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ABSTRACT

This resource book represents shared experiences of Project LINCS' (Lead teachers INvolved in making ConnectionS) teachers and staff in addressing the question, "Are we teaching mathematics so that all students will be empowered to use it flexibly, insightfully, and productively?" and other fundamental questions that are driving change in school mathematics. These questions affect: (1) the mathematics that middle school students learn; (2) the way the curriculum is organized; (3) the way students learn mathematics; (4) the roles of teachers; and (5) the climate and activities of our mathematics classrooms. The major portion of this resource book is devoted to sharing sample activities to assess, build on, and nurture student understanding and progress toward important mathematical goals. The teaching ideas are grouped under the major categories of algebra, alternative assessment, cooperative learning groups, geometry, number and computation sense, probability and statistics, problem solving, and writing to nurture and communicate understanding. Sample activities contain information to help teachers and students to focus, explore, disucss and reflect, and extend the activities. Each section contains references. (MKR)

Teaching and Learning

Middle School Mathematics TODAY

A Resource FOR Teachers BY Teachers From Project LINCS

A project of Illinois State University Mathematics Department
funded by the National Science Foundation

Project LINCS For Grades 4-8: Lead Teachers Involved in Making Connections was a three-year staff development effort sponsored by the Mathematics Department of Illinois State University and funded by the National Science Foundation. The materials of this booklet do not necessarily reflect the views of the funding agency.

Project Staff included:

Dr. Carol A. Thornton and Dr. Jane O. Swafford, Co-Directors
Dr. Beverly S. Rich, Dr. Graham A. Jones, Carol A. McGehe,
Dr. Barbara Wilmot

Project teachers were drawn from twelve school districts in the area surrounding Illinois State University. The name of participating teachers are written around the border of the inside title page of this booklet. Their dedication to the improvement of teaching and learning has been the inspiration of this Project.

* *

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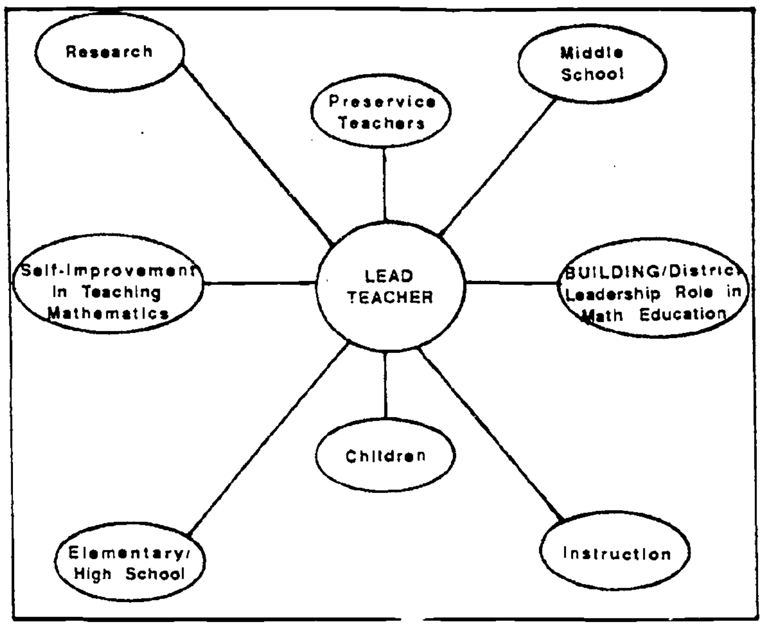
Teaching Middle School Mathematics: A Resource for Teachers by the Teachers of PROJECT LINCS

(Lead teachers INvolved in making ConnectionS)



Major connections emphasized in Project LINCS providing the framework for project activity.

Judy Mason Mary McClintock John Muirhead Eileen Norin Terry Oberhardt Terri Peek Jeanne Petkoff Judy Rocke Annette Rush Tom Schueman Rita Stone Jane Sumrall Dawna Tucci Margaret Watson Barbara Wiegand Alan Wilson Joyce Zeiters

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TEACHING MIDDLE SCHOOL MATHEMATICS

Overview

New Teaching and Learning Thrusts for a Changing Curriculum

For three years, under the auspices of Project LINCS, a middle school staff development effort focusing on mathematics teaching and learning in grades 4 through 8, Project LINCS teachers (highlighted on the inside title page of this book) have wrestled with the following question:

> *Are we teaching mathematics so that all students will be empowered to use it flexibly, insightfully, and productively?*

This Resource Book represents shared experiences of Project teachers and staff in addressing this and other fundamental questions that are driving change in school mathematics today and affect the mathematics that middle school students learn, the way the curriculum is organized, the ways students learn mathematics, the roles of teachers in teaching mathematics, and the climate and activities of our mathematics classrooms.

The National Council of Teachers of Mathematics (NCTM) recently published a landmark document, the *Curriculum and Evaluation Standards for School Mathematics* (1989), that presents a coherent vision for school mathematics and provides standards to guide its implementation. This document and its companion, the *Professional Standards for Teaching Mathematics* (NCTM, 1991), highlight characteristics of quality mathematics programs and features affecting the teaching-learning of mathematics that have become a forum for reflection as Project teachers continue to refine their own responses to questions like that above.

A worthy goal and challenge for teachers addressing this question is to help students become confident "doers" of mathematics. This means that they will be capable and resourceful problem solvers (see pp. 119-124), will develop the ability to communicate and reason mathematically (see pp. 125-133), and will value mathematics as worthwhile and essential. This can only be accomplished if mathematics makes sense to them and if they believe in their ability to make sense out of mathematics--which is more likely to happen if students possess a strong understanding of mathematical ideas, relationships, and applications.

Clearly, a "traditional" curriculum with its repetitive emphasis on arithmetic is inadequate. We tend to believe that the mathematics we learned is the mathematics that students need to know today, even though intellectually, for example, we accept the reasonableness of cutting back (or even eliminating) expectations for long division and mixed number computation.

Indeed, the curriculum content needs to be substantially broader than arithmetic. Access to technology is dramatically changing mathematics. Some mathematics has become more important because technology *requires* it; some mathematics has become less important because technology *replaces* it; and some mathematics has become possible because technology *allows* it. In view of these facts, content thrusts like probability and data analysis (see pp. 100-118), number sense and estimation (see pp. 81-99), algebra, patterns and functions (see pp. 10-31), geometry and measurement (see pp. 57-80) should receive as much emphasis at the middle school level as operations and computation.

Overview

In order to more appropriately address a fuller range of topics, Project LINCS teachers have found great success in starting their school year with a *noncomputational* topic such as probability and statistics or geometry (major focus each day), while systematically reviewing most of the material in the first chapters of their textbooks (minor focus each day). This approach represented a dramatic shift from prior years but was far more successful, enabling teachers to:

- engage students immediately with rich mathematical problems,
- provide greater time for on-going review,
- free up instructional time for more significant mathematics and. overall,
- create a broader and more *balanced* curriculum.

These are exciting times, for we are rejecting an inappropriate, static, and outmoded curriculum and are creating:

- ◆ a dynamic curriculum,
- ◆ a comprehensive curriculum.
- ◆ an integrated curriculum,
- ◆ an applied curriculum,
- ◆ a problem-driven curriculum,
- ◆ a balanced curriculum,
- ◆ an active-learning curriculum, and
- ◆ a curriculum that meets students' present and future needs in a technological age.

PROCESSES for Learning Mathematics

According to Lappan and Schram (1989):

The coin of the realm in the twenty-first century will be ideas. It will no longer be sufficient for students to enter the working world with

> only disconnected rules, theorems, and techniques stored in their mathematical heads. What will be valued in business and industry is being able to think and reason mathematically and to bring the power of mathematics to bear on a problem that needs a solution. The computational aspects of the solution can often be done by a computer, but a human must reason through the situation to decide what techniques need to be applied to solve the problem. (Lappan and Schram 1989, p.20)

Thus, while rethinking their vision for a refocused mathematics curriculum, Project LINCS teachers sought ways to interweave content with the context and *processes* by which it is acquired. Emphasizing problem solving, reasoning, making important connections and communicating mathematically--the first four "standards" (NCTM, 1989) proved to be one way of accomplishing this. These standards provide insight into the *processes* by which mathematics programs can help all students be successful. They highlight what students need to experience in learning mathematics and what they need to be able to do with their knowledge. They also present a dynamic and comprehensive view of what it means to know and to do mathematics.

The central role of *problem solving* in mathematics is highlighted in the first standard. The call is for including many types of problems and the use of a variety of strategies in solving them. Of even greater significance is the notion of embedding the mathematics to be learned in rich problem settings. The *mathematical connections* standard addresses the need for deliberately making connections among mathematical topics, between mathematics and other curriculum areas, and between mathematics and its multiple applications.

The *communication* and *reasoning* standards highlight the means by which students make sense of mathematics -- through the use of models, through expressing ideas verbally and in writing, through listening and collaborating with others to solve problems (see pp. 46-56; 125-133), and through using a variety of approaches to attain and verify solutions. Not only does communicating support children's learning, it provides teachers insight into what their students understand, an important component of on-going assessment (see pp. 32-45).

Another pragmatic approach to focusing on *process* goals in mathematics learning evolved as Project LINCS teachers drew from the list below to adopt short-term goals for themselves.

Learning Actions Consistent with the NCTM *Standards*

Classify	Explore	Collect	Investigate
Predict	Graph	Analyze	Develop
Verify	Create	Estimate	Generalize
Explain	Clarify	Interpret	Validate
Formulate	Evaluate	Apply	Represent
Relate	Express	Reflect	Extend
Read	Judge	Write	Conjecture
Justify	Connect	Believe	Solve
Link	Translate	Determine	Design
Model	Transform	Select	Test
Describe	Draw	Appreciate	Prove
Compare	Contrast	Discuss	Reason

Compiled by Barbara Wilmot

For the time-frame chosen (e.g., one grading period), many teachers. often with student input, selected and especially attended to ONE "action." For example, "During this nine weeks I'll try to encourage students to **state their prediction** before they attempt to work out any solution."

Simply stated, this scheme has proved to be very powerful--an excellent example of "less is more." Teachers also found that by frequently involving students in only one or two small but powerful processes, they often found students actually accomplishing much more!

Rethinking "Teaching" Mathematics

The recommended changes outlined in the *Professional Standards for Teaching Mathematics* (NCTM, 1991) entail far more than using manipulatives, adding a problem solving lesson here and there, teaching probability and statistics or geometry earlier in the year, or otherwise tinkering with the current curriculum. Implementing substantive educational change is complex for it involves *rethinking long established teaching-learning practices*.

This process of change is more difficult because of our view of mathematics resulting from our own experiences with it. Change involves confronting deeply held beliefs that make it difficult to perceive that things could be different. It challenges accepted views of what teachers do as they teach mathematics; it means accepting uncertainties associated with using less structured teaching approaches; and it means dealing with anxieties about current school policies, standard testing practices, and parental acceptance.

There are no easy shortcuts to refining our teaching practices. We change as a result of thinking ideas through for ourselves. We can reflect on arguments for change in our teaching style, become more familiar with the details of change, and interact with colleagues. Perhaps the most powerful stimulus to change is gaining new awareness of our students' capabilities and strengths, and seeing their positive responses and new perceptions of themselves.

Rather than teaching mathematics TO students, today's emphasis is on providing learning opportunities which enable students to actively construct their view of mathematics. Thc new role for teachers as *facilitators* of this construction is to:

- Build on students' intuitive knowledge;
- Provide experiences that enable students to "make sense" of mathematics;
- Establish a climate that respects and utilizes both individual work and collaborative problem solving;
- Require students to describe and justify their mathematical thinking;
- Nurture important connections between what students think, say, do and write as they solve mathematical problems;
- Provide ready access to calculators (or other technology) and to a variety of manipulatives which students can use for modeling and solving problems;
- Challenge students to apply mathematics as a tool for reasoning and problem solving in situations which stem from or relate to their own experiences.
- Use ongoing assessment of student thinking to guide instructional planning.

The challenge for teachers is to be aware of important curricular goals, to be attuned to what students know and can do, and to create appropriate bridges between the two.

In this spirit Project LINCS teachers have made an effort to focus on what students do, how they go about solving problems. Rather than laying out a detailed curriculum for the school year before even meeting the students, we try to map out important instructional goals and then, on an ongoing, day-by-day basis, use our assessment of what happens in the classroom to provide the teacher information for developing strategies and content for tomorrow. Sometimes, even within a lesson, uncalled ...ifts in plans are made in response to student questions or comments.

Overview

Instead of just recording the percentage of right answers in the grade books, an effort is made to examine student's thinking processes behind the answers. As we look for patterns and insights in that thinking and in student approaches to problems that can inform the next question, task, or lesson, we are aware that a wrong answer can often be a lot more informative than a right one. To gain deeper insights into both right and wrong answers, we often ask our students to write about a math concept or problem solution and, most importantly to logically justify or explain their thinking or approach to a problem. Consider these opportunities in the daily lesson:

- *At the beginning* of a mathematics lesson, to step back and ask questions, do webbing or present problem tasks to help determine what individual students already know about a concept or topic. (Sometimes we do not start a lesson as initially planned for, in response to this activity, we sense the need to backtrack or move further ahead than expected.)
- *Within* a lesson, to ask students to state their predictions; observe students as they work; probe for *different explanations and alternative correct approaches;* and strive to create a supportive mathematical climate for conjecturing, inventing, discussing, justifying thinking and refining approaches. (We are learning to try to "understand what students understand" and to continue to build bridges to important mathematical understandings, rather than just "doing" one two-page lesson after another. In other words, the effort is on teaching students, rather than teaching the book.)
- *At the end* of a lesson, to ask the students themselves to summarize "big mathematical ideas" that have been learned. (This activity provides a further opportunity for to observe, analyze and reevaluate the learning that has

taken place as a basis for determining "next steps," for extending and challenging the mathematical thinking and growth of all students.)

Throughout our three years together in Project LINCS, we have reevaluated and refined both our mathematics curriculum and our teaching approaches. A general cycle integrating assessment and reflection on students' thinking and progress toward major curricular goals has been central to our instructional efforts.

The major portion of this Resource Book has been devoted to sharing sample activities which have stemmed from seminar or summer course sessions. We have used activities like these in our attempts to assess, build on and nurture student understanding and progress toward important mathematical goals. The Epilogue (see pp.) shares the soul-searching in relation to "rethinking teaching" which has been part of our three years together. Perhaps some comment or activity will serve as a catalyst for your own reflection as you interact with and empower your students to think mathematically.

References

Lappan, G. & Schram, P. W. (1989). Communication and reasoning: Critical dimensions of sense making in mathematics. In Paul R. Trafton & Albert Shulte (Eds.), *New Directions for Elementary School Mathematics, 1989 Yearbook* (pp. 14-30). Reston, VA: National Council of Teachers of Mathematics.

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Algebra

"To achieve an equitable society, we must change the algebra course from a filter that screens out segments of our population to a pump that propels all students toward opportunity."

-National Research Council (1990)

Making sense of algebra has been a difficult task for the majority of U. S. students. In fact, 85% of those enrolled in algebra courses take no further mathematics (Mullis et al, 1991). Recent recommendations have not only deplored this situation, but have recommended that algebra topics be incorporated into the elementary and middle school curricula much earlier.

In nurturing algebraic thinking, three key prerequisites for success have been identified (National Council of Teachers of Mathematics, 1990):

- Understanding the concepts of variable
- Understanding the concepts of relations and functions
- Understanding the technical language of algebra

While it might be important to begin the study of algebra earlier, the critical question is "How can we nurture algebraic thinking so that these prerequisites will be attained by *all* students?" In Project LINCS, this became a key question for our middle school teachers.

Approaching Algebra With a Problem-Solving Focus

Problems arising from space travel, communication, business, sports data, and other real-world situations abound in variables that are often related to each other. Investigating these relationships provides a natural introduction to algebra. The key is finding problems that are relevant to the students' interests and, at the same time, are at an appropriate level for developing algebraic ideas with middle school students.

The activities which follow have emerged from the LINCS Project, and may be characterized by the following descriptors:

- Find patterns and describe the patterns algebraically using variables
- Construct and interpret relation or function to model a real-world problem in the form of:
  -- a table
  -- a graph
  -- an equation
- Investigate relationships between two variables
- Simplify algebraic expressions
- Construct and solve equations
- Explore the properties of linear, quadratic and exponential functions

The technology of graphics calculators has revolutionized the teaching and learning of algebra. In particular, the graphics calculator enables students to simultaneously analyze the three representations of a function: a table of values, the related graph and its corresponding equation. Further, the graphics calculator empowers students to carry out explorations like:

- analyzing the properties of functions
- determining special points associated with a function or relation
- investigating why different functions have different shapes
- comparing functions on the same set of axes
- testing the effect of changing parameters of a function and
- solving equations and inequalities both analytically and graphically.

The Challenge

Nurturing algebraic thinking in the middle school is a challenge that must be met. In meeting this challenge in Project LINCS, we have come to recognize that it is necessary to:

- believe that our students can successfully learn and use algebraic ideas
- establish an environment in which students engage in cooperative problem-solving.
- encourage students to use and share different approaches
- help students make connections between the problem context, the underlying patterns and relationships, and representations of these relationships in the forms of relevant tables, graphs, or equations
- encourage students to use technology flexibly to enhance understanding and make the solution process more efficient
- share with parents the importance of algebra for their children's future.

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Mullis, I. V. S., Dossey, J.A., Owen, E. H. and Philips, G. W. (1991). *State of Mathematics Achievement: NAEP's 1990 Assessment of the Nation and the Trial Assessment of the States.* Washington, D. C. National Center for Education Statistics, U. S. Department of Education.

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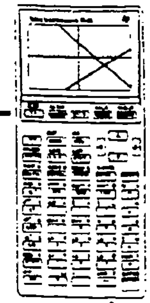
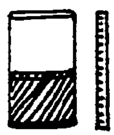


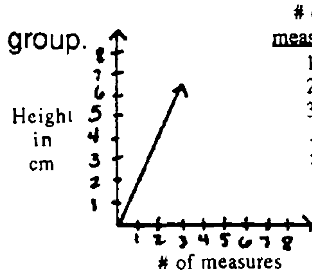
HEIGHT AS A FUNCTION OF MEASURE

Focus: Investigate relationships between two variables (height and volume).

Explore

Break the class into groups of 2 or 3. Give each group 1 straight sided clear container, 1 bathroom size Solo® cup, ruler with centimeter markings, and graph paper. Use the Solo® cup as 1 measure. Fill the cup with water and dump into the container. Measure and record height and number of measures. Repeat (do not overflow). Be as accurate as possible. Transfer data to graph paper and connect points. Discuss results in group.





# of measures	height in cm
1	2
2	4
3	6
⋮	⋮

Discuss / Reflect

- Is there any pattern in the graph?
- Will the pattern continue forever?
- What would the graph look like if we continued dumping in water?
- What would happen to the graph if a different measure was used?

Extend

- Repeat experiment using wide, narrow, and irregular shaped containers. Compare graphs. Do you see any patterns in the graphs depending on the choice of container?

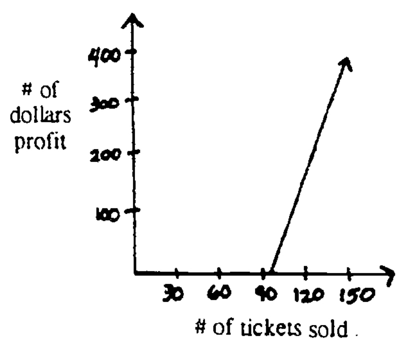
Need: Solo® cup, ruler, cylinder, pencil, and paper

SENIOR PROM

Focus: Look at a graph to find solutions to the problem.

Explore

The senior class of Fairway High School has 247 members. The dance comittee has decided that only members of the senior class can attend the Senior Prom they are hosting. They called many different companies for decorations, catering services, and music. After contacting every company they needed for dance supplies, they estimated the cost of the dance to be $686. How many seniors are needed to buy tickets for the senior class to break even? If they sold tickets for $7 per person, how many tickets would they need to sell in order to make a profit? a $91 profit? Make a graph to represent this situation. What is the maximum profit the class can make?

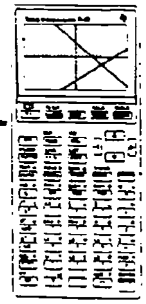


Discuss / Reflect

- How did you solve the problem?
- How did you set up the graph?
- What did the graph tell you about the situation?
- Describe the values for which a loss would occur.

Extend

- What if a single ticket was $7 and a couples ticket was $12? How would that change the graph?
- How many tickets of each would need to be sold to make a profit? Are there any other combinations in which the tickets could be sold?

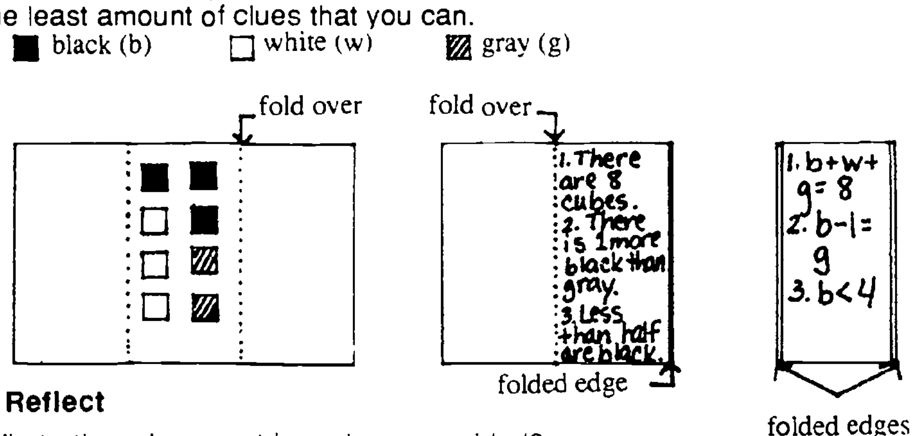


PAMPHLET VARIABLES

Focus: Use variables in algebraic and sentence equations.

Explore

- Choose no more than 25 Unifix cubes in 3 colors. Take a sheet of paper and fold it into thirds.
- On the center space, draw and color squares to match your cubes.
- Fold the right flap over the middle section. On this flap, write three sentences that give clues about the cubes. (1 clue should tell the total number of Unifix cubes)
- Fold the Left flap over the middle section. Write 3 equations to match your sentences.
- Check your clues carefully.
- Trade folded pamphlets with a partner and solve each other's clues, using the least amount of clues that you can.

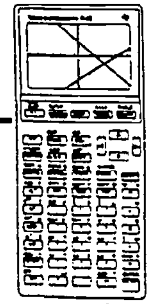


Discuss / Reflect

- What other clues could you have provided?
- Why do your clues work?

Extend

- Describe your classmates' eye colors using variables.
- What other physical characteristics could be described using variables?



VERBAL VARIABLES

Focus: Use variables to solve verbal clues.

Explore

(n represents a positive integer)

1. My number plus 5 is 25. What is my number?
2. My number minus 6 is 17. What is my number?
3. My number doubled is 46. What is my number?
4. My number divided by 7 is 8. What is my number?
5. The product of 4 and my number is 32. What is my number?
6. Seven more than the product of 2 and my number is 21. What is my number?
7. One less than the product of my number and 5 is 14. What is my number?
8. When my number is divided by 8 and the quotient is increased by 1, the result is 7. What is my number?
9. The product of my number and 12, decreased by 10, is 26. What is my number?
10. When the sum of my number and 8 is divided by 2, the result is 9. What is my number?

Discuss / Reflect

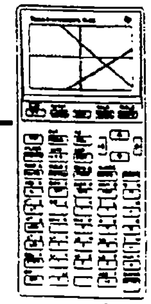
- What strategies did the students use to solve the problems?
- How did they know these strategies would yield the correct answer?
- Were there multiple ways of arriving at the same answer?

Extend

- Write the algebraic equation for each "verbal variable".
- Invent your own "verbal variables" and give them to a partner.

Answer Key:

1. (n+5=25: 20) 2. (n-6=17: 23) 3. (2n=46: 23) 4. (n/7=8: 56)
5. (4n=32) 6. (2n+7=21: 6) 7. (5n-1=14: 3) 8. (n/8 + 1 = 7: 48)
9. (12n-10=26: 3) 10. ((n+8)/2=9: 10)



DIMES AND NICKELS

Focus: Use of variables as a justification of solution.

Explore

Joanne has $2.40 in change. She has 33 coins that consist of dimes and nickels. How many dimes and how many nickels does she have? (Use and define variables in an equation to justify your solution.) *

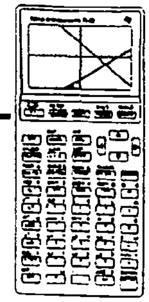
Discuss / Reflect

- How did the value of the coin affect the variables?
- How would the problem change if Joanne had other coins?

Extend

- What are the other possible combinations Joanne could have with 33 coins? How do you know that is all?
- How many dimes and nickels would she have if she had an equal number of dimes and nickels?

* Answer: d represents the number of dimes
n represents the number of nickels
d+n=33
d=33-n
10d+5n=240 or .10d+.05n=2.40
10(33-n)+5n=240
330-10n+5n=240
330-5n=240
90=5n
18=n
therefore d=15 (d=33-18)

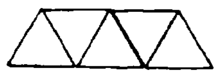


PERIMETER CHAINS

Focus: Express patterns using variables in mathematical expressions.

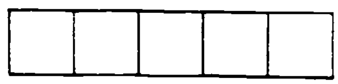
Explore

Perimeter chains of regular polygons. Polygons must be connected by one side and form a chain.



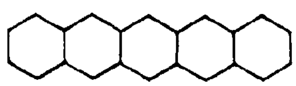
Triangles

# of triangles	1	2	3	4	5	6	7	8	9	10	. . .	n
Perimeter	3	4	5	6								



Squares

# of squares	1	2	3	4	5	6	7	8	9	10	. . .	n
Perimeter	4	6	8	10								



Hexagons

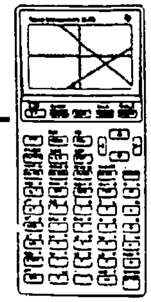
# of hexagons	1	2	3	4	5	6	7	8	9	10	. . .	n
Perimeter	6	10	14	18								

Discuss / Reflect

- What is a variable?
- How is a variable used?
- Why will it work?

Extend

- Find an expression using a variable(s) that would work for any regular polygon.
- Form trains with different polygons and have students predict the pattern the perimeter will form.



SUPREME COURT JUSTICES

Focus: Use variables to find the solution.

Explore

Every year the Supreme Court session begins with each of the nine Supreme Court justices shaking hands with every other judge. How many handshakes will take place? If 1 judge is absent, how many fewer handshakes take place? *

Discuss / Reflect

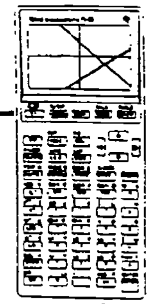
- How did you determine how many fewer handshakes took place? What did you have to know about the problem?
- How can you use variables to represent the number of handshakes? **
- What other ways besides using variables could you use to solve the problem? (circle-see below, chart, act it out, etc.)

Extend

- How many handshakes remain if 3 judges are late? ***
- If 5 of the judges are left-handed, how many handshakes involve at least one left-handed person? ****

Answer Key: * 8 *** 21 **** 35

** n(n-1)/2. The number of judges (n) shakes hands with every other person beside him/herself (n-1), and to eliminate for double handshakes(judge 1 shaking hands with judge 2 is the same as judge 2 shaking hands with judge 1) you divide by 2.



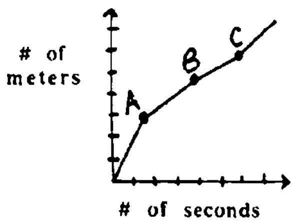
WRITE A STORY

Focus: To provide students with opportunities to interpret graphs.

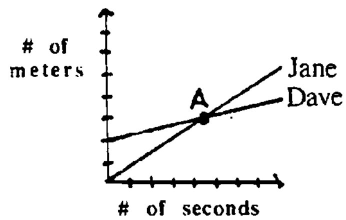
Explore

Write a story describing what is happening in the graph.

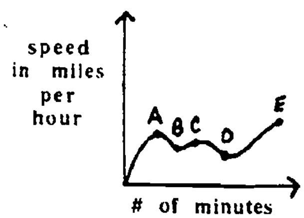
Jane's Race



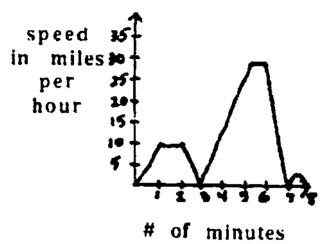
Jane Races Dave



Cross Country Runner



Bike Ride to Dave's House



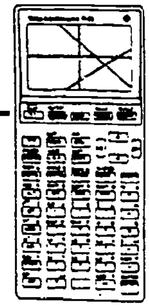
Some suggestions to include in your story are: maximum speed, what happened between values, why does the graph decrease/increase, describe what happened, explain why the graph is/is not not a straight line, who won, etc.

Discuss / Reflect

- Read several stories aloud to the class. Discuss the different interpretations.
- Rationalize why your explanation is acceptable.

Extend

- Create your own graph and story. Share the story with a partner and challenge them to draw a graph for it. Compare and contrast the 2 graphs.

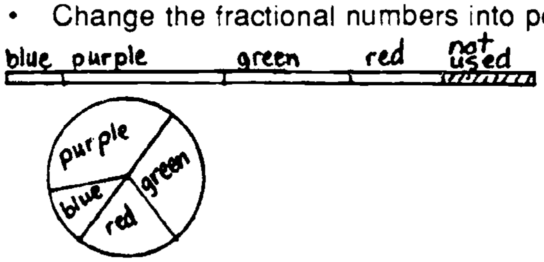


UNDERSTANDING CIRCLE GRAPHS

Focus: To explore the representation of data in circle graphs (comparing part to whole).

Explore

- Begin by brainstorming questions that the class can answer that have at least 3 possible answers. Some possible questions are: What is your favorite kind of pizza? and What is your favorite kind of music or radio station? Choose one.
- Divide the class according to their answers. Arrange them in a circle. Mark the middle of the circle with a garbage can or book. Put a string between each pair of students who chose a different response , thus demonstrating a circle graph.
- Provide students with 100 cm strips of paper. As a class, decide how many cm each student will represent and label sections on the strip according to response (see example).
- Bend the strip in a circle and draw it on paper. Be sure that the students have a point in about the middle of the circle, and draw in the connecting lines.
- Change the fractional numbers into percents.



3 cm represents one class member. Therefore, if 6 people chose red, 6 x 3 or 18 cm would be used as red.

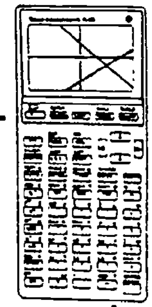
Discuss / Reflect

- Discuss when a circle graph is appropriate and inappropriate to use.

Extend

- Challenge students to find out how to figure the number of degrees at the center of each piece of pie. Use a protractor and compass to draw the circle graph as accurately as possible, using the degree measurements.

Need: 100 cm strip, ruler, pencil, and paper per student



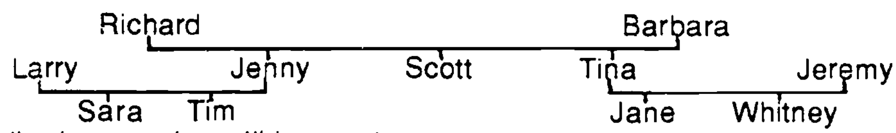
FAMILY TREES

Focus: Introduce function notation:

f(x) is a value which depends on the stated value for x

Explore

The Martin Family Tree



The following notation will be used.

m(x) means the mother of x, so m(Tim) is Jenny

f(x) means the father of x, so f(Scott) is Richard

s(x) means the sister of x, so s(Jenny) is Tina

b(x) means the brother of x, so b(Sara) is Tim

Provide examples for students to determine the person if possible or if more than one person is possible. Have students make up their own problems.

Examples:

m(Jenny)	b(Tim)
f(Jane)	s(Tina)
f(s(Tim))	f(s(Scott))
f(m(Whitney))	b(m(Sara))
s(Barbara)	s(m(Tim)
m(m(b(Sara)))	f(s(Jane))

Discuss / Reflect

- Discuss relationships among family members. Does m(f(Tim)=f(m(Tim)? How do you know?
- The s(m(Sara)) is Sara's aunt. What other relationships can you describe in this way?

Extend

- Reverse the problem, such as fill in the blank problems s( ) is Jenny.
- List all of the different function notations you can think of to represent one family member. Create your own notation with a key.
- If you are given the expression 3x + 5, where the value of the function is a number, what does f(x) equal?

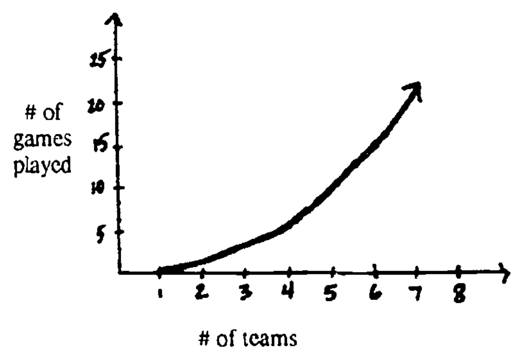
SOCCER GAMES

Focus: Find a pattern from student-derived data, form a written rule about the pattern, and recognize it as a function.

Explore

Four soccer teams are playing in a tournament in which each team must play every team once. What is the total number of games played? In small groups have the students complete the table below by determining the number of games played when there is(are) 1 team, 2 teams, 3 teams...?

Teams	Games Played
1	0
2	1
3	3
4	6
5	10
6	15
7	21
.	.
:	:
t	t(t-1)/2



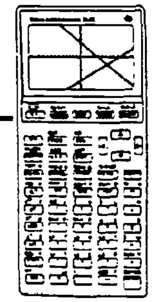
Graph the data. Encourage students to find the pattern.

Discuss

- How did the students find a pattern in the data?
- Assist students in deriving the rule for the function.
  t represents the number of teams
  g(t): the number of games played is a function of the number of teams in the tournament
  $g(t) = t(t-1)/2$

Extend

- How many games are there if there are 15 teams? (105)
- How many teams are in the tournament if 190 games are played? (20)
- How would the function change if each team played every other team twice?



THE RABBIT PROBLEM

Focus: Express a pattern using variables.

Explore

There is a single pair of rabbits at the beginning of January which breeds a second pair of rabbits at the end of the month. Each pair takes a month to mature and then breeds one pair each month. Assuming no rabbits die, how many pairs will there be at the end of June? December? the year?

End of Month	Jan	Feb	Mar	Apr.	May	June	July	...
Newborns	1	1	2	3	5	8		
Adult Pairs	1	2	3	5	8	13		
Total	2	3	5	8	13	21		

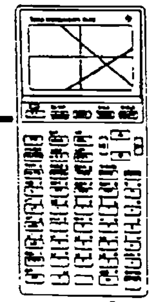
Discuss / Reflect

- What strategies did you use to solve this problem?
- * What patterns did you find?
- Introduce the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, ... The Fibonacci sequence: each number is the sum of the previous two numbers.

Extend

- Compute the number of rabbits in two years.
- If a field could only support a population of 2,430 rabbits, what month would the population exceed this limit?
- Investigate the ratio of various consecutive Fibonacci numbers (i.e. 2 and 3, 4 and 5, ...).

* Each month is the sum of the previous two months.



PASSING A NOTE

Focus: Express patterns using variables.

Explore

Central High School has 2036 students. Ashley decided to pass around a note that read:
"Our principal, Mr. Jones, will be 46 on January 31! There will be a surprise party January 31 at 12:30 in the cafeteria. Please copy this letter twice and give it to two people tomorrow. You can not give the letter to anyone who has already received one."

Day	# receiving note	#who have notes
0	1	1
1	2	3
2	4	7
3	8	15
4	16	31
5	32	63
6	64	127
7	128	255
⋮	⋮	⋮
n	2^n	$2^{n+1} - 1$

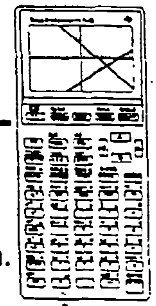
- How many people had a note to start on Day 0?
- How many people had received a note on Day 3?
- How many days will it take before all of the students have read the note?

Discuss / Reflect

- What strategies did you use?
- How did you organize your thinking/information?

Extend

- If Ashley starts the note on January 23, how many students won't know before the party?
- When would she need to start the note to be sure that all of the students will have read it before the party?
- Will everyone get to give their note to someone? If not, how many?



THE HUGE COOKIE

Focus: Examine fractions and infinite processes, as determined through a pattern.

Explore

Jim got a huge cookie for his birthday. He ate 1/2 of it on his birthday. The next day he ate 1/2 of what was left from his birthday. On the third day, he ate 1/2 of what was left from the day before. If Jim continues like this for 8 days, how much of the cookie has he eaten? How much is left? If he keeps doing this, will he ever eat all of the cookie?

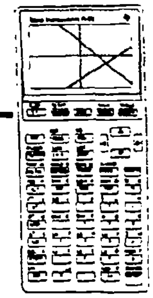
day	amount eaten	amount not eaten
1	1/2	1/2
2	3/4	1/4
3	7/8	1/8
4	15/16	1/16
5	31/32	1/32
. .	.	
: :	:	
n	$(2^n - 1)/2^n$	$1/2^n$

Discuss / Reflect

- Are there any other ways we could look at the data? (graph, picture,...)
- If the sum continues, how close will he get to eating the whole cookie?

Extend

- Solve the problem by reversing the situation (on what day will there be 1/2048 cookie left?)
- Solve the problem using 1/3 instead of 1/2.



TELEPHONE COSTS

Focus: Use a graph to look at the relationship of different parts of a function.

Explore

Let "C" denote the cost of a long distance telephone call. In this case, if the call lasts less than 3 minutes, the cost is $1.85. For each additional minute or fraction of a minute there is an additional cost of $0.30. Under these conditions, the cost "C" of a call can be written as a function of its duration t (time) by

	Cost	Time
	1.85	$0 \leq t < 3$
C(t) =	2.15	$3 \leq t < 4$
	2.45	$4 \leq t < 5$
	2.75	$5 \leq t < 6$

- What is the cost after 2 minutes?
- After 3.34 minutes?
- After 5 minutes?
- After 5.01 minutes?

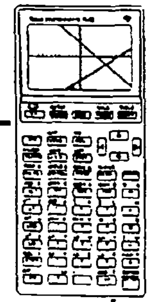
Discuss / Reflect

- What would the graph look like for this problem?
- Can you explain why it looks that way?
- How is it different from other graphs you have worked with?

Extend

- What if the phone company charged $1.85 for the first 5 minutes and an additional cost of $0.80 for each additional minute? What would be the cost after 5 minutes? After 8 minutes? How would your graph change? What would it look like now?
- What would be the cost for a call that lasted m minutes, where m is an integer greater than 3?

  Answer: $1.85 + (m-2).30 = C$

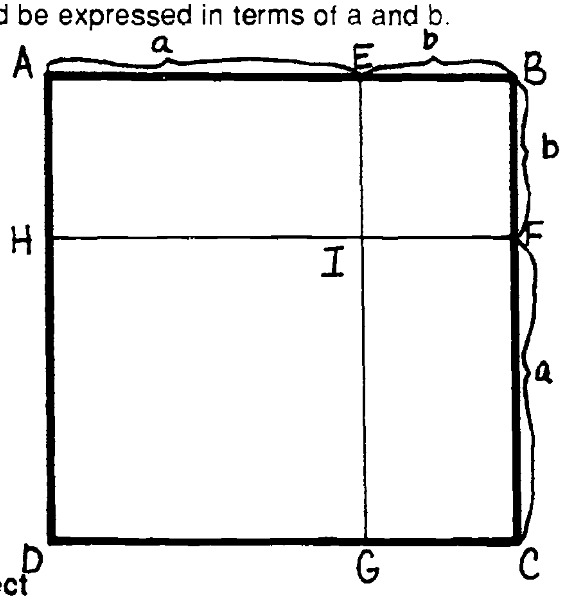


THE RECTANGLES WITHIN A SQUARE

Focus: Use variables to describe a relationship.

Explore

Find the area of every rectangle, including square ABCD. In each case, the area should be expressed in terms of a and b.



Discuss / Reflect

- What relationships among rectangles can be determined using the areas found above, if 1/2a = b?
- Using the relationship 1/2a = b, how does rectangle GIFC compare to rectangle IEBF? What other relationships can be determined among the rectangles?

Extend

- How does the area of Triangle HDG compare to the area of HDGI?
- What other relationships among polygons can be determined?

DECIDE YOUR OWN PAY

Focus: Interpret data through graphs representing the situation.

Explore

You started a new job at the beginning of September. Your supervisor asked you how you would like to be paid. He gave you the following options. Which would you rather receive: $35 per working day, or $1 the first day, $2 the second day, $4 the third day, $8 the fourth day, where the amount you get at the end of the day is double the amount you received the previous day? Graph your results for both cases (how much would you receive each day?)

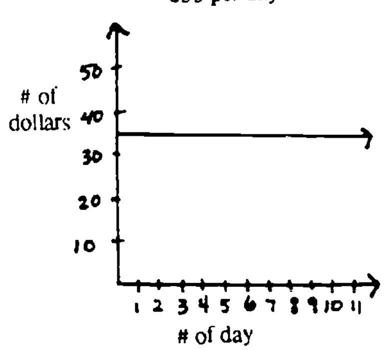
Discuss/Reflect:

- Do you think any employer would double an employee's pay each working day?
- What is the total amount at the end of the month in each situation.
- What do the graphs look like? How are they the same and how are they different?
- Why does the graph increase at such a high rate after a period of time?
- If the job only lasted 9 days, which way would you rather get paid?

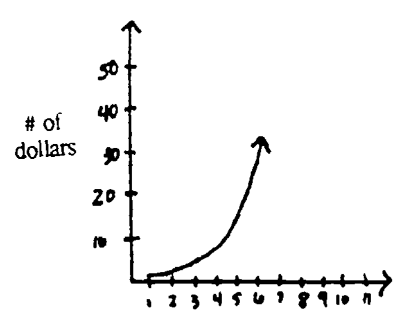
Extend

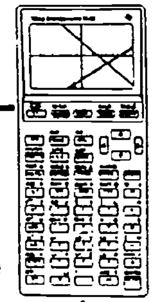
- After 1 year, how much would you have made in both situations?
- How would your results change if you received 1 penny on the first day, 2 pennies on the second, 4 pennies on the third, 8 pennies on the fourth, etc.?

$35 per day



double each day





VIDEO GAMES

Focus: Find relationships between the graph and the equation representing the situation.

Explore

At Vera's Video membership costs $20. Video games can then be rented for $3.50. What would the ***total cost*** be to rent the first video game? The second video game? The third video game? The nth video game?

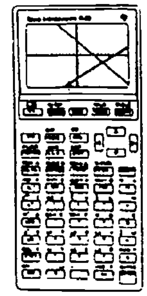
Discuss / Reflect

- Discuss how to figure out how much you will have to pay for a certain number of video games, and how the students thought it out.
- Enter the data in the claculator: Press 20 ENTER (for the membership fee), +3.5 ENTER, explain that for 2 games you press ENTER again.
- How could you find out how much it would cost to rent 10 video games? (2 ways: 3.50*10 + 20, or press ENTER 8 more times.)
- In groups have the students find how much it would cost for 20, 30, and 40 games. Ask how you could express the situation for *any* number of video games? (3.50x + 20 = y: where x represents the number of video games and y represents the total amount of money spent on renting that many games.) *Make sure that students understand what the variables represent!*
- Graph the equation on the calculator (see next page)

  Make the connection between y-intercept and slope.

Extend

- Suppose in 4 months Jamie had spent $181 on video games. How many video games had she rented?
- If Video World charges no membership fee, but charges $4.75 for each game, and Vera's Video charges a one time $20 membership fee and $3.50 per game, which store offers the better deal? (It depends on the total number of games that will be rented. If the total number of games rented is 18 or more then it is better to rent from Video, Video, Video.) Graph both equations on the calculator.



Press Y= CLEAR
Write the function for the equation: $y = 30 + 3.50x$
Make sure that all students have this equation.

Press RANGE
Enter the following values:
xmin=0
xmax=100
xscl=1
ymin=0
ymax=400
yscl=10
xres=1

Press GRAPH

Press TRACE to find the different values
Note: The values for x need to be integers because you can't rent 3.5 video games.

When the graph is drawn, you don't just get integer values, which you need to represent the number of video games.

To get the value for any x:
Example: x=2
Press 2 STO STO ENTER
2nd VARS
1 ENTER
(2 can be replaced with any number of video games)

Alternative Assessment

Recognizing that tests often drive the curriculum in the United States, standards for evaluation were included in the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989). The *Standards* proposes changes in the current processes and methods of assessment and recommend alternative assessment practices. Key to their recommendations is the notion that student assessment should be an integral part of teaching and should be based on multiple assessment methods to ensure that the assessment covers the breadth and depth of the program content. All aspects of mathematical knowledge and its connections should be assessed. The focus should be on a broad range of mathematical tasks rather than on a large number of specific and isolated skills.

Teachers participating in Project LINCS have explored a variety of alternative assessment techniques including alternative pencil-and paper tasks and performance tasks.

Pencil-and-Paper Tasks

Open-Ended Items Open-ended items are student-constructed response items that have more than one correct answer or have multiple paths to a correct solution. In responding to such an item, students not only show their work, but are asked to explain how they got their answer or why they chose the method they did.

Student-Constructed Tests Another way to assess students' understanding of a subject is to have them, individually or in groups, develop their own test for a unit or chapter of study. A test constructed by students identifies for the teacher what content students view as important and the depth at which they perceive their understanding of that content to be.

Performance Assessment

The move to more authentic assessment in education encourages the use of observations of actual student performance in addition to pencil-and-paper assessment tasks.

Performance tasks One form of performance assessment is the use of performance tasks in which students are presented with an open-ended task in an assessment event and their performance observed. Generally, performance tasks involve the use of some physical materials. Specific procedures or protocols are developed for evaluating the performance (e.g. Stenmark, 1991).

Investigations and projects Another form of performance assessment is the assignment of investigations or projects. Investigations and projects often can take 2-3 weeks to prepare and can be either individual or group endeavors. The usual product is a written report and should include a class presentation.

Observations Performance assessment need not be a separate assessment event but can be incorporated into the regular instructional activities through observations. The challenge is to structure observations and to develop a systematic procedure for recording them. The use of an annotated class list or gummed labels on a clip board to note observations, or the use of a check list of desired behaviors are all ways to systematically record observations.

Interviews Interviews and conferences with students are a source of rich information about their understandings and feelings about mathematics. Even though interviews are time-consuming, they can help diagnose learning difficulties in a way no other assessment technique can. Even an informal help session can provide an assessment opportunity for obtaining interview information by using set procedures such as the five-point error analysis suggested by Newman (1983) and by keeping records.

Portfolio Assessment. In addition to student performance, a variety of student products can be assessed. One technique that is becoming increasingly popular is the development of a student mathematics portfolio. Portfolios allow students to document what they can *do* rather than what they cannot do. What goes into a portfolio should depend on the instructional goals of each situation, but might include such things as responses to open-ended questions, reports of individual or group projects, work from another subject area that relates to mathematics, a problem made up by the students, a mathematical autobiography, tests or test scores, or a teacher's interview notes or observation records

Self-Assessment

Since one of the most constructive and empowering goals of education is to equip students to monitor their own progress. Self-assessment is important. The simplest example of self-assessment is a questionnaire following an activity that asks students to evaluate the impact of the experience. A mathematical journal is another form of self-assessment. In a mathematics journal students might record the most important thing they learned since the last entry, what they are having trouble with, what they found easy, how they feel about the topic under study, or what they liked most or least about the mathematics class. Journals can be private or shared with the teacher for comments.

With the development and spread of open-ended items, performance tasks, assigned projects, structured observations and self-assessment instruments, there are now a variety of assessment techniques available to teachers. Many of these are being used widely in pre-college mathematics classrooms or by states in their assessment programs (Stenmark, 1989, 1991). The challenge for teachers is to identify what techniques are best for their own instructional goals and to adapt these alternatives for their own classroom use.

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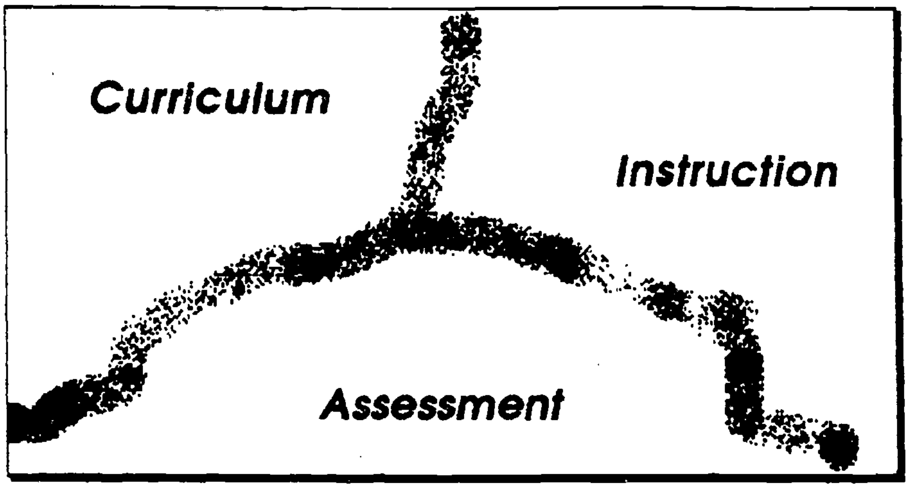


Curriculum - *The ideas, concepts, facts, skills, procedures, and vocabulary which is to be learned.*

Instruction - *the process of facilitating learning.*

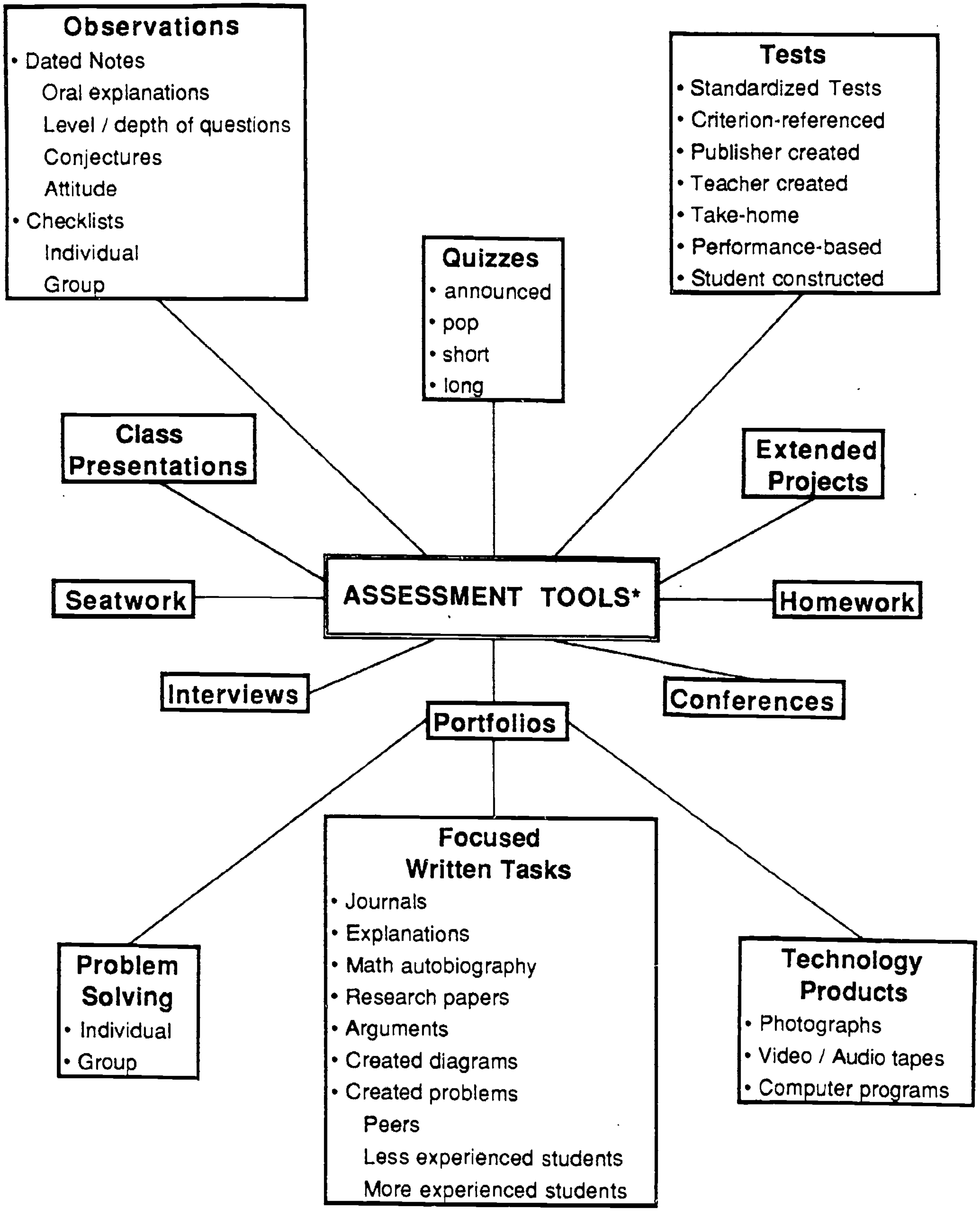
Assessment - *The process of collecting information for decision making about each student's learning.*

Because today's tests largely measure computation and routine procedural skills, they are one of the greatest obstacles to mathematics education reform.

For Good Measure, p.5

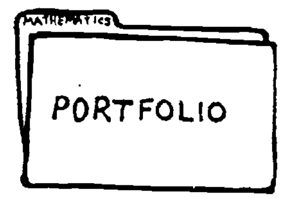
*"Assessment must measure what matters.
We need to get beyond the mechanics.
We need to gauge the ability to use the knowledge,
not just the knowledge itself."*

Ian Ross
President Emeritus,
AT&T Bell Laboratories



* Recent recommendations suggest that teachers place greater emphasis on a variety of assessment techniques beyond traditional paper-pencil tests, such as those presented above

Using Open-ended Questions to Assess



Assessment At the Beginning of a lesson:

- provides the teacher with the opportunity to see what students already know about the topic to be explored that day.

Examples: What does the graph tell us?
Tell your partner what you think of when you see this word: geometry... (then call on students to share ideas exchanged. Create a board list which clumps related ideas.)

Assessment During Instruction:

- allows the teacher to gain insight on the students' understanding of the material that is being taught; of their creative problem-solving abilities.

- provides the teacher with the opportunity to observe students and invite them to *explain* and *justify* their thinking.

Example: Who thought about it / solved it a different way?

Assessment At the End of a lesson:

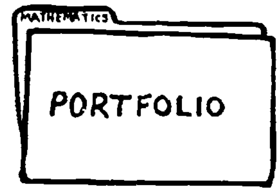
- allows the teacher to gain insights concerning what the students' learned from the lesson. This information can be the basis for further instructional planning.

Example: What "big" math ideas were learned / used today?

Open-ended questions...

- provide opportunities for students to solve problems using a variety of techniques;
- call for students to construct individual responses instead of conforming their reasoning to a single answer;
- allow students the opportunity to reason for themselves and to express personal ideas and concepts that correspond with their mathematical development;
- provide students the opportunity to demonstrate their knowledge and understanding of a problem.

Assessment



With the blocks, show me . . .
as many ways to make 7 as you can.

Draw two or more diagrams which. . .
explain how to tell whether a number is even or odd.

List all the possibilities for . . .
separating 10 counters into four piles.

In a newspaper or magazine, find pictures which have examples of. . .
triangles, rectangles, and trapezoids.
pyramids, prisms, and cones.

Tell how...
prisms and pyramids are alike and how they are different.

Explain why . . .
addition and subtraction are opposites of each other.

Write a paragraph about. . .
telling time.

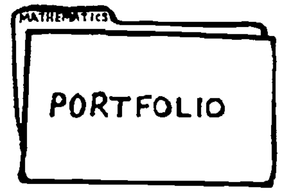
On your geoboard (and dot paper) make figures that . . .
have exactly one line of symmetry.

Draw a picture about how you felt when . . .
you knew you were right and your group would not agree.

Make a spinner that shows . . .
the chance of getting red is twice that of getting blue. And the chance of getting green is the same as that of getting blue.

Make a circle graph for...
each of these bar graphs.

Assessment



Name some things that it would take about 100 of...
to fill up a large suitcase.
to cover the bottom of a shoebox.

Decide. . .
If a flat shape has 4 sides, it is a rhombus.
Always true. __________ Never true. ___________
Sometimes true. Examples of when it is not true are:

What's more likely to happen...
it rains tomorrow or it doesn't rain tomorrow.
Why? How might this be different if you lived somewhere else?

When your are skip counting by 7s...
will you say the number 78?
How did you decide?

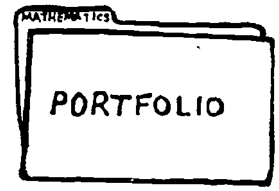
Show me...
If you have 37¢, what coins might you have?

If you were asked to find the answer to each of the following, would you use a calculator, paper and pencil, or do it in your head? Why?

3,675 + 2 = ________
3 x 15 = ________
4 + 8 + 7 + 10 - 3 = ________
300 - 13 = ________
Find the number of inches in 4 feet. ________
Find the number of minutes in 1 and 1/2 hours. ________
It is now 2:30. What time will it be in 2 hours? ________
How much money would I get back from $ 1.00, if I bought something for 89¢? _____

Write a problem situation...
for each type of subtraction.
Show with the blocks how each is different.

Assessment



Make a mind map for...

graphing

numbers

Write some incredible equations for...

20

250

2 1/2

Help Anita...

Anita is trying to find a way to solve two-digit subtraction problems like 75-26 without regrouping. How can she change the problem so that the answer will be the same and she will not have to reqroup?

With your thinking partners, ...

Create a story situation which contains some two-digit numbers and multiplication.

Verify this conjecture...

Sophia is pretty sure that all two- and four-digit palindromes are divisible by 11. Is she right? What about three-digit ones?

Tell how you could decide...

where your class should go on its next field trip.

about how much taller is the ceiling in the gym than the ceiling in our classroom

how to decorate the room for Valentine's Day

Record some possible keys you could press to make your calculator say ...

25 - but you cannot press a 2 or a 5.

Show me at least two ways that you could equally share ...

8 cookies with your best friend

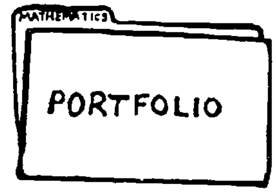
1 candy bar with two other friends

3 apples among 2 children

17 comic books with three other friends

$ 5.00 with three thinking partners

Write two lists. One of some things are are hard to share equally and one of some things that are easier to share equally.



INSIDE A PORTFOLIO

The focus in student portfolios is on these items:
- Student thinking
- Growth over time
(Be sure to date all papers.)
- Mathematical connections
- Student views of themselves as mathematicians
- The problem-solving process

A solution to an open-ended question done as homework—not just a neat set of figures, but showing originality and unusual procedures

A report of a group project, with comments about the individual's contribution, for example:
- A survey of adult use of mathematics in work
- A review of media use of mathematics

Work from another subject area that relates to mathematics, such as an analysis of data collected and presented in a graph for social studies

A problem made up by the student, with or without a solution

A mathematical autobiography

Art work done by the student, such as string designs, coordinate pictures, and scale drawings or maps

Teacher-completed checklists

Work in the student's primary language

Excerpts from a student's daily journal

Notes from an interview—by the teacher or another student

A table of contents

Papers that show the student's correction of errors or misconceptions

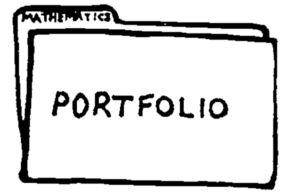
Draft, revised, and final versions of student work on a complex mathematical problem, including writing, diagrams, graphs, charts, or whatever is most appropriate

A photo or sketch made by the student (or teacher) of a student's work with manipulatives or with mathematical models of multi-dimensional figures

A letter from the student to the reader of the portfolio, explaining each item

A description by the teacher of a student activity that displayed understanding of a mathematical concept or relation

(Stenmark, 1991)



Alternative Assessment Techniques

Other alternative assessment suggestions which Project LINCS teachers have found beneficial include:

Have a student teach a process to me (the teacher) as if the roles were reversed. This approach provides the opportunity to LISTEN to and assess the student's understanding of the process.

Allow students to take tests in their cooperative learning teams as well. Great communication and discussion occurs, and during testing I walk around and individually "quiz" each member of the team to make sure everyone is participating and understands. Students, in turn, feel more confident about their knowledge and performance.

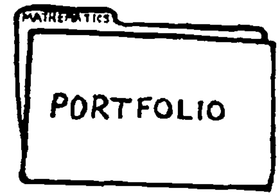
Invite students to tell or write about how a particular mathematics lesson applies to the real world. I encourage them to be as specific as possible, so I can assess the level of their ability to apply math in practical settings.

Have students draw pictures or diagrams to illustrate their understanding of a concept, process, or answer. They can also use manipulatives to do the same.

Give students problem solving activities and have them write out how they would go about solving the problems. I grade the problem them on the "how to" explanation.

Have students be "experts" on problems at the board. Students display their problem at the board and explain it as well as the process for doing the problem.

Using a novel as a basis, I have students write story problems using concepts studied in a particular unit. The students must be able to perform the mathematics involved in the problem they write.



Implications of Authentic Assessment

Classwork and assessment are inseparable.

Assessment takes place in the course of daily work.

Learning does not stop for a test.

The conditions for assessment mirror the conditions for doing mathematics outside of school.

Students should have ample time and access to collaboration with colleagues and to necessary tools.

They must also have the opportunity to revise their work.

The tasks for assessment engage a student's sense of purpose and are rich enough to be multi-dimensional.

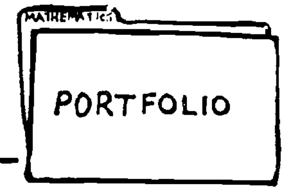
Through these tasks, students demonstrate thinking, understanding, and communication skills as well as mastery of techniques.

Feedback from assessment is concrete and specific to the task.

It informs students of the results of their efforts.

Students participate in the process of assessment.

They help to create and to apply standards for quality work through self and peer assessment activities.



Assessment -- Journal Writing

Many different aspects of problems lend themselves to journal writing. For any given problem. Students might be asked to log their thinking on one or more of the following:

Aspect of the Problem	Students's Thinking
1. Understand the situation / question(s) to be explored.	1. I am trying to find...
2. Analyze the data.	2. I know or don't know...
3. Plan the solution. Choose the strategy, operation, etc.	3. I will probably... because...
4. Estimate the answer.	4. I predict... because...
5. Solve the problem.	5. Do it and show it.
6. Examine the solution.	6. This means...

7. Create a new problem that uses the same type of thinking but has a different topic or situation.

Cooperative Learning Groups in Middle School Mathematics

Promoting cooperation instead of competition is becoming one of the most valuable of our recent paradigm shifts. The concept of cooperative learning groups has many interpretations and facets. But regardless of the 'school of thought,' increased academic learning and improved social skills are fostered when cooperative learning is appropriately used during mathematics instruction (Godd, 1992). Putting students in groups does not automatically insure more understanding and better behavior. However, using teaching techniques specifically focused on those goals produces progress.

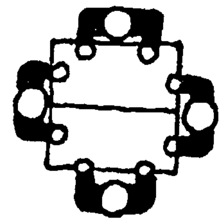
As mathematics educators, we are concerned about increased and broadened learning in mathematics, but we also are in agreement with the American Society for Training and Development which, under a grant from the United States Department of Labor, Employment and Training Administration, developed a set of "NEW BASIC SKILLS". These include: 1) Learning-to-learn, 2) Competence in reading writing and computation, 3) Oral communication and listening skills, 4) Problem solving, 5) Creative thinking, 6) Personal management, 7) Group effectiveness, and 8) Influence skills. Cooperative learning approaches to rich tasks can help enhance all of these skills.

The three sources of organization and research which can provide a basis for cooperative learning in mathematics instruction are Johnson and Johnson (1986), Slavin, Sharan, Kagan, Hertz-Lazarowitz, Webb, and Schmuch (1985), and Kagan (1990). Each represent groups which have a different focus and set of techniques and structures. All suggest approaches which can be used to meet a variety of needs in middle grades mathematics instruction. From each we integrate that which is most helpful.

Across America, teachers' experiences with and knowledge related to using cooperative learning groups for mathematics instruction varies. Some teachers have only vaguely heard the term and still associate it with traditional groups and all the problems inherent with them. Some have read an article or book and tried aspects of cooperative learning. Others have had extensive training and through the help of mentors have established cooperative learning approaches into their repertoire of classroom structures. Some teachers may even have served as trainers or mentors, helping others learn or apply cooperative learning practices in mathematics instruction.

One approach for focusing on cooperative learning techniques in a mixed group of this sort is to use the jigsaw structure to delve into the topic. Four heterogeneous groups based on classroom experience can be formed to brainstorm four different aspects of cooperative learning: 1) classroom management, 2) grouping, 3) the role of the teacher and 4) the role of the student. Each group should address just the one aspect assigned. At the end of a specified time, different groups should be formed -- involving at least one person form each of the four topic groups, so accumulated lists can be shared.

This type of session may not be sufficient to prepare a novice to incorporate cooperative learning, but will serve teachers at various levels of concern -- from awareness to collaboration. Typically some teachers will become convinced that they should seek further training and information; others will receive new classroom management hints, helpful for "smoothing some wrinkles" they already have encountered. All will emerge feeling excited about having experienced and benefitted from cooperation.



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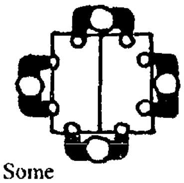


FOCUS ON STUDENTS

Students who are new to cooperative learning groups or who do not understand the benefits may pose interpersonal problems. Some possible problems are listed below. Consider others. Discuss reasons for as well as techniques and ideas to promote cooperation among students who:

WHAT	WHY	SUGGESTIONS
Tend to dominate the group.	*Aggressive, impatient, brighter students, competitive,needs attention, selfish, doesn't value others, excited, know how, mentally skilled, insecure . . .*	*Each student allowed only 1 response.* *Keep response sheet to tally responses. Seek balance.* *Build in rewards to get others to share.* *Use talking chips to promote turn taking.* *Discuss"group therapy."* *No "cross talking."* *Put time limit on talking.* *Assign roles.*

WHAT	WHY	SUGGESTIONS
Prefer to work alone.	*Timid or afraid of making mistake, personality conflict, impatient brighter child wants to get it done, afraid to work with others, doesn't want to be slowed down, frustrated, afraid of appearances, lazy, can do it alone faster, doesn't need motivation, doesn't like peers, didn't want to explain, insecure in math knowledge, poor social skills, not accepted by students, "speed" workers . . .*	*No negative comments.* *Sink or swim together.* *Give group grade.* *Reward for working together.* *Cruelty won't be tolerated.* *Behavior mod approach.* *One on one.* *Value cooperation-talk.* *Don't overdo cooperative learning.* *One job per person with 1 sheet of paper.* *Place student with a friend.* *Put in group with others they can work with.* *Counsel other group members about special needs of non-accepted student.* *Bargain - "You wo... in this group and I will give you a chance to work alone."*

FOCUS ON STUDENTS, continued

WHAT	WHY	SUGGESTIONS
Is reluctant to share.	*Shy, afraid to be wrong, low self esteem, compares self to others and doesn't value, doesn't understand (poor in concepts), insecurity, bad previous experiences with ridicule, never want to be wrong . . .*	*Have an encourager.* *Sink or swim together.* *Promote a positive atmosphere.* *No negative comments.* *Reward.* *Make safe.* *Each member shares one thing about what they did.* *May take time to realize what they say is acceptable.* *Create a non-threatening environment.* *Learn that there are no "wrong answers" - all answers lead to increased learning.*

WHAT	WHY	SUGGESTIONS
Wants to be in a different group.	*Friends in other group(s).* *Others aren't "good enough."* *Behavior things.* *Personality problems.*	*Make a contract (behavioral).* *Choose 1 person to be in with next time.* *Rotate groups.* *Set group time for set number of weeks.* *Try to work out differences.* *Avoid if at all possible changing groups once made - but realize it is O.K. if needed.* *Do not give in to them.* *Assure them groups will change in 6 weeks.* *Tough it out.* *Brainstorm strategies on how to get along with a group.* *Develop problem solving skills - apply problem solving skills.*

FOCUS ON GROUPING

Discuss the process of forming groups and the decisions which need to be made.

CONSIDERATION	ADVANTAGES	DISADVANTAGES
Size: 2	*Don't change seating arrangement, less noise, enough varied input management easier, no one person overwhelmed, more active participation, simplicity, security, . . .*	*Less sharing-knowledge to share* *Socializing* *One tends to dominate* *Large class, too many groups* *Teenage social scene* *Hard to get heterogeneous group . . .*
3	*heterogeneous (high-mid-low), can assign roles, more balanced, requires more social skills, more interaction; begin with this at lower level, more interaction and thinking, ...*	*Tends for odd number to be left out* *Limited ideas/sharing* *Sometimes vote is 2 to 1(1 student is out) . . .*
4	*more diversity and ideas, assigning roles easy, can have them pair up and bring back ideas to the 4-some, more accountability, starting is less intimidating, easier to provide enough manipulatives, easy to move to pairs, more ideas, . . .*	*Easier to get lost. Quiet students hide.* *Harder to reach consensus* *Not all have enough time to participate* *Easier for 1 child to be "free loader* *Noise level management can be tough*
5 or more	*forced to come to consensus, large number of students in a small area easier management, research "large topic," . . .*	*Difficult to assign roles* *Too many, hitchhiker on end* *Confusing*

FOCUS ON GROUPING, continued

CONSIDERATION	ADVANTAGES	DISADVANTAGES
Random vs selected	***Random:*** *Chance of variety, good at beginning of year when you do not know the students, may get students who really clique, if students choose it takes less time to form groups, . . .* ***Selected:*** *Greater control, get a better mix needed for sharing, can tailor the academic levels for more successful groups, can make sure groups are heterogeneous, teacher knows the group "high learner," . . .*	***Random:*** *Chance of a poor group, when students select -- could lead to social consequences, too many low levels together, personality clash (if not considered in grouping), friends might be together, could get all high or all low in a group, may not work together, don't always work . . .* ***Selected:*** *Poor student doesn't get selected or get help needed . . .*
Heterogeneous vs Homogeneous	***Heterogeneous:*** *By gender, race, culture, ability, learning styles think differently, get broader base, not feeling inadequate, blending of abilities, social skills, tolerance, respect for others, higher kids need to think through more, higher level order of thinking skills, no feeling of "smart groups" vs "dumb" groups, encourages low learners to excel, high learners are able to excel . . .* ***Homogeneous:*** *Security . . .*	***Heterogeneous:*** *High learners may disagree with each other, low learners - may never reach an answer, requires more cooperation, . . .* ***Homogeneous:*** *Poor students all together, low learners would have no role models, academic gifted not allowed to use talents, higher learners don't have a chance to be teacher, might be better for projects, . . .*
Length of time group is together per day	*Throughout the day, may not always be doing group work, 25-50 minutes, . . .*	
Length of time group stays together (days, weeks, months..)	*Staying together for longer periods children learn to work together; 3-6 weeks; nine week grading period(base scores); change monthly, weekly, daily; for certain topics . . .*	*Short time - no bonding; long time - tired of each other.*

FOCUS ON THE TEACHER

FROM	TO	RAMIFICATIONS FOR TEACHER	RAMIFICATIONS FOR STUDENT LEARNING
Dispensing information	Planning dynamic lessons for transfer of learning	*Rely less on textbooks, take more of a project approach, develop more of a willingness to brainstorm, planning time, different planning, more time thinking about questions that might come up and how to answer with a question, plan so one can't take over so students can see the need to cooperate, more advanced planning, change physical space, more open-ended problems to be solved, less talking, tolerate more noise, more difficulty keeping kids on task, . . .*	*Take responsibility for planning, see learning as a long-term process, active involvement, adjusting to different expectations, children must learn to trust one another rather than teacher, must be more responsible for 1) time management 2) information gathering 3) sharing information 4) working together 5) evaluating self and others 6) keeping focused . . .*
Performing for and entertaining passive students	Teaching students how to learn Involving students in active learning	*Ask more than tell.* *Developing more open ended questions.* *Less direct instruction, less lecture.* *Less time talking, more time observing.* *Offer manipulatives as alternative.* *Direct to appropriate resource.* *Understand problem and formulate questions needed.* *Teacher is more a facilitator and guides instead of leader . . .*	*Accept responsibility for extending learning.* *More hands on.* *More responsible for own learning.* *More dependent on teammates.* *Difficult for student not to give all the answers.* *Develop critical thinking skills.* *Be flexible and try new approaches when one doesn't seem to work.* *Be more open minded . . .*
Rewarding and punishing	Developing student responsibility	*Relinquishing some authority.* *Focus on group achievement.* *Set goals.* *Provide positive experiences.* *Rewards are more intrinsic so takes the responsibility off the teacher . . .*	*Increased motivation.* *Decreases disruptive behavior.* *More self-control.* *Peer pressure.* *Connect performance with end product . . .*
	Facilitating student self-evaluation	*Monitor student evaluation.* *Provide an instruments for self evaluation . . .*	*Improvement points reward how they as a group improve rather than competing.* *Rewards intrinsic "punishment" from group. . .*

FOCUS ON THE TEACHER, continued

FROM	TO	RAMIFICATIONS FOR TEACHER	RAMIFICATIONS FOR STUDENT LEARNING
Preparing for standardized tests	Encouraging and cheerleading mastery of skills and concepts	*More "kid watching" and note taking.* *Develop more open-ended activities.* *Celebrate small successes.* *Alternative assessment.* *Use more checklists rather than grades (new assessment techniques) . . .*	*Students are expected to positively encourage and question more.* *Kids wiling to accept more than one solution, more than one way to do something . . .*
Grading workbooks and tests for recall	Motivating high-level thinking.	*Better listener.* *Encourage fulfillment of partial achievement.* *Develop alternate evaluation techniques.* *Teacher becomes a more effective questioner with open-end questions rather than 1 answer questions . . .*	
	Extending participation	*Teacher models how to gain student involvement . . .*	*Taking turns and encouraging others to take their turn.* *Kids have more power . . .*
	Building group skills	*Teacher models how to gain student involvement.* *Model and role play . . .*	*Cooperate.* *Kids praise each other, encourage each other, discuss what they can do better . . .*

FOCUS ON CLASSROOM MANAGEMENT

CONCEPT	POSSIBILITIES	ADVANTAGES / DISADVANTAGES
Room arrangement	*Rows which rotate or slide into groups when cooperative learning.* *Rearrange the room into groups of four desks.* *Arrange three desks in a semicircle in front of a chalk board.* *Arrange desks in groups of 3,4,5. Desks should be facing each other so as to facilitate working together, grouping in tables (3 in group best), partner or groups no larger than 4 . . .*	*Ad: Good transition from traditional to groups. Dis: It takes time to get up.* *Ad: Larger work surface. Easy communication with group. Less distraction between groups. Face each other, more space, talk at all times, Dis: Space between the groups. Test situations. Whole group instruction may be difficult.* *Ad:The blackboard is good for recording purposes. Dis: Proximity - too close together. Furniture arrangement not conclusive with blackboard etc/speaker.* *Ad: Larger work surface. Easy communication with group. Dis:Test situations. Whole group instruction may be difficult.*
Equipment needs	*Portable blackboards. Tables. Area of desks for whole class. Overhead. Manipulatives. Tables and chairs instead of 1 piece items. Screens. Transparencies. Management cards of job. Number cards for teams, grouped in packages ready to use. Bell or whistle to signal coming together.*	*Ad: Good for the use of the recorder. Good for manipulatives. Advantage to get their attention or identify the end of the time for the project. Materials divided into sets. Fewer sets needed.Save decision making.* *Dis: Not everyone may 'do' the hands on work. Share items - seating problems, storage problems. Space.*

FOCUS ON CLASSROOM MANAGEMENT, continued

CONCEPT	POSSIBILITIES	ADVANTAGES / DISADVANTAGES
Assignment of roles	*Recorder-rotated job or random choosing.* *Each person has a roll.* *Rotate roles so each person gets to experience differing responsibilities.* *Timer.* *Encourager.* *Smart Kid.* *Disadvantaged learner.* *Disruptive student.* *Chronic absentee.* *Cards with "jobs"* *Could be verbal or chart.* *Let students choose . . .*	*Ad: Having a record gives the students a chance to be mature and solve it on their own.* *Groups are responsible for managing materials.* *Some students responsible for own area of weakness.* *Gives responsibility, greater participation, shared learning, active learning, successful learning, one person dominating, one person doesn't do anything, distracting student passive learning.* *Dis: Argue-time.* *Chaos.*
Record keeping	*Keep track when group changes-who is in the group, grades.* *Group turns in 1 product.* *Collect 1 paper from group.* *Peer checking.* *Student log - Journals.* *Self Evaluation.* *Student Checklist/Evaluation.* *Teacher checklist.* *Final "paper" - check together.* *Reinforce(reward) social skills/cooperating.* *Final product (project test etc.) . . .*	*Ad: Students enthusiastic.* *Quick, variety of techniques, ownership of something they couldn't do on their own.* *Self awareness.* *Focus on specific.* *Dis: Parent complaints.* *Time consuming. Too much to manage.Credit without participation.*

Geometry

Geometry plays a distinctive role in our lives. Its applications are pervasive, emerging in a diversity of physical, business and industrial settings. Geometrical principles allow us to organize, clarify, and refine visual images and build models which important mathematical concepts and relationships.

Further, geometry provides a special forum for incorporating modeling, divergent thinking, spatial visualization, logical thinking and reasoning to solve a wide variety of problems. At the middle school level, students can work in cooperative groups and independently to engage in geometric problem solving in two and three dimensions, make and test conjectures, construct and use geometric materials and available technology, communicate their results and, when appropriate, pursue related connections or applications.

As part of this process, the van Hiele model of thinking, consisting of five levels, provides a framework for characterizing and nurturing the growth of insights and understandings in geometry:

- Level 1 - *Visual:* students judge shapes by their appearance;
- Level 2 - *Analysis*:: students see figures in terms of their components and discover properties of a class of shapes;
- Level 3. - *Informal deduction:* students logically interrelate previously discovered properties;
- Level 4. - *Deduction:* students prove theorems deductively; and
- Level 5. - *Rigor:* students establish theorems in different postulational systems.

For middle grade students, it is appropriate to focus on nurturing students' growth through levels 1, 2, and 3 of the van Hiele Model. This might be accomplished by organizing geometry instruction around the van Hiele sequence of five "phases" of learning:

- The *inquiry* phase in which students are encouraged to discuss and develop questions on a geometric topic to be explored;
- The *directed orientation* phase in which students explore sets of carefully sequenced activities;
- The *explicitation* phase in which students express explicit views and questions about inherent structures of their investigations;
- The *free orientation* phase in which students now encounter multistep tasks and gain experiences in finding their own way of resolving the tasks; and
- The *integration* phase in which students form an overview in which objects and relationships are unified and internalized into a new domain of thought.

In practice these phases are not usually accomplished in sequence. Rather, students often cycle through several of the lower phases more than once before being "ready" to move into a higher phase. It is important to establish a classroom atmosphere that encourages middle school students move through these phases as they explore and investigate geometry problems, ask questions, engage in divergent thinking, and use logical reasoning to develop cogent and convincing arguments (National Council of Teachers of Mathematics, 1992). It is this challenge which LINCS teachers have accepted and the spirit in which they share the geometry ideas which follow.

Reference

National Council of Teachers of Mathematics. (1992). *Geometry in the Middle Grades: Addenda Series, Grades 5-8.* Reston, VA: The Council.

Is It Possible?

Focus: Evaluate students' understanding of the basic elements: points, line segments, rays, lines, and planes.

Explore:

- **Is it possible for:**
  - two line segments to intersect in more than one point?
  - two collinear line segments to intersect to form a ray?
  - two rays (not on the same line) to intersect in a point?
  - two rays (not on the same line) to intersect to form a line segment?
  - two collinear rays to intersect to form a point?
  - two collinear rays to intersect to form a line?
  - the union of two collinear segments to be a line segment?
  - the union of two collinear line segments to be a line?
- **Considering a line and a plane, is it possible for:**
  - a line and a plane to be arranged so they do not intersect?
  - a line and a plane to intersect in exactly one point?
  - a line and a plane to intersect in only two points?
- **Considering two planes, is it possible for:**
  - two planes to be arranged so they do not intersect?
  - two planes to be arranged so they intersect in a line?
  - two planes to be arranged so they intersect in only one point?
- If you answered yes to any of the above questions verify your response by demonstrating the solution.

Reflect: • Discuss with your partner the questions you disagree on and reach a concensus by justifying you responses. Can you reword the question to make it possible?

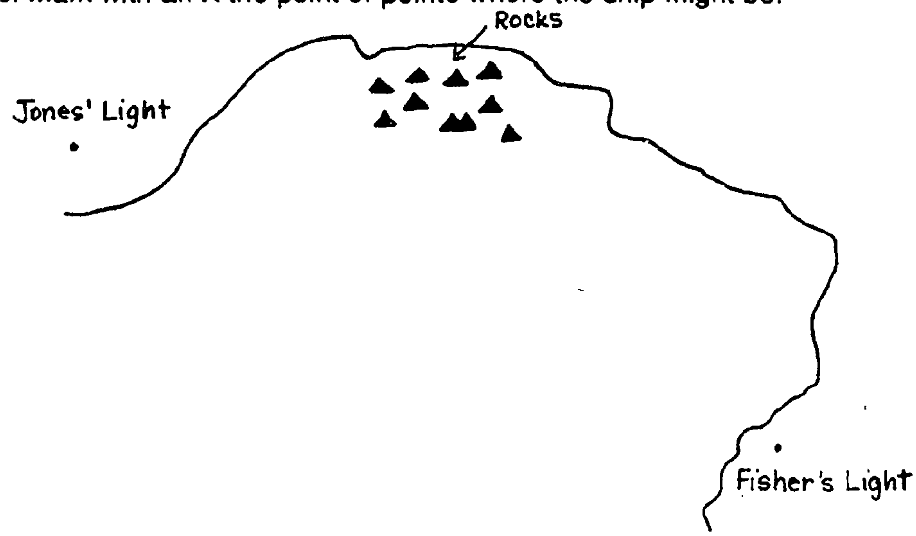
Extend: • Construct four questions similar to those presented above.

Find Your Way

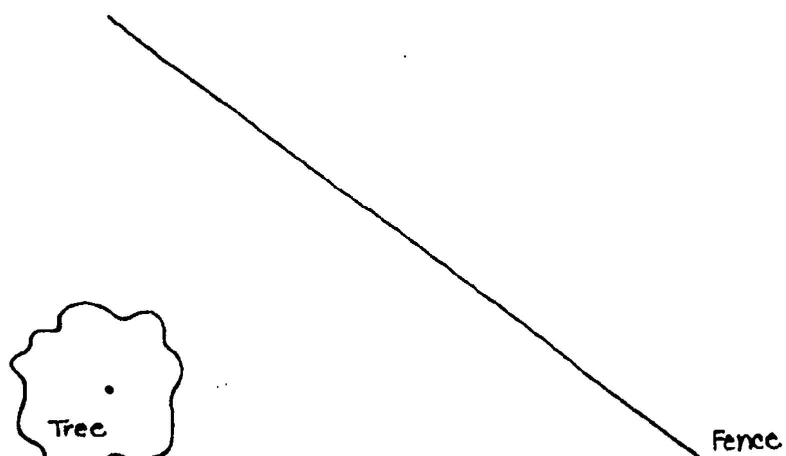
Focus: Develop an understanding of equidistant points.

Explore: • On the picture below, find the location of a ship that is 2 1/2 miles from Jones' light and 4 miles from Fisher's light by following these directions:

a. Use the scale, 1 mile = 1 inch. Locate all points that are 2 1/2 miles (2 1/2 inches on your drawing) from Jones' light.
b. Locate all points that are 4 miles from Fisher's light.
c. Mark with an X the point or points where the ship might be.



• There's a buried treasure located 10 feet from the fence and 15 feet from the tree. Use the scale: 10 feet = 1 inch. (On your drawing, 15 feet will be represented by 1 1/2 inches.)



Extend: Construct a similar problem of your own. Exchange problems with your partner and solve each others' problem.

Polygons

Focus: Allow students to explore the angle measures of polygons through the drawing of diagonals.

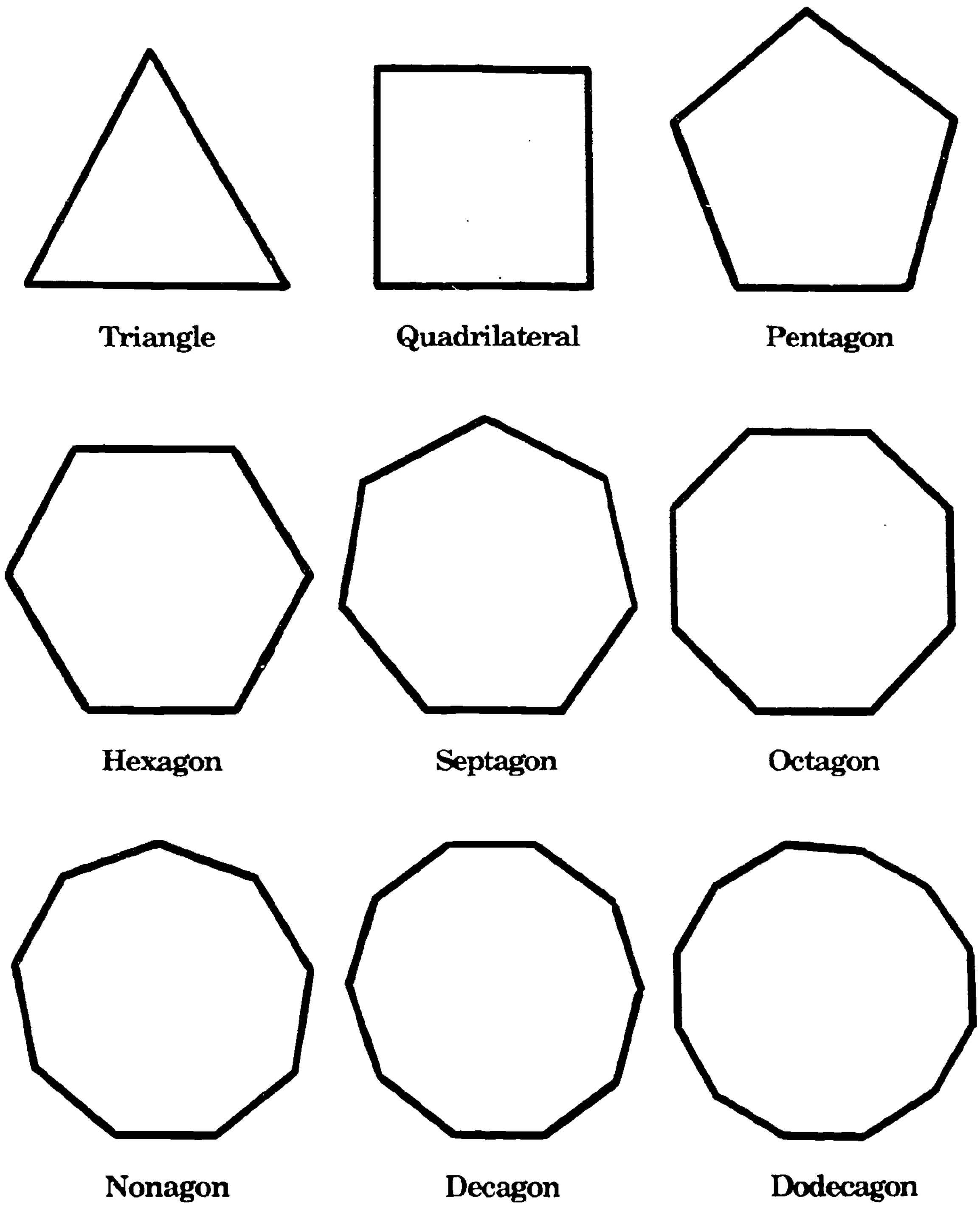
Explore: For each regular polygon, choose one vertex and draw all of the diagonals from that vertex. Then complete the table.

Name of shape	# of sides	# of diagonals	# of triangles	Sum of <u>all</u> the angles	Measure of each vertex angle (regular)
Triangle					
Quadrilateral					
Pentagon					
Hexagon					
Septagon					
Octagon					
Nonagon					
Decagon					
11-gon					
Dodecagon					
100-gon					

Reflect:
- Is there a pattern forming in your table?
- What if the polygon has n sides?

Extend:
- Can the formula be applied to non-regular polygons?
- Can the process be used to find the sum of the vertex angles of a non-regular polygon? Explain your response and illustrate it in the case of a non-regular pentagon.

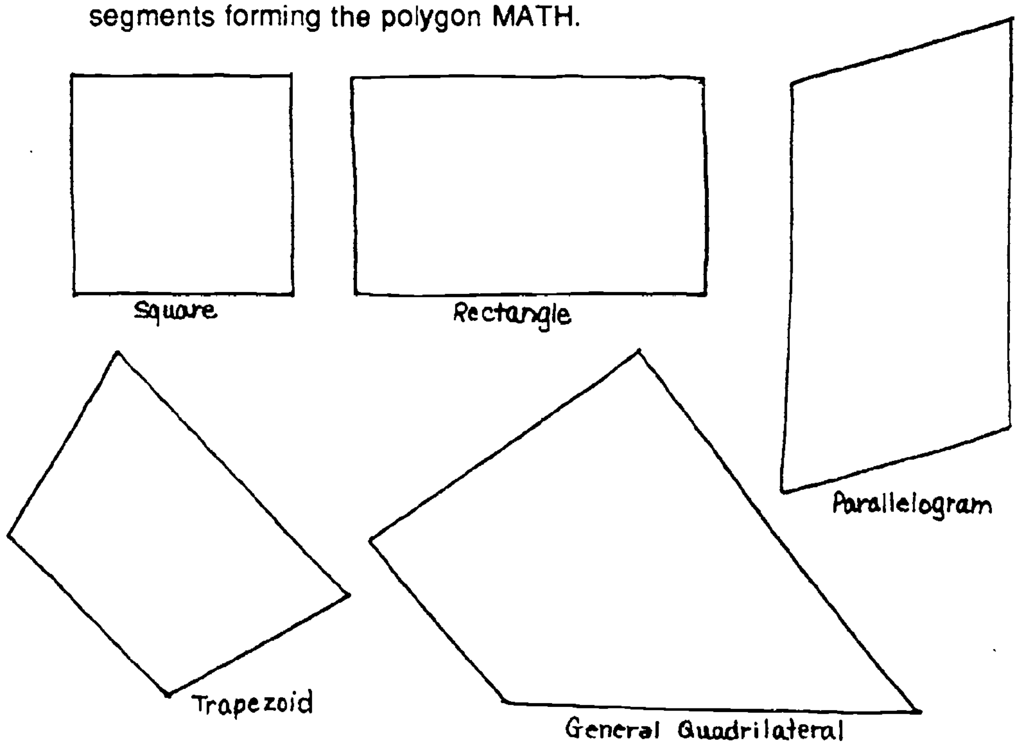
Regular Polygons



Midpoint Mania

Focus: To develop the richness of the concept of midpoint through the exploration of figures formed by joining the midpoints.

Explore: • In each of the following quadrilaterals find the midpoints of the sides and label them consecutively, M, A, T, and H. Then draw the segments forming the polygon MATH.



Discuss: • What do all MATH polygons seem to have in common? Verify your hypothesis with a partner.

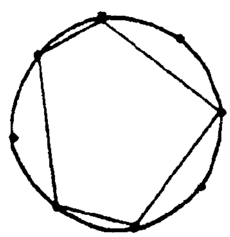
Extend: • Work with your partner to discover the relationship between the area of each MATH polygon and the area of the original quadrilateral. Record your findings in your math journal.

• Are there any exceptions to your hypothesis? Try to design a quadrilateral that does not have the same relationship. Is it possible?

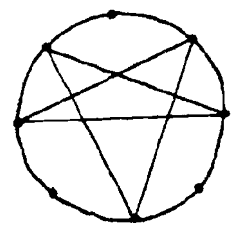
Circle the Stars and Polygons

Focus: To develop an understanding of simple and non-simple (star) polygons through exploration on circle geoboards.

Explore: Polygons of many different shapes can be drawn on a circle geoboard. If no sides cross, the polygon is referred to as simple. If the polygons sides cross then it is referred to as non-simple (see examples below).



simple

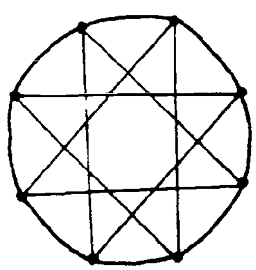


non-simple

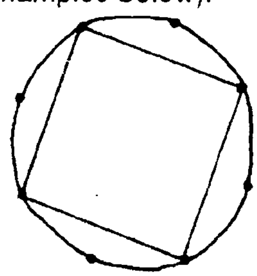
- Determine how many different simple polygons with equal side lengths can be formed on circle geoboards with 6, 8, and 15 nails.

- Is there a relationship between the number of nails and the number of sides the polygon has. How many simple polygons with equal side lengths can be formed in a circle geoboard with 60 nails? 79 nails?

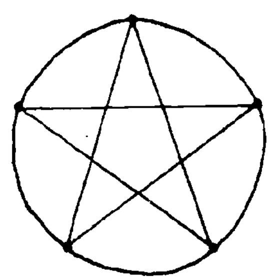
- Non-simple polygons that are formed by joining every nth point are called star polygons (see examples below).



star polygon



not a star polygon



star polygon

- How many star polygons can be drawn on a circle geoboard with 18 points? 24 points?

Extend: • Why do some values of n produce star polygons and other values of n do not?

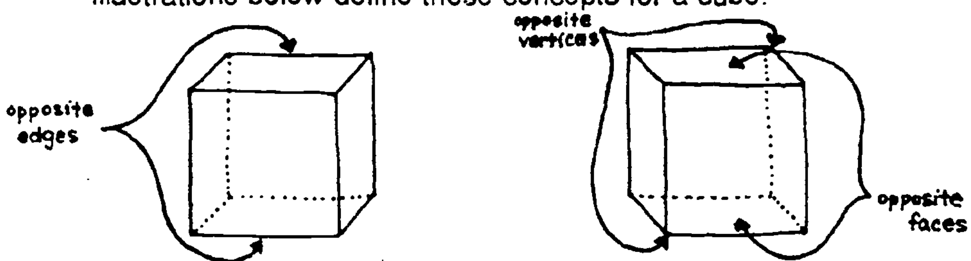
- Predict what values of n would produce a star polygon for a circle with 21 points. Test your prediction.

Modeling with Regular Polyhedra

Focus: To develop an understanding of vertices, edges, and faces of regular polyhedra through the construction and exploration of models.

Explore:

- If a three-inch cube is painted blue and then cut into 27 one-inch cubes; how many one-inch cubes are painted on three faces? two faces? one face? no faces? Can any cube be painted on more than three faces? How many on a four-inch cube? An n x n x n cube?
- Cut out the five templates on the following pages. Fold along the edges and paste to create five regular polyhedra.*
- Using the regular convex polyhedra models complete the chart below finding the number of pairs of opposite vertices, edges, and faces. The illustrations below define these concepts for a cube.



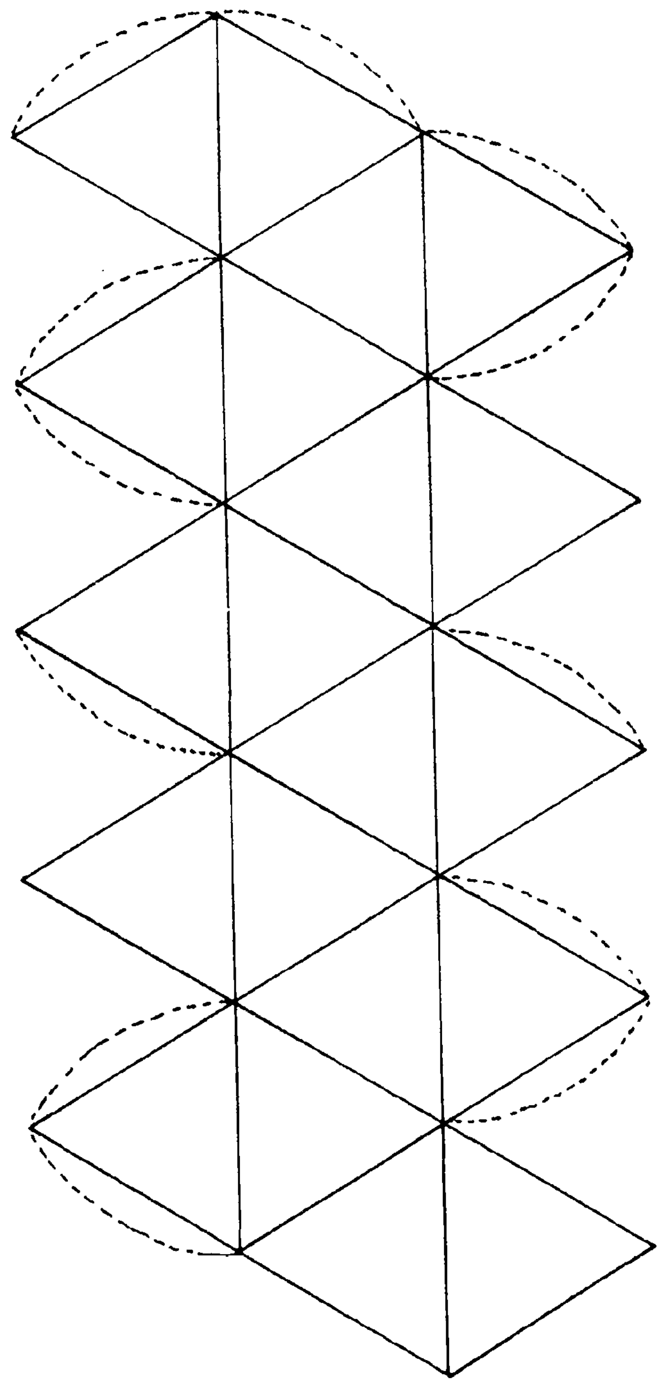
Polyhedron	How many pairs of opposite vertices? (V)	edges? (E)	faces? (F)
tetrahedron			
cube			
octahedron			
dodecahedron			
icosahedron			

Extend:

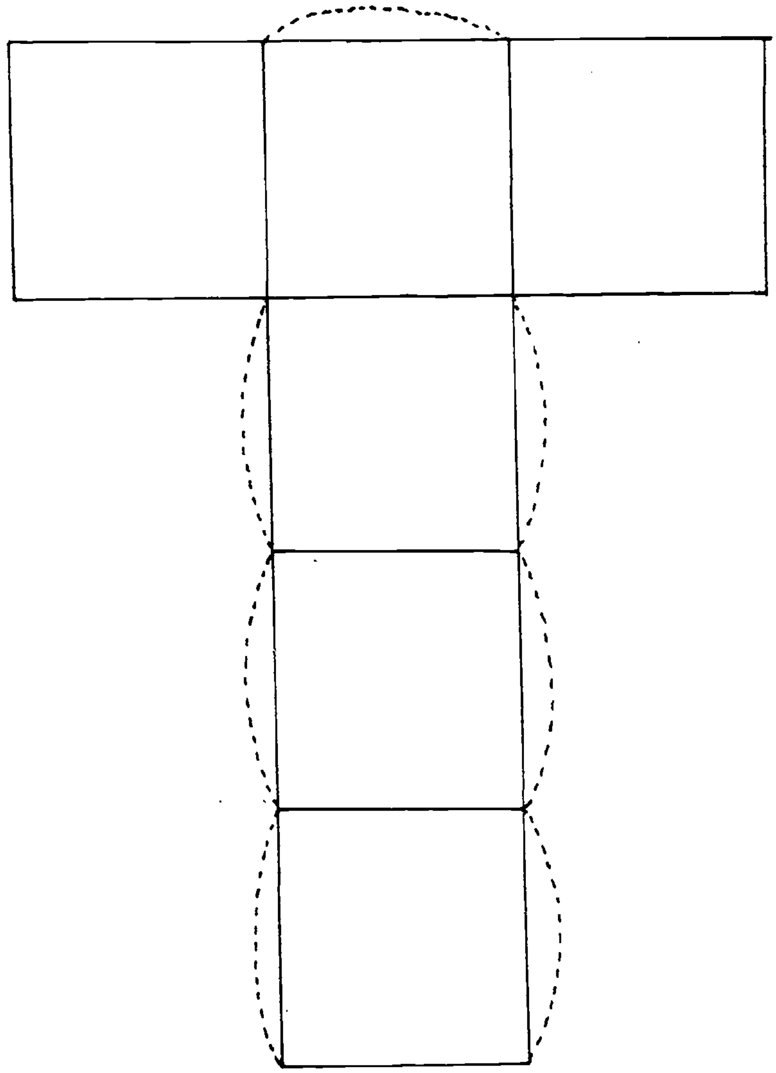
- If V represents the number of vertices in a polyhedron, E the number of edges, and F the number of faces, determine if there is a relationship between V, E, and F. (Hint: F+V is related to E - this relationship is known as Euler's formula.)

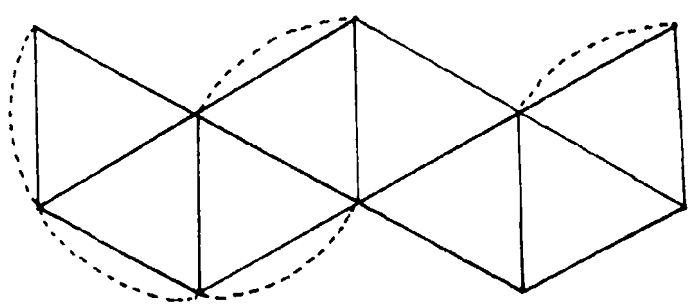
* If available, Polydron work nicely. (Available from some educational resource companies.)

Icosahedron



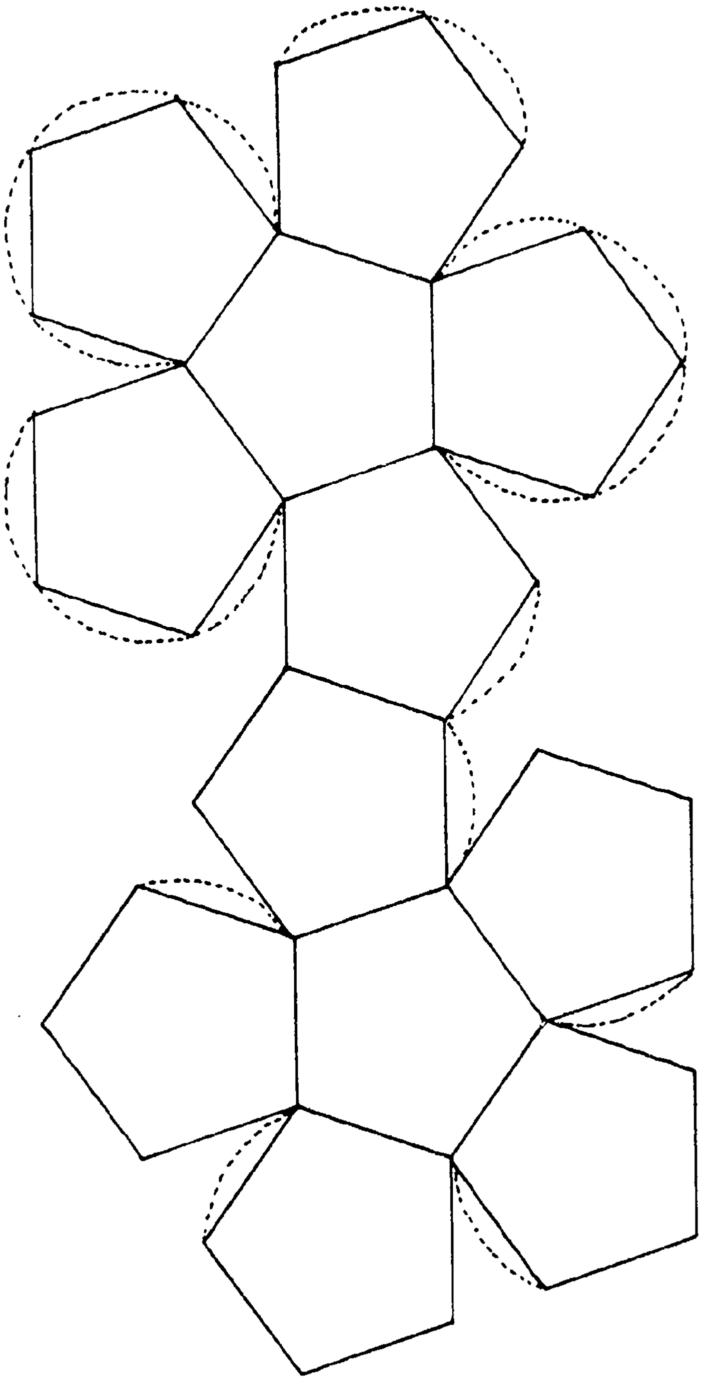
Cube

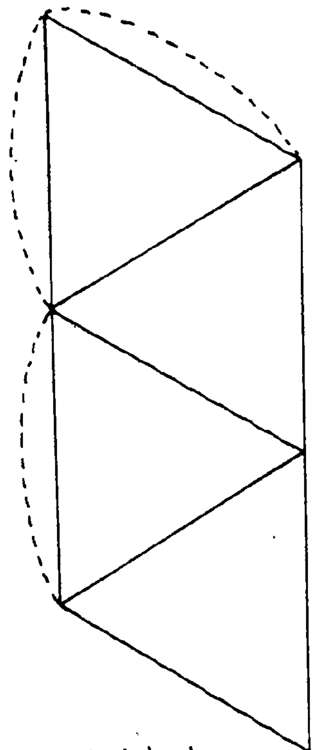




Tetrahedron

Dodecahedron





Octahedron

Logo Turtle, P. I.*

Focus: To explore angle measurements and the formation of polygons through the use of Logo.

Explore: • Pretend that you are a Logo turtle and the set of commands below was given to you, what shape would you draw? Sketch it below.

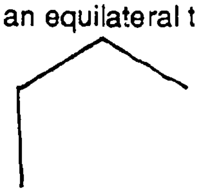
```
ST
FD 45
RT 90
FD 45
RT 90
FD 45
RT 90
FD 45
RT 90
```

Sketch:

• Type these commands into your computer and watch the turtle perform them. Does his shape match your sketch above? What shape has been drawn? What properties do the two shapes have (yours and the turtles)?

• Now think about an equilateral triangle and make a sketch. What properties does this shape have? Write a set of commands for the turtle and have him draw them. Did the turtle draw an equilateral triangle? Are the sides and angles correct? If not, what instructions should you change? Revise your commands and have the turtle draw it again.

• One student gave the turtle a set of commands for an equilateral triangle and this is what the turtle drew.



* What commands did the student give the turtle?
* How would you change the commands so the turtle draws an equilateral triangle?

• Visualize a regular pentagon and then draw one. What do you know about its properties? Write a set of commands for the turtle to draw what you have sketched. Was the turtle successful? If not, try again.

• Repeat the above procedure for a regular hexagon.

Extend: • For each of your polygons above, describe which angle of the polygon the turtle created at each vertex. Was it an interior or exterior angle? What was the sum of these angles for --

the square? the equilateral triangle? the pentagon? the hexagon?

• On the basis of this data, what conjecture would you make about the sum of the measures of these angles for regular polygons?

*** Private Investigator**

From Toothpick to Triangle

Focus: Provide students with the opportunity to develop the basic properties of triangles through investigation.

Explore: Give each student a pile of toothpicks of all the same size. Starting with three toothpicks try to form a triangle using all three toothpicks and placing them end to end. Can a different triangle be formed? What types of triangles are possible? Now answer these questions using four toothpicks. Repeat with five toothpicks, six toothpicks, and so on filling in the table below.

No. toothpicks	3	4	5	6	7
Is triangle possible?	Y				
No. of triangles?	1				
Kinds of triangle?	Equilateral				

Discuss / Reflect: What property of a triangle is the basis for determining if a triangle exists with a particular number of toothpicks?

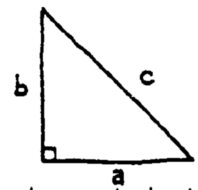
Extend: Develop a chart similar to the above table except for quadrilaterals. What properties can be determined for squares from this method of investigation? trapezoids? rectangles? kites?

Materials: toothpicks (all the same size)

Pythagorean Theorem and Tangrams

Focus: Allow students to develop the formula for the Pythagorean Theorem through the use of tangrams and the examination of the relationships between the tangram pieces.

Explore:
- Using two sets of tangrams per individual have students determine the relationships in the ratio of areas between the triangle pieces in the set (i.e. the ratio of the small to medium to large is 1 to 2 to 4).
- Have students trace the small triangle on a piece of paper (in the middle) and label the sides a, b, and c as well as the right angle.



- Using the tangram pieces, have students construct a square on each side of the triangle they traced and labeled.
- Encourage students to determine the relationship between the pieces used to construct the squares on sides a and b and the square on side c (The squares on sides a and b will cover the square on side c).
- After the relationship is determined have students record their results in their math journal.
- Figure 2 shows one possible solution to this exploration activity. Can other tangram pieces be used to show the same relationship?

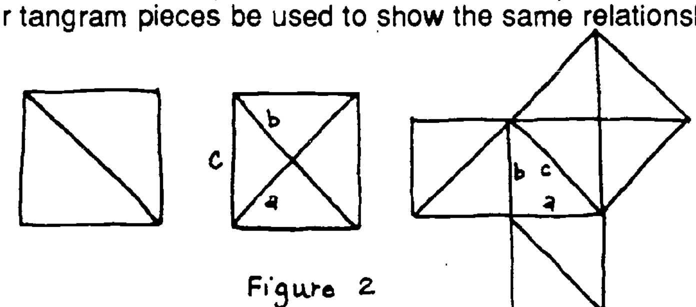


Figure 2

Extensions:
- Can the same relationship be found for the medium triangle?
- How many ways can the squares on the legs be matched to cover the square on the hypotenuse?
- Verify your results for the large triangle.
- Record your results for each extension in your math journal.

◊ ◆ Pattern Blocks ◆ ◊

Focus: Allow students to develop understanding of perimeter and area using pattern blocks.

Explore: Give each pair of students pattern blocks to experiment with the following activities in perimeter and area. The final shape must satisfy all conditions.

Perimeter

Activity One:

- Make a shape with a perimeter of 7 units. Use 4 blocks.
- Use 3 different colors of blocks. No block is orange.
- Two blocks are the same color. They are not red.
- One short side of the red trapezoid equals 1 unit of perimeter.

Activity Two:

- Make a shape with a perimeter of 8 units.
- Use 6 blocks.
- Use only 1 orange square. Each of its sides equals 1 unit of perimeter.
- Two of the blocks have 4 sides each.

Area

Activity One:

- Make a trapezoid with 12 units of area.
- Use 7 blocks in all.
- No green block may be next to another green block.
- One green triangle equals 1 unit of area.

Activity Two:

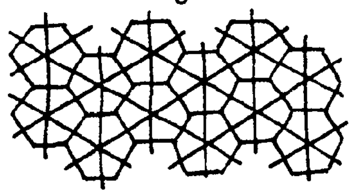
- Make a parallelogram with an area of 2 units.
- One yellow block equals 1 unit of area.
- Use 5 blocks in all.
- Do not use any red blocks.

Extend: Have students create their own activity, listing the restrictions for creating their shape. Exchange activities and allow students to recreate their new shape.

Time To Tessellate

Focus: To develop understanding of tessellations through the exploration of quadrilaterals and curved patterns formed from quadrilaterals.

Explore: Tessellations of polygons are patterns in the plane constructed using polygonal figures that completely cover the plane with no "holes" and no "overlapping." Below is a tessellation of regular hexagons with three of them surrounding each vertex.



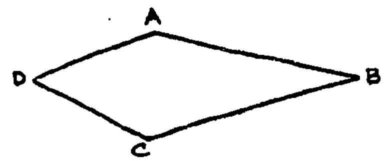
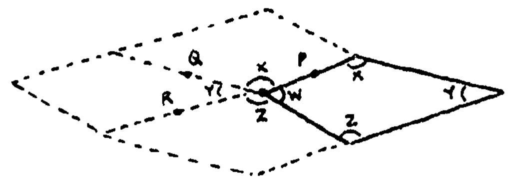


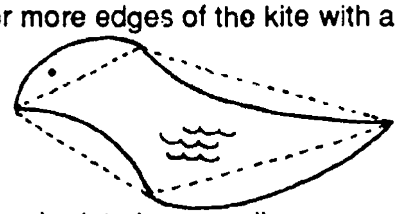
Figure 1

- The quadrilateral shown in Figure 1 is a kite. How would you describe its defining properties?
- Will a kite tessellate the plane? Use your cut out version to try it on a whole piece of paper.
- Here is one way to do it which actually works for any quadrilateral:



- Successive 180° rotations about the midpoints (P,Q,R) of the sides of the kite will generate 4 copies of the kite around the vertex O. Why does the sum of the angles around O add up to 360°?
- Since this process can be carried out at each vertex, it is possible to generate a tessellation of the original kite.
- Will the process described above work for any quadrilateral? Why? Does this mean that all quadrilaterals will tessellate the plane?

Extend:
- Replace one or more edges of the kite with a curved line.



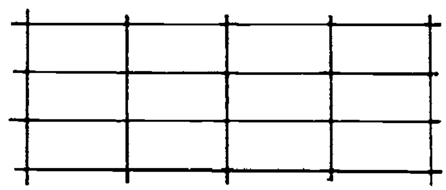
- Put a pattern on the interior as well.
- Now use the transformed kite to tessellate the plane. Draw and describe your print.

Tessellating with Regular Polygons

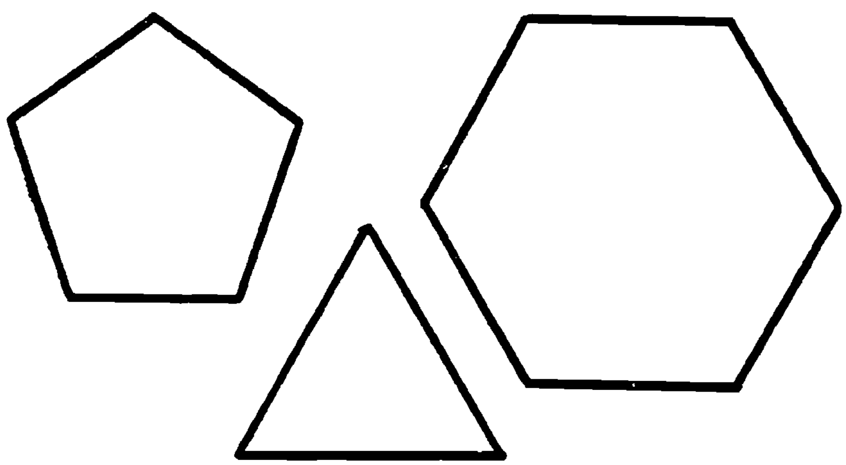
Focus: Allow students the opportunity to explore tessellations and which regular polygons will tessellate a plane.

Explore: Discuss what a tessellation is and what it means when a polygon tessellates the plane (the shape can cover a plane with no overlapping and no gaps).

- Show students that a rectangle tessellates the plane.

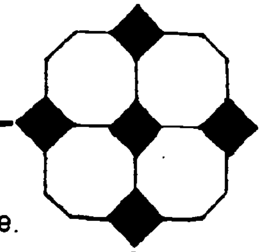


- Involve students by asking them where they have seen any tessellations (tiles in kitchen floor, sidewalk, etc.)
- Cut out 12 copies of each of the regular polygons below and have the students experiment to determine which regular polygons will tessellate the plane.



Extend: • Does the size of the vertex angle of a polygon affect whether or not the polygon will tessellate the plane?

- What angle measures will tessellate?
- Use regular triangles, quadrilaterals, and hexagons to determine the relationship.
- Describe this relationship in your journals.

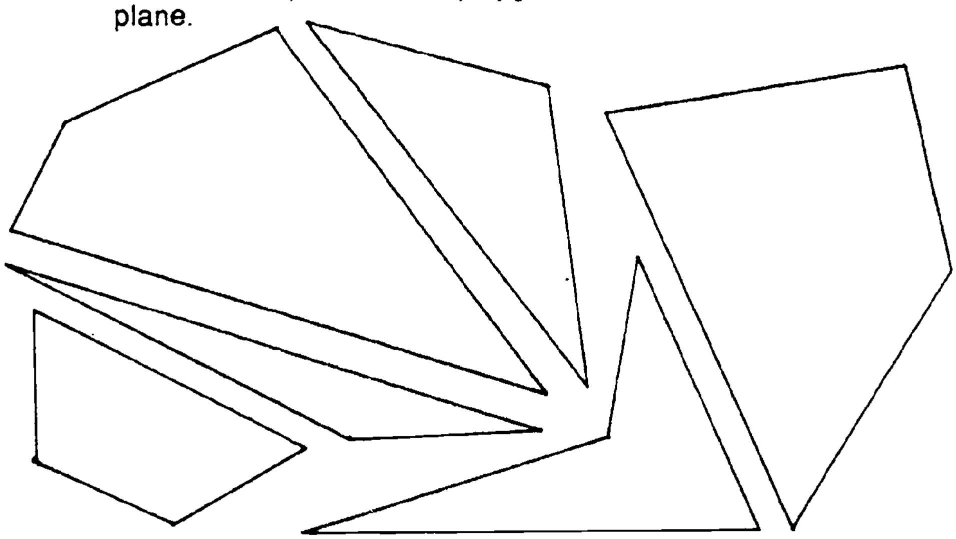


Quadrilateral and Triangle Tessellations

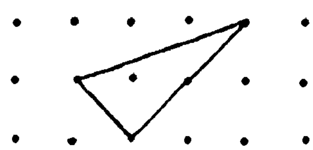
Focus: To explore which triangles and quadrilaterals will tessellate the plane.

Explore: • Two carpenters were designing a family room floor using scraps of carpet that were triangular shaped. One carpenter thought that any triangle would tessellate the plane and therefore could be used in a design. The second carpenter disagreed saying, " You can't use just any strange shaped triangle. Not all triangles will tessellate!" Do you agree with the first or second carpenter? What if they were using quadrilateral shaped carpet pieces? What it make a difference?

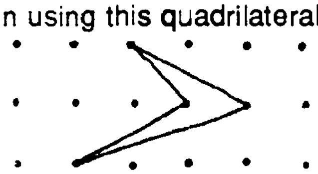
• Cut out 12 copies of each polygon and decide which will tessellate the plane.



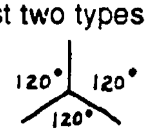
• Draw a tessellation using this triangle below and your own dot paper.

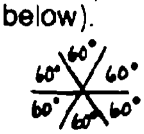


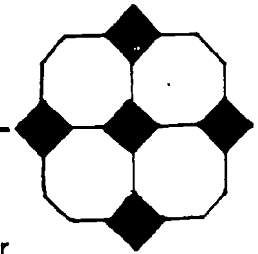
• Draw a tessellation using this quadrilateral and your own dot paper.



Extend: On isometric dot paper, draw a tessellation of a trapezoid shape in which there are at least two types of vertices (see examples below).

Some look like 

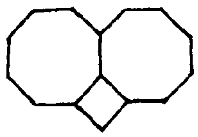
and others like 



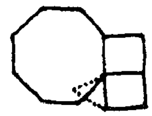
Semi-Regular Tessellations

Focus: Explore semi-regular tessellations through exploration using regular polygons.

Explore: • Some combinations of polygons can be arranged around a point with no overlapping or gaps.

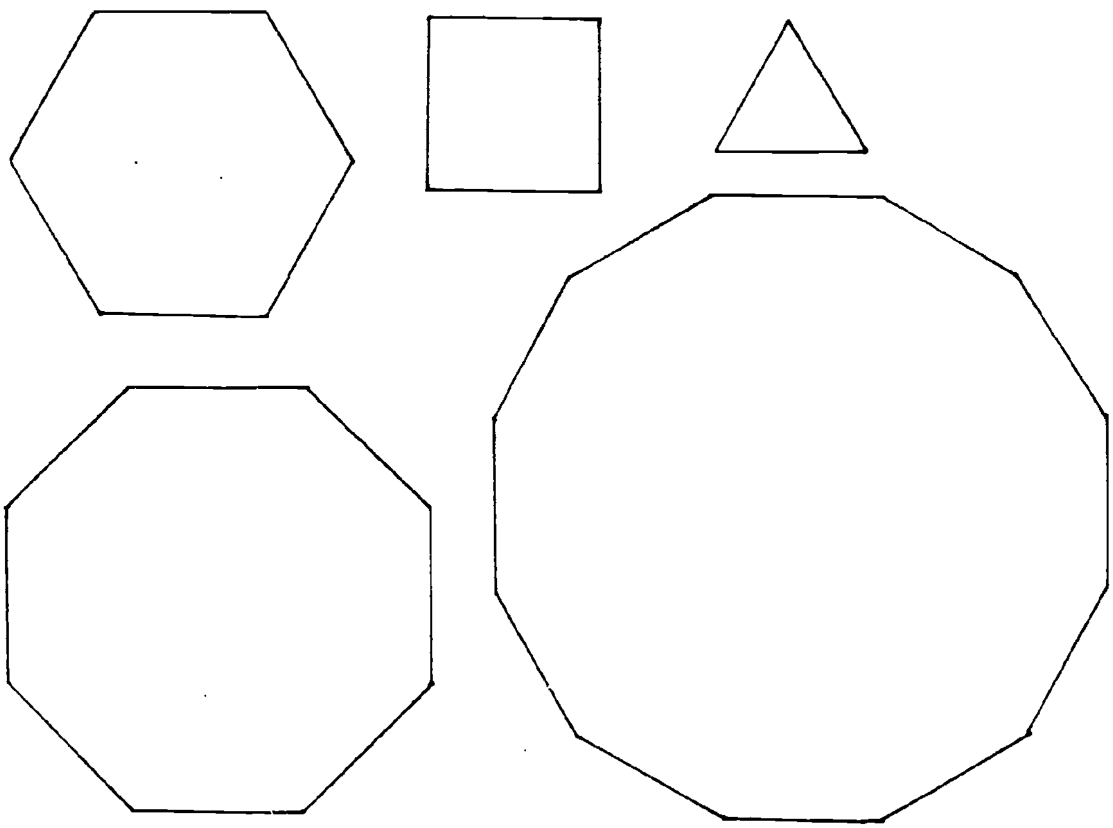


Two polygons that fit around a point.



Three polygons that do not fit around a point.

• Cut out four copies of each of these five regular polygons. How many different ways can you fit at least two of these polygons around a point with no gaps or overlapping?

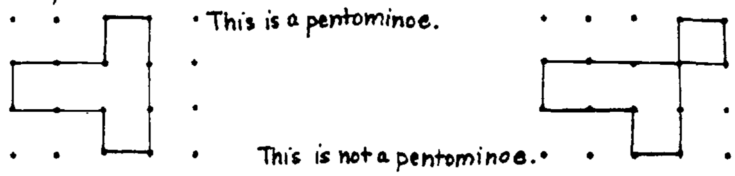


Extend: Which eight of the above vertex arrangements will tessellate the plane?

Pentominoe Who?

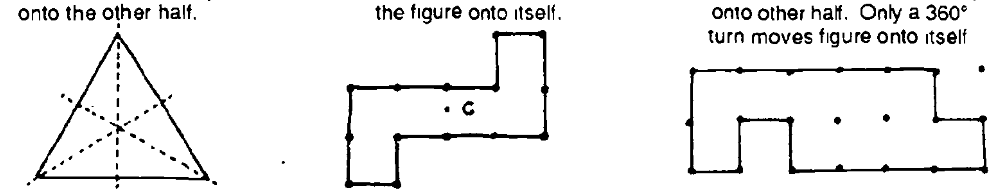
Focus: Develop understanding of line symmetry and rotational symmetry through the use of pentominoes.

Explore: • Describe the characteristics of a pentominoe to your students (a polygon made of five squares with each square touching another along the edge. Show them an example as well as non-example (Possible examples below).



• Provide each student with dot paper to determine how many pentominoes are possible. Remind student that two pentominoes are not different if they can turned, flipped, or rotated to create the same shape. (There are 12 different shaped pentominoes.)

One half folds exactly onto the other half. | A 180° turn about point C moves the figure onto itself. | Does not fold one half exactly onto other half. Only a 360° turn moves figure onto itself



3 lines of symmetry | **C is center of rotational symmetry** | **No line of symmetry. No rotational symmetry**

• Once students have discovered the 12 possible pentominoes have them analyze the pentominoes for line symmetry and rotational symmetry.

• Instruct students to redraw those pentominoes from the above exercise that have either line symmetry and / or rotational symmetry.

• Students should sketch all lines of symmetry and label all centers of rotational symmetry.

Extend: • Challenge students to arrange all 12 pentominoes on an 8 x 8 square with no overlaps.

• Place 6 pentominoes on an 8 x 8 square so that none of the remaining pieces will fit without overlapping.

• Construct a 3 x 5 rectangle using only 3 pentominoes having either line symmetry or rotational symmetry. What did you discover? What if only 2 of the 3 pentominoes had symmetry?

Angles, Symmetry, and Tessellations

Focus: To develop understanding of angle measures, symmetry, and tessellations through the exploration of pattern blocks.

Explore: • Your first task is to figure out the angle measures in each pattern block. Some possible hints are:

- Use what you already know about the angle measures in a square.
- Remember that angles which together form a **full** circular rotation measure 360°.
- Remember that angles which together form a **half** circular rotation (or that lie a straight line) measure 180°.
- Discover a way to use the mirrors to help you.

Trace the pattern blocks below and fill in the angle measures for each:

TRIANGLE **SQUARE** **PARALLELOGRAM (BLUE)**

TRAPEZOID **HEXAGON** **RHOMBUS (TAN)**

- For each pattern block:
  - circle **L** if it has **line** symmetry and indicate how many lines of symmetry;
  - circle **R** if it has **rotational** symmetry and indicate how many turns;
  - circle **P** if it has **point** symmetry.

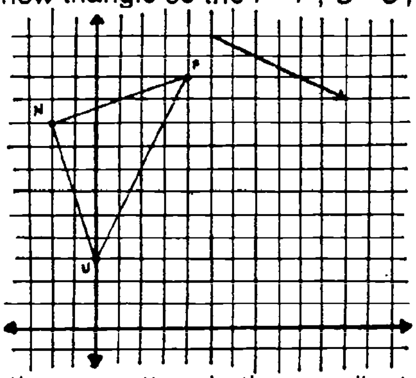
Triangle:	L _____	R _____	P	Parallelogram:	L _____	R _____	P
Square:	L _____	R _____	P	Trapezoid:	L _____	R _____	P
Hexagon:	L _____	R _____	P	Rhombus:	L _____	R _____	P

Extend: • Use at least 2 different shapes of pattern blocks to create a tessellation. By tracing around the shapes, show your tessellation on the back of this page.

Transformations and Coordinates

Focus: To explore slides, flips, and rotations on the coordinate graph.

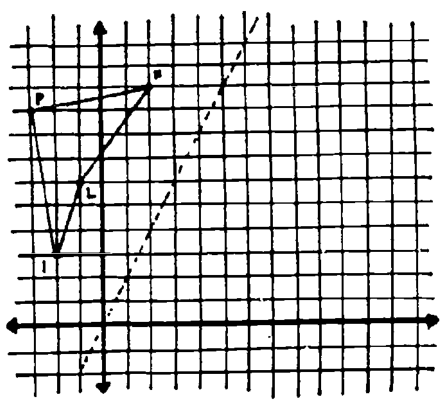
Explore: • A **slide** is shown on the coordinate graph below. Draw the triangle obtained by sliding *FUN* as indicated by the arrow. Label the points of the new triangle so the *F*→*F'*, *U*→*U'*, and *N*→*N'*. Then fill in the table:



Coordinates of the original figure	Coordinates of the image (after slide)
F(4,11)	*F'*
U	*U'*
N	*N'*

- Is there a pattern in the coordinates? Describe it to your partner.
- Describe the pattern using symbols. (I ,)→(,)

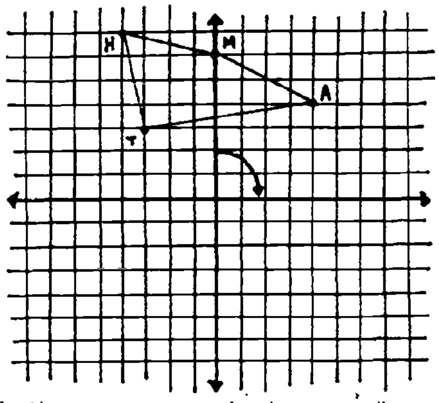
- Try this procedure for a **flip** about a diagonal for the quadrilateral *FLIP*. Draw the quadrilateral obtained by flipping *FLIP* over the dotted line and label the points of the new quadrilateral *F'L'I'P'* so that *F*→*F'*, and so on. Then complete the table below:



Coordinates of the original figure	Coordinates of the image (after flip)
F (2,10)	*F'*
L	*L'*
I	*I'*
P	*P'*

- Is there a pattern in the coordinates? Describe it to your partner.
- Describe the pattern using symbols. (,)→(;)

• Try this procedure for the last transformation, **rotations**. Rotate the quadrilateral *MATH* about the origin. Draw the image obtained by rotating it 90 degrees as shown below. Then fill in the table:

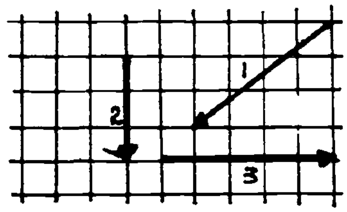


Coordinates of the original figure	Coordinates of the image (after turn)
M (,)	*M'*
A	*A'*
T	*T'*
H	*H'*

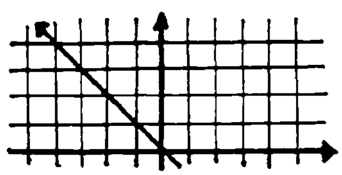
• Is there a pattern in the coordinates? Describe it to your partner.

• Describe the pattern using symbols. (,) → (,)

Extend: • Work with a partner, can you predict what will happen for other slides? Use the same procedure from above to find rules for other slides, such as those shown below.



• Still working with your partner predict and investigate what will happen for other flips. Use the same procedure for flips over the diagonal and x and y axes shown below.



• Finally, what happens for other rotations? Try it for 90 degrees counterclockwise and 180 degrees, both around the origin. Also try it for 180 degrees around the point (1,1).

Number and Computation Sense

The *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989) has identified number sense as one of the major thrusts of the mathematics curriculum and has recommended that school programs nurture the development of number sense among students. As interpreted in this document and in the middle grades Addenda book on the topic (NCTM, 1991), "number sense" refers to:

- An intuitive feeling for numbers and their various uses and interpretations;
- An appreciation for various levels of accuracy when figuring;
- A common-sense approach to using numbers;
- A recognition of whether numbers used in particular contexts are "reasonable."

An important aspect of number sense, specific to computation, is characterized by students' abilities to:

- Recognize conditions in real-world situations which reflect a particular operation;
- Demonstrate insights into the effects of an operation on two or more numbers (e.g., recognizing that adding 7 to 24 produces a far smaller change than multiplying 24 by 7);
- Select and use appropriate methods for computing from mental, paper and pencil, calculator or computer techniques;
- Use a variety of estimation, mental, calculator and written computational techniques to solve problems involving whole numbers, fractions and decimals; and

◆ Detect arithmetical errors and identify "reasonable" solutions.

As far back as the 1930s number sense, an important aspect of "meaningful learning," has been a concern for teachers. William Brownell (1935) believed in helping children to construct mathematical relationships which were sensible to them so they could make intelligent decisions based on their understandings and the context or application of those relationships. Ideally, an emphasis on number sense should permeate all aspects of mathematical learning and teaching in order to insure this understanding.

Certain characteristics can be identified in those students with a high degree of number and computation sense. Addressing a problem holistically before dealing with the details of the problem is one aspect of number sense. Using relationships among operations and numbers as well as accommodating a solution to the situation in which the problem is stated can be considered another use of number sense more specific to computation. Number sense can also be recognized in an individual's ability to detect unreasonable results to a problem's calculations and to calculate relatively simple computations mentally --perhaps in several different ways.

Teachers participating in Project LINCS have explored a variety of ways to develop number sense in their students by promoting it as an underlying theme of all their activities. A special focus has been on activities which engage students in thinking about numbers and number relationships, and on drawing connections to everyday life as a first step to the development of "good" number and computation sense. Nurturing the following kinds of thinking among middle grade students has been viewed as important amplifications of this process:

- Using benchmarks to judge number magnitude (e.g. 3/8 of 49 is less than half of 49);
- Using well-known number combinations to figure out others for which one is not so sure (e.g. 2 x 7 = 14, so 4 x 7 is twice as much (28));
- Estimating results of computations rather than calculating exact answers when approximations are adequate and efficient;
- Mentally calculating computations in any number of *different* ways;
- Judging whether a particular number constitutes a reasonable answer to a particular problem;
- And, generally, wanting to "making sense" of situations involving number and quantity.

A classroom environment which encourages students to question, verify, explore, and make sense of the mathematics around them stimulates the development of number sense. Allowing students to become active participants by sharing their hypotheses, reasoning, and conclusions to certain problems -- through class discussions and through writing -- is part of this process.

Investigating how an answer is obtained, whether it is a sensible one, and whether better alternative solutions exist become appropriate points of focus over just the solution itself. These methods encourage exploration which focuses on the meaning of an answer, invention of strategies, alertness to reasonable solutions and sense making of numerical situations.

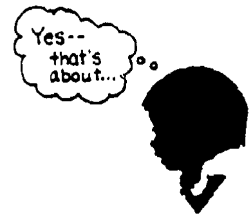
The role of the teacher in this activity assumes the new dimensions of selecting appropriate problems and tasks and moderating, redirecting, propelling, questioning, and clarifying students' thinking. Towards this end, activities and resources like those outlined below have proved useful.

References

Brownell, William. (1935). Psychological considerations in the learning and teaching of arithmetic. In *The Teaching of Arithmetic* (pp. 1-31). Tenth Yearbook of the National Council of Teachers of Mathematics. New York: Bureau of Publications. Teachers College, Columbia University.

National Council of Teachers of Mathematics. (1991). *Developing number sense.* Grades 5-8 Addenda Series to the *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.



COMPUTATIONAL ESTIMATION

1. *Use Front-End Estimation.* Compute with the leading digits in an addition or subtraction computation. Then adjust your estimate by looking at the numbers represented by the remaining digits.

   Example:
   To estimate 356 + 498 + 647 + 829, find 3 + 4 + 6 + 8 = 21 to get a rough estimate of 2100. Then adjust this estimate by noticing that 56 + 47 added to 98 make another 200. The estimate for the sum is 2300.

2. *Use Rounding.* Round the numbers to a selected place to produce multiples of 10, 100, etc. Then compute with the "rounded numbers" and give the estimate.

   Example:
   497 x 32: Round 497 to 500 and round 32 to 30. The estimate for the product is 500 x 30 or 15,000.

3. *Substitute Compatible Numbers.* Replace the numbers in a computation with numbers that are "close" to the original numbers, but which are compatible and make the computation easier.

   Examples:
   a. 26% of 28: Replace 26% with 25%, which is compatible with 28. 25% of 28 is 1/4 of 28, or 7.
   b. 4,367 ÷ 60: First round to 4,400 ÷ 60. Then replace 4,400 with 4,200, a number compatible with 60. The estimate is 4,200 ÷ 60, or 70.

4. *Use Clustering.* Look for one number that is an "average" or representative of each number in a sum. Then multiply this number by the number of addends.

   Examples:
   a. 589 + 617 + 596 + 624: Note that each number is "close" to 600. The estimate is 4 x 600, or 2,400.
   b. 37/8 + 41/5 + 32/3: Note that each number is "close" to 4. The estimate is 3 x 4, or 12.

5. Use a variety of other strategies, including student-generated approaches.

See Reys, B. J. & Reys, R. E. (1983). *Guide to Using Estimation Skills and Strategies (Boxes 1 and 2).* Palo Alto, CA: Dale Seymour, Publications (1-800-235-7566 for order information).

ESTIMATION AND LARGE NUMBERS

Focus: Develop intuitions for large numbers.

Explore

- About how many steps would it take, heel to toe, for you to walk around the school building?
- About how many pieces of paper could be stacked, floor to ceiling, in the classroom?
- About how many times does your heart beat in an hour? in a day?

Discuss / Reflect

Share your thinking and solutions approach with others in your class.

- How many *different* strategies were used?
- When was a calculator helpful?

Extend

- How would you determine the number of blades of grass on the football field or front lawn of the school?
- Write a report that describes your approach and the result obtained.

NO ANSWERS, PLEASE

Focus: Without stating the solution, tell as much as possible about the problem or different ways of solving it.

Explore

What can be said about 1.25 + 1.25 + 1.25?

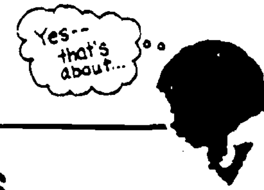
Discuss / Reflect

Share your thinking and solutions approaches with others in your class.

- How many *different* things could be said about the sum?
- How many different ways to solve the problem were found?

Extend

Pose other problems to your partners. Challenge them to come up with different ways of describing the problem without stating the solution.

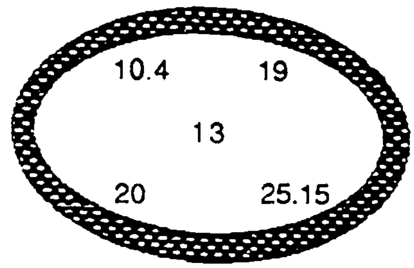


ESTIMATION WARM-UPS

Focus: Look at numbers on the board and orally give estimated answers to the given questions.

Explore

Which two numbers have a:
- sum of about 30?
- difference of about 12?
- difference of about 2.5?
- product of about 400?



Discuss / Reflect
- How many *different* ways did you and your classmates to solve each question?

Extend
- Write down 5 different numbers with a partner *and then* generate questions that have answers to match the questions.
- Do some warm-ups involving fractions or division.

ESTIMATION MAZE

Focus: Explore the effect of addition, subtraction, multiplication, and division with decimals to reach a goal.

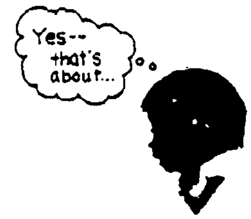
Explore

Start with 100 on your calculator. For each segment on the maze that you choose, key in the assigned operation and number. The goal is to choose a path that will result in the largest number when you finish the maze. You may not do the same segment more than once, and you may not move upward through the maze.

Discuss / Reflect
- Compare with a partner and describe your thinking or strategy for choosing your path.
- What different strategies did you and your classmates have for finishing with a large number?

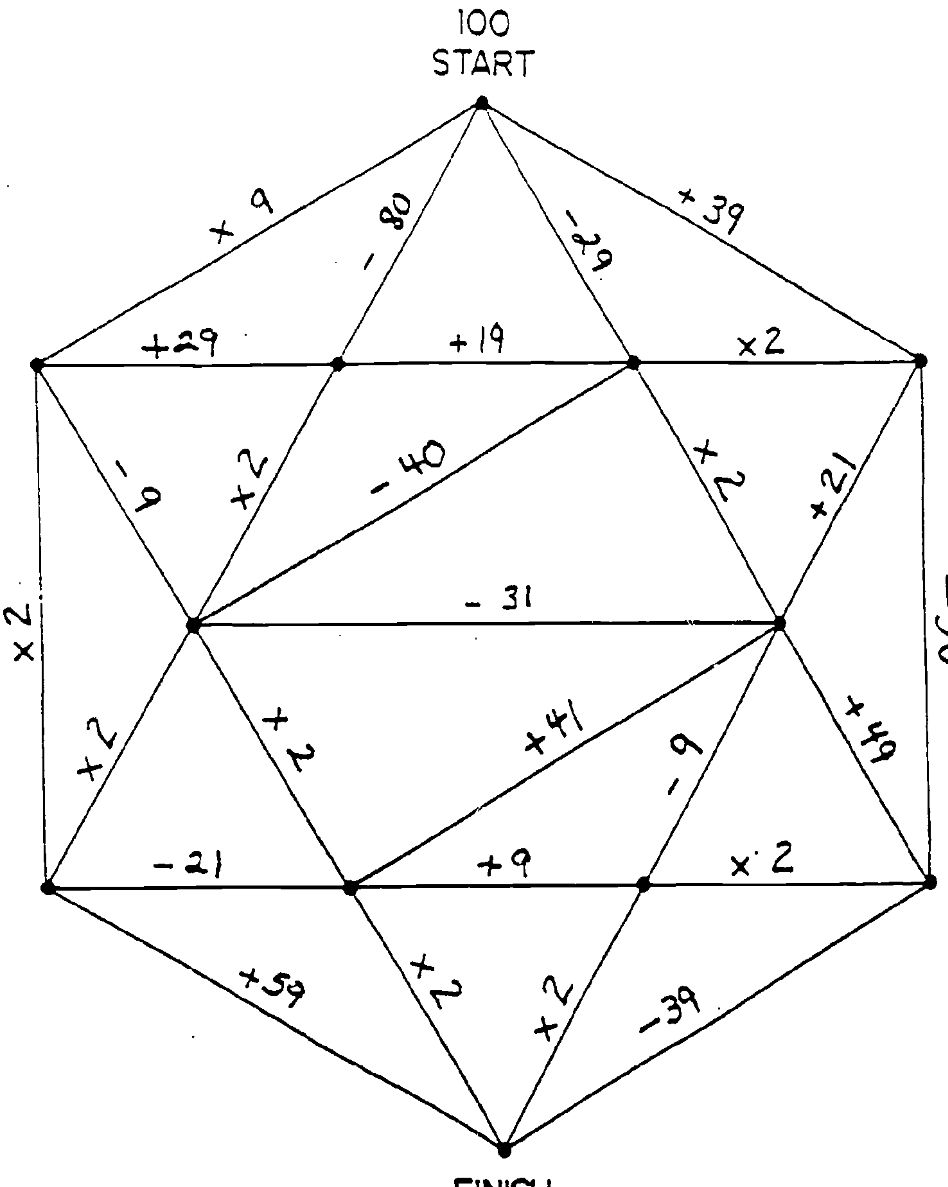
Extend

Make your own maze for your partner to do.

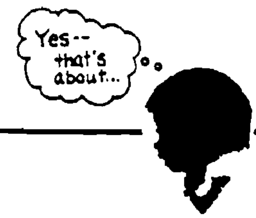


Maze

Playing Board



See National Council of Teachers of Mathematics (1991). *Developing Number Sense* :Addenda Series Grades 5-8. Reston, VA: The Council, p.41. (For a decimal version of this activity and for many other appropriate number sense ideas, call 1-800-235-7566 for ordering information.)



POSE THE QUESTION

Focus: Pose questions for a given numeric solution.

Explore

If the answer is 12, what is the question?

Discuss / Reflect

"What's 10% of 120?
"What's half the number of hours in a day?"
"What's the square root of 100, plus 2?"
"What's the square of 3 added to 3?"
"What's the denominator of 5/12?"
"What's about half of 25?"
"What's 1/3 of 36?"

Extend

Write numbers on post-its and put one on each student's back. Challenge students to determine their number by listening to questions their classmates ask about their number.

WHAT FITS?

Focus: Determine which numbers complete a story based on quantitative data in a reasonable way.

Explore

Where do these numbers "fit" in the paragraph below? 12, 80, 100, 160, 400

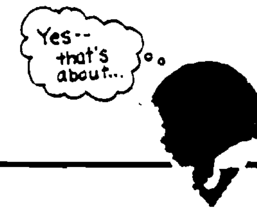
About Rainforests: In tropical rainforests, because it rains almost every day of the year, the chance of rain on any given day is nearly (100)%. The average yearly rainfall is between (160) and (400) inches each year. The temperature is in the (80)s all year and the sun shines (12) hours a day.

Discuss / Reflect

Compare where you placed the numbers with others in your group.
Explain your reasoning.

Extend

Work with other paragraphs in which numbers have been deleted.
Select from a given list to complete the story OR, if numbers are not provided, provide numbers that "make sense" to you.



LESS IS MORE!

Focus: Incorporate estimation into computational assignment exercises.

Explore

Try one or more of the following as you complete written assignments:

- For a designated row of problems, order them low to high, by size of answers. Use estimation rather than figuring out each problem.
- Do those problems having answers less than 250, or more than 5 (or any other target number selected).
- Solve only the first problem in each row of problems, and then create a problem whose answer, compared to the first problem, is:

• less	• 10 more
• about 20 less	• about 50 more
• about half	• about double

Discuss / Reflect

What estimation strategies did you use?
For given problems, how did your strategies differ from those used by others?

Extend

Create *different* "problems" that have the same answer as the first in each row.

ESTIMATE TO TELL WHAT'S REASONABLE

Focus: Give a reasonable range for the answer.

Explore

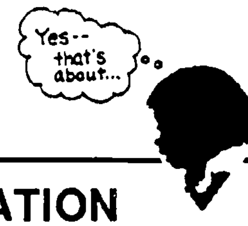
- 5 divided by 1/2
- 3/4 multiplied by 1 1/2
- .66 multiplied by 1.98

Discuss / Reflect

Share your estimation strategies with others in the class.
What different strategies were used?

Extend

Provide a real situation to match the conditions of each problem.



USING REASONING AND ESTIMATION

Focus: Decide which problems to solve with a calculator and which to estimate the answer mentally.

Explore

Choose three problems to solve with a calculator and three to solve using mental estimation. You may not solve any problems using pencil and paper.

- 100 times 799 is ____________
- 6096 divided by 34 equals ____________
- 342 - 219 equals ____________
- Two times 4 million is ____________
- (800 + 233) multiplied by (321 - 56) equals ____________
- 25.40 + 100.60 is ____________

Discuss / Reflect

- Compare with a partner and describe your thinking or strategy for solving each problem.
- What different strategies did you and your classmates have for finding an estimate?

Extend

Generate problems for your partner to do by using both the calculator and estimation.

WHAT NUMBERS FIT?

Focus: Identify two numbers that fit given clues.

Explore

Referring to the next page:

- The first column describes the number in the first circle;
  The last column describes the number in the second circle;
  The center column makes a statement true of both numbers.
- What are the numbers?

Discuss / Reflect

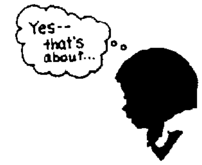
Compare your work with others. Does more than one set of rules fit the clues?

Extend

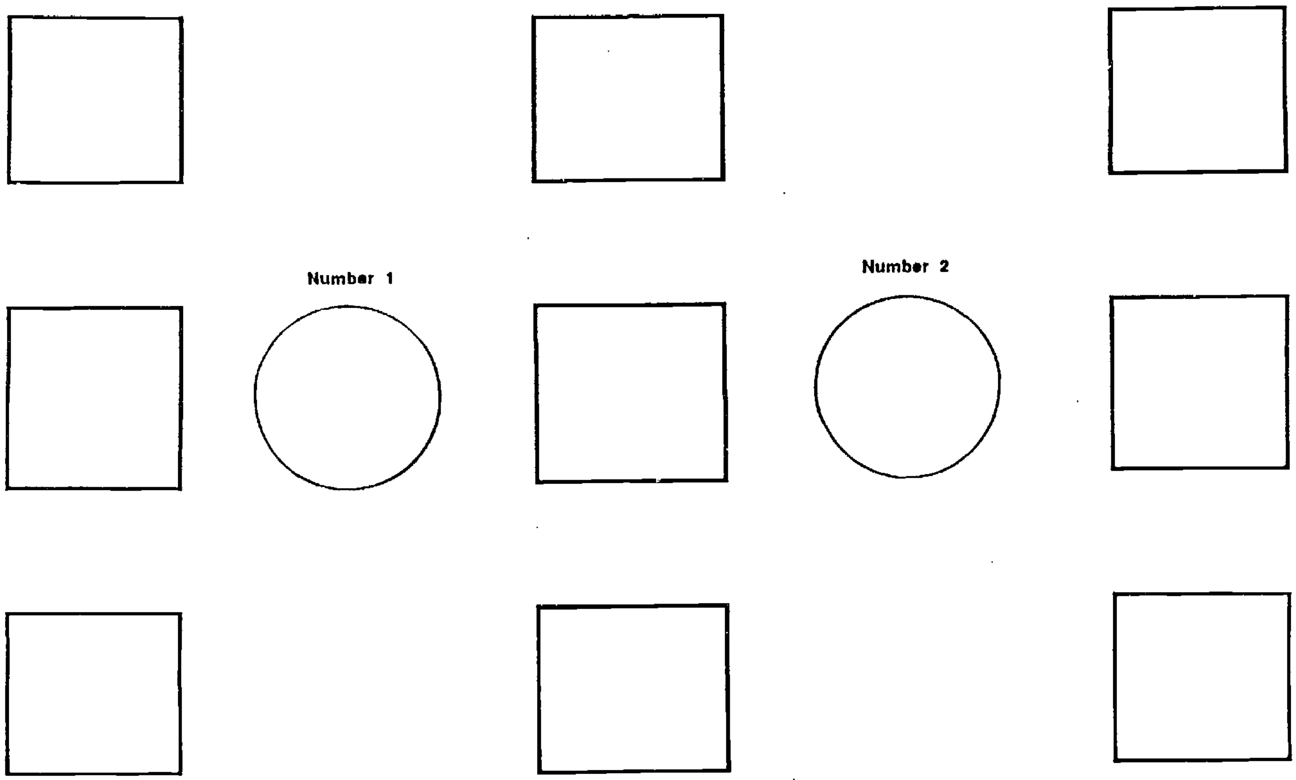
Using the blank grid on p. , create descriptions for other number pairs for your classmates to identify. Use fractions, decimals, percents, integers,...

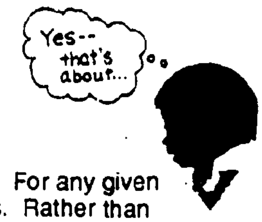
WHAT NUMBERS FIT?

proper fraction		less than 2		mixed number
seen on a measuring cup	Number 1 ( )	common denominator is 12	Number 2 ( )	equivalent to 5/3
less than half		difference is close to 1 1/2		1.66 as a decimal



WHAT NUMBERS FIT?





MENTAL MATH STRATEGIES

Present problems which invite students to use one or more of the following strategies. For any given problem, expect and encourage different students to use different mental approaches. Rather than doing "many" problems, do a few each day and provide time for students to share solution strategies. Strategies like the following have proved to be very helpful.

- Change special fractions to add or subtract.
  1/2 + 1/6 + ?   7/8 - 1/4 = ?
- Subtract fractions from whole numbers.
  3 - 1/3 = ?   6 - 1 3/4 = ?
- Use compatible fractions.
  6 1/2 + 2 1/2 = ?
- Balance to subtract fractions.
  5 1/2 - 3 7/8 = ? (add 1/8 to balance)

- Use compatibles.
  2.29 + 1.75 = 2.25 + 0.04 + 1.75
- Subtract by balancing.
  3.78 - 1.99 = 3.79 - 2
- Tack on trailing zeros.
  700 x 40 = ?
- Cancel common trailing zeros.
  6400 ÷ 80 = ?

- Add (or subtract) from the left.
  134 + 352 = 400 + 80 + 6
- Break it up.
  230 + 365 = 230 + 300 + 60 + 5
- Break up and bridge.
  3.5 + 2.7 = 5.5 + 0.7
- Use compatibles.
  2.75 + 2.99 = 2.74 + 3

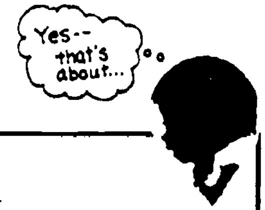
- Break up the dividend.
  315 ÷ 5 = (300 + 15) ÷ 5
- Make compatibles to divide.
  136 ÷ 8 = (80 + 56) ÷ 8
- Use fractional equivalents of percents.
  10% of $16.50
  1/5 = 20%, so 4/5 = ?
- Use simple percents to figure harder ones.
  10% of $20 = $2, so 5% of $20 = ?

- Halve one, double the other.
  5 x 12 = ?   8 x 1.5 = ?
- Think money.
  12 x 25 = ?   8 x 50 = ?
- Use compatible factors.
  25 x 9 x 4 = ?
  6 x 12 = 72, so 18 x 12 = ?
- Slide the decimal point.
  0.247 x 100 = ?
  7.98 ÷ 100 = ?

For other ideas, see:

Hope, J. A., Reys, B. J., & Reys, R. E. (1987). *Mental Math in the Middle Grades.* Palo Alto, CA: Dale Seymour Publications, (1-800-872-1100 for order information.)

______. (1988). *Mental Math for Junior High.* Palo Alto, CA: Dale Seymour Publications, (1-800-872-1100 for order information.)



THE DAILY NUMBER

Focus: Give different ways to form a given number. (If it is helpful, use unifix cubes, base 10 blocks, or a calculator to help.)

Explore

How many ways can you make 100?

Discuss / Reflect

How many *different* ways did you and your classmates find for "100"?

Extend

- Repeat, but select a different number: the date, your age or other whole number, fraction, decimal, percent, or integer.
- Create a class "number" book for which each student contributes on page with many possible ways to "make" the number featured on the page. Allow students to elicit ideas from family, classmates, and friends. Such a book makes a nice display and discussion piece for parent - teacher nights.

MENTAL ADDITION AND SUBTRACTION

Focus: Use a 100s Chart to develop skills for mentally adding and subtracting 2-digit numbers.

Explore

Take turns with a partner.

- One thinks of 2-digit addition problems like 34 + 20, 34 + 21 or 34 + 19.
- The other adds the numbers mentally, using a "peep-hole card" on a 100s Chart (see sheet which follows) to help or check.

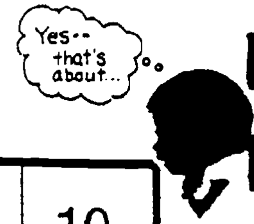
Discuss / Reflect

Share your thinking and solutions approaches with others in your class.

- How many *different* strategies were used?
- Was 100s Chart helpful?

Extend

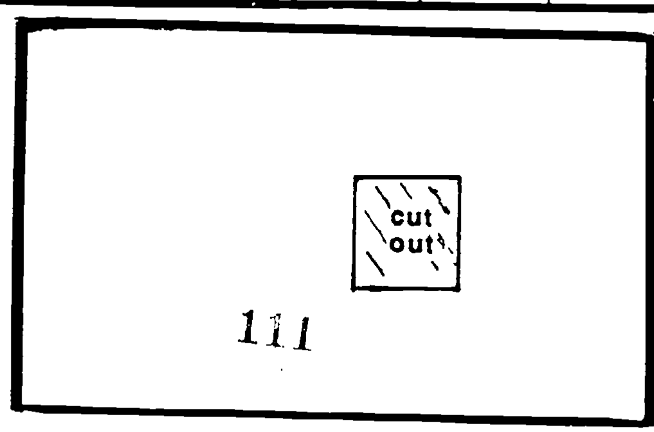
- Include harder addition problems like 34 + 47.
- Repeat with 2-digit subtraction problems.

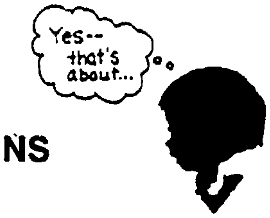


100s CHART

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Peep-hole Card:





THINKING MATHEMATICALLY WITH FRACTIONS

Fraction Kit 5 (different color) sheets of construction paper per student--same colors for all in the class; die or spinner: 1/2, 1/4, 1/8, 1/16, 1/16.

As the pieces are torn, get students to notice that:
- Fractions involve EQUAL-sized parts
- 2/2, 4/4, 8/8...each = 1, and generalize to 3/3, 5/5, 6/6, ...100/100...
- COUNT with the pieces to establish equivalences like 6/4 = 1 1/4...

Sample Games

(Modify target number to 1 7/8 or any other by combining kits)

Game #1: **Make One** with one color.

Teacher against the class. What you roll is what you get. Record to keep a running total of what each person has. Sample reading after 2 rolls by each:

Class	Teacher
1/2 + 1/16	1/4 + 1/8
8/16 + 1/16 = 9/16	2/8 + 1/8 = 3/8

Questions as you play:

"How much covered? How much to win?"
"What would it take to win on the next roll?"
"Can you win with 1 more roll?"
"What are your chances of winning on the next toss?"
"Who has covered more, you or the teacher?"

To win: roll exact amount at end.

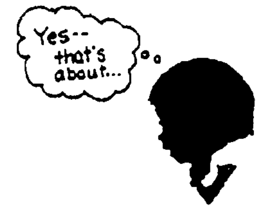
Game #2: **Make One** but keep as few pieces as possible on the board. When you can, trade for larger unit. (Trade equivalent fractions for "lowest terms".) Record the addition.

Game #3: **Go for Broke** . . . (Start with 1 whole then trade down; what you roll is what you take off.) Record the subtraction.

Variation: **Go for 3/16.** (Start with 1 7/8.)

Game #4: (Informal multiplication) Roll and take half of this off/on. Record.

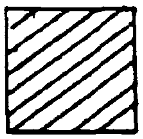
Game #5: (Informal division) Partner fills the board part way and writes the fraction. Roll and estimate about how many pieces the size of the fraction rolled are on the board. Record. Check.



PROBLEM SOLVING WITH FRACTIONS

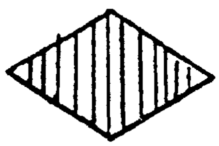
by Dale G. Jungst*

- The region below is one third of the unit region. Draw the unit region.

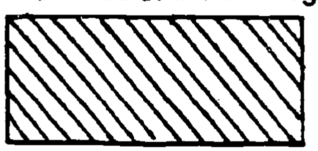


- The line segment below is one third of the unit line segment. Draw the unit line segment.

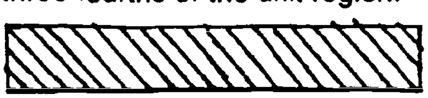
- The region below is two thirds of the unit region. Draw the unit region.



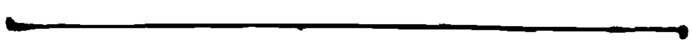
- The region below is three fourths of the unit region. Draw a region that is two thirds of the unit region.



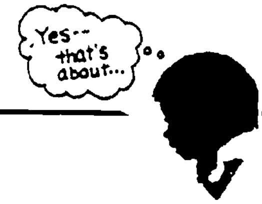
- The region below is three eighths of the unit region. Draw a region that is three fourths of the unit region.



- The line segment below is five thirds of the unit line segment. Draw a line segment that is three halves of the unit line segment.



* Used with permission of the author.



STRING ESTIMATES

Focus: Using a string, show an estimate of the number requested.

Explore

- Let about 7/8 hang over the end of the desk.
- Let about .63 hang over the end of the desk.
- Let about 58% hang over the end of the desk.

Discuss / Reflect

- Compare with a partner and describe your thinking or strategy for gettling an estimate.
  Note: Strings may be of different lengths. Consider this when comparing estimates with your partner.
- What different strategies did you and your classmates have for finding an estimate?

Extend

Create riddles for classmates to solve, show and justify with their strings.

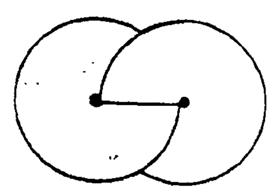
Example: "I'm thinking of a fraction that is less than 3/4 but more than 2/3. Can you name it and show me?

2-COLOR WHEEL ESTIMATES

Focus: Using a two-color wheel, show an estimate of the number requested.

Explore

- Show about 1/3 blue.
- Show about .42 yellow.
- Show about 73% blue.
- Show an angle that is about 48 degrees.



Discuss / Reflect

- Compare with a partner and describe your thinking or strategy for gettling an estimate.
- What different strategies did you and your classmates have for finding an estimate?

Extend

Create riddles for classmates to solve, show and justify using the wheels.

Probability and Statistics

Capturing valuable opportunities to link data and chance is the heart of the new focus for teaching statistics and probability in the middle school. Up to this time statistics has usually involved collecting, organizing, presenting, and interpreting data from one real world setting while probability has involved modeling an entirely different real world context.

The new focus capitalizes on the fact that most real world problems in probability depend on the collection and organization of data to establish baseline probabilities. Consider the following examples:

- A baseball player has a batting average of 0.300:
- The forecast for rain is 25%:
- There is a 30% chance that the traffic light near our school will be green when we reach it; and
- There is a 50% chance that people will hang up when they are connected to an answering machine.

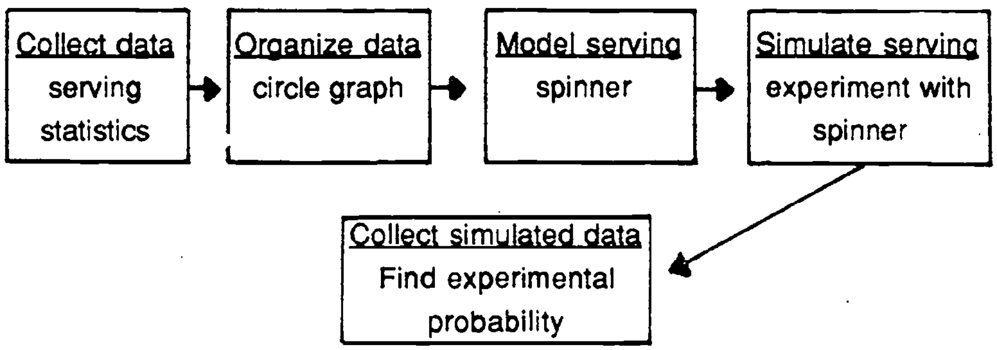
What do these probabilities mean? How were they determined? Who collected the data? Under what conditions was it collected? These are questions which are basic to the emphasis given to probability and statistics at the middle school level.

Data and Chance: Linked Through Simulation

Just as meteorologists, sporting coaches, engineers and scientists collect data to establish probabilities so too can children . 'ing activities from their real world and their own classroom. Moreover, once these probabilities have been established from data collection and calculation, the children can use simulation to solve even more complex albeit more realistic problems.

"Steffi's Serve," which is provided as an activity in this chapter, is an excellent example of such a problem. Data was collected, a probability spinner which modeled Steffi's serve was constructed and used to find the chance that she would serve a double fault.

The diagram below summarizes the process.



Activities illustrating this process, are presented on the following pages. All of these activities have been used by LINCS teachers with their students. Ideas for additional activities can be found in the books referenced below.

References

National Council of Teachers of Mathematics, Commision on Standards for School Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Jones, G.A. and Thornton, C.A. (1993). *Data, chance, and probability: Grades 4-6.* Chicago: Learning Resources, Inc.

About How Long?

Focus: Generate original data and construct a bar graph

Explore

- Have each student estimate the length of their foot with string.
- As the teacher you may wish to demonstrate this procedure.
- Allow students to check their estimates.

- -

- Have students cut the string to the actual length of their feet.
- Measure the strings (in incnes or centimeters).
- Place the strings on the chalkboard with tape as shown above.
- Discuss the differences between the estimate and actual string lengths.

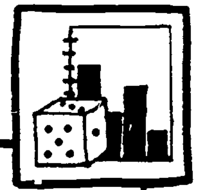
Discuss / Reflect

- Students should be allowed to group the strings into various categories (e.g. shortest to longest and 5 inch strings, 6 inch strings,etc.)
- Graph this information using a bar graph.

Extend

- Discuss the mode, median, range, and mean of the string lengths.
- Students could also construct a double bar graph comparing feet lengths of younger students vs. older students or boys vs. girls.

Need: Scissors, rulers, string, maski ıg tape, and a chalkboard or display board



WHO GROWS TALLER?

Focus: Collecting, organizing, presenting, and interpreting data.

Explore

- Students raised the question, "Who grows taller - boys or girls?" Discuss what data needs to be collected, after making a prediction on the result.
- Collect heights of all the **boys** and **girls** at the **beginning** of the school year and near the **end** of the school year.
- What are the changes in height for each student?
- How should the data be organized? (Data for a grade 6 class is provided.)

- - - - - - - - - - - - - - - - - - - -

- Constuct "box and whiskers" graphs to compare the height changes for boys and girls.
- Discuss: median, lower quartile, upper quartile, upper quartile, range, inter-quartile range, and outliers.

Discuss / Reflect

- Which group grew more?
- Was the spread of growth the same? different?
- What was the largest growth change? the smallest growth change?
- Did anything about the data surprise you?
- Write a story about your **prediction** and how the results confirmed or did not confirm it.
- Will this trend continue next year? indefinitely?

Extend

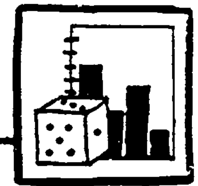
- How would you investigate this question for the population of grade 6 students in the State of Illinois?
- Describe how you would sample and collect the data.
- What do you predict the result would be for the entire State? Why?
- A data base of various measures and preferences of the students in the class can be generated for use in a number of similar activities. Discuss.

Data - Grade 6

Change in height (cm) 1990-91 School Year

GIRLS	BOYS
2	2
8.5	5
5	5.5
5.5	6
6	3
6	3.5
7.5	2.5
5.5	2
4.5	6
5	3.5
5.5	3
6.5	5
	4.5
	4

Need: Measuring tape, graphing paper, and colored pens



HOW HOT IS IT?

Focus: Collecting, Organizing, Presenting, Interpreting Data

Explore

- Find, from the newspapers, the highest temperature in your "Home Town" for January 10, January 24, February 10, February 24, and March 10.
- How would you present the data? What kind of graph would you construct? Construct a graph to present the data.
- Compare your "Home Town" with New York and Sydney. (Graphs below).

Discover

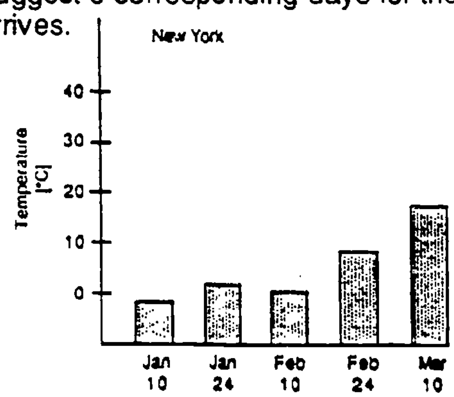
- Which town had the highest daily maximum?
- Which town had the lowest daily maximum?
- Which two towns have temperature patterens which are most alike? Which two towns are most different?
- Is the daily maximum increasing? decreasing for each city?

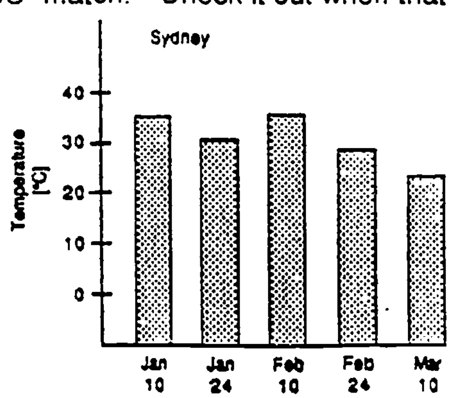
Reflect

- Did anything about the data surprise you?
- What temperature do you predict for each city on March 24?
- Will these temperature trends continue through December 24?
- Write a story about the temperature pattern of each city during this period. (Look at the globe to help.)

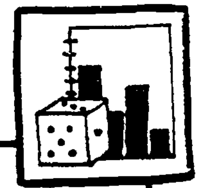
Extend

- Check the latitudes on the globe; find a US city to match Sydney. Discuss other factors in determining your match.
- Would the US "match" have a graph like Sydney for this period? for another period?
- Suggest 5 corresponding days for the US "match." Check it out when that time arrives.





Need: Newspapers, Graphing Paper, and Colored Pens



CHOCOLATE CHIPS IN THE COOKIES

Focus: Make predictions that are based on experimental probabilities.

Explore

You are making a batch of (6) cookies from a mix into which you randomly drop (10) chocolate chips. What is the probabilty that you will get a cookie with at least (3) chocolate chips?

- -

- How many faces does the die have? How could the numbers on the die be used to represent the cookies?
- Roll the die <u>once</u> - this will place the first chocolate chip. On which cookie did it drop? Keep track.
- Roll the die (ten) times to exhaust all the chocolate chips. Keep track of the cookies on which they dropped. (e.g. 3,2,6,5,5,1,2,4,5,3)
- How many cookies had at least (3) chips? What was the chance of a cookie with at least (3) chocolate chips?
- Repeat the process, i.e., randomly roll another (10) chocolate chips into a batch of (six) cookies.
- Group the results from all pairs and find the probability that there will be a cookie with at least (3) chocolate chips.

Discuss / Reflect

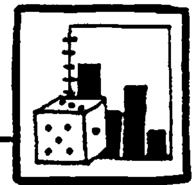
Talk about how the problem was modeled. Write a story about the results.

Extend

The baker makes 100 cookies and randomly drops 300 chocolate chips into the mixture. What is the probability of getting a cookie with at least 4 chocolate chips?

- What could you use instead of a die to represent the cookies?
- Describe how you would simulate the process and if there is time carry it out.
- How could computer generated random numbers be used?

Need: Dice for each group, paper, and pencils



MOVIE MANIA

Focus: Collection and simulation of data

Explore

- One-fourth of the students in a state have seen Movie A; half have seen Movie B. About how many students have seen neither movies?
- Construct two spinners to model the viewing of movies A and B. ,Take turns and perform the experiment to simulate interviewing 30 people in a class.
- Spin each spinner and tally the outcome.
- Repeat until there are 30 tallies.

		Movie A		
		Seen	Not Seen	Totals
Movie B	Seen			
	Not Seen			
	Totals			

Discuss / Reflect

- Describe the results of the simulation.
- What percentage saw neither movie? both movies?

Extend

- Why does the simulation model the movie viewing?
- Is the class result different from the state statistics?
- Have the class choose two popular movies and perform the experiment again. Interview 30 students in another class.

Need: Paper clip for spinners.

WHO'S WHO?

Focus: Interpreting data from a stem-and-leaf plot

Explore

These stem-and-leaf plots each represent the heights (in inches) of two groups of students, a group of first graders and a group of high schoolers.

Discuss / Reflect

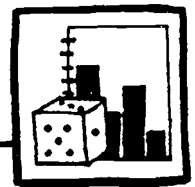
- Which stem-and-leaf plot most likely represents the first graders? Explain.
- What is the height of the tallest first grader? the shortest?
- What is the height of the shortest high school student? the tallest?
- What is the range of heights of the high school students? the first grader? all of the students?
- How many first graders are taller than 4 ft 2 in?
- How many students are exactly 5 ft 6 in tall? 3 ft 8 in?

Extend

- Combine the information from the two stem-and-leaf plots above and construct one ordered plot.
- Write five questions for your partner to answer about your new stem-and-leaf plc

0	
1	
2	
3	7
4	9 4 3 2 6 4 2 1 8 5 5 7
5	
6	
7	

0	
1	
2	
3	
4	
5	7 9
6	4 2 5 1 7 7 8 6
7	2 1 0



WHAT'S A MINUTE?

Focus: Use collected data to create a box plot

Explore

- Who would estimate a minute more acurately, a student sketching a self-portrait or a student doing nothing?
- Flip a coin to determine which partner draws the self-portrait.
- Note the time and tell your partner to START estimating one minute.
- Record the time that has elapsed when your partner says "one mintue."
- Switch roles.
- Construct a class back-to-back stem-and-leaf plot to display the collected data.

- Which group's estimates were least scattered: the portrait or no portrait group?
- Use the class stem-and-leaf plot to make back-to-back box-and-whiskers plots.
- Use an * to mark any outliers.

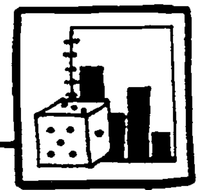
Discuss / Reflect

- Was your estimate correct?
- How do the groups compare?
- Which group was more scattered?

Extend

- Of mean, median, mode, range, and lowest / highest score which could you use to compare the two groups?
- Is it possible to detemine an "outlier" more precisely? How would you go about this?

Need: Watch or clock with a second hand, and 1 coin



JOAN DYER IS UP TO BAT

Focus: Interpreting, Determing Chance, Simulating

Explore

The table below gives the record for Joan Dyer's last 100 times at bat during the softball season.

Joan Dyer at Bat

Home Runs	8
Triples	2
Doubles	14
Singles	22
Walks	10
Outs	34
Total	90

Joan Dyer is now coming up to bat. Based on the data:

- What is Joan's chance of getting a home run? a hit?

- Construct a spinner that models Joan's batting based on the data.
- Use the protractor to determine sectors for home runs, doubles, singles, walks, and outs.
- Use the spinner to find out how many home runs Joan scored in the next 20 times at bat. How does this compare with your expectation from the data?

Discuss / Reflect (Class Activity)

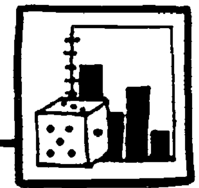
- Average the spinner results (for the next 20 times at bat) across all pairs. What is the average? How does the average compare with the expectation from the data?
- Did "averaging" help?

Extend

What is Joan's chance of getting hits on each of her next two times at bat?

- Use the spinner; simulate her next two bats, collect data for 20 simulations (of two bats).
- What percent produced "both" hits?
- What is Joan's chance of getting hits on both of her next two times at bat?
- Make up another problem like this extension. Share it with your partner. Solve each other's problem.

 Need: Spinner, Protractor, Colored Pen



SHOOTING A "ONE-ON-ONE"

Focus: Collecting, determining chance theoretically, by geometry, and by simulation.

Explore

Have each pair collect data on their basketball "shooting" performance. Each throws 10 shots per morning for a week from the foul line and records the results.

- How many "shots" did you take?
- How many did you make?
- What is your probability (chance) of shooting a basket, if you were to step up to the foul line now?
- Establish the probability for each student. Discuss it.

Discuss / Reflect

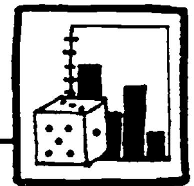
- Each student constructs a spinner that models his/her "free throw" shooting, based on the data.
- Use the protractor to determine sectors for a "make" and a "miss."
- Use the spinner to find how many times you "make" a basket in 50 shots. How does this compare with your expectation from the data?

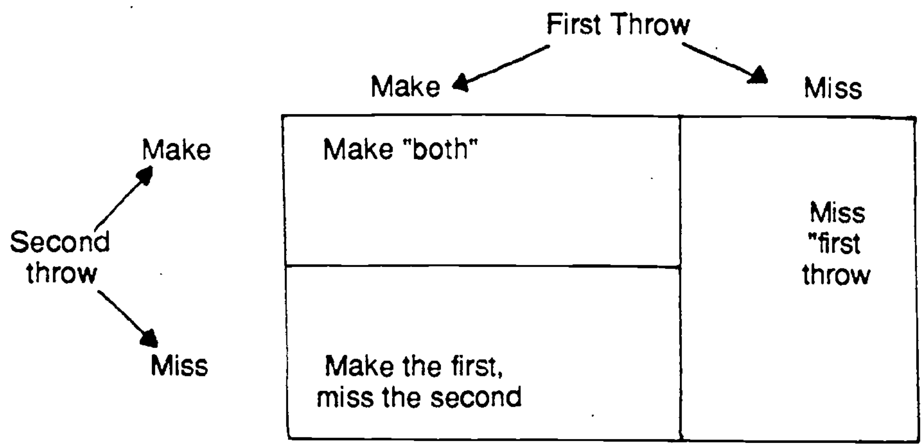
Extend 1: Shooting a "One-On-One" - Simulation

- Use the spinner to simulate a "one-on-one" (i.e., if you make the first you try the second). Record whether you scored 2,1, or 0 points.
- Collect data for 30 simulations (i.e., 30 "one-on-ones").
- What percent produced 2 points, 1 point, or O points?
- What is your chance of getting 2 points on your next "one-on-one"?
- Make up another problem like this extension. Share it with your partner. Solve each other's problem.

Extend 2: Shooting a "One-On-One" - Geometrical Approach

- How could you use area measurement to find the chances of getting 2 points, 1 point, or 0 points on a "one-on-one"? How can the "unit squares shown below help you?

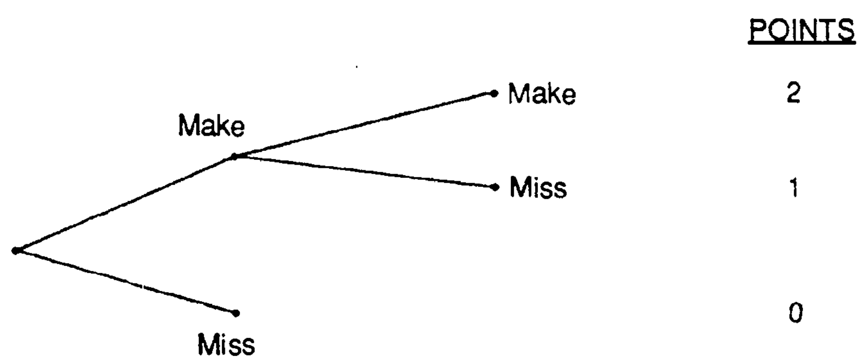




- Use your probability of "shooting" a basket and your probability of "missing" as appropriate lengths on the sides of the unit square.
- Calculate the relevant areas to find your chance of scoring 2,1, or 0 points in a "one-on-one".
- Compare the results with those found in **Extension 1**.

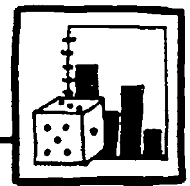
Extend 3: Shooting a "One-On-One" - Tree Approach

- How could you use a tree diagram to model the chance of getting 2, 1, or 0 points on a "one-on-one". Consider the tree diagram below.



- Where will you show the probabilities of a "make", a "miss"? How do you combine the probabilities to determine the chance of getting 2,1, or 0 points? Use the area approach in **Extend 2** to help you decide.
- Compare the results with those found in **Extends 1** and **2**.

Need: spinner, protractor, paper, and pencils

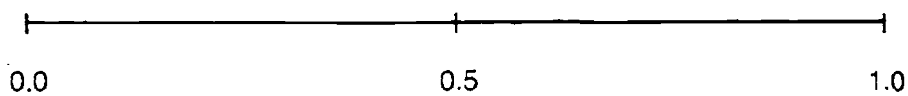


Super Serves

Focus: Modeling of experiments

Explore

- Steffi is ready to serve. Given her serve statistics, what do you estimate are her chances of a double fault? X it on the scale below.



- Design a spinner to appropriately simulate Steffi's serving.
- How many spins are needed to simulate two serves?
- Did a "double fault" occur? Tally to show.
- Repeat until each student has taken 10 turns.
- Create a class graph to display the collected data.

Discuss / Reflect

- Based on the class graph, about how many double faults were there?
- What was the total number of possibilities?
- Determine the experimental probability of two faults.

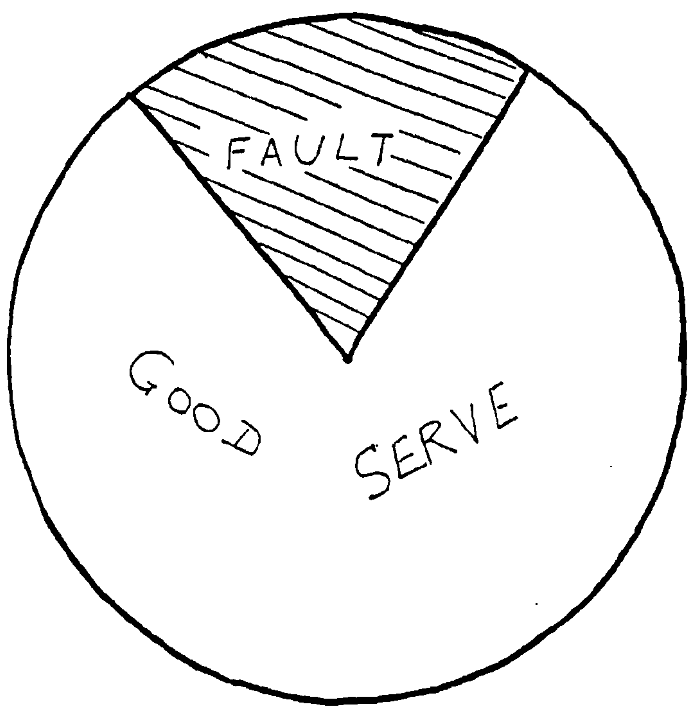
Experimental Probability: $\dfrac{\text{number of successes}}{\text{total possibilities}}$

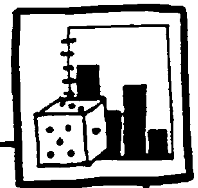
Extend

- If on a bad day Steffi serves 6 faults in 10, how will this affect her chance of serving a double fault?
- Design a spinner to simulate Steffi's serving on a bad day.
- Determine the experimental probability of two faults on a bad day.
- How does this compare with your prediction?

Data for Super Serves

# of Good Serves # of Faults	80 20
Total	100





FLIP THE COIN

Focus: Determine experimental probabilities

Explore

- If a coin is tossed, what are the possible outcomes?
- On the scale, x your probability prediction for a tail.

0.0 impossible	0.5	1.0 certain

- With a partner, flip a coin 40 times and tally the results for tails.
- As the teacher polls the class, enter (x_1, y_1) as (1 (for 1 tail), # of pairs who recorded 1 tail); (x_2, y_2) as (2 (for 2 tails), # of pairs who recorded 2 tails); and so on to (x_{40}, y_{40}): [2nd] [MATRX] [▶] [▶] [1] [ENTER] (TI-81).
- Check the range: [RANGE] (TI-81).
- Make the histogram for the entered data: [2nd] [MATRX] [▶] [1] [ENTER] (TI-81).
- Calculate the mean: [2nd] [MATRX] [1] [ENTER], read and record $\bar{x}$ for later use (TI-81).

Discuss / Reflect

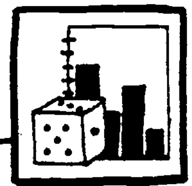
- From the histogram, which numbers of tails occurred most frequently?
- Complete and discuss the information in the table.

	Heads	Tails
$\bar{x}$		
$\frac{\bar{x}}{40}$		

Extend

- If the experimental probability for tails is $\frac{\#\text{ of tails}}{\#\text{ of flips}}$ why does $\frac{\bar{x}}{40}$ represent the experimental probability using the class data?
- Why does the data provide a better probability estimate than data from a single pair?

Need: 1 coin for each pair of students



FLIP THE CUP

Focus: Determine experimental probabilities

Explore

- If a cup is flipped, what are the possible outcomes?
- On the scale, x your probability prediction for each outcome and label it.

0.0 impossible — 0.5 — 1.0 certain

- With a partner, flip a cup 25 times and tally the results for each of the outcomes.
- As the teacher polls the class, enter (x_1, y_1) as (1 (for 1 cup down), # of pairs who recorded 1 cup down); (x_2, y_2) as (2 (for 2 cups down), # of pairs who recorded 2 cups down); until (x_{25}, y_{25}) has been entered: [2nd] [MATRX] [▶] [▶] [1] (TI-81).
- Check the range: [RANGE] (TI-81).
- Calculate the mean: [2nd] [MATRX] [1] [ENTER], read **and record** $\bar{x}$ for later use (TI-81).
- Clear the data and repeat for the other 2 outcomes: [2nd] [MATRX] [▶] [▶] [2] [ENTER] (TI-81).

Discuss / Reflect

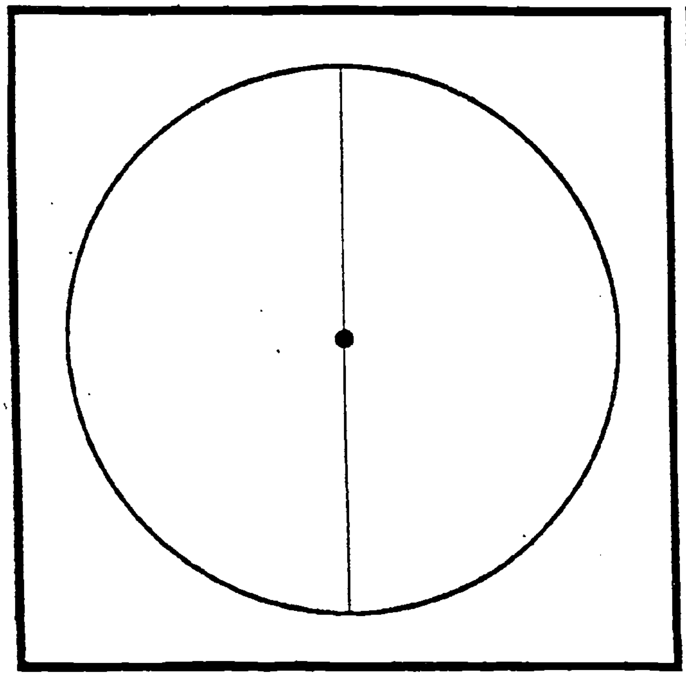
- Complete and discuss the information in the table.

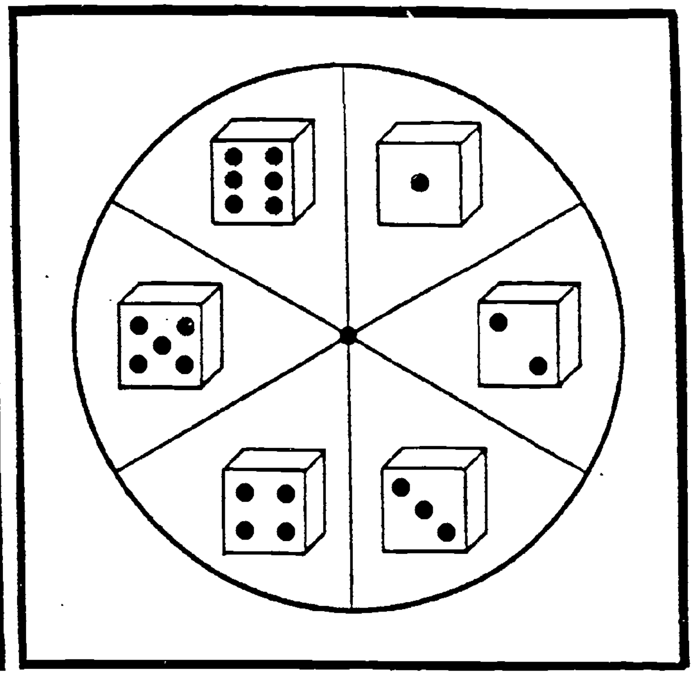
	DOWN	UP	SIDEWAYS
$\bar{x}$			
$\frac{\bar{x}}{25}$	$\frac{\quad}{25}$	$\frac{\quad}{25}$	$\frac{\quad}{25}$

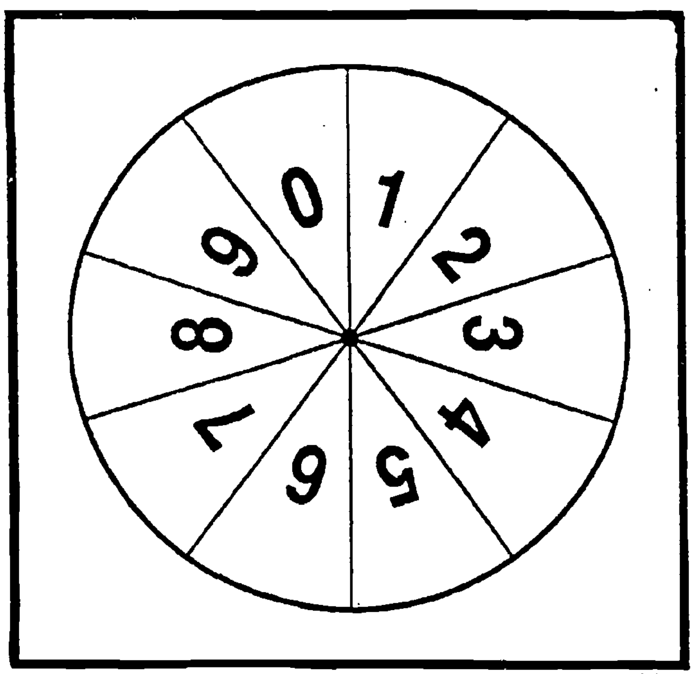
Extend

- Why does $\frac{\bar{x}}{25}$ represent the experimental probability using the class data?
- How does the experimental probability for the cup differ from those for the coin?

Need: 1 paper cup for each pair of students and probabilities from "Flip the Coin"



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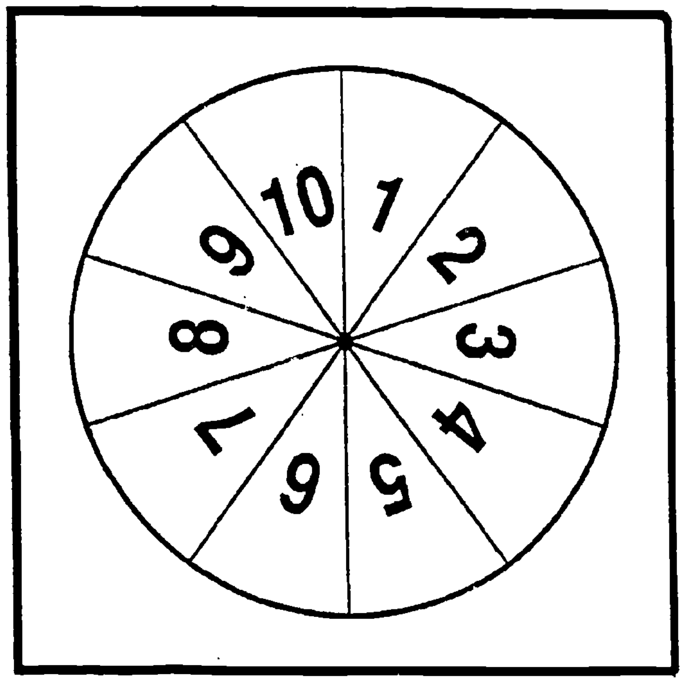
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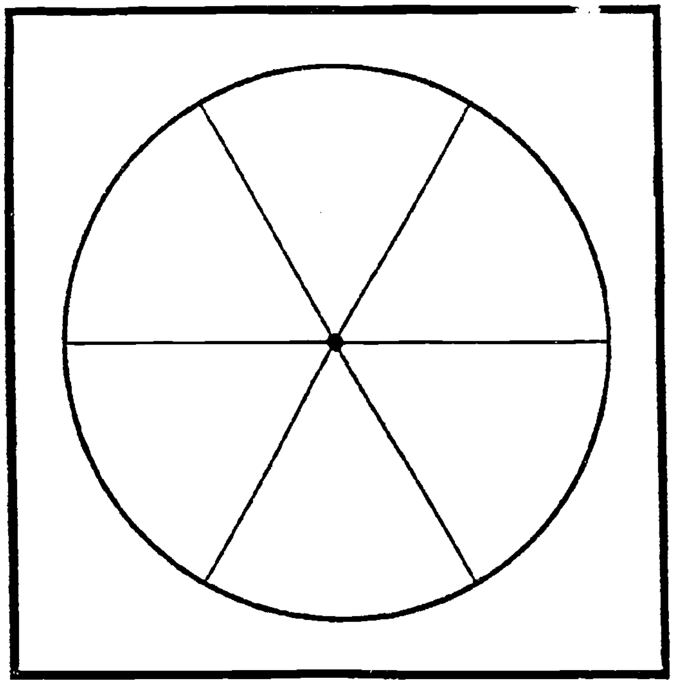

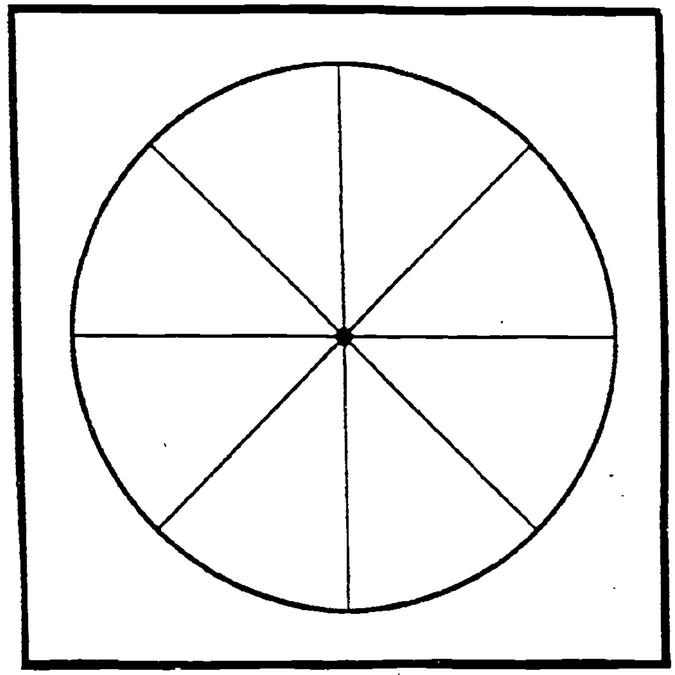

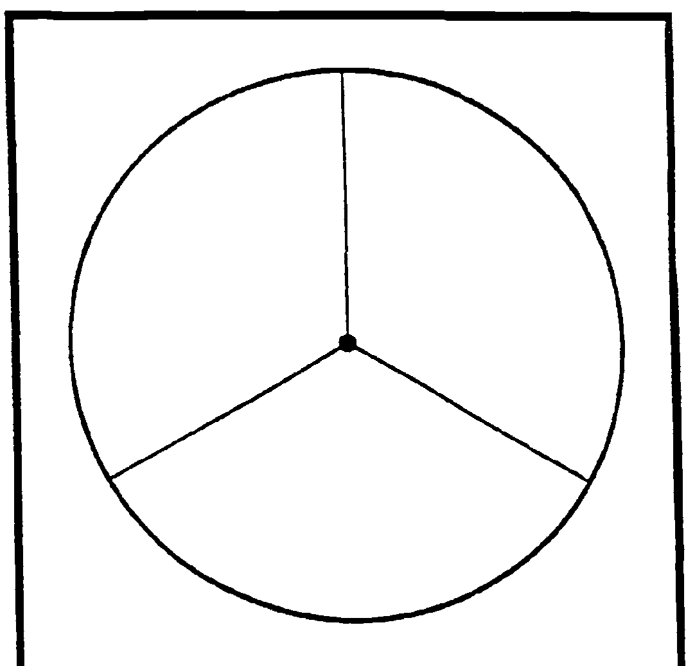

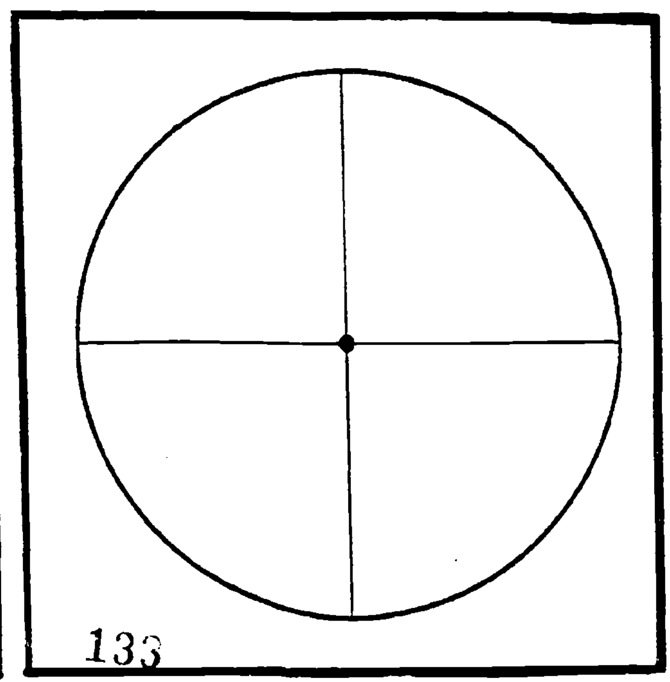

Problem Solving in the Middle School Mathematics Classroom

Recent reform efforts in school mathematics have generated a more focused and pervasive emphasis on problem solving. It is seen to be the key by which students "experience the power and usefulness of mathematics in the world around them" (National Council of Teachers of Mathematics, 1989, p. 75). Part of this new emphasis has highlighted the need:

- to use problem situations to explore and understand mathematical content;
- to use problems as a focus for developing number sense;
- to embed computational review in problem solving settings instead of merely completing routine exercises;
- to develop and apply a variety of strategies to solve problems, especially non routine or process problems;
- to work collaboratively making conjectures about , analyzing and solving a variety of problems;
- to verify, discuss or otherwise communicate and interpret the results of solving problems (e.g., through oral presentations or written reports);
- to evaluate the reasonableness of results;
- to generalize solutions and strategies to new problem situations; and
- to investigate and use mathematical connections within mathematics itself and in problems that are part of students' daily lives.

In particular, it can be argued that there are real payoffs in challenging students with rich problems that raise other questions, that generate natural extensions and that invite different solution approaches.

Five types of problems have been identified (Charles & Lester, 1982):

1. **ONE-STEP PROBLEMS-** These problems can be solved by choosing one of the operations addition, subtraction, multiplication, or division. These are the familiar "story problems" that have been a traditional part of school mathematics programs.
2. **MULTIPLE-STEP PROBLEMS-** These problems can be solved by choosing two or more of the operations addition, subtraction, multiplication, and division. Problems that require the repeated use of the same operation also fall in this category.
3. **PROCESS PROBLEMS-** These problems can be solved using one or more strategies such as:
   - guess and check
   - make an organized list
   - look for a pattern
   - solve a simpler, similar problem
   - use logical reasoning
   - build a model
   - classify
   - draw a picture
   - make a table
   - work backwards
   - write an equation
   - act it out
   - draw a diagram
   - make a scale model
4. **APPLIED PROBLEMS-** These require students to collect data and make a decision. The solution may require the use of one or more operations and/or one or more of the strategies given above for process problems. These problems reflect realistic situations and they often have more than one correct answer.
5. **PUZZLE PROBLEMS-** These problems allow students an opportunity to engage in potentially enriching recreational activities. Solutions to puzzle

problems often require students to look at the problem from a different perspective.

A further dimension of problem solving is identified by Stenmark (1992). She introduces the term "rich problems", and characterizes in three ways:

- The problem leads to other problems;
- The problem raises other questions;
- The problem has many possibilities.

As an example, the "dog pen" problem (National Council of Teachers of Mathematics, 1991, p. 28), modified in the activity, "Maximizing the Dog Pen (p____)," is seen to be rich in many ways. Some of these are:

- It generates other extension problems (e.g., using the house as one side of the pen).
- It raises questions about whether the square shape obtained for the greatest area could be generalized to other lengths of fencing wire.
- It has many possibilities in the sense that students could generate numeric, geometric or graphic solution approaches.

In addition, the problem provides a powerful opportunity to challenge students to identify numeric and geometric patterns and, at the same time, to review computations involving decimals, fractions, and whole numbers.

Teaching For and Via Problem Solving

The preceding paragraphs have focused on the nature of problem solving, characteristics of various kinds of problems and, more generally, the importance of problem solving. LINCS teaches also have focused on problem solving strategies such as those on p. ____, and on important aspects of problem solving associated with *teaching*.

Teaching for and via problem solving is reflected in the emphasis given to making connections between mathematics and the real world, and between various mathematical ideas. Another aspect of this philosophy is to recognize the role of problem solving as pivotal in exploring and developing new mathematical ideas and reviewing others. From such a perspective teaching problem solving encompasses challenging students, encouraging them to take risks, fostering collaborative and cooperative work, getting them to explore unfamiliar paths, emphasizing the need for logical reasoning and justification, and accentuating mathematical communication in both verbal and written presentations.

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MAXIMIZING THE DOG PEN

1. Problem
A farmer who has 42 feet of chicken wire wishes to make a rectangular dog pen to raise a small number of dogs near his house. What should he make the dimensions of the dog pen to maximize the grazing area?

2. Draw up a table of values for rectangles whose perimeter is 42 feet.

Perimeter	Length	Breadth	Area
42	20	1	20
42			
42			
42			
42			
42			

Can you see a pattern? What is your best prediction for the dimensions? Justify it.

3. How can you get a better approximation if you use decimals? What decimal values would you try given the information you have found in the table? How can you extend the table to get a better approximation?

4. What kind of rectangle appears to give the maximum area? If there was 50 feet of chicken wire would the rectangular pen of maximum area still have the same shape? Why?

5. Extension Problem
This time the farmer decides to build the rectangular dog pen using one side of his house as one of the sides of the dog pen. If he has 42 feet of chicken wire what are the dimensions of the dog pen of maximum area now?

Problem Solving with the Calculator

Problem: During July, Judy did not do her chores like she was supposed to do. Consequently, Judy's father said her August allowance would be cut by 12%. During August, Judy worked very hard to get her chores done since she was saving to buy a new bike. She did so well in August that her father said her September allowance would be raised 12% over that of August. Is her September allowance the same as her July allowance? Why or why not?

Understand the Problem

Decide on a Plan

Carry out the Plan

Look Back

Challenge!

Writing to Nurture and Communicate Understanding

Mathematics as Communication is one of the four unifying standards for K-12 highlighted in the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989). The focus and discussion of this standard for grades 5-8 (pp. 78-80) suggest that:

- the ability to read, write, listen, think creatively, and communicate about problems will develop and deepen students' understanding of mathematics.
- opportunities to explain, conjecture, and defend one's ideas orally and in writing can stimulate deeper understandings of concepts and principles.
- emphasizing communication in a mathematics class helps shift the classroom from an environment in which students are totally dependent on the teacher to one in which students assume more responsibility for validating their own thinking.
- writing and talking about students' ideas gives the teacher valuable information from which to make instructional decisions.

As teachers of mathematics begin to explore alternative assessment measures to the paper-pencil test, new ideas for communication in the classroom can contribute significantly.

Teachers in Project LINCS have begun to incorporate a number of writing activities into their classes and report a significant difference in their own understandings and insights into their students' knowledge-base, learning styles and growth in achievement.

In order to link problem solving with oral and written communication, the teachers began by introducing a problem originally written by Jane Loop of Lincoln School in Ypsilanti, MI for her first grade class:

> **Three baby rabbits found seven shamrocks growing by an old tree. One rabbit ate five shamrock leaves. One rabbit ate three shamrock leaves. The smallest rabbit left one leaf on three shamrocks. How many of the shamrock leaves were left growing? [A shamrock is considered to be a three-leafed clover.]**

While written originally for first grade, this problem has been introduced to students in grades 1-16 and to scores of teachers. With its many interpretations and at least fourteen correct answers, the shamrock problem promotes much discussion, debate, and reflection and provides the teacher with valuable information about the students' thinking and attitudes about problems.

Another valuable activity conducted by Project LINCS teachers was engaging their students in writing story problems using a variety of approaches:

- Having students write a problem given certain conditions. For example, it must use two operations or yield a particular answer or range of answers.
- Providing a picture that all students use to write their problems or allowing students to choose their own pictures from magazines.
- Writing and illustrating a problem for younger students to solve.
- Writing a mathematics lesson to teach a fellow student a particular topic.
- Writing a letter to a math pen pal giving them an original problem to solve or explaining a topic the student has just learned.

Additional writing activities Project LINCS teachers have begun to use in their classes include:

- Word Webs
- Guided Response Forms
- Reaction Sheets
- Math Journals
- Independent Projects
- Lesson Writing

Below are a selection of these activities and other valuable resources, including a reference to the *Professional Standards for Teaching Mathematics* (National Council of Teachers of Mathematics, 1991). This practical document devotes three of its six standards to discourse in the classroom (pp. 34-54). They address the Teacher's Role in Discourse, the Students' Role in Discourse, and Tools for Enhancing Discourse. Gaining an understanding of these areas is invaluable for establishing a classroom atmosphere and philosophy that will enable students to share their ideas through a variety of communication strategies, to create appropriate mathematical constructs and to achieve mathematical power.

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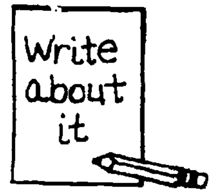


PROBLEM SOLVING REPORT

Problem: <u>A doctor gives a man 8 tablets to take one every 3 hours starting now. How long will it be before the man has taken all 8 tablets?</u>

Diagram (optional) and/or work:

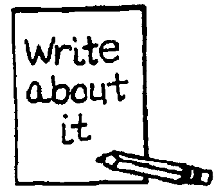
At first I thought the answer to this problem was _______ hours, because_______________

Then I did some more thinking about it, and my ideas were _______________

I decided on the answer of ________ hours, because _______________

Other problems that we've had like this are _______________

They are alike because _______________



Mysterious Patterns

1) In finding the decimal values for fractions, there are patterns. Two interesting examples of this patterning occur in the ninths and elevenths, for example, $\frac{2}{9}$, $\frac{6}{9}$, $\frac{8}{9}$, etc. and $\frac{3}{11}$, $\frac{6}{11}$, $\frac{8}{11}$, $\frac{10}{11}$ etc. Using your calculator, find the values for the fractions listed above, and describe below the pattern you notice: [Test your pattern with other ninth and eleventh fractions.]

Pattern for ninths: ______________________________

____

Pattern for elevenths: ______________________________

____

2) Do you have any suggestions for <u>why</u> the pattern of the elevenths works like it does? Explain: ______________________________

____

____

____

____

3) **<u>According to the patterns</u>,**

$\frac{9}{9}=$ ________ and $\frac{11}{11}=$ ________

What is your reaction to these values, according to the more widely accepted values of $\frac{9}{9}$ and $\frac{11}{11}$?

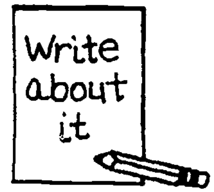
____

____

____

____

____



REACTION SHEET

One thing I learned today was ______________________________

____

____

____

One thing I still don't understand is ________________________

____

____

____

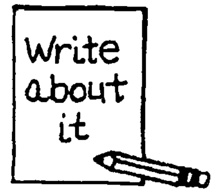
Other comments ____

____

____

____

____



INDEPENDENT PROJECTS AND ORAL PRESENTATIONS

Students are required to complete an independent project and give an oral presentation once during the year. The entire process requires approximately ten weeks, six of which are used by the student for the research, design and writing of the project.

The "process", explained below, provides numerous learning experiences for the students at all six levels of learning and described in Bloom's Taxonomy.

1) TOPIC SELECTION: An initial "brain storming" of possible topics takes place in class with the only restriction being that the topic must involve the use of mathematics. Two types of projects are allowed: a ten-page research paper or a five-page paper with a visual aid or model.

2) CONTRACT WRITING: The student chooses his/her topic and must search out possible resources before the topic is approved by both parents and teacher. A complete description of the project must be included in the contract, encouraging pre-planning.

3) GUIDELINES: A set of requirements and guidelines is given to each student including the "mechanics" for completing the project and due dates. Consideration for detail and promptness is stressed.

4) EVALUATION OF WRITTEN PAPER (AND VISUAL AID): This evaluation is completed by the teacher based on a set of predetermined criteria which has been made available to the student in advance.

5) ORAL PRESENTATIONS: The process for preparing for the oral presentation and then presenting it involves a variety of skills; choosing an appropriate, "catchy" title; abstracting important and interesting information from the original project in order to meet a ten minute time constraint; developing poise and effective delivery; and developing creative methods for presenting the information. The process of evaluation of the presentations also involves learning on the part of the student. For each presentation given, all students in the "audience", as well as the teacher, are given an evaluation form. Six criteria are used as a basis for scoring the presentation. A rating scale is used with additional space for comments. All student scores are averaged, and names are removed before the presenter sees the evaluations. (Prior to the evaluation process, a discussion of the meaning and style of constructive criticism is conducted, and during the process there is further discussion on how to accept such criticism.)

6) FOLLOW-UP DISCUSSION: After the entire process of projects and presentations has been completed, class time is taken to discuss openly the characteristics which mark effective and interesting projects. Negative aspects are also discussed. Students are encouraged to take notes and use this knowledge as they develop their ideas for future projects.

OCCUPATIONAL INTERVIEWS

Requirements:

- You must use your first choice of occupation.
- The person being interviewed must not be a relative.
- The interview is to take place at the person's place of business by appointment.
- The appointment should be made a week in advance (at least).
- A letter of confirmation should be sent the same day you make the appointment.
- Questions are to be prepared in advance in a notebook - one question per page in order of importance.

The written report will be graded on all of the following:

1) It should be in a folder, have a title page (including which occupation you are doing), be written in ink or typed on one side of the paper, have correct spelling and punctuation, and be neat.

2) A list of all questions you prepared should be included with an * beside those you actually asked. After each question asked, write a summary of the person's response.

3) A summary paragraph concerning the uses of math plus other educational requirements is necessary.

4) A paragraph pointing out other interesting aspects of the occupation (possibly something you did not know before) should be included.

5) Give a detailed impression of how the interview went (including embarrassing or especially interesting moments).

6) Include an assessment of whether you are still considering this occupation. Explain fully why you do or do not want to pursue this field.

7) You must include the form filled out by the person you interviewed.

EPILOGUE

Perspectives on Teaching Mathematics to Nurture Mathematical Thinking Among Middle School Students

LINCS teachers took advantage of opportunities offered them in seminars during the final year of the funded project to reflect on teaching and learning in relation to "nurturing mathematical thinking." The following short papers, the work of four working groups, represent their collaborative efforts to articulate their current perspectives on this important topic. In sharing their thoughts in this epilogue, they represent their commitment to continued reflection and growth as facilitators of mathematics learning among students.

WHAT SHOULD BE HAPPENING IN A LESSON THAT NURTURES MATHEMATICAL THINKING?

Rich lessons are a key to developing positive mathematical understandings. A rich lesson may last a day or continue for weeks depending upon its purpose and how it evolves within a specific classroom setting and in the context of student learning. Rich lessons will generally include:

- discourse on mathematical teaching and learning
- construction of mathematical meaning
- extension of mathematical thinking
- applications of mathematical concepts
- teacher and student reflection on mathematical knowledge

No specific order or hierarchy is implied in their implementation. These five key elements are all important strands of a rich lesson which nurtures mathematical thinking.

Discourse On Mathematical Teaching And Learning

Discourse which is needed for strong mathematical thinking evolves in an atmosphere where communication encourages positive interaction between all participants - student, teacher, parents, administrator. This type of discourse should incorporate a variety of techniques - oral, written, graphic, manipulative, and technological. Using these techniques would encourage numerous opportunities for multiple solutions and approaches, as well as time for questions and extensions. The environment which nurtures this type of discourse permits an appropriate amount of time for all participants to analyze, to think independently, and to work in collective groups. The teacher's role becomes one of a facilitator to engage active participation from everyone involved.

Construction Of Mathematical Meaning

In contemplating the principles of mathematical thinking, the construction of meaning implies building from a student's prior and intuitive knowledge. Encouraging the use of manipulatives, real life situations, communication and discovery, multiple solutions, justification of thinking, as well as cooperative problem solving leads to empowerment and a positive attitude toward math. Two example of cooperative problem solving activities which may enhance the construction of mathematical meaning follow.

Perimeter with color tiles. Give each students 5 tiles. Students then line them up in a line, count the sides and arrive at the perimeter which is 12 units. Students are then asked to make as many shapes as possible using the 5 squares to get the same perimeter. . . . An extension of this activity would be to invite students to use graph paper to draw as many garden plots with a perimeter of 24 units as they can. For justification of thinking, remind students that a garden plot must be plowed.

Use of pattern blocks to teach fraction concepts. Give each partner a bag of pattern blocks that contain hexagons, trapezoids, triangles, and rhombuses. The children will discover that using the hexagon as the whole that the other pieces are 1/2, 1/3, and 1/6. Questioning might include: "Can you show when 1/3 is larger than 1/2? (By changing the size of the whole.)

Extension Of Mathematical Thinking

The purpose of extension is to nurture the students' thinking through five constructs. Subject integration is the first of these constructs, it is important to use math problem solving skills to reason through Social Studies and Science problems. A second construct is personal and societal relevance. Lessons should focus on skills using real life problems and situations not just computations. Students need to be encouraged to broaden their affective expression. This third construct would develop and unlock students' feelings, concerns and passions. The fourth construct is a fostering of perpetual questioning. Hopefully, this idea will instill the idea that answers to problems can come from different directions and take many shapes. Finally, in the fifth construct, lessons should be linked to prior knowledge. Classes and lessons should focus on math skills always building upon themselves in order to eliminate the idea that a concept stops with a particular chapter.

Applications Of Mathematical Concepts

Mathematical thinking should be nurtured through application of situations which pertain to the students lives and the world in general. This application can take the form of real life math problems based on student interest and understanding . They should see the relationship between mathematics and other areas of life (e.g. social studies, science,....). Application should involve higher level reasoning and analysis along with multiple methods of approach and solutions.

An example of a rich application problem would be to have students plan a meal for five without any monetary limits using newspaper circulars. Having completed this activity, students would apply the same concept of planning a meal but with a limit of $10.00. The students would discuss and write in their journals how "as head of a family," they would explain to their children why they should be happy with their $10.00 meal even though the children are aware of other families' feasts.

Teacher and Student Reflection On Mathematical Knowledge

A key principle in a lesson that nurtures mathematical thinking is that of reflections, made both by the teacher and the students. This reflecting is an integral part of teaching and learning. The teacher reflects before, during, and after a lesson. In deciding what material should be taught and how it should be presented, the teacher considers what he/she knows about his/her students. This knowledge can be derived from teacher observations or some form of written assessment.

During the lesson the teacher reflects when listening to explanations of students, to conjectures made, of questions asked, of arguments made supporting answers, and observing students' attitudes and work. As a result of these reflections, a teacher becomes a decision maker when deciding how a lesson is to progress.

After a lesson is completed, the teacher reflects a gain on how the lesson went and how he/she could have improved mathematical thinking. Frequently, the teacher has results of multi-dimensional tasks to help in this reflecting process. In addition, the teacher may have records of written notes or a portfolio of the students' work to give an accurate and thorough picture of the mathematics the students know.

Students are also asked to reflect in many different ways, such as writing in a journal about some aspect of the lesson or when choosing a particular project. Frequently, he/she may reflect informally when justifying his/her answer or when making a connection from new to previous knowledge. We, as teachers, believe that reflection is crucial to nurturing mathematical thinking in the classroom.

Concluding Remarks

Developing rich lessons is a challenge well worth the effort. The outcomes are often long term, however not always immediately recognizable. Modification of the lessons through discourse engage students in expanded mathematical thinking. Rich lessons enhance the students' awareness and appreciation of mathematical order in nature and the world around them. Rather than students thinking of mathematics as a series of skills to be learned, rich lessons encourage students to view mathematics as a systematic approach to problem solving that can be used in all aspects of their lives.

. . . Reflections of the following LINCS teachers:

Marsha Andrew *Linda Brook*
Lana Cappitelli *Cynthia Diederich*
Steve Dillon *Mike Judd*
Lynne Krueger *Annette Rush*
Margaret Watson *Barb Wiegand*
Joyce Zeiters

INSTRUCTIONAL PRINCIPLES FOR NURTURING MATHEMATICAL THINKING

Teachers are continually being challenged to find ways of nurturing students' mathematical thinking. While mathematical thinking is not easily defined, it is certainly exhibited by students when solving problems, building mathematical models of real world situations, identifying patterns and relationships, using logical reasoning, making connections with other areas of mathematics and with other subjects, and when communicating mathematical ideas and solutions. Mathematical thinking also appears to have another dimension that includes displaying number sense, showing curiosity about mathematical ideas, and having trust in mathematical solutions. Within this broad context, the following instructional principles are seen to be helpful for nurturing students' mathematical thinking.

Principle 1: The lesson should be consistent with long-term goals and key learning expectations. Teachers should set long-term goals that help them make decisions when planning lessons or projects. Over the course of a semester or year, long-term goals should reflect the BIG IDEAS the teacher has for nurturing mathematical thinking, developing content knowledge, and encouraging students' positive disposition toward mathematics. Opportunities that occur during learning experiences which support the these goals should be fully utilized. Of particular importance are opportunities which help students:

- Develop number sense.
- Value mathematics.
- Gain satisfaction from solving problems.
- Develop confidence in doing mathematics and trust their use of numbers to make decisions.

Principle 2: A worthwhile mathematical task should provide a focus for the lesson. Worthwhile mathematical tasks are lessons that make mathematics more relevant and reachable to students. These tasks incorporate the lesson objectives in such a way that mathematics is a means to the end, not an end in itself.

- Problems should be purposeful and challenging rather than drill and practice. For example, students might use proportions to design a dream house, collect data using surveys to present information visually, collect data to analyze statistically (e.g., deriving mean, median or mode to describe what is "typical;" using stem-and-leaf or box-and-whiskers plots to display the data), use standard or non standard measures to compare perimeter, area, surface area, volume, or mass.
- Problems should not limit students' creativity. They should be open-ended and encourage divergent thinking and multiple solutions which allow students to be creative.
- Problems should initiate or lead to new problems by encouraging students to ask "what if" questions. For example:
  -- "What if a new group were surveyed? How would this affect the mean, median or mode?"
  -- "What another room were added to your dream house? How would this affect the cost?"

Principle 3: A stimulating and supportive learning environment should be created. To nurture students' mathematical thinking, a stimulating and supportive environment should be created. This is especially evident when one can build an accepting atmosphere that allows students to explore problems in diverse ways, make individual interpretations, and even find solutions which are genuinely different.

Using a positive risk-free atmosphere, trust is fostered and students will experience satisfaction and pleasure in their mathematical tasks. Attention to a variety of physical aspects will also contribute to a compelling atmosphere. Use of technology, hands-on materials, a variety of groupings, and attention to providing ample time are essential to nurturing mathematical thinking.

Principle 4: Instruction should facilitate thinking and learning rather than orchestrate it. Students should be encouraged to make and verbalize conjectures and then should be provided with opportunities to interact, justify, and verify their mathematical thinking. Teachers must be willing to accept and explore divergent conjectures and thinking, a process which leads both teachers and students to expect the unexpected. To be successful facilitators of learning, teachers must avoid pre-judging student opinions, ideas, or thought processes and and encourage both peer and self-evaluation. Another extension of students involvement is to encourage students to summarize the key ideas in their thinking. These ideas characterize the teacher as facilitator.

. . . Reflections of the following LINCS teachers:

Christine Biggs | *Jim Cantrell*
Judy Duvall | *Vicki Erickson*
Pam Helfers-Riss | *Nancy Howard*
Jill D. Keller | *John Muirhead*
Eileen Norin | *Thomas Schuerman*
Jane Sumrall | *Dawna L. Tucci*

USING QUESTIONS IN A LEARNING ENVIRONMENT TO NURTURE MATHEMATICAL THINKING

Mathematical thinking involves many things. Teachers need to foster an atmosphere that encourages student thinking. This environment allows students to apply mathematical principles to everyday life. Students can then communicate their reasoning, ideas, and methods through verbal and written means. Students should communicate with each other, with the class, and with the teacher. This communication can also involve others outside the classroom. Students can explain mathematics problems and their solutions with parents or guardians, friends, or siblings. This allows students to extend their understanding beyond the classroom. In order for students to feel free to offer their ideas and solutions, the classroom should be risk free and the teacher and other students should be non-judgmental. The teacher should become the facilitator rather than the focus of the class.

Classroom Environment

A desirable atmosphere can be achieved in different ways. Real-life applications allow students to use the mathematics they learn in a practical way. Open-ended questions that have multiple solutions provide students with an opportunity to be creative and justify the responses they offer. For example, students might be asked to plan a field trip or a pizza party. In cooperative groups, they would need to determine the important aspects of the problem, decide what questions must be answered, and then use mathematics to answer their questions.

Each group could ask different questions, could answer questions in a different way, and could justify their solution in various ways. All the solutions may be valid. Open-ended project options like this encourages different strategies and multiple answers. This type of diverse thinking is important and it shows that it is all right to solve problems in different ways.

Mathematical thinking can also be nurtured in an environment that connects mathematics to other subjects. Social studies, science, literature, and art all provide opportunities to integrate mathematics into the curriculum. Students can see the usefulness of mathematics in other areas. They can create new problems using mathematical ideas tied to the other subjects. For example, students might write and solve mathematics problems that are tied to the literature book currently being used. Other areas of the curriculum provide a rich source of mathematics problems. All of these things help create an atmosphere that fosters mathematical thinking.

Questions to Enhance Thinking and Learning

Questioning is an important aspect of fostering mathematical thinking. Both students and teachers need to use questions to enhance learning. Teachers need to ask questions that invite higher-level thinking and not just simple recall. Students need to learn to ask questions that expand and extend the concepts and ideas under discussion. In addition, students should create their own questions for mathematics problems since most real-world problems that students encounter do not have a nice, complete question as part of the situation.

Questions can be used to encourage students to explain their thinking. The teacher can stimulate discussion and show that unconventional or different strategies are accepted and used by asking such questions as:

"How did you arrive at that answer?"
"Can you explain this part more specifically?"
"Could you share your method with the class?"
"How did you begin to think about this problem?"

"What strategy did you use?"
"Is there another way to find an answer?" or
"What kind of skills, (concepts) other than mathematics skills, did you use?"

Questions can also be used to get students to justify or expand their answers. Questions like those which follow invite students to think about and then justify their solutions:

"Does this make sense/is it a reasonable answer? Why or why not?"
"Does everyone agree/disagree? Why?"
"Why does this method work for you?"
"Explain how your answer is different than ________________'s?" or
"Why do you agree with ________________?"

Letting students explain and justify their solutions to the class, even if it means having them repeat their answer/solution, or asking another student to explain, focuses the learning on student ideas rather than on the teacher telling the students an algorithm or a strategy. The explanation becomes the focus, not the answer. The teacher is no longer the all-knowing sage, but a person who takes an active, communicative role in working problems.

In addition to explaining what they know, students should be encouraged to ask questions about what they do not understand. When knowing what you don't know is valued, students will freely ask questions and be more reflective. This atmosphere can be encouraged by asking "What is the next question that we should ask?"

Another important question is "What if. . .?" Situations and rich problems that encourage students to think and conjecture are essential in helping students think mathematically. These problems allow students to categorize, look for patterns and relationships, predict, test, and revise. They can then justify and revise again as needed. Some questions may take 10 minutes to answer, others may take a day or a week. A variety of tools can also be used in this process from manipulatives to calculators and computers. By using the tools they need, the students focus on the problem and its solution rather than computation or process. The teacher can model this type of behavior by providing situations that ask for conjectures, by using technology when appropriate, and by giving thinking time.

As teachers prepare and teach their lessons, they should consider questions that stimulate multiple strategies (or solutions where appropriate), and encourage student thinking. Questions eliciting factual responses should be mixed with those that encourage higher-order thinking. Questions that will start a discussion or expand an idea should be used to connect ideas and subjects. Using questions in this way may require time, thought, and effort, but the resulting rewards of increased student interest and more complete understanding make it worthwhile.

. . . Reflections of the following LINCS teachers:

Mary Lou Bastion	*Charlotte Brucker*
Mary Buescher	*Theresa Carley*
Diana Coombs	*Lynda Estes*
Lynn Gaddis	*Cathy Greene*
Janet Halsey	*Diane Dachur*
Barbara Malito	*Terry Oberhardt*
Jeanne Petkoff	*Alan Wilson*

About Nurturing Mathematical Thinking

In realigning the curriculum to focus on problem solving, we can respect and nurture the mathematical thinking of individual students. In a broad sense, problem solving is a way of teaching and a way of learning. It involves being sensitive to the potential, needs, and interests and individual students through tasks and activities in which students:

- explore and experience mathematics in "real" applications;
- make connections to other areas and "see math all around them";
- work collaboratively with other students;
- become more flexible in their thinking;
- develop the attitude, "I don't know, but if you give me time I can figure it out."
- learn there is more than one way to solve a problem and, at times, more than one "correct" answer;
- explain and justify their solutions.

Problems should be broad enough in scope to challenge *all* students--while providing a balance between mathematical concepts and skills. Some problems may be readily solved; others may take a day, a week, or several weeks. Whenever possible, problems should highlight interrelationships with other subjects or other mathematical topics. While working on problems, students should be encouraged to be "active learners," to use appropriate materials--from calculators, measurement instruments, or blocks to unifix cubes, geometric shapes, fraction pieces or graph paper--and report their findings in writing and orally.

The teacher's challenge is to provide a non-threatening environment which helps students succeed in problem tasks. In this type of an environment the stigma that has traditionally gone along with having "only one right answer" is replaced by a *listening* atmosphere where different approaches and sometimes different correct answers are elicited, where students know they will be expected to explain their thought processes and have the opportunity to revise when necessary.

The Student's Perspective

Students in a classroom that nurtures mathematical thinking have natural expectations. For example, some students enter the mathematics classroom expecting to participate in motivating activities which help them to "make sense" of mathematics topics they are currently studying. In problems and projects, they value mathematical connections to their world and the option to a select from approaches or materials that "fit" their personal learning styles.

Perhaps just as important, most students anticipate that a sufficient amount of time will be provided for them to reach their personal best. Because students often take considerable time to develop sound understandings and work out meaningful strategies for approaching problems, they become frustrated when insufficient time is provided for completing tasks that are posed. Mathematical thinking, independence and personal responsibility for learning are best nurtured when students are given the time to explore significant problems, to produce strategies that fit their individual ways of doing things, and to interact productively with both classmates and the teacher.

The Teacher's Perspective

The teacher is the turn key to nurturing mathematical thinking and appropriate quantitative decision-making abilities among students. In the new role as *facilitator* of learning, the mathematics teacher of today is challenged to create and implement a curriculum which is "driven" by problems which arise from a variety of contexts--including classroom or current events, literature or scientific themes--along with student-posed problems. This involves identifying (or creating!) problems which have potential to promote a variety of solution strategies and cause important mathematical content and understandings to emerge. Teachers might start with open-ended problems which encourage divergent thinking, which use real world examples, and which make connections between past knowledge and new material.

Teachers also need to recognize that, in order to nurture mathematical thinking of students, traditional curricular activities will need to be adjusted, or even eliminated, in order to provide the needed time for students to conjecture, explore, and justify their thinking and solutions. A variety of classroom structures may need to be tried--to accommodate individual work, as well as for small group, cooperative problem solving and whole-class interaction. Through the nurturing process, the goal is for students to exceed prior expectations to include and assimilate new attitudes into their belief systems. As a result, students may approach new challenges with a new-found confidence and willingness to aspire.

The real challenge for the teachers of today now lies ahead as they try to implement these ideas in the classroom. Not only do experienced teachers need to rethink the teaching and learning of mathematics, but they need to be supported by administrators and parents who are convinced that these new directions are in the best interests of students as future workers in a technological society.

. . . Reflections of the following LINCS teachers:

Pam Bloom	*Chris Bohne*
Greg Buikema	*David Colba*
Jim Eaton	*Tammy Harbaugh*
Kelly Lyle	*Terri Peek*
Julia Mason	*Mary McClintock*
Judy Rocke	*Rita Stone*