will introduce the major forms of representation used extensively by the students and then report on the quantitative data concerning the students' performance on written ratio and proportion assessments.

1. Venn diagrams for LCM and GCF. The children were taught to prime factor numbers and to find their LCM and GCF using Venn diagrams. For two prime factorizations A and B, $A \cap B$ yields the GCF and $A \cup B$ yields the LCM. LCM was explored in the context of clapping rhythms to predict when two clappers would clap simultaneously. The idea behind this exploration was to explore numbers' "multiplicative biographies."

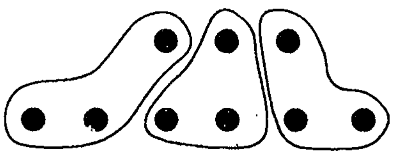
2. Daisy chains. Before beginning the introduction to ratio and proportion, students were asked to explore multiplicative space by creating sequences of operations, only using multiplication and division, to move from one number to another. Thus, to go from 28 to 36, a student might write: $28 \div 7 \Rightarrow 4 \times 3 \Rightarrow 12 \times 3 \Rightarrow 36$. Later this notation would be curtailed to $28 \times \frac{9}{7} = 36$. The students discovered two important methods concerning how to move from a to b (where a and b are rational numbers): 1) divide by a to get 1 and then multiply by b; 2) multiply by b to get ab and then divide by a. We claim that this is the critical meaning of multiplication by a ratio.

3. Contingency tables. The introduction to comparing ratios was undertaken in the context of polling. Students used 2 by 2 contingency tables, often divided into boys' and girls' responses categorized into "yes" and "no." Totals were listed in the margins. This format, in contrast to writing the proportions as $\frac{yes(part)}{no(part)}$, encouraged the students to work flexibly with their data concerning both "numerators" and "denominators." Children described results as part to part or part to total, depending on what they wanted to claim from their data.

	Boys	Girls	Total
Yes	3	5	8
No	7	9	16
Total	10	14	24

4. Tables of values. As the children extended their explorations from the comparison of ratios to the equivalence of ratios, the use of the contingency table was extended to the use of a table of values. Employing the context of a two-ingredient recipe (one ingredient in each column), students easily made larger recipes by doubling or tripling the original recipe. Then they explored halving it and argued for its equivalence. One student recognized the "pace" of the ingredients (water and oranges) claiming, "Just add three (waters) on every time you add an orange" (Confrey, 1995, p. 9). Students used other terms including "basic combination" and "little recipe" (which became the class favorite) to refer to the smallest whole number ratio for a given proportion. We suggest that this be named a ratio unit (Ibid., p. 11), to recognize its importance as a multiplicative unit.

5. Dot drawings. The children used dot drawings in order to find the "littlest recipe" (2 to 1 in the figure below). Rather than finding the littlest recipe numerically using factoring, they used dot drawings and employed a recursive process to check the validity of "little recipes." Validation was determined when the children's search image of a series of circled little recipes left no dots uncircled and each set was identical, or when they regrouped to see the recursiveness in the whole picture. If uncircled dots remained after using the attempted ratio unit, children would recursively select another ratio unit for a further attempt at determining the "little recipe."



6. Graphing on the two dimensional plane. The children were introduced to the idea of a ratio (a:b) as a vector from (0,0) to (a,b). The axes' labels allowed them to distinguish a:b and b:a. Equivalent ratios lie along a vector. The children found this notion extremely generative and connected, and explored its relations to rectangles, stairsteps, triangles and straightness. They were able to make sense of the meaning of steeper and less steep, and learned to interpolate and to extrapolate using data. Later, in the context of falling domino chains, without any formal introduction, the students extended their analyses to include discussions of acceleration and deceleration based on the shapes of curves.

7. Ratio boxes. Ratio boxes encouraged student exploration of the ratio relations both across and down (as an isomorphism of measures and as a functional relation in Vergnaud's terms). Using ratio boxes avoided problems concerning the lack of distinguishing notation between ratio and fraction, and supported a smooth conceptual networking of contingency tables, tables of values, and ratio boxes. Having become a primary tool for the children, the use of ratio boxes led to three significant results: 1) all students in the class believed that for any three values, a fourth could be found; 2) a student recognized that the fourth value could be obtained by multiplying the two numbers in the diagonal positions, and then dividing that product by the number in the third position (9 × 8 = 72. 72 ÷ 12 = 6); and 3) students learned that if they could find a daisy chain to go from one cell to an adjacent cell, that same daisy chain should also work for the other pair of cells (12 ÷ ***4*** ⇒ 3 × ***3*** = 9, so 8 ÷ ***4*** ⇒ 2 × ***3*** = 6).

12	9
8	

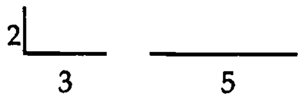
This paper claims that, given appropriate contextual challenges and representational tools with which to approach ratios, an earlier and more robust introduction to ratio can be presented. The table below presents the results of these students' written assessments given at the end of fourth grade and repeated at the end of fifth grade, and the comparative data from the "Concepts in Secondary Mathematics and Science" (CSMS) studies (Hart, 1988).

	CSMS			Belle Sherman	
PROBLEM	Age 13	Age 14	Age 15	Ages 9-10	Ages 10-11
Short/Tall	28.1 (51.4)	29.6 (50.6)	42.0 (39.1)	45 (15)	90 (5)
Ls	7.9 (47.6)	11.0 (39.4)	19.7 (39.7)	35 (30)	65 (10)
Onion Soup:					
Water for 4	94	94	96	90	100
Cubes for 4	95	95	96	90	100
Water for 6	85	84	88	70	70
Cubes for 6	75	75	79	75	75
Cream for 6	24	23	21	35	45

Percentage correct (Percentage using additive strategy)

In Mr. Tall and Mr. Short, students are given, "Mr. Short's height is 6 paper clips or 4 buttons. His friend Mr. Tall's height is 6 buttons" and are asked, "How many paper clips are needed for Mr. Tall's height?" (Karplus R., Karplus, E., & Wollman, W., 1972). Of these fifth graders, 90% answered correctly and only 5% (one student) showed any evidence of using additive strategies. These are striking results in light of the CSMS data for 15 year olds where only 42% answered correctly and nearly as many (39.1%) used additive strategies.

On the Ls problem, students are asked to "work out how long the missing line should be if this diagram is to be the same shape but bigger than the one on the left" (Hart, 1988). Of the fifth graders in this study, 65% solved it correctly and only 10% showed any evidence of additive strategies. This is in contrast to the 15 year olds in CSMS where 19.7% got the item correct and nearly 40% gave evidence of additive approaches.



On the onion soup problem, given a recipe to serve 8 people, 100% of the fifth graders figured out the amount of ingredient needed to serve 4 people. When faced with what seems to be the most difficult problem for students (determining the amount of cream for 6 people given that the amount for 8 people is F(1,2) pint), the data from non-experimental students shows dramatic drops in performance (to 21% among 15 year olds). However, with the children in our study, the drop in performance is considerably less (to 45% among 10 and 11 year olds). In fact, these 10 and 11 year olds more than doubled the accuracy of the 15 year olds in the CSMS studies on this challenging question, demonstrating the robustness of their approaches.

The data presented here suggests that conceptual analyses and developmental studies (Karplus et al., 1983; Noelting, 1980) have tended to underestimate the power of different representational forms in allowing students access to the conceptual understanding of ratio and proportion. In contrast to traditional curricula which have treated ratio and proportion in settings with little (single) or no context, we are embedding the study of ratio and proportion in a rich interactional system with physical and representational tools, and in multiple problem contexts. Our results suggest that higher success may be achieved by many students at an earlier age.

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