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ABSTRACT

Although the average, or arithmetic mean, has a rich conceptual meaning, it is often defined simply as the outcome of a procedure. The purpose of this study was to compare the nature and extent of the procedural and conceptual understandings developed by two groups of students who received different forms of instruction, one based on the traditional numerical algorithm, the other on a visual algorithm. When confronted with tasks varying along several dimensions, students adjusted or extended their basic approach to finding the arithmetic mean in ways that give insight into their understanding of the concept. While both groups of students showed a degree of understanding and flexibility with the procedure they had been taught, students who had learned the visual procedure showed a deeper conceptual understanding of the arithmetic mean. An appendix contains selected tasks presented to subjects. (Author/MKR)

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PROCEDURAL AND CONCEPTUAL UNDERSTANDINGS OF THE ARITHMETIC MEAN: A COMPARISON OF VISUAL AND NUMERICAL APPROACHES

Elizabeth Ann George, University of Pittsburgh

Although the average, or arithmetic mean, has a rich conceptual meaning, it is often simply defined as the outcome of a procedure. The purpose of this study was to compare the nature and extent of the procedural and conceptual understandings developed by two groups of students who had received different forms of instruction, one based on the traditional numerical algorithm and the other on a visual algorithm. When confronted with tasks varying along several dimensions, students adjusted or extended their basic approach for finding the arithmetic mean in ways that give insight into their understanding of this mathematical concept. While both groups of students showed a degree of understanding and flexibility with the procedure they had been taught, students who had learned the visual procedure showed a deeper conceptual understanding of the arithmetic mean.

A growing amount of information in today's world is presented and must be processed numerically. Therefore it is crucial to understand the relationship between a set of numbers and the representative numbers, or statistics, used to describe the set. One commonly used descriptive statistic, the arithmetic mean, is usually introduced in elementary and middle school mathematics classrooms. Traditional instruction on this topic primarily focuses on a numerical algorithm which is executed when a set of numbers is given and determining the average value is the intended goal. The arithmetic mean is rarely taught as a concept, but rather as the outcome of a computational procedure—the result of dividing the sum of the numbers in the given set by the number of numbers in the set.

If a student's sense of the arithmetic mean is too closely tied and limited to the outcome of a procedure, an impoverished understanding of the arithmetic mean is often the result. A series of studies have probed students' understanding of the arithmetic mean. Strauss and Bichler (1988) identified the concept of the mean as having seven different properties and found that it was particularly difficult for children to view the arithmetic mean as representative of the values that had been averaged. Mokros and Russell (1995) further examined the relationship between students' ideas of representativeness of a set of numbers and their understanding of the arithmetic mean and found that students who approached the mean as an algorithm rarely understood the average as a number which represents a data set. These students were limited in the strategies they had available and confused about the meaning of the total sum, the arithmetic mean, and the numbers in the data set. Earlier investigations of students' understanding of the arithmetic mean (Pollatsek, Lima, and Well, 1981; Mevarech, 1983), showed that even college students who relied on the numerical algorithm to find the average of a set of numbers displayed

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misconceptions when confronted with more complex tasks involving the arithmetic mean.

A visual alternative to the traditional numerical approach for finding the arithmetic mean is offered in the middle school curriculum, *Visual Mathematics* (Bennett and Foreman, 1991). Students build a column of wooden cubes to represent each number in a given set, then level-off the columns of cubes; the height of the leveled-off columns is defined as the arithmetic mean. Students are encouraged to move from physically constructing columns of cubes to using diagrams. The authors claim that discussion and practice with this visual "leveling-off" method reinforces the concept of the average as students are forced to consider the relationship between the numbers in the set and the average itself.

The two instructional methods described above both involve finding the arithmetic mean through the use of a procedure, whether numerical or visual. Both procedures move in a linear fashion from a given set of numbers through an algorithmic series of actions taken on those numbers, to produce a numerical outcome which is called the arithmetic mean or average. With either procedural approach students can easily come to interpret average as a "do-something signal", in much the same way that Kieran (1981) described students' view of the equal sign as an operator, not a relational symbol. Just as students must come to understand the equal sign as expressing a relationship of equivalence, students must come to see the relationship between the numbers in a set and the arithmetic mean. Understanding this relationship should allow students more flexibility in solving problems involving the arithmetic mean. With this deeper conceptual understanding, they should be able to move back and forth between the numbers in the set and the average, not simply proceed in one direction from the given numbers to the average.

Both numerical and visual procedures for finding the arithmetic mean have strengths and limitations dependent on the size of the numbers in the set, the size of the set, and the adaptability of the procedure for less straightforward problems. Both instructional methods could potentially produce a limited understanding of the arithmetic mean. Of interest is whether either or both of the procedural based instructional methods can help students construct rich conceptual understandings. Examining the nature and extent of students' procedural and conceptual understanding of the arithmetic mean involves accessing students' understanding of the procedure they were taught, their flexibility with and willingness to extend that procedure, and their ability to move between visual and numerical procedures. The purpose of this study is to compare the nature and extent of the procedural and conceptual understandings developed by two groups of students who have received different forms of instruction, one based on the traditional numerical algorithm and the other on a visual algorithm.

Method

Six students participated in this study during the fall of their seventh-grade year. None of the students had yet covered the topic of the arithmetic mean in their seventh-grade math classrooms, though all had received instruction during their sixth-grade year. Two of the subjects were enrolled in a school whose mathematics instruction was primarily drawn from the *Visual Mathematics* curriculum and had learned the visual "leveling-off" method for finding the average of a given set of numbers. The other four subjects had received instruction based on the numerical "add and divide" algorithm.

Five tasks involving the arithmetic mean were administered individually to each student. The goal of the first task was simply to find the average of a set of four relatively small numbers; the results of this initial task served as a baseline from which responses to the subsequent tasks were examined. The other four tasks varied along several dimensions, such as the size of the numbers, the size of the set of numbers, the initial representation of the task in either visual or numerical form, and the goal of the problem. Tasks 2 and 3 were presented in the context of a story, with the term "average" embedded in the text. Both tasks stated three of four numbers in the set and the average of the set; the goal was to find the fourth number in the set. While both tasks presented information visually, task 2 used discrete objects and task 3 used a bar graph to model the given situation. These tasks were adapted from the QUASAR Cognitive Assessment Instrument (Lane, 1993). The fourth task shared the same goal as the initial task, to find the average of a given set of numbers, but used larger and more numbers. Task 5 asked students to construct sets of numbers having an average of 12. (See Appendix for selected tasks.) To fully capture student thinking, students were asked to think aloud as they completed the tasks. These verbal protocols ranged from 20 - 35 minutes and were transcribed from audiotapes. Coding and analyses of the transcriptions and of students' written work form the basis for this study.

Analysis focused on the approaches each subject used to complete the tasks. The strategy used on each task was first coded as primarily visual versus primarily numerical. Visual strategies were further coded as involving wooden cubes or diagrams. Responses were also coded as successful or unsuccessful in arriving at a correct solution. Both successful and unsuccessful attempts were analyzed for evidence of student understanding of the arithmetic mean concept and sources of errors were identified as computational, counting, or conceptual.

Results

Similarities and differences in students' procedural approaches and conceptual understandings become evident as patterns of behavior appeared for each subject across tasks. When presented with the initial task, most students used the method that had been the basis of their classroom instruction. Examining each individual student's responses across the subsequent tasks revealed that most found ways to adjust or extend their basic approach to finding the arithmetic mean as the format and demands of the tasks changed. Table 1 displays the results of this analysis.

The two students whose instruction had been from the *Visual Mathematics* curriculum materials (S1 and S2) continued to use visual strategies in approaching each task. Understanding the relationship between the heights of the columns of cubes, representing the given numbers in the set, and the height of the leveled-off

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Table 1.
Strategies used and success in solving averaging tasks

	Task 1	Task 2	Task 3	Task 4	Task 5
S1	Visual - cubes Successful	Visual - cubes Successful	Visual - cubes Successful	Visual - numbers representing cubes Successful	Visual - cubes Successful
S2	Visual - cubes Unsuccessful counting error	Visual - diagram Successful	Visual - diagram Unsuccessful - counting error	Visual Unsuccessful - good approx.	Visual - diagram Unsuccessful - good approx.
S3	Numerical Successful	Visual - diagram Successful	Numerical Unsuccessful - computation error	Numerical Successful	Numerical Successful
S4	Numerical Successful	Numerical Successful	Numerical Unsuccessful - conceptual error	Numerical Successful	Numerical Successful
S5	Numerical Successful	Numerical Unsuccessful - no solution found	Numerical Unsuccessful - conceptual error	Numerical Successful	Numerical Successful
S6	? Unsuccessful	Visual - diagram Successful	Visual - diagram Successful	? Unsuccessful	? Unsuccessful

columns, the average, provided these students with flexibility in using the method of transferring of cubes in tasks posed in various ways. S1 consistently chose to actually build the columns of cubes, even when problems were presented in diagrammatic form, and arrived at correct solutions. S1 described how she was modeling the fourth task with imaginary columns and, though she proceeded to work solely with the numerical values, her explanation focused on her actions in moving imaginary cubes. S2 quickly moved into sketching diagrams. Her errors related to difficulties which arose in depicting the movement of cubes in her sketches and in keeping track both visually and numerically of the current state of the problem. The greatest challenge that these students faced was in applying and extending the visual solution process to the fourth task, where the numbers given were too large to be modeled directly with the cubes. The actions of the students demonstrated their ability to adapt the visual procedure in finding the average and to work back and forth between the numbers in the set and the average.

Students whose instruction was based on the traditional numerical algorithm depended on the one-way application of that algorithm to solve a majority of the tasks. S4 and S5 approached each of the five tasks with the algorithm and were successful in solving all but the third task. No mention was made that the fourth task used larger or more numbers than the first task. The second and third tasks did prove more challenging as they utilized a trial-and-error approach to find the missing number in the set. Their preliminary attempts resulted in errors primarily involving the divisor in the algorithm. Neither subject chose to work visually, even though the numbers in the data set for tasks 2 and 3 were given in a diagram.

S3 and S6 did choose to move at least once from the use of the numerical algorithm into a visual solution strategy. S3 employed a visual strategy successfully to solve the second task; he used the same visual method described by S2. He chose to return, unsuccessfully, to the numerical algorithm in the third task. S6 was unable to solve any task which depended on a decontextualized understanding of average. Although she commented that she recognized this type of problem, she did not know the numerical algorithm. But the meaning of the average was implicit for her in the contexts of the second and third tasks and she was able to find the solution to the third task more quickly and directly than any of the other subjects. In fact, neither S3, S4, nor S5 was able to solve this task.

Discussion

The results of this study show that the same method which formed the basis for classroom instruction on averaging was used by students when presented with the initial task of finding the average of a set of numbers. Students overcame the obstacles found in variations on the initial task by adjusting their use of the method learned or by finding a new problem space in which to work. No student whose experience was in Visual Mathematics used any form of the numerical algorithm, while two of the four students whose instruction involved the numerical algorithm did work with the diagrams when tasks were represented in visual form.

Analysis of student responses showed how task demands presented different challenges for students who had learned the numerical versus visual procedures. Students who had learned the numerical algorithm were confident and successful in finding the arithmetic mean when a complete set of numbers was given, regardless of the size of the numbers or the size of the set. When given an average and asked to find a number(s) in the set, they were often successful in identifying a solution, but consistently worked from the numbers in the set to the average, moving unidirectionally and using a trial-and-error approach. Regardless of context, students who had learned the numerical algorithm referred to the average as the outcome of a procedure, "what you get".

Students who had learned the visual approach revealed greater flexibility in moving back and forth between the numbers in the set and the average. Recognition that the sum of the deviations from the average is zero, one component of the average concept identified by Strauss and Bichler (1988), appeared to be attained as students focused on the relationship between the heights of the original columns (the numbers in the set) and the leveled-off height (the arithmetic mean). While both groups of students showed a degree of understanding and flexibility with the procedure they had been taught, those who learned the visual procedure showed a deeper conceptual understanding of the arithmetic mean.

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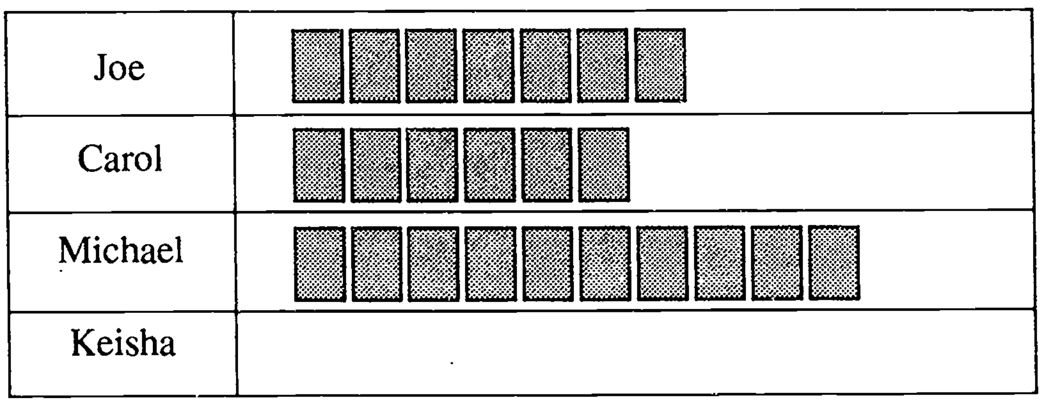
Appendix: Selected Tasks as Presented to Subjects

TASK 2: In order to raise money for a trip, the seventh grade class is selling candy bars. The class is divided into teams of four students. Joe, Carol, Michael, and Keisha make up one of the teams. If the team sells an average of 8 candy bars each day, they win a prize. The picture below shows the number of candy bars sold by Joe, Carol, and Michael.

How many candy bars does Keisha have to sell in order for the team to win a prize?

TASK 4: Jim recorded the amount of time he spent watching television for five days.



Monday - 120 minutes
Tuesday - 100 minutes
Wednesday - 60 minutes
Thursday - 90 minutes
Friday - 180 minutes

What is the average number of minutes Jim spent watching television?