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ABSTRACT

This volume contains the full text of 2 plenary papers and 26 research reports. In addition, brief, usually one-page, reports are provided for 6 discussion groups, 10 technology focus groups, 7 symposiums, 7 oral presentations, and 17 position sessions. The two full plenary reports are: (1) "Problems of Reification: Representations and Mathematical Objects" (A. Sfard and P. W. Thompson); and (2) "Elements of a Semiotic Framework for Understanding Situated and Conceptual Learning" (J.A. Whitson). The twenty-six full research reports are: (3) "Factors in Learning Linear Algebra" (G. Harel); (4) "Articulations Between the Settings, Numeric, Algebraic and Graphic Related to the Differential Equations" (A. Hernandez and F. Hitt); (5) "Image Structures and Reification in Advanced Mathematical Thinking: The Concept of Basis" (L. Krussel); (6) "A Survey of Tertiary Students' Entry Level Understanding Of Mathematics Vocabulary" (L. D. Miller and B. White); (7) "Constructing the Derivative in First Semester Calculus" (B. Speiser and C. Walter); (8) "Visual Salience in Algebraic Transformations" (T. Awtry and D. Kirshner); (9) "Preparing Students for Algebra: The Role of Multiple Representations in Problem Solving" (M. E. Brenner and B. Moseley); (10) "Introducing Algebra With Programmable Calculators" (T. C. Avalos); (11) "Blind Calculators', 'Denotation' of Algebra Symbolic Expressions, and 'Write False' Interviews" (Jean-Philippe Drouhard, And Others); (12) "School Algebra: Syntactic Difficulties in the Operativity With Negative Numbers" (A. Gallardo and T. Rojano); (13) "A Constructivist Explanation of the Transition from Arithmetic to Algebra: Problem Solving in the Context of Linear Inequality" (T. Goodson-Espy); (14) "Multi-Tasking Algebra Representation" (L. P. McCoy); (15) "Assessing Student Responses to Performance Assessment Tasks" (S. Hillman); (16) "Multi-faceted Inferences from an Interview Assessment" (T. L. Schroeder); (17) "Visualization in Mathematics: Spatial Reasoning Skill and Gender Differences" (L. Friedman); (18) "Negative Consequences of Rote Instruction for Meaningful Learning" (D. Simoneaux and D. Kirshner);

(19) "Long Term Relationship Between Spatial Ability and Mathematical Knowledge" (G. H. Wheatley, And Others); (20) "The Role of Context in Mathematical Activity" (D. Clarke and S. Helme); (21) "The Relationship Between Preservice Teachers' Metaphors for Mathematics Learning and Habermasian Interests" (M. J. Fleener and A. Reynolds); (22) "An Investigation into the Development of Student Understanding of the Graphs of Polynomial Functions of Degree Greater Than Two: Results and Implications" (J. E. Curran); (23) "A Modeling Approach to Constructing Trigonometric Functions" (H. M. Doerr); (24) "Students' Conceptions of Functions in a Computer-Rich Problem Solving Environment" (B. R. O'Callaghan and D. Kirshner); (25) "Towards an Algebraic Notion of Function: The Role of Spreadsheets" (T. Rojano and R. Sutherland); (26) "Students' Development of Length Measurement Concepts Using a Specially-Designed Turtle Graphics Environment" (D. H. Clements; And Others); (27) "The Role of Language in Geometric Concept Formation: An Exploratory Study with Deaf Students" (M. M. Mason); and (28) "Development of the Concept of Randomness" (K. E. Metz). (WTB)

Proceedings of the

Sixteenth Annual Meeting

NORTH AMERICAN CHAPTER

of the

INTERNATIONAL GROUP

for the

PSYCHOLOGY OF MATHEMATICS EDUCATION

Volume 1:

Plenary Sessions, Technology Focus Groups,

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and Research Papers, Oral Reports and Posters

PME-NA XVI

November 5-8, 1994

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Editor:
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HISTORY AND AIMS OF THE PME GROUP

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the Group and of the North American Chapter (PME-NA) are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

Editor's Preface

The program for the 16th Annual Conference of PME-NA was established through the active participation of the Program Committee during the year leading up to the 1994 meeting. There were three invited session themes. *Reification in Secondary and K-12 Education* was selected to serve as a bridge between mathematicians and mathematics educators within the PME-NA family. This theme was addressed in a plenary session jointly presented by Anna Sfard, Hebrew University in Jerusalem, and Patrick Thompson, San Diego State University. *Situated Cognition Theory* was selected as a theme to explore philosophical and epistemological concerns that have been important to the PME and PME-NA community for a number of years. This theme was addressed in a second plenary session delivered by Tony Whitson, University of Delaware, with Judit Moschkovitch of the Institute for Research on Learning, in Palo Alto CA, serving as discussant. The final invited theme, *Rational Number Concepts*, was addressed in an invited discussion group organized by Thomas Kieren, University of Alberta. This discussion group used a new format in which the membership was invited to submit one page synopses of positions and views of rational number concepts to Dr. Kieren to be orchestrated into the discussion group session. These one page synopses are included in the Proceedings.

A special feature of the 1994 meeting were the *Technology Focus Groups* organized by James J. Kaput, University of Massachusetts, North Dartmouth. Each conference registrant was invited to participate in one of ten such groups in which developers of technology-based mathematics learning environments led investigations into problems in the psychology of mathematics education that arise in such learning environments. The groups met for a three or four hour block of time in a computer laboratory on the Louisiana State University campus. Our purpose in this session format was to broaden the base of understanding within the mathematics education community of the problems and possibilities of technological innovation. One page synopses of these groups are included.

The 67 research reports, 16 short oral reports, and 28 posters that constituted the main body of the conference program span a variety of issues and are organized topically within these *Proceedings*. This is a change from previous PME-NA proceedings in which only research reports were organized topically. Additionally one symposium and 5 discussion groups were presented, and synopses of these are included in the *Proceedings*. Proposals for all sessions were reviewed by three reviewers with expertise in the area of the submission. Cases of disagreement among the reviewers were resolved in a meeting of the Program Committee in New Orleans in April of 1993. This procedure resulted in the rejection or reassignment of about 20% of the proposals.

Thanks are due to all participants in PME-NA XVI, but especially to the Executive and Program Committees, to the reviewers, to the LSU University community for use of computer laboratories, to Dean Larry Pierce of the College of Education and to Dr. Neil Mathews, Chair of the Department of Curriculum & Instruction for their support of conference activities, to Kathy Carroll for technical coordination of the proceedings, to Doug Bourgeois and Brandy Baechle of LSU Conference Services for their tireless work, and to Clint Kaufmann for his valuable assistance.

David Kirshner
September, 1994

PME-NA 1994 REVIEWERS

Awtry, T
Bebout, H
Behr, M
Berenson, S
Bowman, A
Brenner, M
Brown, D
Bohlin, C
Bohlin, R
Bondoreur, P
Campbell S
Corwin, R
Chapin, S
Curcio, F
Davenport, L
Dawson, A
DeGuire, L
Ebert, C
Edwards, L
Fleener, J
Friedman, L
Ganter, S
Gaulin, C
Gordon, A
Grover, B
Goodson-Espy, T
Golding, T
Graham, K
Gorowara, C
Harel, G
Harik, F
Harnisch, D
Ham, J
Hart, L
Heid, M
Jakubowski, E
Kaufmann, C
Kirshner, D
Kloosterman, P
Koch, L
Krussel, C
Kaput, J
Lamb, C
Lambdin, D
Lamon, S
Lester, F
Maher, C
Masingila, J
Massey, J
Mayer, R
McCoy, L
McNeal, B
Mitchell, M
Meredith, J
Magidson, S
Miller, D
Mukhopadhyam,
O'Shea, T
O'Callaghan, B
Pence, B
Post, T
Presmeg, N
Preston, R
Raymond, A
Romberg, T
Reynolds, A
Risacher, B
Schifter, D
Selden, A
Selden, J
Senger, B
Shealy, B
Smith, E
Sowder, J
Storygard, J
Simon, M
Swafford, J
Schroeder, T
Underhill, B
Watanabe, T
Wheatley, G
Walen, S
Wilson, M
Wandersee, J
Yackel, E
Zazkis, R

Table of Contents

Contents of Volume I

SYMPOSIUM AND DISCUSSION GROUPS

RESEARCH PAPERS, ORAL REPORTS, AND POSTERS

Advanced Mathematical Thinking

12

Assessment

Cognitive Modalities

Geometric Thinking

Probability and Statistics

Contents of Volume II

RESEARCH PAPERS, ORAL REPORTS, AND POSTERS (continued)

Problem Solving

Students' Attitudes and Beliefs

Teachers' Beliefs and Attitudes

Teacher Change

Teachers' Understanding of Students' Understanding

Plenary Sessions

Problems of Reification: Representations and Mathematical Objects

Anna Sfard
Hebrew University, Israel

Patrick W. Thompson
San Diego State University, USA

Had Bishop Berkeley, as many fine minds before and after him, not criticized the ill-defined concept of infinitesimal, mathematical analysis — one of the most elegant theories in mathematics — could have not been born. On the other hand, had Berkeley launched his attack through Internet, the whole foundational effort might have taken a few decades rather than one and one-half centuries. This is what we were reminded of when starting our discussion. Like Berkeley, we were dealing with a theory that works but is still in a need of better foundations. Unlike Berkeley and those after him, we had only a few months to finish, and we had e-mail at our disposal.

Needless to say, the theory we were concerned with, called *reification*, was nothing as grandiose and central as mathematical analysis. It was merely one of several recently-constructed frameworks for investigating mathematical learning and problem solving. The example of Bishop Berkeley taught us there is nothing more fruitful than a good disagreement. Thus, we decided to play roles, namely to *agreed to disagree*. Since we are, in fact, quite close to each other in our thinking, we sometimes had to polarize our positions for the sake of a better argument.

The subject proved richer and more intricate than we could dream. Inevitably, our discussion led us to places we did not plan to visit. When scrutinizing the theoretical constructs, we often felt forced to go meta-theoretical and tackle such basic quandaries as what counts as acceptable theory — or why we need theory at all.

Above all, we enjoyed ourselves. We also believe it was more than fun, and we hope we made some progress. Whether we did, and whether our fun may be shared with others, is for you to judge.

Pat:

The following excerpt appears in *Research in Collegiate Mathematics Education*. In it

I speak about the "fiction" of multiple-representations of function—I do not speak about reification as such. However, I think what I said about functions is applicable in the more general case of reification, too: That we experience the subjective sense of "mathematical object" because we build abstractions of representational activity in specific contexts and form connections among those activities by way of a sort of "semantic identity." We represent to ourselves aspects of (what we take to be) the same situation in multiple ways, and we come to attribute logical identity to our representations because we feel they somehow represented the "same thing."

A number of fuzzies are entailed in what I said above, such as matters of scheme and matters of abstraction. I'm sure these will come out as we go along.

> I believe that the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of representation.[1] Tables, graphs, and expressions might be multiple representations of functions to us, but I have seen no evidence that they are multiple representations of anything to students. In fact, I am now unconvinced that they are multiple representations even to us, but instead may be areas of representational activity among which, as Moschkovich, Schoenfeld, and Arcavi (1993) have said, we have built rich and varied connections. It could well be a fiction that there is any interior to our network of connections, that our sense of "common referent" among tables, expressions, and graphs is just an expression of our sense, developed over many experiences, that we can move from one type of representational activity to another, keeping a current situation somehow intact. Put another way, the core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance.
>
> I do not make these statements idly, as I was one to jump on the multiple-representations bandwagon early on (Thompson, 1987, 1989), and I am now saying that I was mistaken. I agree with Kaput (1993) that it may be wrongheaded to focus on graphs, expressions, or tables as representations of function, but instead focus on them as representations of something that, from the students' perspective, is *representable*, such as some aspect of a specific situation. The key issue then becomes twofold: (1) To find situations that are sufficiently propitious for engendering multitudes of representational activity and (2) Orient students to draw connections among their representational activities in regard to the situation that engendered them.
>
> (Thompson, 1994b, pp. 39-40)

[1] This is entirely parallel to the situation in information processing psychology—no one has bothered to question what is meant by "information" (Cobb, 1987; Cobb, 1990).

Anna:

How daring, Pat! After all, the idea you seem to be questioning is quite pivotal to the research in math education right now. A comparable move for a physicist would be to say that he or she doubts the soundness of the concept of force or energy. Indeed, what can be more central to our current educational project than the notion of representation? What could be more fundamental to our thinking about the nature of mathematical learning than the idea of designating mathematical entities in multiple ways? Your skepticism does not sound politically correct, I'm afraid. But , I'm glad you said this. In fact, I have had my doubts about the "careless" way people use the notion of representation for quite a long time now. Obviously, when one says that this and that are representations, one implies that there exists a certain mind-independent *something* that is being represented. Not many people, however, seem to have given a serious thought to the question what this something is and where it is to be found.

Some methodological clarifications could be in point before we go any further. I remember the discussion that developed in August 1993, when Jim Kaput decided to forward your blasphemous statement to his Algebra Working Group.[2] Many people responded then to the challenge, but my impression was that each one of them attacked a different issue, and everybody was looking at the problem from a different perspective. Somehow, the disputants seemed to be talking past each other rather than disagreeing. For example, David Kirshner interpreted your statement as a rejection of introspection. He said:

> Amen to Pat!
>
> I am deeply supportive of perspectives that challenge the presumed connection between our introspections about our knowledge and the actual underlying representations. Understanding consciousness as a mechanism that allows us to maintain a coherent picture of ourselves for the purposes of interacting within a social milieu, shows introspection as an extremely unreliable guide to our actual psychology ...

Thus, for David your message was mainly methodological: it dealt with internal rather than external representations and with the problem of how to investigate these

[2] The Algebra Working Group is an affiliation of mathematics educators who communicate regularly via Internet on matters pertaining to learning and teaching algebraic reasoning at all grade levels. The AWG is managed by Dr. James Kaput, JKAPUT@UMASSD.EDU, under the auspices of the University of Wisconsin's National Center for Research in Mathematical Sciences Education.

representations rather than with the question of the existence and the nature of their referents. Ed Dubinsky, on the other hand, understood the issue as mainly epistemological. While taking the use of the term "representation" for granted, Ed translated your dilemma into the question how we come to know and how we construct our knowledge:

> I think that Pat raised ... the epistemological question of existence and representation. No one seems to have trouble with various forms of representational activities, but if one speaks of representation as a verb, then its transitivity forces one to ask the question what is being represented. Actually, Pat is asking the deeper question, is anything being represented?

Although these two interpretations are miles apart, they seem to share a tacit ontological assumption. This assumption was also quite clear in the language you used yourself. As I already noted, the very term "representation" implies that it makes sense to talk about an independent *existence* of certain entities which are being represented. The expression "multiple representation" remains meaningless unless we believe that there is a certain *thing* that may be described and expıessed in many different ways. I am concerned about the fact that the discussion whether this implication should be accepted or rejected took off before the disputants explained what kind of "existence" each one of them had in mind. One may agree or disagree with the claim about the existence of mathematical objects, but if the meaning of the world "existence" in this particular context is not made explicit, our discourse will never rise above the level of a mere word game.

Let me present you with two options (by no means exclusive). First, it seems to me that the default interpretation of the whole issue would be as follows: we should view our problem philosophically rather than psychologically, and the Objectivist outlook should be taken as a point of departure. Let me explain.

Objectivism was defined by the American philosopher Putnam as a view grounded in two assumptions:

> 1. there is a clear distinction to be drawn between the properties things have "in themselves" and the properties which are "projected by us;"
>
> 2. the fundamental science ... tells us what properties things have "in themselves." (Putnam, 1987, p. 13)

In this description, Putnam refers to science rather than to mathematics. In the case of mathematics, the problem is somewhat more difficult, as the distinction between the knowledge and the object of this knowledge is much less clear than in the case of physics or biology. Even so, the very fact that representation is a central motif of our discourse shows that we do view the mathematical realm as independent from the way we think or talk about it.

An alternative position would be psychological, and not philosophical. We could concentrate on what people have in mind and disregard the problem of the "objective" existence of mind-independent abstract entities. But then, of course, the question must be answered what we have in mind when we talk about things "that people have in mind." Did you notice the circularity in this last sentence? It seems we cannot escape it, just as we cannot escape talking about mind. A truly sticky issue, isn't it?

Pat:

A sticky issue indeed!

Let me, for the moment, side-step the philosophical matters you raised and speak about my motive for saying what I did. My motive was pedagogical. The multiple-representations movement often translated into a particular kind of instruction or a particular kind of curriculum: Show students several representations and tell them what they mean—or worse, have them "discover" what they mean. To the person doing the showing, the representations always represented something—a function, a structure, a concept, etc. That is, the person doing the showing has an idea in mind, and presents to students something that (to the person doing the showing) has that idea as its meaning. This creates an impenetrable loop—impenetrable by students, that is. So, the background motivation for my opening statement was largely pedagogical, and its thrust was psychological. I was calling for taking students' reasoning and imagery as preferred starting points for discussions of curriculum and pedagogy instead of taking (fictitiously) unitary constructs, like function (or division, fraction, rational number, etc.), as preferred starting points.

I hope you don't interpret these remarks as saying that we must abandon adult mathematics and be satisfied with whatever mathematics children create. Rather, I was saying that we must be more clever. Rather than teaching the mathematics we know, we should understand students' construction of concepts *from their assimilations and accommodations over long periods of time*, and to be open to the realization that what students end up knowing never will be a direct reflection of what we teach. Neither David nor Ed, in the AWG excerpts you presented earlier, picked up on the last paragraph in my opening statement — that we cease our fixation with representations of (our) big ideas and instead focus on having students use signs and symbols only when they (students) have something to say through them (symbols).

I propose that we force ourselves to speak in the active voice—that when we speak of a representation, we always speak of to whom it is a representation and what we imagine it represents for them. When we speak of, say, "tables as representations of functions," we say for whom we imagine this to be true, what we imagine it represents for them (the idea they are expressing in a table when producing it, or the meaning they are reading from the table if it is presented to them), and something about the context in which this is all happening.

What, you ask, does this have to do with reification? It is this: Whenever I observe people doing mathematics and constrain myself to speak in the active voice, and constrain myself to be precise in my use of "representation," I don't see objects in people's thinking. Instead, I see schemes of operations and webs of meaning. Sometimes these schemes are ill-formed or in the process of formation. Sometimes they are well-formed and highly integrated. In the latter case, the people possessing these schemes maintain that they are thinking of "mathematical objects."

Anna, your turn!

Anna:

Easier said than done, Pat.

When you insist that we should "cease our fixation with (our) big ideas and instead focus on having students use signs and symbols only when they (students) have something to say through them (symbols)," you seem to have a more or less clear image of what you want to say and where it all is supposed to lead us to. But it is not that obvious to me. I still feel that this conversation will not proceed if we don't make it clear what this discussion is about.

You already seem to have given your response. If I understood you well, you are questioning the concept of representation (which, in the present context, is meant to refer mainly to an external representation, right?), and you do it on the grounds of a claim that in the eyes of many students, nothing is being represented by a graph or a formula; the abstract objects that would unify the clusters of symbols supposed to refer to the same thing are absent from student's mind only too often. I agree whole-heartedly that we should reconsider the concept of representation. This will force us to focus on the notion of mathematical object, and try to examine its possible meaning and uses. I will stress right away that, for me, the notion of "mathematical object" can only function as a theoretical construct, and it should only be used as such if a good theory may be built around it.

Someone may ask whether anything as elusive as the idea of mathematical object has a chance to turn into a scientific concept at all. The notion of science, however, has greatly evolved in the last few decades, and it became a lot more flexible than it was when cognitive approaches made their first steps toward general acceptance. Many factors brought about this change. One of them was the growing dissatisfaction with the information-processing account of the functioning of human mind. Another was the evolution of philosophers' vision of human knowledge. The notions about what is scientific and what cannot be regarded as such underwent dramatic transformations. In fact, the demarcation line between science and non-science, once so clear to everybody, was irreversibly blurred and in certain domains became almost impossible to draw. Today, the majority of those who view themselves as scientists are prepared to deal with concepts that would once be discarded by them without hesitation. Varela et al. (1993) admit that still

> most people would hold as a fundamental truth the scientific account of matter/space as collection of atomic particles, while treating what is given in their immediate experience, with all its richness, as less profound and true (pp. 12-13).

Cognitive science, however, cannot discard all the elements of human immediate experience any longer, and thus it is

> Janus-faced, for it looks at both roads at once: One of its faces is turned toward nature and sees cognitive processes as behavior. The other is turned toward the human world (or what phenomenologists call the "life-world") and sees cognition as experience (Varela et al., 1993, p.13).

Today, nobody is really afraid anymore of talking about such immeasurable entities like concept images and abstract objects — the entities that can only be seen with our minds eyes. To great extent, it is the growing abandonment of the Objectivist epistemology that made us more daring than ever in our theorizing about the human mind and about its functioning. Indeed, we came a long way since the times when "mind" itself sounded somewhat dirty. Nowadays, people are no longer concerned with the objectivity of knowledge — with the question how well a given scientific theory reflects the "real" state of affairs. There is no belief anymore in the "God's eye view" of reality. The concern about the truthfulness of our representation of the pre-given world has been replaced with pragmatic questions of usefulness (Lyotard, 1992) and of "intersubject agreement" (Rorty, 1991). One of the central criteria for evaluation of scientific theories is the question whether they are likely to generate many interesting ideas:

> ... the justification of scientific work is not to produce an adequate model or replication of some outside reality, but rather simply to produce *more* work, to generate new and fresh scientific ... statements, to make you have "new ideas" ... (Jameson, 1993, p. ix)

Thus, if I somewhat disagree with your critique of the notion of (multiple) representation, it is not because I wish to keep this notion with its traditional meaning intact. On the contrary, I think that the use of the word in the context of cognitive science is somehow misleading. But for me, it is misleading not so much because of the fact that the referent of the symbol may be absent from the student's mind, but because when construed in the traditional way, it seems to reinforce an Objectivist approach. It is misleading because it implies an existence of an objectively given state of affairs even within the human mind itself (like in the case when we say, for example, that such abstract concept as function, represented by a graph and a formula, is inaccessible to a student).

I am not sure whether your protests against the traditional approach to the issue of representations stemmed from the disillusionment with the Objectivist epistemology, but my doubts about this notion *are* the result of such disillusionment. This does not mean that I will not talk about "abstract objects hiding behind symbols." I will. But when I ask whether an abstract object exists or not, it will not be a question about any *real* existence, which can be proved or disproved in a rigorous way. The only criterion I will use will be that of theoretical effectiveness: I shall make claims about existence or non-existence of abstract objects in the learner's mind only if it helps me in making sense of observable behaviors. Indeed, you yourself, in your last statement, gave me a perfect example of a situation in which *I* would say that I can see objects in peoples' thinking — just when you say the opposite. Let me remind your own words:

> ... I don't see objects in people's thinking. Instead, I see schemes of operations and webs of meaning. Sometimes these schemes are ill-formed or in the process of formation. Sometimes they are well-formed and highly integrated. In the latter case, the people possessing these schemes maintain that they are thinking of "mathematical objects."

What else do you need to at least try using the notion of an "abstract object" as a potentially fruitful theoretical construct? Your own description makes it clear that this idea could help us in pinpointing the difference between different mathematical behaviors in a concise and productive way. You seem to me still quite afraid of being accused of making ontological statements (about some kind of *real* existence of the abstract objects). Free yourself from these fears — go theoretical and be brave! After all, theory is the way we speak, not an attempt to say that our abstract constructs mirror reality.

Pat:

I must chuckle. This is the first time I am chided for appearing to *fear* being theoretical. I am often accused of being *too* theoretical. I've even called for members of PME to take theory more seriously (Thompson, 1991a).

Perhaps it will help to make clear on what we agree before going further. I agree with you that modern notions of science no longer are concerned with whether theories are *true*, only with whether they are *coherent* and *useful*. This point is well-articulated in the writings of Kuhn (1962, 1970a), Popper (1972), and Feyerabend (1988) in the philosophy of science and in the writings of, among many others, Brouwer (1949, 1952), Lakatos (1976, 1978), and Wilder (1968, 1981) in the philosophy of mathematics. In earlier publications I, too, have said that what matters most is that we develop useful ways of thinking about aspects of teaching, learning, and experiencing mathematics (Thompson, 1979, 1982, 1991b) — useful in the sense that greater insight into problems leads to more informed and efficacious action. One of my favorite sayings is Dewey's: *There is nothing more practical than a good theory* (Dewey, 1929). We have no quarrel on this matter.

Finally, you object to my criticism not because you disagree with it (I know you don't), but because you see my criticism as being misleading. You said,

> But for me, it is misleading not so much because of the fact that the referent of the symbol may be absent from the student's mind, but because when construed in the traditional way, it seems to reinforce an Objectivist approach. It is misleading because it implies an existence of an objectively given state of affairs even within the human mind itself.

I agree completely that the notion of "representation" as implying a symbol-referent relationship is highly problematic. In fact, following von Glasersfeld's (1991) and Cobb, Yackel and Wood's (1992) examples, I tried being quite careful to make my usage of "representation" reflect the context of someone attempting to convey or impute *meaning*. My criticism is of people using "representation" too loosely, without mentioning a person to whom some sign, symbol, or expression has some meaning. I think you put it quite nicely in another publication:

> While Objectivism views understanding as somehow secondary to, and dependent on, predetermined meanings, non-Objectivism implies that it is our understanding which fills signs and [notations] with their particular meaning. While Objectivists regard meaning as a matter of a relationship between symbols and a real world and thus as quite independent of the human mind, the non-Objectivist approach suggests that there is no meaning beyond that particular sense which is conferred on the symbols through our understanding. (Sfard, 1994, p. 45)

Part of our miscommunication is due, I suspect, to the various stances we take naturally when speaking about mental processes and to the various perspectives we take, again naturally, when speaking theoretically. In regard to the first, Donald MacKay (1969) makes a useful distinction between "actor language" and "observer language." We speak in actor language when speaking for ourselves or in an attempt to speak for another. We speak in observer language when speaking as an observer of another or others. It is very difficult to remain within one or the other. In the previous sentence I adopted neither stance

— and illustrated my point about the dangers of writing in the passive voice. Did I mean that *I* find it difficult to remain within one or the other, or did I mean that *anyone* will find it difficult to remain within one or the other? You cannot tell.

In regard to perspectives we take when speaking theoretically, I find Alan Newell's (1973) discussion of "grains of analysis" quite useful. In one example he compares different analyses of teeth. On one level, teeth are quite structure-like. They are stable biological structures which we use to gnash and to chew. When examined on another level, teeth are constantly changing shape, eroding, and regenerating. We could say they are the same "things," only viewed with different grains of analysis.

It seems that when you quoted my passage beginning "I don't see objects in people's thinking ..." and said that, indeed, you could see objects, we used different grains of analysis. What were the objects you saw me speaking of? Schemes? At the grain of analysis I had in mind, I would say those were *my* constructions (actor language regarding me, observer language regarding the people I observed). At a more distant grain of analysis I could say, yes, those schemes were objects in their thinking. But to whom are they objects? They are objects to me. There may be something in their thinking that are objects to them, but I would not automatically attribute "objectness" to the schemes I identified (speaking in actor language regarding the people I observe). To hypothesize what they took as objects at the time of my observing them would require a different analysis.

Anna:

Shall I let you have the last word on the first question? Oh well, I will. After all, you managed to show that there is more agreement between us than argument. So let me start pondering our second issue.

After all the explanations regarding the non-Objectivist vision of knowledge and of scientific theories, I feel it is my duty now to show that *abstract object* is a useful theoretical construct. For the sake of enlightening the discussion, I invite you to try to make my life difficult also on this point. To put things straight, however, let me precede the defense of the theory of reification with a more thorough clarification of what "abstract object" means to me.

First, may I remind you that my use of the term is quite different from that of a Platonist. For me, abstract object is nothing 'real', nothing that would exist even if we did not talk about it. As Putnam (1981) put it,

> "Objects" do not exist independently of conceptual schemes. We cut up the world into objects when we introduce one or another scheme of description. (p. 52)

In other words, objects — of any kind whatsoever — are, in a sense, figments of our mind. They help us put structure and order into our experience. My approach is no different from Putnam's: for me mathematical objects are theoretical constructs expected to help in making sense of things we see when observing people engaged in mathematical activity. What counts as a good theoretical construct? Something that makes it possible for you to have more insights and generate more knowledge out of a fewer basic principles. Something that helps you to build an effective theoretical model.

I see such notions as *abstract object* in psychology of mathematics — and, for that matter, as *energy* in physics or as *mind* in cognitive science — as a kind of link (a glue, if you wish) we add to the observables in order to make the latter hung together as a coherent structure. These additional "somethings," being our own inventions, cannot be directly observed, and cannot be identified with any specific discernible entities. Their "presence" can only manifest itself in certain well-defined clusters of phenomena — phenomena which, in fact, wouldn't appear as in clusters and wouldn't make much sense if it wasn't for these special "somethings" that we invented.

As it often happens, the nature and function of the special element unifying many different situations may best be scrutinized in pathological cases: in situations in which it is missing. Indeed, Pat, I agree with what you said in your opening piece: for many people, certain "representations" may be empty symbols that do not represent anything. But while saying this, you only made a stronger case for the notion of abstract object! It was thanks to the notion of mathematical object that in my studies on the notion of function and on algebraic thinking I was able to see many kinds of student's faulty behavior as different symptoms of basically the same malady: student's inability to think in structural terms. A failure to solve an inequality, an unsuccessful attempt to answer a question about a domain of a function, a faulty formulation of an inductive assumption on the equality of two sequences of numbers, a confusion about the relation between an algebraic formula and a graph — all these diverse problems combined into one when I managed to see them as resulting from learner's "blindness" to the abstract objects called functions.

Needless to say (but I'll say it anyway, just to be sure that you don't accuse me of overlooking this important aspect), theoretical notions are not stand-alone constructs. One can only justify their use if they become a part of a theory — theory which neatly organizes the known facts into a coherent structure and, maybe even more important, has a power of generating new insights and turning into clearly visible things that would otherwise escape researcher's attention. As Sherlock Holmes once nicely put it, without the special alertness which can only come from a good theory, "you see but you do not observe."

Well, abstract objects did become a part of a theory some time ago. Many mathematics education researchers worked in parallel on theoretical frameworks which took the concept of abstract object seriously. The list is quite impressive: it includes both Piaget and Vygotsky (whose theories may be viewed as incompatible in some respects, but who nevertheless seem to be in agreement on the points which match our present interest), Dubinsky (1991), Harel and Kaput (1991), Gray and Tall (1994), Douady (1985) ... and the list is still quite long. Of course, both participants of this dialogue are among the most devoted members of this school — or, at least, one of them was and still is, and the other one was known to be before this dialogue began; see e.g. Thompson (1985), Sfard (1991, 1992); Sfard and Linchevski, (1994). I will later give an outline of my favorite variant of such theory — the breed which we call here a *theory of reification.*

Right now, however, I can feel it's time to cut this flow of theorizing and meta-theorizing with some enlightening illustrations which would take us back to the mathematical thinking itself. I'm going to present you with two short examples which hopefully will make it clear how the notion of abstract object works as an explanatory device. First, I will present you with a situation which, when seen through the lens of the theory of reification, displays the presence of an abstract object in the learner's thinking. It will then be contrasted with a case in which abstract objects are conspicuous in their absence.

My first example comes from a study recently completed by Carolyn Kieran and myself in Montreal. In our experiment, 12 year old kids made their first steps in algebra. Our approach was functional and the learning was massively supported with computer graphics. I'm far from saying that in the study everything went according to our expectations and that our special approach brought a solution to all the problems the teachers always grappled with. But some nice things did happen. In the final interview, a boy named George was asked to solve the equation $7x+4 = 5x+8$. The children did not learn an algebraic method of solving equations, but they did learn to see linear functions through formulae such as $7x+4$ or $5x+8$. Here is our exchange with George:

G: Well, you could see, it would be like, ... Start at 4 and 8, this one would go up 7, hold on, 8 and 7, hold on ... no, 4 and 7; 4 and 7 is 11 .... they will be equal at 2 or 3 or something like that.

I: How are you getting that 2 or 3?

G: I am just graphing in my head.

For me, it is clear that George was able to see more than the symbols — more than the formulae and the graphs. He was able to imagine abstract objects called functions. Why abstract objects and not just graphs? Because "graphing in one's mind" is one thing, and being able to make smooth transitions between different representations (I hope, Pat, that you agree with my use of the term representation in this case) means that there is something that unifies these representations. What I call "linear function $7x+4$" is such a unifying entity (it is neither the formula, nor the graph — it's an abstract being).

Here comes my second example. A 16 year old girl — let us call her Ella — was asked to solve a standard quadratic inequality: $x^2 + x + 1 < 0$

At this stage, Ella could solve any linear inequality and was quite familiar with quadratic functions and their graphs. The girl approached the problem eagerly and in a few minutes produced the following written account of her efforts:

$$(1)\quad x_{1,2} = \frac{{}^-1 \pm \sqrt{1-(4)(1)(1)}}{2} = \frac{{}^-1 \pm \sqrt{{}^-3}}{2}$$

$$(2)\quad T = \{\ \}\ \ (No\ \ Solution)$$

Was the written solution the only source of teacher's insight into Ella's thinking, he would certainly reward her with a high score. As it happened, however, he talked and listened to Ella when she was working on the problem, and the things he heard prevented him from praising her. Let us have a look at a fragment of this dialogue.

E: [After she wrote line (1) above] There will be no solution for x, because here [points to the number under the radical sign] I've got a negative number.

T: O.k., so what about the inequality?

E: So the inequality isn't true. It just cannot be ...

T: Do you know how to draw the parabola..?

E: The parabola of this [expression]? But there is no y here ... how can one draw parabola when there is no y?

T: Do you know the relationship between a parabola and the solutions of such an inequality as this?

E: Of an inequality? No. Only of an equation. But maybe it is the same.. Let's suppose that this is equal zero [points to the inequality symbol and makes a movement as if she was writing "=" instead of "<"]. But how can there be a parabola if there is no result here [points to the expression she wrote in (1)], no solution?

T: So what is your final answer ? What is the solution of the inequality?

E: There is no solution.

Do I have to add anything to convince you that we are dealing here with the case of a girl who cannot see *through* symbols and can only see the symbols themselves?

Pat:

I fail to see how my argument provides a case for "abstract objects." It seems you are saying if something is meaningless for students, it is meaningless because they do not possess the abstract objects which would give it meaning. This doesn't follow. Chinese characters are meaningless to me. Does that imply that my possession of certain "abstract objects" will render them meaningful? No. My inability to read Chinese characters means only that I do not possess the many grammatical, rhetorical, and perceptual schemes which I need to read Chinese. On a similar note, I recently picked up a physics text which uses notational conventions unfamiliar to me, and on top of that it employed poor rhetorical style. I felt like I was reading Chinese, but I certainly understood the physics about which the text's author spoke. To say that I am blind to certain "abstract objects" is a poor explanation of my inability to understand either Chinese characters or a particular physics text.

Our agreement on the importance of theoretical constructs should be clear, so I'll say nothing further on that. However, not just any theoretical construct is a good one. I find the construct of an "abstract object" problematic in two ways: its internal coherence and your use of it as an explanatory device.

You haven't said what you mean by an *abstract* object. I think I understand what you mean by an object, or at least my understanding is not incompatible with what you've said. It seems that by object you mean what Piaget had in mind when he spoke of children's construction of object permanence (Piaget, 1950, 1976, 1985) — people construct objects by building and coordinating schemes of action or thought to form a locally-closed, self-regulating system which they can re-present to themselves in the absence of the network being activated *in toto* (von Glasersfeld, 1991). As a matter of methodology, to characterize someone's construction of a *particular* object (especially, an object to which we might assign a name like *function*), I would think it necessary to say something about that person's schemes of action or thought which we presume constitute it.

As for "abstract" objects, if a person has constructed an object, then it would seem this object, to that person, will be concrete. I don't know what you mean by an *abstract* object. I know what Steffe et al. mean by an *abstract unit* (Steffe, Thompson, & Richards, 1982; Steffe, Cobb, & von Glasersfeld, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983), but they use this as a technical term to denote something that a child has constructed through reflective abstraction — an *abstracted* unit, so to speak. They do not use "abstract" as a tack-on adjective, as if there are objects and then there are abstract objects. If my characterization of "object" is satisfactory to you, it would help me considerably were you to explain how the adjective *abstract* adds anything to its explanatory power.

If in an explanation of some student's behavior you say "she has constructed function as an object," I would still have to ask what schemes comprise this object for that student, for objectness comes from her possessing coordinated schemes — but not necessarily the schemes you wanted her to construct. Lee and Wheeler (1989) found a large number of students for whom expressions, proofs, and rules were "objects," but they were objects to these students in the same way that this sentence might be an object to my 8 year-old daughter. She knows about sentences — that they are to be read, interpreted, that they have a beginning and an end, they generally communicate a single thought, and so on. But that sentence is not the same object to her as it would be to a linguist who takes it apart according to systems of grammar or pragmatics. Why? Because the schemes which constitute sentence-objects for my daughter are very different from the schemes which constitute sentence-objects for the linguist. You cannot say that a sentence is an object for one but not for the other. Rather, they are different objects to two different people.

Another example that "objectness" cannot be taken at face value is Kuhn's account of a debate between a chemist and a physicist (Kuhn, 1970b, p. 50). The chemist maintained that a helium atom is a molecule, because it behaves as a molecule should according to the kinetic theory of gases. The physicist maintained that a helium atom was not a molecule, because it displays no molecular spectrum. Looked at one way, they were arguing about what label to apply to some object. Looked at another way, their argument reflected that the term "molecule" *pointed to different (i.e., non-identical) objects for these two people*. The mention of "molecule" activated different schemes of operations in them. Is one person correct? No. In fact, the question can be misleading.

Your examples illustrate the difficulty I have with the way you use "abstract object" as an explanatory device.

[*Solve* $7x + 4 = 5x + 8$]

G: Well, you could see, it would be like, … Start at 4 and 8, this one would go up 7, hold on, 8 and 7, hold on … no, 4 and 7; 4 and 7 is 11 …. they will be equal at 2 or 3 or something like that.

I: How are you getting that 2 or 3?

G: I am just graphing in my head.

You said,

> For me, it is clear that George was able to see more than the symbols — more than the formulae and the graphs. He was able to imagine abstract objects called functions. Why abstract objects and not just graphs? Because "graphing in one's mind" is one thing, and being able to make smooth transitions between different representations … means that there is something that unifies these representations.

I agree with you that George "saw more than symbols." But it is a leap I cannot make to say he was able to "imagine abstract objects called functions." What does that mean? Did he imagine a domain? A range? A correspondence? Did he imagine two variables covarying continuously? Did he understand that going 3/11 of the way between x_0 and x_0+1 would correspond to an increase of 3/11 of 7? What is a variable to him? What kinds of operations can George perform on these objects called functions? Can he compare them? Combine them? Compose them? I suspect he can do none of these. If I am correct, then I have a difficult time understanding what these objects called "functions" are to him. I do not mean he is not thinking of objects which he calls "functions." Rather, I mean we do not know what comprises those objects.

It seems a more "explanatory" explanation would be: When he said "Start at 4 and 8, this one would go up 7 ... " he was thinking something like, "I need to find a value for x so that the two expressions have the same value. As I start at 0 and go over 1 in the left-hand expression (as in moving on a horizontal axis) I go up 7 (as in moving on a vertical axis) and as I start at 0 and go over 1 in the right-hand expression, I go up 5. So going over 1 in the left-hand-side is 4+7 ..." [then, to himself, "going over 1 in the right-hand-side is 5+8"]. I have too little information to guess at his reasoning in regard to his saying, "They will be equal at 2 or 3," but what I've postulated certainly fits the information you presented. What constructs would I use to enrich my explanation? Constructs like imagery, scheme, etc. How would I explain the connections he seemed to make? I would appeal to constructs like assimilation and generalizing assimilation (Thompson, 1994a). I see no need to appeal to such a vague notion as his imagining "abstract objects called functions" or to posit that, because he made some connections, that "there is something that unifies these representations." When we appeal too quickly to grand ideas, we lose sight of the richness and intricacy of students' reasoning.

Actually, I would have tried not to be in the position of so boldly guessing George's reasoning. Had I conducted the interview I would have looked to get different information than what you presented. The question asked of George, "How are you getting that 2 or

3?" moved the discussion in the direction of explaining answer-getting actions instead of discussions of what he had in mind when thinking about the task (Thompson, Philipp, Thompson, & Boyd, in press). I suspect the conversation would have produced more useful information had the immediately succeeding question been something like, "You said: Start at 4 and 8, this one would go up 7. What did you mean by that?" with subsequent questions sustaining that emphasis.

Your example involving Ella is even more problematic. At this point I must be brief, so I'll just say that I do not understand how it furthers our understanding of students' mathematics to explain their reasoning in terms of the *absence* of various abstract objects in their reasoning. I can understand attempts to compare where students are with where we would like them to be, but to explain where they are by saying they are not where we want them is a non-explanation. I think a richer explanation of Ella's behavior might be found by speaking about her assimilation of certain figural forms to an action-schema which has "replace '<' with '=' and solve" as its first part. That is, explanations of students' behavior which try to capture students' experience and which posit what students *do* understand add more to our understanding than do explanations which explain their behavior by stating what they do not know.

You end Ella's example by stating "... we are dealing here with the case of a girl who cannot see *through* symbols and can only see the symbols themselves." To a great extent I agree with your statement. I do not agree that Ella's example buttresses your case that "abstract objects" is a useful theoretical construct.

Anna, my complaints might seem methodological, but they are methodological at the level of *research programme* (Lakatos, 1978), for they address basic orientations we bring to our work of theorizing and they raise the question of the kinds of theories we value most.

Anna:

Wow, Pat! You do seem to have taken the invitation to make my life difficult riously! You might even have overdone it a little bit. But it's good. A fight will force us to sharpen our theoretical weapons and to elicit points inadvertently glossed over.

Your reaction to what I said sounds convincing: the often observed "meaninglessness" (I dislike this term) of mathematics is not, *per se*, a proof for the usefulness of the notion of mathematical object. I agree, and the fact is I never made this illogical claim. The only aim of the episodes and phenomena I brought earlier was to exemplify situations in which a person *who looks through the lens of theory of reification* would spot either the presence or the absence of abstract objects. The examples were not, and could not be, meant to show the objective necessity of the notion of abstract objects as means of explaining the phenomena.

Since we seem to agree that there is more to understanding mathematics than knowing the rules of symbol manipulations, the question arises what is this additional something. This may be translated in the question what we mean by "meaning." No, don't expect me to explore the morass of this time-honored philosophical puzzle. Let me tell you one thing, though. You say "Chinese characters are meaningless to me." And you ask, "Does that imply that my possession of certain 'abstract objects' will render them meaningful"? Of course it doesn't; but although "meaninglessness" certainly does not *imply* the necessity of abstract objects, having abstract objects is one way of explaining how people make certain expressions meaningful. If the sentences you are dealing with happen to be built around a noun, such as, say, a chair, a gremlin or a function, then having the ability to think about the objects hiding behind the words is what we call "grasping the meaning." Sometimes, like in the case of a chair, the referent of a noun is a tangible material object. Sometimes, like with gremlins, the existence of such object could, theoretically, be ascertained with our eyes, provided it really existed. Sometimes, like in the case of function, the nature and the existence could not, even in theory, be explored with our senses. In this last case I say that the objects we are talking about are *abstract*. I hope this answers your question what I mean by the adjective "abstract" in this context.

Not quite yet? You might be right. Well, I have more to say about that. You claim for example, and rightly so, that "If a person has constructed an object ... [then] to that person it will be concrete." I can even help you with this. Some mathematicians I have recently talked to used expressions like "concrete," "tangible," and "real" when referring to the things they were manipulating in their minds when thinking mathematically. Here, one starts to wonder what the word "concrete" means. Like in the case of the notion of 'meaning', it is much too loaded a problem to be dealt with in this short exchange. But let me refer you to an insightful essay by Wilensky (1991) in which, in one voice with Turkle and Papert (1991), he suggests a "revaluation of the concrete." The need for the revaluation arises at the crossroads of two current trends: constructivism and emergent-AI. Wilensky analyses the "standard view" of the concrete through new glasses and arrives at the conclusion that "concreteness is not a property of an object but rather *property of a person's relationship with the object*" (Wilensky, 1991, p. 198). If so, it may be helpful here to follow your suggestion and distinguish between two perspectives: that of an actor and that of an observer. The mathematical object may be concrete to the former, and at the same time abstract to the latter.

I once tried to capture the difference between these two perspectives in the metaphor of mathematics as a virtual reality game. Have you ever seen a person wearing a computerized helmet and a glove, engaged in a virtual reality game? Wasn't it quite amusing? Could you make anything of this person's strange movements? Probably not, but I bet that it never occurred to you that the funny fellow might be out of his mind. And if in addition you had been told that, for instance, he is trying to transverse a heavily furnished, messy room, then, quite likely, you instinctively tried to imagine the kind of objects he could be moving around. Like this virtual reality game player, a person engaged in a mathematical activity seems to be dealing with objects nobody else can see. You and I, as observers, do not have a *direct* access to what the actor thinks he is playing with. But assuming that he does see some objects helps us in being tolerant toward his strange movements and makes us believe that the funny behavior has an inner logic. Trying to figure out what the player sees is the most natural way to make sense of what he is doing. Thus, we recognize the existence of the objects the player is dealing with, but while for the latter they are quite concrete, for us they are abstract. Since in this conversation I am speaking mainly from observer's perspective, I refer to the mathematical objects as abstract.

As a side-effect, this parable brings home to us that the notion of mathematical object is a metaphor that shapes the abstract world in the image of tangible reality. Hopefully, it is also more clear now how important a role a perceptually-based metaphor plays both in our mathematical thinking and in our thinking on mathematical thinking. As we both agreed, the actors themselves, when looking at what they are doing, would usually admit that some objects are present in their thought. At least the best, the expert actors would say so. You mentioned it yourself in the beginning of this dialogue, remember? "The people possessing [highly integrated schemes of operations] maintain that they are thinking of 'mathematical objects'." My work with mathematicians brought lots of further evidence that, indeed, the inner world of a mathematizing person may look very much like a material world, populated with objects which wait to be combined together, decomposed, moved and tossed around (Sfard, 1994; Thurstone, 1994).

The fact that according to the actors themselves the metaphor of object is ubiquitous in mathematical thinking is hardly surprising. What can be better known to us than our perceptual experience, than the physical world that surrounds us? The mathematical objects we can see with our mind's eyes are metaphors that *constitute* the mathematical universe in the first place, and then make it possible for us to move around it in ways similar to those in which we move in the physical world. The embodied schemes generated by our physical experience are deeply engraved in our minds and this is thanks to them that we often find our way in this world intuitively, without reflection (Johnson, 1987, p. 102). By using such schemes to help ourselves move in the virtual reality of mathematics we inject mathematical thinking with the meaningfulness of our physical experience.

The metaphoric use of "object" is by no means restricted to mathematics. Here is an account of a physicist:

> [W]hen analyzing physical phenomena, people like to put into play "objects." Beside real objects, they ascribe a realistic character to physical concepts or models. They build their reasoning on these "objects" as if they were material. (Viennot, in press)

Finally, the picture will not change in a substantial way if we climb to the meta-level

— the level of thinking about mathematical thinking, the level of an observer. The metaphor of mathematics as dealing with as-if-material objects has a special appeal for the researcher. Speaking about mathematics in terms of abstract objects and processes on this objects makes mathematics in the image of the world we know best: the material world. Whatever we know about the former — and we know an awful lot about it — has a potential of bringing insights about the nature and function of the latter. Those concerned with the methodology and psychology of scientific innovation have agreed a long time ago that scientist is "an analogical reasoner" (Knorr, 1980) — that resorting to our knowledge of things with which we are familiar and which are somehow similar to those we find in the new domain may be for the scientist the most powerful, albeit "unofficial," way to get moving in untrodden territories.

At a certain point you say, Pat, that "'Objectness' cannot be taken at its face value." I couldn't agree more. Objects have many faces and our knowledge of them can never be "full." What your daughter knows of "sentence-objects" seems to be partial to rather than different from what the linguist knows. Your second example is even more enlightening: you say that for the chemist and the physicist "the term 'molecule' *pointed to different ... objects*." In mathematics, things like that are happening all the time. For example, through one algebraic formula, say $3x$+b, one may see quite a number of different mathematical objects: a number, albeit unknown, a linear function, a family of linear functions. The expression may, of course, be also taken at its face value and treated as nothing more than a string of symbols. You don't have to deal with a number of different people in order to have all this interpretations; on the contrary, it should be teacher's goal to help her students construct a scheme which will include all the possibilities. Such scheme is necessary for the *flexibility* of thinking which was called "the hallmark of [mathematical] competence" (Moschkovich et al., 1993) and which can be described as the ability to match an interpretation to the context in which the formula is used. You also remark that "objectness comes from possessing coordinated schemes — but not necessarily the schemes you want [the student] to construct." I agree again. More often than not, the scheme built by the student is only partial. Worse than that, it often includes only one object: the "opaque" formula, a formula which is taken as an object in its own right and through which no other object can be seen. My colleagues and I once called this kind of conception (scheme) *pseudo-structural* (Linchevski & Sfard, 1990; Sfard & Linchevski, 1994). In our studies, we had a chance to observe numerous phenomena which can be interpreted as adverse effects of the lack of flexibility inevitably accompanying such impoverished conception.

Now to the most important part of your critique: the questions about my interpretation of George's and Ella's behavior. What happened here is a result of talking "from the middle," of bringing just one piece of a greater whole and hoping that this one element will speak for all the rest. Obviously, things do not work in such simple way. It is only when the stories of George and Ella are viewed within the context of a theory that they may become truly meaningful. Talking about the presence or absence of abstract objects without tying the notion to a theoretical framework is like using the term "energy" in physics while talking to a person knows nothing about mechanics. It would be quite futile if one said to this person that one of two stones has much kinetic energy and the other has none. Such labeling is only useful if, by connecting a given situation to a theory, it immediately increases the amount of information — if, for example, it conveys the message that one stone is in motion while the other rests. I'll try now to make up for my mistake by showing how my explanations draw on the theory. Incidentally, I wouldn't like to sound presumptuous when I call this particular framework "a theory." I am using this word only to be brief.

Let me begin. You just said that "to explain where [students] are by saying they are not where we want them to be is a non-explanation." It is certainly not the case when you are within a theory which provides you with information about the possible alternatives to the student's being where we want her to be, about the consequences of being in this other place, about the possible reasons of the situation and about the means that can be taken to change it. I tried to explain George's success by conjecturing that he did construct a certain object, and I ascribed Ella's failure to the absence of this object. This kind of explanation is common in science. When two kids come in contact with a case of chicken pox but only one of them gets infected, the doctor is likely to say that the first one had antibodies whereas the other did not. This statement has an explanatory power, since it ties both cases to the same underlying mechanism; it has a predictive power, because it gives a basis for expecting what will happen to the children if they come in contact with the chicken-pox again; finally, it may serve as a basis for some medical decisions. In a similar way, my interpretation of the two episodes, if supported by a theory, may have an explanatory, predictive, and prescriptive power.

Let me elaborate on the explanatory aspect. Both children were presented with situations with which they were not well acquainted. The parable of the messy room highlights the importance of "seeing" object for a person who is supposed to move in an unfamiliar setting. One salient feature of objects — whether material or abstract — is the fact that they tend to preserve their identity and are easily recognizable in different contexts. You are right in claiming that we cannot say much about the kind of conception George has developed and our information is too scarce to know whether he had the multifarious structural (object-oriented) understanding of the formula I was talking about earlier. But we have reasons to conjecture that he was able to deal with the non-standard situation because he had a good sense of the particular object he was dealing with (a liner function) and thanks to that he could adjust his actions to the new needs. Ella obviously could not see the objects with which she was supposed to deal and, as a result, the only thing she could do was to repeat the standard movements she once learned by watching and mimicking people engaged in the game (e.g. the teacher). To use a description by Dörfler (in press): When the "adequate image schemata have not developed to supply meaning through metaphor [of object], the discourse will instead be used in a parrot like and rote manner and will not be flexible or extensible." Ella's problem was that the standard behaviors she learned (you are right: most probably, it was solving quadratic equations) were inappropriate in the new situation — but, not having access to the "virtual reality" of functions, she could not see the change.

The idea of reification may give us an even deeper insight into Ella's plight: it can help in figuring out the reasons of her inability to think in terms of abstract objects. More generally, the theory provides its own explanation for the fact that the kind of deficient conception displayed by Ella is evidently very common. The first thing that must be explained is how mathematical objects come into being. According to the theory, *these objects are reified mathematical processes*. To understand this statement, one has first to notice an inherent process-object duality of mathematical concepts: such notions as -3, $\sqrt{-1}$ or function $3x$-2, although clearly referring to *objects*, may also be viewed as pointing to certain mathematical *processes*: subtracting 5 from 2, extracting the square root from -1, a certain computational procedure, respectively. Historical and psychological analyses of concept formation led to the conclusion that operational (process-oriented) conceptions usually precede the structural. Let me build the rest of the outline around the testimony of a mathematician:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (Thurstone, 1990, p. 847)

If the "compression" is construed as an act of reification — as a transition from operational (process-oriented) to structural vision of a concept (it doesn't *have* to be construed in this way, but such interpretation is consonant with what was said before about structural conceptions), this short passage brings in a full relief the most important aspects of such transition. First, it confirms the developmental precedence of the operational conception over structural: we get acquainted with the mathematical process first, and we arrive at a structural conception only later. Second, it shows how much good reification does to your understanding of concepts and to your ability to deal with them; or, to put it differently, it shows the sudden insight which comes with "putting the helmet and glove on" — with the ability to see objects that are manipulated in addition to the movement that are performed. Third, it shows that reification often arrives only after a long struggle. And struggle it is! Numerous studies suggest that whether we are talking about functions, numbers, linear spaces or sets, reification is difficult to attain (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Sfard, 1992; Sfard & Linchevski, 1994). The main source of this inherent difficulty is what I once called *the (vicious) circle of reification* — an apparent discrepancy between two conditions which seem necessary for a new mathematical object to be born. On one hand, reification should precede any mention of higher-level manipulations on the concept in question. Indeed, as long as a lower-level object (e.g. a function) is not available, the higher-level process (e.g. combining functions) cannot be performed for the lack of an input. On the other hand, before a real need arises for regarding the lower-level process (here: the computational procedure underlying the function) as legitimate objects, the student may lack the motivation for constructing the new intangible "thing." Thus, higher-level processes are a precondition for a lower-level reification — and vice versa! It is definitely not easy to get out of this tangle.

It seems that Ella's predicament was like that of many other students who fall victim to the inherent difficulty of reification. The explanation provided by the theory presented Ella's story as a special case of a general phenomenon. Understanding the underlying reason of Ella's poor performance on the given task makes it now possible to predict what kind of situations will be most problematic for the girl in the future. Indeed, she may be expected to fail time and again when confronted with tasks that require having a function as an object. Thus, while she may be quite skillful in solving all kinds of equations, she will probably be helpless if asked to deal with, say, a singular equation or a parametric equation (Sfard & Linchevski, 1994).

Finally, the theory has a prescriptive power and it does provide its answers to the question how we should teach in order to cope with the difficulty of reification, in order to prevent the situations like the one Ella got into. Since I already talked longer than I should, I will confine myself to one more remark (see the literature quoted above for more on the didactic implications of the theory). You opened this debate with the statement that we should "cease our fixation with representations of (our) big ideas and instead focus on having students use signs and symbols only when they (students) have something to say through them." I suppose that mathematical objects, such as function, are the "big ideas" you were talking about. In our dialogue we tried to clarify this term, so when we now approach pedagogical questions we hopefully know a little better what these "big ideas" are all about. I agree that it is highly unlikely that the student will construct mathematical objects right away. The theory I just presented points to the great difficulty of the undertaking and explains why reification does take time. I also agree that "finding situations that are sufficiently propitious for engendering multitudes of representational activity" may be very helpful indeed, and once again, the theory supports this view. But if the upshot of what you said is that we should give up striving for fully-fledged structural conceptions in our students, I hope that what I said will make you soften this position. The ideas I just presented support the view that structural conceptions — the ability to "see" abstract objects — are difficult to attain, but having them is most essential to our mathematical activity at all ages and at every level.

After all I said here you may be surprised that I have no wish to argue with your alternative interpretations of the two episodes. I won't do it because I don't think there is a real discrepancy between us. You just chose to look at things from a different vantage point, and I do see the merits of this other approach. After all, accepting my point of view does not necessarily imply rejecting yours. I hope you agree that the same phenomena may admit different interpretations when scrutinized with different theoretical tools, and that such different interpretations should often be regarded as complementary rather than mutually exclusive. I hope you agree that the theory filled the notion of abstract object with meaning just like geometrical axioms fill the primary geometrical concepts (point, line) with meaning.

Let me finish with a few words on the place of theories in our project as researchers. With all my preference for theorizing in terms of abstract objects, nothing could be farther from my mind than claiming an exclusivity — than saying that the resulting theory is an ultimate answer to all the questions about mathematical thinking people have ever asked. Two theories are sometimes better than one, and three are better than two. To quote Freudenthal (1978, p. 78), "Education is a vast field and even that part which displays a scientific attitude is too vast to be watched with one pair of [theoretical] eyes." Like in physics, where a number of ostensibly contradictory theories exist and flourish side by side, so in mathematics education there is certainly a room for several research perspectives. Theory of reification, like any other model, brings in full relief certain aspects of the explored territory while ignoring many others. The only thing I wanted to convince you about is that what it does show is important enough to make this particular theoretical glasses worth wearing, at least from time to time. We have agreed that good theory is a theory which may become a basis of a rich research program. Does the framework built around the concept of reification stand up to the standards? I believe it does. In fact, I *know* it does. It already proved itself in the past, when it spawned numerous studies, as well as many useful pedagogical ideas, and enabled a synthesis of much of the existing research on the development of central mathematical concepts. One day, it will probably exhaust its power to generate new research, like any theory. But not quite yet. At the moment, when the world of cognitive science is more and more fascinated by new theories of perceptual-metaphorical sources of all human thinking, the notion of mathematical object may have more appeal than ever.

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ELEMENTS OF A SEMIOTIC FRAMEWORK
FOR UNDERSTANDING SITUATED AND CONCEPTUAL LEARNING

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When Ross Perot's running mate appeared on the platform with Dan Quayle and Al Gore for the vice-presidential candidates' debate in 1992, he started out by asking, "What am I doing here?" As someone who is neither a psychologist nor a mathematics educator or math ed researcher, I might be asking myself, "What am I doing here, at PME–NA?" In recent years, however, the psychology of mathematics education has become heavily involved in exciting developments with interests and implications extending far beyond the more specialized concerns of any of the particular sub-disciplines within educational research.

In the "situated cognition" movement, for example, we see a developing appreciation for the crucial involvement of specific practical and social situations in the generation and the use of knowledge; and the generation and use of mathematical knowledge in particular has been prominently featured in the situated cognition literature (e.g., Lave, 1988; Brown, Collins, & Duguid, 1989). Yet the situation-specific character of knowledge and learning is being advanced as a general principle, and not just for mathematics in particular; and, while investigation of situated cognition can be seen as a fully psychological enterprise, it is no less fully sociological, anthropological, neurological, philosophical, and linguistic.

Indeed, the multi-disciplinary or inter-disciplinary character of research on cognition and learning has been raised as an issue in the situated cognition movement. While Lave (e.g., 1992) has argued against an individualistic psychological approach in favor of a more anthropological mode of investigation, Walkerdine (e.g., 1988, 1992b; and in Henriques et al., 1984) has criticized cognitive and developmental psychology in favor of an approach that emphasizes discursive practices and semiotic processes in constituting both the subjects and the objects of "knowledge." Other investigators such as Clancey and Roschelle (1991) have been more concerned with developing an interdisciplinary approach capable of dealing with cognition in the full range of its neurological and psychological as well as social and cultural dimensions. St. Julien (1992, 1994) emphasizes the value of connectionism in accounting for the neurological basis of *learning*, while at the same time recognizing *knowledge* as sustained in social practices. Bereiter (1991) likewise sees connectionism as an important contribution in developing approaches that will integrate the insights of "situated cognition," "embodied cognition" (citing Lakoff, 1987, and Johnson, 1987) and other recent departures from classical rule-based views of cognition and learning.

It might seem that the crowded array of "situated," "embodied," "connectionist," "constructivist" and other recently advanced views of teaching and learning have already given us enough to think about, without our also having to deal with semiotics. Semiotics should not, however, be regarded as providing yet another theory of cognition, offered as an alternative to theoretical approaches such as those mentioned above. Semiotics (i.e., the study of signs and their activity in sign-mediated processes) is presented in this paper, rather, as the study of the possibilities for sign-activity (or "semiosis") in general. As such, semiotics provides conceptual resources and vocabulary that are needed in accounting for cognition, teaching, and learning *as* sign-mediated processes. Thus, we should not expect semiotics to provide any kind of over-arching theory of cognition that would incorporate and subsume the insights of connectionism, constructivism, situated and embodied cognition, etc. Indeed, semiotics might be thought of as providing, rather, something of an "under-arching" theory of basic elements and principles by which cognition, teaching, and learning—within the broad rich universe of sign-mediated processes—may be understood to operate.

My own interests are pragmatic; I am personally not much interested in systematic theory-building for its own sake. Accordingly, I believe the value of approaching these issues with a semiotically-informed perspective should be assessed in terms of how semiotics might contribute to dealing with real problems that arise in our efforts to understand cognition, teaching, and learning.

The problem of accounting for representation is central in the effort by Clancey and Roschelle (1991) to provide an understanding of situated cognition that departs from the discredited tradition based on decontextualized rules or algorithms, and that bridges across neurological and social levels. Dealing with such problems in mathematics learning, Nunes et al. (1993) have concluded that "in order to understand the psychological processes involved in street mathematics, we need a theory that allows for analysis of situations and their representations" (pp. 137-8), and that "we may need not only theoretical ideas that overcome the polarization between general and particular knowledge but also ideas that bring to the fore the importance of forms of representation in thinking" (pp. 144-5).

Thus, we see that these investigators find the problem of representation to be bound up with other problems, such as the problems of generality or particularity arising from the problems of conceptual or "transferable" learning, and the problem of accounting for cognitive processes that are at once psychological, sociological, biological, and cultural. Since these are sign-mediated processes, I believe that semiotics, the study of such processes, provides a necessary framework for dealing with these and other problems. Although there is not sufficient space here for all the demonstrations, arguments, and connections that would be required to justify my assessement, I will try to provide some idea of the value of semiotics by very briefly introducing the most basic ideas of different semiotic traditions, and suggesting how these might be used in dealing with the problems in cognition, teaching, and learning.

As noted earlier, the use of semiotics in discussing situated cognition theory has been introduced by Walkerdine (1992b). Her discussion employs terminology derived from Jacques Lacan's radicalized variation on the model of "semiology" introduced by the Swiss linguist, Ferdinand de Saussure. While Lacan's variation is certainly more capable of accounting for the dynamic and *creative* (i.e., not merely static and *representative*) character of sign-activity, I believe that it neglects features of Saussurean or "structuralist" semiotics that make it possible to account for other aspects of semiosis in general, and of cognition in particular.

Semiotics begins with a rejection of the naive understanding of the "sign" as something that simply *denotes* another object in the world. Saussure's definition of *the sign, in general*, is derived from his definition of *the linguistic sign, in particular*. Saussure illustrated his definition with the example of the sign formed by the union of the concept <tree> with the sound pattern *"arbre"* (or *"tree"*). Saussure himself moved beyond the model of concepts united with sound patterns, when he replaced that terminology with his more general definition of the sign as a combination of a "signified" together with its "signifier." Although Saussure explains this substitution as a way of indicating the relatedness of terms within the sign, it also generalizes his definition of the sign beyond his initial reference to *linguistic* signs (with sound patterns as signifiers), so that he could now propose a more extensive new social science of "semiology."

The anthropologist Claude Lévi-Strauss has provided the most influential example of how Saussure's structuralist approach could be generalized for diverse uses in the humanities and social sciences (see, e.g., Howard Gardner, 1981). The influence of the psychoanalyst Jacques Lacan is more important for our purposes, however, since it is Lacan's departures from Saussure's model of the sign that paved the way for a recognition of the semiosic processes discussed by Walkerdine (1992b). At the risk of violently oversimplifying Lacan's notoriously subtle and complex formulations, we can identify two basic steps in the transformation of Saussure's semiotic model which have been adopted in a broad range of "post-structuralist" semiotic analysis.

First, Lacan inverted the priority of "signified" over "signifier" that was at least implicit in Saussure's model of the sign. Lacan pointed out that formulation of the sign as $\frac{\text{Signified}}{\text{Signifier}}$ does in fact tacitly preserve a kind of classical bias (cf. Plato) that accords some kind of priority to the signified—whether the signified is seen as a purely mental concept that can be "communicated" through expressions of a related sound-pattern, or whether the signified is seen (even more mistakenly, from a structuralist point of view) as a *referent* (i.e., an object that exists prior to the sign, and is *referred to* by the signifier). Lacan insisted on inverting this relationship, yielding his formulation of the sign as $\frac{\text{Signifier}}{\text{Signified}}$ and accordingly recognizing far-ranging autonomy for a dynamic and continuously productive play of signifiers that was not so easily recognized when it was assumed tacitly that a signifier was somehow constrained under domination by the signified. The more autonomous play of signifiers can be seen, for example, in a kind of "chaining" process, in which the signifying term ($Signifier_1$) in a preceding sign combination comes to serve also as a signified term ($Signified_2$) in a succeeding sign combination.

In such a "chaining of signifiers," the preceding signifieds and sign-combinations are sometimes described as "sliding under" the succeeding signifiers. Terms which may have originated in relation to certain needs and interests of the "speakers" (or of those engaged in practices using linguistic and/or non-linguistic signs) become displaced from active use by terms of the succeeding signs. Succeeding signifiers may initially be admitted into use as substitutes for the preceding terms, as if the sense and import of those terms has been preserved through the succeeding links along the "chain" of signifiers. Ironically, it is the very ability of succeeding signifiers to appear as sense-preserving substitutes which allows preceding terms to disappear without notice, as the use of succeeding terms gets taken over by the competing projects and practices in which they are introduced and deployed.

Walkerdine reports a dialogue in which "one mother gets her daughter to name people they are pouring drinks for and to work out how many drinks by holding up one finger to correspond with each name" (1992b, pp. 19-20). We begin with the people's names as "$signifiers_1$," within the conversational and mental discourse(s) of the mother and daughter. As Walkerdine observes, however, those names drop quickly to the level of $signifieds_2$ in relation to new $signifiers_2$—the fingers. Subsequently, spoken numerals might be used as $signifiers_3$ in relation to the fingers, which are now $signifieds_3$. "By this time any reference to people or outside the counting string no longer exists within the statement". Walkerdine observes how, at this point, the combination of fingers and numerals starts being used in "small addition tasks of the form: 'five and one more is . . . .'"

Walkerdine calls our attention to the "discursive shift" which has occurred when the numerals and fingers are used to deal with problems posed in forms that "can refer to anything." The same physical fingers and sound patterns might be used in either discourse, but these are merely the "sign vehicles": When they occur in discourses of abstract calculation, the signs in which the numerals serve as signifiers, and fingers serve as signifieds, are not the same signs (and those numerals and fingers are not the same signifiers and signifieds) as those which occur in other discourses (even when the same fingers and numerals are being used in either case). In such cases, the same sign vehicles are conveying different signs, with different semiotic values, when employed in different discourses. All of this might sound like a scholastic or sophistic quibble, except for all that we have learned from Walkerdine and others who have shown numerous and varied examples of how such differing discourses provide very different structural potentials for the positioning of subjects able to participate within those discourses—with dramatic consequences for formation of the very selves and subjectivities of the participants. Such examples help us avoid misunderstanding the "chaining of signifiers" as a process in which originally real and material signifieds are progressively concealed behind illusory or "merely symbolic" signifiers. Instead, we understand sites along the chain as sites of conflict among competing material practices—conflict in which the sign activity produces real and consequential practices even as those practices produce the signs by which they are themselves conducted.

Although Lacan's notion of a chaining of signifiers helps in explaining how signifiers can take on lives of their own, as it were, free from domination by any "true nature" of the "signifieds" that might be presupposed as a realistic basis for the signs in use, Lacan's focus on relations between signifieds and signifiers neglects the relationships of *difference* which have been observed as the basic elements of semantic structures. The differing uses of the word "more" observed by Walkerdine (1992b, pp.15-18) can be used as an example. In school-mathematics tasks, "more" is used for quantitative comparisons, in opposition to "less." "Less" is actually only one of the possible oppositions that would presuppose the negatively-defined contradictory (not-more); but when "more" and "less" are used as antonyms in these discursive practices, then the practices within which *that* opposition is most relevant will pragmatically determine the semantic sense of both terms in their relation to each other.

Walkerdine demonstrates the kind of mistake that researchers can make when neglecting the differences between school-mathematics tasks of this sort, and other tasks, in other situations, in which particular students might be more consequentially familiar with the "same" words (such as "more"), but with very different meanings—as in the example where the opposite of "more" is not "less", but "no more." As in this case, that difference can be even greater than one of differing conceptual opposition: Here, the conceptual or *semantic* opposition between <more> and <less> is contrasted with a *pragmatic* opposition between speech-acts: "More (please)?" and "No more!"

Walkerdine argues that "while [the terms 'more' and 'less'] might be the same signifiers the actual signs, the specific relation with signifieds was made in specific practices" (p. 16). While Lacan's "chaining of signifiers" would help in accounting for the flexibility of sign-relations in accommodating certain social and cognitive requirements of the practices in question, it neglects other structural dimensions of those sign-relations, and the ramifications that can both influence and transcend those practices.

Figure 1 illustrates what Greimas would refer to as "secondary meta-terms" of the square generated by the opposition of "more" (as a demand or request) and "no more" (as refusal or denial). On this level we find oppositions between engagement and non-engagement, and between satisfaction or compliance and discipline or deprivation. The semiotic structures both incorporate and generate the semantic meaning and pragmatic force of terms within the discursive practice here, in sharp contrast to school mathematics or other discourses in which some of the same signifiers might occur.

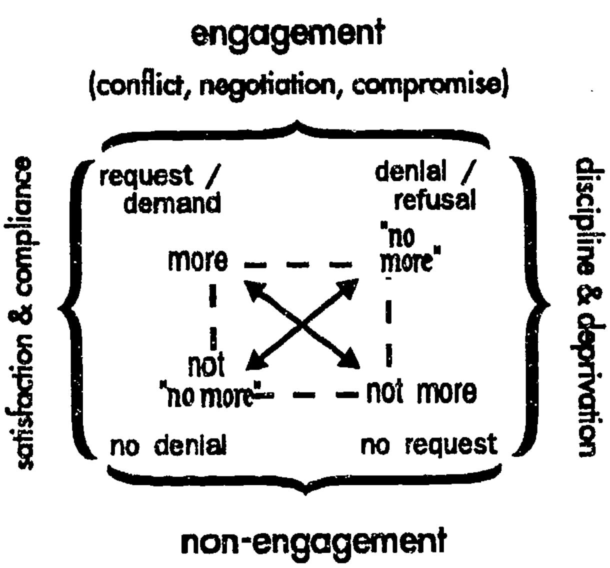

Figure 1. **Elaborated Square: More vs. "No More"**

Walkerdine (1990, pp. 61-81; 1992a) and Walkerdine et al. (1989) report a situation in which, paradoxically (at least from the standpoint of official rationales for schooling), school achievement *by girls* is disparaged, even as non-achievement by boys is regarded in a more positive light—and sometimes even treated as a sign of brilliance! The structural coding of these attributions can be understood in relation to what Walkerdine (1992b) reports as "the concern expressed when poor children appear to possess advanced calculating skills, indeed, sometimes not only more advanced than their school performance would suggest, but actually more advanced than their higher class peers" (p. 6). Having observed that "teachers tend to understand such children as 'underdeveloped and over mature.'" Walkerdine explains that "those children taken to display procedural knowledge or rote learning are taken to have demonstrated an apparent maturity that hides their lack of appropriate conceptual development" (p. 7).

As Walkerdine (1990) explains:

> Girls may be able to do mathematics, but good performance is not equated with proper reasoning. . . . On the other hand, boys tend to produce evidence of what is counted as "reason", even though their attainment may itself be relatively poor. . . . Throughout the age range, girls' good performance is downplayed while boys' often relatively poor attainment is taken as evidence of real understanding such that any counter-evidence (poor attainment, poor attention, and so forth) is explained as peripheral to the real (Walden and Walkerdine, 1983). It is interesting that in the case of girls (as in all judgments about attainment), attainment itself is not seen as a reliable indicator. (p. 66)

One aspect of this discourse addressed by Walkerdine (1990, p. 72) is its articulation with the opposition between "production" and "reproduction" (see Figure 2). Achievement by girls is attributed to rote-learning and rule-following, which *is*

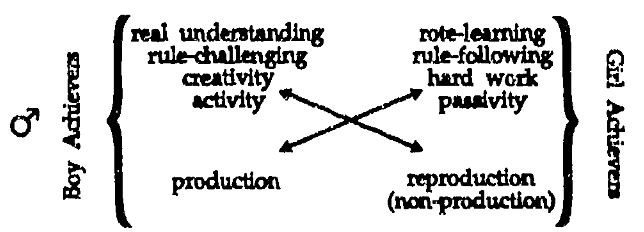

Figure 2. The Production (and Non-Production) of "Real Understanding"

invested with positive value as a kind of *reproduction*, even though this is not credited with the value attributed to the boys' achievement, which is marked, rather, as a *production* of "real" (i.e., "conceptual") understanding. Walkerdine notes, in this connection, that the peculiar combination of (reproductive) attainment along with a purported lack of real (productive) cognitive development

> . . . is precisely that combination which is required for the entry of girls into the "caring professions", in this case specifically the profession of teaching young children. Recruitment to elementary teacher training requires advanced qualifications, but usually a lower standard (poorer pass marks, for example) than that required for university entrance. (p. 72)

In this observation of discursive practice in specific homes and classrooms, we can begin to see how the structures in which terms (such as "achievement," "development," "maturity," "conceptual", etc.) take on their effective meanings in concrete social practices, do so in part by embedding the specific local practices within semiotic structures as far-reaching as the schemata generated by encodings of difference between "production" and its opposites and contradictories.

Considering what we have learned from Walkerdine about how she has seen the distinction between "rote learning" and "conceptual development" used in discourses and practices that systematically disparage the real intellectual achievements of female, minority, and working class students, this raises the problem of how to understand "concepts" and "conceptuality" in relation to the situated nature of cognition. According to Brown, Collins, and Duguid (1989):

> For centuries, the epistemology that has guided educational practice has concentrated primarily on conceptual representation and made its relation to objects in the world problematic by assuming that, cognitively, representation is prior to all else. A theory of situated cognition suggests that activity and perception are importantly and epistemologically prior—at a nonconceptual

> level—to conceptualization and that it is on them that more attention needs to be focused. An epistemology that begins with activity and perception, which are first and foremost embedded in the world, may simply bypass the classical problem of reference—of mediating conceptual representations. (p. 41)

In Peircean terms (see Figure 3), something becomes a sign, or a *representamen* (*r*), in relation to an *object* (*o*), by virtue of the possibility that an *interpretant* (*i*) will be produced, i.e., a singular event, or an habitual or regular response, which responds to the representamen *as* signifying an object (something other than itself) in some respect. The object is interpreted, in some respect, in the interpretant—not directly, or im-mediately, but only through the mediating representamen. (In Figure 3 the horizontal bar and broken line indicate that the object is not immediately present to the interpretant.) The representamen is related to the object, in some way (e.g., symbolically, indexically, or iconically), so that the object "determines" the representamen *as* something having a potential to "determine" something else, in turn, *as* an interpretant, which is indirectly "determined" *as* a mediated interpretation of the object.

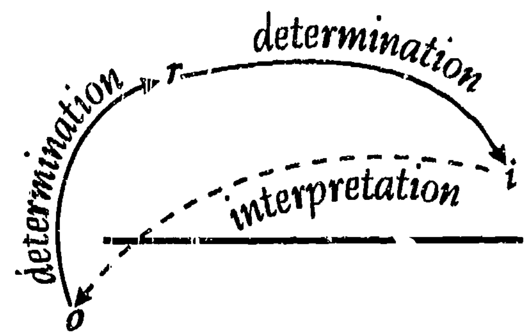

Figure 3. Triadic Sign Relations (Peirce)

It should be noted that this model of continuously productive triadic sign-relations can accommodate relations among the most diverse elements even within a single triadic sign. A verbal utterance or a cultural norm can occur as an interpretant—as can an institutional policy, a connectionist pattern of neurological activity, a sound, a shape, a color, a physical movement, or a social practice. Of course, any of these (or other kinds or combinations) can also function semiosically as an object or as a representamen within other triadic signs; moreover, a single triadic sign might be comprised of widely disparate elements, ranging across physiological, linguistic, and social levels.

Recognition that the most diverse elements can operate within a triadic sign also has implications for the kind of interdisciplinary work needed to account for cognition and other semiosic processes. Instead of seeking linkages, or ways of bridging gaps between social, economic, cultural, linguistic, psychological, neurological, or other "levels" of organization, this approach (first) shows the need to account for processes that actively and intricately cut across such levels (so that it cannot be assumed that order is established first on each of those respective levels, which might then be seen to "interact"), and (second) provides a conceptual and notational vocabulary for investigating such processes. By showing how cognition operates on the "atomic" level through the action of signs that combine elements as diverse as social policies and neurological or even meteorological events into indecomposable signifying triads, this helps to demonstrate how knowledge is always situated in the world, and how knowledge exists as something distributed across diverse aspects of our mental, physical, and social world.

Figure 4 illustrates how multiple triadic relationships can incorporate perception and habituated action in ways that can give rise to concepts as both generalized and situated semiotic practices. Two triads are presented. In both, the action of slicing twenty-five cents worth of cheese serves as an interpretant [i] which, through the mediation of the coins presented (either five nickels [r_1] or one quarter [r_2]), signifies a common monetary value [o]. "Twenty-five cents worth" thus becomes conceptually generalized as a value that can correspond not only

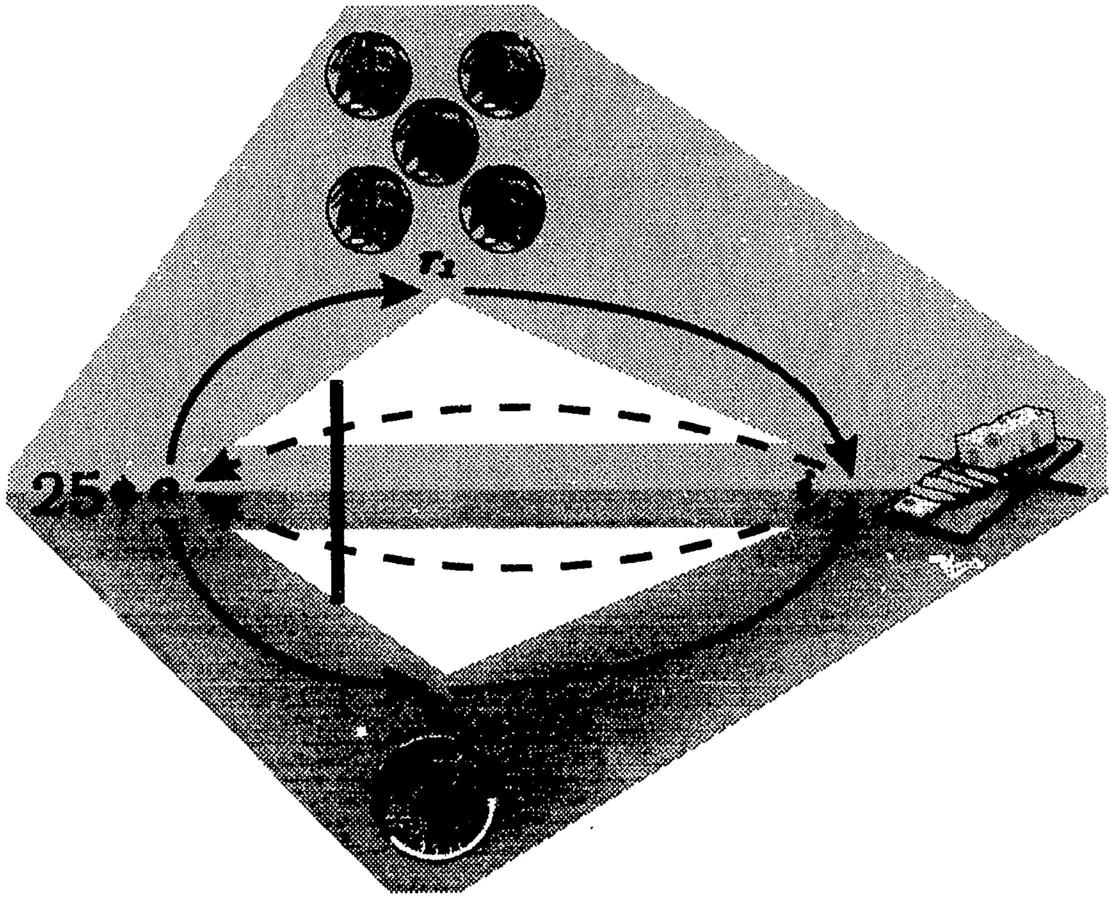

Figure 4. Slicing Two Bits Worth of Cheese (A Concept as a Generalized *and* Situated Semiotic Practice)

to various coin combinations, but also to specific quantities of cheese or bread or other goods. Although this might be described as an "abstract" value, we should note that it has not become established in this illustration through the formal logical procedure of abstraction. Instead, it has been *conceived* as a general sign in a manner very much like that in which five nickels came to *perceived* as the sign of an equivalent value.

At least in the case of such regularly encountered quantities, the value of the nickels will not be *calculated* by the "expert" cheese vendor, but simply *recognized*. Every time five nickels are encountered they will differ in their physical arrangement. There might be a darkened dirty nickel, a Canadian nickel, or an old

"Indian head" nickel, variously showing "heads" or "tails". The vendor is confronted with a different visual image every time. But this does not mean that, each time, the vendor must go through an algorithmic rule-governed procedure to ascertain the monetary value of the coins. In the case of five nickels, the expert does not even execute the rudimentary procedure of counting them; she simply *recognizes* them *as* 25¢. Experts might recognize the value of five nickels more readily than that of five dimes or even four nickels. If so, this is because of the repeated and familiar practical relationship between five nickels and the frequently encountered monetary value 25¢. In that case, the *perception* of five nickels is no less abstract than the *conception* of 25¢; the concept is not derived from the concrete objects through rule-governed processes of sensation, information processing, and calculation. Instead, in the manner described by John St. Julien (1994), the recognition of five nickels is itself arrived at through the unruly but reliably regular processes of (socially supported and constrained) perceptual pattern completion, and the pattern of five nickels, in particular, is more readily perceived because of its relation to an "abstract" quantity (25¢) which may be semiosically more *solid* than the metal coins themselves, by virtue of the density of practical transactions and communications in which the value of that quantity is so well established. Quantities of cheese, coins, and monetary value are sustained in practical cognition through the habituated relationships among them, and among them and the terms of countless networks of other triads in which they are also involved.

A semiotic framework enables us to see how knowledge and learning that are embodied in and distributed across specific concrete situations may, at the same time, have the conceptual range and generality to transcend those specific situations in potentially empowering and critical ways. Since the sign activity in which mathematical learning takes place brings the most diverse social, psychological, and biological elements together within dynamic sign-relations, semiotics may provide a common general framework for practitioners and researchers from diverse disciplines seeking better understanding of these matters.

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Invited Discussion Group

REFLECTIONS AND INTERACTIONS ON RATIONAL NUMBER THINKING, LEARNING AND TEACHING:

An introduction to the discussion group

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Prospect: There has been a considerable body of theoretical development and research on rational number thinking over the last 20 years. The purpose of this session is to allow for a broad discussion of this work with input from a variety of persons who have and are playing a role in this work. [See the brief reports which follow this introduction for examples of such input.] It is hoped that audience questions as well as "panel" interaction will highlight the major contributions of the work done to date but also make it "problematic" in hopes of pointing to new directions for the work on rational number thinking.

There are two intertwined thrusts that can be observed to underlie thinking and research on rational number thinking and learning over the last 20 years or so. The first of these is an epistemological but also a phenomenological thrust (if those two are not contradictory terms) and responded to the question: "what would a person know and be able to do if that person was observed to know rational numbers?" (knowing being identified with doing from the very beginning of this aspect of thinking about rational number thinking). In response to this question the various sub-constructs of fractional number thinking have be identified; e.g. quotient, measure, operator, ratio number. (The work of Kieren; Freudenthal;Vergnaud; Behr, Post, Harel and Lesh; and Mack, for example, are relevant here). The second thrust is psychological in nature and responded to the question: "are there certain schema or knowledge structures which a person builds and then uses to construct that person's rational number knowledge structures?" (This question carries with it a constructivist flavor which also underlies much of the research and thinking in this domain.) In response to such a question Vergnaud developed very general related schemes: multiplicative structures; others have looked for more specific constructive mechanisms which a child might use in building up rational number knowledge. (See the work of Kieren, Confrey, Harel, Lamon, Figueras, Steffe etal in this regard.) Of course both of these thrusts were not divorced from other recent work on number learning and proportional reasoning; none-the -less there is a body of work which looks explicitly at fractional number learning through these two general approaches.

Interaction 1.: What have we learned from this work of the last 20 years?

There have been two kinds of work which follow from the consideration, implicitly or explicitly, of the sub-constructs of rational numbers. Streefland, (reflective of a general Dutch interest) in particular, has been very active in developing curriculum material for elementary school children which engage them in tasks which promote practical human actions and informal symbolic expressions of those actions in the various sub-constructs particularly aspects of ratio and quotient numbers . The Germans (e.g. Griesel, Padberg, Hasemann) (in a way following the earlier work of UICSM as well as Dienes) focused particularly on operator numbers in a more formal, mathematical program and studied the effects of such a program. These are only two of a number of broad ranging curriculum, instruction and evaluation efforts which relate to a construct view of rational number knowledge and knowing.

The second body of work related to a subconstruct view has related to verification, use and formalization of subconstruct knowledge. Variations of Kieren's Rational Number Thinking Test have been used and studied in an attempt to identify and find evidence for the sub-constructs of rational numbers in measured mathematical performance of students of many ages. While some doubt the existence of sub-constructs or at least the particular form described above (e.g. Ohlsson), to the extent that contemporary factor analytic techniques are useful, the four subconstructs or elaborations of them have been observed in several different studies with varied populations (e.g. Giminez, Bockbrader, Rahim, Brindley, Bye and Harrison). Relationships between performance on such sub-constructs and particular instructional schemes for fractions exists as well. In addition, there are a number of anecdotal reports which support the fact that students tend to act differently on items from the different constructs - although to be sure their actions in these different fractional settings are necessarily highly related.

Behr, Post, Harel and Lesh have attempted to provide a formal symbolic language as well as a formal, logical description of how students would function using the various sub-constructs of fractions with respect to equivalence and computations. They have provided an extensive body of evidence which shows differences in hypothetical formal models of fractional action between the various subconstructs.

Interaction 2. What is the nature of and relationship among performance on items or tasks related to various rational number sub-constructs?

Interaction 3. What is the nature of and what are the potential uses of the various formal models of rational number thinking?

Interaction 4. What are the implications of this body of work for curriculum makers or for the assessment of rational number thinking and learning?

Vergnaud and others have offered both models and evidence from student work as to how rational number thinking is part of a more general set of multiplicative schemes. Other researchers have looked at the act of dividing up equally as a critical proto-mathematical mechanism for building up fractional ideas (e.g. Kieren, Pothier and Sawada, Shell). While this latter work focused on the effect of such partitioning of continuous quantities, Hunting and his colleagues, in a number of different studies including several clever computer settings, have looked at discrete sets and acts on them as a "way in " to fractions. Others (e.g. Figueras) have considered the very early basis for partitioning and its relationship with pre-number activities.

This stress on "mechanisms" reflects the influence of radical constructivism on this body of work. Following both the work on partitioning, and the idea of multiplicative structure, Confrey has studied the nature of evidence for a more general mechanism "splitting" and sought through it to relate the basis of fractional number thinking to the basis of more general multiplicative and particularly exponential thinking. There is evidence (e.g. Kieren, Mason and Pirie) that a folding version of "splitting" or partitioning provides an effective introduction to fractions for children and that they are highly aware of the multiplicative (as opposed to additive) nature of such actions.

Steffe and his colleagues, using another set of clever computer "worlds", have studied children's fractional mathematics which they generate using the mechanism of iterated fractional units. This work highlights the relationship between (as well as subtle differences between) whole number and fractional number thinking in children.

The formal work of Behr, Post, Harel and Lesh above is also a source for identifying possible mechanisms in children's fractional number thinking in action. Two that are very evident in their work might be called "combining fractional quantities" and "unit reconfiguration". The former seems related to fractions as numbers for extensive quantities while the latter allows persons to function with proportional equivalence and fractions as numbers which for quantities which are simultaneously extensive and intensive. With respect to the latter, Lamon has studied items which pushed students to use various kinds and levels of proportional reasoning with fractions as ratio (or intensive quantity) numbers.

Interaction 5. What are the identifiable "mechanisms " used by students in building up fractional knowledge structures? Do those mentioned above form an exhaustive list?

Interaction 6. How does mechanism use lead to or function within rational number thinking?

Interaction 7. Because such mechanisms should be considered as embodied phenomena, how are they inter-related in the actual fractional number work of students? From what other more basic human schema might they arise (see the work of Mark Johnson)?

The focus of the discussion above has been on the less formal "core" of rational number thinking. How does such thinking change and grow to be more formal? Mack, and Kieren and Pirie have suggested different ways of modeling such growth especially as it takes into account the various subconstructs and some of the mechanisms mentioned above. While Mack argues for more or less direct "transferability" from knowledge under one subconstruct to another with special attention to partitioning, Pirie and Kieren point to the necessity of folding back to less formal activity as one broadens ones rational number knowledge to include new subconstructs.

Interaction 8. How does a person develop more formal rational number understandings which are not disjoint from less formal ways of fractional knowing?

It is hoped that the open discussion of rational number thinking research will extend our knowledge base with respect to the above aspects (and others) of fraction learning and will allow a broad but informed dialogue on questions such as those above. Such a dialogue may serve to trigger a new phase of the study of fractional number, thinking and learning.

IMAGINATION AND MATHEMATICS: THE SIZE OF A RATIO

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One of the best documented impediments to children's progress in the rational number domain is the failure to conceptualize fractional numbers, quotients, and ratios as single entities. An instructional intervention was designed to help children focus on the relationship between the two quantities composing a ratio, rather than on the quantities themselves. The intervention is consistent with current philosophical theory which holds that understanding has a basis in bodily movement, forming imaginative structures which are gradually elaborated to allow conceptual understanding (Johnson, 1987).

If k is a ratio number such that $a/b = k$, $a,b \in I$, $b \neq 0$, then the relationship between the values a and b is the size of k, and varying the size of a or b or both effects changes in k. By assigning qualitative values of increase (+), decrease (-), or no change (0) to the size of a and b, there are nine possible pairs of change values for the two variables, and knowing these, one can reason qualitatively about the size of k. These change pairs were used to construct the problems that 6th-8th graders solved in clinical interviews. Ex.: Yesterday you shared some candy bars with your friends. Today you shared fewer candy bars with more friends. How much candy did each person receive today? {more, less, the same amount, can't tell} Student responses were poor except in the cases represented by change pairs (0,+) and (0,-), where 85% responded correctly. Only 10 % were correct in the ambiguous cases (+,+) and (-,-).

In the second phase of the interview, the students were taught an imaging processing to facilitate their reasoning. Both hands, one representing candy bars and the other representing their friends, were extended at the same level in front of them. The hands were then raised or lowered as indicated by the change pairs in a given problem, and the students were told to focus their attention in the space between their hands. It was impossible to obtain correct answers merely by moving the hands, but the movement created an "imaginary space" in which the students could focus on the relationship. 73% -100% of the students reasoned correctly on each of the candy bar problems. One week later, 74%-100% responded correctly under each change condition in another structurally similar problem .

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CLASSROOM ACTIVITIES TO PREPARE EARLY ELEMENTARY STUDENTS FOR PROPORTIONAL REASONING

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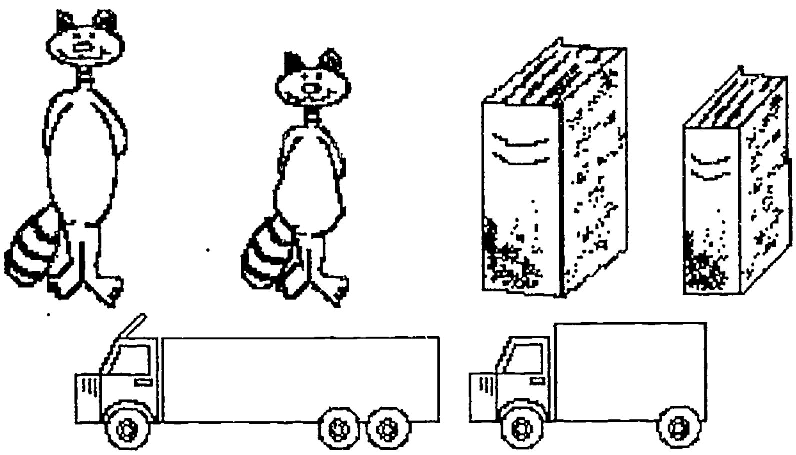
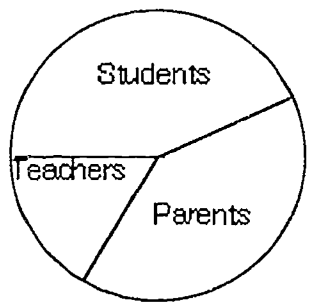
Ratio and proportion is a topic that has traditionally been reserved for the middle school mathematics curriculum. However, because most students do not reason proportionally by the time they leave middle school, instruction may need to play a greater role in mediating the learning of critical ideas. The following activities have been designed to increase the interaction of early elementary school children with some learning sites for critical aspects of ratio and proportion. Activities such as these help students to develop a visual sense of proportion and to verbalize about relationships.

1. The student is given a circle with a diameter of about 12 cm. This circle represents a dinner plate. Draw a knife, a fork, and a spoon in their usual positions next to the plate.
2. If the first object were to go through a shrinking machine, would it look like the second one?





3. Mr. Pete Sauce from the World's Largest Pizza Company gave out samples to all of the people at school last night. There were 45 students, 15 teachers, and 30 parents. The picture above shows the parts of the pizza that each group ate. Did Pete give everyone a fair share?
4. Students are shown a tunnel drawn in perspective (the far end appears very small). If a freight train enters the tunnel, will it be able to get out the other end?

RATIO SENSE

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One prerequisite to proportional reasoning is having a "sense" of ratio. This means that the student should have an intuition that informs qualitative judgments about numerical relationships, appropriateness of operations, and reasonableness of answers, while at a more primitive level, it may help the student to identify situations in which it is appropriate to apply a ratio model.

In some mathematical domains, it is thought that interactions with the environment build intuitive knowledge that forms a foundation for later conceptual understanding and formal methods. In the case of ratio and proportion, we need to more fully investigate the existence and role of intuitive knowledge. For example, one might expect that by the time students are in middle school, they would understand something about proportions in the human body. It was disconcerting, then, when a majority of seventh grade students responded as Robert did in a recent interview:

I: How tall would a person be if he has arms that are 6 feet long?

R: He would have to be 6'1" tall so that his fingers don't drag on the ground.

Such answers suggest that informal experiences may not be sufficient to enhance understanding of the ideas of ratio and proportion. Instruction may need to help students consciously explore and expand informal experiences.

153 eighth grade students participated in a study designed to describe their ratio sense. They were given 15 statements involving proportional, inversely proportional, or non-proportional situations, and containing correct or incorrect numerical information. They were asked to tell whether each statement made sense or not. If a statement did not make sense, they were to make it sensible by changing one or more numbers in the statement or to tell why the statement could not be fixed by changing numbers.

Ex. 1 If an orchestra can play a symphony in 1 hour, 2 orchestras can probably play it in 1/2 hour. [non-proportional; presented incorrectly; change number(s)]

Ex. 2 If 1 basketball player weighs 175 pounds, then 2 players probably weigh 350 pounds. [non-proportional; explain the difficulty]

Each correct proposition was identified by between 85 and 95% of the students, but each incorrect statement was identified by only 27 to 71% of the students. Students showed a strong tendency to treat inversely proportional or nonsensical propositions as if they were proportional and their numerical substitutions showed a strong preference for halving and doubling.

A COMPARISON OF TWO APPROACHES TO DEVELOP PARTITIONING SKILLS AS PRECURSORS TO LEARNING BASIC FRACTIONS

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This study examined Kieren's (1975) hypothesis asserting that partitioning a unit is critical to all rational number interpretations and alluded to Mack's (1990) recommendation for a teaching strand on fractions based on partitioning.

Two teaching sequences for basic fractions were developed and contrasted in a four-week experiment with two fourth-grade classes (N=40) receiving 10 hours of instruction. One teaching sequence (direct construction) began with activities involving partitioning of area models (geometric shapes) in which pencil and straight-edge were used to draw lines and to show equal parts. Fraction terms and concepts of part-whole, ordering, equivalent fractions were developed by having students construct appropriate unit partitioning. The alternative teaching sequence (indirect construction) initially included activities with pattern blocks in which students covered different geometric shapes and recorded the number of blocks (of same color and shape) required. Following the pattern block activities, fraction lessons from a traditional mathematics textbook were taught in which the area units presented were already partitioned.

Analysis of covariance (repeated measures) showed no significant mean differences in achievement between the two classes when student CAT score was controlled. While pre-, post- and retention tests revealed a gradual but consistent improvement in fraction understanding for both classes, the two-week instruction was inadequate in promoting fraction understanding necessary to succeed on the paper-pencil assessment that measured 7 fraction concepts. Further analyses of written tests and videotaped interviews revealed: (a) students in the direct construction class showed significant improvement in representing fraction given an unmarked area unit but their awareness of the equal-part condition was not evident when asked to name a fraction of a unit that was pre-marked, (b) unfamiliar partitioning strategies for area models like a regular pentagon can be taught, and (c) during the four-week period, fraction knowledge of students in both classes was unstable but evolving into a more cohesive and mature level.

CHILDREN'S IDEAS ABOUT PARTITION AND SHARING

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For the last five years a group of Mexican researchers has been focusing on children's ways to solve partitioning and sharing tasks. At a first phase, individual interviews were carried out with children -six to ten years old- throughout different type of communities (urban, rural and indian) within three states of the country.

The interview protocol was structured considering various aspects related with rational number concepts and the dealing scheme, using concrete and discrete settings. Ideas of various researchers as Kieren, Steffe, Streefland, Pothier and Sawada; Hunting, Davis and Pithkeley serve as a fundamental framework to design the tasks for the interview.

From these work major hypotheses were derived, among them are: a) one to one correspondance is an early precursor of rational number thinking related with sharing situations, b) small children generate verbal expresions to distinguish fractional parts and c) the selection of a procedure to equally distribute a quantity is linked with the setting of the task.

Case studies based on a longitudinal observation during a year constitute the second phase of the investigation; ten children from four to eight years old are been individually interviewed. The main purposes of the work carried out is to further understand the partitioning constructive mechanism and child's conception of equalness, as well as to identify more precisely verbal expresions for communicating ideas linked with equality, part-whole relationships and the results of sharing tasks.

Evidence to sustain more strongly our hypotheses has been derive from the case study analyses and it is linked with the nature of mechanisms that children use in building up rational number constructs.

Technology Focus Groups

VISUALIZING ASPECTS OF NUMBER THEORY THROUGH SPREADSHEET MODELING

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Within the past decade, the mathematics education community has been occupied with the challenge of finding ways of integrating technology in school mathematics curriculum, and in particular, enhancing and improving learning in technology-rich environments. One of the tools which has been recommended for teaching and learning mathematics through modeling is a spreadsheet. The true potential of a spreadsheet, as a learning environment where students can exercise their own creativity, reveals when the concept involved is formulated in the language of discrete processes depending on two positive integral variables. One such content area which represents enjoyable mathematics with little previous knowledge is elementary number theory. In the past the use of computers in number theory investigations required skills in programming languages. The use of spreadsheets, however, allows investigations for many topics in number theory, particularly, in Diophantine analysis. Due to the spreadsheet capacity for immediate recalculations participants have the opportunity to review a broad range of problems that have challenged mathematicians throughout centuries: exploring sums of powers of integers, discovering arithmetical properties of Pythagorean triangles, multiple representation of integers as the sum of two squares.

We argue that flexibility of the spreadsheets in the context of independent explorations of problems, accommodates learners of different abilities. Students can easily modify problems under investigation, and change the degree of the complexity of the ideas being explored. These create an open interactive environment allowing individual constructions of situations based on individual interests. The presentation includes a demonstration of these features, and a discussion of how a computer spreadsheet allows learners to recognize patterns and regularities, to make and test conjectures through numerical evidence, and in some cases stimulates the development of mathematical proofs.

LINKING THE PARTICULAR AND THE GENERAL IN ALGEBRA

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Visual representations are particular, but are often useful for thinking productively about the general. In geometry, we reason about triangles in general while sketching a particular triangle on a napkin. The trick is not to be caught in the particularity of the diagram when attempting to make general conclusions. In algebra, the interpretation of literal symbols allows for slippage between the particular and the general. Literal symbols can be interpreted as a particular, though unknown, number, as is traditionally the view when solving equations. Alternatively, they can be used to make generalizations about the behavior of numbers as is traditional in number theory.

The session will raise questions for discussion about complexities which arise when linking the particular and the general in algebraic representations with educational technology in 7-12 curricula. For example, in the symbolic expression $ax+b$, variation in x for particular values of a and b creates a function. How can technology help students come to see a particular expression of an arithmetic calculation (e.g, 2x+3) as a procedure which generates a function whose graph is made of infinitely many points all sharing a connection to the same arithmetic calculation? Variations in a and b for a set of values of x create families of functions. Unlike the set of points which make the graph of the particular function, these families can not be pictured simultaneously on the standard Cartesian plane. How can functions of the form $ax+b$ be represented so their relation to the family is highlighted? Similarly, in to be able to study a particular family of functions (e.g. cubics), it might be useful to be able to explore the members of this family which share some property (e.g. that their graph goes through 3 points). Yet, each graph with this property is but one such graph. How can technology make this point salient, while allowing us to think productively with the particular?

GEOMETRY TECHNOLOGY FOCUS GROUP*

Douglas H. Clements
Julie Sarama Meredith
State University of New York at Buffalo
Michael T. Battista
Kent State University

Computers, especially with their graphic capabilities, may facilitate the construction of geometric concepts. Decades of pre-computer research reveal few differences between media. There are, however, certain functions computers can perform that other media cannot easily duplicate. Do these functions affect the learning of geometry? This focus group will address this question, hypothesizing the following unique characteristics of computers as facilitators of geometry learning.

1. Given their graphic capabilities, computers appear to have substantial potential to facilitate the construction of geometric representations.
3. Nevertheless, they can be concrete, in the most significant sense—concrete for the learner.
4. Computer representations are manipulable and interactive.
5. They encourage a manipulation-based, empirical approach to knowledge construction that may be consistent with the way students reason.
6. Mistakes in reasoning are more readily detectable than in other environments.
7. Thus, computers can aid the transition to more abstract settings.

Our plan is to have groups of researchers using different software environments lead discussions of the issues involved in the educational application of their software. Each group will: (1). Present the main functions, or capabilities, of the software environment. (2.) Describe how these functions have the potential to make a unique contribution to students' learning of geometry. (3.) Review or present new research that addresses these contributions. (4.) Lead a discussion among all participants on these issues.

TOOLS FOR KNOWLEDGE CONSTRUCTION: TOOLBOOK AND SPPPC IN CLASSROOM SETTINGS.

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Delwyn L. Harnisch, University of Illinois at Urbana-Champaign

We present a five phase method in which students construct mathematical ideas via physical materials and computer technology. The initial two phases use physical materials to pose problems which require active student involvement with physical materials to model mathematical situations, define symbols, and develop solution strategies. The third phase require student use of sketches of these materials and situations, constructed on the computer, to encourage moves toward abstraction. These computer sketches then serve as the basis for additional problems. Due to the ease with which a computer graphic is manipulated, they can be powerful tools for thinking. In the fourth phase, mental images are developed through imagining actions and situations. These experiences culminate when students construct strong generalizations and problem solving skills by scripting their understandings using ToolBook.

Each of the outlined phases is viewed as steps along the path toward eventual mathematical abstraction. For example, the computer based sketches draw much of their power from the earlier experiences with objects. In a similar fashion, the student generated computer representations and solutions reflect their developing mental images. The final abstractions, rather than being based upon a single demonstration of rules, rest upon a tightly woven network of understandings.

An integral part of this approach relies upon dynamic evaluations based upon Sato's student - problem curve theory. This approach, together with modeling of continuing interviews and group assignments, help foster the best match of cognitive styles to create effective learning groups. The computer role in this Focus Group will be much different than that usually associated with CAI and AI based models. Rather than using the computer for its speed, the computer's patience and need for exactness of logic and clarity of expression will be utilized. The computer is used as an active listener that does exactly what it is told, as opposed to a pre-programmed instructor requiring a specific type of answer.

PACESETTER: MATHEMATICAL MODELING CURRICULUM

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Michigan State University and Battle Creek Central High School

Pacesetter, an innovative curriculum written by The College Board, calls for three main practices -- group projects, the use of technology, and student communication of their observations and findings. This interaction of technology and groups through mathematical modeling enhances mathematics learning as prescribed in the NCTM Curriculum and Evaluation Standards. In particular, the importance of communicating within the group embellishes mathematics problem solving, strengthens mathematics reasoning skills, and proposes connections that might be unseen to some participants. "It's a class in which you have to listen to each other in group or you might miss something," commented a Pacesetter student remarking on the importance of communication in the curriculum. This emphasis on communication takes a potentially isolating learning environment and turns it into a laboratory of peers. With the addition of technology, in the form of graphing calculators, this laboratory conducts powerful experiments with complicated mathematical models with a high measure of success. The social environment created by the use of technology in the classroom provides an impetus for group work that results in a deeper student understanding of mathematics. How Pacesetter achieves this delicate balance is with mathematical models from a wide range of subjects. These models evoke discussion and essentially cannot be solved individually, but instead calls for a pooling of resources and knowledge. This relationship will be the main focus of our discussion.

CHILDREN'S CONSTRUCTION OF FRACTIONS USING TOOLS FOR INTERACTIVE MATHEMATICAL ACTIVITY (TIMA) MICROWORLDS

John Olive
University of Georgia

Children's construction of operations necessary for building the rational numbers of arithmetic has been an important issue in the psychology of mathematics education for several decades (Kieren, 1988; Behr et al., 1992). Some attempts have been made to use computing technology to aid children's learning of fraction concepts but this has most often been in the form of tutorials or drill and practice software. Few research efforts have attempted to use the computer as an integral part of the shared learning environment for the teacher/researcher and students. The computer microworlds that will be used in this investigation are being developed as part of an NSF research project on children's construction of the rational numbers of arithmetic. They have been designed as tools for the children to develop and enact their operations on discreet and continuous quantities. But they are also tools for the teacher/researchers to construct situations in which they can test their emerging models of the children's mathematics. As such, they may offer the constructivist researcher a powerful, dynamic medium for investigating children's constructive itineraries.

Participants will be introduced to the three microworlds (TOYS, STICKS and BARS) in much the same way that we introduce them to children. After some initial exploration of the possible actions of the microworlds, the participants will be encouraged to play with these actions, creating pleasing designs on the screen. They will then be asked to think of possible questions that could be asked of the results of the play activity (or challenges posed within the play activity) that might transform this play into mathematical activity.

Following this introduction, example activities from the project will be posed for the participants to investigate. These activities will be focussed on the children's construction of *Iterative and Unit Fraction Schemes*, *Units-Coordinating Partitioning Schemes*, *Recursive Partitioning Schemes* and *Co-measurement Schemes for Fractions* (Olive, 1993).

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TEACHING AND LEARNING GEOMETRY WITH THE AID OF DYNAMIC CONSTRUCTION TOOLS SUCH AS THE GEOMETER'S SKETCHPAD.

John Olive
University of Georgia

The use of dynamic visualization as an aid to learning and teaching is a modern issue arising from the recent availability of such tools as the *Geometer's Sketchpad*. Research concerning the use of graphing utilities as aids for learning function concepts and ideas in precalculus has been ongoing for several years. Lampert (1988) posed questions concerning teachers' thinking about students' thinking about geometry after using a computer tool called the *Geometric Supposer*. The *Geometer's Sketchpad* and similar tools (Cabri Geometre, Geometry Inventor) provide a level of dynamic visualization that goes beyond the capabilities of the *Supposer* used in Lampert's study (the most recent editions of the *Supposer* software incorporate similar dynamic features to those found in the *Sketchpad*). The dynamic aspect of these new tools provides the user with ways of modeling and testing conjectures that are not possible with any static medium.

The *Sketchpad* provides the user with a set of "mouse" driven geometric construction tools (point, line, segment, ray, and circle tools) and menu driven constructions such as a perpendicular line given a selected point and straight object. The mouse interface permits the user to make continuous, dynamic transformations of geometric constructions simply by dragging part of the on-screen construction with the mouse pointer. All geometric relations embedded in the construction are maintained during these continuous transformations, thus providing the user with visual confirmation of what is and what is not invariant in these geometric constructions. Measurements of lengths, areas, angles and ratios can be obtained. These measurements are "active" in the sense that they automatically update as constructions are manipulated. Recording and play-back features built into the program provide the user with a written record of the constructions that can be saved and used to recreate the construction (on a new set of "givens") when needed. This recording feature also provides teachers with a means for evaluating student work with these tools.

Images of transformations can be obtained that are interactively linked and determined by their pre-image. Thus, properties of transformations can be experimentally determined by changing the pre-image and watching the effects of this change on the transformed image. Version 2 of the *Geometer's Sketchpad* provides the ability to define transformations based on constructed objects. A rotation of a figure based on a constructed angle can be changed dynamically by simply changing the angle. Similarly, a translation based on a segment or a dilation based on the ratio of two segments can be changed dynamically by simply altering the segments. Custom transformations can be built from combinations of transformations and used iteratively on a construction.

Elements of a construction can be animated in controlled ways so that loci of points or other objects can be traced through the animation. This capability opens up a whole new field of investigation for students. Conics can be constructed based on their locus definitions. Constructions can be used to simulate circles rolling on lines or even on other circles, thus generating epi-cycloid curves as the trace of the locus of a point on the rolling circle. Trigonometric and other parametric relations can be modeled using the animation and trace locus features.

Given these powerful capabilities to investigate topics that were previously thought of as being in the domain of the research mathematician, we need to seriously question the current role of geometry in our precollege curriculum. With such tools, geometry can become an experimental science for students and provide a bridge to more advanced topics in algebra and calculus.

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JUMPING AND RUNNING: USING TECHNOLOGY TO CAPTURE THE REAL WORLD

Andee Rubin and Nathan Kimball

TERC

Mathematics as many students experience it often has only tenuous connection to their everyday lives. Yet the current injunction to "connect mathematics to students' lives" requires deep thought about just what parts of students' activity are amenable to mathematization and which pieces of mathematics are easily embodied in children's experiences. One approach is to use measurement to mediate between the world and mathematical representations and meaning. Because real-world experience can be messy and disorganized and the underlying mathematics blurred by the complexity of the phenomenon. The challenge is to mathematically harness the environment without sacrificing its authenticity. This focus group will deal with two attempts to develop technology that gives students the power to capture motion so it is amenable to analysis. The two projects we will explore are "Playground Physics" and "VIEW: VIdeo for Exploring the World".

The Playground Physics project is building new playground equipment to turn the playground into an environment in which children's play leads naturally to mathematical questions. In the session, we will use a different tool - a light track that flashes a linear set of lights in sequence at a rate which can be controlled interactively. The lights may be followed or raced and the velocity displayed.

The VIEW project is developing software tools that allow students to measure event occurrence (like individual jumps in jumping rope), positions, lines, or angles over time, and to graph the data, its velocity, and other transformations. In this session, participants will design a structure so that a small object moving through it will create certain patterns of velocity, acceleration, and deceleration.

THE ADVENTURES OF JASPER WOODBURY AND *THE ADVENTUREMAKER*: TECHNOLOGIES THAT SUPPORT MATHEMATICAL PROBLEM SOLVING AND PROBLEM POSING FOR REALISTICALLY COMPLEX PROBLEMS

Susan M. Williams, Mitchell J. Nathan, Joyce L. Moore, Sashank Varma, Susan R. Goldman, and The Cognition and Technology Group at Vanderbilt
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The Adventures of Jasper Woodbury is a video-based mathematical problem-solving series that provides opportunities for activities that are part of complex problem solving, e.g, planning, cooperative problem solving, evaluation of multiple solutions, and communication of mathematical ideas. In addition to the videos, we have developed *The AdventureMaker,* a computer microworld for exploring the complex, open-ended problems presented in the Jasper videos or similar problems posed by students.

Development of *The AdventureMaker* was stimulated by research showing that even after mastering the solution of the specific problem posed in the video and transferring this understanding to an isomorphic problem, students still needed to improve their understanding of the underlying structure of the problem. The goals of *The AdventureMaker* are:

- To support an inquiry-based approach to learning mathematics in which students generalize the results of a known problem by posing variations of that problem and receiving feedback via an animated simulation.
- To promote discussions about mathematics and its role as a tool for analysis and problem solving. As part of these discussions, students can reflect upon the idea that different situations can be addressed using similar mathematics.
- To increase engagement by encouraging students to pose their own problems and explore mathematical avenues of their own by creating problems for others to solve.

Symposium and Discussion Groups

LOOKING AT ASSESSMENT IN MATHEMATICS: BUILDING A FRAMEWORK

Kathy Kelly-Benjamin, Florida Institute of Technology
Paula S. Krist, Florida Institute of Technology
Marie Revak, Florida Institute of Technology
Carol Kehr Tittle, City University of New York

This presentation describes ways of investigating teachers' use of assessment within a framework proposed by Tittle (in press). Tittle's model for looking at assessment provides a conceptualization that is responsive to current cognitive constructivist (Cobb & Steffe, 1983) and interpretive (Moss, 1992) perspectives on mathematics teaching and learning.

The T^2M^3 Project (Teachers Using Technology to Measure Mathematics Meaningfully) was designed to investigate teachers' capabilities to create and use formal instructional assessment in the math classroom, with special emphasis on integrating technology. This project applies current research in mathematics education to assessment (Carpenter & Fennema, 1991; Leinhardt, 1989; Livingston & Borko, 1990).

An Overview of Tittle's Framework for Looking at Assessment

Tittle (in press) proposes an underlying framework for examining assessment, responsive to issues in educational psychology as well as mathematics education, was needed. She suggests that to fully study mathematics assessment we must examine three dimensions: (1) epistemology and theories; (2) interpreters and users; and (3) assessment characteristics. Together, these dimensions provide a comprehensive view of the context and implications of assessment in math. The model gives equal status to all dimensions thus defining assessment within the context of practice, that is, the learning and teaching of mathematics.

Tittle's model also suggests lines of inquiry for examining assessment within the context of the classroom. It broadens the discussions of validation theory (Shephard, 1991; Messick, 1989) by placing the interpreter and user as the central dimension. Further, by including the users and interpreters, and their beliefs, it enhances the structure of Webb's (1993) five features of assessment. Using the three dimensions: theories, interpreters, and tasks, we have begun to describe the similarities and variations of the 15 T^2M^3 teachers as they create and use various assessment tasks and techniques.

Epistemology And Theories

Tittle's first dimension takes into account: (1) theories of knowledge, (2) teaching and learning, (3) curriculum, and (4) the development and change of both the system of assessment and its interpreters/users. It has provided a way of identifying the epistemologies and theories held, implicitly or explicitly, by the teachers in the T^2M^3 Project. It focused our attention on the underlying beliefs of the teachers as they use, interpret, and create assessment items.

This dimension allowed us to develop a systematic investigation of the components that influence teachers as they develop and use various forms of assessment. It also offered a way to declare the epistemologies and theories that underlie and drive the T^2M^3 Project. Our presentation illustrates the ways of identifying teachers' theories and epistemologies and discusses the relationship of these theories to changes in their use of assessment. The theories and epistemologies that underlie this project , and the means of determining the participants' theories and epistemologies, are elaborated. These will be shown through exemplars from our research, such as videotaped vignettes, that highlight the categories within this dimension of Tittle's model.

Interpreters and Users

The T2M3 Project began from the premise that teachers have beliefs about learning, instruction, and assessment which influence their understanding, interpretation, and use of alternative assessment in mathematics. In addition, we made the assumption that no aspect of instruction, learning, or assessment can occur independently from the context within which it is embedded.

To test these assumptions we selected participants for the T2M3 Project who represented a broad range of beliefs and practices. In particular, they were selected to represent a range across three major characteristics: (1) familiarity with technology and use of technology in the math classroom, (2) knowledge and incorporation of the NCTM Standards, and (3) expertise in mathematics education, defined in terms of possession of the mental set usually associated with experts (see for example, Leinhardt, 1989). To increase the degree to which the findings could be generalized to other teachers, we included teachers from the primary, middle, and junior high school levels and diverse school populations, from inner city to upper middle class suburban.

This presentation tracks the course of change in the participants using videotaped comments and descriptions from the teachers themselves. The T^{2}M^3 teachers reflect on ideas, understandings, conflicts, and difficulties they have experienced as they developed assessment tasks and techniques and disseminated the assessments to their colleagues.

Assessment Characteristics

"The assessment occurs at the intersection of the important mathematics that is taught with how it is taught, what is learned, and how it is learned" (NCTM, 1993, p .5). The assessment repertoire of the T^{2}M^3 participants varied widely at the beginning of the project. The teachers' facility with assessment techniques, such as performance assessment, portfolios, scoring rubrics, ranged from novice to expert. Some had elaborate systems for formalizing their observations and interactions, while others recognized only their pencil-and-paper tests as assessment tools. Changes for all teachers are being measured throughout the project.

The teachers are currently pilot testing their mathematics curriculum modules which have assessment as the backbone of the lesson. Early in the project all the teachers participated in several assessment training workshops.

These included an introduction to videotaping, benchmarks, scoring rubrics, interviewing techniques, performance assessment, portfolios, and teacher-as-researcher. They were exposed to several assessment systems including the Toronto Board of Education *Benchmarks* and the *New Standards Project*.

Through the T^2M^3 Project both the assessment measures explored by the participants and the measures they have developed are being examined, with a focus on the changes occurring in their assessment tasks and techniques. This presentation follows the evolution of the teachers' curriculum modules and how they conceptualized, reconceptualized, and incorporated alternative assessments into their modules.

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Discussion Group: Emerging Standards for Research

Group Leaders: Thomas J. Cooney, University of Georgia; Frank K. Lester, Jr., Indiana University

One of the major goals of PME-NA is to promote interdisciplinary research. This goal poses a difficulty for the community of researchers who make up the membership of PME-NA. The difficulty stems from the fact that researchers within any discipline set standards for scholarly discourse that often are not functional outside that discipline. Recently, various groups have begun to discuss standards for assessing the quality of research—in particular, standards that can be applied to reports based on a wide variety of ideological and methodological traditions. This discussion group will propose a preliminary set of standards for research that seem to be emerging within mathematics education.

Session Description

The session will begin with very brief commentaries by a group of panelists made up of a moderator, the group organizer, and members of groups who have been involved in discussions of standards for research. Specifically, the session will have the following format:

1. (20 minutes) A preliminary set of "standards" that have resulted from deliberations among various groups of researchers over the past four or five years will be presented by the panelists. A brief elaboration on these standards will be provided.
2. (30 minutes) Participants will be asked to arrange themselves in small groups to consider a series of questions about the appropriateness and adequacy of the set presented by the panelists. Also, each group will be challenged to come up with its own set of standards.
3. (25 minutes) Groups will share their sets of standards with the entire group.
4. (15 minutes) Next steps. The session will close with a consideration of suitable mechanisms for extending/promoting the discussion of standards.

Mathematics classrooms as complex adaptive systems

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Current writings on the nature of mathematical learning directs attention towards cultural and social aspects of human life. This is exciting and challenging work for researchers in mathematics education. Equally provocative work is occurring in the areas of biology, economics, complexity theory, chaos theory, computer science and AI, and the theory of coevolution and codetermination. The purpose of this discussion group would be to explore key ideas from these fields with a view to broadening research programs in mathematics education. Some of those key ideas are:

Evolution is not just a result of random mutation and natural selection, but equally if not more importantly, it is the product of emergent behaviour and self-organization. Self organizing systems are *adaptive*. *Emergence* is the incessant urge of complex adaptive systems to organize themselves into patterns. Living beings and their environments stand in relation to each other through *mutual specification* and *codetermination*.

Complex behavior need not have complex roots. Complex behavior can and is generated from simple roots: it is not possible to predict what the outcome will be from any set of roots. Whatever behavior appears at a particular time is dependent upon events which have preceded it--yet the emergent behavior is *not* predictable from those events.

Life at the edge of chaos is an internally driven dynamic of complex adaptive systems. This phase describes a region located between the state where a system *dissolves into chaos*, where conditions are so fluid that no patterns are discernible, and a state where a system has become completely stable and in which it exhibits no fluidity at all. The conjecture is that this region--*the edge of chaos*-- is where learning takes place.

The discussion session will (1) provide a brief overview of the key ideas noted above, and (2) through a series of focused questions and by working in small groups explore the possible relevance of these developments for research in mathematics education, so as to (3) share the outcome of small group discussions with the goal of providing directions for further explorations .

Discussion Group: Developing Training and Research Paradigms for the Preparation of Elementary Mathematics and Science Lead Teachers

Susan N. Friel, University of North Carolina — Chapel Hill, NC.
Rebecca B. Corwin, Technical Education Research Centers, Cambridge, MA.
Robert G. Underhill, Virginia Tech, Blacksburg, VA

It is increasingly apparent that if we want to impact change in how mathematics and science are taught at the elementary levels (K-6), we need to work with practicing teachers to update and improve their content and pedagogy knowledge and skills. This creates serious needs for increased capacity in mathematics and science at the school level. An emerging paradigm—the use of lead teachers—responds to the increased emphasis on site-based management and local control and adaptation.

As programs for lead teachers are developed and implemented, we need to refine our assumptions based on the results of evaluative research. We must document not only the impact of such professional development programs on the lead teachers themselves but the impact over time of the *second tier* professional development programs and support provided by the lead teachers to their colleagues and peers. The central issue is:

What paradigms best capture the important elements needed in developing and researching the impact of lead teachers in mathematics and science, K-6?

There are a number of questions that must be addressed in order to respond to this issue (See below for examples). These will be refined to focus the work of the discussion group.

1. What does the literature tell us about ways to evaluate lead teacher development programs as they relate to mathematics and science at the K—6 level?
2. How do we identify people who will be successful in the role of lead teachers? Do the characteristics of the role of lead teacher interact with the characteristics of the individual? If so, in what ways?
3. How do we evaluate the content and pedagogy knowledge and skills of teacher leaders before, during, and after their involvement in a development program?
4. How do we assess "teacher leader impact" on students with whom they may work directly and on peers and colleagues with whom they work in their capacity as teacher leaders?

CONCEPTUAL UNDERSTANDING OF COMMON AND DECIMAL FRACTIONS

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One of the reasons students lose interest in mathematics in school is that they see mathematics as a series of unrelated meaningless rules and symbols. In designing instructional tasks we tried hard to insure that the tasks would be meaningful to the children. We used manipulatives in order to base the concepts in something the students could touch and sense and could relate to the words and symbols. When using word problems, we set the tasks in contexts we believed would be familiar to the children. We gave the children opportunity to integrate manipulatives, oral fraction language about the situations, and common and decimal fraction words and symbols. We focused on helping children develop concepts before procedures.

The first part of the discussion session will be a brief description of the teaching tasks and interview tasks and responses. At the end of the presentation, participants will be invited to express reactions to the content, materials, examples, student responses, and some issues and questions: How can we best design instructional tasks on fractions to provide experiences for children which will be in the realm of their experience? When we use manipulatives and other contrived tasks, how do experiences with these tasks relate to understandings in contexts? How does the school setting or culture affect students' performance when presented the tasks in a one-to-one interview setting? How does the school setting constrain the application of principles learned in school to settings outside the school? How can we best help youth to integrate the concepts of common and decimal fractions, percent and ratio? What is the impact of technology on the curriculum and teaching of rational number concepts? Participants will be invited to react and express their own issues about the curriculum, teaching or student understanding of rational numbers.

INTERACTIVE MULTIMEDIA: ITS IMPACT ON RESEARCH

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The purpose of this discussion group is to investigate interactive multimedia as a research tool in mathematics education. This technology consists of a CD-ROM disk with computer access, and involves the creation of QuickTime sequences from collected videos and the creation of a HyperCard or HyperCard-like interface.

CD-ROM currently has the potential to store huge amounts of information, both graphic and textual as well as voice, sound, and video. When combined with an appropriate computer interface, the contents of the CD-ROM data source may be accessed in a non-linear fashion. This non-linear access feature allows researchers to explore different paths through this new kind of data, raising the possibility of addressing a broad range of questions with the particular research domain.

The nature of interactive multimedia influences the type of research questions asked and the nature of the information collected, the ways in which data are displayed and analyzed, and ways in which research reports are presented. For example, decisions regarding the allocation of the CD-ROM space among the elements to be included in the final project must be made and evaluated as the project develops. Current CD-ROM disks hold approximately 640 MB of material. Storage of different multimedia objects utilize vastly different amounts of space. A Quick Time video clip of one minute duration requires about 4 MB of memory whereas other data sources such as text require far less. The importance of different elements such as video clips, voice-over, text, pictures, etc. must be evaluated to determine a "best mix" of these elements to create a CD-Rom data source of maximum utility within the available space.

At the beginning of the discussion, information will be presented on the basic functioning of the technology. A CD-ROM disk made from video clips of children engaged in active construction of mathematical ideas will be shown. The use of such disks for teacher training and classroom research will be described. Participants will be encouraged to examine the impact of this technology on mathematics education research.

Children's Mathematical Knowledge and Construction of Units

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The concept of units as mental constructs has been of interest to mathematicians, from the time of Euclid. In the current era, the concept of units was used extensively by Steffe and his colleagues in explaining how children learn to count. Furthermore, a number of researchers have reported that the nature of units constructed plays a significant role in an understanding of a variety of mathematics concepts, for example, fractions, ratio and proportion, exponential function and geometric problem solving.

Because children's unit-related notions influence their mathematical understanding of so many topics, some researchers have suggested that a theoretical framework with a specific focus on children's units would be productive. However, it is clear that the development of such a theoretical framework is still in its infancy, and requires an open and vigorous discussion by researchers. The purpose of this discussion group, is to begin such a discussion, and promote the interchange of ideas on units by researches studying diverse topics. Specifically, the following two questions will begin our discussion.

1. What will we mean by the term unit? Specifically, is there any difference among the notions of a unit, a unity, a whole, etc.?
2. What are some unit-related concepts that may transcend specific topics?

The first question is designed to bring coherence to the study of units and develop some uniformity in the language researchers use. In order to develop a theoretical framework, we must first come to a consensus on what we mean by the term "unit." The second question assumes the view that children's unit-related notions are fundamental mathematical knowledge construction schemes. Among the ideas that have been suggested are the notions of composite units and children's units coordination schemes. During the discussion, we will examine the viability of these suggestions and identify other important ideas.

Advanced Mathematical Thinking

FACTORS IN LEARNING LINEAR ALGEBRA

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The Linear Algebra Curriculum Study Group (LACSG) have generated a set of recommendations for the first course in linear algebra (Carlson, Johnson, Lay, & Porter, 1933). These recommendations have highlighted the need for a first course in linear algebra that would give students a solid understanding of this topic. This paper points to factors essential to the building of an effective concept image of linear algebra. The factors that will be discussed are:

1. *The appropriateness of the time allocated to linear algebra.*
2. *Students' background and readiness for the kind of course called for by the LACSG, in regard to linear algebra ideas and language.*
3. *Students' background and readiness for the kind of course called for by the LACSG, in regard to the concept of proof.*

Appropriateness of the Time Allocated to Linear Algebra

For most students, the construction of an effective concept image (Ala Vinner, 1985) is a long and painstaking process. It is not always easy for us, as teachers, to realize this fact, for, as Piaget (1960) pointed out, a concept is deceptively simple when it has reached its final equilibrium, but its genesis is much more complex. The building of an effective concept image in linear algebra requires a major effort and sufficient time on the part of the students as well as their teachers. Yet, we allocate only one course in the entire undergraduate mathematics curriculum to linear algebra. In comparison, as Alan Tucker (1993) has pointed out, we devote an entire year-and-half of the lower-division core mathematics to calculus. Even with this amount of time, calculus is still difficult for students, a fact which raises doubts on the sufficiency of the time allocated to linear algebra.

In the case of calculus, we understand that students must build solid concept images for one-variable calculus concepts and, rightly so, we devote two courses to this goal, before we introduce multivariable calculus. For example, we understand that students must gradually abstract the idea of derivative by first dealing with it extensively in the case of one-variable functions, then abstract it into higher, yet spatially imaginable, cases (i.e. real-valued functions f: $R^2 \to R$ and $R^3 \to R$), and only then move to general functions $f:R^n \to R^m$. On the other hand, we do not seem to have the same patience for the abstraction process in linear algebra. Nor do we take into consideration the obvious fact that linear algebra concepts are indispensable for

[1]The complete version of this paper will be published in the *College Mathematical Journal* under the title "The linear algebra curriculum study group recommendations: Moving beyond concept definition."

understanding many multivariable calculus ideas. In most cases, ideas that require linear algebra background are shuffled under separate sections or exercises labeled "Optional."

Students' Background and Readiness In Regard to Linear Algebra Content

The Imbalance between the time allocated to calculus and that allocated to linear algebra is, in fact, even greater because high-school mathematics is geared toward calculus needs more than linear algebra needs. This argument may not be true if examined solely from the viewpoint of content. High-school curricula does include topics such as systems of linear equations, analytic geometry, and Euclidean space; all are part of linear algebra. But these topics are taught in high-school in ways that have little to do with the basic ideas of linear algebra. High-school students are not prepared for the objects, language, ideas, and ways of thinking that are unique to linear algebra.

From the students' point of view, calculus is a natural continuation of the mathematics they learned in high-school. After all, students deal with real numbers and functions of real numbers in high-school, and continue to deal with these objects in calculus. Also, they are often impressed by the power of calculus tools to help them solve problems in familiar domains, such as finding the area of non-standard figures, or modeling projectile motion. In contrast, students make little or no connection between the ideas they learn in linear algebra and the mathematics they learn in high-school. In the current situation, the only connection that potentially exists between high-school mathematics and linear algebra is the study of systems of linear equations. But even this connection is superficial. High-school students' Involvement with systems of linear equations amounts to learning a solution procedure for 2X2 and 3X3 systems. They do not deal with matrix representations of these systems, questions about existence and uniqueness of solutions, relations to analytic geometry of lines and planes in space, geometric transformations, matrix algebra and determinants, etc. Evidence that students place a low value on the relevance of linear algebra for high-school mathematics can be derived from a recent survey of mathematics education graduates. In this survey, 45% of the respondents believed that the value of linear algebra to their profession is marginal or useless, in contrast to an average of only 13% who thought so about calculus (OAS, 1992).

To demonstrate what the discontinuity between high-school mathematics and linear algebra entails, let me focus on one aspect of this discontinuity. Students in high-school deal with real numbers and continue to deal with the same type of objects in calculus. Real numbers, for all purposes of high-school mathematics and elementary calculus, represent either ratio quantities, such as, speed, density, price, and probability, or magnitude quantities, such as time, weight, length, and cost. Accordingly, the symbolic representations for these objects are one-dimensional. In linear algebra, on the other hand, new types of objects are added to the play: n-tuples, matrices, and functions as entities of a vector-space. These, in contrast to real numbers, represent multidimensional quantities, such as, probability vectors and price vectors, directed graphs, and solutions of a differential equation that models the effect of temperature change. According to the LACSG recommendations, the first course in linear algebra should be matrix-oriented; therefore, students would have to deal with vectors and matrices right at the beginning of the course. For this, students would be required to develop, in a relatively short period of time, a spatial symbol manipulation ability they never acquired before. For example, consider the statement:

> $RX=0$, where R is a row reduced echelon matrix with r non-zero rows in which the leading entry of row i occurs in column k(i) and X is a column vector. This system consists of r non-trivial equations in which the unknown $x_{k(i)}$ occurs with non-zero coefficient only in the i-th equation.

To comprehend this statement, we need to carry out several mental activities, among which (a) we visualize the matrix R and the positions of the leading entries, (b) mentally carry out the product of R with a column of unknowns, (c) visualize the corresponding positions of the unknowns in the system of equations $RX=0$, etc. Even when we express each of these steps on paper, we must first imagine and carry them out mentally; otherwise, they become entirely mechanical for us without our seeing the overall structure. In the same manner, take the useful formulas for computing a row $C_{(i)}$ and column $C^{(i)}$ in the the matrix products $C = AB$,

$$1. \quad (AB)_{(i)} = A_{(i)}B = \Sigma A_{(i)k}B_{(k)} \qquad 2. \quad (AB)^{(i)} = AB^{(i)} = \Sigma A^{(k)}B^{(i)k}$$

Students may be able to verify these formulas by a direct computation of the expressions involved. But to make these formulas part of their concept image so that they can apply them on their own and appreciate their usefulness, they need to develop a feel for the relations expressed by them. That feel involves, in part, spatial symbolic manipulations of the different components in these formulas.

The above statement and the latter formulas may not seem to be difficult to us. Even so, experience shows that the language of linear algebra and the new way of symbol manipulation take time to become part of the student's repertoire. But the problem is that in the midst of their struggle to adapt to this new environment, students are introduced to complex ideas, such as linear independence, spanning set, and subspace. That is when, using David Carlson's words, the "fog begins to roll in" and students lose track of what they are learning (1993).

Students' Background and Readiness in Regard to the Concept of Proof

The LACSG recommendations have set forth the standard for the first course in linear algebra to be an intellectually challenging course, with careful definitions and statements of theorems, and proofs that enhance understanding. From a cognitive and pedagogical viewpoint, a linear algebra course that stresses proofs is both a necessity and a challenge. It is a necessity because the emphasis on proofs is indispensable for the development of rich and effective concept images in linear algebra. Without understanding the reasoning behind the construction of concepts and the justification of arguments, students will end up memorizing algorithms and reciting definitions. It is a challenge because, as we all know, proofs are a stumbling block for many students. Research has shown that many students carry serious misconceptions about proofs. For example, students do not understand that inductive arguments are not proofs in mathematics; they do not see the need for deductive verifications; they are influenced by the ritualistic aspect of proof; and they do not understand that a proof confers on it a universal validity, excluding the need for any further checking (see, for ecample, Harel & Martin, 1989; Fischbein and Kedem, 1982). This situation requires, therefore, careful considerations and a special attention to the teaching and learning of mathematical proof.

In the current situation, the first course in linear algebra, if it emphasizes proofs, would be students' first experience with algebraic proofs, because calculus often is being taught proof-free and, traditionally, the idea of proof, as a deductive process, where hypotheses lead to conclusions, is stressed in the teaching of geometry but not in the teaching of algebra. Philip Davis and Reuben Hersh (1982) pointed out that "as late as the 1950s one heard statements from secondary school teachers, reeling under the impact of the 'new math,' to the effect that they had always thought geometry had 'proof' while arithmetic and algebra did not." The death of the "new math" put an end to algebra proofs in school mathematics.

In the last few years, I have been working on the epistemology of the concept of mathematical proof with students at various levels. One of the conclusions coming from this work is that a major reason that students nave serious difficulties producing, understanding, and even appreciating the need for proofs is that we, their teachers, take for granted what constitutes justification and evidence in their eyes (Harel, in preparation). Rather than gradually refine students' conception of what constitutes evidence in mathematics, we impose on them proof methods and implication rules that in many cases are extraneous to what convinces them. This begins when the notion of proof is first introduced in high-school geometry. We demand, for example, that proofs be written in a two column format, with formal "justifications" whose need not always are understood by a beginning student (e.g., Statement: AB = AB. Reason: Reflexive property of segment congruence). Also, we present proofs of well stated, and in many cases obvious, propositions, rather than ask for explorations and conjecturing. As a consequence, students do not learn that proofs are first and foremost **CONVINCING** arguments, that proofs (and theorems) are a product of human activity, in which they can and should participate, and that it is their responsibility to read proofs and understand the motivation behind them.

No one can expect to remedy students' misconceptions and "fill in" other missing conceptions about proofs in one single course. To meet the challenge to teach a linear algebra course that emphasizes proof, we must succeed in educating our students throughout the mathematics curriculum in school and college to appreciate, understand, and produce proofs. The movement towards this important goal cannot start in the first course in linear algebra; it must begin in the high-school years and continue into the calculus courses. In fact, with a careful approach and a suitable level, we should begin educating students about the value of justification (not mathematical proof, of course) in the elementary school years. Despite this, I believe that an emphasis on proof in the first course in linear algebra, as was recommended by the LACSG, is vital.

Summary

In this paper, I have discussed three factors essential to the building of effective concept images in linear algebra: The appropriateness of the time allocated to linear algebra; students' background and readiness in regards to objects, language, and ideas that are unique to linear algebra, and students' background and readiness in regards to the concept of proof. In Harel (in press), I make several suggestions for instructional treatments that address each of these factors. These suggestions address the need to prepare students for the unique environment of linear algebra prior to their first exposure to this topic in college. Specifically, three ideas are presented: (a) the need for and feasibility of incorporating linear algebra in high-school; (b) a suggestion how, prior to their first course in linear algebra, students can be acquainted with the environment of linear algebra, and (c) instructional treatments for the concept of proof.

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ARTICULATIONS BETWEEN THE SETTINGS, NUMERIC, ALGEBRAIC AND GRAPHIC RELATED TO THE DIFFERENTIAL EQUATIONS

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Abstract

Some obstacles in the learning the Ordinary Differential Equations are, in part, due to the traditional treatment that emphasizes an algorithmic-algebraic approach (since Euler, 1768). In this work, we are taking as theoretical support the notion of didactic transposition (Chevallard, 1985), and the idea of setting (Douady, 1986, 1991) to analyze the numeric, algebraic and graphic approaches in the solution process of Ordinary Differential Equations. In this study the computational software plays an important role in the three treatments.

Introduction

The traditional teaching of the Ordinary Differential Equation's course (EDO in what follows), basically is concerned with an algebraic approach on the resolution process, leaving out of the numeric and graphic treatments (Artigue, 1989, Hernández, 1993a). On this direction, we do an analysis of the situation, with the purpose of having a theoretical view of the problem, from which we can modify the strategies of actual teaching. Thus, on the one hand, we identify the different variables immersed in the process of didactic transposition related to the study of ODE's teaching; and on the other hand, we analyze the mathematical dimension in the context of Douady's idea of 'setting' (1991, p. 117), and their articulations (translations from one system to another, preserving meaning). To promote the articulations between settings we propose the use of the software DERIVE. This tool allows to implement the three treatments in the same screen: The classical numeric algorithm (Euler, Runge-Kutta, etc), the sketch of slopes field with the goal is to have a global vision of the behavior of the solutions, and finally, the construction of strategies to obtain the solutions of the first order equations (separable, homogenous, linear, exact, etc.) and the second order (linear).

Didactic Transposition related to the Ordinary Differential Equations

The historical development of the ODE shows a clear predominance of the algorithmic-algebraic approach, this was impulsed by Euler (Institutiones Calculi Integralis, 1768-1770). Hernández (1993b, pp. 8-27) uses the didactic transposition (Chevallard, 1985, Arsac, 1992), in order to study the ODE. From this analysis, he conclude: "...The algorithmic-algebraic character is determined, on the one hand, by the close relationship that exists between the algebraic development (as the search of roots of a polynome in terms of radicals) and the linear differential equations (cuadratic integration, Demidov, 1982), on the other hand, by the fact that the integral transforms, in particular Laplace transform, were built on the study of the linear differential equations (Deakin, 1984, Lützen, 1979)". This phenomenon has influenced the design of the curriculum. For example, the actual syllabus in the universities in Mexico has the two historical components, which determine the algebraic approach.

This dominant treatment in the syllabus (as well as in textbooks) for engineering students in Mexico, is due, in addition to the two historical components mentioned above, to other factors, from which we highlight the following: a) The algorithmic-algebraic process of routinary problems, are easy to develop by the students as it is shown by the studies of Orton, Seldon, and Artigue (1991), b) The orchestration of the graphic treatment (or numeric) in the classroom, provokes necessarily the use of the microcomputer, if not it is hard to visualize the slopes of the fields and isoclines curves, and c) with the incorporation of Laplace's transform to the syllabus of the engineering schools, in the middle of this century, the algebraic proccesses at the universities became stronger.

Didactic variable in the three treatments using the microcomputer

Nowadays, with the theory of dynamic systems and computation development, there is the possibility of involving the numerical and graphical treatments in teaching. In fact, during this century, propositions such as that of Brodetsky (The Graphical Treatment of Differential Equations, 1919-1920)[1] did not influence teaching, because of the avoidance of visual considerations in formal mathematics. Brodetsky's work, emphasized the role of the geometrical treatment as he showed a differential equation that did not have, an algebraic solution. However, he showed particular solutions graphically to that equation. Nowadays, computers have produced a change in teaching. Firstly, the ODE have been worked out in computers curses. And secondly, with the development of microcomputers, the graphical approach suggested by some authors (SMP, 1983, Sánchez et al., 1984, Artigue, 1992, Hubbard y West, 1991).

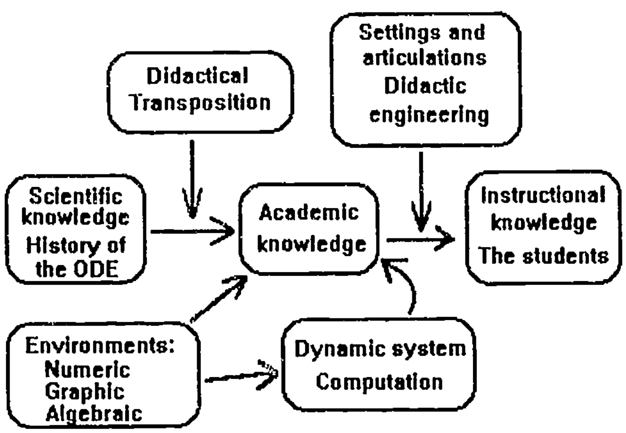


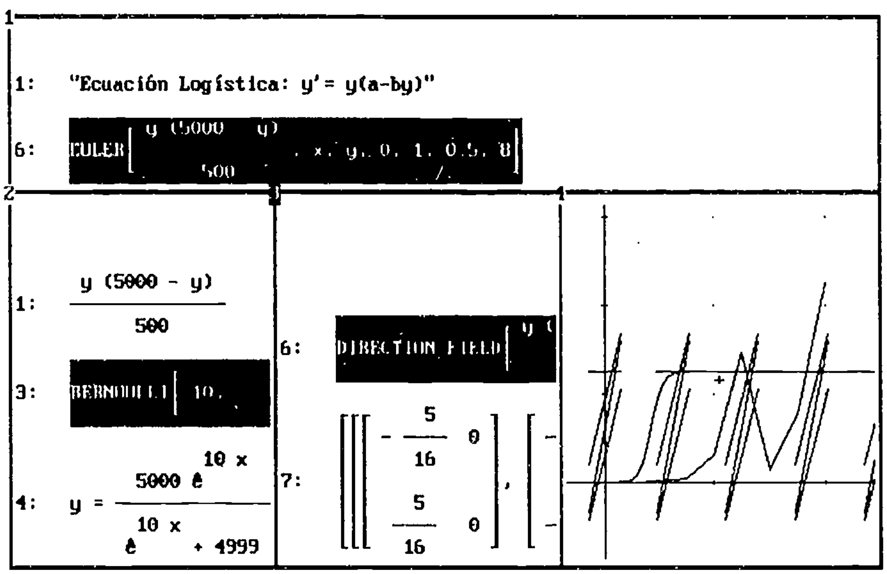
Figure 1

1 It was forgotten about in the studies of visualization of the ODE, it is perhaps the first try to embody a graphical approach to the resolution of the ODE.

Hirsch (1984), Selden et al (1989) and Artigue (1991) have pointed out that the teaching of mathematics emphasizes the algorithmic-algebraic approach and leave out the numerical and treatments. In this direction, the microcomputer plays an important role as a didactic variable (Brousseau, 1984). That is, it creates an equilibrium between the numeric and algebraic treatment with the graphic. Using the microcomputer, we can have on the screen the three representations cited above. We resumed these considerations with the diagram of the figure 1.

A proposition: Change and interplay of settings in the teaching the ODE

To construct didactical situations as those suggested by Douady (1986, 1991), we have taken the notions of setting and the relations of change and interplay of settings, in order to have a teaching proposition, which embodies and provokes the articulations between the numeric and graphic approach, and to give a complementary vision of the algebraic treatment. This proposal has been worked out in Hernández (1993b), and some of the characteristics are:



Numeric, algebraic and graphic representation using *Derive*.

Figure 2

I) The ODE are deeply concern with the modelling of deterministic processes (i.e. those where the final trajectory has been determined by the initial conditions) and that exist basically three settings of the ODE's resolution: numeric, graphic and algebraic. In other words, it means that there are three different methods to understand, describe and calculate the solutions of one differential equation, where the final representation of the solution can be an algebraic expression, a table with numbers or the graph of a curve.

II) Incorporating the numeric and algebraic approach does not mean that we are giving up the algebraic treatment, on the contraire, we are extending the ways of ODE's resolution and having at the same time, connections between the different representations, and giving a better framework to generate models that are described by an ODE. In this way, we need to change the algorithmic approach, because in textbooks, the exercises are sketched to drill using a "recipe" instead of doing a research work on the solution. Also, the models that are studied at this level, almost the differential equations are classified in linear and separable variables.

III) The numeric treatment, that in general does not belong to the traditional course of the ODE, it is worked out in the courses related to programming or numerical analysis. From our point of view, it is important to work with, at the beginning of an ODE's course. Emphasizing, on the one hand, the transition from a particular solution to the general solution, and on the other hand, connecting the tabular representation with the graphical one.

IV) From the graphic representation system's resolution, we use basically slopes field with the microcomputer, adding Brodetsky's method, and the isoclines curves. Fundamentally, we propose the students to sketch on the slopes field the integral curves. Also, we search the connections between this system of representation with the concept of funnel and antifunnel, to study the stability or not of the solutions.

Finally, we propose the use of the software DERIVE on the classroom, it allows us to provoke the articulations between the three systems of representations.

Final comments

The historical analysis related to the ODE leade to the conclusion that there was a development of the ODE, from Euler up to now where visual considerations were avoided (excepting perhaps Brodetsky's work, 1919-20). This development, in his didactical transposition, influenced the teaching of the ODE, in these curses, the geometrical representations were almost out of the educational context. Also, the numerical aspect was transferred to the computations courses and the students were conducted to work mostly with the algebraic representation system.

Douady's ideas related to interplay between settings, have induced the work to elaborate strategies to work within the classroom, as those exposed in I, II, III and IV, of this document.

Related to the resolution of the ODE in a computer environment the software DERIVE seems to be one between others that can provoke the articulations between the different settings. The experimental work with engineering students is in process at this moment.

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IMAGE STRUCTURES AND REIFICATION

in ADVANCED MATHEMATICAL THINKING: The Concept of Basis

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This is part of a larger study which examined nine advanced mathematical thinkers for evidence of visual image structures, links among structures, and reification events in their understanding of 21 mathematical concepts. In this paper, the concept of basis is analyzed.

Associated with any given mathematical concept, Tall and Vinner(1981) define a *concept image* as 'the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes.' However, one's mathematical understanding is determined as much by the *structure* of the concept image as by the sheer quantity of images, formal definitions, facts and propositions associated with the concept. Milestones in the development of one's mathematical understanding come from radical new structures or a restructuring of existing structures, but in a radical way.

Intuition plays a central role in advanced mathematical thinking. Fischbein (1987) asserts that the main factor contributing to the immediacy of intuition is visualization. He discusses three types of visual models used in intuitive reasoning:

- **Diagrammatic model** - This is comprised largely of graphical representations, and other diagrammatic schemas that pull out the essential aspects of a problem situation, and offer a global perspective.

- **Analogic model** - Two distinct conceptual systems are analogic if there are systematic similarities between the two which would lead one to assume further similarities. For example, electric current is sometimes described as being analogous to fluid flowing through a narrow tube.

• **Paradigmatic model** - This is more than just an example, but a class representative, through which one may view the entire concept.

Lakoff(1987) provides details of similar structures for a concept image in his discussion of the imaginative structures employed in reason, two of which, important in mathematics, are:

• **Propositional model.** In contrast to the other three, this model does not use imaginary devices of any kind. In a mathematical context, this will be referred to as *algebraic symbols.*

• **Image schematic model.** Mental categories structured by image schemas may be understood in terms of *container schemas* of various types, providing the scaffolding by which concept images may be structured.

Sfard(1991) discusses an operational/structural duality in the understanding of mathematical concepts. She identifies three stages in the progress towards a structural understanding: *interiorization*, *condensation* and *reification*. Attainment of the reification stage requires a quantum leap, an earthquake-like event in one's understanding, providing a radical restructuring of one's concept image.

Sfard makes the case that the cognitive schemas used in an operational approach are significantly different from those employed in a structural one, that the images associated with an operational understanding - linear, sequential, verbal - are different from those associated with a structural one -wholistic, integrated, visual. Sfard notes that, in the structural approach, these related compact, wholistic, visual images are used as *pointers* to more detailed information, and that as such are also useful in bridging the gap between an operational conception and a structural one, serving as a '*way-station* in our intellectual journey.'

To document the use of these different types of cognitive structures in advanced mathematical thinking, I conducted an in-depth study of nine individuals who were advanced mathematical thinkers, three advanced undergraduates participating in a Research Experience for Undergraduates program, four mathematics graduate students, and two faculty. Each individual was extensively interviewed, with the interviews being audiotaped, and then transcribed. These transcriptions were analyzed for evidence of the use of these different schemas, for pointers to more detailed information, as well as for evidence of reification, through the types of images employed, the links among ideas, and also through the language used to describe the concepts. In all, twenty-one concepts were examined. I will illustrate the results by examining their responses to the concept of basis.

The Concept of Basis

What follows is a short summary interpretation of the nine individuals' responses.

- **Beth U.**

Beth has only a tentative link between a basis and the general area of algebra, but she provides no details as to how the concept of basis fits into this area, nor does she elaborate on any further details about what a basis is. She has not yet built much of a conception of basis.

- **Adam U.**

Adam links this concept with the area of linear algebra. Once he makes this connection, he is then able to provide a formal verbal concept definition of a basis as a linearly independent, spanning set of vectors. When asked about a visual image, he recalls trying to prove things, rather than evoking any visual image.

This language, and the absence of any visual image, is indicative of an operational understanding of basis. He gives a prototypical example of a basis - the standard R^3 basis, described using the algebraic symbols $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

- **Craig G.**

Craig links the idea of a basis with Euclidean 2 or 3 space and its standard basis vectors. He describes having a visual image, but does not elaborate, so it is difficult to know exactly how he is envisioning this basis. His understanding has broadened somewhat since he has encountered more general n dimensional space, as opposed to his initial experience with 2 and 3 dimensional vectors in physics. However, even though he has experienced more general vector spaces, he first evokes the familiar 2 and 3 space prototypical model, indicating this prototype occupies the foreground in his concept image.

- **Calvin U.**

Calvin links this concept with the areas of linear algebra and topology, indicating he may be thinking of a basis in two different ways. His visual image of a basis is of the defining parameters in the basis itself, the algebraic symbolic notation for representing a basis.

- **Doris G.**

Doris links this concept with linear algebra. She offers several properties of bases, that any vector in the space can be written as a linear combination of the basis vectors, and that bases are not unique. Her visual image is of algebraic symbols $\{e_1, e_2\}$. She describes her understanding as having 'come together' as a result of a graduate class. Rather than this being a reification event, it would seem that she has condensed much of the information she has gathered concerning basis, but has not reached a fully structural understanding yet. She does not describe any sudden shift in her understanding, but rather a gradual consolidation of ideas. Her language is not particularly indicative of any one stage, but her lack of visual imagery other than algebraic symbols and her lack of any indication of intuitive understanding, signify that her understanding may still be operational.

- **Bill G.**

Bill links this concept with that of a vector space, in fact in his mind they are interchangeable. He describes a basis as a set of points, but accompanying that is a visual image of how they emanate from the origin, and that linear combinations of them fill out the space. He also links this concept with Galois theory and Galois extensions. He describes his understanding as having matured, and having split in two different directions with the introduction of Galois theory. This he describes as a sudden change, but from what he offers it is difficult to know whether this constitutes a reification of the concept of basis. He may have previously reified the concept, being so far removed from that event that it is no longer of significance to him. He is rounding out his understanding and consolidating facets of this concept from different areas of mathematics.

- **Andy G.**

Andy's prototypical image of a basis is the standard R^3 basis, represented symbolically by $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, $\vec{\mathbf{k}}$. He also has a diagrammatic visual image of this standard basis, analogous to the idea of toothpicks emanating from the origin. Notice that he is using an analogy with everyday objects so that he has a simple language with which to describe a basis. He also gives an intuitive definition of a basis as 'the minimum number of objects you need to identify a point in space,' and suggests that this understanding of a basis has remained unchanged.

- **Brad F.**

Brad links the concept of a basis immediately with a vector space basis. He describes it intuitively as a way of representing or 'seeing' vectors, of naming vector spaces. He has links to many areas of mathematics where this concept is used - in group theory, in the construction of a non-measurable set, in number theory and in analysis. He links this with Zorn's lemma in the proof of the existence of a basis. He describes his understanding as having expanded and changed from his experience with the concept in different mathematical content areas. This would appear to be a gradual broadening rather than a sudden shift. Apparently Brad has understood this concept both operationally and structurally for some time.

- **Alan F.**

Alan clearly demonstrates the most intuitive, structural understanding of this concept, describing a basis as a 'very fundamental thing'. He describes the concept using an *analogic* model, comparing a basis to a set of building blocks sitting on a shelf, which can be taken down and used to build more complicated objects, or at least approximations to more complicated objects. It is clear from his use of language that a basis is an object, used in a process to build more complex objects, clearly a structural understanding. Note that the image is wholistic, and clearly connected to some everyday experience, providing a way-station between the very intricate and complex mathematical structural concept of a basis as an object, and the operational understanding of a basis as a linearly independent set of vectors used as a process to represent vectors. This important analogy then provides the respondent with a rich language with which to discuss the structural nature of a basis. His prototypical image of a basis is of the polynomial basis used for approximating functions.

He describes his understanding as having developed, and that he has come to realize the importance of a basis. However, it is not possible to point to any reificaiton event, especially since he explains that he was first introduced to the concept of basis through the 'building block' analogy. He may well have had a dual operational structural understanding from the outset.

Conclusion.

For some respondents the concept image of the canonical basis in R^3 consists of a formal concept definition of a basis as 'a linearly independent spanning set' and its algebraic symbolism. This does not necessarily contribute any image schema to the concept image. Those who *only* evoke this type of understanding would seem to have a predominantly operational understanding of a basis. A more structural understanding of the concept of a basis requires some *intuition* about the concept, which is evidenced in the language used to talk about the concept, and in the intuitive, experiential images accompanying the concept.

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A SURVEY OF TERTIARY STUDENTS' ENTRY LEVEL UNDERSTANDING OF MATHEMATICS VOCABULARY

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Abstract

Mathematics teaching and learning at all levels is an interactive process dependent upon the understanding of carefully defined terms and symbols. Particularly at the tertiary level where instruction is traditionally teacher-centered and dominated by oral discourse, students' understanding of mathematics vocabulary is vital to their meaningful construction of mathematical knowledge. This study surveyed 443 students' understanding of mathematics vocabulary used in entry level tertiary mathematics. Students in 3 classes were asked to define, in writing, 15 words and 5 symbols. The results indicate that the students were inadequately prepared for their studies in respect to their knowledge of mathematics vocabulary and symbols .

Introduction

Mathematics teaching and learning is an interactive process dependent upon the understanding of carefully defined terms and symbols which name fundamental concepts. Understanding, in turn, is dependent upon a student's knowledge of the vocabulary of mathematics. Teachers use mathematics vocabulary routinely in the instructional process, assuming students have previously constructed meaningful definitions for words which may have been introduced several years previously. However, the results of studies conducted by Garbe (1985), Hanley (1978), and Nicholson (1977, 1980, 1989) confirm that some students have an alarmingly poor command of mathematics vocabulary.

An increasing body of mathematics education research indicates that one of the crucial roles of teachers of mathematics is to assist learners to acquire, in both receptive and expressive modes, the formal language of mathematics (Ellerton & Clements, 1991). This responsibility is reflected in *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990) and the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989).

Both documents suggest that a command of mathematical terminology is essential in learning mathematics and is a part of numeracy. Without a personally constructed knowledge base of mathematics vocabulary, the task of reading a mathematics textbook, interpreting a teachers' instructional commentary, solving a word problem, or communicating ones own knowledge about mathematics to others becomes extremely difficult for the student.

Background and Purpose of the Study

The failure rates of entry level mathematics classes at Curtin University of Technology in Perth, Western Australia, are a reported 25-33%. Because instruction at the tertiary level is traditionally teacher-centered and dominated by oral discourse, a lack of understanding of mathematics vocabulary may be a contributing factor to this percentage of failures. The results of a study surveying secondary students' knowledge of mathematical vocabulary (Miller, Malone, & Karmelita, 1992) suggested that several students were entering tertiary studies in mathematics without having an understanding of the vocabulary used routinely by their lecturers and tutors. The purpose of the study reported in this paper was to examine entry level, tertiary students' knowledge of the mathematics vocabulary used by their lecturers in both oral and written instruction.

Methodology

The three courses surveyed in this study were: MATHS 101 - Calculus and Analytic Geometry taken by students majoring in mathematics, computer science, or a science like physics or chemistry; MATHS 171 - Calculus and Analytic Geometry taken by students majoring in a field of engineering; and, STATS 121 - Statistics taken by students majoring in the health sciences or applied sciences. After attending the first 3 classes of the 1992 academic year for each of the three courses identified and reviewing the textbooks used in the classes, the researcher presented a list of vocabulary commonly and frequently used in oral and written discourse to the lecturers of the classes and an indepentent consultant in the Mathematics Department. In group consultation, fifteen words and 5 mathematical symbols were selected for each course which students were asked to define in writing. The vocabulary and symbols selected were representative of knowledge students should have had prior to enrolling in the tertiary class. They had been used by the lecturers in oral and written discourse with the assumption that the students did know what they meant. After the survey instruments were designed, the lecturers for each of the three courses identified one class to participate in data collection. Approximately 20-30 minutes were provided at the end of a lecture during which the students wrote their definitions to the words and symbols.

One hundred twenty-five students completed the survey in MATHS 101, one hundred ninety-eight in MATHS 171 and 120 students completed the STATS 121 instrument. The students' responses to the items were read by a masters student in the Mathematics Department and coded as acceptable, unacceptable, the students wrote "I don't know" or the item was left blank. After the responses were coded by the graduate student, the researcher and the independent consultant from the Mathematics Department reviewed the codes on each instrument to check for errors and/or discrepencies in acceptability of responses. When differences of opinion arose as to the acceptability of a response the researcher and the consultant discussed the student's response and reached a consensus for coding the item. Another review was conducted by a different graduate student to check for consistency of coding between and among classes.

Data Analysis and Discussion of Results

The instruments were scored by counting the number of acceptable responses. The researcher simply wanted to know how many of the twenty items students could acceptably define in writing and to compare the scores by various subgroups represented within the sample. Thus, means, standard deviations and t-tests were used to compare the subgroups. The choice of the t-test is defended on the basis that, even though the various student subgroups selected may demonstrate the characteristics of a non-normal population, the sample sizes, together with the robustness of the test ensure that the values of the statistic obtained approximate "t" very closely.

Because the MATHS 101 and 171 curricula are so similar, the researcher, lecturers and independent consultant decided to use the same 15 words for both instruments. Two of the five symbols were also the same. Eight of the 15 words were also used on the STATS 121 instrument so statistical comparisons could be made between the MATHS and STATS classes. Tables 1 and 2 (following Conclusion) reflect the words and symbols used on the instruments with the percentage of acceptable responses shown in the table.

Since it was assumed that the students had prior knowledge of these words and symbols, the percentage of acceptable responses should have been high for every item. However, in each of the three classes only about half of the 20 words were acceptably defined by 75% of the students. Table 3 (following Conclusion) reflects a t-analysis between classes. Only the words that were common to both survey instruments were analyzed. The comparison of MATHS 171 to MATHS 101 is statistically significant at the .01 level of confidence with the MATHS 171 students acceptably defining more of the words and symbols than the MATHS 101 students. In reality, the results can be summarized as the MATHS 171 students acceptably defining approximately 12 of the 15 items while the MATHS 101 students acceptably defined 11 of the 15 items.

Statistically the results may represent a difference in the abilities of these students to acceptably define mathematics vocabulary in writing. However, this one word difference may not be meaningful to tertiary lecturers.

The differences between the two MATHS classes and the STATS 121 class are statistically significant and, perhaps, more significant in a "real world" sense as well. There were 8 words common to the MATHS instruments and the STATS 121 instrument. In both MATHS classes, the students acceptably defined roughly 2 more words than the STATS students. It should be reiterated that the STATS 121 students may not have had the same mathematics at the secondary level as the MATHS 171 and 101 students; however, equally important is that the STATS 121 lecturer and the textbook used in this class assumed that the students had a prior knowledge of these words.

Conclusion

A *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990) and the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989) promote the need for students to develop common understandings of mathematical ideas, including the definitions of specialised mathematical vocabulary. These documents also promote the need for students to be able to communicate their knowledge of mathematics in writing. This researcher is not suggesting that vocabulary is the only aspect of language important in mathematics instruction. However, its importance is recognised in the *National Statement* and the *Standards*. The results of this study indicate that entry level tertiary students have difficulty expressing their understanding of mathematics vocabulary in writing.

Table 1
Words by class showing percent of acceptable responses

↓ Word/Class →	MATHS 171 n = 198	MATHS 101 n = 125	MATHS 121 n = 120
tangent	95	85	
product	97	97	70
absolute value	87	82	46
Pythagoras' Thm	95	87	
area	73	74	54
polynomial	67	67	
set	67	58	
function	45	46	13
symmetric	75	67	13
proportion	66	44	
sum	97	94	85
quotient	79	67	
denominator	95	87	85
variable	64	71	47
parallelogram	81	89	
$(x,f(x))$		76	
$A(R) = \pi R^2$		87	
$\lvert \underline{a} \rvert$		62	
$\lim_{x \to 0} \frac{1}{x} = \infty$	64	80	
$y = f(g(x))$	90	80	
$\overrightarrow{AB}$	78		
$\hat{\underset{\sim}{i}}$	26		
(x_1,y_1,z_1)	91		

Table 2 also reflects the words by class showing the percent of acceptable responses. It differs from Table 1 in that it shows the words and symbols which complete the STATS 121 survey.

Table 2: Words by class showing percent of acceptable responses

↓ Word/Class →	MATHS 121 n = 120	MATHS 101 n = 125	MATHS 171 n = 198
mean	95		
product	70	97	97
absolute value	46	82	87
probability	86		
area	54	74	73
median	91		
inference	23		
function	13	46	45
symmetric	13	67	75
range	67		
sum	85	94	97
standard deviation	50		
denominator	85	87	95
variable	47	71	64
mode	77		
p(A)	97		
$\overline{X}$	95		
$\sum_{i=1}^{N} x_i$	81		
A U B	92		
$\frac{\sum (x_i - \bar{x})^2}{n-1}$	59		

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Constructing the Derivative in First-Semester Calculus

Bob Speiser and Chuck Walter[1]

Summary. This paper is about what happens in first-semester calculus when we take mathematical modeling seriously. Surprising issues surface, about what functions are, about what we can and cannot learn from real-world data, about how we choose to represent data and models graphically. The issues we found cut so deeply into our ingrained assumptions that we needed to rebuild almost completely the way we taught and pictured the derivative, in order to maintain contact with our students' thinking and maintain a common discourse. This paper is a short version of a longer work which will appear elsewhere.

Listening to our students pushed us into this. In a nutshell, to take real-world modeling seriously in the first calculus course, we may need to introduce the basic concepts of calculus with much more subtlety and depth than may be usual.[2] Our rethinking springs from classroom discourse, triggered by the motion of an actual cat, seen in a historic sequence of time-lapse photos by Eadweard Muybridge[3].

We discussed the cat in a first-semester calculus course based on the Harvard-Arizona model.[4] We had just completed the first chapter of our text, a review of the elementary functions through a sequence of mathematical modeling examples. On this particular day, we planned to motivate the derivative as a rate of change, exemplified by speed. On the one hand, we expected Muybridge's photographs to provide a modeling problem which involved more complex decisions than the ones we had already treated. On the other hand, we wanted to explore the extent to which the concept of instantaneous speed may represent an idealization, reaching significantly beyond the necessarily discrete data we observe.

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[2] We develop this idea more fully, both epistemologically and in terms of pedagogical implications, in R. Speiser and C. Walter, Catwalk: first-semester calculus, J. Math. Behavior, to appear.

[3] Horses and other animals in motion, Dover Publ. (1985). We learned from David Lomen and David Lovelock how powerful a stimulus these photographs can be, at this point in the course. For their version, see Cushing, J. , Gay, D., Grove, L., Lomen, D., & Lovelock, D. , The Arizona experience: software development and use, in Proc. 3rd Intern. Conf. on Technology in Collegiate Mathematics (F. Demana, B. K. Waits, J. Harvey, eds.) Addison-Wesley (1992) 41-47.

[4] Our text was Calculus, by Hughes-Hallett, Gleason, et. al., Wiley (1993). Our classes, at Brigham Young University, took place in September, 1993.

To begin, we projected overheads of the cat photos and distributed photocopies of them. Next, after a review of average speed, we asked our students to work together on two questions: How fast is the cat moving in Frame 10? How fast is the cat moving in Frame 20?[5] Because predators which follow their prey visually, such as cats, are likely to have an evolutionary advantage if they keep their heads steady as they run, we suggested measuring distance along the grid from the tip of the cat's nose.[6]

How fast is the cat moving in Frame 10? We can measure time in seconds from Frame 1, position in centimeters from the heavy grid line just to the left of the cat's ears in Frame 1 to the tip of the cat's nose. Here are one investigator's numbers:

frame	t=time	s=position
9	.248	12.5
10	.279	15.0
11	.310	22.5

Between frames 9 and 10, the average speed is approximately 2.5/.031 = 80.7 cm/sec. Between frames 10 and 11, however, the average speed is roughly 7.5/.031 = 241.9 cm/sec. Gven these average velocities, can we reasonably say *anything* about the speed in Frame 10? Our feeling is that we can't. We could suggest that the speed *might* have been between 80.7 and 241.9, but we can't really be sure of that. As we go from Frame 9 to frame 10, the cat's hindquarters, for example, seem to be moving much more slowly than it's nose. In particular, how might someone marshal convincing evidence against the hypothesis that the cat, at the moment captured in Frame 10, was not moving at all?

Contrast this with the situation in Frame 20. There, the average velocities between frames hardly vary, so we're much more willing to assign an instantaneous speed. One glance at the photos, several students emphasized, offers further confirmation: the cat appears to be in the air. Graphing distance against time provides additional evidence in favor of a linear model for this part of the cat's motion. Returning to Frame 10, where the average velocities differ so strikingly,

[5] At this point we strongly recommend that the reader do precisely what we asked our students to do, as preparation for the discussion to follow.

[6] This decision simplifies. At a discussion of the cat measurements during a talk by Speiser in early November at Sacramento State, one participant measured from a spot on the cat's hindquarters, and obtained a wonderfully complex motion. Near Frame 10, it's possible that the cat's rear end was actually moving backwards.

the cat's hind legs are firmly placed beneath the body, ready to spring. Between one bound and the next, what's happening? How can we tell?

If we think of Muybridge's photos as a sequence of motion picture frames, we do have an unmistakable impression that the cat has a well-defined instantaneous speed at Frame 10. To sharpen the questions above, what status do impressions like this have? After all, impressions may well be all our nervous systems give us, though we typically take them to be real. We had a lively class discussion, contrasting what the pictures showed, what we seemed to think was happening, and what our numbers might be telling us.

As teachers, we base major decisions on what we prefer to think is a clear sense of what is obvious. Sometimes this construction, too, breaks down. Two classes after having shown his students the cat data and discussed its uncertainties, one of the authors (Chuck Walter) walked into his classroom, drew two points on a curve, and, for the first time, joined them with a secant line.

Following tradition confidently on the basis of twenty years' teaching experience, Chuck then asked the students to imagine what happens to the secant when one of the two points approaches the other.

They said they could not do so.

After reflecting for a moment, Chuck asked his students to imagine that the picture above, instead of being on the blackboard, was now on their calculator screens, and then to imagine zooming in on the two points as they come together. Wouldn't the curve look straighter after zooming in? Here is the subsequent exchange, as Chuck remembers it.

"No," said the students, "the curve looks straighter, but we still can't tell you what will happen, because the curve has *thickness*."

"Thickness," Chuck asked?

"Yes, thickness," the class responded.

"Is this thickness due to *uncertainties*?" Chuck asked.

"Yes."

"As in the motion of the *cat*?"

"Yes," said the students, "we don't really know where the cat is, so we can't say how the *points* will come together."

We decided to take Chuck's students seriously, because their doubts were founded, we felt, on a firm although perhaps not fully explicit perception of the difficulties surrounding the hypothetical function f(t) and its derivative. What would happen, in this classroom, if we took the uncertainties in the definition of f(t) more carefully into account?

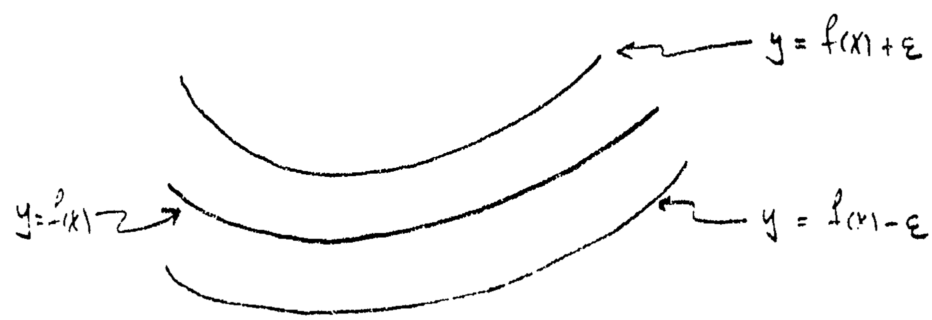
The first thing we did was to rethink the construction of the tangent line. To take significant uncertainties about the choice of the model f(t) into account, we first reframed the discussion by regarding f(t), not as the *actual* position of the cat, but, instead, as one of a range of possible mathematical *models* of that position. In this reading, f(t) is a well-defined function, perhaps even given by a formula. Rather than conflate the cat's actual motion with a particular model of it, we would attempt precision only *after* a given function f(t) has been selected from a collection of reasonable possibilities. In this way we supported our students' preference, based on earlier class discussion, for keeping explicit the distinctions between motion, measurement and model.

Next, to avoid zooming, we found a more global way to relate the tangent to the difference quotient. Our work here depends crucially on a new perception, which our students supported, about what was obvious about tangents and secants, and what wasn't, in the unfamiliar psychological world which had unfolded after the cat experiment.

To enter this new world, let's follow our students' suggestion, and imagine a thick curve.

It's important to remember that we typically expect our students to believe that a diagram not very different from this, for example, represents a parabola. For a parabola, if we move away from the point of tangency along a given tangent line, we should move off the curve. The picture we have drawn, however, does not support this "obvious fact."

Is it always best, however, to imagine a curve to be *infinitely thin*? Let's explore some possibilities. Here is a road, with y = f(t) as centerline. The road, bounded by the two curves y = f(t) ± e, determines a ball of radius e in the topology of uniform convergence. To be precise, a function y = g(t) is in the ball exactly when its graph is on the road.



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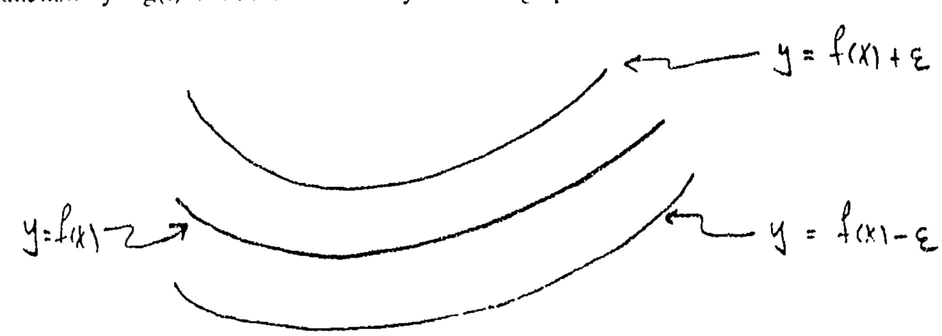
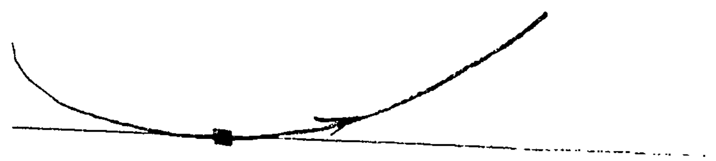
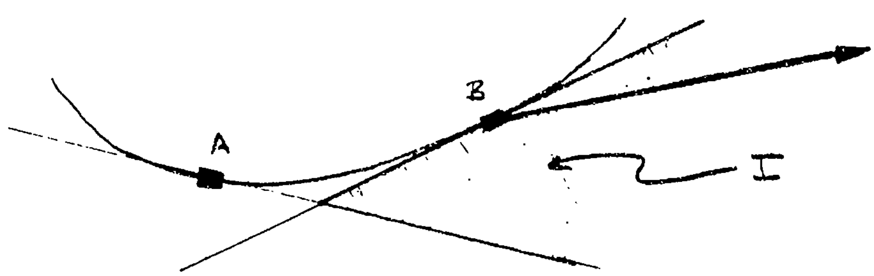


A car moves steadily forward, following the centerline. It's tangent line is shown.



We regard the tangent line as *obvious* here, because it is.[7] Indeed, if we're actually *driving* the car, the tangent is simply the line between us and the point on the horizon we are driving toward. That point on the horizon, which is also obvious, represents our direction. We may not be sure where the road goes as it turns ahead of us, but we are quite sure, at each given moment, where the car is pointing.

Now imagine two cars, driving toward each other along the centerline, each with its tangent line. When the cars collide, the tangent lines will, at that moment, coincide. We can use this simple observation to show that the secant limits to the tangent. Consider the secant as well: the line joining the two cars, as shown in the next illustration.



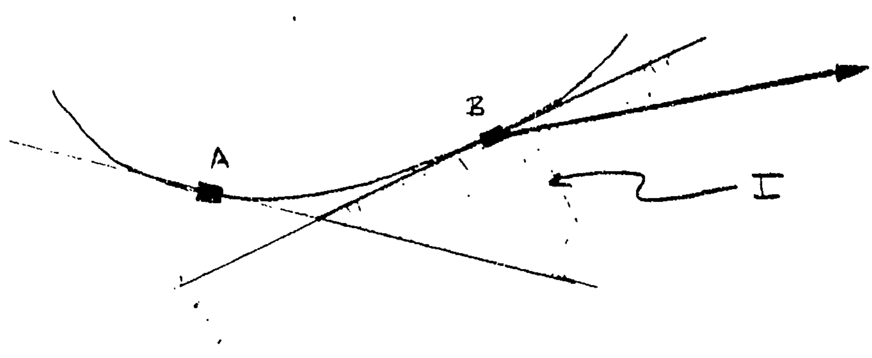
Here we have shaded the interior, denoted I, of the angle between the two tangents, as shown. Now I is the intersection of the halfplane above the tangent at car A and the halfplane below the tangent at car B. Because the road is on the same side of both tangents (it is concave up) we see easily, by examining the relevant halfplanes, that one ray from B of the secant line must lie in I. Because, by the same reasoning, the given ray will *remain* in I as the cars approach their collision, it follows that the limit of the secant must be the common limit of the two tangents, the tangent at the point of collision. This geometric reasoning shows that the difference

[7] Thin ice! From what *perspective* is the tangent obvious? Watch carefully.

A car moves steadily forward, following the centerline. It's tangent line is shown.

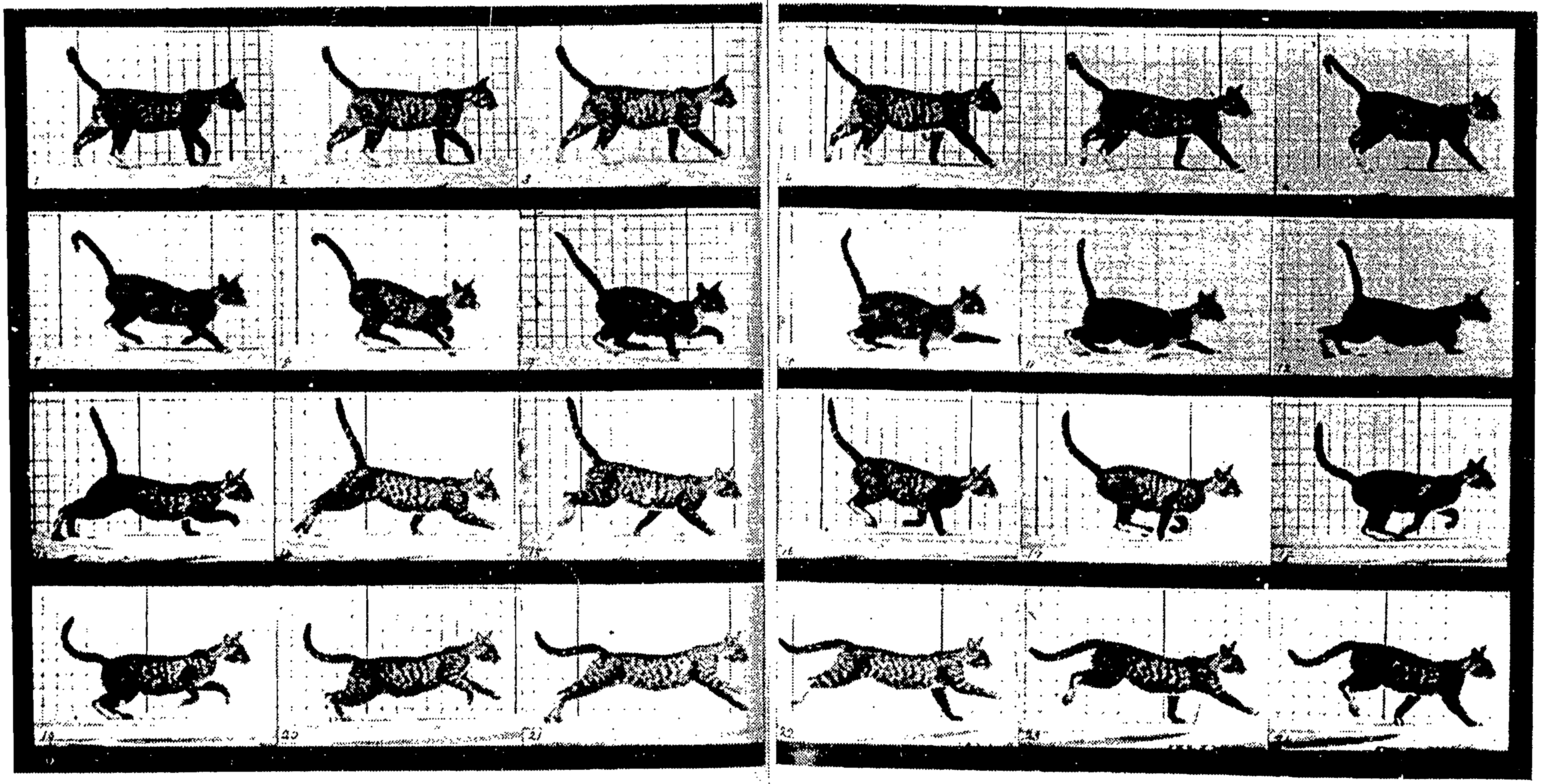
We regard the tangent line as *obvious* here, because it is.[7] Indeed, if we're actually *driving* the car, the tangent is simply the line between us and the point on the horizon we are driving toward. That point on the horizon, which is also obvious, represents our direction. We may not be sure where the road goes as it turns ahead of us, but we are quite sure, at each given moment, where the car is pointing.

Now imagine two cars, driving toward each other along the centerline, each with its tangent line. When the cars collide, the tangent lines will, at that moment, coincide. We can use this simple observation to show that the secant limits to the tangent. Consider the secant as well: the line joining the two cars, as shown in the next illustration.



Here we have shaded the interior, denoted I, of the angle between the two tangents, as shown. Now I is the intersection of the halfplane above the tangent at car A and the halfplane below the tangent at car B. Because the road is on the same side of both tangents (it is concave up) we see easily, by examining the relevant halfplanes, that one ray from B of the secant line must lie in I. Because, by the same reasoning, the given ray will *remain* in I as the cars approach their collision, it follows that the limit of the secant must be the common limit of the two tangents, the tangent at the point of collision. This geometric reasoning shows that the difference

[7] Thin ice! From what *perspective* is the tangent obvious? Watch carefully.



CAT IN WALK CHANGING TO GALLOP (.031 second)

Time of action: .71 second

quotient Q_h, the slope of the secant, limits to the slope of the tangent as h->0. After working these ideas out on Speiser's board, we both tried them in our classes. The students agreed, significantly, that taking a given model f(t) as centerline gave well-defined tangents, and that the road represented a zone of uncertainty into which the data may be expected to fall.

In other words, if we chose to describe the cat's position by a function f(t), then we must admit that f(t) is undefined for many values of t. This would lead to conceptual problems, because a function's values ought to be defined. Hence we reject this view, and emphisize that the cat's position, for general t, is unknowable. A further choice now follows: to regard the centerline f(t) as one of a range of possible mathematical models. Hence we take y = f(t) as the center of a road of appropriate size, and imagine other models on the road, along with our data. Each model is a well-defined function, which might even be given by an explicit formula. At the same time, we must be clear that the cat's motion took place significantly outside the universe of models. We regard both the universe of mathematical models and our psychological perception of the cat's universe to be somewhat independent constructs, which arise together and most strongly interact.

After choosing a model f(t), and working with the secant globally, we found that our students now felt that they understood what happens to the secant, in Chuck's original picture, when the two points come together. Returning to the uncertain data which compelled us to reformulate our presentation of derivatives, our students could finally compare, explicitly, what they now felt that they understood to what their calculator zooms appeared to show them.

Photo caption: Cat photos by Eadweard Muybridge, at Gentlemen's Riding Park, Philadelphia, 1885. Twenty-four cameras, 5cm grid in backdrop. Time interval: .031 sec. between frames.

USING A COMPUTER LABORATORY SETTING (CLS) TO TEACH COLLEGE CALCULUS

Nkechi Madonna Agwu

Mathematics Education, Syracuse University

This oral report will summarize the data collection process and initial findings of the presenter's on-going dissertation study, titled: "Using a Computer Laboratory Setting (CLS) to Teach College Calculus". For this study, a CLS is an environment where students learn calculus through explorations on the computer and discussion about their explorations and a Graphing Calculator Setting (GCS) is an environment where students learn calculus with the use of a graphing calculator. Students will be assigned randomly to work individually or in pairs on class projects. The research methodology will comprise of survey (questionnaires that address teaching, learning, and student attitudes) and experimental design (Randomized Block Factorial design) methods.

This study is designed to answer the following questions:

(i) What are the differences in learning outcomes between the CLS and the GCS approaches and between different CLS approaches (individual or cooperative)?

(ii) What are the differences in student attitudes between the CLS and the GCS approaches, and between different CLS approaches (individual or cooperative)?

(iii) What are the differences in CLS strategies for teaching college calculus to students working individually or in cooperative groups?

The objective of this study is to provide information for curriculum design, policy-making, and teaching of undergraduate calculus, with the use of calculators and computers.

A CONSTRUCTIVIST APPROACH FOR HELPING STUDENTS LINK THE POLAR PLANE TO THE CARTESIAN PLANE WITH A GRAPHING CALCULATOR

Yvelyne Germain-McCarthy
University of New Orleans

Introduction

I guided students on a journey from the Cartesian plane to the polar plane by helping them graph polar equations, $r = p \pm q\sin\theta$, from their hand sketched auxiliary Cartesian graphs of, $y = p \pm q\sin x$. The method provided a good review for sketching the circular functions from the coefficients p and q and helped students to connect the two topics.

I extended the method to graphing the polar conics

$$r = 1/(p \pm q\sin\theta) \text{ and } r = 1/(p \pm q\cos\theta) \qquad (1)$$

directly from the auxiliary Cartesian graphs,

$$y = 1/(p \pm q\sin x) \text{ and } y = 1/(p \pm q\cos x). \qquad (2)$$

This time, students were encouraged to hand sketch only the polar form (1) from the graphing calculator graphs of (2). My objectives were to teach students that the applications of related problems to new ones are conducive to richer understanding and that in the polar plane, the graphs of (1) are conics with focus at the pole and axis on one of the coordinate axes; When $|p| = |q|$, the conic is a parabola; $|p| > |q|$ is the ellipse and $|p| < |q|$ is the hyperbola. I also wanted to help them find differences and similarities between the planes. Rather than tell students the above information, they worked in cooperative groups to discover the results. During this process, students unexpectedly came to terms with contradictions very much like those within a constructivist framework. The results were students' discoveries which provided clear examples of the linkages between the planes.

STUDENT UNDERSTANDING OF DERIVATIVE IN A TECHNOLOGY ENHANCED CLASSROOM

Michelle J. Zandieh

Oregon State University

What kind of understanding of derivative do high school calculus students have and how does this understanding develop? In particular, how does it develop in a classroom in which each student has a powerful graphing calculator and uses a text designed to take advantage of the multiple representations of functions that the calculator provides?

Interviews with each student in a nine-member class at five different points in the school year as well as audiotapes of daily classroom discussions and copies of written work provide information on student understanding. The understanding described is not a matter of right or wrong answers to specific questions about derivative, but rather an attempt to provide a description of the cognitive structure a student has developed with regard to derivative.

Two particularly interesting threads in student understanding are emerging from the data. The first concerns the interpretations of the derivative concept that a student possesses and chooses to use in a particular context. Some examples of these interpretations are derivative as slope, derivative as rate of change, derivative as velocity, derivative as a symbolic manipulation and derivative as a procedure that allows for certain applications. A further question is whether a student can relate a model of derivative such as slope to the limit definition of derivative and the details of its notation. The latter entails a fine grained understanding of the operational and structural nature of several concepts -- difference, rate, limit and function -- as well as how these four concepts are combined to form the concept of derivative.

AN EMPIRICAL INVESTIGATION OF GRAPHING CALCULATOR USE AND GUIDED DISCOVERY STYLE TEACHING IN DIFFERENTIAL CALCULUS

George L. Emese

Rowan College of New Jersey

This report describes a three-group experimental study conducted in an introductory university differential calculus course with the following design. Group 1: Use of graphing calculators and (guided) discovery approach, Group 2: Use of graphing calculators without discovery, Group 3: Traditional instruction.

The two major objectives of the study were to verify that students *can* discover a significant portion of differential calculus and to investigate the effects of the use/non-use of graphing calculators and the instructional technique (lecture/discussion or guided discovery style teaching). The development of interactive graphing technology resulted in a renewed interest in discovery learning since it facilitates student experimentation and discovery.

In the discovery section, part of the new material was covered using worksheets, where a chain of questions/problems led to the new concept, relationship, or technique. Students worked in groups, pairs or individually. They could get help from *hint-sheets, solution-sheets,* their classmates and the instructor.

According to a questionnaire students in the discovery group completed after the final exam, they found the answer on their own to 47% of those questions on the worksheets where the answer was not previously known to them. They found the answer to an additional 22% of the questions with hints from the *hint-sheets,* from classmates or the instructor. 88% of the students suggested that some classtime (in average 30%) be spent on discovery style teaching. This shows that discovery style teaching is a viable alternative to traditional teaching for at least part of the new material.

Analyses of covariance were used for student achievement comparisons. The scores on the corresponding subtest of the pretest served as covariates. Statistically significant differences were not found between the groups on any of the variables. No instructional method proved superior to the others on comparison.

Poster Session

Calculus for All: Simulations for calculus Learning in Middle School

James J. Kaput
Department of Mathematics
UMass Dartmouth

We will demonstrate and discuss with interested parties a new series of software simulation environments and activities intended to help students in grades 5-8 (and beyond) develop the fundamental ideas of calculus, the relations between change and accumulation of quantities. We will concentrate on motion-based simulations that begin with simple directed motion in an elevator that makes possible an approach to the Fundamental Theorem of Calculus, as well as other basic theorems such as the Mean Value Theorems for derivatives and integrals, in the context of whole number arithmetic. We will discuss various activity structures and the place of these in a revised "strands approach" to the mathematics of change that begins in the elementary grades and runs through high school. For those who are interested, we will make the software available for testing with students.

Title : Informal Approaches to Limits Through Topics in K-12 Mathematics Curriculum

Presenter: Brenda Lee

Institution: Wu Feng Instutite of Technology and Commerce

The limit concept is a fundamental concept in mathematics in general and calculus in particular. However, the limit concept is seldom introduced before students take the calculus. Students usually first encounter the limit concept in calculus. One or two weeks lectures on the limit concept does not adequately prepare students for calculus. Whenever one is doing differentiation or integration, one is finding limits of some functions. The traditional ways of ε–δ definition often bring confusion and non-internalization in learning. Thus, calculus becomes the last mathematics course for most university students. Students who do not complete calculus are lost to further study in science, mathematics or engineering.

The teaching and learning of limit seems to cause problems for both teachers and students. However, the mathematics curriculum can provide early limit experiences and the abstractness of the limit definition will then be made more concrete. As a matter of fact, the concept of limit is embedded in different topics of mathematics such as numbers, fractions, decimals, functions, graphs, etc. Many ideas in early mathematics topics can be integrated into activities to enable students to informally understand the limit concept.

This presentation shows several examples of weaving the notion of limit into the different topics of the mathematics curriculum. One can show, for instance, how to link fractions to limit by an informal approach. When teaching the conception cf the size of unit fractions, we can ask students to fold the different unit fraction bars. This activity can provide the visualized comparison of the size of unit fractions. When listing the unit fractions as follow:$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$, the notion of sequences can be mentioned. Since the lengths of the unit fractions in the sequence get smaller and smaller, students can challenge to think the following questions: what will be the next term? Will this process goes on forever? How to interpret this infinite processing? What is "..."? Will the students recognize the limit is zero?

In the above activity,we can also let the students shade the unit fraction bars and place them successively in decending order, observing that the smaller the unit fractions, the smoother is the shape of the curve. This not only informally shows students the notion of limit, it also produces an excellent piece of art work. At the same time, the fraction bars' activity (1) allows students to estimate the the sums of the shaded areas, thus the notion of partial sum of a series is embedded here; (2) introduces the upper sums and the lower sums as different ways of finding the shaded area under the curve which leads to integration; (3)cutting the unit fraction bar into two equal parts, then cutting one part into two equal parts, and so on, leads to the notion of infinite process; (4) the continuation of this half division process also leads to the notion of infinity; and last but not least (5) when the denominators of the unit fraction bars get larger and larger, the notion of arbitrary smallness could be introduced as well.

The difficulty of accepting the existence of the "final product" (if this "final product" exists) of an infinite process and the actual infinity is universal. However, if the proper activities were investigated; the arousal of curiosity of minds would be created; the experiences of something that could get nearer and nearer to a fixed something, no matter how close you wish it, would be internalized; and students probably would be more comfortable when they enter the calculus classes.

Algebraic and Quantitative Thinking

VISUAL SALIENCE IN ALGEBRAIC TRANSFORMATIONS

Thomas Awtry and David Kirshner

Louisiana State University

This study establishes that beginning algebra students are dependent on visual aspects of algebra rules, and that this tends to inhibit the development of a propositional base for the rules. A qualitative analysis suggests that the visual salience of rules depends on two factors: repetition of symbols from the left to the right side of the equation; and dynamic visual tensions created by the parse of the left-hand expression resolved in the right-hand expression.

An aesthetic analysis of transformational rules in algebra might lead to the observation that some rules are visually coherent and appealing (e.g. $(x^y)^z = x^{yz}$ and $(\frac{x}{y})(\frac{w}{z}) = \frac{xw}{yz}$), whereas others are not (e.g. $(x - y)^2 = x^2 - 2xy + y^2$ and $x^{-1} = \frac{1}{x}$). In the past we have proposed that there is a continuum of visual salience in which transformational rules fall (Awtry, 1993) and that visual salience is relevant to problems of students' learning of rules (Awtry, 1993; Kirshner, 1989b). This approach to the visual structure of students' algebraic knowledge diverges from traditional perspectives in cognitive science which hold that rules are acquired in propositional form (e.g., Anderson, 1983) and only subsequently automated through visual adaptations. This paper will focus on a qualitative analysis of visual salience using students' interview protocols as support.

Methodology

In order to investigate whether visual salience is related to students' learning of rules, a replication of Awtry (1993) had two groups of algebra novices (grade 7 students) memorize eight rules, four of which we judged to be highly visually salient (visual rules):

$$2(x - y) = 2x - 2y \qquad (xy)^2 = x^2y^2$$

$$(x^y)^{-1} = x^{y \cdot -1} \qquad (\frac{x}{y})(\frac{w}{z}) = \frac{xw}{yz}$$

The other four rules lacked visual salience (propositional rules):

$$x^2 - y^2 = (x - y)(x + y) \qquad (x - y) + (w - z) = (x + w) - (y + z)$$

$$(x - 1)^2 = (x^2 - 2x) + 1 \qquad x(y^{-1}) = \frac{x}{y}$$

We used specialized versions of ordinary algebra rules to balance crucial characteristics of the two rule types (i.e., number of parentheses, constants, and operations).

The rule set was given to one group of students in ordinary notation to learn. The second group of students was given the same instructional unit using tree diagrams instead of ordinary notation. Tree diagrams present the same propositional content of the rules, but distort the visual characteristics that might lead to visual salience in ordinary notation. For instance, $(\frac{x}{y})(\frac{w}{z}) = \frac{xw}{yz}$ and $(x - y) + (z - w) = (x + z) - (y + w)$ are nearly identical in their structure; however, this similarity is apparent only in tree notation (Figure 1):

```
      M           D                 A           S
     / \         / \               / \         / \
    D   D   =   M   M             S   S   =   A   A
   / \ / \     / \ / \           / \ / \     / \ / \
   x y w z     x w y z           x y z w     x z y w
```

Figure 1. Tree Notation Representations for $(\frac{x}{y})(\frac{w}{z}) = \frac{xw}{yz}$ and $(x - y) + (z - w) = (x + z) - (y + w)$

The capital letters indicate the operations of the expressions, and the hierarchy of the operations indicates the parse. The difference in visual salience between the rules that is so apparent in ordinary notation is lost in tree notation.

Students' mastery of the rules was assessed at two different levels. *Recognition tasks* presented the student an expression with five other expressions, only one of which could be derived from the given expression by a lawful application of one of the taught rules (e.g., for the rule $(xy)^2 = x^2y^2$ the left-hand expression $(5x)^2$ was given with the choices $(5^x)^2$, $5x^2$, 5^2x^2, 5^{x2}, 5^2x , and "none of these.") The subjects' response of 5^2x^2 may reflect a (relatively superficial) pattern matching understanding of the rule in question.

Rejection tasks also presented an initial expression with five alternatives; however, in this case none of the alternatives was derivable from the original expression by lawful application of an algebra rule. Rather, each rejection task presented a near deviation from a lawful rule. For example, the initial expression $x^2 + y^2$ together with an alternative expression $(x - y)(x + y)$ constituted the rejection task item for the valid rule $x^2 - y^2 = (x - y)(x + y)$. Choosing the correct "none of these" alternative for such items would reflect a deeper understanding of the limits and constraints of algebra rules.

Students in the two groups were given identical instruction and tests, except for the notational form in which expressions were represented. A nonparticipant observer was present to record extraneous factors which might discriminate outcomes, but no such biases were reported.

Results

Posttest and retention test results showed that visual salience is a proactive feature in algebra learning. A table of mean percentages for recognition tasks on the posttest and retention test for the rule type by treatment interaction is provided in Table 1.

Table 1.
Recognition Task Posttest and Retention Test Percentages

	POSTTEST		RETENTION TEST	
	VISUAL	PROP	VISUAL	PROP
ORD	73	40	69	35
	(56)*	(56)	(52)	(52)
TREE	49	56	47	53
	(58)	(58)	(53)	(53)

* Values in parentheses represent number of subjects within cell

Although in tree notation the visually salient rules and propositional rules were equally difficult to master, in ordinary notation the visually salient rules were significantly easier to master than the propositional rules at the superficial recognition (pattern matching) level ($p < .0001$ on both the posttest and retention test).

A table of mean percentages for rejection tasks on the posttest and retention test for the rule type by treatment interaction is provided in Table 2.

Table 2.
Rejection Task Posttest and Retention Test Percentages

	POSTTEST		RETENTION TEST	
	VISUAL	PROP	VISUAL	PROP
ORD	13	20	8	16
	(56)*	(56)	(52)	(52)
TREE	15	11	13	7
	(58)	(58)	(53)	(53)

* Values in parentheses represent number of subjects within cell

The visually salient and non-visually salient rules were again equally difficult to master when presented in the tree notation, but in ordinary notation the visually salient rules were more difficult to constrain given the opportunity for overgeneralization presented by the rejection tasks ($p < .005$ for the posttest).

We conclude that visual rules are more easily apprehended at a superficial recognition level because of their visual salience. But, because visual rules are learned more easily, learners have not had to wrestle with their underlying propositional meaning. Thus visual rules are less easily understood in the propositional terms that might constrain their overgeneralization.

These results, obtained in a brief treatment with algebra novices, help to explain the persistence of school curricula that provide students with endless repetition of algebra tasks, but with little meaningful discussion of structural categories. Such curricula provide the opportunity to assimilate the visual patterns of algebra, but unfortunately, without the mastery of structural categories that might forestall the epidemic of rule overgeneralization errors (Carry et al., 1980; Davis and McKnight, 1979; Matz, 1980; Wagner et al., 1984).

Qualitative Analysis

The rest of this paper will focus on a qualitative analysis of visual salience. Whereas several past studies (Davis, 1979; Kirshner, 1989b) have explored visual aspects of algebraic transformations, our discussion extends these previous explorations to the mechanisms for visual salience of transformational rules.

This replication of Awtry (1993) suggests that visual salience is intrinsic to certain rules, but not others. We suggest two basic characteristics underlie visual salience, both of which must be met: (1) repetition of elements; and (2) some sense of a dynamic visual displacement of elements. Subjects in an extensive pilot study were interviewed about the characteristics of the rules that they found easy to learn and those they found difficult to learn.

Repetition of Elements

Repetition of elements is the reoccurrence of elements on the left and right sides of a rule. This repetition is necessary (but not sufficient) for a sense of visual continuity for the rule. Repeated elements can include alphanumeric symbols (representing operations, variables, and constants) as well as the visual characteristics like wide spacing and vertical or diagonal juxtaposition that Kirshner (1989b) found to be related to parsing competence. For example, the repeating elements of the visual rule $a(b + c) = ab + ac$ are, "a," "b," "c," "+," horizontal juxtaposition, and wide spacing.

Consider $(x^m)^n = x^{mn}$ and $x^m x^n = x^{m+n}$. Both are visual rules, but the second rule violates the repetition of elements condition by containing an additional plus sign on its right-hand side. This yields the prediction that the second rule would have less visual coherence than does the second. We propose there is a continuum of visual salience on which transformations rules fall.

The absence of the repetition of elements condition seems to stimulate propositional reflections about the structure of some rules. For instance, one subject pointing to the rule $(x - 1)^2 = (x^2 - 2x) + 1$ said, "I [keep thinking] you're to multiply one and two . . . and *subtract one* [italics added]," indicating the desire for repetition of the subtraction. Referring to the same rule, another subject said, "You have to remember to add the one and *not subtract* it and not multiply it." Another student found the difference of squares rule difficult "because you don't put any twos in it. I always think there should be some twos in there [on the right side of the equation]."

Indeed, more elaborate semantic connections spring up to deal with absence of repetition of elements in a transformation. For example, one subject thought the difference of squares rule was easy to remember, since "the two [square] tells both of them [the x and y] to double each other and [you] just put minus and plus and put them in parentheses." The subject in trying to learn this (propositional) rule has found it useful to refer to the meaning of the element "2," (though this reasoning is unrelated to valid mathematical deductions).

Repetition of elements is not sufficient, however, as can be observed by comparing $(\frac{x}{y})(\frac{w}{z}) = \frac{xw}{yz}$ and $(x - y) + (z - w) = (x + z) - (y + w)$. Both of these rules involve element repetition, but only the first appears to have visual salience.

Visual Reparsing of Elements

We suggest that the first of these rules contains the more elusive second characteristic of visual salience - some sense of a dynamic visual displacement of elements. That is, a visual reparsing of elements is at work here. The parse of the left hand side of a rule serves to create boundaries that are broken down on the right hand side. A resolution of tension permits access to previously cloistered elements of the left-hand expression.

Several of the distributive rules we observed in the algebra repertoire seem to obtain a sense of visual cohesion from such visual reparsing. For example, in $c(a + b) = ca + cb$ and $(ab)^n = a^n b^n$ there is a tension created by the cloistering of the "a" and "b" on the parse of the left-hand expression that is resolved in removing the parentheses on the right.

One subject when interviewed said $2(x - y) = 2x - 2y$ was easy to learn because "when you take off the parentheses, you just push ... put it together with the things inside and keep the minus sign." The same subject also said $(xy)^2 = x^2y^2$ (a form of $(xy)^z = x^zy^z$) was easy, because "you remove the parentheses and leave one two where it is and put another one between the x and y." When asked how he knew to put a "2" on both variables, he said, "because its already close to the y and I put it on the x because its automatic when I put it on the y. I know its suppose to be on the x when I put it on the y." The subject seems to have a visual reparsing algorithm that dynamically transforms the left side of a rule into the right side.

The rule $(x^y)^z = x^{yz}$ presents a somewhat different type of visual reparsing. Although there is an incursion into restricted territory, the attachment stops with the closest element, instead of bonding with all cloistered elements. Thus the "z" bonds with the "y," but not the "x." Note that this notion of visual reparsing is distinct from ordinary reparsing, in that it is not the actual *operations* that are preserved in the reparse, but only their *visual character*. Thus $(x^y)^z = x^{yz}$ changes the exponent operation that connects "x^y" with "z" for a multiplication between "y" and "z," but the visual characteristic of horizontal juxtaposition is retained (Kirshner, 1989a).

One subject marked $(x^y)^{-1} = x^{y \cdot -1}$ as easier to learn than $x(y^{-1}) = \frac{x}{y}$, "because the y is *up* with the negative one and all you have to do is *bring it up* (italics added) a little bit more and put a decimal between them as multiplication." Another subject referring to the former rule said, "On this one all you do is just take out the parentheses and in this case, multiply." Both subjects seem to be describing the visual reparsing feature of the first rule.

The rule $(\frac{x}{y})(\frac{w}{z}) = \frac{xw}{yz}$ is another rule in which elements bond only with those elements closest to them. One subject claimed it to be the easiest rule, because "all you have to do is connect that line and take off the parentheses." The subject has verbalized a visual reparsing algorithm to dynamically transform the left side of the rule into the right side by removing the parentheses elements and horizontally juxtaposing the numerators and denominators.

Rules with an absent visual reparsing of elements are harder to learn. One subject working in ordinary notation stated some rules were harder because "you had to change so many things," whereas for easier rules "you just had to do the same thing. You hardly do anything." The difference of squares rule was rated as being hard, for example, because "you have to change all of it" and "whenever you get finished with the answer, it doesn't look anything like the rest of it." Although $x(y^{-1}) = \frac{x}{y}$ is short in length, it was said to be hard, because "you have to change it," in comparison to the slight visual change of $(x^y)^{-1} = x^{y \cdot -1}$.

Discussion

The notion of a visually-based cognition of algebra contradicts most cognitivist approaches to mind that work backward from the meaningful contexts of expert knowledge to a hypothesized conceptual ontology. Instead, this perceptually based approach is consonant with the challenging parallel distributed processing models (Cohen, Servan-Schreiber, & McClelland, 1992; Rumelhart, 1986) which see pattern completion as the fundamental cognitive building block. This qualitative analysis of visual salience attempts to contribute to an understanding of the perceptual substrate of cognition.

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Preparing Students for Algebra: The Role of Multiple Representations in Problem Solving

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This project was designed to examine the proposition that conceptual understanding of basic mathematical ideas is enhanced when students see the concept embodied in many different ways in the context of problem solving. A two week curriculum was developed to introduce prealgebra students to basic concepts about variables, expressions and one variable equations. The data collected included pre and post-test think aloud protocols from a sample of students in experimental and control classes and a final written test from all students. Experimental students showed more use of representations and equations and fewer syntactic translation errors. The control class students were better at solving equations and equal or better in general problem solving.

Background and Rationale

The study of algebra entails a major cognitive transition for students who have previously only studied arithmetic and related areas in the elementary and middle school curriculum. A major difference between arithmetic and algebra is that algebra requires students to use new and more abstract forms of representation of mathematical ideas and operations. The transition to this new form of representation is seldom directly covered in mathematics classes (Kieran and Chalouh, 1993) and appears to underlie many of the misconceptions that impede student progress in algebra (Matz, 1982).

Research in the early learning of algebra reveals four major areas in which students exhibit misconceptions (Booth, 1988), three of which are particularly important to this project: arithmetic relationships, algebraic symbols/conventions, and focusing on answers to the exclusion of meaningful relationships. For example a student might have difficulty in any one, or several of these areas such as: undoing the arithmetic operations, or not recognizing that '4b' means 4 times b, or interpreting algebraic expressions as meaningless because they do not fit the student's concept of 'an answer'. Highlighting these common cognitive obstacles (Herscovics, 1989) has led to theories about the underlying mechanisms that can be implicated in the formation of misconceptions.

Increasing attention has been focused on the possibility that these three areas of misconceptions stem from a tendency to perform direct syntactic translations of words into mathematical symbols. Students using this approach would exhibit an algorithmic orientation, primarily focusing on a simple left to right mapping of each word to its corresponding symbol. Reliance on this method has the potential to impact each of the three areas of cognitive obstacles. It is hypothesized that such an orientation would foster a lack of understanding as to how mathematical symbols function within the structure of algebra, and therefore lead to misconceptions (Herscovics, 1989).

Another contributor to students' formation of misconceptions may be a lack of skill at translating between different representations of a problem, either mathematical representations or real world situation. Resnick and Omanson (1986) found that many students lack comprehension of the connection between arithmetic algorithms and physical representations of the number system such as base ten blocks. At higher grade levels word problems are included in textbooks as the primary way in which students are expected to relate mathematical concepts to real world problems. Word problems are difficult for older students because they are not sure how to translate a verbal problem into a mathematical situation (Mayer, 1987). A contributing reason for the difficulty is the disjunction between the everyday experiences of students and the examples and applications used in algebra and prealgebra classes (Mellin-Olsen, 1987; Moses, Kamii, Swap, & Howard, 1989).

Research design and methodology

The project took place during the Fall of 1993 within 3 existing seventh grade classes at a junior high school in southern California. The same teacher taught two classes with the experimental curriculum and one control class with a traditional prealgebra curriculum based upon a textbook.

Students in the experimental class worked in cooperative groups of approximately four students, sometimes to produce a joint product, sometimes on individual tasks. The experimental curriculum featured problems based on the theme of running a candy store. Algebraic concepts and notation were introduced after the students had spent some time exploring a situation in which the algebra was appropriate. For instance, on the third day of the curriculum students were given a simile of a store's order form for candy making supplies and a simile of the invoice form for what was received. Each ingredient was ordered at least five times. For each ingredient there was a systematic error which was attributed to a computer malfunction. Each cooperative group was assigned the task of discovering the systematic error for one ingredient and writing to the supply company to describe the error. In their letters the groups had to present the evidence in the form of a table, graph or equation that demonstrated the nature of the computer malfunction. After each group presented its results to the whole class, the teacher introduced the notation for variable expressions as a parsimonious way of representing the patterns described in the letters.

The textbook used in the control class emphasized symbol manipulation and the translation of English phrases such as "five times eight n equals thirty-six" into mathematical equations. The students did only a few word problems.

All students were given a written post-test which covered the traditional symbol manipulations skills as well as items on translations between different forms of problem representation. The written test took two class periods. In addition, a sample from each class consisting of eight students balanced for gender, prior achievement and ethnicity was interviewed at the beginning and end of the unit, for a total of 24 students. The questions probed students' comprehension of the nature of variables and equations and asked them to solve a number of problems using a 'think-aloud' procedure. Each student was interviewed individually by a researcher and all interviews were audio taped.

Data Analysis and Results

In this presentation, the post test interview data and written final examination data are used to address the following questions: 1) Do students from the treatment and control groups understand the meaning of symbols (e.g. variables, equations) differently? 2) Are the students able to link the symbols of algebra to different representations? 3) Are the students in the treatment group better able than the control students to represent word problems with symbols? 4) Is the treatment group able to manipulate symbols as well as the control group?

Question 1 Students' understanding of the written symbols were assessed in two ways. On the written exam students were asked to translate phrases and sentences from English into mathematical symbols. Two of these questions were written so that syntactic translation would result in errors and more meaningful interpretation of the words was necessary to accurately answer the questions. Although both groups of students made equal numbers of errors, it was almost exclusively the control class that used syntactic translation on both questions. On question 7 (three less than twice b) 71% of the control class errors were due to syntactic translation as opposed to 28% of the experimental class errors, a statistically significant difference (Pearson's chi=8.06, $p<.005$). A similar result was found on problem 9 (two less than the product of x and 9 is 16) in which the relative proportions of syntactic translation were 65% and 24% (Pearson's chi=7.4, $p<.01$).

A qualitative analysis of the post-test interviews also showed more sense making efforts on the part of the experimental class. Although both groups of students displayed misconceptions about the structure of algebraic notation, substantive differences were observed when the types of misconceptions and depth of responses were analyzed. Members of the control class generally responded with terse, answer oriented responses, which emphasized algorithms for solving equations. One control class student, when asked to describe the meaning in an equation said "I guess the you would need another number...the number that you would multiply to get that, the answer." In contrast a student exposed to the experimental curriculum unit responded saying "this is one that you would have to find out what the x means."

The greatest differences were found as the students attempted to interpret the meanings of variables in equations and expressions. Members of the control class were more likely to respond that they "would have to have the answer" before they could be comfortable ascribing meaning to a variable. Although the experimental classes also showed hesitancy in assigning meaning to the variables, they were able to give examples that included real world situations. For '4b' a student said "(It) could be four of a certain thing, like, 'b' could be books. You could have four books of a certain size..maybe it could stand for the area of a book." Students in the experimental classes showed a greater tendency to use objects in their responses, and to use the letters as labels for entities, as opposed to numbers.

Question 2 In answering two word problems, the experimental group showed a larger tendency to employ other forms of problem representation, e.g. graphs, tables, etc. in their solution. On the first problem 20% of the control class used a representation while 50% of the experimental students did (Pearson's chi=4.77, $p<.05$). A similar trend was found on the second word problem with 55% of the control class using a representation and 74% of the experimental class (Pearson's chi=1.95, p<.20) although this difference was not staticstially significant. In addition to using other representations more often, the experimental classes used a wider variety of representations including drawings, linear graphs, number tables, pie graphs, and number lines. The control class only used tables with the problem information on the first word problem and drawings on the second problem.

Question 3 The experimental classes were also more likely to use equations when solving these two word problems, although the difference did not reach significance. The experimental classes used equations on 74% of the first problem and 50% of the second problem while the control classes used equations 55% and 35% of the time respectively.

Question 4 Although the experimental classes showed more apparent efforts at sensemaking on the measures described above, the control class was better at solving equations. The control class successfully solved 87% of the equations while the experimental class only solved 55% of the problems. The experimental classes poor performance was due in part to a lack of practice and in part because one topic (inequalities) was not covered fully due to time limitations.

Conclusions

The experimental curriculum showed some impact upon students' sensemaking in basic algebra topics. However, students' efforts to contextualize their algebraic problem solving through the use of other representations and equations did not result in improved performance on these questions. In addition, the students in the experimental classes showed markedly poorer performance in traditional symbol manipulation skills.

The curriculum itself and its implementation need improvement. Some of the activities did not flow smoothly while others overestimated students' graphing skills. The students themselves needed time to become accustomed to working in groups on more open-ended activities. In addition it was very difficult to design meaningful activities that embodied very simple uses of variables and equations.

In addition, the problem solving approach totally de-emphasized direct instruction and practice in symbol manipulation. We don't feel we found the right balance between problem solving and exploration of a new topic, and opportunities to practice new skills.

Finally, this study did not attempt to explore which kinds of representations actually facilitated student understanding of the algebraic concepts. Each new problem employed a different representation and did not build directly from the prior day's work. A more extended curriculum on this topic is being designed in which students will work for a week or more with one kind of representation , perhaps in the context of problem solving situations which take more than one or two days to complete. This new curriculum will also be used with a larger number of students taught by more teachers and there will be more control classes.

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INTRODUCING ALGEBRA WITH PROGRAMMABLE CALCULATORS.

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Abstract. *The research is focused on the role of programmable calculators as a tool to introduce the algebraic code. It was implemented in a class with 23 children (11-12 years old) who had no previous algebra instruction. Empirical evidence shows that calculators help to create a mathematical setting that allows children to acquire the algebraic language as a tool both, to negotiate problem solutions and to explore and justify generalizations. The report addresses issues concerning the theoretical approach and its influence on children's achievements.*

Introduction

This research is developed around the following conjecture: Computing devices which use as programming code a symbolic language like the algebraic symbolism, may be used to create an Algebra Acquisition Support System (in Bruner's sense). A system that helps children learn different uses of algebraic language, without previously having to learn its syntax rules and structure, just by using it, as one learned the mother tongue and most of its different uses (Cedillo, 1992).

In terms of this research, let us suppose that we have a sensible way to make children experience programming, so several questions arise. Whether the arithmetic background the children have before facing algebra is either constitutive of the language they are about to learn, or whether such knowledge even provides any clues to the aspiring learner about the formal structure of the new language? To which extent such experience may help children learn rudiments of algebra and to use them to explore and justify general number relationships? These questions served as guiding lines for this research. The way in which the learning environment was arranged is what I have called an *Algebra Acquisition Support System*. Its relation with the learning of natural language is discussed in the following section.

Theoretical background

I will first discuss the points that led me adopt the acquisition of mother tongue as theoretical framework. The calculator programming code offers two modes of representing a numerical relationship. One is the analytic representation of the programming expression. The other takes place when one runs a program, doing this for several values produces a table formed by inputs and outputs. These calculator features may be exploited to design tasks connecting arithmetic with algebra. For instance, once the children have grasped that a program consists of a series of arithmetic operations performed on a given value to produce an output, we can ask them to make a program so that it reproduces a given table.

In doing this, children are *using* the programming code as the language the calculator "understands". Here arithmetic plays the role of a clue-giving source. These tasks provide an environment within which the language used is so strongly attached to the content that it can permanently be checked by means of the content. This strong link between form and content constitutes as well a major feature in learning the mother tongue. For the intended learner, it would be even impossible to handle the linguistic form without content support.

The approach taken in this research is heavily influenced by the pragmatics facet of natural language acquisition. Pragmatics entails quite different processes from those involved in being master of a set of syntactic or semantic codes. Pragmatics may be seen as the study of how speech is used to accomplish social ends. Its elements "do not stand for anything: they *are* something" (Bruner, 1982, p.7). The view that language acquisition depends upon interaction as the clue-giving source has several variants. To this respect Bruner (1983) proposes that the adult plays a major role. The adult him/herself may arrange the environment and him/her encounters with the child in ways to *scaffold* language input and interaction to make it better fit the child's natural way of proceeding. These ideas roughly describe what Bruner calls a Language Acquisition Support System. Bruner's investigations led to the hypothesis that in order for the young child to be clued into the language, *he must first enter* into social relationships of a kind that function in the manner consonant with the uses of language in discourse. That relationships are called *formats*. A format is a rule-bound microcosm in which the adult and child do things with each other. Since formats pattern communicative interaction between infant and caretaker before lexico-gramatical speech begins, they are crucial vehicles in the passage from communication to language. Bruner's investigations suggest that formats "eventually migrates from their original situational moorings" and are generalized to new activities and settings (p. 121).

Method

Setting. A group with twenty three children (11-12 years old) took part. According to the Mexican Curriculum algebra is taught until the next course, so these children are not even supposed to have algebra instruction. Each child was given a calculator from the beginning of the course where the researcher played the role of the teacher throughout the course. The experimental stage took six weeks, three sessions per week (50 minutes each). The activities were delivered as worksheets trying to let children work at their own pace.

Subjects. Eight children were chosen as case study subjects according to their prior mathematical attainment. They were chosen as follows: (i) above average (one boy and one girl), (ii) average (two boys and two girls) and, (iii) bellow average (one boy and one girl).

Tasks. The programming code structure determines its content and the sequence tries to mirror the Bruner's concept of format. The tasks are grouped in six sets that I call "formats". Format 1 contains the "raw material" which in Formats 2-6 is further elaborated. Format 1 introduces the children to the use of expressions containing letters as a mathematical language that makes them "control the calculator so it do what they want." The activity is delivered as a game like "guess my rule" (Rojano and Sutherland, 1993). Its global structure is the following: Given a table (simulating the calculator screen) children are asked

?	
2	
	5
?	
5	
	11
?	
9	
	19

to (i) find out the rule and somehow express it, (ii) make a program that produces the same table and, (iii) fill in another table using their programs. All the proposed tables were generated by linear functions. The task's sequence is briefly described bellow.

Format 2	Children are asked to design items like the ones in Format 1. The tasks are aimed at making children think of a program before visualizing a number pattern.
Format 3	To find out two or more different expressions for the same rule.
Format 4	Given either a table or a programming expression, children are asked to find out both the direct and the inverse programs.
Format 5	Problem situations that involve whole-part relationships (e.g. to find out the box of maximum volume built up from a given squared piece of cardboard)
Format 6	Problem situations involving related variables (e.g. "all the windows in certain museum have these characteristics: the wooden frame costs $12 per meter and the with is 50 cm. less than four times the length ...¿how much ...?")

Data collection. The eight case study children were interviewed four times each. One prior to the study, two at the middle and one at the end. Other sources of data were everyday written children's work and notes wrote down ...er each session..

Results

Children's achievement was observed according to the following categories. (i) *Syntax*: how children come to make utterances that conform to algebraic syntax rules, (ii) *Semantics*: how children refer and mean when using algebraic symbolism, and (iii) *Pragmatics*: how children get what is requested or grasp deeper information about a problem situation by using the algebraic code.

Syntax. A remarkable outcome, I think, concerns the children's *acceptance* of priority of operations and the use of parenthesis and as *conventions*. That is, as constraints imposed by the formal code they were using. I will try to make this clear. The use of brackets appeared as a children's need when they were creating number patterns for a fellow to guess (Format 2). Several children asked questions like: "I want the calculator to add 1 first and then multiply it by 2 ... I typed A+1×2, but it does not work, Why?". I told them why and how to do it by using brackets. After a while a number of children were using brackets too. Later, dealing with inverting programs, they rarely used brackets to explain me what they did. Contrastingly, they always used brackets when programming the calculator. I asked them why. The following is a representative answer: "I do not think as a calculator does... neither you do ... so you can understand me ... if I want the calculator to understand me I must use parenthesis ... otherwise it makes a mess."

Even more, it seems that as the children gain further insight into the language as a codified system of representation, they come to operate not on concrete events, but upon possible combinations derived from operations on the language itself. The following episode illustrates this. The child was trying to invert the program **A×2−1**. She made two programs: **A÷2+0.5** and **(A+1)÷2**. When I asked why she said: "I found A÷2+0.5 because I new it was needed to divide by two, then I just adjusted the result adding 0.5. Then I saw that using brackets I could use the same numbers as the ones in the first program ... I thought it was better ... I did not need to adjust anything." From then on she used brackets to face similar items. I want to emphasize here how naturally the children made sense of conventions. For them using letters or considering the priority of operations to insert parenthesis are just means to *communicate* formally their own way of reasoning.

Semantics. Due to the nature of the activities the notion of variable was always present. Children showed from the very beginning they have grasped that a letter serves to represent "any number," that the choosing of a letter does not affect the essence of a program, etc. However, children's notion of variable and of the sign system in which it is embedded reaches a much higher level when they were "pushed" to *operate* on the programming expressions. It seems that it is the fusion of the dual nature of symbols in algebra (to represent and operate) that makes children gain awareness of the role of letters as variables and its power to deal with generality.

During an interview the child wanted to type the program A×4. Unwittingly he typed A×3. I asked him to correct without deleting what he had typed. His first attempt was A×3+1. He realized that "it works only for 1" (for A=1). "If I put 2 it only works for 2 ... If 3 ... for 3. Then he shyly said "may it be A×3+A? He then run the program and saw it was right. When I asked why he was so insecure he said "I thought it was not possible (it was the first time he dealt with this)... I already had 3×A... it changes as I put a number ... I needed to add a number but it changes too ... at once as 3×A ... so it must be another A. Later he proudly showed me programs he "had made shorter" (e.g. A×10001+B×1010+C×100 as a simpler form of A×10000+B×1000+C×100+B×10+A, a program to produce five digits palindromes). It suggests that with the development of a sign system a second feature is added: language can then operate intralinguistically in the sense that signs can point to or be related to other signs.

Pragmatics. The following questions were asked to the eight case-study children during interviews. **1**. What do you think about? A pupil from other class says that: **(a)** $A^2+B^2 = (A+B)^2$; **(b)** every time he sums two consecutive numbers he gets an odd number; **(c)** Observe the sequence 5, 9, 13, 17, if you continued putting numbers in that list, would you find the number 877? **2**. Choose any number between 1 and 10. Add it to 10 and write down the answer. Take the first number away from 10 and write down the answer. Add your two answers, may I guess your final answer? It is 20. Can you explain how could I know it?

These questions do not, or not explicitly, relate to number relationships arranged in a table. It is supposed that the only recourse the children can use to face the questions are the notions they may developed about variation and letters as computing symbols.

Question 1a: The eight children made sense of the expression and faced it using numerical examples. **Question 1b**: The eight children explain it by specific cases. When asked for a more general argument, six of them made a program (like B+B+1). Two from these six children used the expression to explain: e.g. "see, B+B gives always an even number ... then you add 1, it gives always an odd number". **Question 1c**: Only four of the eight children could give an answer. They did it by programming the calculator. One of these children first gave an answer by finding the pattern ("I saw that 5/4, 9/4, 13/4 leaves 1 as remainder ... 877/4 too). Then she made two programs. A×4+1 "to go ahead the sequence" and (A−1)÷4 "to find out the place a given number has in the sequence".

Question 2: The eight children explained verbally "the trick". Four (two distinct from the four children in question 1c) made a program to convince me: "you are adding nothing... see ... A+10 + 10–A (pointing at the A's) ... it will always give 20.

Final remarks.

This research draws a promising sight about the use of calculators but there are various aspects that deserve attention. Although the experimental tasks and the novelty of using calculators seemed to be motivating, the case is still far from being on the case of learning a natural language. Its acquisition process is characterized by a children's willing receipt, which, in general, is no the case of mathematics. Children's success strongly depend on their arithmetic background. Despite the computing support that calculators provide, children with a weak arithmetic could hardly face the proposed tasks. Finally, I think is interesting to investigate whether these children present or not the arithmetic-algebra dissociation found by D. Wheeler (1989) and new possible implications derived from an approach like this.

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'BLIND CALCULATORS', 'DENOTATION' OF ALGEBRA SYMBOLIC EXPRESSIONS, AND 'WRITE FALSE' INTERVIEWS.

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SUMMARY

Algebra takes its power from the ability to manipulate algebraic expressions without referring continuously to their meaning. But, this power is also the very source of the main difficulty to teach algebra: some students never refer to any meaning. We call them 'blind calculators'.

In the first part of this paper, we will propose a theoretical framework for a better understanding of these students' attitudes, based on the notion of 'denotation' (Frege, 1892).

In the second part, we will present a key to start up an actual dialogue on meaning of symbolic expressions with a blind calculator, called the 'Write False' interview, based on the "symptom prescription" method (Watzlawick, Beavin & Jackson, 1967).

BLIND CALCULATORS AND DENOTATION OF ALGEBRA SYMBOLIC EXPRESSIONS

As many authors (Bell, 1993; Boero, 1993; Drouhard, 1992b; Kieran, 1991...) noticed, algebra takes its power from the ability to manipulate ("transform") algebraic expressions without referring continuously to their meaning (Leibnitz: "blind calculation"). But, this power is also the very source of the main difficulty to teach algebra: some students never refer to any meaning and become 'blind calculators' forever. This phenomenon is also described by Linchevski & Sfard (1991) as "pseudostructuralism".

Most studies (see for instance Sutherland & Rojano, 1993) deal with the following problem: how to help students to avoid such a pitfall when (or just before) reaching algebra? By contrast, we focus our attention on remedial activities. In particular, we try to make 'blind calculators' aware of the meaning of the expressions they manipulate, and the role it plays.

We came up against a cumbersome problem, which we suppose is rather familiar to all those who try to teach algebra. Blind calculators are quite unconcerned by all you might say about the meaning of the expressions they produce. Once they have wrongly transformed an expression, its value does not remain the same; if that shocks you, it is your problem, not theirs!

For example, many students make the well-known mistake:

$$(a+b)^2=a^2+b^2$$

The point is that the argument: "if a is 2 and b is 3, $(a+b)^2$ is 25 whereas a^2+b^2 is 13" is rarely convincing. Some students reply "*so what?*" For them, algebra appears as a question of *rules*. They do not care whether $(a+b)^2$ may have the value 25 or 13 or anything else: values of expressions are not relevant criteria.

For some authors (e. g. Healy, 1993; Vergnaud, 1988), blind calculators do not bind together expressions and corresponding problem situations. Others (e. g. Gray, 1993 or Linchevski & Sfard, 1991) think that blind calculators have difficulties relating expressions and underlying concepts.

However, we think that these various approaches deal with the "so what?" attitude only in an indirect way. On the one hand, 'fluent algebraists' (Kirshner, 1987) do not need to refer to a *problem situation* whenever they have to perform a calculation. On the other hand, in the previous example $(a+b)^2$, the difficulty (for blind calculators) is not the*concept* of square itself, but rather that the *value* of this square has to remain the same when it is developed.

So, these observations have led us to focus our attention on the logical (neither situational nor conceptual) aspect of the meaning, in terms of the 'value' of expressions. In order to define this, we used the distinction Frege (1892) established in between *Sinn* ("Sense") and *Bedeutung* ("Denotation" or "Reference") (see also Arzarello, Bazzini & Chiappini, 1992). "*François Mitterrand*" and "*The President of the French Republic*" have the same denotation (*Bedeutung*): the real man whose name is *François Mitterrand*. On the other hand, these two phrases do not have the same sense (*Sinn*): the second phrase emphasises the official role of the man, while the first stresses his name. In algebra, Frege explaned that:

"2+3"	denotes a number:	5
"2+3=7"	denotes a Boolean:	false
"$2x+3$"	denotes a function f:	$f(x)=2x+3$
"$2x+3=7$"	denotes a Boolean function b:	$\begin{cases} b(x)=\text{true if } x=2 \\ b(x)=\text{false otherwise} \end{cases}$

A more detailed description of algebraic denotation may be found in Drouhard (1992a).

We claim that a student has to know that expressions denote, even if he is not able to express it; i. e. he must know that $2x+3$ has a value (his 'instantiation' in model theory) and that this value depends on the value of x . Moreover, he must know that formal transformations between formal expressions (as to transform $x+3$ into $3+x$) do not change their denotation. Without denotation, algebra is just a question of rules. Denotation is a keystone: it is the exact difference between raw symbolic computation (as computers do) and actual algebra.

Precisely, 'blind calculators' ignore that expressions are denoting, and a fortiori ignore that denotation remains when transforming. It is very difficult then to discuss with them. When the teacher disagrees with a transformation (because the denotation changes) they believe that he just prefers another rule. Even a numerical instantiation proposed by the teacher (e. g. $a=3$ etc.) is not relevant for them, for they ignore that transformations retain values. For them, there is a difference, not a contradiction ("You made a transformation and I made an other transformation: obviously the values are not the same, as we did not do the same thing!"). *Contradiction requires denotation.* Blind calculators ignore denotation, and teachers may ignore that blind calculators ignore it: it is a total misunderstanding.

WRITE FALSE INTERVIEWS

Background

Many students we interviewed were complaining of always writing wrong things in algebra. This suggested to us to use the "symptom prescription" method (Watzlawick, Beavin & Jackson,1967). A man, who suffered from total insomnia and claimed that no therapist could help him, was ordered not to sleep the next night ("in order to better observe the phenomenon"). Of course, the man could not stay awake! So, in algebra, we found that asking the student to 'always write false' was a good way to break the vicious circle.

We propose an expression as: $\frac{6a+3b}{2a+b}=\ldots$ or: $(a+b)^2=\ldots$ and ask him to write something which is always false. In the latter case, most students begin with something like: $(a+b)^2= a^2+3ab+b^2$.

In general, it is not *aways* false (as for blind calculators, "false" is synonymous of "incorrect"). We ask: "how do you know that it is false?", then "is it always false?" or "how do you know that?". Afterwards, an actual dialogue on 'false' and 'true' in algebra may start, as there is no formal rule available to produce always false statements. The only solution consists in dealing with the denotation of expression. So (for example) he may produce a solution equivalent to $a=a+1$.

On the contrary, 'traditional' interviews where a true answer is asked are ambiguous: it is difficult to know if a good answer is due to a student who is aware of the truth of the expression, or just to a blind calculator who fortunately used the correct transformation. Indeed, in algebra correct transformations lead to true expressions! But, you cannot obtain *always* false expressions just by changing correct transformations.

Example

Let us provide an example of a 'write false' interview, which was conducted in december 1993. The interviewer was one of us (letter "M"; her interventions are in **bold**) and the student is a 16-year old girl who was very aware of her low level in algebra. The interview begins with explanations about what will be done (setting of a 'contract').

17 M **I will give you something which is certainly familiar to you, I say you "write $(a+b)^2$"** [N Writes]. **Well, write "=" and write something wrong**

18 N Mm [she writes a^2+b+b]

19 M **Well, now tell us what makes it false for you**

20 N Well, that is $b+b$ doesn't make b^2, that makes $2b$

21 M **And how do you know that it makes $2b$?**

22 N [laughs] Just like that, I don't know, I learned it

23 M **You learned it where?**

24 N In middle school

One cannot just say that, for 'blind calculators', rules have no reason (cf. Linchevski & Sfard (1991): "Rules without reason and processes without objects"). They just have no *intrinsic* justification. Things are done so because it is the way they are to be done. Next, N. tries then to show 'with numbers' (i. e. particular examples) that b^2 is not $2b$. She chooses $b=2$ and then $b=3$, calculates and eventually says "I believe that this relation ($b^2 = 2b$) is always false for, er..."

99 M. **And what makes you believe that it is true for all numbers**

100 N. Ah all numbers, no, because, well, I think that, as it is an even number, maybe, I believed that maybe there were differencies between even and odd numbers

101 M. **Yes**

102 N. And, well, as here, it is an even and odd number, for both it is, it is not equal then. In fact, what one needs is, each time one needs proof by numbers, not by

103 M. one needs proof by numbers, what does it mean?

104 N. Because, well, when one see with letters, one cannot know very well, maybe it's false, but

105 M. What does it mean, one cannot know very well, what do you see there, when seeing letters?

106 N. When I see letters, that means nothing to me

107 M. That means nothing to you

108 N. No, I can't prove that it's equal to

109 M. Ah. When you see letters you can't prove...

110 N. No

111 M. ... that it's equal to

112 N. No. Not at all because, maybe, after, well, when one takes numbers

113 M. So, how do you do, when you are in classroom and you have letters and you have to prove that it's equal?

114 N. Well, I never thought [she laughs] to substitute with numbers!

It could be hazardous to trust N. when she assumes that she never thought to substitute letters with numbers. However, it is clear that, at the moment of the interview, she is far from the idea of denotation. Then **M.** asks N. to remember what did she did, the last time when she has to know whether two expressions were equal or not. N. gives (!):

$$(a+b)(a-b)=a^2+2ab+b^2$$

and later gives as an example of the latter formula: $(3x+2)(4x-8)$

197 N. Well then, I developed and I found...

198 M. But I, I see there $a+b$, $a-b$, so a, what would it be, for you?

199 N. a? ah, well, uh...

200 M. I don't know, uh...

201 N. Mm a that would be $3x$

202 M. Yes

203 N. and that one, would be $4x$

204 M. So the a, then its value is not the same

205 N. No

206 M. Its value is not the same number in the two parentheses

207 N. Ah no [pause 3 sec] no.

208 M. One do this, in maths?

209 N. Uh, at the moment?

210 M. No no. Does it happens, in maths, that one has *a* here, whith the value 3*x* and *a* there, whith the value 4*x* ?

211 N. I don't know, however, it's possible

212 M. It's possible

213 N. Yes

A letter may represent two different expressions in the same line: this may be regarded as an evidence for the lack of denotation. Later N. finds an equation and its solution, $x=2$. **M.** asks if there is a way of validate this solution. N. proposes to replace x by 2 in the equation, calculates and says "8-8=0, that's it!"

424 M. When you say "that's it"...

425 N. Uh, I see that, well, this relation, it's equal to 0, therefore there, it's equal to 0. In fact, I've never got the idea of replacing x by...

426 M. You've never got the idea...

427 N. No, never, never!

428 M. So there, you think that $x=2$, it's alright?

429 N. Yes

430 M. Yes, because if one replace here that give 0

431 N. [laughs]. It's the first time I see that!

Actually, one may suppose that N. was said this many times. However the point here is her shift of attitude towards the letters.

Conclusion

Write False Interviews are nothing but a universal remedy against algebraic miscomprehensions, neither the prescription of the symptom makes insomniacs sleep. However, these interviews are a powerful way to break the misunderstandings, lead students to be aware of the role of the denotation and then let them start an actual work on algebra.

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SCHOOL ALGEBRA. SYNTACTIC DIFFICULTIES IN THE OPERATIVITY WITH NEGATIVE NUMBERS

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ABSTRACT

In this article, we report results from a clinical study, carried out with 12-14 year old pupils, in which we analyse 25 protocols of individual interviews. The main purpose is to investigate the role played by the operativity and the different levels of conceptualization of negative numbers, in the resolution of algebraic equations as well as in the interpretation of algebraic expressions. The conflicts which arise with the elementary operations when numerical domain is extended from natural to whole numbers are also analysed. The results point out in the direction that such a numerical extension, during the processes of acquisition of algebraic language by secondary school children, constitutes a crucial element for achieving algebraic competence in the methods of resolution of problems and equations.

Introduction

Various studies on the teaching and learning of whole numbers coincide in recognizing the importance of these numbers in the comprehension of elementary symbolic algebra (Thompson & Dreyfus, 1988; Vergnaud, 1989; Freudenthal, 1983; Gallardo & Rojano, 1990). Furthermore the algebraic character of the historical origin of negative numbers has been pointed out (Freudenthal, 1973; Glaesser, 1981). This is also true of the close relationship between the operative and conceptual evolution of the latter and the evolution of algebraic methods for solving equations and problems (Gallardo, 1993). This suggests that symbolic algebra might be a significant context for the analysis of the difficulties manifested by secondary school students when they carry out mathematical tasks involving whole numbers.

After revising the research literature on whole numbers in the process of teaching and learning mathematics at secondary school level, we carried out, within the project *The Status of Negative Numbers in the Solution of Equations*[1],

[1] Ongoing research project (Department of Matemática Educativa of Centro de Investigación y de Estudios Avanzados, IPN, Mexico).

a historico- epistemological study of negative numbers in the solution of algebraic equations (Gallardo, 1993). At the same time we undertook parallel research in the field of algebra teaching (Gallardo & Rojano, 1990, 1993). These studies showed that the extension of the numerical domain from natural to whole numbers, during the process of acquisition of algebraic language by secondary school children, constitutes a crucial element for achieving algebraic competence in the solution of problems and equations. At the beginning, our research formulated the following questions: In equations and problems, which numerical domain do the secondary school students confer on the constitutive elements of the equation when they try to arrive at a solution? Which numerical domain do they accept for the solution? What is the relationship between the numerical domain assigned to the equation and the type of language associated with it? Which methods or strategies obstruct or lead to the evolution of the notion of negative numbers?

In order to carry out a clinical study to seek plausible responses to the former questions, we designed a questionnaire which was answered by 25 students aged 12-14 years. We also analyzed the protocols of individual clinical interviews which were video-recorded. The issues dealt with within the interview were: 1) numerical operativity; 2) resolution of linear equations; 3) resolution of word problems.

The results concerning aspects 1) and 3) have already been reported (Gallardo, 1994 and Gallardo & Rojano, 1993). Here we will focus on the results corresponding to aspect 2).

Some Results of the Clinical Study

It is important to point out that in both the historico-epistemological sphere and in the didactic sphere we found that negative numbers pass through different levels of conceptualization before becoming the formal mathematical notion of whole number. These levels are as follows: **subtrahend**; **signed number**; **relative or directed number**; as an **isolated number** two levels can be observed, the **result of an operation** or as the **solution to a problem or equation** . Finally there is the formal mathematical concept of **negative number** where the latter acquires the same status as positive numbers. (see Gallardo, 1994, for a full description of these levels). Furthermore, it can be seen that the consolidation of algebraic language is determined in a fundamental way by evolution towards more advanced levels of conceptualization of the negative number.

As antecedents of this part of the analysis, the interpretation attributed by the students to the notion of symmetric of a number, order of whole numbers and the use of numerical operativity are examined. Some results of the clinical study follow.

Symmetric of a number. Mechanisms inhibiting the construction of the symmetric are found because $-(+a)$, $-(-a)$, $a \in N$ are not recognized as numbers. Some subjects operate these expressions subtracting them from zero: $0-(+a) = -a$, $0-(-a) = a$. This allows "familiar" numbers to be obtained, a, $-a$, (in the interview protocol there are specific numbers in all items). Other students consider symmetry as "half and cut the number in two". Thus, the symmetric of **-2** will be **-1**. The subjects who recognize the symmetric of a number are those who spontaneously resort to the model of the number line and say that "the number line expresses symmetry". Moreover, they possess a fluid handling of the rule of signs $(-)(+)=-$

Order of the whole numbers. Pupils respond incorrectly when they associate the negative with the positive magnitude or use the model of goods and debts to justify their answers. On the other hand, they respond correctly when they use the number line model or its more concrete version, the temperature scale.

Numerical operativity. The majority of pupils frequently resort to the invention of rules which lead to both correct and incorrect results. Here are two erroneous cases:

Rule of too many signs. "This sign **[-a-(-b)]** is to take away, the other **[-a-(-b)]** is not required".

Multiple rule of signs. "Minus **[-a-(-b)]** with minus **[-a-(-b)]** plus and plus with minus **[-a-(-b)]** is minus".

Resolution of equations. In the sphere of algebra, the interview protocol consisted of algebraic expressions and linear equations.

Algebraic expressions. Open statements of the form **x+a-b** were presented to the students. The following situations were found:

1) They closed the expression x+a-b=c; 2) An arbitrary numerical value was assigned to x; 3) They decode the expression as the equation **x=a-b**; 4) Inhibitory mechanism: impossibility of equating **x+a-b** with "any expression" since this would be to invade the place of the "result". At most two simultaneous expressions are considered: **x+a-b; a-b=c**; 5) Inhibition of known operativity: "I can't add or subtract because I haven't got the result, nor do I know what 'x' is worth"; 6) Conjunction of dissimilar terms: **x+a-b=(a-b)x**
The students were also presented with expressions of the form **a-x-b**, and they proceeded in a form analogous to the above case **x+a-b**.

Linear Equations. As regards the solution of equations, the findings are grouped in relation to the nature of the solution and with respect to the nature of the coefficients.

Nature of the solution.

• In the equation **x+a=b; a>b**, the following was observed:

1) Inhibitory mechanism. "It can't be done. There is no number which when added to **a**, gives **b** [abbacus type reading]. This language inhibits the negative solution.

2) Inhibitory mechanism. Faced with a possible negative solution,

previously known methods of equation resolution are **not** brought into play: trial and error, inversion of operations, transposition of terms.

3) A scheme of quasi-equality is resorted to (carrying out operations ignoring the sign of equality) in order to obtain a positive solution: $x=a-b$.

4) The structure of the proposed equation, $x+a=b$, is altered and it becomes $x.a=b$ or $x-a=b$.

• In the equation $ax+b=cx+d;\ c>a$:

1) The solution $-x=e$ is reached which inhibits any action by the student, or else leads to the erroneous positive solution $x=e$. The student does not decode $-x=e$ as $-1x=e$.

2) Rule of Arithmetic Result: "You can't do it because you haven't got the result (the second member of the equation is not a "known number".

3) The polisemy of the x is presented (different values are assigned to the two occurrences of x).

• In the equation $2x+b=x+b$:

1) Inhibitory mechanisms when faced with the null solution. The reduced equation $1x=0$ leads to $x=1$. Thus, also, $2x=x$ leads to the solution $x=1$.

2) When $x=0$ is obtained, the student continues to seek another value since the null solution represents "the absence of value".

Nature of the coefficients.

• In the equation $a-x=b$, the following is found:

1) The unknown is taken as positive: "if x were negative, it should be written $a--x=b$".

2) The opposite situation is presented: the equation $a-x=b$ is transformed into $x-a=b$, "because $-x$ is a negative number".

• In the equation $ax-b=c$:

1) A partial inversion of the operations is carried out: $ax=c+b$. Division is not carried out, an abbacus type reading is used, "Which number multiplied

by **a** is equal to **c+b**?"

2) Defective operation of the unknown: **ax-b=c** gives **x(a-b)=c**.

3) Duality of the minus sign. Given the equation **ax-b=c**, the question arises "Is it a subtraction [**ax-b=c**] or is it a negative number?"

4) Place value reading: **ax** is understood as a number where the figure of the units is unknown.

Conclusions

1.- The pupils of the study spontaneously resort to the use of teaching models for negatives (number line, thermometer, goods and debts, etc) in order to justify their responses. This leads to errors in the interpretation of the symmetric of a number and the order of whole numbers.

2.- The triple nature of subtraction is found. The students with an advanced level of conceptualization of the negative number recognize the triple nature of subtraction (*completing, taking away, and the difference between two numbers*) and the triple nature of the minus sign (*binary, unary and the symmetric of a number*).

3.- The invention of rules arises in the operativity with negatives, both correct and incorrect.

4.- The domain of multiplication is found in additive situations.

5.- There are mechanisms of inhibition when faced with double signs [**-(+a), -(-a)**] and expressions which involve both negative numbers and letters.

6.- When faced with a possible negative or null solution, school methods of equations resolution are not used.

7.- Erroneous interpretations of the negative number are found. and negative or null solutions are not accepted [difficulty in accepting the negative or the zero as isolated numbers].

8.- The phenomenon known as blind algorithm of subtraction is found. It consists of the unthinking use of the syntactic rules which prevent the comprehension of the concepts.

This analysis shows the conflicts which arise with the elementary operations when the numerical domain is extended from natural to whole numbers. The majority of pupils tend to charge the syntactic expressions with meaning, in the same way as they will concentrate only on the syntax of the numerical relations when they work with concrete models. The dialectical interaction semantics-syntax is always present, although sometimes in an implicit form. It is worth mentioning the surprise shown by the subjects when they find that addition does not always make bigger; subtraction does not always make smaller; multiplication is not always repeated addition; and, that addition can be seen as subtraction and vice versa.

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A Constructivist Explanation of the Transition from Arithmetic to Algebra: Problem Solving in the Context of Linear Inequality

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Five case studies of college students solving a set of constructivist learning tasks provided the basis for the development of an explanation of the transition from arithmetic to algebra. This paper focuses on comparing the cognitive activities of the solvers of two of the case studies. One of these solvers completed a transition from arithmetic to algebra while the other was unable to do so. The case studies reveal that a successful transition requires that a student: (a) have a robust understanding of the concepts of variable and equality and, (b) must attain higher levels of reflective abstraction if they are to be able to reify their arithmetic processes into abstract objects that can be the focus of algebraic symbolism.

THEORETICAL FRAMEWORK

Filloy and Rojano (1984) defined the didactical cut to express how students' arithmetic experiences can interfere with their transition to algebra. In the theory of reification, Sfard (1991) and Sfard and Linchevski (1994) suggested an explanation for the problems students encounter as they attempt to develop mathematical concepts. They defined interiorization, condensation, and reification as stages in students' concept development. Interiorization was described as the stage where the learner performs operations on lower-level mathematical objects. During this stage, the learner is focused on the processes that he or she is involved with. As the learner becomes more familiar with performing these processes, he or she arrives at a point where he or she can think about what would happen without actually carrying out the process. The process is said to have been interiorized when the learner no longer has to perform the operation in order to think about the process. Condensation was described as the stage where a complicated process is condensed into a form that becomes easier to use and think about. Sfard (1991) explained the stage of condensation as being where a new concept is actually "born" (p. 19). It is not until the phase of reification that a student becomes able to conceive of a mathematical process as an object and becomes able to use this abstract object as an input for higher order processes. If at any point this developmental cycle is interrupted, the student may resort to activity that is no longer meaningful. Such activity is referred to as being pseudostructural (Sfard and Linchevski, 1994).

In a study that examined the role of reflective abstraction as a learning process, Cifarelli (1988) described several levels of reflective abstraction that may be used to specify the details of the theory of reification. These levels of reflective abstraction include: (a) recognition, (b) re-presentation, (c) structural abstraction and, (d) structural awareness. In relation to the present study, the level of recognition can be thought of as the ability to recognize that one can solve the current problem by doing again what one has done before. Solvers operating at this level would not be able to anticipate sources of difficulty. Solvers who are able to mentally run-through a solution and who can anticipate sources of difficulty when using previously developed methods are referred to as operating at the level of re-presentation. The next level, structural abstraction, is said to occur when a solver becomes able to mentally run-through a procedure and can reflect on potential, as well as prior activity. The highest level of reflective abstraction is structural awareness. At this level, the problem structure created by the solver has become an object of reflection. The solver would not have to conduct mental run-throughs of solution methods in order to make judgements concerning the solution of the problem.

Sfard and Linchevski's (1994) theory and Cifarelli's (1988) levels are closely related in several respects. First, these researchers agree with the Piagetian assumption that knowledge is rooted in the activity of the learner. Next, they acknowledge the role that mental imagery plays in the development of conceptual knowledge. Finally, Cifarelli's (1988) levels of recognition and re-presentation precisely describe the cognitive activities that are required to progress through the stage of interiorization. The level of reflective abstraction referred to as structural abstraction provides the bridge from the stage of condensation to the stage of reification. Structural awareness is the level of reflective abstraction that allows the solver to attain reification. In the present study, the levels of reflective abstraction defined by Cifarelli (1988) provided a means of illustrating the theory of reification in action as solvers attempt to make their transition from arithmetic to algebra. This theoretical framework provides a basis for analyzing the activity of solvers who make a transition to algebra as well as those who are unable to do so.

THE STUDY

Procedure

Five interviews were selected for the development of case studies that examined the transition from arithmetic to algebra and the solver's understanding of the concept of linear inequality. During an individual interview, the student solved a set of nine learning tasks, that for the expert, involved the concept of linear inequality. The students were allowed an unlimited amount of time to solve the problems and were not told that the problems involved linear inequality (See Table 1). The interview typically lasted two hours. The data included the videotapes, transcripts of the tapes, and the written work of the students. The analysis of the case studies fell into three categories: (a) students who used purely arithmetical methods that were not based on situation-specific imagery, (b) students who utilized charting methods that were based on situation-specific imagery and, (c) students who used formal algebraic methods.

Table 1
Some Examples of the Learning Tasks and a Characterization from an Expert's Perspective

Task 1: Horatio has decided that instead of purchasing a car, he wants to lease one. He is considering two cars. Horatio can lease a Mazda for two years for \$300 per month with no additional charge for mileage. He can lease a Toyota for the same period of time for \$200 per month, but there is a mileage charge of 20 cents per mile. How many miles would Horatio have to drive during the two years in order for the Mazda to be the best choice?

Task 1 & 2: $a < bx + c$ Note: Task 2 presents a similar situation

Task 3: You can rent a 15 foot moving truck from I-Haul rental for \$100 per day plus 10 cents per mile or you can rent a comparable truck from Spyder rental for \$75 per day plus 20 cents per mile. How many miles would you have to drive the truck for it to be cheaper to rent from I-Haul?

Task 3 & 4: $ax + b < cx + d$ Note: Task 4 presents a similar situation

Task 5: Similar problem with extraneous information
Task 6: Similar problem with insufficient information
Task 7: Problem describes a contradiction
Task 8: Student asked to write symbolic statements for the previous problems
Task 9: Student asked to write a problem similar to one of the tasks

The precedent for using a set of learning tasks such as the ones described in Table 1 to observe and analyze cognitive activity rests with the work of Yackel (1984) and the work of Cifarelli (1988). According to Cifarelli, working through carefully crafted learning tasks provides the solver with opportunities to reflect and reorganize his current conceptual understanding and to develop more powerful representations. The following section compares the cognitive activity of Solver #6, who used a charting method but was unable to make a transition from arithmetic to algebra, to the activity of Solver #12 who was able to make such a transition.

Results and Discussion

Solver #6 could not complete a transition to algebra because she was working with a statement such as (X • .20) + 75 as a process only. She was unable to conceive of it as a quantity that could be compared to another quantity. This incapacity to cope with the process/object duality as described by Sfard (1991) was a result of Solver #6 being unable to attain levels of reflective abstraction higher than re-presentation. In terms of Sfard's (1991) constructs, the solver remained at the level of interiorization. The level of structural abstraction was required to enter the phase of condensation and the solver was unable to reach this level. The solver needed to pass through the phases of condensation and reification in order to develop the ability to consider processes as objects. Furthermore, because the solver remained at the stage of interiorization, she held process-oriented conceptions of variable and equality.

Solver #6 found answers to all of the tasks by using arithmetical methods and charts that were grounded in situation-specific imagery. She provided algebraic representations, with difficulty, when she was asked to do so on Task 8. She wrote, X = [(300 • 24) - (200 • 24)] ÷ .20, as her symbolic representation for her work concerning the first task, referring to it as, "my algebraic!". In her representation for the second task which follows, she altered her use of the letter X, allowing it to represent the number of miles to be traveled.

S: X = Now I can't write an algebraic equation for this one. . . maybe. . . (X • .22). Well, maybe I can. (X • .22) . . . No, I can't, wait, no. . . (X • .22) + 189 =, O.K. there's my algebraic equation. Equals? Um, I don't know, I can't do that one.

While the solver used charts to solve some of the problems, because she was operating at lower levels of reflective abstraction, she was unable to think of the processes used to create the charts as abstract objects. The following episodes illustrate this:

Task 3

S: I'm comparing the two [costs for renting the vehicles] and I don't see how I could write an algebraic equation to compare them. Cause I mean, I'm sure I could write down the algebraic equation easily like, (X • .20) + 75 and then (X • .10) +100, but then I would have to compare them.

I: Well, what was the comparison? What was the question asking? What are you doing at each stage of the chart?

S: I have to compare and I don't know.

Task 4

S: (X • .10) + 28 and (X • .16) + 14, I can write it, I just can't use it!

These examples illustrate that for Solver #6, a phrase such as (X • .10) + 28, exists only as an arithmetic process. She was unable to think of it as the total cost of renting a truck and, lacking the ability to view this phrase as an abstract object, she literally had no thing that she could compare to something else. While she referred to (X • .20) + 75 and (X • .10) + 100 as an equation, she never wrote an equals symbol between them and her comments reveal that she did not consider doing so.

Solver #12 was able to complete a transition to algebra because he was able to attain the levels of structural abstraction and structural awareness as he solved the problems. The solver was able to solve the first two tasks by using an arithmetical method that was based on situation-specific imagery. On Task 3, the solver encountered mileages charges for both vehicles (See Table 1). This additional information caused the solver to create a chart to organize his arithmetic activity. The solver was able to realize higher levels of reflective abstraction by mathematizing his earlier activity. Achieving the level of structural abstraction allowed the solver to think of his arithmetical statements in terms of what they meant to him in the context of the situation-specific imagery he had used to solve the problems. This activity permitted his conceptions to undergo the condensation phase. Attaining the level of structural awareness allowed the solver to complete this phase and to think of these arithmetic processes as abstract objects that could be manipulated. This activity illustrated reification in action.

On Task 8, the solver was able to develop algebraic representations for the tasks by mathematizing what he had done to create the charts. Through his activity, he became able to think of an arithmetic process, such as $100 + (X \cdot .30)$, as an abstract object, such as the total cost of renting a truck, that could be symbolically compared to another abstract object. Developing this capability allowed the solver to write equations and inequalities to symbolize the comparisons he had made while using his charts.

During his work on Task 8, the solver developed algebraic representations for problems that he had solved earlier by using arithmetic or charting methods. For the first task, the solver wrote, $300 = 200 + (.20)X$. He wrote, $255 = 189 + (.22)X$ to represent the second task. Solver #12 carefully analyzed what he had done in the chart to solve Task 3 before he wrote his algebraic representation.

> S: Um, I think you could do it like, um, put the 100 per day, add that to your 10 cents on the mile, times X, and X is your number of miles, and that's going to have to equal your 75 per day plus 20 cents also times X.
> [Writes] $100 + (.10)X = 75 + (.20)X \quad X = \text{miles}$

While the solver wrote, X = miles, his verbalizations indicate that he is thinking of it as representing the number of miles. This demonstrates that a more robust understanding of the concept of variable is developing out of the solver's activity. After working the problem to check his solution, he decided that he wanted to change the equation to an inequality. In the two case studies where a transition to algebra occurred, the solvers first represented their work with an equality and only later translated it into an inequality. This results supports Sfard and Linchevski's (1994) observation that equality precedes inequality in conceptual development. Solver #12 changed the representation to $100 + .10X > 75 + .20X$. The reader will observe that the inequality symbol is reversed. In the transcript of the solver's charting, it was apparent that the solver was using the notion of finding a critical point and adding one. Thus, during his algebraic representation he bases the direction of the inequality on his prior activity. While this error is interesting, it does not detract from the solver's ability to make a transition to algebra.

CONCLUDING REMARKS

The results of all five case studies revealed that if a solver was to make a successful transition to algebra they needed to attain post-re-presentational levels of reflective abstraction. The first three solvers, who operated at the levels of recognition and re-presentation, typically held weak conceptions of variable and equality. A letter simply served to name a value they were looking for or it served to name an arithmetical process that they were describing. Solvers operating at these levels did not use equality sentences at all or used the equals symbol in an arithmetic sense– to announce the answer to a computation. These solvers were unable to conceive of arithmetic processes as objects and thus their transition to algebraic methods was blocked. The solver of the third case study, Solver #6, made some progress through the use of charting and became able to use a letter to represent a varying quantity, such as the number of miles traveled. She was able to describe the arithmetic process that she had used to create her charts by using algebraic symbolism, such as $(X \cdot .16) + 14$, but this phrase still represented a process to her. The solvers of the last two case studies were able to attain the levels of structural abstraction or structural awareness. These solvers held or were able to develop robust conceptions of variable and equality. Significantly, the attainment of the level of structural awareness allowed the solver to view mathematical statements such as $(X \cdot .10) + 25$ as both a process and an object. Being able to cope with the process/object duality (Sfard, 1994) made a transition to algebra possible for these solvers.

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Multi-Tasking Algebra Representation

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This study examined the skills of students in translating among four different representations of algebraic situations: words, tables, graphs, and equations. Results revealed a general weakness in translations involving words and equations. Also, experts demonstrated a greater facility in more complex translations.

INTRODUCTION

The new methodology of teaching and learning mathematics is based on understanding. In the past, many students "learned" mathematics by memorizing definitions and/or procedures. In lessons recommended by the *Standards*, students learn by actively participating in mathematical activities. They do mathematics in many different contexts and representations. They communicate mathematics by reading, writing, listening, and speaking mathematics. This includes representations that are described in words, as well as concrete, iconic and symbolic models.

This constructivist view of cognition, which is based on Piaget's theory of assimilation and accommodation, is the basis for constructivist teaching methods. Students must *experience* the concepts in order to internalize them and achieve understanding (Davis, Maher & Noddings, 1990). When the internal representation approximates the external representation and appropriate connections are made to other related material, then understanding has occurred (Hiebert & Carpenter, 1992).

Students are better able to "make sense" of a concept as they discuss the mathematics with peers and/or teachers (Corwin & Storeygard, 1992; Lodholz, 1990). The activity of having to form one's understanding into words forces metacognitive activity and, thus, improves thinking. Research studies have reported an increase in mathematical learning as a result of requiring students to share their thinking (Russell & Corwin, 1991). The recommended model is that students experience the mathematics and then seek understanding by discussion, including conjecturing, arguing, and justifying (Peterson & Knapp, 1993).

In traditional algebra courses, students learn to manipulate symbols by simplifying algebraic expressions and solving equations. This type of exercise has little or no connection to real world applications. Recent research and curriculum reform in mathematics education encourage us to make algebra more accessible to all students by making it more application-based. Contemporary algebra instruction includes integration of algebraic concepts in real-world contexts using tabular, graphical, symbolic, and verbal representations (Glatzer & Lappan, 1990). This multi-tasking approach means that students should be able to translate freely among multiple representations: words, table, equation, graph. For example, if a student is given an equation such as $3X = 18$, he or she should be able to describe a problem situation for which the equation would be used. Similarly, given a graph, the student should be able to write "the story of the graph", translating to words. In traditional courses, we ask students to translate FROM words, but the new emphasis is to have students demonstrate understanding by translating TO words from other representations. Wagner and Kieran (1989) identify problem representation in an algebraic system as a key feature in algebra learning.

METHODS

Subjects were eighteen college students (all of whom had completed at least one course in college calculus), and thirteen high school freshmen who had just completed the first semester of Algebra I.

Each participant was given the *Algebra Representation Assessment*, a twelve item open-ended pencil-and-paper test requiring the translation among the four modes. Each item was scored independently. No reliability data was computed, as this measure was essentially twelve separate tasks. Validity of tasks to measure the specified translations was verified by two math educators.

RESULTS AND CONCLUSIONS

Chi-square Tests of Goodness of Fit results revealed that overall five items of the twelve had a significant number of correct responses. Further, the translation task scores of the expert and novice groups were compared. The experts (college students) scored significantly higher on nine of the twelve translation tasks.

Table 1.
Correct Responses to Algebra Translation Tasks.

Task	Total Correct	Chi-Square
W-->T	29 (94%)	23.516 *
W-->G	18 (58%)	0.806
W-->E	14 (45%)	0.290
T-->W	25 (81%)	11.645 *
T-->G	30 (97%)	27.129 *
T-->E	15 (48%)	0.032
G-->W	27 (87%)	17.065 *
G-->T	27 (87%)	17.065 *
G-->E	9 (29%)	5.452
E-->W	10 (32%)	3.903
E-->T	21 (68%)	3.903
E-->G	20 (65%)	2.613

W= words, T= table, G= graph, E= equation

**Chi-Square* (1, $N = 31$), $p < .01$

Number correct was significantly greater than 50%

In examining the total group, there were six translation tasks involving words. A significant number of participants could translate only three of these: words to table, table to words, and graph to words. This indicates a deficiency in ability to relate graphs and equations to real world situations in words. The results of this study indicate that there may be a weakness in students' ability to connect the algebraic representation to the real world.

There were mixed results in translations involving tables and graphs. These representations may be considered intermediate stages in the translation of concrete, real life representations and abstract algebraic symbols.

Table 2.
Expert/Novice Differences in Responses to Algebra Translation Tasks.

Task	College Students	High School Students	Chi-Square
W-->T	17 (94%)	12 (92%)	0.057
W-->G	16 (89%)	2 (15%)	16.749 *
W-->E	14 (78%)	0 (0%)	18.438 *
T-->W	18 (100%)	7 (54%)	10.302 *
T-->G	17 (94%)	13 (100%)	0.746
T-->E	15 (83%)	0 (0%)	20.990 *
G-->W	17 (94%)	10 (77%)	2.062
G-->T	18 (100%)	9 (69%)	6.359 *
G-->E	9 (50%)	0 (0%)	9.159 *
E-->W	9 (50%)	1 (7%)	6.183 *
E-->T	17 (94%)	4 (31%)	14.006 *
E-->G	16 (89%)	4 (31%)	11.138 *

W= words, T=table, G=graph, E=equation,

**Chi-Square* (1, N = 31), $p < .01$,

Number of correct answers and group were related.

The students could translate tables to words and graphs and also could translate graphs to words and tables, but they were not able to translate either tables or graphs to equations. Similarly, they were not able to translate equations to any of the other representations. This is further indication of students' inability to connect the abstract algebra symbols to real world situations.

In comparing the experts and novices, the three tasks that were uniformly high were the simple translations of adjacent levels: words to tables, tables to graphs, and graphs to words. The experts were significantly better at all six tasks involving equations and on three other translations that were somewhat less logical: words to graphs (requires intermediate step of table or equation), tables to words (requires backward thinking), and graphs to tables (again, requires backward thinking). In short, the expert group demonstrated skills indicative of a deeper understanding of the underlying concepts represented.

IMPLICATIONS

In conventional algebra classes, the curriculum consists of pencil-and-paper manipulations with algebraic expressions, equations, and graphs and little application to real world situations. In the *Standards* ' recommendations, connection to the real world is a key part of the algebra curriculum. Given the weaknesses identified in this study, teachers are encouraged to provide experiences for students that connect the algebraic representation in equation form to the real world in words. These connections should be communicated in various forms. Algebra students should actively experience mathematical concepts and be able to listen, speak, read and write the mathematics.

The mixed results involving tables and graphs are consistent with the stages of representation suggested by Bruner (1977). The iconic, or intermediate, stage is a table and/or a graph, and this is a transition between the concrete real-world situation and the abstract algebraic representation. In the constructivist classroom, students actively participate in experiences that assist them in constructing their own concepts of these representations and the transitions between them. The emphasis in algebra classes should be on understanding and communicating these translations.

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QUANTITATIVE MENTAL MODELS OF LINEAR FUNCTIONS

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A core unresolved issue in situated and constructivist theories of learning is the nature of the relationship between physical and symbolic understanding. To explore this issue, I compared how students coordinated their physical and symbolic activities in the contexts of several learning environments. A physical device, a computer simulation of the physical device, and a computer-based numerical representation were used to examine reasoning and learning about linear functions. The physical device is a pair of winches that consists of blocks that are pulled by strings winding around spools as a handle is turned. The position reached by a block after some number of turns can be represented by y in the formula $y = b + mx$, where b is the starting position, m is the size of the spool, and x is the number of turns. The computer simulation involves a simulation of the winches that is linked to a table of numerical values and a pair of equations. The computer-based numerical device is analogous to the winch device but operates like a pair of "function machines." My expectation was that the physical device and computer simulation would ground understanding of symbolic representations of function in students' implicit understanding of the functional relations among physical quantities. Therefore, I predicted that students using the physical device and computer simulation would be more likely than the number machine students to integrate their quantitative and symbolic reasoning, and that their resulting understanding would generalize to other situations. Twelve pairs of seventh graders were videotaped using one of the devices while learning about variables, linear functions, and equations. Analyses of the videotapes confirm my predictions that the physical device and computer simulation are environments in which students understand the quantitative structure of the devices, and that students' quantitative models can provide a foundation for the use and manipulation of equations.

Climbing the Incline of the Slope Concept

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Theoretical Rationale

The concept of slope is fundamental to understanding the study of calculus. Students are introduced to the idea of slope in their algebra classes in a restricted domain, that of linear functions, and from this limited exposure often develop misconceptions which later impact their construction of calculus concepts. This study investigated students conceptualizations of slope, and also investigated the components of the slope in terms of van Hiele level characterizations.

Method

Students ranging from those studying Algebra I through graduate students and mathematics instructors at junior colleges were asked to draw concept maps of their idea of slope. The differences and similarities in the concept maps were examined to find "common cognitive paths", as well as the identification of misconceptions which would impair understanding. Additionally, levels of behavior have been formulated according to the van Hiele characterizations.

Results

The van Hiele Levels of Student Behaviors for slope are described. At the previsualization level of slope, the student can: use words such as "rise over run" (without meaning); show the slope between two points graphically; calculate the slope between two given points. At the visualization level, students can: recognize that "slope" refers to the steepness; see similarity between horizontal and vertical lines (no slant); identity the slope in the equation y=mx+b (but not graph the line); relate "rise over run" to $\Delta y/\Delta x$. Students at the analysis level can: recognize positive and negative slopes from graphs; connect graphical and algebraic ideas of slope; recognize that the slope of a vertical line is undefined. Students at the abstraction level can: recognize that the graphical representation of slope can be misleading (because of scale); relate slope to rate of change; connect procedures of finding maxima and minima to the concept of a slope of zero. Students at the deduction level can relate the concept of slope to vectors.

THE DISTRIBUTIVE LAW IN ARITHMETIC AND ALGEBRA AND THE COGNITIVE PROCESSES OF COLLEGE LEVEL REMEDIAL STUDENTS

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RESEARCH GOAL: TO DETERMINE THE NATURE OF COGNITIVE SKILLS NECESSARY FOR THE MASTERY OF THE DISTRIBUTIVE LAW (DL) IN ARITHMETIC AND ALGEBRA. THE HYPOTHESIS WAS DERIVED FROM THE COMPARISON OF MENTAL OPERATIONS INVOLVED IN APPLYING DL WITH THE SKILLS DISPLAYED DURING SUCCESSFUL PERFORMANCE ON THE 2-DIM PIAGETIAN CLASSIFICATION TASK (BLUE, YELLOW, AND RED CIRCLES, TRIANGLES AND SQUARES WITH THE INSTRUCTION: PUT TOGETHER THOSE WHICH ARE ALIKE). THE HYPOTHESIS WAS TESTED BY COMPARING THE PERFORMANCE OF 29 REMEDIAL STUDENTS AT HOSTOS CC ON DISTRIBUTIVE TASKS WITH THEIR PERFORMANCE ON THE CLASSIFICATION TASK. **HYPOTHESIS:** THE NECESSARY COGNITIVE SKILL NEEDED FOR THE SUCCESSFUL PERFORMANCE ON DISTRIBUTIVE TASKS IS <u>THE ABILITY TO RECOGNIZE AND TO HOLD-ON-TO A PARTICULAR SIMILARITY CRITERION IN THE 2 DIMENSIONAL CONTEXT</u> OF THE CLASSIFICATION TASK.

DISTRIBUTIVE TASK #1 (ARITHMETIC): A SUBJECT WAS GIVEN TWO RECTANGLES WITH CONGRUENT WIDTHS OF 5 UNITS AND DIFFERENT LENGTHS OF 5 & 7 UNITS AND ASKED TO COMPUTE THE TOTAL AREA. THE RECOGNITION OF TWO WAYS TO FIND THE AREA: 5*5+5*7 & 5*12 WAS COUNTED AS THE SUCCESSFUL ANSWER.

DISTRIBUTIVE TASK #2 (ARITHMETIC): A SUBJECT WAS GIVEN A COLLECTION OF 15 CUBES WITH NUMBERS 2, 3 OR 4 PRINTED ON THEM (FIVE CUBES OF EACH DIGIT) AND ASKED TO FIND THE SUM OF ALL DIGITS. THE RECOGNITION OF TWO COMPUTATION TECHNIQUES: 5*2+5*3+5*4 AND 5*(2+3+4) WAS COUNTED A SUCCESS.

DISTRIBUTIVE TASK #3 (ALGEBRA): THE SUBJECTS WERE GIVEN A PEN AND PAPER TASK CONSISTING OF 3 CSMS PROBLEMS TESTING THEIR KNOWLEDGE OF DL.

RESULTS: THE HYPOTHESIS WAS FULLY CONFIRMED IN THE ARITHMETIC CASE; ALL SEVEN SUBJECTS WHO WERE SUCCESSFUL ON TASKS 1 & 2 DISPLAYED THE HYPOTHESIZED COGNITIVE SKILL. THE RESULT IN THE ALGEBRAIC CASE WAS WEAKER: ONLY TWO STUDENTS DISPLAYED THE HYPOTHESIZED COGNITIVE SKILL AMONGST THE THREE WHO WERE FULLY SUCCESSFUL ON TASK #3 .

FOURTH GRADERS INVENT WAYS OF COMPUTING AVERAGES

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Fourth graders who were not taught the conventional algorithm for getting the average were encouraged to invent their own procedures. Below are some of the methods they invented.

<u>The Average of 40 and 100</u> (two scores)

1. Half of 40 + half of 100 = 20 + 50 = 70

2. 100 - 40 = 60, 60 ÷ 2 = 30, and 40 + 30 = 70

3. 40 + 10 = 50, 100 - 10 = 90; 50 + 10 = 60, 90 - 10 = 80;

   60 + 10 = 70, 80 - 10 = <u>70</u> (the average)

<u>The Average of 2, 9, 3, and 6</u> (four scores)

The midpoint between 2 and 6 is 4, and the midpoint between 9 and 3 is 6. Since the midpoint between 4 and 6 is 5, the average is 5.

	2	
4	9	6
	3	
	6	

<u>The Average of 150, 125, and 200</u> (three scores)

1. Starting with the median (150), equalizing the three scores, and then distributing the 25 left-over points.

   150 + 0 = 150 (then add 8)
   125 + 25 = 150 (then add 8)
   200 - <u>50</u> = 150 (then add 8)
   25 left over
   <u>158 with 1 left over</u>

2. Starting with the lowest score (125) and distributing all the points above it. 150 - 125 = 25, 200 - 125 = 75, and 25 + 75 = 100. Since 100 ÷ 3 = 33 r.1, the average is 125 + 33 (158) with a remainder of 1.

3. Guess-and-checking by starting with an estimate of 160, equalizing the three scores, and realizing a shortage of 5. The child then took 2 from each score of 160, got the average of 158, and had 1 more than the 5 he needed.

   150 + 10 = 160 (then take 2 away)
   125 + 35 = 160 (then take 2 away)
   200 - <u>40</u> = 160 (then take 2 away)
   5 short
   <u>158 with 1 left over</u>

SEE AND SAY: AN ANALYSIS OF VERBAL AND VISUAL REPRESENTATION IN FIRST YEAR ALGEBRA STUDENTS

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Purpose: The purpose of this study was to determine if algebra students connect external meanings, both visual and verbal, to algebraic expression and to investigate the relationship of gender and success in algebra to the ability to represent expressions verbally and visually.

Procedures: Subjects were all students enrolled in first year algebra in a mid-sized suburban high school. The subjects were identified only by age and gender. Each subject was provided a set of ten algebraic expressions. For the first five expressions, subjects were asked to draw a picture, graph, or diagram that represented each expression. For the second five expressions, subjects were asked to describe a real world situation for each expression.

Analysis: The responses were scored for appropriateness by two algebra teachers. Each subject was given a score for verbal and visual representation. The visual and verbal components were compared using a t-test. Analysis of Variance was also performed to determine the effects of gender and success in the demonstration of verbal or visual representation.

Findings: An analysis of the data suggests that students tend to correctly represent algebraic expressions in a verbal modality more often than in a visual modality. There are some differences according to gender. Success in algebra as determined by grade received does not seem to be tied to ability to represent algebraic expressions in either mode.

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MISCONCEPTIONS OF VARIABLE

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This Poster presentation shows initial results of a research project in course of development in México, by staff members of the the Department of Mathematical Education CINVESTAV and the Department of Mathematics of ITAM.

The project is concerned with the difficulties beginning College students have, in using an interpreting different realizations of variable,namely variable as specific unkwon, variable as general number and variable in a functional relationship.

The poster will display results of a diagnostic questionnaire applied to beginning College students, and sample items whose corresponding answers show striking similarities with the misconceptions of pre-algebraic students and algebra beginners when dealing with variable.

A further step in the research project, will investigate how difficulties to deal with different realizations of variable are affected by instruction.

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A.C.T. IN ALGEBRA

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In fulfillment of the University of Northern Colorado's dedication to curriculum enhancement, we have participated in the development of the ACT in Algebra project. This innovative approach to teaching college algebra was generated by Dr. Robert Mayes to focus the course upon Applications, Concept, and Technology.

Applications The view of mathematics as a stagnant body of facts prevails among college freshmen. The ACT in Algebra project attempts to diminish this view by motivating the mathematics with real world applications. Each topic is introduced in a "need to know" manner, motivated by interesting applications that are as true to life as possible. In addition to driving the curriculum, applications are also studied from a data analysis perspective. Students are taught the techniques *finite differences* and *linearization* to derive modeling functions for real world data.

Concept ACT in Algebra emphasizes the formation of mathematical concepts over manipulation skills. The text is specifically designed to aid in this endeavor by engaging students through the use of embedded exercises. Students are expected to pre-read the text and discuss it with their teammate before whole class discussion. Many topics of study also feature in-class team explorations to further build concepts. Although traditional exercises are assigned, greater emphasis is placed upon the completion of non-trivial, conceptually-based team projects.

Technology Computer laboratory explorations utilizing the DERIVE computer algebra system occur weekly and allow the analytic, graphic, and tabular investigation of both applications and theoretical mathematics. Students work with teammates during computer labs and submit joint lab reports, several of which are written, requiring students to organize their findings and draw conclusions.

STUDENTS INTERPRETATION OF VARIATION PROBLEMS IN GRAPHICAL SETTING

MARÍA TRIGUEROS

ITAM

It has been shown that students have difficulties to deal with the concept of function, particularly when it is presented in a graphical context. This problem becomes more apparent when they face situations that involve variation. A study was designed to explore how students deal with variation problems in a graphical setting in order to analyze their interpretations and to relate their difficulties with their conceptions. We worked during three semesters with a group of mathematics major students taking a course in differential equations. The sizes of the groups were 34, 36 and 31 students. Cooperative learning was employed. We analyzed group's reports, homework's, evaluations and the results of a small interview with each of the students all dealing with first order differential equations and first order autonomous systems of differential equations. It was found that before instruction started students had difficulties relating tangent fields with local variation: There was a generalized tendency to consider different curves in the same graph as representing solutions of different equations. We related this behaviour with a lack of understanding of the Fundamental Theorem of Calculus. They know it from previous courses but when faced with new situation they fail to use it. The analysis of local variation and its relationship with tangent fields proved to be a difficult task even for students that had experience with graphical manipulation from previous courses. It seems that the abilities developed while working with functions cannot be easily transferred when one level of abstraction is added. These problems seem to be related with the students having a process conception of function and derivative. In dealing with systems of equations students had difficulties with the interpretation of parametric representation and the meaning of the phase space. Difficulties persisted even after instruction. Students that don't have a strong concept of function and of derivative as an object cannot overcome the difficulties. In the presentation the equations used and some examples of student responses will be shown, to benefit from comments of the participants.

Assessment

ASSESSING STUDENT RESPONSES TO PERFORMANCE ASSESSMENT TASKS

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The purpose of the study was to describe what happens when open-ended, real-world problems in the form of performance assessment tasks are used as a part of mathematics instruction in middle school mathematics. This paper focuses on the teacher's interpretation of students' responses to two tasks. The relationship between the teacher's interpretations of student responses to real-world problems and her expectations about what students might or should do with respect to using such problems as a part of instruction will be discussed.

Introduction

The reform efforts in mathematics education have called for the use of performance assessment tasks as a new source of information to assist teachers in making informed decisions about students' mathematical knowledge and understanding. As teachers begin to use performance assessment tasks based on open-ended, real-world problem situations in their classrooms, there is a need to understand how teachers use these tasks as a source of information about students. This paper will focus on describing a teacher's interpretations of student responses to such problems and three factors that seemed to influence the teacher's construction of knowledge about students.

As students construct mathematical knowledge, teachers collect information and construct knowledge about students. Multiple sources and methods for collecting information about students' mathematical knowledge provide a clearer picture of what students can do and understand than a single source or method (NCTM, 1993). Even though teachers may collect information about students from many sources, they cannot possibly notice everything that happens in their classroom.

Teachers' schemata, knowledge and beliefs about mathematics, and the teaching and learning of mathematics, seem to influence instructional and assessment practices (Anderson-Levitt, 1984; Fennema and Franke, 1992; Thompson, 1992). Specifically, teachers' views about using performance assessment tasks in the form of open-ended, real-world problems may influence expectations and interpretations of student responses. Anderson-Levitt (1984) proposes a model of teacher thinking that suggests teachers' teaching schemata (ideas about teaching, learning and assessing), social schemata (ideas about appropriate communication), and related individualized theories of students influence the information teachers gather and interpret about students. This model will serve as a framework to discuss how a teacher's ideas about using performance assessment tasks based on open-ended, real-world problems interact with the interpretations of student responses.

Method

Participants. A seventh grade mathematics teacher, Ms. Turner (not her real name), was selected because of her interest in using open-ended, real-world problems as a part of instruction, and her involvement in the Teacher Enhancement Partnership[1] project. One of the goals of the project is to support teachers' implementation of real-world problem situations. The school is located in an urban-rural school district with about 40% of the student population including African Americans, Hispanics, Asians, and other minorities. The class reflected the diverse population of the school and included 28 seventh grade students from the pool of general mathematics students not selected for pre-algebra, as well as several mainstreamed special education students.

Performance Assessment Tasks. The teacher chose the problems from several activities available for field testing from the *PACKETS® Program: Performance Assessment for Middle School Mathematics* [2] developed by Educational Testing Service (Katims et al., 1995). Each problem activity is based on a newspaper article that sets the context for a model-eliciting problem (Lesh and Lamon, 1992). Ms. Turner chose the CD Toss problem to accompany a short unit on probability, and the Million Dollar Getaway problem as part of an interdisciplinary unit on the Holocaust where the mathematics classes focused on the magnitude of large numbers and particularly the size of one million.

The CD Toss problem involves designing a game board for a school fund raising event. Players toss a CD onto a grid, trying not to land on any lines. The problem also includes making decisions about the price of throwing two sizes of compact discs with 5 inch and 3.5 inch diameters. A letter is requested by the carnival planning committee that describes the game board, and justifies why it would be a successful game for the fund raising event.

The Million Dollar Getaway problem activity involves analyzing the situation where a report is made about a person who robbed a bank and apparently walked away with about one million dollars in a large leather bag, where the majority of the cash is one, five, and ten dollar bills. A report explaining the plausibility of this situation is requested by a local news editor who plans to talk about the robbery on the evening news (Katims et al., 1993).

Data Collection and Analysis. Sources of data included interviews with Ms. Turner and classroom observations. Ms Turner's class was observed a total of 29 days, where 12 of the days were spent implementing the real-world problem situations. For each problem, one day was spent introducing the problem by reading the accompanying newspaper article and discussing the context of the situation, 3 to 4 days involved working in groups,[3] and 1 to 2 days involved group presentations and some discussion. The data were examined for emerging patterns and themes using the process of decontextualization and recontextualization (Tesch, 1990).

Results

Expectations related to reasons for choosing to use real-world problems

Ms. Turner chose to use the two real-world problem activities for specific reasons that included (1) making connections between the mathematics involved in the problems and real-world events in the lives of her students, (2) developing mathematical reasoning, and (3) ensuring the inclusion of several topics in the curriculum such as probability, area, and volume.

Real-world connections. The CD Toss problem was viewed by Ms. Turner as an activity that might motivate her students to participate in the school's first Math Fair since one of the other mathematics teachers had planned to have a booth at the fair with a similar game. Consequently she scheduled time for the students to work on the problem during the same week as the Math Fair. Later in the spring, when Ms. Turner's team (including the Social Studies, Science, and Language Arts teachers) was planning an interdisciplinary unit about the Holocaust, Ms. Turner thought the Million Dollar Getaway problem would "... fit in absolutely perfect!" to help the students understand the magnitude of the large number of people who were involved.

Ms. Turner was interested in trying new ways to teach mathematics by involving her students as much as possible (rather than a sit and absorb approach), and thought it was important that the context of the problems were such that students would be able to relate to the situations. Because she hoped the context of the problems would connect with students' understanding of events in the real-world outside of school, she expected that they would be interested, motivated, and able to at least make an attempt to construct a solution to each problem.

Mathematical reasoning. Since both problems allowed for more than a single approach or answer, Ms. Turner saw an opportunity for her students to be exposed to situations that would allow them to begin to develop mathematical reasoning skills. The implementation of the problem activities afforded potential experience in reasoning, justifying, and explaining both orally through presentations by student groups to the rest of the class of what they had done or were trying to do, and through written products in the form of a letter or report as requested by the respective client in each problem. While students were working in their groups, Ms. Turner communicated her expectations through comments such as the following: "... but one of the things you're going to have to do is justify your answer, this is why we did it this way ..."; "I wanna know **why**, the **reasoning** behind it ..."; and "as long as you can justify mathematically that would be a fine answer." During the students' first round of presentations on the CD Toss problem, Ms. Turner provided feedback such as: "Excuse me, you're supposed to be justifying this!" when a group responded "I don't know" to the question "Why does the smaller [disc] cost less?" asked by another student in the class. To another group Ms. Turner responded "I would still like you to make some prediction as to how many times people would win, based on mathematics not based on 'I think' or 'maybe'."

While Ms. Turner seemed to indicate that learning to reason, explain, and justify solutions to problems in mathematics is important and worth the time it takes to do extended projects such as these problems, she realized that it was difficult for these students who were unaccustomed to this type of activity. In fact, she indicated her students generally resisted having to write or explain what they did and that "they seem to be ditto worksheet oriented, short problems, short answer ... they don't want to do anything, they don't want to put the time into it, and they see no reason to go through all that, a quick answer, a quick job ...." In spite of the struggle Ms. Turner felt as she encouraged the students to write explanations and justify their solutions, she still indicated that the experience was important for the students particularly because the state testing was being changed to include performance assessment tasks requiring written explanations.

Mathematics content. Ms. Turner chose to do the CD Toss problem within a short unit on probability since she viewed probability as the main mathematical concept involved in this problem. Ms. Turner decided to use this unit and the CD Toss problem activity as a way to extend the previous unit on fractions, but within a new context--a context that she thought would be more interesting and meaningful to the students. As she introduced the CD Toss problem to the students she commented "Remember the probability we've just been working on, and see if there's a reason that [I] chose to go over some probability with you before we did this." The unit on probability included performing experiments pulling colored chips out of bags, rolling two dice, recording experimental probabilities, and calculating theoretical probabilities of the same events.

Since the CD Toss problem was the first activity of this kind that Ms. Turner had implemented, knowing her students did not have much experience with solving real-world problems, and thinking that an analysis of the problem using ratios to compare winning area to losing area on the game board might be difficult for the students, she was not sure what to expect. After the students gave their first round of presentations where only three of the seven groups reported the chances of winning (based on their experience of playing with cut out paper discs and their constructed game board, an approach perhaps suggested by their previous classroom activities in the probability unit), Ms. Turner decided to discuss how they might be able to predict the number of winners. After showing the students an example of how to calculate a ratio comparing winning area to losing area, she said to the students "... now see if working something out like that will help you with the probabilities here." In a second (and final) round of presentations after another class period for making "revisions" with explicit direction from Ms. Turner to try the approach she had demonstrated, only one group successfully explained how they used the approach to analyze their game board.

Ms. Turner thought that the Million Dollar Getaway problem was more straight-forward, and would be easier for the students to understand and construct reasonable solutions involving a combination of bills that sum to a million dollars, some statement about the volume (dimensions) or weight of the money, and a mathematical justification of their solution. Although she had not yet taught a unit including area and volume, Ms. Turner scheduled this problem activity during the part of the school year when she normally covered those topics. She felt that she was running out of time (it was near the end of the school year), and thought that doing this problem would provide the opportunity to discuss these topics, as well as the more general topic of measurement.

During the time that Ms. Turner observed students working in groups, she frequently asked individual students or groups such questions as "How big would the suitcase have to be?"

and "Could [the robber] carry it?" During the first round of presentations, all of the groups reported how they were trying to decide how much room the money would take up and Ms. Turner responded "... so what are we missing? In addition to this we have to be concerned with what? How heavy the thing would weigh." Although she had originally expected the students to make some statement about volume **or** weight, she seemed to indicate to the students that their final solutions should include something about both volume **and** weight.

Information gathering and interpretations of information gathered

The implementation of these two problem activities provided opportunities to gather information from class discussions, observations of group work, listening to oral presentations, and looking at final student products. Ms. Turner seemed to gather most of her information from the students' presentations and final products. Part of the explanation might be that she was absent during 3 of the 7 days while groups were working; however, even when Ms. Turner was present during other days that groups were working, her interaction with the groups was minimal. Ms. Turner did not take notes on her observations of students except during the presentations when she sat in the back of the room and recorded some of the content of the presentations. Ms. Turner also had a system of recording checks, pluses, or minuses for student effort or participation which she used during the presentations and class discussions. Before the students presented in front of the class, Ms. Turner indicated her concern and expectation that all group members should participate by saying "We'd like to see what you've worked on. We'd like you to present it, and we'd like everyone in your group to have something to say, ok? Just to show us that you all worked on it."

Having something concrete such as a student product to justify a certain grade, at least on the surface, seems to be less subjective than a "gut feeling" from observations; however, Ms. Turner seemed to have definite theories about her students and what they were capable of based on information from grades and her system for documenting effort and participation throughout the school year. Ms. Turner used her observation time to confirm what she thought she already knew about her students as indicated by her statement "... at this point in the year, their grades are pretty much, I mean, I know what everybody is capable of ... who would do it right and who would do it wrong, who wouldn't do it at all ...."

Although Ms. Turner only implemented two problems in the last half of the school year, it was possible to notice a shift in Ms. Turner's information gathering. During the second problem, not only was she present during most of the group work time, she interacted more with the groups (i.e. listening to explanations and asking questions) rather than using the time to sit at her desk doing paper work. Consequently she was able to talk more confidently about the different approaches used by the student groups and participation levels of certain students.

As a final evaluation of using both problems, Ms. Turner thought that her students put a lot of effort and time into both problems. However, in terms of what the students "got out of it," she commented for the first problem "I don't see that they got much out of it at all, at least not from what they presented ... they're interested in a nice diagram and a pretty picture ... very little mathematics ... there was very little change, very few corrections made" and for the second problem "... they have the general idea ... at least they figured out how many stacks make a million ... the size of one million was there... I think most of them had answers that made sense whether they were right or wrong, at least they had a reason." Although Ms. Turner would have liked the students to be more careful with their calculations (referring to "right or wrong"), she was encouraged that they had made some progress from the first problem to the second problem in their efforts of attempting to explain and justify what they had done.

Conclusion

Teachers' attention to gathering information about students' mathematical knowledge and understanding may be influenced by what they consider appropriate sources and evidence based on established assessment practices. Ms. Turner's assessment practices initially emphasized information from final products and presentations. After the first problem generated what she considered poor quality presentations and as Ms. Turner became more involved in observing students during the second problem, she seemed to realize that the final products and presentations did not capture the full extent of the mathematical ideas, reasoning, tools, and approaches used by her students. Ms. Turners' experience with using these performance assessment tasks seemed to provide an opportunity to recognize the limitations of gathering information from a single source as well as the opportunity to gather information from several sources.

Teachers usually have specific reasons for choosing to use problem activities in instruction. Certainly it is appropriate for teachers to have an agenda for accomplishing objectives of the curriculum and ways of assessing whether those objectives have been achieved. However, gathering and interpreting information about students' mathematical knowledge and understanding is not a trivial task. Since most teachers are used to evaluating student knowledge from some culminating event or product rather than assessment integrated with instruction, it will take time and experience to change their assessment practices. Ms. Turner has taken a first step.

Notes

1. This study was partially supported by the Teacher Enhancement Partnership project, funded by NSF (grant number TPE9155307).

2. It should be noted that Ms. Turner used only the newspaper article, readiness questions, and Focus Project (model-eliciting problem) from each of the two activity units; current versions of these activity units include additional components that support classroom implementation.

3. It should be noted that the authors of the PACKETS® Program greatly value teacher observations of students during the time that students are working in groups as a part of the total assessment process. However, Ms. Turner was unable to be present during two of the four days of group work for the CD Toss problem and one of the three days of group work for the Million Dollar Getaway problem.

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Multi-faceted Inferences from an Interview Assessment

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Task-based interviews involving a problem with two conditions were conducted with students in Grades 7 and 10. The original focus of the project was on students' mathematical problem solving and their use of various strategies and approaches to the problem, but the data that were collected have also been analyzed from the point of view of the values, beliefs and expectations that students held concerning mathematics, problems, and problem solving strategies. The extent to which the students engaged in "number grabbing" gives an indication of their disposition towards sense-making . Students' views on the appropriateness of various strategies was also apparent.

The work presented and discussed in this paper is part of a small-scale qualitative evaluation of students' mathematical problem solving (Schroeder, 1992) that involved the use of a series of non-routine problems in interviews with individuals and pairs of students. This paper goes beyond the project's original focus on problem solving to explore additional facets of the students' performance and to demonstrate how evidence gained in this kind of interview assessment can be used to make inferences about students' values, beliefs, and expectations concerning mathematics, problems, and problem-solving strategies.

The importance of these aspects of students' performance is underscored by the National Council of Teachers of Mathematics (NCTM) Evaluation Standard on "mathematical disposition" (NCTM, 1989, pp. 233-236) which is stated in terms of confidence, flexibility, willingness to persevere, interest, curiosity, inventiveness, and so on, but which is also explained in terms of the beliefs and conceptions which students develop through their experiences in mathematics classes and which in turn influence their future mathematical development. Schoenfeld's review (1992, pp. 358-359) of the extensive and expanding body of research on students' beliefs about mathematics has shown important interactions among problem-solving, metacognition, and sense-making in mathematics.

Procedures

The problem "A farmer has some pigs and some chickens. He finds that together these animals have 19 heads and 60 legs. How many pigs and how many chickens does he have?" was presented to 41 volunteers, 20 females and 21 males, 35 in Grade 7 and 6 in Grade 10. Twenty-six of the Grade 7 students were interviewed in pairs, the remaining nine individually; all six of the tenth graders were interviewed individually. In the introduction to the interviews students were informed that the activity was not a test and that their performance would not affect their standing in their school mathematics course. They were told that the interviewer was most interested in how they went about solving the problem, and that they could ask questions of the interviewer at any time. Students were urged to think aloud as they solved the problem so that the interviewer could understand how they worked on the problem, and they were told that the interviewer might ask them questions for clarification or give them hints or help if they wished.

As the students worked on the problem, the interviewer observed them closely and made field notes. At the conclusion of each interview the interviewer collected the students' written work, annotated it, and also completed a record sheet designed to be a convenient and standardized way of summarizing and reporting students' work on the problem. The sections of the record sheet headed "Understanding," "Strategy Selection," and "Monitoring" reflect Polya's (1945) description of the phases of problem solving and the importance of metacognitive activity in problem solving; they list a number of anticipated features of students' work which can be checked off as appropriate, and they provide spaces in which to describe the students' work and to note any comments they made as they worked and the answers they gave to the interviewer's questions. In the final section headed "Overview" the interviewer is to indicate whether the students solved the problem essentially on their own, or solved the problem with needed help from the interviewer, or did not solve the problem even with help from the interviewer. The amounts of time spent reaching a solution, extending the problem or looking back, and using different approaches are also to be noted.

In preparing to use this problem, consideration was given to the variety of ways in which students could approach and would be likely to approach and solve the problem. Indeed, an important reason for selecting this problem was that it can be solved by both seventh and tenth grade students in a variety of different ways. The different approaches that can be used in solving this problem include pictorial approaches, guess-and-test approaches of several types, and algebraic approaches. A pictorial approach might involve the use of drawings or diagrams to represent the problem, for example stick-figures to represent the chickens and pigs or groupings of tally marks, some containing four strokes and others with two strokes. By representing the bodies or heads first and adding the legs later, the problem solver can reach a solution quite directly, especially if each animal is initially given two legs and the remaining legs are added two at a time, essentially changing chickens into pigs. Guess-and-test approaches can range from just a series of "random" guesses, to fairly sophisticated strategies for choosing later guesses on the basis of previous guesses. For example, the student may start by considering 15 pigs and 4 chickens, and note that the number of legs on 15 pigs and 4 chickens is more than 60. Then the student may "trade in" one or more pigs for the same number of chickens, reducing the number of legs while leaving the number of heads unchanged. A perceptive user of this process might even reason that there with 15 pigs and 4 chickens there are 8 too many legs, so that 8 legs must be removed from 4 of the pigs yielding the correct solution 11 pigs and 8 chickens. Without capitalizing on this insight, the problem solver may solve the problem by repeating the process of trading pigs for chickens or vice versa as many times as needed. Making a table can also help the problem solver select the successive guesses systematically.

Algebraic approaches can also lead to a solution. For example, if x = the number of chickens and y = the number of pigs, then the given fact that there are 19 heads is represented by the equation $x + y = 19$, and the fact that there are 60 legs is represented by $2x + 4y = 60$. This system of equations can be solved by expressing one variable in terms of the other and substituting, or by multiplying one equation by a constant and adding or subtracting it with the other. Writing and solving an equation in one variable is similar to the substitution method for solving two equations in two unknowns.

Results

In the discussion which follows, the unit of analysis is the interview. Because of the small numbers, results are not reported separately for males and females nor for students interviewed individually or in pairs. However, the results for Grade 7 students are considered separately from the results for Grade 10 students because of the substantial differences in their respective mathematics backgrounds, which influenced not only their problem-solving strategies and mathematical approaches, but also certain of their beliefs and expectations. The overall results for Grade 7 were that in 16 of the interviews (72%) the students solved the problem on their own, and that in 6 interviews (28%) the students solved the problem with help from the interviewer; in none of the interviews did the students fail to solve the problem. Five of the six Grade 10 students solved the problem on their own; the remaining student solved the problem with help from the interviewer. The total length of time spent in each interview varied from 5 to 35 minutes; the time to solution ranged from 3 to 29 minutes.

In both grades by far the most widely used strategy for reaching a solution was guess-and-test. Guess-and-test was used in all of the Grade 7 interviews, but in eight of these interviews (44%) a pictorial approach was used together with or instead of a verbal or numerical form of this strategy. In two interviews (9%) the students chose on their own to keep track of their guesses in a clear table or list, and in several other interviews the students did so when the interviewer suggested it. In half of the Grade 10 interviews guess-and-test was the initial strategy that led to a solution; in the other interviews the initial solution was algebraic. All three Grade 10 students who initially used algebra showed they could use guess-and-test, which they referred to as "the long way," when they were asked by the interviewer for a different way of solving the problem, but two of the three tenth graders who initially used guess-and-test were unsuccessful using algebra to solve the problem when it was suggested that they do so. Interestingly, the average time to initial solution was no greater for those students whose initial solution was by using guess and test, than for those students whose initial solution was algebraic; in fact several students' solutions by guess and test were faster than any student's algebraic solution.

In about half of the interviews with students in each grade the students used the pseudo-strategy which has been called "number grabbing" (Szetela, 1991, p. 197), that is, using the numerical values given or implied in the problem with arithmetic operations to get an answer, without regard to whether it makes sense to carry out those operations using those numbers. Several students began working on the problem by dividing the two given numbers, 60 and 19, and some of these students tried to use the result, 3.15 (the average number of legs per animal), as they proceeded with a guess-and-test strategy, but others ignored this result and attempted other meaningless computations. For example, one student divided 60 by 4 and 19 by 2 and answered that there were 15 pigs and 9.5 chickens. In a particularly blatant instance of number grabbing, another student added 60 and 19, divided this sum by 2 to find the number of chickens (38.5), and divided this sum by 4 to find the number of pigs (19.75).

Discussion

The prevalence of "number grabbing" as an initial approach to this problem suggests that many students have the expectation that the verbal problems to be solved in mathematics classes are simple ones in which the numeric data given in the problem will be used with a few arithmetic computations to find the answer directly. While some students relatively quickly concluded that this was not the case with this particular problem, some other students were quite persistent in trying computations that they thought might work without bothering to consider whether those computations were sensible ones. In these interviews the extent to which the students continued in the "number grabbing" mode gives a good indication of the extent to which their mathematical disposition inclined them to sense-making and checking of results.

Virtually all of the Grade 7 students interviewed seemed to believe guess-and-test was a legitimate and appropriate strategy for solving this problem. One seventh grader, however, wrote nothing on paper but seemed to be concentrating intensely on unspoken mental computations. After about three minutes of effort, the interviewer asked him whether he wanted to write anything down, but he declined. A short while later the interviewer asked if he could explain what he was thinking, but the student's only reply was to give the correct solution to the problem and a justification of that answer in terms of the number of animals and the number of legs. This student, unlike most of the other seventh graders, seemed embarrassed that he did not know the answer immediately, and unwilling to admit that he used guess-and-test as his solution strategy.

An interesting incident that sheds light on students' views about what approaches and strategies are appropriate and valid ones occurred during the period of data collection. It happened that two siblings in different grades and different schools were both interviewed on this problem a few days apart. The seventh grader later reported to us that she had told her older brother about her interview on the problem and her solution of it by guess and test. She reported further that her brother had told her that "it was OK for her to solve the problem that way," but that when she got to tenth grade she would "have to use algebra to solve it, because algebra was a better method." This comment may well reflect what students have been told by their algebra teachers, but if time-to-solution is the criterion for a "better" method, then this viewpoint is not supported by the data that were collected.

The priimary focus of the interviewers' observations was on the students' problem solving, their choice of strategies, and use of mathematical processes. At the same time, howver, their behavior can be analyzed in terms of the beliefs and dispositions that are reflected in it. In the development of alternative assessments in mathematics, interview tasks such as this one have potential to give insight into several dimensions of students' mathematical power and their mathematical disposition.

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ASSESSING STUDENTS' MATHEMATICAL ACTIVITY IN THE CONTEXT OF DESIGN PROJECTS

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Research in cognition and learning has pointed out the need for closing the gap between learning mathematics in and out of school (Carraher, Carraher, and Schliemann, 1985; D'Ambrosio, 1991; Lave, 1988; Saxe, 1991). This perspective redefines what mathematics is and extends mathematical activity to include more than using rote algorithms. Following this perspective, current curriculum guidelines and standards for mathematics call for engaging students in "real world" mathematics rather than mathematics in isolation of its applications. However, it is not clear how application projects will change students' activity or how these projects will affect assessment practices.

The Middle-school Mathematics through Applications Project (MMAP) is designing curriculum materials in line with these standards and investigating students' activity when using these materials in the classroom. In this alternative learning environment students explore mathematical concepts in the context of design projects. One of the 4-6 week units, Antarctica, puts middle school students in the role of designers who are creating a research station for a scientific expedition to Antarctica. This unit guides the students, working mostly in small groups, through the design and analysis process. Tools include ArchiTech, IRL-designed software that allows students to create floor plans and analyze information on their station's heating and building costs.

This poster presents the preliminary analysis of research undertaken in one MMAP classroom using videotapes of classroom interactions. The analysis examines the concept of "authentic assessment" as a way to clarify the goals and motivations for classroom assessment practices. Two senses of authentic assessment are explored from within two theoretical frameworks, ethnomathematics (D'Ambrosio, 1985) and the didactical contract (Brousseau, 1981). The paper also addresses several questions encountered in the design and investigation of assessment practices: How does a design project affect classroom assessment practices? How are the assessment needs different than in a traditional classroom or in a work setting? What is a reasonable focus for assessment practices: the design process, uses of mathematical tools, use of mathematical argumentation, or other areas?

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A MODEL FOR ASSESSING CHILDREN'S MATHEMATICAL THINKING IN CLASSROOMS WHERE REFORM IS TAKING PLACE

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The constructivist position maintains that 1) mathematical learning involves the active manipulation of meanings; 2) individuals learn by building understanding and knowledge through acting on objects (which may be mental objects) and interacting in a variety of social contexts; 3) activities that emphasize inquiry and exploration help students develop greater conceptual knowledge; and 4) learning mathematics for understanding requires that students experience mathematics as a subject that can be understood. A constructivist perspective places emphasis on learning from experience, for teachers as well as their students. With this in mind, reform movements such as the New Jersey Statewide Systemic Initiative, have been initiated to go beyond traditional inservice, to develop in teachers a high degree of mathematical and pedagogical competence, focusing on students' thinking and reasoning as they actively construct mathematical ideas.

As we begin to implement reform, it is important to develop appropriate assessment models to analyze their effects on children's mathematical thinking. Beyond merely examining standardized test scores, we believe that performance must be examined according to a number of cognitive and metacognitive dimensions. These include: 1) processes by which the students build their solutions to a problem; 2) students' use of heuristics; 3) models they build; 4) language used to communicate solutions; 5) the nature of the representations that are constructed; 6) the ability of the students to be metacognitive; 7) their ability to generate descriptions, explanations, and predictions for other mathematical problems; 8) their ability to reflect on their own problem-solving capabilities; and 9) the richness and depth of their solutions.

We have developed a model for assessing the impact of a long-term teacher development partnership between an urban school district and Rutgers University. The model involves a comparison of students taught for a period of three years by teachers considered successful in incorporating the philosophies and perspectives of the project with other students taught by non-project teachers. The problem-solving behaviors of both groups of children were carefully observed and videotaped during in-class activities in which they and their classmates were given authentic mathematical problems to solve. This experience was followed by two task-based interviews conducted by nationally and internationally recognized experts in the fields of mathematics and mathematics education. Profiles of the children were developed and comparisons are being made according to a list of dimensions indicative of higher-order mathematical thinking.

The results of this assessment study should provide important implications for studying future programs designed to impact instruction, in particular for initiatives for systemic reform. The model for assessing student learning will be shared in this session offering, a forum to consider the multi-dimensional aspects of this research.

Cognitive Modalities

VISUALIZATION IN MATHEMATICS: SPATIAL REASONING SKILL AND GENDER DIFFERENCES

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Summary: *The relationship of spatial and mathematical abilities has been the subject of both speculation and empirical investigation. This work reports on a meta-analysis of correlations of spatial and mathematical tasks, with particular attention to the possibility that spatial ability is responsible for gender differences in mathematical performance. Early research indicates that overall correlations of tests of these abilities are not high. The research reported here focused on areas in which mathematical gender differences appear. Female space-math correlations were higher than males' on college entrance examinations. Studies were separated according to academic level of sample: the more select the sample, the larger the difference in correlations.*

Introduction

This work summarizes correlational evidence on a proposed explanation for gender differences in mathematics. The explanation was suggested by Julia Sherman (1967), who speculated that the socialization of females towards verbal and away from spatial activities might be at the root of their inferior performance in many cognitive areas, including mathematics. Small gender differences are still sometimes found on spatial and mathematical tests, particularly in high school age youth, though these are decreasing (see, e.g., Friedman, 1989; Hilton, 1985).

The relationship of mathematical and spatial reasoning had intrigued researchers well before Sherman made her conjecture. In the 1960's, several psychometricians proposed that spatial ability, or another even more fundamental trait producing it, enabled those who possess it to reason more effectively (e.g., Smith, 1964; Witkin et al., 1962). Smith particularly singled out mathematical reasoning as requiring spatial ability: others have supported the same view more recently (e.g., Battista, 1994; Burnett, Lane, and Dratt, 1979). A large body of research has developed on the relationship of spatial and mathematical skills.

A good portion of this research deals with mathematical gender differences. Schonberger (1976) and Tartre (1990) have reported evidence which is somewhat inconsistent with Sherman's conjecture: Schonberger found no evidence that gender differences in mathematical problem solving were a result of differential spatial skill. Tartre (1990) found that spatial skill did not contribute to better mathematical performance in males. However, she did conclude that spatial ability was better related to mathematical performance for females. Many researchers before Tartre had suggested that the relationship of spatial and mathematical skills appeared to be gender-specific: several early researchers had found different factor structures for females and males; later work in correlations found differences (e.g., Weiner, 1984).

To give an integrated picture of the research, Friedman (in press) used meta-analysis to combine correlations of spatial and mathematical tasks. She combined zero-order correlations, as these are the most frequently reported by those doing empirical investigations on the topic. She found that, overall, space-math correlations were not high. However, if space-math correlations calculated separately by gender were to show different patterns, particularly in the mathematical areas in which gender differences appear, or if there were direct gender differences in correlations, spatial skill might still be their source.

This work reports on comparisons of correlations calculated separately by gender. Study variables such as age and selectivity of sample, type of spatial or mathematical test used, and year of publication were explored. Gender patterns of verbal-math and space-math correlations were contrasted, to search for evidence of differences in clusters of skills.

Neither mathematical nor spatial skills are considered unitary today. With regard to mathematical tasks, a relatively complex categorization of tests into computational, conceptual, and problem-solving and mixed categories was first undertaken. However, this division yielded no more information than a simpler categorization into computational or reasoning tasks.

Spatial skills ordinarily designated "orientation" and "visualization" were targeted, as these are most often considered reasoning processes. Researchers vary widely in their description of these constructs: however, they generally all involve mental transformations of objects or parts of objects. Most commonly, orientation is taken to mean an holistic imagining of simple rigid motions of whole objects. Visualization, is then taken to involve multi-step reasoning about parts of objects. Putting the pieces of a tangram together to form a certain shape is a typical two-dimensional visualization task. Recognizing the shape that would be formed by folding a thin sheet of material along given lines is a typical three-dimensional visualization task. These descriptions of orientation and visualization, articulated by Michael, Guilford, Fruchter and Zimmerman (1957), are used here. In addition, following Schonberger (1976), we divide spatial tasks by dimension. Tasks not conforming to these descriptions of orientation or visualization, such as map-reading or Gestalt Completion exercises, were not considered.

Method

Seventy-five studies reporting gender-specific information on correlations between mathematical and spatial skills were gathered using searches of bibliographies of books and journals known to the author and searches of three computerized data bases, ERIC, PSYCHINFO, and Dissertations Online. Criteria for selection of studies primarily involved the specific spatial skills they tested.

The meta-analytic techniques used were based on those developed by Hedges and Olkin (1985) for combining correlations from several studies, testing for homogeneity, and fitting random effects and linear models. Two general types of analyses were carried out. The first was a direct comparison of male and female correlations: when both were reported in a study, differences of Fisher *z*- transforms of the correlations were calculated. These differences were then combined, and the average mean difference reported. The second type of analysis involved contrasting different types of correlations from the same samples. This latter type of analysis was more complex, as correlations of correlations must be considered.

Results

Verbal-math versus space-math correlations. For both genders, verbal-math correlations were almost always numerically higher than space-math ones; when a statistically significant difference was found, it favored verbal-math correlations. Samples were divided by age: when the average age of the sample was less than 14 years, the sample was considered "younger." Statistically significant differences were more often found in combined correlations of older samples than younger ones. In younger samples, the difference between verbal-math and space-math correlations had a tendency to be larger for females than males.

Area of mathematical test. Small gender differences in achievement favoring males are often found on geometrical and problem-solving tests. College entrance examinations also show gender differences in the same direction. No gender differences in space-math *correlations* were found when the mathematical test was geometrical or of the type explicitly designated "problem solving". Correlations were small in these areas. The SAT-M, however, did produce relatively substantial correlations with spatial skills, and these correlations did show a gender difference: females' space-SAT-M correlations were higher than males. This led to an investigation of selectivity of sample.

Selectivity of sample. In more selected (elite high school or college) and highly selected (young gifted or elite college) samples, females' space-math correlations were higher than were males'. This was not true in unselected samples. The table below gives these results.

TABLE. Gender differences in space-math *z*- transforms for students at varying age levels in samples divided according to selectivity.

	Mean *r*	Number of Studies	Mean Difference	CI
		Average samples		
Females	.44	26	-0.01	(-0.03,0.02)
Males	.44			
		Moderately selected samples		
Females	.39	14	0.07	(0.01,0.13)
Males	.31			
		Highly selected samples		
Females	.46	7	0.15	(0.02,0.28)
Males	.35			

NOTE: The mean *r* is given for the groups of correlations to illustrate their level: the mean difference is not the difference between these two numbers or their *z*-transforms, but the mean of the differences computed in studies; CI = 95% confidence interval for the mean difference.

Implications

The first result indicates that verbal skills are better related to mathematical skills than are spatial skills throughout the school curriculum. It suggests that verbal skills may be more related to mathematical skills for young females than they are for males: However, there are no gender differences in space-math correlations for these students. As the verbal-math relationship is stronger than the space-math one, it is unlikely that emphasizing spatial reasoning will have much effect on test scores in mathematics as mathematics is tested today. The overall low spatial correlations also suggest this. We cannot predict from this evidence, however, what would happen should the mathematics curriculum change or be tested differently.

The second result indicates that gender differences in correlations do not appear in the types of mathematical tasks which ordinarily show small gender differences in achievement. College entrance examinations show gender differences in correlations, but these tests are not of a special kind of mathematical content: they differ from other mathematical tests primarily in the subjects who take them. Students taking these exams are academically select. The third result substantiates the second: non-verbal skills cluster together more frequently in females than males in selected samples. Age is a factor in these samples: they are all of junior high school age or older. Because gender differences in correlations do not appear in younger samples or as function of the mathematical task, environmental and socialization factors are more likely to explain this difference than are cognitive processes. Students must become interested in non-verbal reasoning and convinced of its value at an early age: encouragement in these directions may be particularly beneficial for females.

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NEGATIVE CONSEQUENCES OF ROTE INSTRUCTION FOR MEANINGFUL LEARNING

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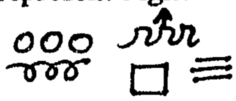
This research investigated the negative consequence of rote (relationship-poor) learning preceding meaningful (relationship-rich) learning, a classroom sequence frequently resulting from conflicting administrative and professional directives. Two similarly designed studies were conducted: a generic and a mathematics specific. In both studies Group 1 was assisted in rote then meaningful learning, whereas Group 2 received only the briefer meaningful instructional unit. On post-tests Group 1 scored significantly less well than did Group 2 suggesting an interference had been developed due to prior rote learning.

Although terms and definitions are variously given, there is a long-established distinction in the literature between meaningful and rote teaching-and-learning (Ausubel, 1963; Brownell, 1935; Bruner, 1960; Gagne, 1985; McLelland & Dewey, 1895; Skemp, 1987; Thorndike, 1922). A common theme in this literature is that meaningful indicates a richness of relationships (within the mathematical sphere or as extension to external domains); rote indicates a relative absence of relationships (Hiebert, 1986).

This distinction is especially important in view of ongoing tensions in the field. The professional mathematics education community strongly advocates models of meaningful learning and teaching (NCTM, 1989). Administrative branches are more closely wedded to the sorts of easily measured gains that rote learning effects (Baker, 1990; Brockett, 1992). The conscientious teacher wishing to explore the practices advised by the professional leadership, but also to heed legislative and administrative directives, will be tempted to adopt a two-track policy: invest some instructional time in relational teaching; but maintain instrumentalist practices to assure students' test-readiness (Romberg, Wilson, Khaketla, & Chavarria, 1992). Underlying this compromise is an additive model of learning: If the total teaching time, T, is apportioned as R and I to relational and instrumentalist teaching, respectively, then the total student learning L(T) should be equal to L(R) + L(I). Indeed more optimistic educators may even hope that the two approaches to teaching will mutually reinforce each other, producing more than just the sum of the parts.

This paper calls into question the viability of this two-track compromise by challenging the underlying additive model. Our study investigated the interaction of these teaching modalities by providing one group of students with an instrumentalist (rote) curriculum followed by a briefer relational (meaning-rich) curriculum. A second group of students was provided only with the relational instruction. This basic design was implemented in two separate investigations using two separate contents: a generic content stemming from a picture grammar devised by Skemp (1962); and a mathematical content involving perimeter and area of simple figures. Previous studies have investigated such sequencing effects (Whitman, 1976; Kieran, 1984; Hiebert & Wearne, 1988; Mack, 1990); however, none was set up for the specific interference hypothesis studied here, so none controlled the rote treatment rigorously.

Design of the Studies

Generic Content Study (49 eighth graders): Skemp's (1962) picture grammar employs pictographs to represent concepts. For instance ○ represents container,  represents dries, ≡ represents apparatus, □ represents cloth,  represents electricity, ○○○ represents heats, etc. These pictographs can be clustered to represent higher level concepts. For instance, □ ○ represents clothes hamper, and  represents clothes dryer, etc.

In the rote treatment, Group 1 was given a class period to memorize six symbol clusters and their translations. Various instrumentalist teaching techniques were used to facilitate memorization including repetition, flash cards, and extrinsic reward. In the meaningful treatment students in Group 1 and Group 2 were instructed on the meanings of the individual symbols and shown how the meaning of a symbol cluster is related to its discrete symbols. Group 2 received only the single period of meaningful instruction.

Mathematics Content Study (99 fifth graders): The mathematics content involved perimeter and area of simple geometric shapes (rectangles, squares, triangles, and parallelograms). In the rote treatment, Group I received five days of instruction to memorize and apply the formulas for area and perimeter of the simple shapes. Instrumentalist strategies including small group practice, flash cards, and repetitive drills were used to aid students in memorizing the formulas as isolated pieces of knowledge. In the meaningful treatment, Groups 1 and 2 were actively involved for three days with a variety of manipulative materials (geoboards, square tiles, grid paper) to assist them in constructing a relationship-rich knowledge of area and perimeter, and in developing methods for calculating area and perimeter measures of simple figures. Group 2 received only the three days of meaningful instruction.

Analysis and Results:

Because its content was entirely unfamiliar to all subjects, the first study used a posttest/retention test design. The second study used a pretest/posttest/retention test design. For both studies the critical dependent variable was the students' comprehension of the material including their ability to transfer their learning to new items and tasks. The results for both studies (See Tables 1 and 2) substantiated the unproductive consequences that can develop from the two track approach to learning. The students who were given initial exposure to rote methods prior to meaningful instruction (Group 1) scored significantly less well than their counterparts exposed to the meaningful instruction only (Group 2).

Table 1

Mean Results for Transfer Items in Generic Content Study

Analysis	Posttest*	Retention Test**
Treatment		
Group 1	38.29	21.24
Group 2	63.96	35.89

* $F_{1,41}=10.03$, $p=.0029$

** $F_{1,41}= 4.87$, $p=.0329$

Table 2

Mean Results for Mathematics Content Study (Pretest Used as Covariate)

Analysis	Pretest	Posttest*	Retention Test**
Treatment			
Group 1	9.80	14.31	13.68
Group 2	10.69	16.35	15.13

* $F_{1,85}=4.78$, $p=.0315$

** $F_{1,85}=4.37$, $p=.0396$

Learning Interference

Videotaped interviews with selected students in the second (mathematics content) study were used to gain more insight into the mechanisms of learning interference. Several students who received both instructional modes felt they learned more during their rote session. Somehow the experience of memorizing material and being able to regurgitate it equated to learning for these students. Perhaps just this expectation of what learning should be like in schools, contributed to the interference effect.

Students receiving the initial rote instruction repeatedly confused area and perimeter in transfer problems - an effect almost entirely absent for students experiencing only meaningful instruction. For instance a problem about the surface area of the wall of a room was interpreted as a perimeter problem because of the enclosing aspect of walls. Or to determine the amount of lumber needed to build the walls of a dog house, students said one needs to know the perimeter "because the wall goes around."

Group 1 also tended to overgeneralize. They applied area and perimeter to liquid measure and length, and otherwise were more concerned about what mathematical operations they were using, than why they were using them.

Generally, the memorization of formulas to obtain area and perimeter measures seemed to inhibit the free, open-ended, creative explorations of ideas and materials observed in students receiving only meaningful instruction. For example, a student in the meaningful-only group explained that she could get the area of the room by taking "those books and start putting them out (on the floor) and (counting) how many books I put out." Skemp (1987) lists the ability to adapt to new tasks a consequence of meaningful learning. This kind of creative thinking was not observed in students receiving prior rote instruction.

Interference resulting from initial rote learning can be understood in terms of Piaget's (1967/1977) notions of disequilibrium. Rote learning sets up superficial associations related to solution procedures. These may conflict with subsequent meaningful instruction. In such cases, either prior structures remain, thus making new relationships impossible; or, structures have to be unlearned and new relationships constructed. This unlearning and relearning creates unnecessary obstacles (interferences). Thus when initial mathematics instruction of a concept focusses on memorizing procedures, facts, and definitions, subsequent meaningful learning may be impaired. This study therefore problematizes the simplistic solution of combining rote and meaningful instruction that teachers may be tempted to adopt as a means of mediating conflicting professional demands.

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LONG TERM RELATIONSHIP BETWEEN SPATIAL ABILITY AND MATHEMATICAL KNOWLEDGE

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In this investigation we studied the use of imagery and mathematical understanding in students who had previously been studied in fifth grade. These students were in tenth grade at the time of the investigation. Thirty two of the original students who had been given the WSAT and five of the six interviewed were available for retesting. The results showed a moderately strong correlation between fifth and tenth grade scores on the WSAT. The qualitative analysis of clinical interviews showed a strong consistency of orientation to mathematical problem solving.

There is a long history of interest in the relationship between spatial ability and mathematical knowledge. Kruteskii (1976) identified two distinct casts of mind, the analytic and the geometric. According to Kruteskii, a person with a geometric cast of mind makes extensive use of imagery in his or her mathematics learning. In our work we have focused attention on the role of imagery in mathematics learning. Currently, there is increased interest in this topic. For example, the National Council of Teachers of Mathematics in the Curriculum and Evaluation Standards has called for more attention to what they refer to as spatial sense. The evidence supporting a major role of imagery doing mathematics is mounting (Brown, 1993; Brown and Wheatley, 1989; 1990; Reynolds, 1993; Reynolds and Wheatley, 1992; Wheatley, 1990; Presmeg, 1985; 1986).

The studies mentioned above, however, have concentrated on students of a single age. Brown, Reynolds and Wheatley studied students at the elementary school level, while Presmeg's studies were limited to high school students. Brown and Presmeg (1993) studied students in both fifth and eleventh grades and found that they used the same types of imagery and in many of the same ways, but this study used different students in the two groups. We do not know if students who make extensive use of imagery in elementary continue to do so as they move to high school.

The purpose of this study was to reinvestigate the role that imagery plays in the mathematical activity of students who imagery had previously been probed. By doing so we hoped to see if long term consistency could be found. That is, if students who scored well on a test of mental rotation and who showed a good relational understanding of mathematics in elementary school continued to do the same while students who had performed poorly continued to do the same. In order to do this students who had previously taken the WSAT and been interviewed in fifth grade (Brown and Wheatley, 1989) were retested and interviewed as tenth graders.

METHOD

Of the 54 students who had been given the WSAT in this group as fifth grade students 32 were still at the same school and enrolled in mathematics classes. As tenth grade students most were enrolled in geometry classes, although some were enrolled in Algebra I or II. Of the six girls who had been interviewed as fifth grade students five were still at the school and consented to be interviewed.

For the quantitative analysis students were administered the Wheatley Spatial Ability Test (WSAT), a test of mental rotations. The WSAT is a 100-item two dimensional test which has been described previously (Brown and Wheatley, 1989; 1990; 1991). Previous data indicated that the 8-minute time limit was too long with older students so in this administration the time limit was changed to seven minutes. All other procedures and scoring remained the same, however.

The test was administered to the tenth grade students in three groups. For two of the groups the tests were administered during their mathematics classes and all students in the classes received the tests. For the fourth group the test was administered in the school auditorium.

Data for the qualitative analysis was obtained in a single clinical interview with each of the five students which lasted about 60 minutes. Tasks for this interview were chosen to provide more information about the students' use of imagery and mathematical reasoning. Tasks were assembled which required a wide range of mathematical sophistication. As the interview progressed, tasks were chosen by the investigator which were felt potentially problematic for that individual student, but not beyond his range. Some tasks were given to all students. The interviews were video recorded for later analysis.

Three tasks were used to probe students' imagery. The Tangram Pattern Task was an adaptation of one that we have used with elementary grade students and which had been used previously with high school students (Brown and Presmeg, 1993). In this task students are given a blank form and a set of tangrams. They were then briefly shown a pattern which they were to use to fill the form. For use with these older students two more complex patterns, which used six and seven Tangram pieces were added if the students performed well on the simpler problems.

A surface development task was also designed. In this task the student was shown: 1) a flat template for a cube with some sides shaded and the letter "E" on one side and 2) a drawing of a cube. The student had first to decide if the template could be folded to make any solid cube. They were then asked to decide if it could make the particular cube shown based on the position of the shaded sides and the orientation of the letter "E." The other three dimensional task was a bisection of a cube task. In this task students where shown a solid regular cube. They were then asked to think of ways in which they could cut the cube so that resultant face made different geometric shapes and what these shapes were.

Two of the mathematical problems were given to all of the interviewees. They were:

THE PAINTED CUBE PROBLEM. There are 343 small cubes arranged in a 7 by 7 by 7 large cube. If the large cube is completely painted on the outside how many small cubes will not have paint on them?

TIGERS IN CAGES. There are 15 tigers and 4 cages. There must be a tiger in each cage. No two cages can have the same number of tigers. How many ways can the tigers be put into cages?

The interview also included at least one task which involved proportional reasoning and least one area problem.

RESULTS

Quantitative analysis. Figure 1 shows the frequency distribution of scores for the

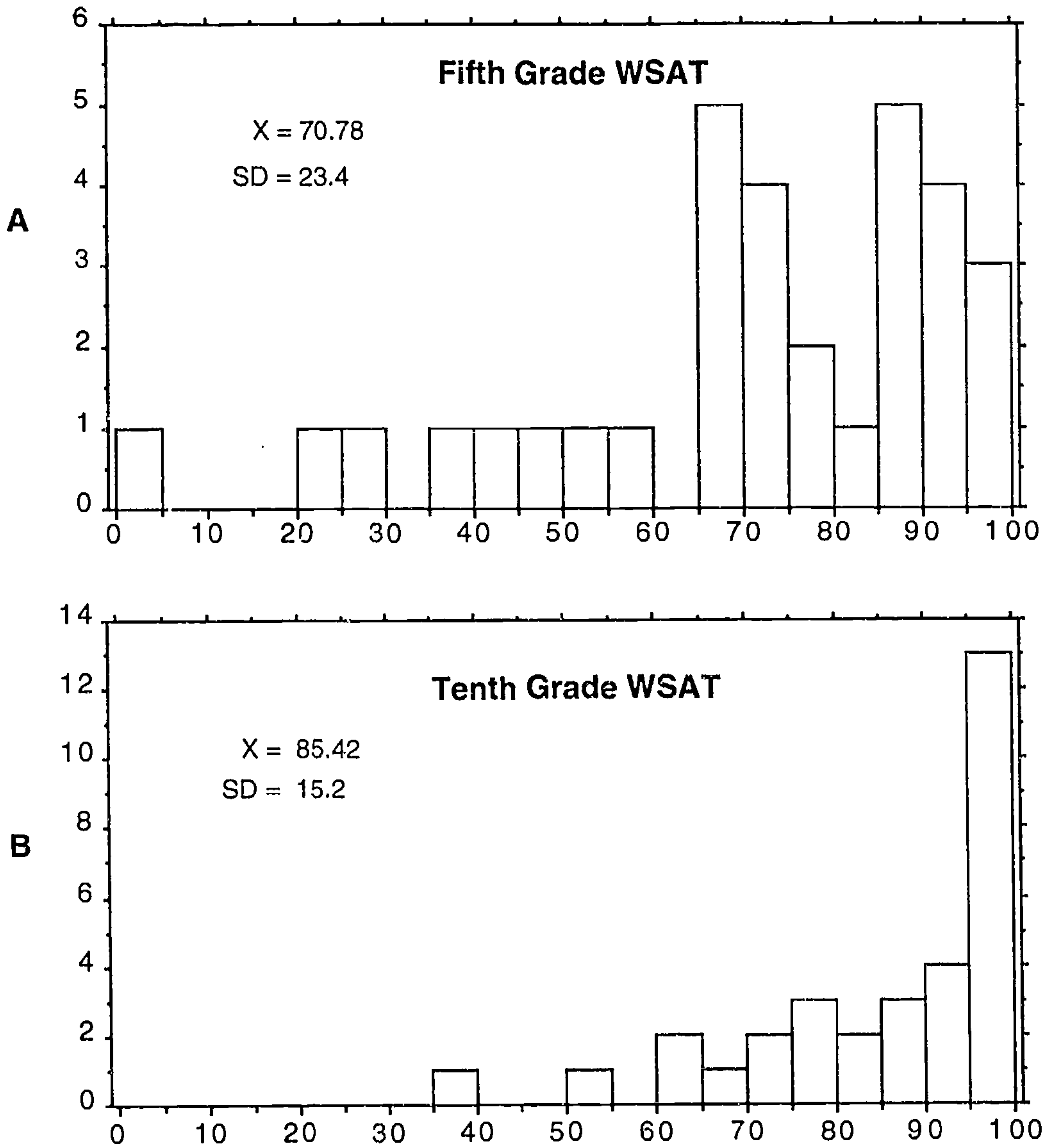

Figure 1. Frequency distribution of WSAT scores for fifth (A) and tenth (B) grade students. Scores are on the abscissa, number of students on the ordinate.

32 students as well as group means and standard deviations. A) is the WSAT scores as fifth graders, B) the WSAT as tenth graders. There is a distinct negative skew in both sets of WSAT scores although this effect is most clearly seen in the tenth grade scores. Almost half of the tenth grade students scored between 95 and 100 indicating a ceiling effect was operating. Any quantitative analysis, then is limited by these results. Most of the students were able to complete the test well before the time limit, indicating that the shortened time was still too long for the test. The WSAT has been administered to many groups of students and the reliability has always been shown to be high (KR-20>.91).

A correlational analysis using Pearson's r showed a good correlation between the fifth and tenth grade WSAT scores (r=.48). This is a moderate relationship, and was probably limited to the extreme negative skew of the tenth grade WSAT scores. If this effect had not been operating it is possible that the correlation may have been much higher.

Qualitative analysis. The analysis of the interview data revealed some strong consistencies in students' use of imagery and mathematical reasoning. In particular, students who tended to rely heavily on imagery in their mathematical reasoning as fifth graders also tended to do the same as tenth graders while students who were more analytic in their mathematics activity at grade five also tended to analytic at grade ten. In the paragraphs below each of the student's use of imagery and mathematics activity is described.

Amy is perhaps the most interesting of the students interviewed. She has very strong imagery and uses it well in her mathematics, when she chooses to do so. At both the fifth and tenth grade levels she was not considered by her teacher to be a good mathematics student because she rarely performed well in classroom situations. At the fifth grade level she scored high on the WSAT. At the tenth grade, however, she was distracted during part of the test and her score was not high. She did however, get all but one of the WSAT items she completed correct. At fifth grade Amy was highly creative in her use of imagery in mathematical understanding. She did not know her multiplication facts, but had invented an image-based procedure for multiplication which involved repeated addition. At tenth grade she was the only student who successfully solved the painted cube problem and did so very efficiently using her imagery and without the aid of any diagram. She visualized the internal cube of unpainted cubes, saw that it was a 5 by 5 by 5 and then multiplied. The pattern is quite clear. Amy has powerful imagery and is capable of high level mathematical reasoning but does not always attend to the tasks presented.

Tiffany's use of imagery in mathematical reasoning was quite consistent in the two interviews. She scored high on the WSAT in both the fifth and tenth grades and her performance on the spatial tasks during the interviews was consistent with these scores.

She also tended to use imagery and visual reasoning in doing mathematics at both the fifth and tenth grade levels. At both these levels she was able to use decomposition/recombination (Brown and Wheatley, in press) successfully, in the conservation of area task in the fifth grade interview and a trapezoidal area problem in the tenth grade. She also used visual reasoning to solve the Mr. Short-Mr. Tall problem in the fifth grade and the painted cube task in tenth grade, although her imagery failed to produce a successful solution in either case. In trying the painted cube problem she attempted to construct an image of the interior of the large cube but was unable to do so, even with the aid of a diagram.

Karen's performance was also consistent in her reliance on analytic, rather than visual methods. In both the fifth and tenth grades she scored high on the WSAT and her performance on the spatial tasks indicated a use of imagery. In most tasks, however, she chose to use analytic reasoning. In the fifth grade she used her knowledge of the relationship between multiplication and area of a rectangle and factors to find the dimensions before building the rectangle. She also used analytic methods in the tenth grade tasks, but was less successful. She attempted to solve the area problem by using geometric theorems and reasoning, but confused area and perimeter relationships.

The weakest mathematics student in the group was Laura. She scored low on the WSAT in both fifth and tenth grade and her performance on the spatial tasks confirm this score. Her mathematics also tends to reflect her low imagery at both levels. At both levels she gave answers which indicated that she had no image which was helping her mathematical reasoning. In the tenth grade interview we gave her several tasks which had been designed for younger students, but she also attempted to do these in a fairly mechanical fashion. In particular we gave her a geoboard area task which could have been solved by counting squares and decomposition/recombination. Laura began this task by counting the lines on the grid instead of the squares. She then counted every partial square as a half square. In the tigers and cages problem she found only one solution and thought this was the only one possible. This is also consistent with her fifth grade performance in which she thought a mathematical task could have only one solution. Laura's mathematics may best be described as instrumental (Skemp, 1987) at both levels. She knows procedures and algorithms, but has little relational understanding.

In both the fifth and tenth grades Helen scored low on the WSAT but her performance on spatial tasks during the interviews indicated her imagery was better than her score suggested. Of the many students we have interviewed, Helen is the only one to show this pattern. In fifth grade her mathematics showed very little use of imagery and could be described as highly instrumental. By tenth grade, however, she did show some relational understanding. For example, on the geoboard area and one proportional task she used an image based solution but was unable to generate a useful image for the more complex tasks. It seems that Helen has been able to construct some useful images in mathematics during the intervening years, although her use of imagery was still limited.

DISCUSSION

These results indicate a consistent orientation to mathematics over a five year time span, suggesting that a student's use of imagery at one age is predictive of use later in school. Students who used imagery extensively in fifth grade also made use of imagery in doing mathematics at the tenth grade level while students who had made little use of imagery in grade five showed the same pattern five years later. It is somewhat surprising that the use of imagery is so robust. During the five years between assessments, the students had experienced school mathematics courses in which imagery was not encouraged and in some cases discouraged. Conventional texts and instruction emphasize analytic and procedural methods over spatial methods. These findings support Kruteskii's contention that the use of imagery, which is so powerful in doing mathematics, is a deep seated characteristic of individual students.

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THE ROLE OF INDIVIDUAL RECONSTRUCTIONS OF MATHEMATICAL CONCEPTS IN DYADIC PROBLEM-SOLVING

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The present work focuses on the role of the students' personal reconstructions of mathematical concepts in the process of instruction in mathematics at the university level. The assumption that each student enters the instruction/learning process as an active thinking individual with a set of previously existing schemas--both academic and real-life, which aid her/him in the mastery of the provided mathematical concepts--leads us to suppose that each student constructs and employs throughout this process a number of personalized heuristic devices which are not reduceable to a number of predefined approaches and which are productive in facilitating communication and joint constructions between partners in problem-solving.

Our approach is based on the theoretical assumption that individual knowledge is constructed personally, yet within and on the basis of interactional contents and contexts. This is close to a Vygotskyan theoretical approach, with a stress on the co-construction of mental contents.

Students were presented with a mathematical problem and asked to think about it for a while. Then they were asked to discuss the problem with a partner who had been presented with the same problem. The partners were chosen for having demonstrated different levels of performance in a previous problem-solving situation. Finally, they were asked to write up their answers to the problem individually.

The analysis focused on the differences in the individual contributions to the discussion of the problem, on the way that the differences were negotiated between the partners during the discussion, and on the way that the results of this negotiation were incorporated (if at all) in the individual write-up of the problem solution.

The practical interest in this study was to support the view that the instruction of mathematics has to take into account the fact that individual understanding of mathematical concepts is aided by the individual's continuously ongoing reconstructions of the problem and the concepts involved in it, in ways which cannot necessarily be foreseen by the teacher. A consequence in terms of preparing instructional material is, for example, the necessity to provide the flexibility for the students' own study heuristics to be practiced.

A CATEGORY-THEORETIC DESCRIPTION OF THE MECHANISMS UNDERLYING SHIFTS OF COGNITIVE STAGE AND SHIFTS IN LEVELS OF GEOMETRIC THOUGHT DEVELOPMENT

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The purpose of this paper is to reflect on the relationships between stage theories of cognitive development and the van Hiele theory of geometric learning. Taking as starting points the expositions of Piagetian theory (Davidson, 1988) and the van Hiele theory (Hoffer, 1983) from a category-theoretic perspective, the mechanisms of how a stage of cognitive development is attained or how a level of geometric thought is reached is examined in detail. The models suggested are modified herein and thereby improved. The concepts of morphism category, functor category, and especially of universal element and adjoint pair (MacLane, 1971) are defined and used in the context of the formalism introduced to model the Piagetian and van Hiele theories. Thus an exposition of basic category-theoretic concepts is provided (MacLane, 1988). This, in turn, is accompanied by examples of how shifts of level or stage are manifested in students' work and attitudes in the learning of geometry (Denis, 1990).

LEGO-LOGO, SPATIAL REASONING AND GIRLS

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This poster presents data from the first year of a three-year NSF project whose goal is to encourage girls' interest and persistence in mathematics, science and technology fields. A central activity of the project consists of a two-week mathematics, science and technology summer workshop offering both Lego Logo and hands-on construction activities in mathematics. One research question investigated in the context of this project concerns the short-term development of the girls' understanding of mechanism, and the question of whether intensive activity with Lego Logo and hands-on construction and visualization activities can lead to improvement in spatial reasoning.

The participants in the project were 48 girls from 5th, 6th, 7th and 8th grade classes, who spent two weeks at the summer workshop (called Project SAME, for "Science and Mathematics Equity"). During the workshop, the girls, divided into two groups roughly by age, attended two morning sessions of activities. One set of activities involved learning Lego Logo. Lego Logo is an activity in which learners build devices from standard and specialized Lego pieces, and then create programs to control the devices. Lego Logo draws on, and may help in the development of, a range of skills, including problem-solving, calculation, programming and planning, and spatial/visual reasoning.

In addition to the sequence of Lego Logo activities, the girls also carried out a series of constructive mathematics activities, including paper cut-and-fold constructions, geometric drawings and other activities. Previous research suggests that spatial reasoning and mathematical ability are strongly associated, and that women often perform more poorly than men on tests of spatial ability. Other research suggests that practice can improve spatial reasoning. One question addressed in this study is whether the activites which the girls carried out over approximately 25 hours during the summer workshop were effective in improving the girls' performance on a standard measure of spatial reasoning (the Differential Aptitudes Test, Bennett, Seashore & Wesman). In addition, a measure of the girls' understanding of mechanical reasoning was administered.

Results from a pre-and post-test comparision on the two measures will be presented, as well as a broader overview of the project and plans for further analysis of videotaped data.

TURTLE MATH: A LOGO ENVIRONMENT GROUNDED IN RESERACH

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Research suggests that Logo can be used to develop students' geometrical concepts and aid them in progressing to higher levels of geometric thinking. The research corpus provides directions for designing a Logo environment fine-tuned for the learning of geometry. We will describe briefly five features of *Turtle Math* based on principles that have been abstracted from several research reviews.

1. Both measurement tools and the overall structure of *Turtle Math* (to be described in the next section) encourage the construction of the abstract from the visual.
2. The dynamic link between the commands in the command center and the geometry of the figure maintains close ties between representations. Any change in the commands leads to a corresponding change in the figure, so that the commands in the command center precisely reflect the geometry in the figure.
3. *Turtle Math* facilitates examination and modification of code through ease of editing and repeating constructions and operations, along with "undoing," "stepping" and similar functions. The rationale for such functions goes beyond simple convenience; the tools embody the critical Piagetian concept of reversibility.
4. *Turtle Math* structure encourages use of procedures from the beginning. First, a tool provided on a palette walks students through the steps of defining procedures. Second, changes made to procedures within the teach window are immediately reflected on the graphics screen upon exiting that window.
5. *Turtle Math* provides freedom within constraints in that the environment allows students and teachers to pose and solve their own problems, encouraging exploration and conjecture. In addition, students can solve problems at a variety of levels. Students and teachers can enable or disable the tools through an options menu. Students can use the enabled tools to analyze figures or to work in a visual, empirical manner via measurement.

Epistemology

THE ROLE OF CONTEXT IN MATHEMATICAL ACTIVITY

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ABSTRACT

In the study which forms the basis of this paper, nine students, in two separate interview and testing sessions, were presented with three tasks from each of six different content areas in mathematics. The test items presented in one session consisted of six decontextualised tasks, and the test items presented in the other session consisted of twelve tasks which were identical with respect to mathematical content and conceptual difficulty, except that the mathematics for each problem was embedded in a 'real world' context.

This study hypothesised that an individual's response to a mathematical task is constrained or facilitated by perceptions of familiarity, interest and difficulty. The research reported here addressed each of these attributes in relation to success and solution method.

Students on average were no more successful with the contextualised tasks than the decontextualised tasks, suggesting that embedding mathematics tasks in context does not necessarily enhance performance. However, individual differences in the way students responded to task context need to be taken into account if the results are to be fully appreciated. The results suggested that students who performed better on contextualised tasks appeared to benefit from being able to understand more precisely the requirements of the question and visualise the situation.

Introduction

The location of mathematical tasks in meaningful contexts for either instructional or assessment purposes derives its contemporary justification from continuing attempts to engage the student's interest, and in the recent recognition of the importance of acknowledging and utilising the situated nature of learning (Lave, 1988). From this perspective, it is claimed that learning is facilitated where students are able to find points of connection between their own experience and what they are trying to understand (Belenky, Clinchy, Goldberger, & Tarule, 1986).

The use of elaborated task contexts in mathematics instruction has been argued on at least three grounds:

- The encountering of new mathematical content in a familiar context facilitates student understanding of the new content since individual representations of that content can be constructed using contextual elements already present in the learner's cognitive framework;
- The utilisation of meaningful contexts facilitates student engagement with the problem, and enhances motivation;
- The learning of mathematical content in familiar contexts holds the promise of increased transfer, application or use of that content in other contexts within the student's immediate or anticipated experience.

The term "meaningful" might be interpreted as "familiar"; that is, a context of sufficient familiarity for the respondent to attach meaning to it. Alternatively, "meaningful" might be taken as suggesting some sense of purpose. From this second perspective, a context might be unfamiliar but sufficiently engaging for the student to seek a resolution of the problem. In this paper, "familiar" and "interesting" are addressed as separate aspects of the student's perception of the problem context. It might also be inferred that unfamiliarity with a context would be associated with the perceived difficulty of a task. Each of these task attributes - familiarity, interest, and difficulty - warrants investigation for its contribution to the mathematical activity stimulated by a particular task.

For our purposes, mathematical activity is taken to consist of both overt behaviours and associated mental processes. Student overt behaviours (for instance, the drawing of diagrams, or the completion of written calculations) may be readily apprehended, while any corresponding cognitive process can only be inferred, either from an analysis of explicit physical behaviour or artifact, or from the student's spoken account of their thought processes. The aspects of mathematical activity of interest here are the methods of problem solution and the consequent degree of success. Student accounts provided data on context familiarity, task interest, perceived task difficulty, and method of solution.

The research study

All of the data reported in this paper arose from a study of adult learners' responses to contextualized and decontextualized mathematics tasks (Helme, 1994). In this study, nine students, in two separate interview and testing sessions, were presented with three tasks from each of six different content areas in mathematics. The test items presented in one session consisted of six decontextualised tasks, and the test items presented in the other session consisted of twelve tasks which were identical with respect to mathematical content and conceptual difficulty, except that the mathematics for each problem was embedded in a 'real world' context. This enabled a comparison to be made between solving a decontextualised problem and a corresponding problem where the mathematics was embeded in context, in terms of success with the task, student perceptions of the task (how interesting and how difficult they found it) and methods used to solve it. Students also rated familiarity of task context. As some contexts were rated as more familiar than others, it was possible to ascertain the extent to which familiarity of task context was related to performance, perceptions and the form taken by the solution. Semi-structured interviews enabled students to describe their solution methods and comment freely about the tasks.

Subjective responses to task context and content

An individual's response to a mathematical task will be constrained or facilitated by perceptions of familiarity, interest and difficulty. The following discussion addresses each of these attributes in turn.

Familiarity

In the research which forms the basis of this paper, contexts were selected with the expectation that some would be familiar to students and some unfamiliar. For example, in the content domain of linear equations, one task was constructed in the context of buying food at a market, whereas another was concerned with a leakage at a chemical plant.

Mathematical isomorphism was central to the research design, if comparison was to be made across tasks in a given content domain. Tasks within a content domain were amenable to solution using precisely the same mathematical processes and calculations and numbers of the same type and order of magnitude. This requirement constituted an operational definition of mathematical isormorphism for the purposes of this study. Although the tasks were mathematically isomorphic, the wording of each version of the task was necessarily different and introduced a further source of variation, which could not be controlled. Obvious sources of variation included the number and difficulty of the words used in the tasks and their sentence structure.

Respondents were asked, upon the completion of each task, to indicate the degree of familiarity of the "situation described in the problem" on a five-point scale, from "not at all familiar" to "extremely familiar".

I

nterest

Participants were asked, upon the completion of each task, to respond to the question, "How interesting did you find this problem?" on a five-point scale, from "not at all interesting" to "extremely interesting".

Difficulty

Respondents were asked, upon the completion of each task, to respond to the question, "How easy or difficult did you find this problem?" on a five-point scale, from "extremely easy" to "extremely difficult".

The categorization of task completion

Success

Each task solution was assessed and rated according to a three-point scale:

Satisfactory:

Complete or reasonable solution using mathematics/reasoning appropriate to the task (2 points).

Partially satisfactory:

Incomplete solution using some appropriate mathematics/reasoning (1 point)

Not satisfactory:

No attempt made at solution or engaged with task with little success (0 points).

Thus students' total scores for the 18 tasks could range from 0 to 36 points, for the decontextualised tasks from 0 to 12 points and for the contextualised tasks from 0 to 24 points.

Scores for each student were recorded as a percentage of the total possible score. This enabled a comparison to be made between performance on contextualised tasks with performance on decontextualised tasks, as well as a comparison between performance on different task contexts.

Solution method

Examination of the existing research literature did not suggest a predetermined scheme for categorizing solution methods. Instead, comparisons were made between solution methods, based upon a post-hoc analysis of the data. This was done in an attempt to identify the extent to which particular solution methods were associated with the presence or absence of task context and variations in task context.

Results

Task context and performance

Students on average were no more successful with the contextualised tasks than the decontextualised tasks, suggesting that embedding mathematics tasks in context does not necessarily enhance performance. However, individual differences in the way students responded to task context need to be taken into account if the results are to be fully appreciated. The results suggested that students who performed better on contextualised tasks appeared to benefit from being able to understand more precisely the requirements of the question and visualise the situation. There was evidence that some students also made use of contextual cues to assist in metacognitive processes such as checking and revising their results. Those who were less successful at the contextualised tasks faced prohibitive language and cultural barriers (in the case of the NESB student) or were confused by the context and unable to extract from it the information necessary for solving the problems. Students who did equally well on both types of problems were generally high achieving students who understood the content so well as to be unaffected by different task formats, or those students who had difficulties with particular content domains and for whom task context in these domains neither assisted nor impeded performance.

Contrary to what was expected, performance on tasks set in familiar contexts was no better over the entire sample than that on tasks set in unfamiliar contexts. Again, individual differences dominated the results. For some students, performance was clearly superior on tasks located in familiar contexts, but for others this was not the case.

An unexpected finding was the absence of a relationship between performance and perceptions of task interest. Arguments put forward in support of the benefits of contextualised tasks emphasise that they generate enhanced motivation, and therefore better performance. However this study found that when performance on high interest tasks was compared with that for low interest tasks, there were no significant differences. In fact, an interesting contradiction was observed in that the task which received the *highest* interest rating was the *least* well performed of the entire set of tasks.

Because individual differences were the most characteristic feature of performance on the tasks, it is inappropriate and uninformative to comment further on the group trends. Instead, the discussion which follows addresses cases which illustrate the distinct ways in which individuals related to task context. An explanation of the results may then be possible in terms of individual differences or subgroups of students exhibiting similar response patterns. This discussion will refer to the interview transcripts as well as the statistical findings. For the purposes of this discussion, the students have been divided into three distinct categories; those who performed better on the contextualised tasks, those who performed better on the decontextualised tasks, and those for whom there were no performance differences.

Within the restrictions on the length of this paper, a full discussion of the data relating to each group is not possible. For full details of all results, the reader is referred to Helme (1994).

Students who performed better on contextualised tasks

Emma, who obtained 67% on the decontextualised tasks and 88% on the contextualised tasks, also rated contextualised tasks as less difficult. She also appeared to be sensitive to context familiarity, as she performed better on familiar contexts and rated them as less difficult than unfamiliar contexts. Her comments on the area question revealed how useful she found task context in solving a problem. It seems that the presence of context provided metacognitive clues which assisted Emma in solving the problem.

Students who performed better on the decontextualised tasks

Janet obtained 58% on the decontextualised tasks and 42% on the contextualised tasks. She was also the lowest performing student in the study. Although she rated both types of tasks as equally difficult (average 3.8), she commented that the decontextualised items seemed more straightforward than the contextualised items, and, indeed, less abstract:

For the students in this category, context served to impair performance in two distinct ways. First, the greater language demands of contextualised tasks clearly made them more difficult for the NESB student. This effect was compounded by unfamiliar task contexts, most likely because of the unfamiliarity of the words as well as the situations. Second, the low-achieving student was less successful with the contextualised tasks due to her difficulties in extracting from the contextualised task the information relevant to its solution. This difficulty was more evident when the context was unfamiliar. Familiar contexts gave students access to the metacognitive cues which assisted the problem solving process in much the same way as for the group of students discussed in the previous section.

Students who performed equally well on contextualised and decontextualised tasks

For the students in this category, contextual influences on performance could not be detected. For some students the test items were not challenging enough for different forms of task presentation to make an impact on performance, as the content was well understood and integrated into existing conceptual frameworks. For other students, tasks in particular content domains, rather than the presence or absence of contextual information, were the major source of impaired performance. The content domains in which students demonstrated difficulties irrespective of the presence or absence of task context were area, gradient and percentages. When students had misconceptions in a particular content domain, the presence of familiar contextual cues was not a guarantee that such misconceptions could be overcome.

Conclusions

That students on average were no more successful with the contextualised tasks than the decontextualised tasks suggests that simply embedding mathematics tasks in context does not enhance performance. However, individual differences in student performance in relation to task context and content suggested that students react individually and uniquely to mathematical tasks and that the individual characteristics determining their responses need to be taken into account. The results suggested that when students performed better on contextualised tasks it was because they clearly understood the requirements of the question and were able to visualise the situation. In some cases contextual cues enabled them to access metacognitive strategies such as checking and revising their solutions.

There was no clear relationship between the presence or absence of context and solution methods for all students or across all content domains. However in the content domains of directed numbers and linear equations it was clear that presenting the problem as an equation stimulated a formal algebraic approach whereas a worded context task triggered an informal approach. It appears from these results that more complex and lengthier tasks may trigger an informal approach.

Perceptions of task interest were not systematically related to the presence or absence of task context, nor was familiarity of task context a significant predictor of perceptions of task interest. Thus there was no evidence to suggest that all students engaged with or were motivated more by contextualised tasks, or that familiar contexts were found more or less interesting than novel contexts.

Differences in performance and ratings of difficulty were found for different content domains, but little variation was found in perceptions of task interest and context familiarity (for the contextualised tasks). These results suggest that some content areas were better understood than others, and were also perceived as less difficult.

A major outcome of this study was the recognition of individual differences in responses to contextual factors. The idiosyncratic and often unexpected ways in which students learn mathematics or respond to mathematical tasks, for instance, can only be explored and understood by supplementing larger scale studies with in-depth small scale research which focuses on particular individuals or groups of individuals. Adoption of research paradigms which allow greater attention to individual differences is a growing trend in mathematics education and one which should continue.

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THE RELATIONSHIP BETWEEN PRESERVICE TEACHERS' METAPHORS FOR MATHEMATICS LEARNING AND HABERMASIAN INTERESTS

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Abstract

This study examines the relationship between preservice teachers' metaphorical language to describe the teachers' role in an ideal mathematics classroom and Habermasian interests in control, understanding and emancipation. Cartoons and verbal prompts were used to elicit metaphors and probe student thinking about the ideal mathematics classroom. Results indicate a predominant control interest and conflict between the socially negotiated role of the teacher as facilitator and student personal histories which include an image of the teacher as the director of student learning. These findings contribute to the body of literature that suggests methods classes do not address the prior histories or epistemological orientations of preservice teachers.

Attempts to interpret preservice teachers' efforts to make sense of professional teacher preparation programs often focus on beliefs, conceptions and knowledge (Pajares, 1993; Kagan, 1992; Pajares, 1992; Fennema, 1989; Hollingsworth, 1989; Thompson, 1984) or cultural and community norms (Cobb, Wood, Yackel, & McNeal, 1992). Alternatively, grounded theory (Glaser & Strauss, 1967) has been used to explore both individual and social influences affecting preservice teachers' attempts to make sense of their learning experiences in the field setting (Tobin, 1990). Metaphorical language organizes and reveals individual personal realities (Lakoff & Johnson, 1980) and is central to a grounded theoretical approach to understanding the dynamic nature of learning to teach.

The historical relation teachers and those preparing to teach have had with their own mathematics learning can be characterized as routine practice of procedures and skills with an emphasis on the production of right answers (Lappan & Even, 1989; Romberg & Carpenter, 1986). Ernest (1991) delineates a range of beliefs associated with mathematics learning from absolutist/authoritarian to social constructivist orientations and describes how traditional personal histories are most often associated with the authoritarian perspective.

Habermas (1971) describes three fundamental human interests (technical, practical, emancipatory) related to the empirical-analytic, historical/hermeneutic, and critical orientations to knowing (Grundy, 1987). A person functioning from a technical orientation is primarily interested in controlling and managing the environment. An individual oriented by a practical interest concentrates on clarification or understanding rather than verification of rules and social norms. The emancipatory interest, according to Habermas, is predominated by abhorrence of convention and commitment to independent action. The emancipatory perspective is often associated with reflection and critical analysis of both self and societal norms. A person's orientation towards a particular fundamental interest intercedes human experience and action.

Habermasian interest categories structure individual interpretations of experiences and may affect how preservice teachers view their future teaching roles and construct meaning from methods class experiences. The idealism and vision communicated during preservice teacher preparation programs often are abandoned and prior orientations, shaped by personal histories, predominate as students join the teaching ranks (Kagan, 1992; Kay, 1992; Bullough, 1991; Veenman, 1984).

> We need to regard the views we hold about teaching not as idiosyncratic preferences, but rather as the product of deeply entrenched cultural norms of which we may not even be aware. ... Teaching becomes less of an isolated set of technical procedures and more of a historical expression of shaped values about what is considered to be important about the nature of the educative act (Smyth, 1992, pp. 298-299).

The question arises: Are teacher preparation programs in conflict with the epistemological needs of preservice students?

This study examines the relationship among preservice teachers' metaphorical language to describe the teacher's role in an ideal mathematics classroom and Habermasian interest categories. Guiding questions for this study are:

1. What are the metaphors for teaching mathematics that preservice teachers express?
2. What is the relationship between these metaphors and Habermasian interests?

Description of the Investigation

Students participating in this project were in their last semester before student teaching. Although most were in the four-year transitional teacher preparation program, all had had extensive experiences in classrooms as part of the developmental field experience component of the newly adopted 5-year program. Approximately two-thirds of the students were traditional students under the age of 25 and all but one were women. Prior to their methods block, consisting of mathematics (4-8), language arts, social studies, and reading methods classes, most had taken a primary grades (PK-3) mathematics methods course and three mathematics content courses. The knowledge base of the teacher preparation program in which they were enrolled includes a commitment to philosophical approaches to teaching and learning consistent with the various branches of constructivism (Prawat, 1993).

Metaphors describing the role of the mathematics teacher were elicited from 63 preservice elementary education and early childhood majors in two intermediate/middle school mathematics methods classes. Students were asked on the first night of class to "Describe what teaching is." Later during the same class, students responded to the 29-item Attitude Instrument for Mathematics and Applied Technology-Version II (AIM-AT-II) (Fleener, 1994a) which was used to determine Habermasian interest orientations for each student. On the second night of class, students were shown 6 non-human Far Side cartoons with the captions removed and asked to pick and describe the one that best illustrates an ideal mathematics classroom. They were then asked to describe which animal was the teacher and what the role of the teacher was in that classroom. During the fourth week, students in one class were given one of the six cartoons and asked to describe the mathematics lesson or activity depicted. These data were used to determine consistency of earlier definitions and descriptions of the ideal mathematics classroom and to compare metaphors elicited by later prompts with earlier language used to describe the role of the teacher.

Student responses to visual and verbal prompts were examined to determine the metaphors used to describe the role of teachers. Students were then categorized into Habermasian interest categories and metaphors were compared across Habermasian orientations. Prior research suggested response patterns to the AIM-AT-II are related to control, hermeneutic, and emancipatory Habermasian perspectives (Fleener, 1994b). Teachers (Fleener, 1994b) and preservice teachers (Fleener, 1994c) were found to be split on whether mastery of skills or concepts was necessary before students were allowed to use the calculator. Analyses revealed MASTERY=YES teachers (i.e. teachers who believed conceptual mastery should be achieved before calculators are used) had stronger interests in control than MASTERY=NO teachers who disagreed that conceptual mastery should occur before calculators are introduced and MASTERY=MAYBE teachers whose position on the mastery issue was unclear. Furthermore, the MASTERY=NO group displayed a stronger interest in hermeneutic and emancipatory matters than the MASTERY=YES group. Preservice teachers in this study were categorized into these same mastery groups based on responses to the AIM-AT-II survey to indicate Habermasian orientations. Teaching metaphors of the three groups of preservic~ teachers were compared to address the second research question.

Results and Discussion

The social influence of constructivism as expressed through prior methods classes was apparent as students chose the 'active learning' visual prompt as most indicative of the ideal mathematics classroom. Initial descriptions of the role of the teacher revealed control, hermeneutical and emancipatory interests as typical metaphors "Teacher as Guide," "Teacher as Facilitator," "Teacher as Manager," "Teacher as Role Model" and "Teacher as Encourager" were expressed. Although "Teacher as Facilitator" was a metaphor used by many of the participants in this study to describe the role of the teacher and the visual prompt selected as the most indicative of the ideal mathematics classroom, examination of written responses to visual and verbal prompts revealed a move towards more controlling language as students were asked to delve deeper into what the teacher was doing with the students in the presented cartoon and to elaborate on their thinking about the role of the teacher. The initial use of the facilitator metaphor and the subsequent shift to more traditional ways of viewing the role of the teacher further indicate the influence of the social setting of methods classes encouraging the facilitative role and the conflict between the idealism of the meth class and the underlying beliefs and experiences of the preservice teachers.

Mastery categorization of participants defined three groups of preservice teachers: MASTERY=YES, MASTERY=MAYBE, and MASTERY=NO. Prior research (Fleener, 1994b) indicates these groupings correspond to Habermasian interests with the MASTERY=YES group exhibiting a stronger control orientation and the MASTERY=MAYBE and MASTERY=NO groups expressing more hermeneutic and emancipatory perspectives. There were 43 participants categorized in the MASTERY=YES category, 12 in the MASTERY=MAYBE group, and 8 in the MASTERY=NO grouping. Twenty seven of the 43 MASTERY=YES preservice students used metaphors depicting the role of the teacher is to instruct, teach, impart or convey knowledge. Ten of the MASTERY=YES students characterized the role of the teacher to help students understand mathematics content; 4 felt the teacher should make mathematics fun, and 2 suggested the role of the teacher was to make a difference in the lives of their children. These findings are consistent with prior research which suggests MASTERY=YES teachers have a fundamental interest in controlling the learning environment, including controlling the content of what is learned.

Of the 12 MASTERY=MAYBE students, 8 indicated the role of the teacher was to guide students in their learning of the curriculum or help students obtain their personal goals and function in society. Although guiding student learning indicates a control orientation (Fry, in press), these students did reveal a more student oriented focus than did the MASTERY=YES group by expressing concern with student personal fulfillment.

The MASTERY=NO group was similar to the MASTERY=MAYBE group with the majority of students indicating the role of the teacher is to guide student learning, help students fulfill their goals, and create an atmosphere where students could pursue and enjoy learning. Only 2 of the MASTERY=NO teachers revealed a strong controlling interest by describing the role of the teacher as the director or instructional leader of learning.

Implications

The influence of prior experience was evident as students' controlling metaphors expressed their personal histories with and, in many cases, fears about mathematics learning. Initial responses to visual and verbal prompts elicited the socially negotiated view of the "Teacher as Facilitator." Continued probing of students interpretations of the Far Side cartoons and verbal prompts, however, revealed the conflict most of the students felt between viewing teachers as facilitators and their concern over controlling the classroom and the curriculum. This conflict was especially apparent for the MASTERY=YES group who expressed a majority interest in control. Although students in the other two groups expressed more hermeneutic or emancipatory Habermasian perspectives, they too exhibited controlling interests. These results suggest the mathematics content and methods classrooms do not address the epistemological needs of preservice teachers. Even though the methods classes in which these students were enrolled included opportunities for reflection and active learning, it may be students need to be more actively engaged in inquiry based content and methods classes to construct meaning for, rather than merely adopt the language of, inquiry based learning.

Further investigations based on this research need to address the role reflection on personal metaphors plays in encouraging preservice teachers toward a more emancipatory or critical approach to their teacher education experiences. Bridging the epistemological gap between empiricism and critical approaches to teaching is necessary before preservice teachers can be transformed through the methods class negotiations. Critical awareness is elemental to change (Freire, 1973).

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WHEN A SOLUTION TO A MATHEMATICAL PROBLEM BECOMES AN ACCEPTABLE PROOF

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Mathematical proof is considered as an indispensable element of mathematical practice. Proving and refuting conjectures are continuously interwoven. Furthermore, the basis for the new discoveries and applications are often not the theorems themselves but the fruitfulness of the ideas included in these processes.

In a traditional calculus classroom, problems are clearly defined and what is to be proven is explicitly stated. Then the formal proof is either omitted as too tedious or its logical necessity is conveyed by an instructor. Students' lack of awareness of the need to support mathematical facts by any proof and their logical immaturity is well documented by researchers. This study investigated whether it is possible to go around students' pragmatism and have them not only produce answers to a problem but also reason about an argument they made.

A clinical case study of second semester college calculus students will be reported. Students solved the calculus problems in small groups, were obliged to offer additional explanations when others did not understand or disagreed, and finally, defended their solutions against the critique of the researcher. In their attempts to convince different discussants of the appropriateness of the solution, varieties of ideas and modes of justifications were evoked.

Function Concepts

AN INVESTIGATION INTO THE DEVELOPMENT OF STUDENT UNDERSTANDING OF THE GRAPHS OF POLYNOMIAL FUNCTIONS OF DEGREE GREATER THAN TWO: RESULTS AND IMPLICATIONS

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Through the use of clinical interviews, this study investigated the development of student understanding with respect to the graphs of polynomial functions of degree greater than two. Of primary interest is information relating to how the students' ability to interpret the graphs of polynomial functions of degree greater than two depends and builds on their understandings of the graphs of linear and quadratic functions. Teaching episodes were developed in an attempt to enhance connections between classes of polynomial functions and to enrich the students' understandings of the graphs. In this paper, excerpts of responses from an 11th grade student, Mark, are analyzed.

Introduction and Rationale

The typical path of instruction in high school algebra courses for the various representations of functions has been from algebraic expression, to ordered pairs, to graphs (Philipp, Martin, & Richgels, 1993). *The Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989, p.148-153) imply that, due to the availability of technology in the classroom, the emphasis of instruction should be on the graphs of functions rather than on the learning of algebraic algorithms. This shift in emphasis is supported by the belief of many mathematics educators that graphical representations will elucidate the function concept and make it easier to learn for most students (Romberg, Fennema, & Carpenter, 1993).

With these changes, it is appropriate for new research that investigates how students' understanding of graphs develop. This paper reports a small piece of a larger study (Spring, 1994), that was based on the hypothesis that an adequate understanding of the graphs of polynomial functions requires that the students develop some sense about the effect of the degree of a polynomial function on its graph, as well as an understanding of the relationship between the factors of the algebraic representation of the polynomial function and its graph. These associations are "assumed" background knowledge for calculus and necessary for such common applications as finding the area between the x-axis and a curve given by a polynomial function.

The theoretical framework for the proposed study is a constructivist position on conceptual change. This position infers that the process of conceptual change is affected by the active participation of the learner in the construction of his or her own knowledge and can be facilitated or hindered by the learner's beliefs and motivation, as well as contextual and situational factors (Pintrich, Marx, & Boyle, 1993).

Method

This study took place within an upper level Algebra II class at a public high school in northern New England. The traditional sequence of instruction, and the sequence that was adopted in this class, was from linear functions to quadratic functions to polynomial functions of degree greater than two. Six students from the class volunteered to participate in the study. Personal interactions with these students occurred in three phases: 1) clinical interviews, 2) teaching episodes, 3) an evaluation of the teaching episodes by repeating the clinical interviews.

The clinical interviews: Clinical interviews were used in an attempt to identify each students' previous mathematics background, their attitudes and beliefs about mathematics, and their attitudes, beliefs, and understanding about elementary functions and graphing. These interviews were also used to explore the students' thought patterns and investigate how and to what extent their understanding of the graphs of polynomial functions was developing. The interviews and relevant classroom sessions were videotaped.

Later clinical interviews focused on student understanding of the graphs of cubic polynomial functions with 3 real roots. Tasks focused on both pointwise and global understandings. They included describing a point's location on the curve, and addressing whether the location of the point inferred anything about the graph's algebraic representation.

The teaching episodes: As a result of these clinical interviews, it was decided that the purpose of the "teaching episodes" (Steffe, 1984) would be to experiment with how building polynomials by taking products of linear functions can, with the aid of technology, enrich and extend student understanding of the graphs of polynomial functions. My hypothesis was that allowing the students to build polynomial functions by taking products of linear functions would make the connections between the classes of polynomial functions more salient. Furthermore, the use of software that allowed the student to see the algebraic representation alongside the graphical representation would perhaps foster the formation of these connections.

Data Analysis

The data included videotapes of class sessions, student and teacher interviews, and teaching episodes, as well as student class work, quizzes, and tests. The students were asked to write some reflections in a journal following each clinical interview and teaching episode. For this paper, excerpts of responses that were made during the clinical interviews and the teaching episodes by one student, Mark, are analyzed.

Results

Initial reviews of the clinical interview tapes indicated that, upon completion of a unit on polynomial functions in Algebra II, Mark, a B student, failed to connect the x-intercepts as seen on the graph to the factors of the algebraic representation of the polynomial function. The class had used synthetic division to find the roots of polynomial functions, and had factored and graphed polynomials of degree greater than two using the Factor Theorem. When shown the graph in Figure (a), Mark determined that the equation was "something like $x^3 + x^2 + x + 1$" and then tried to make sense out of the attributes of the graph by reaching back to his knowledge of quadratic functions. For example, when asked if the turning points (Mark referred to them as vertices) gave any information about the equation, Mark replied, "Um..you know, if it was like a parabola..you could plug in the..Oh, yeah, let me see..yeah, you could plug in the numbers the..$a(x-h)^2+k$..you could plug them in there and that would help you get the equation..".

When asked if the x-intercepts had anything to do with the equation of the cubic, he replied, "I'll guess that those are like..I'll take a guess and say that's the..um..coefficient of the x terms..but I don't know...that's a guess."

Mark did understand that substituting these values for x in the equation caused f(x) to equal 0. His answers suggested, however, that the link between the roots and the factors of the polynomial was not salient. When asked for the *factors* of the polynomial, he gave the x-coordinates of the x-intercepts.

Concerning the y-intercept, he said, "That is where I got the +1..even though it's not a parabola or a swivel if you add 1 it moves up or down...it moves up one." This point on the graph seemed to have the most clarity for him, perhaps because he had done so much work with linear equations and the form $y=mx+b$.

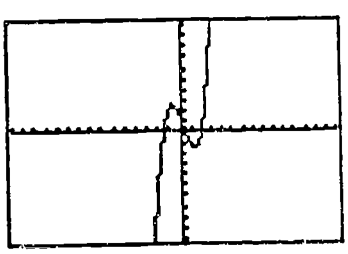

Figure a

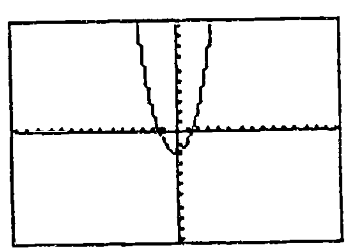

Figure b

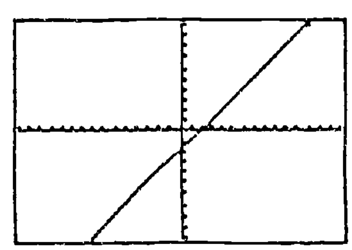

Figure c

After turning his attention to the graph of a quadratic function, Mark was asked if the x-intercepts would help him find the equation of the graph (Fig. b). Mark responded, "Would it help me find it?.. no." The only point Mark thought *would* help was the vertex, (h,k), which could be "plugged" into the form $y=a(x-h)^2+k$. Even though a lot of work had been done with factoring quadratic expressions, and finding the roots using the quadratic formula, the only relation that the x-intercepts had to the equation was that it made the function value 0.

Mark was then shown the graph of the line in Figure (c) and asked whether the x-intercept had anything to do with the equation of the line. He answered, "I forget if they have factors or not. I think so..I think that's a factor of it..but, I wouldn't know how to get it [the equation]." He tried to link this linear function with what we had just done with quadratic and cubic functions, but was not sure how to do it. Mark was able to get the equation of the line, however, by determining the y-intercept and the slope and "plugging" these values into the form y=mx+b.

In summary, Mark's responses in these clinical interviews indicated that his methods for finding the equations of linear and quadratic functions were strictly mechanical. This result was not surprising since this was the emphasis in class instruction and homework.

The "teaching episode" went beyond observing Mark to intentionally intervening in his knowledge construction. Using *The Function Supposer: Explorations in Algebra* (Educational Development Center, 1990), Mark was asked to enter the equations for two linear functions, which were graphed in boxes on the same screen (Fig. d). When asked if he knew what the graph of the product of these two linear expressions would look like, Mark multiplied the linear expressions in his head. The result was a quadratic expression and

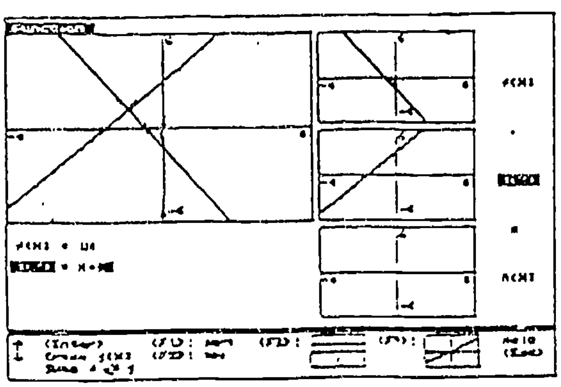

Figure d

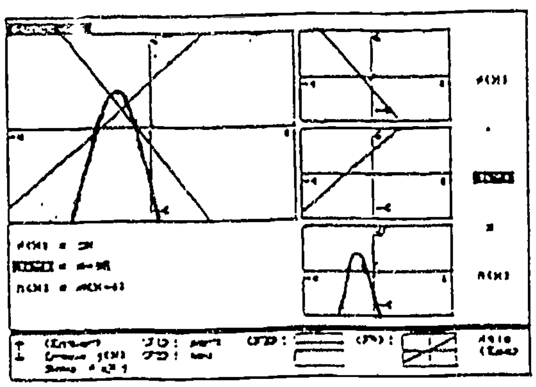

Figure e

therefore, he said, the graph would be a parabola. He also determined that the coefficient of the x^2 term was -1/2 and that meant that the parabola was wider and would open down. After seeing the graph of the product on the same axes (Fig. e), he noted that the parabola "had moved sideways" and was not centered on the y-axis, and that the vertex of the parabola was near the intersection of the two lines. Not until his attention was directed to the x-axis and the intercepts did he note that the parabola had the same x-intercepts as the lines.

> JC: ..and how does that correspond to the linear graphs?..the linear expressions?..
> MARK: Oh!.. Yeah, that's right where they cross... Huh...Huh...

The fact that the graphs cross the x-axis at the same point was new to Mark as evidenced by his surprise. He continued to transfer this idea to future tasks of this sort, showing he understood that the parabola would have the same x-intercepts as its linear factors.

In order to evaluate his understanding, he was shown the graph of the cubic polynomial that was used in the clinical interviews (Fig. a) and again asked if the x-intercepts had anything to do with the equation.

> MARK: you take that [the x-intercept] and you put x..wait..what is it..I don't know..1.8?..X-1.8 and multiply it by these to get the equation.
> JC: Multiply it by those..what do you mean?
> MARK: Multiply it by whatever that is .. I think that's 1/2 and 1 and 1/2..so, x+1/2 and..
> JC: OK, what are those things called?
> MARK: Factors? x-intercepts, too..

...and when looking at the parabola (Fig. b) again:

> JC: So, if I asked you to find the equation of the parabola..what would be your way of finding it?
> MARK: it looks like 1.5...x-1.5 and x+1.5 and multiply them out.

(We discussed after this that the actual equation of the parabola may be this product multiplied by a "stretch factor".)

Mark seemed to recognize at this point that the x-intercepts *did* have something to do with the equation. He wrote in his journal that these ideas were new to him and very interesting. As seen by his responses, though, his understanding of the terminology (factor, x-intercepts) was still blurred.

Conclusions

Due to the emphasis of his instruction, Mark had viewed classes of polynomial functions as independent from each other. The algebraic representation, y=a(x-h)2+k, used when studying the graph of a parabola did not extend to the other classes of polynomial functions. The results seem to indicate, however, that the method of building polynomials from linear expressions used in the teaching episodes, did foster the connections between these classes of polynomial functions.

It is possible that as the data from this study continues to be analyzed the results may provide some direction relative to the creation of classroom modules and/or suggestions for curriculum change with respect to the study of the graphs of polynomial functions.

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A MODELING APPROACH TO CONSTRUCTING TRIGONOMETRIC FUNCTIONS

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This paper presents an approach to mathematical modeling that integrates physical experimentation, computer simulation, and multi-representational analytic tools. Using this approach, a curriculum unit was designed to develop and explore the trigonometric functions through the relationship between uniform circular motion and its vertical and horizontal components. In this paper, the investigation undertaken by a small group of students in an integrated mathematics and physics classroom is reported. In developing a model for uniform circular motion, the students linked the trigonometric relationships to the concepts of horizontal and vertical distance. The results of this classroom study suggest that this modeling approach is productive and empowering for students.

Introduction

Modeling, simulation and discrete mathematics have all been identified by the National Council of Teachers of Mathematics (1989), the Mathematical Sciences Education Board (1990) and other professional mathematics education organizations as important areas for secondary school study. However, there is still a need for a better understanding of the processes involved in modeling and how these processes might be most effectively used to improve students' problem-solving skills and content knowledge. The modeling process is typically described as having several components: understanding the particular phenomena to be modeled; defining the context and constraints; identifying and explicitly defining the relationships between the key variables; translating those relationships to an appropriate computer implementation; and analyzing and interpreting the results (Edwards & Hansom, 1989).

The modeling process is thus distinguished in three ways from the typical problem-solving activity in the mathematics classroom today. First, the building of computer models by students forces them to make explicit their own understandings and to explore the consequences of those understandings. Second, the kinds of mathematical representations that are possible with computer-based tools extend far beyond algebraic equations to include tables, graphs, calculator algorithms, flow and control diagrams and animations. Third, the iterative nature of the modeling process provides the opportunity for approximate solutions which are refined through analysis and evaluation by the problem solver. This iterative process includes both reflection on the part of the individual ("what do I think") and the communication of that understanding among peers ("what do others think"). This process of iteration is not necessarily linear. But rather, as Bell (1993) argues, students as modelers need to spend time in each of several activities or "nodes": confronting and defining the problem, deciding on a model and operating, evaluating and interpreting, and gathering data and information.

Description of the Study

In this study, students engaged in a modeling process which integrates three component activities: the action of building a model from physical phenomena, the use of simulation, and the analysis and validation of potential solutions using multi-representational tools. In particular, we examined how and to what extent these components of the modeling process can lead to the improvement of student skills in solving mathematical problems in the content areas of trigonometry. This paper addresses the investigation of the trigonometric functions, preliminary to a larger research project on an integrated modeling approach for building student understanding of the concepts of force and vectors and for enhancing student problem-solving skills.

Design of the Curricular Unit The overall curricular unit was designed around three activities: (1) the gathering of data from a physical experiment, (2) the development and exploration of a computer simulation, and (3) the mathematical (algebraic, graphical, and tabular) analysis of the data. The second and third component activities were supported through Interactive Physics© (Baszucki, 1992) and Function Probe© (Confrey, 1992a), respectively. This approach to the design of the unit builds on earlier work of the Mathematics Education Research Group at Cornell University (Noble, Flerlage, & Confrey, 1993). Critical to the overall philosophy of the researcher and the high school teachers, this unit was designed to include extensive student discussion and reflection, collaborative work, small and large group tasks, and individual assignments.

Setting, Data Sources and Analysis The setting for this study was an alternative school within a public school system in a small city. The classrooms within the school are an open, flexible environment where small group work is common and the expression of student ideas is encouraged and nurtured. This study took place in an integrated algebra, trigonometry, and physics class with 17 students in grades 9 through 12, who had elected to take the course. Seven computers were available for student use. The class met for a double period of 1.5 hours for four days per week. The class was team taught by two experienced mathematics and physics teachers, who were familiar with computer technology and the particular software. The teachers had been trained and supported in the use of the Function Probe software by the Mathematics Education Research Group at Cornell. One of the most important aspects of the classroom was the role that the teachers took as guides and facilitators for student inquiry. Students were consistently encouraged to explore their own ideas and to make sense of physical phenomena in a context of interactions with their small group, the entire class and their teachers. The students were familiar with Function Probe from earlier course work; the simulation environment, Interactive Physics, was introduced with this unit.

The class was divided into five small groups of 3-4 students. The small groups provided a setting within which to analyze and observe how the students went about interpreting the questions, generating and negotiating their conjectures, devising their strategies for analyzing the data, confirming the sense of one or more conjectures, and using the tools and their data. Each class session of this unit was video-taped, including both the whole class discussion and the small group interactions. Written work and computer work done by the group were collected for analysis. Extensive field notes were taken by the researcher during the class sessions. The video-tapes of class sessions were reviewed and selected portions were transcribed for more detailed analysis. In this paper, the results of the analysis of the approach to the investigation taken by one small group are presented.

Description of the Curricular Unit The unit began with a simple physical experiment: the rotating motion of a wheel with a fixed hub. A bicycle wheel was mounted on the edge of a table with a sheet of poster board behind it marking various angles from zero through 360 degrees. As the wheel rotated through 360 degrees, the students were asked to describe the relationship between the height above or below the table for a fixed point on the wheel and the angle of rotation. This experiment allowed the students to extend the definition of the sine, cosine and tangent functions to include angles greater than 90 degrees and to develop a graphical representation of the functions that was grounded in the circular motion of the bicycle wheel.

The students then applied these extended definitions of the trigonometric functions to the analysis of a Ferris wheel ride (Confrey, 1992b) using the multi-representational analytic tool, Function Probe. The students created tables, graphs and equations to describe the motion of a Ferris wheel of varying diameters, at different speeds, and from different starting positions. Finally, the students created a simulation of a Ferris wheel ride using the tools and objects of Interactive Physics. They used the meters in the simulation environment to "measure" the vertical distance above the hub of the wheel of a point on the wheel. This simulation "data" was then brought into the analytic environment for comparison to the earlier analysis. Since this was the first use of the simulation environment, the objectives of this use of the environment were to familiarize the students with the tools and actions in the environment and to confirm that the results or data from a simulation made sense to the students in terms of the physical phenomena under study.

Results

The unit began with measuring the vertical and horizontal distance between a point on the rim of a bicycle wheel mounted on the edge of a table and the hub of the wheel. In a whole class demonstration, the students created a table of values for angles varying from zero through 180 degrees, recording the vertical distance, the horizontal distance and the radius. This led to a discussion of measuring the horizontal distance as negative for angles between 90 and 180 degrees. The students agreed that the vertical distance should remain positive, but several students questioned whether the measurement of the radius should be considered negative as well. The class agreed to accept as a convention that the radius would remain positive throughout the 360 degree rotation. The students extended their data table to angles between 180 and 360 degrees through an argument from symmetry and then calculated and graphed the ratios for the sine, cosine and tangent functions.

The students then investigated the Ferris Wheel problem (Confrey, 1992b) as follows: Suppose a Ferris wheel with an 80 foot diameter makes one revolution every 24 seconds in a counterclockwise direction. The Ferris wheel is built so that the lowest seat on the wheel is 10 feet off the ground. The boarding platform for the Ferris wheel is located at a height that it is exactly level with the hub of the Ferris wheel. You take a seat level with the hub as the ride begins.

The students were asked to create a table and a graph to represent the relationship between the time and the height above or below the platform for at least two revolutions of the Ferris wheel. The small group of students, which was the focus of this study unit, began with a diagram of the wheel and identified the critical relationship between the time and the angle of revolution at a 45 degrees. For this particular angle, they used the sine function to calculate the height above the hub at three seconds or 45 degrees. Arguing from symmetry, they then completed a table that had time from zero to 72 seconds, in increments of three seconds, and the height above or below the platform. They sent these table values to the graph window of Function Probe and then puzzled over how they could create an equation for this, recognizing that the values of their table did *not* increase and decrease linearly. One of the students suggested using the sine function and from there the students encountered the problem of how to reconcile the use of time and angle as an argument to the sine function. They reasoned that they could calculate the angle by multiplying the time by 15 and confirmed this argument for the 45 degree and 90 degree angles. The teacher encouraged them to make a column for degrees that would correspond to their column for time, which they did. However, throughout the remainder of the problem, the students used time and multiples of time to generate equations and graphs that corresponded to the event of the Ferris wheel traveling twice as fast and half as fast as the initial rotation. They used both vertical and horizontal "stretches" of the graph of y=sinx to confirm the correspondence between their graph from their calculated values and the graph of y=sinx.

The last activity of this unit consisted of creating a simulation of the motion of the Ferris wheel using the tools of Interactive Physics. This activity was intended to introduce the students to the simulation environment so that they would be familiar with it for later units in the overall research project and to verify that the results of simulation "experiments" corresponded in some way to the results that they were able to create from other types of analysis. The students created a simple mass on a rope, with an initial vertical velocity, to simulate the motion of a Ferris wheel. Using the built-in meters of the software, they observed the elapsed time and that the graph of the vertical distance above the center of rotation was a sine curve. They did not systematically confirm the details of these graphs for the range of situations that had been created in the analytic environment. The students did, however, move the "data" from the simulation for the initial conditions of the first ride into the analytic environment and confirm that these "results" were the same as their earlier analysis.

Conclusions

The bicycle wheel demonstration provided the students with an initial experience which allowed them to extend the sine, cosine and tangent functions to include angles greater than 90 degrees. The analytic tools provided the flexibility of tables and graphs, enabling the group to define both equations and related graphs that described the relationship between the angular motion of the wheel and the vertical distance above the hub. The students were able to move the data from their simulation experiments into the analytic environment for analysis. The use of a physical experiment, computer simulation, and analytic tools enabled the students to convince themselves through multiple and varied approaches of the validity of these relationships. This study provides evidence that an integrated approach to modeling is both productive and empowering for students.

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STUDENTS' CONCEPTIONS OF FUNCTIONS IN A COMPUTER-RICH PROBLEM SOLVING ENVIRONMENT

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This report presents a description of certain aspects of a research project which examined the effects of the Computer-Intensive Algebra curriculum on students' understanding of the function concept. The paper begins with a overview of the experiment and a brief summary of the results. It then focuses on one facet of the investigation, namely, the theoretical framework delineating a conceptual knowledge of functions. In particular, the problematic of the reification construct is discussed at that time.

In response to calls for reform in the teaching and learning of mathematics (NCTM, 1989; NRC 1989), an innovative curriculum, Computer-Intensive Algebra (CIA), was developed as an alternative approach to traditional algebra (TA). CIA is a function oriented curriculum characterized by (a) a problem solving approach based on the modelling of realistic situations, (b) an emphasis on conceptual knowledge, and (c) the extensive use of technology. Whereas previous research (Boers-van Oosterum, 1990; Matras, 1988) had shown the beneficial effects of CIA on students' problem solving abilities and on their understanding of variables, the project described here (O'Callaghan, in press) was aimed at examining the curriculum's effects on the students' understanding of functions.

Overview of Experiment

The experiment was conducted at Southeastern Louisiana University during the spring semester, 1993. The CIA curriculum was implemented in one section of the College Algebra course, and this class was then compared to the TA classes. Overall, there were 802 students included in the study; but the primary focus was on the experimental class (CIA) and two control classes (TA1 & TA2). Both quantitative and qualitative data were collected from these groups in an effort to probe their conceptions of functions as thoroughly as possible.

Many instruments were used in the quantitative portion of the experiment, most notably, a pretest and posttest on functions. These tests were designed by the researcher based on the theoretical model for the function concept as described later in this report. Other important instruments were the departmental final examination and two attitude measures, which were also analyzed for the possibility of differential effects resulting from the two curricula. The qualitative aspects of the project consisted of two sets of interviews conducted with students from the CIA and TA classes.

Summary of Results

Both the quantitative and qualitative analyses revealed that the CIA students were better than their TA counterparts in every competency described in the function model except for reifying. The indications for that component were that this level of abstraction was beyond the reach of both groups. Furthermore, the interview data suggested that the two groups were forming very different conceptions about functions. While the TA subjects thought of functions as formulas or equations, the CIA students described them as dependency relations involving input and output variables. Also, those in the CIA group were more inclined to use functions in their problem solving attempts; and they had clearer conceptions of domain and range.

Relevant to the affective domain, the CIA subjects showed significant improvements in their overall attitudes toward mathematics and significance decreases in their anxiety in regard to the subject. They also expressed the opinions that their class in general, and functions in particular, were important and relevant to their lives both in and out of the classroom.

Theoretical Framework

The basic foundation for this research was the theoretical framework proposed to describe a conceptual knowledge of functions. It is a synthesis of the ideas of the researcher and other mathematics educators (Fey, 1992; Kaput, 1989; NCTM, 1989; & others) as expressed in the literature on this topic. Comprised of a theoretical basis and a function model, the framework represents this author's attempt to impose some structure and organization onto the complex web of meanings that is the function concept.

Theory

Thompson (1985) says that the purpose of education is to develop intelligence through problem solving. He further states that the essential feature in the construction of mathematical knowledge is the creation of relationships, and creating relationships is the hallmark of problem solving in mathematics. Since functions are the mathematical tools used to describe the relationships between variable quantities, they are at the core of the mathematical problem solving process.

Kaput (1989) describes four sources of meaning in mathematics, which he divides into two complementary categories. The first category is called referential extension and is accomplished via (a) translations between mathematical representation systems and (b) translations between mathematical and non-mathematical systems, such as physical systems or natural languages. The second category, named consolidation, consists of (a) pattern and syntax learning through transforming and operating within a representation system and (b) building conceptual entities through reifying actions and procedures.

Another important source for the proposed function model is the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989). According to the NCTM, an understanding of the function concept involves the ability to (a) model real world phenomena with functions, (b) represent and analyze relationships using equations, tables, and graphs, (c) translate between these three different representations, and (d) understand operations on and properties and behavior of certain families of functions.

Function Model

Firmly rooted in a problem solving environment, the proposed function model is formulated in terms of the uses of functions to solve problems. The rationale for this approach is the nearly universal agreement that problem solving should be the focus of school mathematics and that functions are the primary tools in that process. The model consists of four component competencies as described in the following paragraphs.

Modelling. As shown in Figure 1, the process of mathematical problem solving involves a transition from a problem situation to a mathematical representation of that situation. This process entails the use of variables and functions to form an abstract representation of the quantitative relationships in that situation (Fey, 1992). The ability to represent a problem situation using functions is the first component of an understanding of the function concept and will be referred to as modelling.

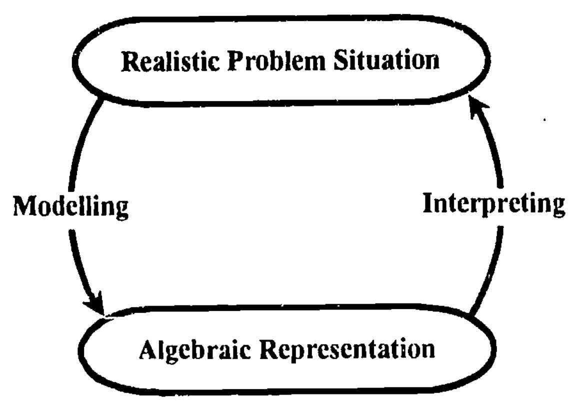

Figure 1. Translations between the real world and algebraic representation.

This modelling component can be divided into a number of subcomponents depending on the representation system used to model the situation. The three most commonly used representations for functions are equations, tables, and graphs. For example, a task could require students to write an equation, compute pairs of values, or sketch a graph corresponding to a particular situation.

Interpreting. The reverse procedure is the interpretation of functions (Fey, 1992) in their different representations in terms of real-life applications (see Figure 1). This ability, labelled interpreting, will be considered the second component of a conceptual knowledge of functions. This component can also be analyzed at a finer grain size and partitioned into subcomponents. These would again correspond to each of the three main representation systems for functions.

Translating. As mentioned previously, the mathematical model may be represented in various ways within the algebraic representation system. The three forms that are most commonly used are (a) symbolic - equations or formulas, (b) tabular - pairs of values for the related variables, and (c) graphical (see Figure 2). These have been called the three core representation systems (Kaput, 1989). Verstappen (1982) refers to them as the algebraic, arithmetic, and geometric categories for recording functional relations using mathematical language. The ability to move from one representation of a function to another, or to translate, is the third component in the function model.

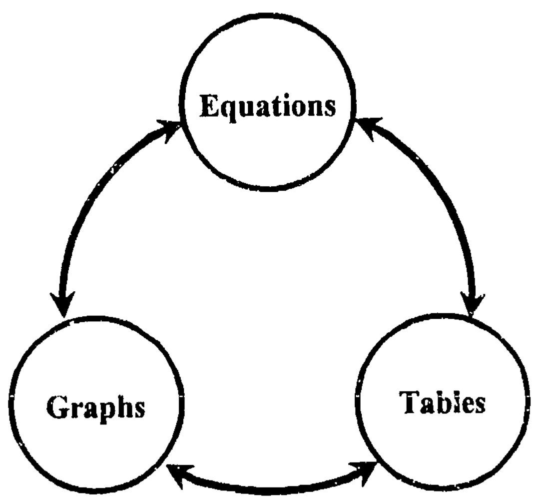

Figure 2. Three core representation systems for functions.

Reifying. The final component in the theoretical model for functions is reifying. Reification can be defined as the creation of a mental object out of what was initially perceived as a process or procedure. This mathematical object is then seen as a single entity that possesses certain properties and that can be operated on by other higher-level processes, such as transformations or composition.

Reification has been described as the final stage in the acquisition of the function concept (Thomas, 1975) and as one of the most essential steps in learning mathematics (Dreyfus, Artigue, Eisenberg, Tall, & Wheeler, 1990; Thompson, 1985). Based on Piaget's notion of "reflective abstraction," it has been said to represent the ultimate goal in the instruction and learning of functions (Kaput, 1989; Kieran, 1990). Yet, reifying is a very difficult process; and very few students ever achieve this conceptualization of functions (Sfard, 1989).

Evaluation of Model

With some qualifications, the proposed theoretical framework for the function concept worked quite well in the study. The model made it possible to organize and operationalize this abstract and complex concept. It provided a well organized and clearly defined structure for the research questions of the study, for the instruments used to investigate those questions, and for the analyses of the results from those instruments.

The individual components of modelling, interpreting, and translating were all rather straight-forward to operationalize and assess. However, there were some areas of overlap and ambiguity even for these components. For example, the processes of modelling and interpreting probably operate more in a cyclical pattern than in isolation within a problem. Thus, a question that was intended to focus on the modelling component necessarily involved a certain amount of proficiency in interpreting. Relevant to the translating component, there was a potential problem involving questions aimed at probing the students' ability to translate from an equation to a graph. In response to these questions, a student might actually perform two other translations, first from equation to table and then from table to graph.

Reifying was unquestionably the most troublesome component for the researcher as well as the students. Its definition leads quite naturally to the operationalization used for reifying in this experiment. The probes for this component investigated precisely the two characteristics described, namely, the students' knowledge of the properties of certain families of functions and their ability to perform operations on them.

The problematic encountered here was that these probes were not sensitive to the nuances of the reification construct. A knowledge of the properties of functions, even when expressed in terms of their various representational systems, was not a convincing indicator that abstraction had been achieved. Similarly, the ability to correctly combine functions could be based on a procedural, rather than a conceptual instantiation of that process. In fact, the evidence from the interviews strongly suggested that this was the case for most of the students.

Conclusion

The ultimate goal of this research is to improve the teaching and learning of algebra, particularly in relation to the concept of function. The theoretical model proposed here provides some structure and organization for the complicated fabric of interrelated ideas comprising this notion. It is hoped that other researchers will continue and extend the investigation of functions within this framework. In particular, the reifying component has manifested itself as an appealing area for further analysis and exploration. A more complete and refined understanding of this and other aspects of the function concept and its acquisition is key to designing ways to help students develop powerful conceptions about this most important mathematical entity.

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TOWARDS AN ALGEBRAIC NOTION OF FUNCTION: THE ROLE OF SPREADSHEETS

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The idea that simultaneous variation of two variables in a numeric function table is governed by a general rule is one of the basic notions upon which the concept of mathematical function is developed. In this paper we report the results from a Mexican/British collaborative project, which investigated the potential of a spreadsheet enviroment to help pupils develope the algebraic notion of a functional relationship. The results suggest that: 1) when working with a spreadsheet pupils are faced with the need to use a language which captures the generality and variation aspects of a function; 2) this, in turn, influences their capability to express general function and inverse function rules.

Introduction

This paper presents the results of a Mexican/British collaborative project which investigated the ways in which students use a preadsheet to solve algebra problems. Within the project we worked with groups of 10-11 year old and 14-15 year old pupils. The results of the study with 14-15 year old pupils are discussed in this paper. These pupils were chosen from pupils who throughout their schooling had all experienced extreme difficulties with learning mathematics. We particularly wanted to investigate the potential of spreadsheets for pupils who are normally denied access to working with algebraic ideas. Many secondary school pupils find it difficult to conceive that the simultaneous variation of two variables in a numeric function table is governed by a general rule. Nevertheless, this idea is one of the basic notions upon which the concept of mathematical function is developed. This paper focuses on the ways in which the pupils used a spreadsheet to express the algebraic notion of a functional relationship. Results of the work on algebra word problems are reported in Rojano and Sutherland (1993) and in Sutherland and Rojano (1993).

Theoretical Background

There are a number of interrelated aspects of Vygotsky's work which influenced the research. The first is the idea that it is "the person-acting with mediational means" which is the focus of the study and analysis (Wertsch, 1991). Sign systems are considered to be mediators of action an these include "various systems of counting; algebraic symbol systems; works of art; writing schemes; diagrams, maps, and mechanical drawings, all sorts of conventional signs" (Vygotsky, 1962, pp 137). So, from the point of view of this study, we have focused on the ways in which pupils make use of the spreadsheet-algebraic symbolism to solve function problems and how this relates to their use of other mediational means, such as natural language, arithmetic language and algebraic language. Vygotsky stressed that mediational means are sociocultural in the sense that mediated action can not be separated from the social setting in which it is carried out. From the point of view of this study this suggests that studies of pupils can not be separated from influences such as the classroom setting which includes the teacher's use of language. Central to Vygotsky's work is th eidea that instruction supports development which he elaborates in terms of the zone of proximal development (Vygotsky, 1978). We have used this idea when carrying out the individual interviews with pupils, taking account of whether or not they answered a question without support from the teacher, with a nudge from the teacher, with considerable support from the teacher or with the support of a spreadsheet.

Computer developments are rapidly changing what it is possible to manipulate on the screen, which has implications for a theory of mediated action. Results from previous and ongoing projects point to the importance of computer-based algebra-like sign systems as mediators of pupil's algebraic problem solving processes (Sutherland, 1992). Much of the work in mathematics education has taken a theoretical position in which the emphasis is more on "what goes on in the student's mind" than on "the student acting with mediational means" with a suggestion that notation systems *constrain* pupils' thinking. This tends to have more of a negative connotation than the idea that a sign system mediates thinking. Also there has been a tendency to assume that pupils will first solve a mathematical problem with natural language before they can solve it with algebraic language. The results of this project suggest that development is more complex than this.

Methodology

During the experimental work pupils engaged in a sequence of spreadsheet activities and were interviewed individually at the beginning and end of the study. The sequence consisted of two blocks of spreadsheet activities[1]. The first block focused on the idea of function and inverse function and equivalent algebraic expressions and the second block on the solution of algebra story problems. The interview consisted of questions related to the solution of function tables (see for example Fig.1) and algebra story problems. Here we focuse on our analysis of

[1]The teaching sequence was carried out over a period of approximately 5 months with the pupils engaging in 10 hours of "hands on" spreadsheet work.

the function and inverse function activities carried out by the 14-15 year old pupils. Both British and Mexican pupils were low matehmatical attainers. The British pupils had been taught very little algebra and had no knowledge of the algebraic language. The Mexican pupils had previous experience of solving algebraic equations and word problems and a number of them were resistant to use the algebraic language.

Question 2c This is a table which tells you the value of y if you know the value of x.

x	y	
-2	-6	There is a rule connecting the y value to the x values,
-1	-3	i) What is the rule? ....
0	0	ii) Can you express the rule with a formula of the form y= ?
1	3	iii) What is the value of y if x is equal to 50? ....
2	6	iv) What would be the undo rule? ....

Fig. 1 Example of one of the four function questions administered to pupils in pre and post interview.

Case studies were written for each of the British and Mexican pupils. These case studies consisted of an analysis of pupils' developing approaches to solving the function table problems. The analytic framework focused on the nature of language used by pupils (natural, arithmetic, algebraic, spreadsheet) and the nature of the scaffolding provided (for example nudge from the interviewer or use of the computer).

Teaching sequence

After being introduced to some spreadsheet and mathematical ideas such as: entering a rule; replicating a rule; symbolising a general rule; decimal and negative numbers; and equivalent expressions (for example **5x** and **2x+3x**), pupils were involved in function and inverse function activities. Figure 2 shows one of the worksheets provided to the students for the function sessions. The main purpose of this part of the teaching sequence was to help pupils to develope the idea that simultaneous variation of two numeric columns can be governed by a general rule. which can be expressed in the spreadsheet symbolism. Control of variation of the independent variable (**x**) as well as predicting tasks through manipulation of such a symbolic representation of the rule, led some students to explore interesting approximation processes. The purpose of increasing/decreasing questions was to introduce students to the mathematical idea of analysing the behavior of a functional relationship within an interval[2].

[2]Function tables involving addition, substraction and multiplication rules were used during the teaching sessions. Negative numbers and decimals were included in these activities. The activities were presented in a slightly different way in Mexico and in the UK.

FUNCTION TABLES

Use the spreadsheet to reproduce the following tables.

A) Copy the rule of the function that appears in the figure. Find the value of the function, when x=30 ___________.

B) In **B4**, change the value of x to 100. Find the value of the function, for **x=115** ________.

C) Make the **x** values vary in steps of 2. Copy down the function rule, when **x** is between 1 and 200. Find the value of the function, when **x=150** __________.

When **x** is between 1 and 200, the function increases decreases remains the same.

D) Try to vary the **x** values in steps of 0.5

	A	B	C
1			
2		x	Function of x
3			
4		1	6
5		2	7
6		3	8
7		4	9
8		5	10
9		6	11
10		7	12
11		8	13
12		9	14
13		10	15
14		11	16
15		12	17

Fig.2 Example of one of the worksheets provided for the teaching of function and inverse function.

The spreadsheet activities consisted of constructing general rules to reproduce fuctions and their inverses in the form of function tables. This involved entering and copying rules of the form "=A2 * 3". The computer work involves synthesising the expression of the general rule in symbolic form and predicting the numeric values produced by this rule. In this respect there is a focuse on the general (expressed algebraically and usually in the form of a table) and the specific (expressed numerically). The spreadsheet activity provokes pupils to actively construct general rules to fit given function tables. Pupils debug the general rules on the basis of the feedback from the computer and are not reliant on the teacher for support.

Pre and Post-interview Results

The function table questions used in the pre and post - interview were presented in a form similar to Question 2c) in Figure 1 and consisted of the following functions:

2a)		2b)		2d)	
x	y	x	y	x	y
0	3	0	-4	0	0
1	4	1	-3	6	2
2	5	2	-2	9	3
3	6	3	-1	12	4
		4	0		
		5	1		

Table 1 shows pupils' responses to question 2b), in the pre and post- interview, in both groups, UK and Mexico (See **Fig.1** for items i-iv). (C) indicates *use of the computer*, (N) indicates *nudge*; and (S) is used to indicate *substantial support*..

		2bi)		2bii)		2biii)		2biv)	2bv)
UK	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Post
Eloise	-2	plus 4	"4+2=3"	plus 4	-40	54	——	- 4	x-4
Sally	——	- 4	——	——	——	46	——	——	y+4 (S)
Carla	——	- 4	——	x=0-y=4	——	46	——	plus 4	x=0+4 =y
Lucy	take 4 each time	- 4	x-4	x-4	46	46	add 4 each	plus 4	x=y+4
James	always 4 (N)	take 4	——	x-4	——	46	——	add 4	"=y+4" (C)
Lee	——	x=0-y=A1-4 (C)	——	0+4	——	46	——	B2+4 (C)	——
Anthony	all 4 apart (N)	x is 4 bigger than y	——	"-4x=0"	——	46	——	y=4-x=0	y=4+x=0
Dennis	——	"+4, -4"	——	"y=x-4"	46	——	plus 4	y=x-4	——
Mexico									
Enrique	y=x- 4	x - 4	y=x-4	y=x-4	46	46	y= x+4	y+4	D2+4 (C)
Zazil Ha	subtracting all with -4	D2-4 (C)(S)	——	y=D2-4 (C)	——	46	——	E2+4 (C) (N)	E2+4 (C)
Alda	——	E1-4 (C)(N)	——	y=x-4 (N)	——	46 (C)	——	x=y+4	F1+4
Alejan	——	A6 - 4 (C)	——	——	——	46 (C)	——	B6+4	B6+4
Edgar	x > y	E2 - 4 (C)	y=4<x	x-4	-53	46	y > x	y=x-4	E2-4
Gisel	take off from x	substract x from y (N)	x minus 2	y=-4, x=0	——	46 (C)	——	J3-4 (S)	J3-4
Carmen	substract -4	it goes by -4 (N)	y=x-4	-x-x-x-x	46	46=x	a diffnt one, one that adds	4+4	B8+4 (C)

Table 1. Children´s responses to Question 2b). In 2bv), pupils were asked in the post-interview to express with a formula the undoing rule they found in 2biv).

In the pre-interview more than half of the pupils did not answer the function table questions. Most of the pupils who did answer these questions expressed the function rule in natural language, either in a more quantitative form (like Lucy, in Table 1: "take four each time) or in a more qualitative form (like Edgar, in Table 1:"x is always grater than y"). These results are similar to the findings of MacGregor and Stacey (1993).

Concerning the inverse function questions, some students, in the pre-interview showed an understanding of the meaning of the "undo" rule but were unable to express it in a precise way (for example, Carmen´s response to question 2biv): "it is a different one, one that adds"; and Edgar´s: "y is greater than x"). Some others could give a correct answer to this item, expressed in natural language (like Lucy: "add 4 each", or Dennis: "plus four"). But in the pre-interview the majority of pupils could not answer the inverse function problems.

In the final interview, after the spreadsheet experience, the majority of pupils could answer correctly the function and inverse function questions. This was the case both when expressing a general rule and when using the rule to calculate specific values. Also, the majority of the students no longer expressed the function rule in natural language.

There were some differences between the ways in which the Mexican and British pupils expressed these general rules. For example, the Mexican pupils moved towards being able to express functions and their inverses in algebraic language whereas the British pupils moved to being able to express the same functions in arithmetic and spreadsheet language (see, for example, results in Table 1). We suggest that this relates to the students' previous experiences of the algebraic language.

In relation to the numerical aspect of the items, questions 2b) and 2c) presented difficulties for some pupils who had a poor knowledge of negative numbers. This numerical deficiency obstructed in these students their evolutionary process towards the construction of a more general rule. The case of Alejandra (a Mexican girl) illustrates well this issue.

Alejandra

As many of the pupils, in the pre-interview, Alejandra couldn't answer any of the function table questions, while in the final interview, in general, she could answer the questions without support and mostly correctly. She is a typical algebra resistant pupil. At the beginning of the post-interview, she answered the first thing that came into her mind related to school arithmetic:

Interviewer: "*....there is a rule or formula that relates the values of **y** with the values of **x**, What rule are we talking about?*

Alejandra: " *.........the rule of three?*

Prompting her to read the table, she finds the rule and expresses it in natural language. From this point on, she uses spreadsheet language and adheres to it constantly for all the items. Her answers in paper and pencil are all expressed with the spreadsheet language. It is noticeable in the post-interview how Alejandra evolved from specific thinking and natural language towards general thinking and symbolic expression (with spreadsheet symbolism). Whereas in question 2c), she gives a specific rule for each row of the table. This is a clear regression to thinking with specific cases and is probably due to an underlying presence of the "signs rule" for whole numbers.

With regards to the connection of the spreadsheet symbolism to algebra code, it was noticeable in some cases how children could move towards a "sensible" use of algebra code without any explicit teaching. The most significant cases are those found in the British pupils, who had had very little experience with the algebraic language. The case of Eloise (a british girl) illustrates this issue.

Eloise

Eloise, as many of the pupils of the study was disaffected with mathematics and disaffected with school. She was able to answer two of the simple function questions of the pre-interview although she did not know how to describe the rules in a conventional way. She used a mixture of natural language and arithmetic to describe the general rules ("The rule is +3"). In the pre-interview before the spreadsheet work, Eloise could sometimes be nudged into expressing a general rule which suggests that we were working within her zone of proximal development. By the final interview, Eloise was able to answer all the function and inverse function questions correctly and was well aware of the need for a general rule. She was able to use the algebraic language to describe the rules (direct function: =y+4; and its inverse: =x-4). When she was asked what she had learned from the computer work she said "formulas.....it's easier to work it out on the computer". When writing the rule as y=x+3 she said that she thought of **x** as a column. She was also able to find the inverse of simple functions without the support of the computer and expressed these inverses in algebraic code (something which she had never been explicitly taught).

CONCLUDING REMARKS

Given the widely reported difficulties which most pupils have with function table problems and given that the pupils in our study were particularly weak mathematically the results from the study suggest that work with a spreadsheet can produce substantial changes in pupils capability to express general function and inverse function rules. We beleive that this is because within a spreadsheet pupils are faced with the need to use an algebraic-like language which captures the generality and variation aspects of a function. In this sense the spreadsheet language takes on a mediating role in the pupils construction of the algebraic notion of function.

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STUDENTS' CONCEPTIONS OF GRAPHICAL MODELS OF LINEAR AND NON-LINEAR FUNCTIONS

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The objectives of this research were to identify students' conceptions concerning "rate of change" models of linear functions and non-linear functions, and to examine the relationships that exist between students' abilities to represent and interpret graphical models of events involving rates of change.

Students in middle grades, high school, and college (N=250) completed a paper and pencil assessment of open-ended questions relating to interpreting graphical models of rates of change and creating graphical models from rates of change information. Whole-class data were collected in mathematics classrooms during regular instructional time of volunteer teachers/instructors. Data were anonymous from the students, with whole-class data being identified only in reporting by grade/math course levels. Qualitative methods, including multiple-sorts will be used to analyze the data.

The theoretical framework of this cognitive research is constructivism. Constructivists believe that knowledge is constructed by the learner in interaction with the world and while these constructions make sense to the individual, they may not match the constructions of other students, teachers, and experts. With this research we expect to find a variety of constructions concerning linear and nonlinear functions. Initial analysis shows that students have more alternative conceptions concerning non-linear relationships than linear relationships. There were many instances of internal representations of graph as picture and misinterpretation of zero slope. Other alternative ideas to be discussed will concern fixation upon large graphical features, concavity, and across-time relationships.

Geometric Thinking

STUDENTS' DEVELOPMENT OF LENGTH MEASUREMENT CONCEPTS USING A SPECIALLY-DESIGNED TURTLE GRAPHICS ENVIRONMENT[1]

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We investigated the development of linear measure concepts within an instructional unit on paths and lengths of paths, and the role of noncomputer and computer interactions in that development. Data from case studies indicated three types of strategies for solving our different length problems: (1) Some students did not partition lengths, but also did not integrate the number for the measure with the length of the line segment. (2) Most students drew hash marks, dots, or line segments to partition lengths; they needed to have perceptible units such as this to quantify the length. (3) A few other students did not use partitioning; however, they did use quantitative concepts in discussing the problekkms and drew proportional figures.

Our goal in the present study was to investigate the development of linear measurement concepts within an instructional unit on paths and lengths of paths, part of a large-scale curriculum development project. We also investigated the role of noncomputer and computer interactions in that development.

The basis of the study's theoretical underpinnings, the instructional unit, and the software is that children's initial representations of space are based on action, rather than on passive "copying" of sensory data (Piaget & Inhelder, 1967). An implication is that noncomputer and Logo turtle activities designed to help children abstract the notion of path—a record or tracing of the movement of a point—provide a fertile environment for developing their conceptualizations of simple two-dimensional shapes. Combined noncomputer and Logo experiences also may affect young students' development of knowledge concerning linear measurement, including knowledge of the effect of unit size (Campbell, 1987). This is significant, given that these students have difficulty dealing with quantitites measured with different units (Carpenter & Lewis, 1976; Hiebert, 1981).

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Method

Participants in the first fiest test were two girls, Beth and Stephanie, and two boys, Ryan and Chris, from a rural town, all 9 years of age. Participants for the second test were students in two third-grade classes from inner-city schools, 85% African-American and most of the remainder Caucasian. As was typical for the school, 80% of the students qualified for Chapter 1 assistance.

The Turtle Paths[2] unit engages third-grade students in a series of combined geometric and arithmetic investigations. The unit teaches about geometric figures such as paths (including properties such as closed), rectangles, squares, and triangles; geometric processes such as measuring, turning, and visualizing; and arithmetic computation and estimation. Throughout the unit, students explore paths and the lengths of paths. At first they walk, describe, discuss, and give commands to create paths. Students give Logo commands to specify movements that create such paths, starting the development of a formal symbolization that is built up during the remainder of the unit. Arithmetic and geometry continue to be linked as students find the missing lengths and turns in given paths. In doing so, they must analyze geometric situations and apply addition and subtraction in a meaningful setting. Finally, students command the Logo turtle to draw many different shapes of a given perimeter, or overall path length. Students complete the unit by designing and programming a face picture, for which each part (e.g., ear, mouth) has to have a predetermined perimeter.

A modified Logo environment, Geo-Logo[3], is an intrinsic component of the instructional unit. Geo-Logo's design is based on curricular considerations and a number of implications for the learning and teaching of geometric concepts with turtle graphics (Clements, Meredith, & Battista, 1992). One critical feature of Geo-Logo is that students enter commands in "immediate mode" in a command center (though

[2]Published by Dale Seymour.
[3]Geo-Logo™ copyright, Douglas H. Clements and Julie S. Meredith. Development system copyright, Logo Computer Systems, Inc. Called Turtle Math™ as a stand-alone environment published by LCSI. All rights reserved.

they can also enter procedures in a "teach" window). Any change to commands is reflected automatically in the drawing. For example, if a student changes fd 20 to fd 30, the change is immediately reflected in a corresponding change in the geometric figure. The dynamic link between the commands and the geometry of the figure is critical; the commands in the command center always precisely reflect the geometry in the figure. Other features include a variety of icon-based tools for defining procedures, measuring, and so on.

In the pilot test, case studies were performed with all four students. In the second field test, two students were studied intensely (Jimmy and Susan); when students worked in pairs, the case study student's partner was also observed (Gina was Susan's partner).

Results[4] and Implications

Three types of strategies were observed. Some students did not partition lengths, but also did not connect the number for the measure with the length of the line segment. These students tended to be those identified by their teacher as being low in mathematical ability. We never observed them making statements that would indicate that they were operating on quantities. Furthermore, when they were asked to draw a figure such as a rectangle with certain dimensions, there was no discernible use of those dimensions in their drawings. They interpreted the problems as numerical problems, rather than as measurement problems.

The second type of strategy was most common among these third graders. These students drew hash marks, dots, or line segments to partition lengths (often not maintaining equal length segmentations). A turtle step is a small unit (1 mm or less on most monitors); moreover, children's experience is probably such that objects 100 units in length are substantial in size. These factors may have made the abstraction of the turtle step difficult for students who wished to assign numbers in a meaningful, quantitative manner. Therefore, they marked off lengths in units that made sense to them, usually units of 10. They needed to have perceptible units

[4]Only general conclusions can be presented here; full results can be obtained from the authors.

such as this to quantify the length (Steffe, 1991).

A few other students did not use partitioning (or ceased using partitioning at some point). These students, however, used quantitative concepts in discussing the problems and drew proportional figures. Therefore, we assume they had interiorized units of length and had developed a measurement sense that they could impose mentally onto figures. These observations substantiate Steffe's argument that these children have created an abstract unit of length (Steffe, 1991). This is not a static image, but rather an interiorization of the process of moving along an object, segmenting it, and counting the segments. When consecutive units are considered a unitary object, the children has constructed a "conceptual ruler" that they can project onto unsegmented objects (Steffe, 1991).

We hypothesize that once children using the first type of strategy have had sufficient physical measurement experience iterating and partitioning into units, they construct schemes that allow them to partition unsegmented lengths, but only on the figurative level. That is, they need to use physical action to create perceptual partitions. In solving problems, these partitioning schemes develop the constraint that equal intervals must be maintained. This constraint leads to the construction of an anticipatory scheme, because the equal-interval constraint can be realized most efficiently when it is done in imagery, in anticipation, without forcing perceptual markings. At this point, strategies of the third kind emerge.

Finally, some students, such as Jimmy, were skilled with numbers and computation, including mental computation. Jimmy, however, gave few indications of performing operations on spatial representations and his computations did not appear to be linked to the quantity in the situation. This should not be construed as implying that Jimmy used no imagery in performing arithmetic, but only that, when doing exact computations, he did not image the length of the segments constituting the geometric shapes in the situation we had created . In more complex situations, he used mental computations that were connected to these lengths, as he iterated another length to estimate the missing length; however, he did not consider other relationships within the geometric figure. Susan, on the other hand, is representative of students who do connect, at least in some situations, their knowledge of numbers and measurement quantities. Susan used arithmetic to solve all missing lengths tasks. Jimmy preferred to use visually-based guesses when figures were more complex, even though his arithmetic skills per se were as (if not more) sophisticated than Susan's. Thus, students who have connected numeric and spatial representations may evince different problem-solving strategies in geometric situations than those who have forged fewer such connections.

As teachers, we might view students who did not connect spatial and numerical schemes, and used only the latter to solve problems, as having different but equally effective solution strategies. We believe, however, that such students would benefit from synthesizing these two schemes. First, students need to pay attention to the scale that is provided for certain figures. This is important for their future learning of geometry, but also in many other situations, such as using maps and graphs. Second, this study's data indicates that those students with connected schemes had more powerful and flexible solution strategies for solving spatial problems at their disposal.

There are five characteristics of the computer environment that aided construction of units for these children. First was a change in problem situation. The small steps (and thus larger numbers) on the computer leads to conceptualizing and counting superordinate units (it is difficult to even image a single turtle step). So, work in this environment may encourage the creation of a more elaborated measurement scheme.

Second, the computer provides feedback that children can use to reflect on their thinking. One pair, Beth and Stephanie, had planned a rectangle with sides labeled 70 30 60 40. Beth used dashed lines to keep track of the "10-length segments" of each side, however, noncongruency of these small measurement segments allowed the drawn figure to be rectangular. When she tried this solution on the computer, however, the resulting open path immediately led her to relate her side lengths tc the properties of a rectangle. The girls changed their procedure to a rectangle of sides of 60 and 30...and believed at first that this was 200 steps in total length. With further discussion after a prompt by the teacher, they changed to a rectangle with sides of 70 and 30. In another discussion, the teacher asked children if these commands would produce a rectangle: fd 40 rt 90 fd 65 rt 90 fd 20 fd 20 rt 90 fd 50 fd 15 rt 90. Both Susan and Gina agreed that it would be a rectangle until they tried drawing it on paper. Susan's drawing was not a rectangle; Gina's was. They still disagreed, even after Gina explained to Susan that "20 and 20 added up to 40 which is actually equal to the other 40" and that the teacher "was only tricking" them. Susan felt the need to check this on the computer but then was convinced; she explained the 50 + 15 = 65 to others. The class ended with Gina and Susan giving each other "tricky" lengths.

Third, the computer context was motivating for these students. The flexibility and dynamic connections between symbol and graphic may have combined to engender a fourth advantage. Beth and Stephanie never combined their commands (e.g., combining fd 60 and fd 20 to be fd 80) when they wrote Logo commands by hand, but did combine them when they typed the commands into the computer. Only later did they begin to combine commands on paper as well.

A fifth advantage relates to the integration of the spatial and numeric. This occurred rarely, especially in noncomputer situations. When it did occur, the statement was often not about spatial extent, or length, but about dynamic movement. For example, Gina and Susan used arithmetic to solve problems, but their language was about the turtle: "It's going on 50, then 30 more, so that's...". The emphasis here is not on the geometric figure, as much as it is on the turtle's movements. Thus, the emphasis on physical action and the dynamic connections between the symbolic and graphic representations in Geo-Logo facilitated children's development of such connections for themselves. This conclusion must be tempered with a recognition that these connections were tenuous and situation-bound in many instances. It is significant, however, that we never observed students using a direct, point counting process to define units (Cannon, 1992); rather, they imaged and counted line segment units.

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THE ROLE OF LANGUAGE IN GEOMETRIC CONCEPT FORMATION:
AN EXPLORATORY STUDY WITH DEAF STUDENTS

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This exploratory study examines geometric understanding and misconceptions among five deaf students, ages 7 - 10, enrolled in a residential state school for the deaf, and their deaf teacher. During interviews the subjects visualized a triangle and then described it; sorted quadrilaterals and triangles, and answered questions dealing with triangles, squares, rectangles, and circles. After eight days of instruction, the six subjects were interviewed again. Prior to instruction, the subjects were operating at van Hiele Level 1 (Visual) in the topics examined with many misconceptions such as all triangles have two or three sides the same. Some difficulties which appear to arise from the limited exposure to mathematical language and the use of particular symbols in sign language were identified.

The van Hieles postulated that language plays an important role in the acquisition of geometric understanding. O'Neill (1968) feels that geometry is a particularly difficult subject for deaf students because of the need for language. Indeed, Kemp (1990) found the performance of deaf college age students in geometry to be low, with many misconceptions present. Hillegeist and Epstein (1989) hypothesized the cause of the comparatively poor performance of deaf students in geometry to be a combination of the increasing complexity and abstractness of the mathematical concepts and "the difficulty of finding an effective language in which to teach and learn those concepts" (p. 704).

This study examines geometric understanding and misconceptions among a deaf teacher and her five deaf students, ages 7 -10, enrolled in a residential state school for the deaf. They used a combination of American Sign Language (ASL) and Signed Exact English (SEE) for communication. Additional background information on the students may be found in Table 1.

The five students and their teacher were interviewed individually by a mathematics educator who is also a qualified interpreter. Initially, the subjects were asked to visualize a triangle and then describe it. They were also shown a set of cut out quadrilaterals and, later, a separate set of cut out triangles and

Table 1

Students' Background Information

Student	1	2	3	4	5
Age	7.8	9.3	9.5	10.10	11.1
Cause of Deafness	hereditary	hereditary	hereditary	hereditary	unknown
Degree of hearing loss	BEA 101 dB profound	not reported	BEA 96 dB severe	BEA 80 dB profound	BEA 58 dB severe
Communication method	sign & speech	sign & speech	sign & speech	sign & speech	sign & speech
Age of onset of deafness	birth	birth	birth	birth	unknown
Add'l handicaps	none	none	none	none	ADD, hyperactive, tremors on left side
Academic achievement*					
Level test	P2	P2	A1	P3	-
Vocabulary	2.2/nn	3.0/nn	7.4/nn	4.7/nn	-
Reading	2.0/nn	3.6/95	4.5/92	5.7/97	-
Level test	P1	P2	P2	P2	P2
Math concepts	2.1/27	4.1/96	3.5/79	***	2.5/44
Computation	2.7/35	3.8/94	2.8/52	4.2/68	3.2/47
Application	1.4/17	3.1/93	4.0/88	3.8/81	2.5/52
IQ (WISC-R)ß	superior	high average	average	average	average
Other		deaf parents	deaf parents	deaf parents	knew little sign entering school keen on speech

*The scores reported were obtained from the Stanford Achievement Test - 8th Edition administered approximately three months prior to the beginning of the study. The first number reported is the Grade Equivalent. The second number is the Hearing Impaired Percentile. Level P measures content commonly taught hearing students in grades 1.5-2.5; Level P2 = grades 2.5-3.5; Level P3 = grades 3.5-4.5. Level A1 = grades 7.5-8.5. The students were tested at the levels deemed most appropriate for their level of work.
nn = not normed for this age group. *** above measurable range

were asked to sort the shapes in any manner they wished, explaining their criteria for sorting. They repeated these tasks until they were unable to devise any new criteria for sorting. In addition, they were given questions developed by Mayberry (1981) dealing with triangles, squares, rectangles, and circles. After eight days of instruction by the researchers in which both the five students and their teacher participated, these six subjects were interviewed again.

Pre-Instruction Findings

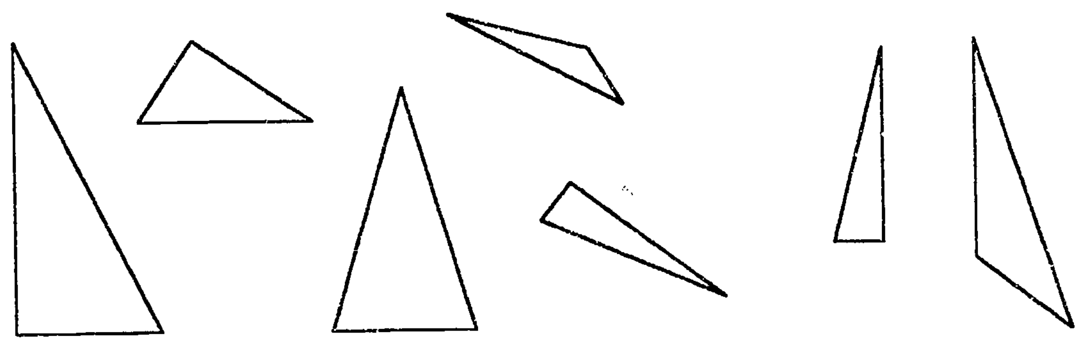
Analysis of the protocols indicates several patterns. Prior to instruction, all of the subjects, including the teacher, appeared to be functioning at van Hiele Level 1 (Visual) in the topics examined. They compared the shapes they were describing to known shapes. For example, various triangles were described as a ramp, a point on a star, a roof, a giant's slide, a pirate's hat, a tent, a wedge, a pyramid, an ice cream cone, and part of a see saw. When asked to think about a triangle and then describe it, three of the students mentioned that it had three corners. Only one student said that her triangle had three corners **and** three sides. The word "angle" was never used. None of the students displayed knowledge of any relationship between the parts of the figure. Another indication that the subjects were functioning at Level 1 is that they seemed to generally view figures holistically such as sorting shapes into categories of small, medium and large.

Some difficulties appear to arise from the limited exposure to the language of mathematics and the use of particular symbols in sign language. For example, the sign for "triangle" is roughly in the shape of an equilateral or isosceles triangle. All subjects except one sorted the isosceles and equilateral triangles into one group and identified those as being "triangles". The other three sided polygons were not recognized as triangles.

Although they did not have the correct label for right angles, the subjects did focus on the right angle in both triangles and quadrilaterals. For example, most subjects sorted the non-isosceles right triangles into a separate category. Although not recognized as triangles, the right triangles were described as shapes having "corners" or looking like ramps. (Ramps had just been studied in science.) The hypotenuse was placed horizontally, parallel to the edge of the table by the students who said the shapes looked like ramps.

Words with multiple meanings sometimes lead to incorrect but generally logically consistent answers. For example, when the subjects were asked to sign the question:

Which of these are right triangles?



every subject signed the word "right" using the sign meaning "correct". All but one subject indicated the third figure from the left as being the "right (correct)" triangle. These answers are consistent with the finding that only isosceles and equilateral three-sided polygons are triangles. With their concept image of a triangle, this would be the only correct triangle in the group.

The Interviews After Instruction

After the initial interviews, all subjects participated in an eight day geometry unit, based on the phases of learning hypothesized by the van Hieles. After the instruction, the subjects were again interviewed. All but Student 4 mentioned 3 sides and 3 angles or 3 corners as they described the triangle they visualized in their minds. For the most part, the way they sorted the shapes had changed dramatically. Now all but Student 2 identified all the three sided polygons as triangles. They were, however, somewhat tentative in their identification of non-isosceles triangles as triangles during the question and answer portion of the interview. Only Student 1 persisted in use of her criteria of "small, medium, and large," but now she said "small, medium, and large triangles." Student 3 first sorted according to the categories "equial - I forgot the rest", "their sides are uneven", and "isosceles", but she then sorted into "look like ice cream cones... don't look like ice cream cones" for her second sort.

The teacher's reasoning appeared to change more than the students' reasoning did. During the post instruction interview, the teacher often moved her hands in the air to trace the shapes she was thinking about, and apparently tried out various possibilities in this manner. However, when asked if a right triangle always has a largest angle, the teacher answered "What is meant by a largest angle? You haven't taught us." One conclusion is that she still seemed to feel that she should remember answers rather than reason them out.

Discussion

Although the students and the teacher shared many of the same misconceptions (e.g. all but one of the students thought a triangle had to have two or three congruent sides, while the teacher thought a triangle had to have all three sides congruent), it is likely that the students did not get these ideas from this teacher, who had not taught geometry to these students.

The students became aware of the parts of figures during their eight day experience with geometry, but seemed only to progress within van Hiele Level One or to the lower stages of Level Two. The teacher, however, made rapid progress and was operating at Level Three (Abstraction) in the Post-Intervention Interview, using informal reasoning and inclusion relationships frequently. The teacher, who is a fully certified teacher of the hearing impaired, admitted that math was not her strong point. She had never taught any geometry. She said that in college she took only high school level algebra and geometry - no college level math at all. This preparation is consistent with

Goodstein's findings (1981) that teachers of the deaf at the elementary level have significantly less mathematics content courses than the 12 hours of collegiate mathematics recommended as a minimum by the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America and significantly less than those taken by public school teachers. However, after only eight days of instruction, this teacher was now extremely confident of her ability to think through problems in geometry, where previously she had been hesitant.

The van Hieles' supposition that language plays an important role in the acquisition of geometric understanding was confirmed. Originally, most of the subjects did not identify scalene triangles as triangles. One factor which is hypothesized to contribute to this misconception is the iconic nature of the sign used by the subjects for "triangle." They used an initialized sign, i.e., each hand made the letter "t", and while maintaining the letter shape, outlined a triangle beginning at the apex and ending at the midpoint of the base in a standard "gravity based" configuration. The shape thus outlined is roughly an equilateral or isosceles triangle.

It would appear that prior to the intervention, all subjects were treating the sign for triangle as a picture of a triangle and not as a symbol representing the broad class of triangles. Perhaps this is due to the nature of American Sign Language or to the low van Hiele level the subjects were operating at, or a combination of both factors. As Hillegeist and Epstein (1989) noted, the property of sign language which tends to associate specific signs with specific concepts is important. The researchers did not specifically discuss with the students how the sign for triangle represented all triangles, even though not all triangles looked like it. The concept image held by the subjects was persistent, and they tended to fall back on it when in doubt. Several of the students and the teacher spontaneously finger spelled the word "triangle" in the Post-

Intervention Interviews rather than using the sign, perhaps indicating a differentiation in their minds between their new definition of the word "triangle" and what they had previously associated with sign "triangle".

The multiple meanings for the same English word such as "right" meaning "the opposite of left", "correct", and "90°" also apparently caused misinterpretations. These subjects exhibited the same difficulties with mathematical terms which also have English meanings (e.g., similar as "almost the same") as hearing students. The students should be specifically taught a mathematical vocabulary and to make distinctions between mathematical usage and common usage of words if they are to successfully solve mathematical problems.

In conclusion, the low geometric knowledge and understanding of these deaf subjects may be the result of several factors including lack of instruction, limited exposure to mathematical language, and the use of certain symbols in sign language. However, as demonstrated in this study, deaf students can learn basic geometric concepts, provided that the instruction is tailored to their particular needs in areas such as specific language development with hands-on experience following the van Hiele phases of learning.

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GEOMETRY AND VISUALIZATION IN DIFFERENT ENVIRONMENTS

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The main purpose of this study is to get a better understanding about the role of geometric contexts, different mathematical processes and specific environments in the development of student's notions about: distance, plane and dihedral angle, parallelism and flatness

A case study methodology was chosen, to find out student's meanings and strategies, their evolution and the role of the different contexts Two categories were used to look, from a semiotic perspective, at the meanings and strategies used by students. They had been classified as extrageometrical when the meanings and strategies were generated and heavily dependent on the context; and intrageometrical when the meanings and strategies were grounded in presupposed geometrical objects or relations.

Upon what could motivate intrageometric meanings and relations to appear, or why and how students may progress through these categories, Vygotsky attributed this development to the demands placed on the students by communication with adults or more competent peers.

So far most of the students, when confronted with new geometric objects or relations, rely more heavily in extrageometric features to clarify reference. It is not until later stages that, in cases of ambiguity, students endeavor to make use of geometric relations. This has not been the case in the Logo environment, where some students found almost from the very beginning symbolic demands and visual aids formally structured.

An explanation for the former could be that the mastery of pragmatically presuppositions based on not explicit geometric relations, demands to operate on a context where sign-sign relationships are essential. The distinction between finding an explicit geometric relation or making auxiliary constructions to point to the relation, requires a change where a new type of geometric object or relation becomes part of mental representations and mental processes.

It became one of our aims to understand under which constrains we could create a geometrical culture where students could assign a functional power to the geometric objects and relations. For some students systematic variants were used and tasks were gradually made more complicated, starting with more familiar objects changes were systematically introduced ending with tasks where the geometric object or relation was not explicitly embedded.

This approach became more difficult in the case where students choose to solve the problems almost exclusively with concrete based materials and found difficulty to work in the Logo environment.

Probability and Statistics

DEVELOPMENT OF THE CONCEPT OF RANDOMNESS

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Abstract

The study analyzed the development of the concept of randomness, from kindergarten to third grade to adulthood from the standpoints of: (a) the extent to which subjects evoke chance or determinism in the context of a physical phenomena with a significant aspect of randomness; and (b) the particular interpretations underlying judgments of chance or determinism. Thirty-six subjects, even divided by age-level and gender, participated in the study. We used a modification of Piaget's marble tilt box problem, which he posited as a relatively transparent example of randomness. All sessions were videotaped to enable three levels of microgenetic analysis. Just 33% of the kindergartners evoked chance, as compared to every third grader and undergraduate. However, with frequencies increasing with age, subjects at all ages evoked determinism as well. Finer levels of analysis revealed interpretations, at radically different levels of sophistication, underlying both chance and determinism.

The research literatures concerned with the cognition of chance are marked by blatant contradictions concerning what students of different ages can understand and the nature of the challenges in grasping and appropriately applying this core idea. For example, most of the developmental literature assumes that the idea of chance, at least in the form of the construct of randomness, emerges in the preschool years (*e.g.*; Fischbein, 1975; Kuzmak & Gelman, 1986). Piaget and Inhelder (1975) concluded that the idea of randomness emerged around seven years of age. However, Green (1988) concluded that the eleven to sixteen year olds' concept of randomness is, at best, fragile.

From the perspective of much of the adult judgment and decision-making literature, adequate understanding and utilization of the concept of randomness tends to looks nontrivial (Ayton, Hunt & Wright, 1989; Tversky & Kahneman, 1971). In this vein Lopes asserts, "to conclude . . . that naive person's conceptions of randomness are poor in general implies that randomness is clearly defined and well-documented by those who are not naive. Nothing could be further from the truth" (Lopes, 1982, p. 628).

In the midst of this considerable confusion, many countries have recommended that chance and probability be incorporated in the mathematics curriculum beginning in the primary grade years. The NCTM (1989) calls for children in grades K-4 to collect and interpret data, and "explore concepts of chance". A deeper knowledge of how children interpret situations involving significant aspects of chance, including the aspects they do and do not grasp and the reasoning underlying their judgments, could support our successful implementation of NCTM's statistics and probability stand.

This study addressed this need by closely examining kindergartners' and third graders' thinking in interaction with one of the classic physical randomness task domains, Piaget and Inhelder's (*ibid.*) problem of the mixing of marbles in a tilting box. Given the contradictions between the developmental and adult cognitive literatures and the significant gaps in understanding and utilizing chance that have been identified among adults, we included undergraduates in the study. This design enabled analysis of those aspects of the concept and its utilization that may or may not be a function of cognitive development or domain-specific knowledge occurring in subsequent grades. We assume that some of the contradictions in the research literature concerning what aspects of chance emerge at what age reflect the methodological tangle Newell (1972) has described as "averag[ing] across strategies". To address this potential pitfall, the study employed three levels of microgenetic analysis, as enabled by videotapes.

METHODOLOGY

Subjects

Subjects consisted of 12 kindergartners, 12 third graders, and 12 undergraduates. Subjects at all age-levels came from multi-ethnic populations and were randomly selected from those who volunteered. Each age-level had the same number of males and females.

Experimental Procedure

The experimenter showed the subject the tilt box with marbles in a row across the lower side, six of one color on the left and six of another color on the right. She asked the subject to try to predict and explain what the arrangement of marbles would look like after a tilt on its axis (straight up and back). She also asked the subject to consider whether or not one could know for sure and his or her reasons why. After the subject observed the tilt, (s)he was prompted to reflect on what happened and why. Then the experimenter elicited the subject's ideas about future events, including whether or not the arrangement would ever return to its original state and, if so, after approximately how many tilts. Finally, the experimenter asked whether identical initial conditions and identical actions on the part of the agent doing the tilting would result in the same transformation of the marble arrangement.

Data Analysis

The marble tilt box was analyzed according to a three-level analytic framework. The top two levels are considered in this paper. Level I analysis consisted of the top-grain attribution of *Deterministic, Presence of Chance*; *Other* , or *Uncodeable*. Level II consisted of specific interpretations underlying the Level I judgments. (See Table 1.) One interpretation, *Data Driven,* involved neither chance or determinism. Two interpretations involved determinism: *Order as the Natural State* and *Deterministic Physical Model.* Three interpretations involved chance: *Affordance of Movement Among Inanimate Objects, Internal Attribution of Uncertainty,* and *Indeterministic Physical Model.* Probability could be added as a modifier to an interpretation. Level III consisted of a reexamination of the videotapes to elucidate cross-age differences and commonalities concerning how and when the Level II interpretations were evoked. Two coders independently analyzed the complete data set at Levels I and II. Level III was conducted collaboratively. The results presented in this paper are limited to the first two levels.

Table 1
Abbreviated Version of Interpretations

Interpretation	Abbreviated Coding Criteria
Data-Driven Reasoning	* predictions & explanations formed on patterns the subject has observed in this situation or related task * no consideration of how these patterns are generated
Order as the Natural State	* natural state as ordered * tendency of elements to return to ordered state expressed in teleological or animistic terms
Deterministic Physical Model	* subject bases predictions on analysis of the physics of the marble tilt box apparatus and marbles movement within the tilt box * inference that physics of the situation supports precise predictions
Affordance of Movemnt among Inanimate Objts.	* chance attributed to assumption that the marbles, as inanimate objects, have no intentionality or internal controls
Internal Attribution of Uncertainty	* uncertainty stems from subject's perception of personal ignorance of the system
Indeterministic Physical Model	* subject bases predictions on an analysis of the physics of the marble tilt box and marble movements within the tilt box * the physical model does not support precise predictions
+ Probability Modifier to chance iinterpretations	* patterns in the randomness * some outcomes more likely than others * distant possibility of eventual return
Other	To be specified by the coder
Uncodeable	

RESULTS AND DISCUSSION

Level I analysis resulted in 157 coded episodes, with an interrater reliability of 92%. 111 (77%) of these episodes were also codeable at the level of specific interpretation underlying chance or determinism, resulting in an interreliability of 88%.

Level I

Analysis of evocation of chance revealed marked development from kindergarten to third grade. Whereas only 33% of the kindergartners ever evoked chance, 100% of the third graders did and 100% of the undergraduates. These findings are consistent with the Piaget's conclusion that children construct the concept of chance, as the difference between predictable and unpredictable events, about seven years of age.

However, the data concerning evocation of determinism diverges from Piaget's model. In Piaget's model, if subjects "have" the concept of chance, they are assumed to evoke it here, in what is presumed to be a particularly transparent instantiation of randomness. This study found deterministic interpretations at all age levels, manifested by 50% of the kindergartners, 25% of the third graders, and 42% of the undergraduates.

Level II

Development underlying the attribution of determinism

Analysis of reasoning underlying the attribution determinism reveals fundamentally different interpretations, at radically different levels of sophistication. Where kindergartners evoke determinism, it always stems from a belief in *Order as the Natural State*, a sense of the ordered state of the marbles as their natural state and consequently the state to which they will return. However, when undergraduates evoke determinism, it stems from *Deterministic Physical Model*, the inference that the physical apparatus can support precise predictions (either in the form of conservation of the arrangement or lawful transformations). The third graders constituted a middle case, with 17% arguing for determinism on the basis of Order as the Natural State and 8% arguing for determinism on the basis of a Deterministic Physical Model.

Development underlying the attribution of chance

Reasoning underlying attributions of chance was even richer and more varied than attributions underlying determinism. Each of the four bases for chance interpretations appears sensible, albeit from another representation of the situation. The most primitive basis for chance was *Affordance of Movement Among Inanimate Objects*, the sense that because the marbles are inanimate they have no intentionality or internal controls and thus their transformations will be chance. This manner of thinking (manifested at least once by 8% of the kindergartners and 17% of the third graders) appears to be a kind of negaticn of the position that order constituted the nature state.

Chance could also stem from their *Internal Attribution of Uncertainty*. Kahneman & Tversky (1982) have argued that this distinction between attributing uncertainty to the external world versus our state of knowledge is fundamental from an epistemological perspective. A minority of subjects at each age level exhibited this sense of chance (8% of the kindergartners, 8% of the third graders and 17% of the undergraduates).

Two other bases for chance appeared with increasing frequency with age. Chance could arise from a analysis of the physical system as not supporting precise predictions, an interpretation referred to as *Indeterministic Physical Model*. This basis was manifested by no kindergartner, 33% of the third graders and 83% of the undergraduates. Alternatively, chance could be grounded in a mathematical representation of the situation in conjunction with probability (manifested by 8% of the kindergartners, 25% of the third graders and 67% of the undergraduates). From a mathematician's perspective, it is only at the point, where subjects appreciate that patterns emerge across the course of events that are in themselves unpredictable, that they have grasped randomness. The most complex representation, basing chance and probability on an analysis of the physics, was manifested only by a minority (25%)of the undergraduates.

CONCLUSIONS

This study used the marble tilt box, a situation which the developmental literature has assumed to be a relatively transparent instantiation of randomness, to unpack the emergence of randomness and the challenges in its interpretation. We employed three levels of microgenetic analysis, in conjunction with videotaped data, to get at the meanings and thought processes underlying children and adults' attributions of chance and its absence.

Chance in the sense of the unpredictability of a single event enters in well before an appreciation of randomness, as the unpredictability of single events combined with a sense of the patterns that emerge across a large number of repetitions of the event. Interpretations at fundamentally different levels of sophistication underlay both attributions of chance and determinism. In addition to differences in conceptual development, domain-specific knowledge and epistemological dispositions come into play.

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PATHS TO INTERPRETING STATISTICAL INFORMATION

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By the time they leave high school, all Americans should be able to make informed interpretations of statistical information such as is commonly reported in the mass media. One requirement is some understanding of probability, which grows from an elementary-school recognition that some events are more likely than others, through middle-school quantification of likelihood as a number, to high-school ability to estimate probabilities either by tabulating empirical data or by enumerating possible outcomes. Another requirement is knowledge of the features of samples, from simply recognizing that a lot may be learned about something from a small piece of it, through recognizing the many ways that samples can be misleading, to appreciating that the larger an appropriate sample is, the better the information it provides. And a third requirement is familiarity with statistics that summarize group data, beginning with displays of all the data, through description of central tendencies and variation, to correlations between different variables. Obviously there are requirements along the way for mathematical ideas -- at first graphing, then fractions and proportions, and finally alternatives in mathematical modeling. Also required, however, are notions of variation within groups, of experimental control, and of bias in doing and reporting studies. A map is presented for connecting all of these ideas as students progress in sophistication from primary to high school, drawing from AAAS's research-based *Benchmarks for Science Literacy*. Participants are encouraged to suggest revisions.

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