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ABSTRACT

The Proceedings of PME-XI has been published in three separate volumes because of the large total of 161 individual conference papers reported. Volume I contains four plenary papers, all on the subject of "constructivism," and 44 commented papers arranged under 4 themes. Volume II contains 56 papers (39 commented; 17 uncommented) arranged under 9 themes. Volume III contains 53 papers arranged under 17 themes, and 4 Research Agenda Project papers. Due to space limitations, the subject content of these volumes will be represented by listing the 30 themes used to categorize the papers. Volume I: (1) Affective Factors in Mathematics Learning; (2) Algebra in Computer Environments; (3) Algebraic Thinking; (4) Fractions and Rational Numbers; Volume II: (5) Geometry in Computer Environments; (6) In-Service Teacher Training; (7) Mathematical Problem Solving; (8) Metacognition and Problem Solving; (9) Ratio and Proportion; (10) Number and Numeration; (11) Addition and Subtraction; (12) Rationals and Decimals; (13) Integers; Volume III: (14) Cognitive Development; (15) Combinatorics; (16) Computer Environments; (17) Disabilities and the Learning of Mathematics; (18) Gender and Mathematics; (19) Geometry; (20) High School Mathematics; (21) Effect of Text; (22) Socially Shared Problem Solving Approach; (23) Didactic Engineering; (24) Curriculum Projects; (25) Affective Obstacles; (26) Instructional Strategies; (27) Measurement Concepts; (28) Philosophy, Epistemology, Models of Understanding; (29) Pre-Service Teacher Training; (30) Teritary Level. Each volume contains an author index covering all three volumes. (MKR)

PROCEEDINGS OF THE ELEVENTH INTERNATIONAL CONFERENCE

ACTES DU ONZIEME CONGRES INTERNATIONAL

PSYCHOLOGY OF MATHEMATICS EDUCATION

P M E - X I

Volume I

EDITED BY / EDITE PAR

JACQUES C. BERGERON
NICOLAS HERSCOVICS
CAROLYN KIERAN

MONTREAL

JULY 19 - JULY 25

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PREFACE

A FORUM FOR RESEARCHERS

This eleventh annual meeting of PME can be singled out for the largest number of scientific communications ever contributed and for the widest geographic distribution of its participants. One of the reasons for this success must be attributed to a constant concern for improvement that can be traced back to the early beginnings of PME. The founding members will remember that following the first meeting in Utrecht in 1977, it was decided that research reports would be called for and that these would be published in Proceedings. At the very next meeting, in Osnabrück, this tradition was started and has been maintained ever since.

This concern for establishing a forum for research in mathematics education was also reflected later on when the aims and objectives were formalized in our constitution adopted at the Berkeley meeting in 1980. Two of the major goals mentioned in that document are:

(1) to promote international contacts and exchange of scientific information in the psychology of mathematics education, and
(2) to promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers.

The constitution also emphasizes the importance of research in its membership qualification, membership being "open to persons involved in active research in furtherance of the Group's aims or professionally interested in the results of such research".

Over the years, several efforts have been made to change the philosophy of PME. At different times there have been pressures to transform it variously into a more teacher oriented organization, or into a general discussion group for mathematics educators. The objectives pursued in these attempts were quite laudable, for indeed serious thought must be given to the problem of bringing research to the teaching profession. Equally important is the realization that some very serious issues exist in mathematics education which are beyond the research domain. But there are many other groups whose vocation is precisely the discussion of these questions. On the other hand, in mathematics education, there are no international groups other than PME where researchers can gather and discuss their work among themselves. Evidence that our association answers such a need can be found in the very impressive number of research reports in the PME-XI scientific program. Thus, it seems essential that PME should continue to be primarily a forum for researchers.

This is not to say that we can ignore the more general issues, such as the significance of constructivism for mathematical didactics, witness the fact that this happens to be the theme of our plenary sessions. Indeed, the discussion of such issues proves to be essential, for it provides us with an opportunity to situate our own research in a broader perspective. And it is against this enriched backdrop that we can exchange more profitably the results of our individual research.

Improving the quality of our scientific exchanges has been an ongoing concern for many years. This has been discussed at several meetings of the International Committee (I.C.). More recently, at the London meeting of the I.C., there was general approval of the suggestion that the PME-XI Program Committee formulate criteria for the selection of research reports. Following this, the President, Pearla Nesher, mandated us to carry out this recommendation. At its October 1986 meeting, the Program Committee (Behr, Bergeron, Herscovics, Kieran, Nesher, Romberg) agreed to the following criteria which were published in the first announcement:

> To allow for a broad range of research issues, both empirical and theoretical papers will be included. Papers reporting empirical research ought to deal explicitly with the following: objectives, theoretical framework, methodology, data source, results, conclusions, and importance of the study for the psychology of mathematics education. These contributions need not be limited to completed research. Ongoing studies may be reported; however, preliminary results must appear in the paper. Papers stating merely that results will be presented at the meeting will not be accepted.

> Theoretical papers are equally important. They can be quite varied and deal with questions of an epistemological nature, methodological problems, a new theoretical approach, a synthesis of the literature in a specific domain, etc. These papers must relate the issues under consideration to the existing relevant literature, indicate how their perspective differs from others, and how they contribute to the psychology of mathematics education.

The criteria we proposed were aimed at improving the readability, coherence and significance of the research reports. The need to provide a theoretical framework and to relate issues to the existing literature was considered essential in order to establish a continuity indispensable for scientific progress. The formulation of some minimal criteria for theoretical research reports was to prevent mere "armchair reflection" from being passed off as research. Our intention was to provide a forum for as many ideas as possible and to encourage a spirit of disciplined inquiry.

Some Innovations

Formulating criteria for research reports was not the only innovation carried out this year. For the first time, research report proposals were subjected to a blind review process. Each one was sent to two reviewers with experience in the given domain. They were asked to use the criteria for research reports as guidelines in evaluating the proposal and to recommend one of the following:

> In evaluating these proposals, please keep in mind that it is not always feasible to cover all the criteria in the required 500 to 700 words.
>
> **Unconditional Acceptance** indicates that the proposal deals with significant issues in a coherent manner reflecting the suggested criteria.
>
> **Acceptance with reservation** indicates that either the proposal deals with an issue of questionable importance or that it does not adhere to the suggested criteria. Please make your remarks sufficiently detailed so that we can make explicit suggestions to the author for improving the research paper.

Rejection indicates either that the issues dealt with are considered insignificant, or that the proposal is totally incoherent, or that it cannot qualify as empirical or theoretical research. Please make your remarks sufficiently detailed so that the Organizing Committee can provide a reasoned rejection.

That the implementation of criteria and a blind review process did not have a discouraging effect is evidenced by the record number (185) of research report proposals received. The review process was carried out by 52 colleagues, time constraints limiting their selection to North Americans. Those proposals which received unconditional acceptance (44) by both reviewers were so accepted by us. Where one of the reviewers recommended acceptance with reservation or rejection, we gave the proposal a conditional acceptance (132). Authors were provided with a copy of the reviewers' comments and were asked to take their remarks into consideration when writing the final version of their paper. Where both reviewers recommended rejection of the proposal, we in turn studied each one very carefully. Only 9 proposals were not accepted as research reports. Their authors were provided with the reviewers' comments and were urged to submit their contribution in the form of a poster presentation or as part of a working/discussion group.

The 176 accepted proposals resulted in 155 research reports, since 20 proposals were withdrawn for a variety of reasons (such as lack of travel funds, conflict with summer schools, etc.) and one paper was rejected for it did not develop the themes announced in the proposal. We would like to have been able to read the final drafts of the research reports to see if the suggestions of the reviewers had been taken into account, but time did not allow it. Thus, every paper that was not withdrawn or rejected appears in the Proceedings.

In order to continue improving the quality and scope of discussions surrounding the paper presentations, another innovation was planned. While in the past many contributions were grouped into subthemes (early arithmetic, geometry, problem solving, etc.), no attempt was made at bringing the reported research into perspective and suggesting future directions. Such syntheses are included in this year's program. Whenever the content of papers was sufficiently related, they were grouped into subthemes warranting a synthesis. We solicited nine established authorities to comment on these grouped sets of papers. Their task was to prepare a written response to appear in the Proceedings, present it at the Conference, and lead the ensuing discussion. In their commentary, they were asked to address more specifically the following questions:

- How has each paper contributed to this area of research?
- Are there common threads to be found in the papers? (e.g. research questions, methodologies, results, etc.)
- What are the major questions in this area of research that still need to be answered?
- Are there any indications in these papers on how to tackle them?

More than half the research papers (83 out of 155) were grouped into the nine commented subthemes. The syntheses of these papers should prove to be highly valuable. On one hand they provide the person unfamiliar with a given domain with a broad overview of the current research in that area. On the other hand, for those researchers in a given domain, they provide an opportunity to relate their individual work to that of others in the same field. Furthermore, these commentaries should stimulate a higher level of discussion at the Conference.

OUTLINE OF THE PROCEEDINGS

The Plenary Papers

As a theme for the plenary papers, we selected a broad topic of general interest in the psychology of mathematics education: the theory of constructivism. Current issues involve questions of definition and distinction from other psychological theories, the status of constructivism as a theory of knowledge acquisition, its implications for research on teaching and learning in general and for research on mathematics education in particular. These issues are addressed by four eminent scholars: Professor Hermine Sinclair who has written from the perspective of a psychologist, and Professor Jeremy Kilpatrick, from that of a mathematics educator. These two perspectives are also reflected in the two reactions given by Dr Gérard Vergnaud and Professor David Wheeler.

The commented research reports

The commented research papers have been grouped into the following subthemes and commented by:

Affective factors in mathematics learning	Douglas B. McLeod
Algebra in computer environments	David Tall
Algebraic thinking	James J. Kaput
Fractions and rational numbers	Thomas E. Kieren
Geometry in computer environments	Celia Mary Hoyles
In-service teacher training	Michael Shaughnessy
Mathematical problem solving	Edward A. Silver
Metacognition and problem solving	Frank Lester Jr.
Ratio and proportion	Merlyn J. Behr

The uncommented research reports

These papers were sometimes difficult to group since a given report could be classified under different topics. We tried as much as possible to retain the authors' preferred classification. The papers have been grouped under the following headings:

Arithmetic
Cognitive development
Combinatorics
Computer environments
Disabilities and the learning of mathematics
Gender and mathematics
Geometry
High school mathematics
Mathematics instruction
Measurement concepts
Philosophy, epistemology, models of understanding
Pre-service teacher training
Tertiary level mathematics

Papers on the N.C.T.M. Research Agenda Project

The North American Chapter of PME (PME-NA) has sponsored the reporting of the Research Agenda Project, a two-year project aimed at developing conceptual frameworks and research agendas in four critical areas of mathematics education research -- middle school number concepts, the teaching and learning of algebra, the teaching and evaluation of problem solving, and effective mathematics teaching. Papers reporting this project are the following:

The Research Agenda Project : An overview	Judith Threadgill-Sowder
Effective mathematics teaching	Thomas Cooney and Douglas A. Grouws
Learning in middle school number concepts	Merlyn J. Behr and James Hiebert
The teaching and learning of algebra	Carolyn Kieran and Sigrid Wagner

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    - La Section d'éducation préscolaire et d'enseignement primaire
- Concordia University
    - Vice-Rector, Academic
    - Arts and Science Faculty
    - Department of Mathematics
- L'Université du Québec à Montréal
    - Décanat des études avancées et de la recherche
    - Département de mathématiques et d'informatique
- The North American Chapter of PME.

We also wish to express our heartfelt thanks to the many people who contributed to the success of this Conference. To begin with, the commentators who in a very short period of time (about three weeks) produced a synthesis of the papers they received, the reviewers some of whom handled as many as 12 research report proposals, and the many committee members listed below:

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With many thanks,

Jacques C. Bergeron

Nicolas Herscovics

Carolyn Kieran,

The Editors

HISTORY AND AIMS OF PME

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany, in 1976. PME is affiliated with the International Commission for Mathematical Instruction (ICMI). Its past presidents have been Professor Efraim Fischbein of Tel Aviv University, Professor Richard R. Skemp of Warwick University, Dr Gérard Vergnaud of the Centre National de la Recherche Scientifique in Paris, and Professor Kevin F. Collis of the University of Tasmania. The ten previous annual meetings have taken place in The Netherlands (Utrecht), West Germany, the United Kingdom (Warwick), the United States, France, Belgium, Israel, Australia, The Netherlands (Noordwijkerhout), the United Kingdom (London).

The major goals of the Group are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

International committee members

Present officers of the group :

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Gender and mathematics

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High school mathematics

Mathematics instruction

Effect of text

Measurement concepts

Philosophy, epistemology, models of understanding

Pre-service teacher training

Tertiary level

PLENARY PAPERS

What Constructivism Might Be in Mathematics Education

Jeremy Kilpatrick

University of Georgia

It is tempting to begin by comparing the constructivist movement in mathematics education, at least as it is being manifested in the United States, to any of the waves of religious fundamentalism that have swept our society in its three-and-a-half-century history. A siege mentality that seeks to spread the word to an uncomprehending, fallen world; a band of true believers whose credo demands absolute faith and unquestioning commitment, whose tolerance for debate is minimal, and who view compromise as sin; an apocalyptic vision that governs all of life, answers all questions, and puts an end to doubt—these are some of the parallels that might be drawn.

I shall not begin with such a comparison, however; it would be unfair. Instead, I shall discuss what constructivism might be for mathematics educators. I was invited to examine what constructivism is from the point of view of mathematics education, but as one who stands outside both constructivism as a belief system and philosophy as a profession, I have decided that it would also be unfair for me to claim that I know, let alone could tell you, what it is. As Jere Confrey (1986) recently noted, presenting "constructivism in all its glory" (p. 347) is a contradiction, presumably because an understanding of constructivism must itself be constructed from the inside out; it cannot be simply displayed or presented. (I am tempted to add that it sounds as though an initial commitment is prerequisite to that construction, but I shall forgo that temptation too.)

In this paper, I discuss first what constructivism seems to be, to judge primarily from the writings of some authors who claim to know. Then I consider various claims that, from the outside, do not seem essential to constructivist doctrine. Finally, I explore some directions that educators who consider themselves constructivists might take.

Each of these topics is examined from the point of view of mathematics education. Although constructivism has had some influence on literary studies (von Glasersfeld, in press), it seems to be having an especially strong impact on the thinking and activities of mathematics educators. Much of this impact is undoubtedly due to our views of mathematics and the learning of mathematics. We seem to have little difficulty adopting such language as "Eddie has constructed rational number" or "Sally has constructed the fundamental theorem of calculus." Our colleagues in other subject fields, however, probably find it awkward to make such assertions as "Eddie has constructed osmosis" or "Sally has constructed the Monroe Doctrine." The claim that there is an independently existing world "out there" that can be known by the cognizing subject is explicitly avoided by constructivism. That avoidance leads some mathematics educators to reject the language of discover in favor of construct when referring to the genesis of mathematical ideas—a rejection that might seem rather easy and harmless. One can describe the recent proof of the four-color theorem, for example, as having been constructed rather than discovered without doing much violence to the ideas involved. A corresponding rejection in other fields, in contrast, might lead to such distortions as "Priestley constructed oxygen" or "Cartier constructed the Saint Lawrence River." The mutual attraction between constructivism and mathematics is an intriguing theme that cannot be developed fully in the paper but that is touched on again at the conclusion.

What Constructivism Seems to Be

Mathematics educators have arrived at their views on students' construction of mathematical knowledge by many routes, including genetic epistemology, information science, and symbolic interactionism. A comprehensive analysis of constructivist positions held by contemporary mathematics educators would undoubtedly reveal many points of agreement and many divergencies. In North America, the major exponent of constructivism, as known by mathematics educators, is Ernst von Glasersfeld (1983, 1984, 1985, 1986, in press), who through his writings and his work with Les Steffe and colleagues (Steffe, von Glasersfeld, Richards, & Cobb, 1983) has argued for a instrumentalist theory of cognition in which the mind is modeled as organizing experience so as to deal with a real world that cannot itself be known. Although von Glasersfeld's theory is far from being accepted in its entirety by all who march under the constructivist banner, it offers the most coherent and elaborated basis for an initial analysis.

An ancient, unresolved epistemological problem for Western philosophy concerns how an independent objective reality can ever be known by a cognizing subject who has no way to check what his or her knowledge is knowledge _of_. Any attempt to test the truth of what is known must itself be an act of knowing and hence subjective. Any knowledge of "objective truth," therefore, is impossible. Constructivism cuts the Gordian knot by separating epistemology from ontology and arguing that a theory of knowledge should deal with the _fit_ of knowledge to experience, not the _match_ between knowledge and reality. The only reality we can know is the reality of our experience.

The constructivist view involves two principles:

1. Knowledge is actively constructed by the cognizing subject, not passively received from the environment.

2. Coming to know is an adaptive process that organizes one's experiential world; it does not discover an independent, pre-existing world outside the mind of the knower.

As von Glasersfeld (1985, in press) and Cobb (1986) have noted, the first of these principles is much more widely accepted than the second by people who think of themselves as constructivists. The first principle is one to which most cognitive scientists outside the behaviorist tradition would readily give assent, and almost no mathematics educator alive and writing today claims to believe otherwise. The second principle is the stumbling block for many people. It separates what von Glasersfeld calls _trivial_ constructivism, what Cobb calls _empiricist-oriented_ constructivism, and what Davis and Mason (1986) call _simple_ constructivism from the _radical_ constructivism that is based on the acceptance of both principles.

Radical constructivism is radical because it rejects the metaphysical realism on which most empiricism rests. It requires that its adherents forgo all efforts to know the world as it truly is. In what von Glasersfeld (1985) terms "an even greater effort of decentration" (p. 82) than humanity needed to give up the view of our planet as the center of the universe, radical constructivists claim that we need to abandon our search for objective truth.

Constructivism appears to have been given its first formulation by Vico in the 16th century (von Glasersfeld, in press), whose motto "verum est ipsum factum" (the true is the same as the made) encapsulates his claim that we can only know what we have constructed. God can know his creation because he created it; we, however, can only know what we ourselves have created. Modern constructivism dispenses with any consideration of God's creation. It focuses instead on the clash between, on the one hand, the Kantian argument that experience can teach us nothing about things in themselves and, on the other hand, the evidence of our own experience, which says that we live in a fairly stable and reliable world (von Glasersfeld, 1984, p. 27). The developmental side of constructivism, first developed by Baldwin and by Piaget, attempts to give an account of how human beings, with access only to their own sensations and to the operations of their own minds, construct such a world (von Glasersfeld, in press).

The mechanism that constructivism postulates as driving development comes from the theory of evolution: just as the physical organism adapts to its environment, so cognition develops through adaptation. Adaptation is coping with the possible, not representing the actual. The mind constructs knowledge that adapts to the world in much the same way as one might construct a key for a lock. The key is not the image of the lock; it is, rather, one of many keys that might open the lock (von Glasersfeld, 1983, p. 95). Or, to use another metaphor, the captain sailing a ship through a channel on a dark and stormy night with no navigational aids, never actually comes to know the channel. If his ship wrecks, he learns something about what his course should not have been, but if he passes through the channel successfully, he cannot know whether his course might have been improved.

His course fit the channel, but he cannot know how well it matched the channel's topography (Watzlawick, 1984).

Radical constructivism adopts a negative feedback, or blind, view toward the "real world." We never come to know a reality outside ourselves. Instead, all we can learn about are the world's constraints on us, the things not allowed by what we have experienced as reality, what does not work. Out of the rubble of our failed hypotheses, we continually erect ever more elaborate conceptual structures to organize the world of our experience.

We are, therefore, self-organizing, self-regulating, self-contained systems (von Foerster, 1986; von Glasersfeld, 1986). Neither knowledge nor information flows in or out of us; we are informationally closed. Because we are also self-reproducing systems, we are sometimes termed autopoietic (Maturana & Varela, 1980). This conception, or rather this set of related conceptions, rests on a cybernetic analogy between human cognition and the behavior of independent effectors in protozoa and metazoa, neurons in the mammalian central nervous system, chemical reactions, insect societies, lasers, superconductors, and other systems that are far from equilibrium and to some extent self-organizing (Haken, 1977; Nicolis & Prigogine, 1977).

Because we are closed systems, language and other forms of communication entail not the interchange of ideas between us but the construction of subjective realities to fit the experiences we have had of situations we have shared. Each of us constructs meaning for the language we use as we build our experiential world, and the meaning in turn shapes that world. Meanings cannot be communicated; they are necessarily subjective.

A full account of constructivism would consider such questions as how do we know others, what is objectivity, and is there a constructivist ethics (von Glasersfeld, 1985, 1986). The constructivist view of what others see as socially constructed knowledge is in particular need of exigesis. Unfortunately, however, space does not permit an exploration of these issues.

In summary, radical constructivism seems to be an epistemology that makes all knowing active and all knowledge subjective. Following modern physical sciences in its rejection of the possibility of coming to know ultimate reality, it treats the cognizing subject as the organizer of his or her own experience and the constructor of his or her own reality. It views coming to know as a process in which, rather than taking in information, the cognizing subject through trial and error constructs a viable model of the world.

An experiment at Stanford University by Alex Bavelas captures well the essence of constructivism (see Watzlawick, 1984): The experimenter read to each subject a long list of number pairs (e.g., 31 and 80, 77 and 15). The task was to say whether or not the two numbers "fit." After each response, the experimenter would indicate whether or not it was correct. The subjects invariably wanted to know in which sense the numbers were to fit and were told that the discovery of those rules was precisely the point of the task. The subjects then assumed that they were engaged in a typical trial-and-error experiment and proceeded to make random "fit" and "do not fit" responses. At first, the subjects were wrong every time, but as they formulated hypotheses as to how the numbers were related, they gradually began to improve, and eventually every response they made was correct. Their hypotheses, though not perfect, received increasing support.

What the subjects did not know was that the experimenter's responses followed a predetermined sequence from all incorrect, through a varying mixture of incorrect and correct, to all correct. The sequence had no connection to the choices the subjects made. When the experiment was over, however, and the subjects were told of the deception, they refused to relinguish their assumption that there was an order in the number pairs. Some subjects even claimed that there was a pattern in the numbers that the experimenter had not been aware of.

In an objective sense, there was no order in the number pairs. That did not stop the subjects, however, from claiming that they had discovered an order. They constructed a reality to fit their experience, and they can serve as models of how all of us—in the eyes of constructivism—organize our experiential worlds.

What Constructivism Seems Not to Be

As a theory of knowledge acquisition, constructivism is not a theory of teaching or instruction. There is no necessary connection between how one views knowledge as being acquired and what instructional procedures one sees as optimal for getting that acquisition to occur. Epistemologies are descriptive, whereas theories of teaching or instruction must necessarily be theories of practice (Kerr, 1981). Nonetheless, constructivists have sought to derive implications for practice from their theory, and in some writings the implication seems to be drawn that certain teaching practices and views about instruction presuppose a constructivist view of knowledge. That implication is false.

Von Glasersfeld (1983, in press) has identified five consequences for educational practice that follow from a radical constructivist position: (a) teaching (using procedures that aim at generating understanding) becomes sharply distinguished from training (using procedures that aim at repetitive behavior); (b) processes inferred as inside the student's head become more interesting than overt behavior; (c) linguistic communication becomes a process for guiding a student's learning, not a process for transferring knowledge; (d) students' deviations from the teacher's expectations become means for understanding their efforts to understand; and (e) teaching interviews become attempts not only to infer cognitive structures but also to modify them. All five consequences fit the constructivist stance, but they appear to fit other philosophical positions as well.

Teaching and Training

The contrast between teaching and training is an old one in educational philosophy. Most people would probably argue that although the two concepts are different, training is a part of teaching when aimed at actions that display some intelligence (Green, 1968). The essence of the distinction between the two seems to hinge on whether the action involves explanations, reasons, argument, and judgment—presumably the sources for the teacher to conclude that the student has understood. Making the distinction into a dichotomy ignores the contexts in which the two terms are used interchangeably but may be useful if it can be defended. Certainly the understanding/behavior contrast fits with the traditional view of teaching as giving instruction, aiming at the acquisition of knowledge and belief, as opposed to forming habits and engendering repetitive behavior.

Inside Versus Outside

The attention to processes inferred as going on inside the learner's head rather than to the learner's overt responses seems to be a hallmark of the constructivist position. On the one hand, it is difficult to imagine any teacher—even Skinner, when he is teaching—looking at a student's behavior only as uninterpreted behavior and not using it to make inferences about what the student was thinking. Any effort aimed at detecting signs of thinking, which teaching most assuredly is, must assume that the teacher makes such inferences. On the other hand, the most radical constructivist, lacking direct access to the student's mind, is forced to fall back on overt responses as the only constraints the world provides for making inferences about internal processes. What else is there? The contrast, then, seems truly one of focus. The behaviorist teacher attempts to see in the overt behavior; the constructivist teacher attempts to see through it. The ensuing teaching actions, however, may not be any different.

Constructed Versus Transferred

The metaphor of knowledge being constructed by the learner, like the metaphor of knowledge being transferred during teaching, is only a metaphor. Both metaphors seem to have some utility for describing what goes on when one person is teaching others. When constructivists shift their attention from students to teachers, they observe that many teachers quite happily use the transport metaphor:

"I got the ideas across." . . .

"Some students pick things up quickly."

"Why can't we chop this section of the content out?"

"The teacher is a medium for delivering curriculum to students." (Davis & Mason, 1986, pp. 8-9)

The teachers quoted evidently have constructed a model of the world in which the transport metaphor provides a viable way of talking about instruction. That model is apparently wrong (I am not sure how the constructivists have come to know that it is wrong, but assume they have), so the task facing the constructivists is to change the teachers' model. The strategy they have adopted is to deny the validity of the metaphor ("knowledge cannot be transferred to the student by linguistic communication," von Glasersfeld, in press) and to attempt to change the metaphor by changing the language used to talk about instruction ("teachers with a constructivist leaning are likely to see themselves not as delivery agents of an educational system, but more as gardeners, tour guides or learning counsellors," Davis & Mason, 1986, p. 9). Whether teachers can be moved to revise both their language and their conception of instruction remains to be seen. Cobb (1983), conceding that constructivists "often manage to tie ourselves in linguistic knots" (p. 1), attributes the problem to a quest for precision. A plausible alternative hypothesis is that it stems from an aversion to common language forms that ordinary people find viable but that signal dangerous thoughts to constructivists.

Cobb (in press) has argued for a constructivist analysis of mathematics instruction over a transmission analysis because (a) mathematical objects and structures that the teacher can "see" are unlikely to be apparent to students, (b) students' misconceptions are better understood when seen as arising from alternative constructions of meaning than as failures in communication, and (c) theories of instruction ought to be consistent with theories of learning and conceptual development. It is not clear how the abstract nature of mathematics fits a constructivist analysis better than a transmission analysis. People who conceive of teaching as, at least sometimes, transmission ought to be just as puzzled as the constructivist over how to put mathematics into a tangible form that can be examined, talked about, and symbolized. Contrary to Cobb's argument, one need not claim that mathematical structures are somehow visible in the environment in order to hold that ideas about those structures can be communicated to students. If you doubt that, ask the next instructor of collegiate mathematics you encounter. The case of misconceptions is similar; one can model misconceptions as arising from alternative constructions or from a breakdown in communication. Either can lead to attempts to find out what the student is thinking. The issue of consistency is a different matter. It becomes an argument for employing constructivism as an approach to teaching only if one accepts constructivism as an adequate description of the acquisition of knowledge. It is not by itself an argument for constructivism.

Unexpected Errors

The attention that constructivists have paid to teachers' expectations and students' deviations from those expectations as clues to students' thinking is one of the most attractive and promising aspects of constructivist work. Many models of the learner treat the learner as someone who is attempting to make sense of the teaching encounter. Consistency demands that they also treat the teacher as someone who is attempting to make sense of that same encounter. Constructivists have drawn our attention to the teacher's view of the student's knowledge as a phenomenon worthy of investigation. But again, one need not be a constructivist to be interested in, or to study, the errors students make that are contrary to the teacher's expectations.

In fact, the constructivist view may turn out to be something of a liability. Whereas the transmission view of teaching takes successful communication as what Cobb (in press) terms its "paradigm case," the constructivist view takes as its paradigm case the situation in which communication breaks down and students and teachers "talk past each other." This argument may yield a conception of communication in teaching as a process that fails most of the time. Of course, we learn from the errors that we and others make, but a full view of cognition suggests that we also learn from our successes. One cannot deny that the world is full of classrooms in which much miscommunication about mathematics is taking place. To take miscommunication as the paradigm, however, is to ignore the role of successful communication in promoting learning. The negative feedback model may be useful in describing self-organizing systems that do not mind having negative feedback models of themselves, but its utility in describing teachers and students to themselves may be limited. Few people respond well to claims that they are failing most of the time, especially when their own models of their communication are signalling success. It may be more productive in the long run to show teachers and students that the glass is not just half empty but also half full.

Teaching Interviews

Steffe and his colleagues (1983) have pioneered an extension of Piaget's clinical interview in which a child is set a mathematical task, the response is analyzed in terms of a model of the child's understanding of the task constructed from an interpretation of that and other responses, additional tasks are given to test the model, and instruction is provided by the interviewer in an effort to develop the child's conceptual structures and to model that development as it occurs. The term teaching experiment is often used to describe such an interview, but that term refers more appropriately to a procedure from the Soviet Union in which a class is instructed by their regular teacher and an experimenter uses their responses, together with data from interviews with selected students, to guide, in consultation with the teacher, the course of subsequent instruction. Teaching interview seems a more appropriate term for what Steffe and his colleagues do.

History, not logical necessity, links the teaching interview to constructivism. Interviews in which instruction occurs have never been popular in research traditions that demand a high degree of control because the instruction would likely be quite variable and comparisons would be difficult to make. Nonetheless, teaching interviews have for some time been populer in Europe end the Soviet Union as a means of studying cognition. They have made their way to North America independently of constructivism.

The impulse to adopt the learner's point of view when one is teaching is a worthy impulse. Successful teaching, like successful communication, depends on having a good model of the other. Constructivists, however, do not have a monopoly on the view of teaching that sees students and teachers negotiating as they develop shared meanings. They are not the only people who believe that teachers should listen to students and attempt to understand what they are thinking. They are not the only ones to encourage investigative work by students—any more than behaviorists are the only ones who give lectures.

What Constructivism Needs to Be

Connected to Ontology

A central problem with constructivism seems to be its relation to ontology—what is. Von Glasersfeld (1985) claims that constructivism "deliberately and consequentially avoids saying anything about ontology, let alone making any ontological commitments. It intends to be no more and no less than one viable model for thinking about the cognitive operations and results which, collectively, we call 'knowledge'" (p. 100). Nonetheless, constructivists seldom behave as though they have made no ontological commitments, let alone that their view is only one among many. To reject "metaphysical realism" is to take an ontological stand. Cobb's (1983) eschewal of "realist language" expresses an ontological view. Contrasting radical constructivists with realists (Davis & Mason, 1986), by saying what constructivism is not, contributes to the construction of a constructivist ontology. Furthermore, such arguments as those given by von Glasersfeld (1985, in press), Cobb (1986), and Davis and Mason to the effect that the only good constructivist is a radical constructivist implicitly reject the viability of alternative views within constructivism, let alone outside.

Constructivists need to clarify and develop their ontological commitments. Cutting epistemology loose from metaphysics as a way of solving the epistemological dilemma does not provide a satisfactory resolution of our problems as educators. We need an epistemology that takes ontology into account. "We must keep metaphysics and epistemology tied together so that (a) our explanation of Knowledge does not leave us committed to there being things we cannot account for in our theory of Being, and (b) our theory of Knowledge (thus restricted) can accommodate our claim to know what Being is" (McClellan, 1981, p. 265).

Connected to Mathematics

I referred at the outset of the paper to the affinity between constructivism and mathematics, so it might seem inconsistent to suggest that constructivism become more or better connected to mathematics. What I mean is that constructivists need to think through and spell out more clearly than they have done thus far the relationships between constructivism and both mathematics as a discipline and mathematics as a school subject.

Von Glasersfeld (in press) has noted that "constructivism has as yet only an implicit relation with the constructivist approach to the foundations of mathematics (Lorenzen, Brouwer, Heyting)." The foundations of mathematics may not pose as much of a problem to constructivism as the practice of mathematics. As Davis and Hersh (1980) contend, "the activity of mathematical research <u>forces</u> a recognition of the objectivity of mathematical truth. The 'Platonism' of the working mathematician is not really a belief in Plato's myth; it is just an awareness of the refractory nature, the stubbornness of mathematical facts. They are what they are, not what we wish them to be" (p. 362). Or, as Gardner (1981) put it, "the existence of an external world, mathematically ordered, is taken for granted. I have yet to meet a mathematician willing to say that if the human race ceased to exist the moon would no longer be spherical" (p. 37). Constructivism needs to come to terms with mathematical realism.

Moreover, constructivism needs to address the claims of a new approach to the philosophy of mathematics, "quasi-empiricism" (Tymoczko, 1985), which studies the practice of mathematics in a sociohistorical context and which appears to be compatible with both realist and constructivist mathematics. Mathematics seems to be wearing a more human face these days; one hears of "mathematics as a humanistic discipline." If indeed it is a humanistic discipline, then perhaps radical constructivism can find a voice to speak to all of the humanities and not just the ones seen as the most abstract and subjective. Vico claimed: "Mathematics is created in the self-alienation of the human spirit. The spirit cannot discover itself in mathematics. The human spirit lives in human institutions" (cited in Davis & Hersh, 1986, p. x). As Davis and Hersh (1986, p. 305) observe, perhaps some day the shade of Vico will look down from Elysium and acknowledge that mathematics is a human institution. And perhaps other constructivists will some day acknowledge that their view of mathematics has not dealt adequately with mathematical practice.

Nor has it dealt adequately with school mathematics. Epistemology alone cannot answer the question of what mathematics to teach. An analysis of knowledge cannot yield a curriculum. The curriculum depends on our purposes, on what we value, about which epistemology is necessarily silent. To think otherwise is to commit what Martin (1981) terms "the epistemological fallacy." Some constructivists (Kamii, 1984; Steffe, 1987; Thompson, 1985) have attempted to build curriculums on a constructivist foundation. Martin argues that we first need to determine the moral, social, and political order we believe to be desirable, then set out our educational purposes, and in the light of those purposes choose curriculum content and objectives. An epistemology may be useful to us at that point in dealing with cognitive objectives, but other theories will be needed in dealing with noncognitive objectives. We need to be careful not to put the constructivist cart before the values horse.

Connected to Reality

If Bauersfeld's (1987) analysis is correct, each scientific theory in the human sciences deals with its own reality from its own perspective. Competing theories cannot judge one another, and metatheories are impossible because there is no external fulcrum on which to hang a common perspective, framework, and language. Therefore, a theory such as constructivism should be seen as having a limited domain and perspective; it cannot become a metatheory that drives all of education, let alone mathematics education. Nonetheless, there is a need for people working within one theory to communicate with people working within other, necessarily incompatible theories. A common technical language is not possible, but a common less-technical language is not only possible but essential.

Constructivism needs to become more connected to reality. Not the inverted commas "reality" about which one reads so much in constructivist writings, but the reality of everyday scientific activity, mathematical investigation, and classroom practice. People live in that reality, and they try to communicate with each other within its constraints. If constructivism has something to say about what it means to come to know mathematics beyond the mathematics of the elementary school, about how teachers might work with pupils in groups, about how indirect guidance of learning can be handled through the grades, then it needs to find a language with which to speak to teachers on those matters. Condemning everyday language by terming it "realist" or "reification" and then putting sanitizing quotation marks about each usage of such words as discover, problem structure, and error may preserve one's theoretical virtue but at the expense of reaching, and keeping, one's audience.

The virtue some constructivists need most is that of humility. It is unbecoming, if not ludicrous, for the adherents of a relativistic theory to treat it as though it were absolute and final. A theory that claims to be only one of many possible viable theories ought to be more tolerant toward competing theories. People who claim there are many possible ways to construct knowledge ought to be more friendly and understanding toward people who have failed to construct their theory.

There is a moment in the film "Let Us Teach Guessing" in which George Polya is asking a student whether, now that another case has been confirmed, she believes the hypothesis they have been exploring. She replies, "Sort of," and Polya seizes on that wording to convey the stance one ought to take toward all knowledge. We "sort of" believe—much more when we think we have a proof, much less when all we have verified are a few specific cases. The researcher and the teacher need to take a "sort of" stance toward what they are doing—having enough faith in and commitment to their knowledge to keep going forward, but keeping an open mind and being willing to reject a position when disconfirmation is found. True believers make neither good researchers nor good teachers. Mathematics educators who are not ready to become born-again constructivists may well find they can live viable lives as sort of constructivists.

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CONSTRUCTIVISM AND THE PSYCHOLOGY OF MATHEMATICS

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I am neither a mathematician nor an educator of children, and accordingly not well versed in the literature on epistemological questions and teaching, even in the field of elementary mathematics. My references to this voluminous literature will be to authors with whom I happen to have had personal contact, knowing full well that there are others who have had equally important things to say. My remarks on constructivism will be almost uniquely based on Piaget's writings, and my examples of children's mathematical reasonings mainly from authors who have some link with Piagetian constructivists thinking. My talk concerns the beginnings of mathematical reasoning, i.e. until the age of seven or so : not because I think this is the most crucial period, but because I have some experience of working with children in the pre-school age. The latter part of my talk will be devoted to what are called "story-problems", first because such problems are often treated as presenting a link between "real-life" situations and mathematical reasoning, and, secondly, because I am particularly interested in language. Finally I feel that the main purpose of my paper is to raise some questions which grew out of my study of constructivism and my, admittedly limited, knowledge of present day teaching of mathematics in kindergarten and first and second grade.

CONSTRUCTIVISM

Constructivism, as a theory of knowledge, is not easy to define or even to describe. Piaget himself gave several descriptions at different times, no doubt because certain aspects of the theory were important within particular contexts. Thus I will not try to give a full account of what Piaget meant by "interactive" or "dialectical" constructivism, but shall only touch on some points that seem to have particular relevance to mathematical thinking.

According to Piaget, the essential way of knowing the real world is not directly through our senses, but first and foremost through our actions. In this context, action has to be understood in the following way : all behavior by which we bring about a change in the world around us or by which we change our own situation in relation to the world. In other words, it is behavior that changes the knower-known relationship. From the baby who laboriously pushes two objects together or who attracts his mother's attention by crying, to the scientist who invents new ways of making elementary particles react and the child or adult who tries to convince his friends of his opinions, new knowledge is constructed from the changes or transformations the subject introduces in the knower-known relationship.

The quality of the knowledge gathered in this way is partly determined by the ways in which reality reacts to our interventions and by its correspondence to the knowledge other people have constructed. As von Glasersfeld (1983, pp. 50-51), who may be an even more radical constructivist than Piaget, puts it : "From an explorer who is condemned to seek 'structural properties' of an inaccessible reality, the experiencing organism now turns into a builder of cognitive structures intended to solve such problems as the organism perceives or conceives... What determines the value of the conceptual structures is their experimental adequacy, their goodness of fit with experience, their viability as a means for the solving of problems...".

In other words, at all levels the subject constructs "theories" (in action or thought) to make sense of his experience; as long as these theories work the subject will abide by them. Since human beings tend to push their ideas as far as they will go and actively seek novel experiences, they will partly conserve and partly transform their ideas when this experience widens, and new questions arise for which the theory is not adequate.

As Piaget, who saw himself as a realist of a rather special kind, expresses it (1980, pp. 221-222) : "With every step forward in knowledge that brings the subject nearer to his object, the latter retreats ... so that the successive models elaborated by the subject are no more than approximations that despite improvements can never reach ... the object itself, which continues to possess unknown properties..." This does not mean that the knowing subjects are forever living in a world of their own making; but it does mean that they can never get absolute knowledge of reality as it is. According to Piaget, this is applicable to children as well as to adult scientists and to science as a social enterprise.

Not only is science a social enterprise, but all humans are social beings; and it is the sharing of approximate models of theories that assures the objectivity of the knowledge gained (vs. "subjective" belief).

The fundamental constructivist view thus postulates changes in the relation between subject and object; and the movement towards better - though never perfect - knowledge of the object has as its concomitant another movement whereby the subject obtains better knowledge of his own actions or thought processes. There may not be perfect synchronicity, but sooner or later every new conquest of the world of objects will lead the subject to restructure his action- or thought operations system, just as new deductions and inferences derived from the internal system will lead to new interrogations of reality.

These movements towards ever more viable knowledge lead to different kinds of knowledge : on the one hand, the subjects' reflection on their own action-coordinations leads to logico-mathematical knowledge, and on the other hand reflection on the properties of objects and the changes actions produce leads to the natural sciences, such as physics and chemistry. However, these different types of knowledge are not symmetrical. Knowledge of the world of objects cannot be constructed in the absence of some kind of logico-mathematical framework, whereas logic and mathematics can become pure, in the sense of being free from particular contents. Clearly, this confers a special status on logic and mathematics inside the edifice of human knowledge in general. All activities imply general coordinations from the lowest to the highest level; they can all be seen as leading to mathematization. However, on this particular point I have encountered a difficulty which I have not been able to solve, and which I think important for psychologists and educators.

In certain passages (cf. Beth and Piaget, 1961, p. 251), Piagèt refers to "actions that are destined to become interiorized as operations". Actions such as combining and ordering can be performed on many different objects, but more importantly, they are, so to say, realization of the most general coordinations of schemes. In any activity, from the simple reflex pattern to learned actions such as picking flowers or solving an equation or lighting a fire, actions have to be combined and carried out in a certain order. A one-year old who collects several objects and puts them into a container one by one materializes in actions on objects coordinations that are needed for almost any action - picking up a spoon, plunging it into a carton of yoghurt and stirring, or pushing a block with a stick along the edge of the carpet, etc. etc.

The complexity of even the simplest intentional actions is enormous. As Minsky (1985, p. 21) puts it : for all of us "it once seemed strange and wonderful to be able to build a tower or a house of blocks. Yet, though all grown-up persons know how to do such things, no one understands how we learn to do them!" Minsky's analysis of the act of building a tower shows the intricate organization of ordered actions necessary for this purpose. The builder has to choose an adequate spot to start the tower, add new blocks, decide whether it is high enough. But to add a block, a new block has to be found, the hand must get it and put it on the tower top. To find a new block, it has to be seen, to get it, the hand has to move and grasp; to put it on top the hand has to move and release.

But what kinds of actions are particularly "destined to become interiorized" as mathematical operations? What are pre-mathematical activities, i.e. activities that prepare what Bergeron and Herscovics call intuitive mathematics? There is, of course, one activity that has a "mathematizing" role, and that is counting, in itself a highly complex activity (cf. Steffe & al, 1983). Gréco (1962) showed the importance of counting for numerical invariance : counting, and the one-to-one correspondences it implies transform the spatio-physical reality of a collection of objects into a numerical mathematical reality. But surely there must be other activites as well, that lead to mathematical concepts and operations?

I have yet another problem concerning constructivism and the psychology of mathematics, and that is the difference between mathematics and logic. Piaget always joins the two, and discusses logico-mathematical operations as one entity. I have found several passages (Beth and Piaget, 1961, p. 233-236) where intuitive geometrical, and more generally, spatial concepts are contrasted with classes and numbers, and more generally logico-arithmetical entities ("êtres"), but none where logic and mathematics are distinguished. Somehow or other this question also seems to be important for psychologists; maybe I will find some answers in this meeting.

A last question which has often been raised about constructivism is worth mentioning, though in contrast with my other questions it has been extensively studied : i.e. the place of social interaction in this theory. It is true that Piaget only rarely studied social interaction and that he did not carry out any studies on social interaction as a factor of cognitive progress. But since "successive models or reality constructed by the subject remain approximations" (Piaget, 1980), one needs some way of distinguishing between subjective beliefs and objective knowledge. This is where Piaget adds : "Objective knowledge is only attained when it has been d' ussed and checked by others" (Piaget, 1965). Thus it is only when our ...els or systems correspond to those of others that they become an objectively valid base for further progress. Sharing ideas, discussion and argumentation, or more simply collaboration in constructive or pretend play, are essential ingredients for the growth of knowledge, at all developmental levels. Moreover, the mastery of all conventional symbolization systems, from spoken language to spoken numerals, arithmetic and algebraic notation, depends to a great extent on interaction of young children with other people : educators, parents and older children.

After this brief sketch of constructivism and the questions this theory raises for me as a psychologist, I shall discuss some more specifically psychological concepts that belong to the theory and that, in my opinion, are fairly directly applicable to education.

1/ Normative facts

The elaboration of gradually more "viable" models leads to the construction of at first very limited systems of reasoning which in turn lead to what Piaget calls "normative facts" or "norms". Normative facts are ideas, concepts or modes of reasoning that are immediately available for the construction of new inferences or hypotheses. The subject feels such ideas to be both evident and necessary, and often can no longer imagine that at some earlier time they were not present in his mind. For example, the commutativity of addition is a normative fact from the age of about seven : 3 added to 5 gives the same result as 5 added to 3, and the same goes for 15 and 3 and 1,000,005 and 3, though if one has 3 dollars and gets 5 more that certainly makes a difference, whereas if one already has a million dollars the increase is imperceptible. Though seven-year olds may not be able to reason as far as millions, commutativity of addition is a normative fact for them, it is felt as something that is "necessary" and it can immediately serve as a base for further reasoning.

In another field, the link between volume and the amount of water displaced by an object that does not float is a normative fact from about the age of ten. According to Piaget it is the task of the psychologist to study the gradual construction of such norms by the subject, i.e. what is necessary and evident in the subject's eyes, but not whether the subject's "norms" are true in the scientific sense. Neither of the above facts is normative for four- or five-year olds, but what is more surprising is that a concept such as the commutativity of addition may first be limited in scope; e.g. that the child may use it as an immediate base for problem solving as long as the numbers do not go beyond 10, or as long as one of the numbers is either 1 or 2. It is also important to note that in the constructivist view the commutativity of addition and the relation between volume and water-displacement have a common source, i.e. in the organization of the interactions between the subject and the world of objects. To many adults, scientist as well as laymen, mathematical "truths" appear to be a priori, Platonic ideas, that emerge at some point in development, whereas physical "truths" are rooted in learning through experience, and thus fit into empiricist theories of knowledge. This is contrary to the constructivist view.

2/ Instruments of knowledge

In "Psychogenèse et histoire des sciences" (1983), Piaget discusses another task for psychologists : to find out "which kinds of instruments the subject uses for problem-solving, what their origin is, and how they are elaborated" (p.22). These instruments or processes constitute the link between the knowing subject and the objects of his knowledge, logico-mathematical objects as well as physical objects, and their study belongs to epistemology and to psychology. Piaget proposes that these instruments fall into two categories : correspondences which imply comparison on the one hand, and transformations on the other. These processes are totally general and operate at all levels of development. Every action scheme is a source of "correspondences", since it can be applied to new objects or situations that are thus compared without further transformation, and every coordination between schemes is a source of transformations, since the coordination can result in a new type of action with its particular result. There is thus right from the beginning a duality but also a linkage between the two types of processes. But since children (and adults, and scientists) first become aware of their transforming actions and their results and only later of their comparisons between the objects as such, in a static state, corespondences and comparisons remain independent of transformations, often for a long time. At different levels of development, the cycle repeats itself : subjects compare, choose objects to transform, transform, take note of the results of their transformations, and only later become aware of links between the transformations and the correspondences they establish when making comparisons.

Interestingly enough, immediately after having discussed comparison and transformation (Piaget et Garcia, 1983, p. 23-34), Piaget speaks about mathematical "beings" ("êtres") - what are they and where do they come from? (p.25). This is, of course, the 64-dollar question. Though Piaget's answer has not changed since his earlier works on number, (i.e. they derive from the subject's actions and his reflections on these actions), the two points I have just discussed seem to clarify the problem to a certain extent. Comparing and transforming in some kind of quantitative sense, as much in measuring as in counting, appear to be activities that lead to reasoning systems (even if of small scope) which imply normative facts. Measuring certainly deserves to be mentioned as much as counting (cf. Vergnaud, 1979, p. 263-274, and the discussion in Steffe & al., p. 19-20). Studies on very young children (Sinclair & al., 1982) certainly seem to show that the roots of actions that will lead to measuring and counting go back to a very early age.

Around the age of twelve months, we observed many spontaneous activities as in the two following examples (Sinclair & al., 1982, p. 63-80). The children pull little bits of cottonwool from a big ball, until it is reduced to many tiny flecks. They carefully observe the way the cottonwool stretches and then breaks. Then they make them stick together again; and then they start all over. At a slightly later age (around one-and-a-half), we observed long sequences of activities with a string : they take the ends into their hands, stretch them apart as wide as possible, touch the string with their nose in the middle, let it go slack and start again. They also put the string around their neck, pull on one side, the other hand goes up, they pull the other side down again, observe, etc. etc. It does not seem too audacious to see in these activities the very beginnings of counting and measuring, i.e. the very beginnings of what Piaget calls the slow construction of mathematical objects. Certainly such behaviors are a good example of the processes of comparison and transformation with a different focus of attention : either on separate bits (which will become countable) or on continuous lengths (which will become measurable).

In short, I think that indeed constructivist psychology and its related hypotheses lead us to see the construction of mathematical objects and operations as a slow construction, deeply rooted in all human endeavors to make sense of their world. As I have already said, however, psychologists do not yet seem to have taken advantage of constructivist theory to show how this construction proceeds in the specific domain of mathematics between the ages of two to six or seven, with the sole exception of counting behaviors. But much research and many observations of classroom behavior show that some important mathematical constructions have already taken place.

Apart from numerical conservation, Genevan research brought to light many mathematical solutions in specific tasks that do not involve counting (Gréco & Morf, 1962; Gréco, Inhelder, Matalon & Piaget, 1963) and that may precede, accompany or follow success on the numerical conservation task. Five-year olds already know that if one person always takes one object when another takes two, the former will at any stage of the proceeding have half as many objects as the latter; somewhat later children begin to understand the connexivity of number, etc. etc.

In educational settings one can also observe typical examples of mathematical reasoning, already in first grade. Kamii and DeClark (1985, p. 233) report behaviors such as the following. Ann, asked about 9 plus 9 inquires: "What is the 8 plus 8 one?" When told it was 16, she says "18. If you know that 8 and 8 is 16, you know how to skip another one and it <u>has to be</u> 18".

During a discussion about what should be brought to a party for the 26 children in the class, Mary announces : "If five people bring five apples and someone else brings one, there will be enough for everyone". These first-graders had the benefit of a special program devised by Kamii and DeClark, but they were otherwise an ordinary class - not a selected group. They certainly were reasoning mathematically; their remarks show moreover the depth and spontaneity of their reasoning, and Mary demonstrates an excellent formulation of what could be a story problem.

Evidently, during the years that precede formal instruction in arithmetic, ordinary, everyday experiences lead to important mathematical achievements by the age of six or seven, though the children may not be able to apply them to classical pen-and-paper school arithmetic. What are these experiences, and how could their stimulation help children that do not seem to have mathematized them?

Krutetskii (1976, p. 217) reports parents' and caretakers' remarks about children that turned out to be brilliant in mathematics : they were observed to be fascinated by counting from the age of three onwards. Somehow or other I suppose that this was the behavior that struck the adult observers, but that there must have been others. Kamii and DeClark, and many other researchers in mathematics education think so, too, and they often consider the early introduction of story-problems as a way of capitalizing on the children's comprehension of daily events that occur without any explicit mathematical context. However, the examples given always concern counting, addition or subtraction. It seems as if it is tacitly assumed that the only spontaneously occuring activities during the pre-school years that are "mathematizable" are those that imply counting. In other school programs the introduction to mathematics is, by contrast, limited to logic (set-theory, class-inclusion, etc.), but the problems in this framework do not seem to have any link at all with the ordinary activities of four- or five-year olds. For the moment, and as far as I know, the only way education tries to build onto such activities is the presentation of addition and subtraction story-problems.

In the last part of my paper I will make a number of critical remarks about story-problems as a psycholinguist, knowing full well that many researchers in mathematics education have made similar remarks and that my knowledge of the literature is limited.

First, in a trivial sense, story-problems are not stories, because stories tell you something, they don't ask you something : the solution to a problem is more like a story than the problem itself. But in a less trivial sense, I feel that one has to find a solution (apart from some calculations) before one can construct the problem, or maybe at the same time, but certainly not after. Formulating a problem clearly and mathematically is not a step towards its solution but part of the solution itself. The already mentioned example of Mary (p. 11) can maybe illustrate this point, which I am afraid remains rather intuitive. When Mary announces : "If 5 people bring 5 apples and someone else brings one, there will be enough for everyone", she is asked how she figured that out so quickly. And she answers : "I counted by fives. S+?5 - 10 - 15 - 20 - 25, and then one more is 26". Though I cannot explain how she came to this computation, I do think that it was the computation that allowed her to formulate a story-solution. Moreover, by saying "5 people... and someone else..." instead of "5 persons (as an adult might do)... and one..." she made perfectly clear what is in fact a problem of quantifying in natural language : someone else, not one of the five already mentioned. Precisely because story-problems only pretend to be stories, most of them continually transgress the rules of natural language usage. Natural language quantification, for example, does not directly correspond to quantification in logic.

In an interesting article Freeman and Stedmon (1986) start from the observation that English, a natural language, has at least three words that can be seen as universal quantifiers : each, every and all, and can be used in affirmative quantification with reference to an exterior reality of countable objects. This gives rise to sentences such as "All the dogs are aggressive", "Each dog is aggressive" and "Every dog is aggressive" when talking about a particular collection of dogs. If one adds universal statements such as "All dogs are mammals", "Dogs are mammals" or "The dog is man's best friend", and quantifiers such as some, as in "Some dogs are aggressive", "Some of the dogs are agressive", "There are some agressive dogs in your garden", etc. etc., the variety of natural language quantifiers may easily bewilder the subject who has to evaluate the truth of any such expression, and, we may add, any subject who has to take such an expression as the basis for a calculation.

The authors of the article I have quoted examine the case of all the, each and every, and argue that these words have both a determinative and a quantifying function : Before one can decide whether "All the dogs are aggressive" is a true statement or not one has to know which dogs are being talked about. Moreover, though as far as quantification goes all and each are equivalent, as far as meaning in ordinary language goes they are not : all is collective, each is distributive, and every is somewhere in between. After discussing several studies carried out with young children, Freeman and Stedmon conclude that children clearly have trouble coordinating the determinative and quantifying functions of expressions such as all, all the, every etc., and that it is unjustified to consider tests using these expressions as tests of logical reasoning. A similar conclusion should, in my opinion, be drawn about story-problems : when such expressions (as well as some) are used in story-problems, the problems are not necessarily tests of mathematical reasoning. Not only quantifiers, but also verb-tenses and pronominalization are used in story-problems.

In ordinary narrative discourse, a succession of the same tenses (he thought... he said... he went... he bought...) indicates a succession of reported events identical to the order of utterance; whenever the speaker intends a different order he indicates this by other markers : conjunctions such as before or after, or adverbial expressions such as already, or contrasting tenses, or combinations thereof. Simultaneity is expressed by special markers such as meanwhile, while, etc. Often, however, the addressee's knowledge of normally occurring events allows him to interpret temporal order or simultaneity as intended by the speaker, without precise indications : Mary and Anne came to visit us (i.e. together at the same time); Mary put on her socks and shoes (i.e. socks first, shoes afterwards). The linguistic and pragmatic rules for the use of tenses, pronouns and other coherence-providing devices in story-telling and understanding are not easy, and as many researchers have shown, they are still being worked out by children between the ages of 6 and 9 or 10, but many of the rules they have already constructed are to a greater or lesser degree transgressed in story-problems. In other words, there is not only a logico-mathematical graduation of story-problems according to whether they concern change of state, reunion, comparison, part-whole relationships etc. which make some problems harder than others even though they demand the same operation with the same numbers, but also a graduation in the degree these problems violate in their wording discourse rules that children have already mastered.

De Corte and Verschaffel ( ) in a detailed analysis of the strategies children use in solving elementary addition and subtraction story-problems clearly demonstrate "that for large number of children the main difficulty does not lie in selecting the proper arithmetic operation but in a prior stage, namely the construction of an appropriate problem representation". I wholeheartedly agree, and would simply add that the trouble with certain story-problems is not so much that they are "very condensed and, in a sense, even ambiguous" (p. 13) as that they treat quantifiers as logicians treat them, neglecting their natural language functions and that they transgress ordinary discourse rules especially story-telling rules. De Corte and Verschaffel give several examples of "wrong", but as they say, quite coherent representations of a problem; let me just quote one as an illustration. The problem is :

> Pete has 3 apples; Anne also has some apples;
> Pete and Anne have 9 apples altogether;
> how many apples does Anne have?

The children were given two puppets and a stock of "apples" (blocks). They were asked to act the problems and their answers. One child proceeds : He gives Pete 3 apples; then he gives Anne 3 apples ("also some") and he places nine blocks in the space between Pete and Anne. His answer to the question "How many apples does Anne have?" is to count the apples he had put in front of Anne and to say : "Three". This child clearly follows the rules of ordinary discourse : in the second sentence, following the first description, the quantifier some is interpreted in its usual meaning of two, three or four; and because of the word "also" three is the obvious choice; the third sentence then describes the next event : somebody gives the two children 9 apples which are intended for both of them.

Clearly, as De Corte and Verschaffel argue, the child in the example had not constructed a correct representation of the problem, and the acting-out modality demonstrates where the problem-representation went awry. Additionally, it seems to me that the acting-out method reinforced the idea of the problem being a story, which it is not; in fact, most story-problems, except those that concern a change of state problem such as

> Peter had 3 apples; his uncle gave him 2 more apples.
> How many has Peter now?

violate discursive story-telling rules (cf. also Escarabajal and Vergnaud, Congrès de Rome, juin 1986). To be able to solve arithmetical "story-problems" children have to learn a new set of rules before they can even think about what numerical operation to perform. It seems highly improbable that such problems are "less abstract", "closer to real-life experience" than simple mental arithmetic without apples, marbles, children, cars and what-not. Of course, solving similar problems when they are presented in horizontal notation with +, -, = and missing addends etc. may be even more difficult, since it implies another set of symbolising rules to be learned, but once again, this would not show that story-problems are closer to real-life situations.

CONCLUDING REMARKS

Mathematics and logic occupy a special place in the edifice of human knowledge, and, in my opinion, constructivism theory clarifies their particularity. Concepts related to the theory, such as abstraction, comparison, transformation, and the gradual elaboration of normative facts give us at least some idea of what is implied by the capacity to conceptualize mathematical aspects of actions and events - a capacity which provides the very foundation for mathematics learning.

It is certainly possible to assume that this capacity develops spontaneously, without direct intervention of parents or teachers. At all times, some children have developed mathematical thinking in essentially similar and creative ways despite inadequate education programs. Unfortunately, not all children - not even the majority - do so. Constructivism, as a psychological theory of knowledge, has already contributed to the elaboration of methods that can guide the majority of children through the complex landscape of mathematics. It has led psychologists and educators to question some of their own "norms" and refocused their thinking about mathematics teaching. I hope that the constructivist point of view can still do far more, and that this conference will be a step in that direction.

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ABOUT CONSTRUCTIVISM

A reaction to Hermine Sinclair's and Jeremy Kilpatrick's papers

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Hermine Sinclair's paper and Jeremy Kilpatrick's paper are quite different both in the tune they adopt and in the questions they address. Hermine Sinclair gives a summary of the piagetian views on constructivism and shifts to an analysis and a critique of story-problems. Jeremy Kilpatrick's paper is rather organised as a reaction to contemporary american researchers like Von Glasersfeld, Steffe, Cobb, Confrey and others ; I have to react to a reaction.

I find both'papers most interesting ; although they might be more specific of mathematics education and research on mathematics education. As a matter of fact, our job, as researchers, is to understand better the processes by which students learn, construct or discover mathematics so as to help teachers, curriculum and test devisers, and other actors in mathematics education, to make better décisions. This is our practical burden. Theory is essential, as it is also our burden to organize our knowledge on mathematics education in coherent systems of description and in powerful concepts.
It is essential to understand how individuals develop or fail to develop mathematical knowledge, therefore to discuss alternative interpretations of constructivism and other theoretical frameworks.

The reference to Piaget is unavoidable as he was, in his days,the most systematic theorist of constructivism. To understand his views, one needs to relate them to the questions he was addressing, in a fashion that parallels the idea that we have to relate the acquisition of mathematical ideas by children to the problems they are faced with.The term "epistemology" covers a large range of meanings ; one of these meanings concerns the relationship of knowledge to the practical and theoretical problems to which is tries to provide an answer.

This aspect of epistemology enlightens the way we may approach the development of new mathematical concepts, procedures and representations in the child's mind. It may also enlighten Piaget's views about constructivism, as Piaget developed his framework as an answer to the general question "How does knowledge develop" : he tried to make this philosophical question a scientific one by studying the development of children's intelligence and knowledge. For him, constructivism contradicts both empiricism and a priori rationalism. His critique of empiricism is probably better known than his critique of rationalism, but one must never forget that his work on the representation of space and time (and speed) is a direct response to Kant's theory that space and time would be a-priori categories of the pure raison.

Piaget was most influenced by the neo-kantian french philosopher Leon Brunschwig : his concept of scheme was originally borrowed from Kant, but his framework is aimed to be different from Kant's views as much as from Hume's views. The reason for this must probably be traced in his background as a biologist and an evolutionist.

In the field of psychology, this framework led him both to the empirical study of development, and to the critique of both associationism and gestalt theory.

As Hermine Sinclair reports, Pieaget's "Interactive constructivism" or "dialectical constructivism", stresses the fact that, on the one hand, children do not simply "read" experience but have schemes and categories to interpret experience, and that these schemes and categories are not a priori schemes and categories but derive from inborn schemes and experience. Action is essential as children accommodate their schemes through action upon the physical (and social) world, in order to assimilate new situations : nearly in the same way as scientists develop new procedures and concepts from former knowledge to understand and master new phenomena. The relationship between Piaget's "genetic epistemology" and the historical epistemology of science is obvious, although Piaget rejected the theory of parallelism between ontogenesis and phylogenesis.

Hermine Sinclair shows very clearly that, for Piaget, "new knowledge is constructed from the changes or transformations the subject introduces in the knower-known relationship", but that "the quality of knowledge is partly determined by its correspondance to the knowledge other people have constructed, and partly by the ways in which reality reacts to our interventions". If the subject's knowledge of reality develops by successive approximations, this does not mean that reality does not exist : Piaget was not interested in this metaphysical question and was not a "radical" constructivist in Von Glasersfeld's meaning of the word "radical". He did not either pretend that the undividual could develop his knowledge through his lonely experience ; on the contrary, confrontation and decentration are important processes that take place in social situationsand help children develop new and better schemes and ideas. Sinclair's quotation "Objective knowledge is only attained when it has been discussed and checked by others" (Piaget, 1965) shows clearly that Piaget did not deny objectivity and social interaction.

Hermine Sinclair also shows that, for Piaget, action is not only a way to transform the outside world but also a way to question it. I agree with her.

I am not so happy when she takes for granted Piaget's distinction between logico-mathematical knowledge, abstracted from action (reflective abstraction), and physical knowledge, abstracted from the properties of objects (empirical abstraction). As physical properties of objects are also abstracted though action and experience, it is not so easy to follow Piaget's views on this point. And they may have wrong consequences upon the theoretical frameworks of research on mathematics education , for instance on the concrete-formal debate. This debate does not concern only early school mathematics, but all levels of mathematics learning and teaching, including University. But before I go into a deeper analysis of this question, I would like to mention that, in Kilpatrick's paper, some comments refer to the very same problem, although with different words.

One of the arguments used for Kilpatrick against radical constructivism is that one may adopt such language as "Eddie has constructed rational number" but not "Eddie has constructed osmosis".

"The claim that there is an independantly existing world 'out there' that can be known by the cognizing subject is explicitly avoided by constructivism", Kilpatrick says, and he purposedly uses examples outside mathematics, developing this idea with historical examples : one might say that the four-colour theorem has been "constructed", whereas one would not say that "Priestley constructed oxygen" or " Cartier constructed the Saint Lawrence River". The language of <u>discovery</u> is opposed to the language of <u>construction</u>. "The mutual attraction between constructivism and mathematics is an intriguing theme" says Kilpatrick ; it parallels the piagetian distinction between two kinds of abstraction, leading one to mathematics and the other to physics.

Kilpatrick mentions two principles as the basis of constructivism.
1 - knowledge is actively constructed by the cognizing subject, not passively received.
2 - coming to know is an adaptation process that organizes one's experiential world. It does not discover an independant, pre-existing world.

As far as I can see, there are two independant ideas in the second one, as the adaptative process is one thing, and the radical constructivist's denial of an independant pre-existing world another thing. This last idea might just as well be considered as trivial solipsism, rather than radical constructivism. One may accept the first principle and the first part of the second one, and not the last part of the second one ; and this is just as radical as radical constructivism, which fails to provide a theory of objective knowledge. Adaptation does cope with the actual world, and not with a purely imaginary fantasy. There is no random reinforcement that can give us the confidence and the feeling of necessity that we have in using our knowledge of spatial relationships and transformations, or our knowledge of numbers : within the social and scientific knowledge that we call mathematics. Students are invited to share that knowledge, and eventually contribute to produce it if they become mathematicians. What is their problem as students ?

Kilpatrick reports on five different questions, that have been raised by Ven Glasersfeld himself.

- teaching versus training
- inside processes versus overt behavior
- linguistic communication and transfer of knowledge versus construction
- interpretation of errors
- teaching interviews as a powerful method.

I will not repeat here what is very well reported in Kilpatrick's paper. I agree with him that some consequences of radical constructivism are not specific of radical constructivism. But I also would like to say, in defence of constructivism, that some of them do contradict empiricism and other widely accepted information-processing models of cognition, especially those which see knowledge as an additive combination of rules, or as a purely symbolic calculus, or as a net of static structures.

At this point I feel the need to change my way of discussing Sinclair's and Kilpatrick's papers. I need to start from examples and from my own point of view.

Let us start from the analysis of the competence to count a set of objects. This requires a one-to-one correspondance between objects, finger movements, eye movements and number words ; it also requires the cardinalization of the whole set, using the last word twice or with two meanings, one for the last object (ordinal), the other for the cardinal of the whole set.

Counting a set is a scheme, a functional and organized sequence of rule-governed actions, a dynamic totality whose efficiency requires both sensori-motor skills and cognitive competences : cardinal, exhaustion, no repetition... There are important "normative facts" implicit in it, (to follow Sinclair's vocabulary), or "invariants", or "theorems-in-action" as I usually call them.

Many different schemes are involved in the solving of the different subclasses of additive and subtractive problems : they consist either of finding the adequate operation and the adequate data, or using a counting procedure that simulates the structure of the problem,or transforming adequately the structure of a problem into another one...

Children also develop important and complex schemes to coordinate different motor-skills and different rotations and translations in space, and still recognize the invariance of the objects and relationships under control. Some of these skills appear quite early in the child's development, others appear later and only through a mathematical or quasi-mathematical analysis : think of technical design for instance.

Some of these schemes are rather spontaneously shaped by children, in the sense that they are not really taught by adults,and depend heavily on the recognition by children of their function and organization. Yet we must never forget that these activities are not purely invented by children as most of them exist in their social and physical environment, and require practice : children spontaneously train themselves and repeat the same scheme under the same circumstances or under diverse circumstances, to master it and delineate its scope of validity.

Whatever the influence of the social and physical environment may be, I consider that the development of such schemes relies essentially on the <u>construction</u>, by the child, of adequate cognitive invariants and skills.

Neo-behaviorists might say that the concept of scheme is not necessary and that the concept of skill (as overt behavior produced by rules) is sufficient. For me the recognition and representation of cognitive invariants such as objects, properties and relationships are essential components of schemes, as the hierarchical development of schemes is tightly associated with the recognition of more and more complex invariants.

This is true for sensori-motor schemes and for intellectual schemes like those involved in mathematics.

The true cognitive task of the child is to "conceptualize" the world, so as to act upon it efficiently. This process is not easy, and it usually goes with all sorts of fancy "conceptions". But the feed back of the physical and social world truely helps the child to shape his schemes : for instance, the conceptions of addition and subtraction are shaped by the first situations mastered by children (addition as increase and subtraction as decrease), but these conceptions have to change when children deal with other cases of addition and subtraction, although there are always sequels of their primitive conceptions.

In this process of recognizing invariants in the world and developing schemes, there is no difference, at the beginning, between mathematics and physics. The differentiation comes later. Mathematics deals essentially with number and space. There would be no meaning for the concept of number if there were no physical quantities, discrete or continuous. There would not even be any primitive conception of addition and subtraction if transformations, that occur in time, did not take place. Time is not usually viewed as a mathematical concept but rather as a physical one. But all we know about children's mathematical schemes shows that we must make room for the representation of time in children's mathematics. Space is also both mathematical and physical, as there would be no representation of space if it was not full of physical objects. The concept of number is tightly associated with the concept of measure (cardinals are measures) and it is only when the concept of number is already well developed that children are able to think about properties of pure numbers.

It is not the distinction between abstraction from action and abstraction from objects that enables us to understand the distinction between mathematics and physics, but rather the level and the kind of objects we are dealing with. The concept of whole number is a good example : for young children it is tightly associated with the measure of discrete quantities and the ranking of physical objects ; but as it is used for many different kinds of quantities (even continuous magnitudes), and for different rankings, it can be abstracted from the specific physical properties and give birth to the concept of pure number, many properties being invariant upon all different kinds of physical properties : the truth of 3 + 3 = 6 does not depend on marbles, sweets or steps.

Fractions and ratios are tightly connected with physical objects and could be viewed as ways of conceptualizing the physical and social world (think of sharing) just as well as mathematical concepts. It is through high-level abstraction that the concept of rational number develops, also through the synthesis of different properties of fractions and ratios (Vergnaud, 1983, Kieren, 1987), namely operators, quantities or magnitudes, scalar relationships, mappings and rates.

At nearly all levels, there are specific mathematical activities, as many activities concerning number are independant of the physical context, but mathematics is rooted in physics. This is true even for high-level mathematical concepts, who would have never come to birth if physics had not raised new problems : think of vector-spaces, of differential equations and calculus. There is some research work in France, at the University level, examplifying the collaboration of physicists and mathematicians, that show the profit students can draw from a better connection between mathematics and physics.

Of course there are also some strong specificities of mathematics. The irrational character of the measure of the diagonal of squares of side 1 or n, is a purely mathematical discovery, although its meaning is rooted in the study of space and measure. Also the fact that the sum of two successive uneven numbers is a multiple of 4 :

$$(2n + 1) + (2n + 1 + 2) = 4n + 4 = 4 (n + 1)$$

Abstraction from action is as assential in physics as in mathematics : think of movement, speed, mass, density.

But I would like to point at three different aspects of abstraction :

- invariance of schemes
- tool-object dialectics
- role of symbols.

Invariance of schemes : The fact that the same scheme, or sub-scheme, operates on different situations, is an essential way of recognizing invariant propetties and relationships, and relies upon this recognition.

Tool-object dialectics : (see Douady, 1985) a new concept is at first a tool to identify invariants and work out operational schemes. Working with objects of any level children discover (or construct) some of their properties and relationships : these are tools. But such properties and relationships can in their turn be considered as objects, having their own properties and own relationships with other objects. Our representation of the world is made of different-level objects. This is true for all sciences, but especially in mathematics : number is first a tool to compare, add and substract, it becomes an object quite rapidly, although not with all its properties. Operations are first tools, they become objects. The same is true for functions and variables, for geometrical transformations. Transforming tools into objects is an essential way of conceptualizing the world.

Role of symbols : natural language, schemas and other mathematical symbols play a crucial part in this process of transforming cognitive tools into objects, as symbolizing is a way of cutting invariants out of their contexts. It is also a way to point at them and discuss about them with other persons.

Actually a concept is not a concept until it has a name and one or several symbolic representations : lingistic symbols are a necessary means for communicating and debating about a concept with other people : about what it is (definition) and what its properties are (theorems). Communication and debate with others are crucial in the development of concepts. This is why it is important that students work together, also why teaching interviews are a powerful method.

But one must never forget that concepts are rooted in the experience of students with different kinds of situations, and in the schemes they use to deal with these situations. Before being objects, concepts are cognitive tools ; and many theorems had better be theorems-en-action before being explicit theorems, especially at the primary and early secondary level : if not before, at least immediately after.

The social character of learning, discovering and constructing does not concern the symbolic aspects of communication only, it also concerns the cooperation of different students on the same task, problem or situation. A natural language problem is not a story and does not have to be analysed as a story, it is a way of refering to a situation : natural language is a way to convey referents : objects, properties, numbers. The analysis of natural language problems is, first of all, a mathematical one. The cognitive task for students facing natural language problems includes understanding words (relationships, quantifiers...), but their understanding depends heavily on the mathematical tools by which they can make sense of this sequence of words and represent it to themselves as a situation and a problem to be solved.

Teachers must explain a lot, and show a lot. But it is also their burden to choose good situations, a large variety of them, and to understand clearly which properties of the concepts involved are necessary for students to make sense of each of these situations.

Didactic situations play many different parts in teaching : help students develop new invariants and schemes, train and confort their existing skills, contradict wrong or narrow conceptions. Concrete and abstract are all relative concepts, as what is abstract at one age, may be very concrete and as real as a wood table a few years later. "Concrete" conveys mainly the idea that teaching situations should make meaningful a new concept. This is true at all levels, and in many different ways. As the choice of these situations cannot be made without reference to mathematics as a science, and to the developmental process of mathematical schemes and concepts in students' minds, I see constructivism as the best way to consider the process of appropriation by which a student makes mathematics his own knowledge. Rather than a pure and lonely construction, the learning of mathematics is for me the difficult appropriation of a social knowledge.

An individual's knowledge is necessarily his own business and his own part is crucial. But there are so many social and physical incentives and feed-backs in the learning process that individuals never think, except when they are radical constructivists, that their knowledge is totally different from other individuals' knowledge.

This is not pure illusion, or science does not exist.

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THE WORLD OF MATHEMATICS: DREAM, MYTH OR REALITY?

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At the PME-NA meeting at Michigan State University last October, Gaea Leinhardt remarked that it seemed to her that people who began statements with the phrase, "Speaking as a constructivist", felt able to complete their sentences in arbitrary ways. The suspicion that constructivism is too opaque to serve as a framework for enquiry was strongly voiced by Jill Larkin at the meeting of the NCTM Research Agenda Project at the University of Georgia in March. Plenty of mathematics educators have similar reservations. Yet constructivism is in fashion in mathematics education research circles. It is so fashionable, indeed, that it has laid claim to several precious plenary hours at this crowded conference. I hope the outcome of this extensive attention will not be seen as an endorsement of fashion but as a recommendation to subject all references to constructivism to critical scrutiny and to refrain from adding to the loose talk. Our efforts may be useful to the extent that they help (in Yeats' phrase) to purify the language of the tribe.

Before looking very specifically at the papers by Sinclair and Kilpatrick*, I want first to set up some divisions in the reference field, in the contexts within which constructivism appears. They occurred to me during my first conscious attempts to think about my views on constructivism when I wondered what, if any, was the connection between constructivism in mathematics and constructivism in mathematics education. This also leads into a subsidiary question about whether mathematics and mathematics education have some special feature that favours consideration of the constructivism option. A second clutch of questions came to me because I found myself agreeing with some of the propositions some constructivists were making while disagreeing strongly with other propositions from the same people. Arrogant enough to suppose that I already knew what I know about mathematical activity and learning, I thought there was a possibility that some constructivists had managed to mix some independent systems of propositions together.

*Which I have read in first draft only.

The divisions I will make are between constructivism as a philosophy of

- mathematics
- psychology, particularly the psychology of learning
- philosophy, particularly ontological questions
- education

Constructivism in mathematics education is likely to feed on some or all of the above.

Although I include philosophy as one of the contexts, constructivism in any of the contexts is itself a philosophy. It is not a theory because it is not formulated in terms that could lead to refutation. At the heart of discussions about constructivism is the difficulty that its espousal and its rejection are more products of taste than of evidence.

Having laid out this challenging conspectus, I find myself incompetent here and now to make sensible observations about each part of the whole picture. I make some brief remarks under each heading: at least this will indicate how much still remains to be understood.

Constructivism and mathematics

There are two strands here, one fairly general and the other special and technical. The general strand is characterized by the classic question: is mathematics invented or discovered? This is often interpreted as a straight choice between the platonist and constructivist positions, but there are other options available that may answer the question: how does mathematics come into being? The empiricist position that treats mathematics as much like any other science, argued by Locke and recently reviewed by Kitcher, probably does not have many adherents, but there seems to be growing acknowledgement of the influence of social forces, as in the anthropological views of L.A. White and their elaboration by R.L. Wilder. There are other positions, and in any case enough alternatives to make it possible to answer "Neither" to our classic question.

A better answer may be "Both". The platonist-constructivist dichotomy puts us in the position either of denying that we have any choice in the directions in which mathematics develops or of denying that the inner coherence of mathematics ever takes us in directions different from those we intended to follow. Both denials are contrary to well-documented aspects of the mathematical experience.

Constructivism in its more specialized sense begins at the end of the 19th century with Kronecker's objections to the analytic methods of Weierstrass and Cantor. In the 1920's Brouwer tried to recreate analysis on non-set theoretic foundations, and more recently Bishop has begun the hard task of "constructing" constructive proofs of the important theorems of the calculus.

The intuitionist programme, as Brouwer called it, is based on attractive assumptions, but it makes doing mathematics extremely difficult since it denies mathematicians some of their most powerful tools. The programme is strong on internal coherence but very weak on external referents. Goodman remarks that Brouwer's mathematics is dream-like. "In a dream ... there are no errors. Everything is arbitrary, and so everything is correct. (Goodman, 1983)

Constructivism and psychology

The psychology of perception has always been dominated by constructivist philosophy. Most psychologists would agree that "our seemingly unified view of the world around us is really only a plausible hypothesis on the basis of fragmentary evidence." (Blakemore, 1973) The recent adoption by psychology of computational metaphors, in information processing and cognitive science, which might be expected to favour mechanistic explanations of other kinds of psychological phenomena in fact do not. "These psychologists agree that thought and behaviour must be conceptualized as meaningful action on the part of a subjective agent rather than a causal process in a natural world." (Boden, 1979) In psychology, it seems, we are all constructivists now. And hence the exalted place that theories of representation are beginning to occupy.

Theories of representation often seem unable to tell whether the thing represented is present to the senses or not, yet most of the time we have no difficulty in knowing whether what is in our mind corresponds to something that is present or is an image evoked in the thing's absence. Theories of representation also postulate certain a priori powers of the mind or the brain that enable us to select invariants from the flux of sensory data - as the baby, for example, recognizes his mother although the constellation of energy impacts on his sensory receptors from the photons reflected by her face and body is different at each moment. (There are, of course, other means for him to know her, but similar observations apply to these as well.) The latter assumption seems potentially much better adapted to dealing with our movement through an ever-changing world.

Constructivism and philosophy

Rather than outline here the standard philosophical positions on constructivism versus the other isms, I want to say that philosophers of mathematics should take more note of Piaget's work. This also puts Piaget in the right place. "Genetic epistemology is essentially an experimental philosophy which seeks to answer epistemological questions through the developmental study of the child." (Elkind, 1968)

Piaget's contribution to the philosophy of mathematics lies in his explanation of the phenomenon of mathematical evidence, i.e. the evidence on which mathematical theories and knowledge can be based (the "fundamental criterion of demonstrative force", in Beth's phrase).

"Piaget takes a decisive step ... by observing (not positing) that evidence develops in parallel with the emergence of mathematical "structures", that is, with the recognition of abstract relations independently of the particular "objects" between which the relations hold. Evidence matures with the progressive acquisition of structures, with the increasing objectification of the components of these structures, with the growing awareness of the automony of the operations performed on these components relative to the particular "objects" which at first are considered to constitute them, these "objects" themselves being structures already previously elaborated at a lower level of conceptual organization. Definite acquisition of evidence ... is associated with completion, or "closure", of the corresponding structure." (Castonguay, 1972)

It is particularly interesting to notice how Piaget avoids the naive platonism of Thom ("mathematical structures exist independently of the human mind that thinks them" (Thom, 1971) and yet can agree with Thom that "the important mathematical structures (algebraic, topological) appear as data fundamentally imposed by the external world." (Piaget, 1970)

In Piaget's constructivism, meaning is cumulative and the evolution of mathematical structures is towards increasing comprehensiveness and rigor. Logico-mathematical structures build on those that came before, integrating them while overcoming their inadequacies. Mathematics therefore moves towards increasing objectivity - which Piaget understands as a process and not as a state.

Constructivism and education

The institutionalization of education leads to a necessary abstraction and formalization of knowledge. Whereas in an apprenticeship the knowledge that is dealt with is normally immediately useable and closely related to the specific intentions of the learner, knowledge in the context of schooling has to be organized and generalized so that it can serve the needs of students with widely different origins and widely different goals. This objective knowledge, while powerful and so potentially liberating, is also regrettably depersonalised and depersonalising. The individual student is offered the chance to appropriate this knowledge but is not given the chance to shape it.

In this situation it is particularly important to recognize that the students also need to see themselves as originators and modifiers of knowledge. Only this awareness can save them from alienation and only this experience can give them a basis for shifting their attention in all their school subjects from what is correct to what is true. Independently, then, of the above arguments, one can argue on educational and moral grounds that schooling should include some component designed to involve the students in the generation - the construction - of their personal knowledge.

Comments on Hermine Sinclair's paper

The most stimulating section of the paper for me is the centre portion where the writer presents Piaget's concept of normative facts and his hypothesis concerning the instruments of knowledge. I'm stimulated because the two ideas are new to me, both are expressed briefly so that I am not sure I have grasped their meaning, and both make my intuition sit up and say, "Something is wrong here!" Without reading the original sources and reflecting at length on them I cannot have anything very cogent to say, so I will just ask a question concerning the second of these ideas. How is it possible to make a comparison of two objects or two situations without employing an action scheme (turning the head from one to the other, moving one thing next to the other, placing one on top of the other, crouching so as to view them in line, or whatever) which brings about a transformation the instant it is applied? Are not the comparison and transformation processes necessarily coordinated and simultaneous since neither can (logically) happen without the other? In the special case of objects or situations judged to be equivalent the coordination of the happenings seems clear. Since no two distinct objects or situations are identical (i.e. have completely identical properties), a judgement of equivalence must involve the awareness that there is some transformation which will carry one into the other. The judgement of difference, while a little more difficult to analyze, must, I think, work in essentially the same way. Perhaps the delay to which Piaget refers is not a delay in the events but a lag time between the subject's knowing what to do and knowing what it was that he did.

I am glad that Mme Sinclair, following Piaget, takes care not to fall into the solipsist position of supposing that because the world cannot be completely and absolutely known it cannot be known at all (and so may not even be there). Indeed, as I have indicated earlier, Piaget has given us one of the most convincing accounts to date of how the subjective intelligence comes to know the objective world. It is true that one "can never reach the object itself", that we are in a state of ignorance that can be modified but not essentially reduced (for the more we come to know, the more we find there is to know ...). Knowledge and ignorance are complementary not incompatible. Human beings strive to know, they thrive on knowing, yet remain in a condition of irreducible ignorance.

To say, as some radical constructivists seem to, that we cannot know anything that goes on outside our own heads is solipsism - a position that may be fair enough in church, great fun in academia, but intolerably irresponsible in connection with, let us say, medicine, politics, or education.

A few words about the three questions.

1) What are the actions "destined to become interiorized as operations"? Counting, of course, is anything but a primitive action scheme in spite of its being mastered at the beginning of formal schooling. It already

contains in coordinated form several action schemes that will later become independent operations that will have nothing to do with counting. The linear permutation of a set of objects will become a mathematical operation long after its emergence within the counting process where one of the most important discoveries that children make is that the cardinal of a set is invariant for different orderings of the set.

When permutations become the subject of attention, the mathematical operation of interchanging becomes important (since certain permutations can be composed of a succession of interchanges). The origin of this quite simple operation may be found in the earliest years of childhood, in action schemes in which two objects are picked up, one to each hand, put down, picked up again, but each now in the other hand, and so on. The temporal gap here between the action scheme and the operation - or at least the operation in conscious use - is very great: a slow construction indeed!

It may be worth considering the possibility that in this case, as in a number of others, the time lag is not only a function of the difficulties that have to be overcome before the operation can become operational. Counting and adding are brought to children's attention very early mainly because certain social criteria say that this knowledge is important and fundamental enough to be mastered as soon as possible. Society cares a great deal less about the mastery of permutation, so some mathematical operations which are at least as easy to master as those involved in learning to count and to add are not detached from their originating action schemes and objectified into autonomous operations until much later. It is my suggestion, which I admit is not very Piagetian in spirit, that children do not interiorize operations until they need to, for whatever reason, and that sometimes this reason will be that the operations are required for the mathematical curriculum and for nothing more.

2) Piaget's compounding of logic and mathematics reflects, I think, the relatively narrow range of his mathematical interest. He is really not at all interested in what people use mathematics for, or why they have developed this extensive repertoire of skills and concepts and theorems. He doesn't show much awareness of mathematics as an activity, as

something to do as well as something to know. He has decided to try to identify in the actions of children the steps they take in getting to the point where they know, say, number or area as well as a mathematician does. So he focuses on the epistemological foundations of these sophisticated ideas, and in order to know what mathematicians say about them he talks to the only mathematicians who have given the matters any thought, and these turn out to be philosophers or logicians, and sometimes both, e.g. Beth.

There may be another reason why "mathematics" carries the "logico" prefix. Taking the example of simple whole number addition and subtraction, for instance, we see that the mathematician will be more interested in the properties that distinguish the operations from each other - that addition is associative, say, while subtraction is not - while the logician will be more interested in their interdependence. For the logician, addition and subtraction entail each other and are therefore logically equivalent. An addition which cannot be "undone" - e.g. the addition of two raindrops - is not a mathematical addition. It is an essential ingredient of the meaning of mathematical addition that subtraction should be possible. And vice-versa. In such ways the logician's insights contribute to the epistemology of mathematical concepts.

3) The educational importance of discussion, argumentation and collaboration is undeniable but does not, I think, have more than a marginal influence on the development of objectivity. The definition that Piaget gives here seems to me to give only a weak meaning to objectivity, viz. approval by others. I cannot help thinking of Copernicus battling it out with a roomful of priests. There is, I am sure, a stronger sense of objective that doesn't depend on anyone's good fortune in finding someone else who agrees.

The final section of Hermine Sinclair's paper takes me out of my depth, although I like what she says. Story problems really are such extraordinary things! They are pedagogical devices, that is clear (since they are not a part of mathematics nor a part of experience outside school), but devices for doing what? If their goal is to link mathematics to everyday experience, then they go about it in the clumsiest way imaginable. They interpose between the two things they are intended to link, something else, the "story", which rather than facilitating the application of some simple mathematics to some simple problems, introduces considerable additional interpretative difficulty. The problem of a story problem is not the mathematical problem but the problem of deciding what the mathematical problem actually is - a difficulty that never arises in a genuine problem situation. What is more, these stories bear superficial resemblances to events in everyone's experience, yet the solution of the problem is not a matter of moment to anyone involved, not even the characters in the story. It seems worth considering what children learn from exposure to things called problems which no one needs to solve, from stories which (as Mme Sinclair remarks) don't tell anything, and from tasks which seem designed to conceal rather than reveal what one is supposed to do. Over the schoolroom door we might as well write, "Alienation begins here!"

Comments on Jeremy Kilpatrick's paper

There are some observations that I hardly need to make. The most obvious is that I share Jeremy Kilpatrick's wary and cautious approach to constructivism, especially the radical variety. What will become obvious is that I have not studied all the sources to which he refers and I will be responding to his use of the sources, not to the sources themselves (some of them not yet in the public domain).

The opening of the paper makes me ask what it is about constructivism that has made some mathematics educators into such passionate converts and many more into fellow-travellers. There is an undoubted appeal about the approach in spite of what seems (to me) to be basic incoherences in its belief system and in spite of the fact that there is no evidence (to my eyes) that the theory has necessary consequences for educational practice. I hazard that the attraction resides in such features as:

i) the theory is generous in its estimate of students' powers, making it seem humane and potentially humanizing; students are seen as in active control of their own learning, not pictured as greedy pigeons nor as attentive but passive listeners;
ii) the theory is realistic about the (generally) out-of-synch process of lecturing and of schoolroom "presentations"; it makes it clear that teaching in the "telling mode" is not only undesirable because it is authoritarian but also ineffective

because it cannot possibly produce any consensus in the students' responses to it;

iii) the theory seems to hold out the possibility of realizing the classic instructional maxims: "start from where the student is", and "do not try to do the student's learning for him";

iv) the theory resonates with personal experiences that frequently show us to ourselves as engaged in the activity of making sense of the things we encounter, making "an effort after meaning" in Bartlett's words, demystifying some of the random and arbitrary-seeming significations that surround us.

Readers will be able to note other factors that contribute to the appeal of constructivism. If it is this and more, how can it possibly be resisted?

Constructivism, however, as Jeremy Kilpatrick says, needs to improve its connection to educational reality. Formal education includes elements of prescription if society is to have a say in what is learned in schools. It was a weakness of the progressive movement of the 1930's that it was never quite courageous enough to face down the dilemma posed by a curriculum, any curriculum. Being an educator or being a teacher may be, in part, to have accepted the responsibility of seeing that students learn what society wants them to learn. Some of the progressives hoped that students would eventually realize for themselves that it would be in their own interest to learn to read, to qualify for a certificate, to graduate - i.e. to do what society wanted: to volunteer, as it were, to follow the curriculum that for ideological reasons could not be imposed. Some progressives were not above trying to achieve these ends by manipulation, consciously or unconsciously.

Straightforward instruction stripped of rewards and punishments is at least not manipulative, and I would rather give students direct instruction than to try to "guide their learning" or "attempt to modify their cognitive structures" (to quote from von Glaserfeld's "five consequences"), both of which smack to me of manipulation. How difficult it is to discuss this matter without a very much clearer idea of exactly what pedagogical techniques are used to pursue these ends!

Finally, a remark about the Bavelas experiment. This kind of phenomenon, which might be called the exclusion of the random or the rejection of the arbitrary, is quite well known (some experiments in the early 1960's brought it to my attention). I suggest the example bears to a theory of constructivism about the same relation as examples of visual illusions bear to a theory of visual perception - that is, there is a connection, but only "at the edges". I don't think the fact that, say, some punters make their living at the track is clinching evidence for the radical constructivist case.

Conclusion

Mathematics invites the attention of constructivists because it has no external referents. There is nothing we can point to, even in a figurative sense, and say that mathematics is about "that", or is the study of "those". People at all times, from the Pythagoreans onwards if not before, have spun stories to establish what mathematics is about. Whereas it is possible to revise and improve, say, the stories that comprise Aristotelean physics by undertaking some critical experiments to refute one or more of its tenets, no one can subject the stories that make up mathematical platonism or mathematical empiricism or mathematical constructivism to the test of critical experiments. The propositions within each framework are testable but not the frameworks themselves. Perhaps, then, we should choose whichever stories, which of the available myths, happen to suit us best.

I dislike leaving my story at that point, though I am unable to see what else I can usefully say. Perhaps just this. Myths are OK when we know that is what they are, but myths that get taken for reality, not as stories about reality, are potentially dangerous. The perverse and impoverished platonism which is the traditional school-based myth about mathematics has poisoned minds and destroyed confidence on a large scale. Are we quite sure that our more sophisticated myths are really less harmful? In the last resort I dislike and distrust radical constructivism applied to mathematics education because it denies students access to any independent path to knowledge and to truth and so gives teachers power over what students learn that I know some will abuse.

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COMMENTED PAPERS

Affective factors
in
mathematics learning

CHILDREN'S IDEAS ABOUT WHAT IS REALLY TRUE IN FOUR CURRICULUM SUBJECTS: MATHEMATICS, SCIENCE, HISTORY AND RELIGION

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ABSTRACT

Pupils ideas about whether what they learn in mathematics, science, history and religion is really true were investigated in two urban secondary schools Pupils were given a questionnaire and asked to make a judgment about the truth of a subject and to justify it. Analysis showed that mathematics and science have a similar profile in the two schools, both subjects being considered by the majority as "true" in all years. Judgments about history changed in one school with age and remained stable in the other, whereas religion does not give an easily recognisable pattern. Qualitative analysis provided categories: nature of subject, relation between theory/practice, evidence through proof, constuctivism, pragmatism and authority of teacher. Evidence through empirical proof was the most popular category of explanation in both schools.

INTRODUCTION

In the last few years great interest has been shown in the relationship between the teaching of a subject and the philosophical ideas which are held by teachers about that subject. This arose from the recognition that teachers' beliefs about their discipline and how pupils perceive it are somehow linked.

Teachers develop strategies to cope with a wide range of classroom situations and these strategies are a result of conscious or unconscious notions, preferences, attitudes, beliefs, and what remains of their "education". Brown and Cooney (1982) suggest that these strategies shape a teacher's behaviour and constitute a sort of "theoretical state" which more or less defines the way in which they teach. Thus it is reasonable to expect that teachers' views of the subject and their instructional practice could be significant in influencing pupils' attitudes towards the subject.

There is not the space to develop the various philosophical views of mathematics, science or history. Suffice to say that many mathematicians will have heard of ideas such as Platonism and "pre-existing structures", Logicism and the reduction of mathematics to a number of logical concepts, to Empiricism where knowledge comes from experience, or Constructivism where mathematics is seen as a construction of man. Similar analyses can be made for Science and History but this is for a longer paper. So, the goal of this study was to see how pupils perceived the various school subjects, and whether or not they actually believed them to be true.

METHOD

A questionnaire was given out in two schools. School A was an inner city single sex independent school, School B was an inner city mixed comprehensive school. Pupils in the sample were taken in School A from the first, second, third and four year (covering ages 11 -15) and in School B, from the first, third and fifth year (11-16). The questionnaire read as follows: **WHAT DO YOU THINK?**

We would all like to know what is really true (well, most of us) What do you think about whether these subjects tell you things that are really true? -what you learn about science, what your learn about religion, what you learn about history, what you learn about mathetmatics. Say what you think here: (choose one for each of the following subjects) **SCIENCE** TRUE because ....

NOT REALLY TRUE because ....

CAN'T DECIDE because.... Similarly for Mathematics, Religion and History.

RESULTS

The least complex model worth testing is that of independence of age and judgment in GLIM or (equivalently) ln alpha =0 uniformly across the table (Osborn 1983). This model was fitted, in turn to the data for each subject areas, and for the schools A and B as shown in tables 1 and 2 (given below). When the model of no interaction was rejected, best fitting values for uniform ln alpha were found and fitted. The results were as follows:

MATHEMATICS: School A $G^2 = 6.2$ d.f. 6 $p. \approx 0.40$

School B $G^2 = 0.5$ d.f. 4 $p. \approx 0.96$

For school A it is a good fit, and for school B it is a more than excellent fit so in both cases the model cannot be rejected, thus there is no association between age and judgment.

SCIENCE: School A $G^2 = 6.2$ d.f. 6 $p. \approx 0.40$

School B $G^2 = 2.7$ d.f. 4 $p. \approx 0.60$

The fit in both cases is good so the model cannot be rejected thus again there is no association between age and judgment.

HISTORY School A $G^2 = 10$ d.f.6 $p. \approx 0.10$

School B $G^2 = 5.5$ d.f.4 $p. \approx 0.25$

The fit for school B is a fairly good fit and so the model cannot be rejected, thus no association between judgment and age. In the case of school A while the fit is far from good, it is close to being acceptable. In this case the best fitting uniform ln alpha, value 0.6, was fitted and this gave a $G^2 = 5.4$ d.f 5, $p \approx 0.4$ which is a good fit, although the strength of association is not very strong, thus in school A there would seem to be some development of children's judgments.

RELIGION School A $G^2 = 16.6$ d.f. 6 $p. \approx 0.01$

School B $G^2 = 30.1$ d.f. 4 $p. < 0.001$

In the case of school A the fit was not good, for school B it was extremely poor, so in both cases the model of no interaction can be rejected. The best fitting uniform ln alpha, value 0.4 was fitted to data for school A, giving $G^2 = 13.7$ d.f.5 $p. < 0.02$, this is still not a good fit. There is a negative development for school B.

Summarising, mathematics and science show similar trends with the majority of children judging them both to be "true" throughout all the ears of school. For history in school A, there is a change in judgments over the years but with school B the judgments stay the same. Religion for school A would have a similar picture to mathematics and science if there were no an unexpected increase in frequency of truth judgments for third years. In school B, "not really true" judgments increase and "true" judgments decrease with age.

TABLE 1. SCHOOL A, JUDGEMENTS ABOUT TRUTH OF SUBJECT

	MATHEMATICS			SCIENCE		
	NRT	CD	T	NRT	CD	T
1st N =49	2% (1)	14% (7)	84% (41)	6% (3)	23% (11)	71% (35)
2nd N =44	7% (3)	9% (4)	84% (37)	7% (3)	18% (8)	75% (33)
3rd N =33	3% (1)	9% (3)	88% (29)	0	9% (3)	91% (30)
4th N =25	0	28% (7)	72% (18)	0	24% (6)	76% (19)
Total	3% (5)	14% (21)	83%(125)	4% (6)	19% (28)	77%(117)
	HISTORY			RELIGION		
	NRT	CD	T	NRT	CD	T
1st N =49	28% (14)	33% (16)	39% (19)	57% (28)	41% (20)	2% (1)
2nd N =44	18% (8)	48% (21)	34% (15)	41% (18)	50% (22)	9% (4)
3rd N =33	12% (4)	42% (14)	46% (15)	30% (10)	43% (14)	27% (9)
4th N =25	16% (4)	20% (5)	64% (16)	52% (13)	44% (11)	4% (1)
Total	20% (30)	37% (56)	43% (65)	46% (69)	44% (67)	10% (15)

TABLE 2. SCHOOL B, JUDGMENTS ABOUT TRUTH OF SUBJECT

	MATHEMATICS			SCIENCE		
	NRT	CD	T	NRT	CD	T
1st N =26	4% (1)	23% (6)	73% (19)	0	12% (3)	88% (23)
3rd N =23	4% (1)	17% (4)	79% (18)	0	22% (5)	78% (18)
5th N =20	5% (1)	25% (5)	70% (14)	5 (1)	25% (5)	70% (14)
Total:	4% (3)	22% (15)	74% (51)	1% (1)	19% (13)	80% (55)
	HISTORY			RELIGION		
	NRT	CD	T	NRT	CD	T
1st N =26	0	27% (7)	73% (19)	15% (4)	27% (7)	58% (15)
3rd N =23	17% (4)	22% (5)	61% (14)	56% (13)	40% (9)	4% (1)
5th N =20	5% (1)	15% (3)	80% (16)	70% (14)	30% (6)	0
Total:	7% (5)	22% (15)	71% (49)	45% (31)	32% (22)	23% (16)

COMPARISONS BETWEEN SUBJECTS

In order to understand better the relationships between subjects six comparisons were made between: mathematics and science; mathematics and history; mathematics and religion; science and history; science and religion; history and religion. The scoring was as follows: X truer than Y (or equal) gives three possible scores:

X truer than Y: score of 1 for X

Both subjects equally true: score of 1 for equality

X less true than Y: score of 1 for Y

Tables of comparisons were constituted and a model of no interaction fitted to each but the commentary will be restricted to comparisons involving mathematics (totals and statistics are given below) As might be expected, apart from the first year, the profiles of the comparisons between judgments about mathematics and science do not change over the years of secondary school. The same is true for mathematics and history but for all four years of secondary schooling. The fit is less good but still adequate for comparisons of judgments about mathematics and religion, because of an increased number of "true" judgments for religion in the third year.

TABLE 3: COMPARISONS OF JUDGMENTS OF SUBJECTS

d.f.6	Maths-equal-Science			Maths-equal-Hist.			Maths- equal- relig.		
	wins		wins	wins		wins	wins		wins
Totals	26	104	21	76	62	13	127	20	4
	$G^2 = 7.8$ $p \approx 0.30$			$G^2 = 4.8$ $p \approx 0.60$			$G^2 = 9.6$ $p \approx 0.15$		
	Science- equal-Hist.			Science-equal-Relig.			Hist.-equal-relig.		
	wins		wins	wins		wins	wins		wins
Totals	76	53	22	124	18	9	81	48	22
	$G^2 = 16.1$ $p \approx 0.01$			$G = 4.0$ $p \approx 0.60$			$G^2 = 12.0$ $p \approx 0.05$		

Mathematics is now compared with the other subjects in terms of its overall chances of winning. First, when mathematics and science are compared, a large majority of children (69%) tend to see these subjects as "equally true". Mathematics wins very substantially over religion (84% of the comparisons). for mathematics and history the picture is not quite so clear. Mathematics wins 50% and history wins 9% of the cases but for 41% of the comparisons children perceive mathematics and history to be equally true.

QUALITATIVE ANALYSIS OF EXPLANATIONS AND CONCLUSIONS

The majority of children attempted some explanation for any given judgment and quite a number gave two. This section will describe the categories for children's responses, indicate frequencies of responses per category. The categories described below refer solely to the explanations for mathematics and science, an analysis of history and religion is in the process of being carried out.

1. **Nature of the subject** Children perceive the subject to be logical or coherent, and by definition true. (Some children argued that because the subject was logical it was not necessarily true)

2. **Relation between theory and proof.** Children perceive the subject to be "true" because it is constantly trying to find, or work on proof for its theories.

3. **Evidence through proof.** There are three sub-categories in this category: a. Children argue that "it can be proved", that is, there is some very general way of prooving the truth of the subject. b.Children specify that there are formulaes or special methods of proving the subject true. c. Children argue that experiments can be done, or that when using the subject it can be shown that "it works", an "empirical" type of proof.

4. **Constructivism.** Children perceive the subject to be true because it is constructed, made-up, invented by man, they often add "intelligent" men, this reason is sometimes used for lack of veracity.

5. **Pragmatism.** Children perceive the subject to true either because it is commonsense and can be found out from one's own experience or it used in the real world.

6. **Authority of the teacher** The explanations in this category simply state the subject is true because "the teacher told us/me".

As shown in table 4 evidence through empirical proof, that is, by experiments or "because it works" is the most popular category of explanation in both schools. Also the outside category of authority of the teacher is not all frequent in either school. For school A, the nature of the subject, that is, its logical nature, and evidence through general proof. are the two next most popular categories. The remaining four categories are all in a similar range of frequency of rosoonse between 7% and 10%. In school B the second most popular category is that of pragmatism, none of the remaining categories exceed a frequency of 12% of the response and Constructivism is the lowest with only 5% of the reponses.

TABLE 4: FREQUENCY OF CATEGORIES OF EXPLANATIONS

	School A	School B
1. Nature of subject	16 %	8 %
2. Relation between theory and proof	8 %	12 %
3. Evidence through proof		
a. general	18 %	6 %
b. through formulae, etc.	17 %	6 %
c. empirical	29 %	38 %
4. Constructivism	9 %	5 %
5. Pragmatism	8 %	29 %
6. Authority of teacher	3 %	4%

Concluding, children on the whole tend to see mathematics as "true" and similarly for science, their judgments not changing from first to fifth year of secondary school. Their reasons for their beliefs are mainly to do with empirical proof: for mathematics, "it works" and for science, "experiments work". The second most popular set of explanations was either the nature of the subject, its "logicality", or, its pragmatic nature. Constructivist reasons appear but very infrequently.

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VARIABLES INFLUENCING BEHAVIORAL SUCCESS AVOIDANCE IN MATHEMATICS

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This study permitted measurement of high and low success avoidant 9th grade females based on post tested observed mathematics performance data. Analysis of Variance was conducted using trained observer ratings and personality scale scores. Based on observed and measured scores, this investigation identified significant personality trait differences on Defendence and Autonomy between high and low success avoidant females. High success avoidant females scored low on Defendence and Autonomy. In contrast to the high success avoidant females, low success avoidant females are self-protective, self-reliant and independent.

Introduction

Morgan and Mausner (1973) and Mausner and Cubit (1979) have developed a paradigm for studying the degree to which females (and males) might hold back their performance in dyadic settings even under conditions when the subject was clearly superior in ability. Related to this paradigm is the question of what background factors and personality variables influence females' behavioral success avoidance in mathematics.

Personality Factors

Horner (1968, 1969) and similar studies by Alper (1971), Lavach and Lanier (1975), and Romer (1975) have hypothesized a process involving motivation to avoid or to be fearful of success. Although Horner reports a variety of her own studies in support of female behavioral success avoidance, a number of questions remain in terms of differential socialization and in terms of personality characteristics of females who manifest avoidance

This research attempts to identify significant underlying processes influencing success avoidance in mathematics. When we began this research, we did what many researchers do in the beginning steps of exploring a research problem. We observed behavior and asked questions. This study reflects our interest in the basic psychology underlying the observations.

Method

Sample

One hundred and twenty students, 60 males and 60 females, were drawn from district secondary schools. These secondary school students were in grade 9 mathematics classes.

Data Collection Instruments

The Personality Research Form E. This personality inventory is designed to provide a set of scores for personality traits widely relevant to the functioning of individuals in a large number of situations. Form E consisted of twenty-two 16-item scales (Jackson, 1984). These scales may be defined as personality variables. They are listed here alphabetically: Abasement, Achievement, Affiliation, Aggression, Autonomy, Change, Cognitive Structure, Defendence, Desirability, Dominance, Endurance, Exhibition, Harmavoidance, Infrequency, Impulsivity, Nurturance, Order, Play, Sentience, Social Recognition, Succorance, and Understanding.

The Canadian Achievement Test. This achievement test (mathematics section, 1981) consisted of 45 questions which tested the subject's ability in problem-solving. In addition, a revised form of this test also was developed for use in the present study.

Procedure

Subjects were tested in groups of 30 during regularly scheduled classes. Subjects completed the Canadian Achievement Test (CAT) and the Personality Research Form E.

Prior to the second session, the Canadian Achievement Test (1981) was scored. Mixed sex dyads of unequal ability were formed on the basis of the group's median score. When the subjects arrived at the second session, they were given their score as well as their partner's score on the CAT.

In the second session, which took almost one hour, students were asked to work cooperatively with their partner on the revised form of the Canadian Mathematics Achievement Test.

While the subjects were working, a trained observer was present and evaluated each individual and the dyad on a number of dimensions. Due to space limitations, all dimensions will not be reported in this paper.

When the subjects' work was finished, they were thanked for their participation and then debriefed.

Results

This study uses data from females who were paired with males according to mathematical ability from two experimental groups. The first group consisted of 28 dyads each containing a high ability female and a low ability male. The second group consisted of 32 dyads each containing a high ability male and a low ability female. The median score based on these subjects' mathematical performance data from the Canadian Achievement Test, taken during the first part of the study, determined each subject's group membership.

Observer Ratings

Analysis of Variance was conducted using observers' ratings as the independent variable and selected Personality Research Form scale scores as the dependent variable. In particular, scores from observers' ratings, taken during the second part of the study were analyzed. Data from females were used for this analysis. Observers' ratings of females' behavioral success avoidance were split at the median. Two groups were formed: Group 1 was defined as high success avoidant females and Group 2 was defined as low success avoidant females.

Personality Scales

Analysis of the personality data revealed that on two of the Personality Research Form scales (Defendence and Autonomy) there were significant differences between females who were rated low success avoidant and those who were rated high success avoidant by the trained observers.

Defendence Scale. The description of a high scorer on the Defendence Scale is as follows: Ready to defend self against real or imagined harm from other people. Does not accept criticism readily. Females who were rated as low success avoidant scored high on Defendence. Low success avoidant females are self-protective. Females rated high success avoidant scored low on Defendence. High success avoidant females are not self-protective, F (1, 26) = 4.10, p>.05. Thus, significant differences at the .05 level were found between high and low success avoidant females on Defendence.

Autonomy Scale. The description of a high scorer on the Autonomy Scale is as follows: Tries to break away from restrictions; self-reliant, independent, autonomous. Females who were rated as low success avoidant scored high on Autonomy. Low success avoidant females are self-reliant. Females who were rated as high success avoidant scored low on Autonomy. They are not self-reliant, F (1, 26) = 4.37, p>.04. Thus, significant differences at the .04 level were found between high and low success avoidant females on Autonomy.

Conclusion

This study permitted initial evaluation of personality variables in high and low success avoidant females. There were significant differences in the personality traits of high and low success avoidant females. Differences are also anticipated for certain background variables reflecting differential socialization in conformity with the above findings regarding personality. Further analyses will be conducted and presented. Discussion of additional analyses in relation to the results given here will follow the PME-XI presentation.

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MEASURING BEHAVIORAL SUCCESS AVOIDANCE IN MATHEMATICS IN DYADIC SETTINGS

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This study pairs one hundred and twenty 9th grade males and females in combinations of high and low pretested mathematics performance. Analysis of Variance was conducted using mathematics and other performance data. This investigation permitted an evaluation of whether previous results of behavioral success avoidance in high ability females would occur. Reduced scores of high ability females working with a lower mathematics ability partner suggested deference to the male and behavioral success avoidance in the high ability female.

Introduction

Women's achievement behavior has become a topic of interest to many researchers. In the last decade, there has been particular interest in women's mathematical achievement and in women's avoidance of mathematical achievement. Reference can be made to excellent research in such areas as sex differences in mathematics and ability and in the mediating effect of sex role orientation on mathematical performance.

Mathematics and Achievement Motivation

It has been generally assumed, according to Maccoby and Jacklin (1984), that male students are more achievement oriented than female students. However, girls generally achieve better grades than boys throughout their school years. Girls are also reported at an earlier age as being more interested in school related skills. According to deWolf (1981), male students have been found to do better on mathematical aptitude and achievement tests because males have chosen to take more mathematics related courses than female students. When differential course taking has been taken into account the sex differences disappeared (deWolf, 1981; Becker, 1982; Fennema, 1980; Pallas and Alexander, 1983).

According to Maccoby and Jacklin (1974), although males may be more achievement motivated under directly competitive conditions than females, they do not appear to have generally greater achievement motivation than females.

Interestingly, Spender (1982) reported that young girls in elementary school indicated that they liked and enjoyed mathematics. The boys, on the other hand, indicated that they did not believe that girls could do mathematics competently. Somewhere in adolescence the attitudes of many females change and girls begin to state that they are not capable of doing mathematics. This occurs when girls reach an age at which boys' opinions are important to them. No doubt many socio-cultural variables impact on females lowered self-regard for the study of mathematics. Variables such as lack of cultural reinforcements and few female mathematically oriented role models appear to be highly influential factors.

The negative attitudes that females have toward mathematics is further revealed by high attrition rates of females in senior level mathematics courses. According to Leder (1982), more females in grades 10 and 11 than males in the same grades intended to discontinue taking mathematics altogether. It was further revealed that girls high in mathematics performance who continued taking mathematics seemed to experience an increase in amount of anxiety as they went through school. According to Becker (1982), sex typing of mathematics as a male domain may inhibit female achievement and interest in mathematics. It has been found by Swanson and Tjosvold (1979) and Morgan and Hauser (1973) that high ability females cooperating with low ability males on a task, when influenced by self presentation and compliance concerns, subsequently lowered their performance level.

It was with many of these research studies in mind that we began a research project which could examine the mathematics performance or decrements in performance in high ability females working cooperatively with low ability males. In addition, incorporated into this investigation was mathematics performance or decrements in performance in high ability males working cooperatively with low ability females.

Method

Sample

One hundred and twenty students, 60 males and 60 females, were drawn from district secondary schools.

These secondary school students were in grade 9 mathematics classes.

Data Collection Instruments

The Canadian Achievement Test. This achievement test (mathematics section,1981) consisted of 45 questions which tested the subject's ability in problem-solving. In addition, a revised form of this test also was developed for use in the present study. Attitudinal, attributional and developmental instruments were also administered in this study but these instruments will not be reported in this paper.

Procedure

Subjects were tested in groups of 30 during regularly scheduled classes. Subjects completed the Canadian Achievement Test(CAT) and the other instruments.

Prior to the second session, the Canadian Achievement Test (1981) was scored. Mixed sex dyads of unequal ability were formed on the basis of the group's median score. When the subjects arrived at the second session they were given their score as well as their partner's score on the CAT.

In the second session, which took almost one hour, students were asked to work cooperatively with their partner on the revised form of the Canadian Mathematics Achievement Test.

While the subjects were working, they also were responsible for determining the following: (a) who had started the problem; (b) who had contributed what percentage to the problem-solving; and (c) who had actually completed the problem. The procedure followed by subjects was that of writing their names next to each of the appropriate categories for each of the problems. When the work was finished, subjects completed an attributional questionnaire.

At the conclusion of the study, subjects were thanked for their participation and then debriefed as to the nature of the investigation.

Results

This study pairs males and females in two dyadic experimental groups: Group 1. This group consisted of 28 dyads each containing a high ability female and a low ability male. Group 2. This group consisted of 32 dyads each containing a high ability male and a low ability female. The groupings were determined on the use of the median score as a cutting score based on these subjects' performance data from the Canadian Achievement Test (mathematics section) taken during the first part of the study. The Canadian Achievement Test is also considered as an ability test.

Mathematical Performance

Analysis of Variance was carried out on the mathematical performance data from the second part of the study. There was no overall difference between the two experimental groups in terms of the actual number of questions answered correctly. Both groups (Group 1 and 2) were equally effective in their mathematical performance.

Closer inspection of the data within each group revealed differences associated within each of the three categories related to problem-solving behavior.

Group 1. In terms of the number of times a subject initiated problem-solving in Group 1, it was found that high ability females appeared to indicate that they had initiated more problem solving than their low ability male partners, F (1, 54) = 3.34, p>.07. The result was not significant at the .05 level.

High ability females also indicated that they had offered a higher precentage of help towards problem-solving than their low ability male partners, F (1, 54) = 11.07, p>.001. This result was highly signficant at the .001 level.

High ability females also appeared to indicate that they had more frequently solved the problems than their low ability male partners, F (1, 54) = 3.38, p>.07. The result was not significant at the .05 level.

Group 2. In terms of the number of times a subject initiated problem-solving in Group 2, it was found that high ability males appeared to indicate that they had initiated more problem solving than their low ability female partners, F (1, 62) = .86, p>.35. The result was not significant at the .05 level.

In regard to who had offered a higher percentage of help toward the problem-solving, the high ability males indicated that they had offered a greater percentage of help than their low ability female partners, $F(1, 62) = 10.96$, $p > .001$. This result was highly significant at the .001 level.

Higher ability males also indicated that they had solved the problems more frequently than their low ability female partners, $F(1, 62) = 11.90$, $p > .0001$. This result was highly significant at the .0001 level.

Conclusion

This study paired a high ability female with a low ability male on the basis of pre-tested mathematics performance. This study permitted an evaluation of whether previous results of behavioral success avoidance in high ability females paired with low ability males is a function of deference to the perceived "dominant role" of the male. The reduced scores obtained by high ability females working with a lower mathematics ability partner suggests deference to the "dominant role" of the male and also suggests behavioral success avoidance in high ability females.

(Discussion of this paper will follow after the PME-XI presentation).

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ANXIETY AND PERFORMANCE IN PRACTICAL MATHS AT TERTIARY LEVEL : A REPORT OF RESEARCH IN PROGRESS

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This is the chronicle of a study which aims to study adults' use of maths in various contexts, and such barriers to this as 'maths anxiety'. One particular interest was how maths anxiety is used to explain women's allegedly poorer performance. Beginning with the standard literature and self-report questionnaires, I produced some results, e.g. some 'truths' about gender differences in maths anxiety. Not entirely convinced, however, I also produced interview data, thus aiming to specify more fully the contexts of using numbers. This raised questions about the usual concepts and methods for studying maths anxiety.

OBJECTIVES OF THE STUDY

(i) to discuss the usefulness of various notions of 'maths anxiety', as a block to numerate activities, among adults;
(ii) to study expressed maths anxiety(MA), both from questionnaires and interview situations, to contrast this with MA exhibited in interviews, and to consider the relationship of these with performance;
(iii) to produce accounts of the origins and nature of MA experienced by a group of 1st year college students;
(iv) to consider gender differences in (ii) and (iii).

THEORETICAL FRAMEWORK(1)

A contemporary psychological definition of anxiety is " a palpable but transitory emotional state or condition characterised by feelings of tension and apprehension and heightened autonomic nervous system activity" (Spielberger,1972,p.24). Since the 1950s, types of anxiety have been distinguished, according to:

(i) the context of the anxiety:general vs. specific ; test anxiety and maths anxiety are examples of the latter.

(ii) how measured: by physiological/overt behavioural means, or by self-reports;

(iii) when measured: a transitory 'state' - immediately after being experienced vs. a chronic 'trait'.

Occasionally, some interesting relationships between levels of anxiety and performance were found such as the "inverted U", but, for the most part, reviews of results are contradictory (e.g. Biggs,1962).

The notion of "mathematics anxiety" has been highlighted, since 1970's researchers were seeking to explain women's apparently lower performance, and 'participation', levels in maths courses, other than by innate differences. Prominent among the measures of MA proposed were the Maths Anxiety Rating Scale (MARS)(Rounds and Hendel,1980), and the Maths Anxiety Scale (Fennema and Sherman).

METHODOLOGY(1)

A suitable setting was a Polytechnic with a relatively high proportion of 'mature' students (over 21 years of age, returning to study after some years of work or child-care), some of whom are admitted without 'standard' H.E.

qualifications (2 A-levels).
Over 1983-85, entrants to two degree courses were asked to complete a questionnaire. This included items about their previous maths experiences, and a maths 'performance scale', followed immediately by a version of the MARS .
Our adaptation selected 26 items, brief descript ons of situations such as "adding two three digit numbers while someone looks over your shoulder", seeking responses on a 7-point scale from "very relaxed" to "very anxious". Half of these items were related to each of two factors proposed by Rounds and Hendel (1980); namely, maths test anxiety(TA), about maths courses or tests, and numerical anxiety(NA), relating to everyday concrete contexts.

RESULTS(1)

(The following relate only to the 1984 Social Science entrants; n=84 Females + 52 Males.)
1. In the questionnaire , the level of anxiety expressed by women was substantially higher than men's.

	NUMERICAL ANXIETY		MATHS TEST ANXIETY	
	F	M	F	M
MEANS	3.07	2.76	4.43	3.80
S. D.	0.89	1.01	1.22	1.14

2. Correlations between results on the maths performance scale and scores on the two MA subscales were negative and low (approx. -.2), with a hint of an inverted-U scatterplot (or at least the right-hand half of one). The pattern was essentially the same for males and females.

PROVISIONAL CONCLUSIONS(1)

1. Women express a higher level of MA than men, on this self-report measure, in the particular conditions (at the end of the first Psychology lecture of the year).
2. A simple linear correlation between MA and performance is not very informative.(Non-linear modelling is underway.)
3. So far, MA is still seen as a personal characteristic, on which one can be assigned a quantitative score. But are there any qualitiative differences in experiences of maths anxiety across social groups,e.g. between women and men ?
4. Does not the context in which a person experiences and reports maths anxiety need fuller description: is it the mathematical features, the social interaction, and/or past experiences which are meaningful?

THEORETICAL CONSIDERATIONS(2)

To address these points, I drew on studies that emphasise 'context' by seeing the use of numerate "skills" as an integral part of some "activity" or "practice", as follows (e.g.Lave et al.,1984; Walkerdine, forthcoming):
1.Context and activity mutually influence; for example, the consumption of food etc, and the regulation of children in so doing produces meanings (e.g. of 'more'); these meanings will also be conditioned by the family's material situation.
2. Most, if not all, activities/contexts support quantification; thus, "sharing" as a child gives meaning to size and distance relations.
3. Practices and their meanings are emotionally charged Thus, buying things may be related either to pleasure or to anxiety, or sometimes to both.
4. Practices are often specific to particular social groups - or cultures. Thus, "going out for dinner" may have a different meaning for men and for women.
5. A particular task may call up one - or several - practices as relevant at one time. Thus looking at a

pie-chart may remind someone both of a school maths topic, or of sharing food "fairly" with siblings - or both.

METHODS(2)

Towards the end of their 1st year a small subsample of the Social Science students were interviewed (1984:n=9, chosen by a mix of random and "volunteer" methods; 1985:n=16, chosen randomly), and asked to describe:
(a)the way they were thinking about solving a set of 'practical' problems; e.g. reading graphs, deciding how much (if at all) they would tip after a meal, deciding which bottle of tomato sauce they would buy; plus
(b) the sorts of practices "called up" by the interview; and
(c) past situations in which they had experienced MA.

RESULTS(2)

(These results refer only to the 1984 cohort, and are currently being tested and developed with the 1985 sample.)
1. As expected, more women (3 of 4) than men (2 of 5) expressed anxiety clearly during the interview, often about the interview itself, sometimes about outside situations; e.g.85/8 (Working Class M), thinking about having to do mental sums, if he were to take a pub job: "I think 'panic' because of people in front ot me waiting to be served".
2. As for exhibiting anxiety, I began by using rough indicators for anxiety, such as: (a) speaking unusually fast, or slowly, or quietly, (b) "mind going blank" or nervous laughter, (c) wanting to discuss the answers (to the problems posed). Initial analysis shows all (5 of 5) men exhibited anxiety, including 3 of 3 who had not expressed

it. For example, apparently confident about using numbers, 85/4 (Middle Class M) feels his mind go blank for a moment while calculating a 9% pay rise exactly, explained as "a sudden block, I guess through not doing tables...through not using it".

3. Interview problems called up a wide variety of practices, sometimes requiring numeracy, sometimes related to maths in a surprising way. An illustration: for 85/5 (MC F), a graph showing changes in gold prices over a day's trading, recalled for her growing up in a stockbroker's family:

"...as a stockbroker, your home and your material valuables are on the line all the time... most of the time, it was like living under a time bomb ... especially if you don't quite know how the time bomb's made up or when it's going to explode...." When I asked how she saw his work, to pick words, adjectives to describe his work, she replied: "capitalist, corrupt, business-like,..um, mathematical, calculating, devious, unemotional..."

PROVISIONAL CONCLUSIONS(2)

1. In the questionnaire , the level of anxiety <u>expressed</u> by women was substantially higher than that expressed by men. This difference is observed in interviews, too, but there men seem to <u>exhibit</u> more unacknowledged anxiety.

2. Interviewees' accounts indicate that experiences formative of maths anxiety include those with teaching at school, but also those to do with relationships with parents and siblings. This suggests new ways to produce a fuller account of maths anxiety.

IMPORTANCE OF THIS STUDY

1. This study explores the concept of 'maths anxiety', and also particular 'truths' about it, e.g., 'females have more of it than males' - by drawing on two theoretical frameworks, and by using questionnaire and interview data.
2. This work is possibly the first(?) to use the Mathematics Anxiety Rating Scale outside North America. Because of the high proportion of 'mature students', this sample is closer than most to being representative of the population of adults at large.
3. This study uses the idea of a 'practice' to describe the contexts of doing maths. It attempts this by interviewing (rather than by more time-consuming observation), and thereby elicits indications of a relatively large number of such practices (though not described in detail).
4. This study aims to understand the fluency and ease with which adults use numbers within particular contexts, not only in terms of cognitive familiarity(as is largely so, say in Lave et al., 1984) - but also in terms of the emotional associations of the practices concerned.

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A COMPARISON OF TWO PALLIATIVE METHODS OF INTERVENTION FOR THE TREATMENT OF MATHEMATICS ANXIETY AMONG FEMALE COLLEGE STUDENTS

by

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Self-efficacy theory (Bandura, 1978) provided the theoretical underpinnings for two mathematics anxiety interventions, cognitive restructuring (CR) and modified progressive relaxation (MPR). When mathematics anxiety was measured with a paper-and-pencil inventory, the difference between the mean levels of self-reported anxiety for CR and MPR subjects was not statistically discernible. When anxiety was operationally defined as skeletal muscle tension and measured with an electromyograph, CR subjects as a group experienced significantly lower levels of anxiety than MPR subjects as a group ($F = 2.81$, $p = .036$). Physiological and paper-and-pencil measures of anxiety were minimally correlated.

Since mathematics anxiety is one of the factors contributing to the problem of underrepresentation of females in scientific and technical fields (Betz, 1978), there is a need to pursue at least three levels of investigation: (1) to understand the etiology of mathematics anxiety, (2) to develop intervention strategies which help individuals who exhibit this affective problem, and (3) to compare the relative effectiveness of these interventions. This investigation focuses on the latter need, and is important for at least two reasons: (1) the problem of mathematics anxiety among female college students is of national concern, but appears to be endemic among students enrolled in southern liberal arts colleges (Thompson and Levin, 1977), and (2) the majority of previous math-anxiety research relied only on paper-and-pencil assessment data (Tobias and Weissbrod, 1980). This study was conducted at Mary Baldwin College, a private liberal arts school for women, and it utilized a two-dimensional response, i.e., a paper-and-pencil instrument and a physiological measure of anxiety.

The math-anxious individual must struggle with a combination of at least three negative elements: (1) undesirable physiological responses, (2) certain pernicious features of the math-environment, and (3) maladaptive thoughts (Heller and Kogelman, 1982). Bandura's (1978) social learning theory seeks to incorporate these three components into an integrated framework. Expectations of personal efficacy play an important role in Bandura's theory. Efficacy expectations are perceptions of personal mastery, i.e., subjective estimates regarding one's ability to cope successfully. The relationship between self-efficacy and attitudes toward mathematics was studied by Collins (1982) who reported that they are positively correlated, i.e., those who regard themselves as highly efficacious approach potentially threatening tasks nonanxiously. Further, Hackett (1981) reported a significant relationship between perceived self-inefficacy in dealing with numbers and mathematics anxiety, i.e., those who regard themselves as inefficacious experience varying degrees of anxiety and stress. Since Bandura (1978) argues that anxiety is the product of perceived inefficacy, social learning theory provides a useful framework for the study of mathematics anxiety.

Operating from different theoretical viewpoints, behavioral therapists have developed a variety of interventions for the treatment of anxiety disorders. Corresponding to the environmental component of Bandura's model are direct action methods of intervention designed to alter anxiety-eliciting environments. Corresponding to the cognitive and behavioral components of Bandura's model are intrapsychic and symptom-directed modes of intervention, respectively, which are aimed at reducing the level of anxiety once it has been aroused. Palliative methods such as these are used to soften or moderate anxiety, thus helping individuals function adequately within anxiety-eliciting environments.

Interventions aimed at modifying the mathematics learning environment are plentiful and achieve positive results. However the impact of direct action interventions is limited, i.e., treatment-produced improvement is not sustained, because the math-anxious individual is not provided with a set of coping skills. Intrapsychic and symptom-directed modes of intervention equip math-anxious individuals with coping skills, and unlike direct action techniques, focus primarily on efficacy-based anxiety. To the extent that math-anxious individuals continue to use these coping skills, long-term or durable improvements are achieved. Modified progressive relaxation (MPR) is a symptom-directed mode of reducing anxiety, whereas cognitive restructuring (CR) is an intrapsychic mode of alleviating anxiety. These two palliative techniques were chosen for comparison since considerable evidence exists which indicates that both MPR and CR are effective as therapeutic interventions for a wide range of stress-related problems.

Subjects for this investigation were sixty-two Mary Baldwin College students enrolled in three mathematics courses of differing levels of mathematical rigor during

Fall semester, 1985. Participation was a course requirement. A review of twenty-three mathematics anxiety intervention studies revealed that only seven dealt exclusively with female subjects. Consideration of this statistic, together with the issue of differential treatment of female and male students by mathematics teachers (Becker, 1980), and the belief that mathematics anxiety is more common and severe among females (Betz, 1978), contributed to the decision to limit this study to female subjects.

Subjects assigned to CR were taught to replace maladaptive thoughts with more positive rational thoughts. During these sixty-minute sessions the underlying assumptions of CR were explained. Subjects learned to identify distorted cognitive styles (e.g., emotional reasoning, overgeneralization, personalization, and all-or-nothing thinking). During these sessions the counselor played the role of devil's advocate. The subjects were to assume that the counselor actually held certain maladaptive beliefs and then generate as many reasons as possible why it may be irrational or unreasonable to hold onto such beliefs. During the last few minutes of each session, while working a series of math-related problems, participants were instructed to use this list of positive coping self-statements to practice changing their own maladaptive cognitions.

Subjects assigned to MPR met individually with a counselor once each week for six weeks. Subjects were informed that the purpose of each thirty-minute session was to help them learn to inhibit dysponetic activity, thereby increasing their performance in mathematics. MPR was presented as a coping skill for dealing with unwanted physiological arousal. At the beginning of each session, the counselor assisted each subject in identifying and locating twelve major muscle groups (e.g., frontalis, trapezius, rectus abdominis, and gastrocnemius). When a subject was comfortably seated, she was instructed to breathe easily and smoothly, tightening only the muscles that she is directed to tighten, letting the rest of her body remain relaxed. The counselor then guided the subject through a fifteen-minute tape recorded script. During the last few minutes of each session, the subject was given a series of math-related problems to work and instructed to use progressive relaxation to cope with unwanted physiological arousal.

In addition to the treatment variable, there were two other independent variables: level of achievement in mathematics (SAT), and level of participation in mathematics (remedial, intermediate, or advanced). Four research questions were investigated: (1) When administered over a six-week treatment period, are CR and MPR equally effective in reducing mathematics anxiety among female college students? (2) Are any combinations of treatment and level of achievement in mathematics characterized by lower levels of anxiety than other combinations? (3) Are any combinations of treatment and level of participation in mathematics characterized by lower levels of anxiety than other combinations? (4) To what extent do physiological indicators of mathematics anxiety and paper-and-pencil assessments measure the same construct?

Data were collected in two stages. The first stage occurred at the end of a six-week treatment period, at which time Sandman's (1973) Mathematics Attitude Inventory (MAI) and an electromyograph (EMG) were used to obtain self-report and physiological measures of mathematics anxiety. The second stage occurred eight weeks later, at which time the MAI was readministered. Initial descriptive statistics suggested that: (1) subjects at remedial levels of participation in mathematics tend to experience higher levels of self-reported mathematics anxiety, (2) subjects at more advanced levels of participation experience a greater degree of skeletal muscle tension than subjects at intermediate and remedial levels of participation, (3) paper-and-pencil and physiological measures of mathematics anxiety are minimally correlated, (4) CR subjects as a group experience lower levels of self-reported mathematics anxiety than MPR subjects as a group, (5) MPR is least effective with students at advanced levels of participation in mathematics. Inferential methods revealed that: (1) when mathematics anxiety was measured with Sandman's MAI, for both the immediate and delayed posttests, the difference between the mean levels of self-reported anxiety for CR and MPR subjects was not statistically discernible, (2) when anxiety was operationally defined as skeletal muscle tension and measured with an electromyograph, CR led to significantly greater reductions in anxiety than MPR ($F=2.81$, $p=.036$) , (3) there was no interaction between type of intervention and level of achievement in mathematics, (4) when anxiety was operationally defined as skeletal muscle tension and measured with an electromyograph, a statistically discernible ($F=3.925$, $p=.027$) synergistic effect was detected between type of intervention and level of participation in mathematics, indicating that CR is superior to MPR for subjects at intermediate and advanced levels of participation in mathematics, whereas MPR is superior to CR for subjects at remedial levels of participation, and (5) there was insufficient evidence to indicate that a linear relationship exists between paper-and-pencil (MAI) and physiological (EMG) measures of mathematics anxiety, implying that the two instruments may be tapping different dimensions of the mathematics anxiety construct.

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STUDENTS' PERCEPTIONS OF MATHEMATICS AS A DOMAIN

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ABSTRACT

This research identified junior high students' (N = 451) perceptions of what is and isn't mathematics. Perceptions were documented by five 11-item questionnaires reflecting six major strands of K-6 content. Students were asked to indicate whether mathematics was used or involved and supply their rationale for each choice. The results were compared to K-6 children's answers from a previous study (N = 1202). The results showed that junior high school students' percentages of YES/NO responses paralleled the K-6 sample in both order and magnitude. Differences in rationales between samples occurred in use of counting, emphasis on the format of problems, and need for an identifiable operation and explicit number pairs. Common elements from both samples included that mathematics is a fluid domain, isolated from other subject areas, active and school related.

This is the second in a series of studies investigating students' perceptions of the domain of mathematics. The underlying assumption of this line of research is that the perceptions that students and teachers have of what mathematics is (and isn't) may affect their concepts of specific topics within mathematics, their attitudes toward mathematics, their performance in mathematics and other related aspects such as confidence, choice of courses/careers and perceived usefulness. However, before looking at how perceptions of mathematics affect other aspects of learning and teaching mathematics, we need to develop a reliable system for identifying, describing, classifying and, ultimately, "measuring" these perceptions. This is the intent of the current series of studies. The data from these studies will provide the necessary foundation for further research investigating the effects of perceptions of the domain of mathematics on those other possible aspects. Of equal importance are the data this research may provide toward the identification and documentation of misconceptions that students have about certain specific aspects of mathematics such as subtraction, division, and the materials or actions that are involved in doing mathematics.

The data from the previous study of 1202 children in grades K-6 showed that children's perceptions of mathematics, are not quite what might be expected. While adults may consider mathematics to be a well-defined subject matter (Ginsburg, 1983), kindergarten through sixth grade children do not see it as so (McDonald & Kouba, 1986a, 1986b). For them, the domain of mathematics, while being narrow, is also not constant. Rather, it is upwardly shifting. To many children when something becomes easy, it is no longer mathematics. Kindergarten through sixth grade children also see mathematics as being isolated from other subject areas, active, and school-related. For these children, whether a situation involves mathematics is influenced by developmental factors, the presense of explicit numbers and operations in the situation, and idiosyncratic aspects of the particular situation.

The major purpose of the current study in this line of research was to identify whether seventh and eighth grade students' perceptions of the domain of mathematics were parallel to those of kindergarten through sixth grade children. Do developmental trends identified with elementary school children continue through junior high? Do explicit cues to numbers and operations continue to affect students' identification of the kind and the extent of mathematics involved in a situation? Does counting continue to play a major role in students' justification for the presence or absence of mathematics in a situation? Will mathematical operations and concepts continue to "drop out" of the students' perceived domains as a result of their becoming more automatic and "easy?" Are there gender differences in students' perceptions which were not identified in the previous research?

METHOD

Subjects. The subjects were 219 7th graders and 232 8th graders from five public or private schools, representing small, medium and large districts in rural, suburban or urban settings.

Procedure. During their mathematics class, the students were given a questionnaire consisting of eleven situations. They were instructed by their classroom teachers to quietly read each situation and indicate by circling YES or NO whether mathematics was being done or was involved in the situation. They then were to indicate in writing why they chose YES or NO. Five different questionnaires were constructed in a stratified random manner from a pool of 55 items (see Figure 1 for sample items). The forms were distributed randomly within each class. The questionnaire items included the majority of

D10. Melanie had to tell the teacher which was greater, 5 or 3.

C3. Melanie had to tell the teacher which number was greater.

C4. Dave played soccer yesterday afternoon.

B4. Billy looked at the clock to see how long a nap he could take before the soccer game.

A3. Betsy made Valentine cards by cutting out hearts using folded paper.

E6. Betsy made paper dolls by using symmetry.

A5. Julie kept track each day of how many miles she rode on her bike.

B1. Alan took out his ruler and measured his desk.

E1. Julie arranged three different colored chips in a line in as many ways as possible.

B3 George cleaned up room number 7 which was really messy.

Figure 1. Sample questionnaire items

the items used in the K-6 study, as well as some new and some revised items. The items were designed to reflect the six major content strands of the New York State K-6 syllabus: number and numeration, operations with whole numbers and fractions, probability, statistics, geometry and measurement. Several items included paired explicit and implicit use of cardinal and ordinal numbers as both facilitating and distracting elements (e.g. D10, C3 and B3). The situations varied from ones where the operational process was clear to those where it was necessary to infer the mathematical process involved (e.g. D10 and B4). Situations in which the protagonist was not using or doing mathematics were also included (C4 and B3).

RESULTS AND DISCUSSION

For each item, students' YES or NO choices were tabulated and matched with the syllabus-specified designation of whether the item involved mathematics. The percent of students agreeing with that designation was recorded by grade level and sex. Significant gender differences appeared on only eight of the 55 items. On five of the items boys were in greater agreement with the syllabus than the girls. On the remaining three items, girls were in greater agreement. The items were then ranked by percentage of agreement. The percent agreeing from grades 7 and 8 combined was correlated with the percent agreeing from grades K-6 for 43 of the items which were identical across samples. A Spearman's rank-order correlation was determined comparing the relative ranks of the items based on the percentages of students' agreement with the syllabus. The resulting rho of .8390 (41, N = 451), $p < .001$, indicated that in general, items which were easily identified by K-6 children as mathematical were equally easy for junior high students to identify. The same was true for difficult items. A Pearson correlation was also calculated on the two sets of percents of agreement with the syllabus. An r of .687 (41, N = 451), $p < .001$, indicated that in addition to a relatively stable order of items, that the individual percents of agreement on each item were also very similar. Agreement with the syllabus was calculated for the entire set of questions for each grade level. In comparing these means with the means of each of the grade levels of K-6, it was determined that the generally increasing trend of agreement in the K-6 data, did not continue through the 7-8 data (see Table 1).

Table 1

Percent of Agreement with Syllabus Designation of Whether Mathematics was Involved

Grade Level	K	1	2	3	4	5	6	7	8
Percent	54	61	61	71	76	80	80	73	77

The comments explaining the students' YES/NO choices were sorted into rationales for responding YES, and rationales for responding NO. These two types were classified and tallied in order to identify relative frequency of response categories. An examination of Table 1 might suggest that students are reasonably adept at classifying mathematical items. However, this table shows only the students' ability to see math in a given item, not identify the appropriate area of skill involved. The analysis of student rationales revealed that, in many cases, the students either designated a skill or concept at a much lower level than the syllabus, or identified an inappropriate or tangential skill or concept. For example, for item E1, students who identified it as being math included those who gave reasons such as, "it has to do with colinear stuff, etc.," "you can count the colors," "to arrange them in as many orders you would divide," or "you have to use numbers... 3 chips X 1 row." As with the K-6 sample, students often misapplied the operations of division and subtraction.

As might be expected, junior high students gave "counting" as a reason for a situation being mathematical much less often than K-6 children. For the K-6 children, counting appeared in the top three reasons for a YES response on 15 of the items. For the junior high students counting appeared in the top reasons on only eight of those same items.

Junior high students seemed to view mathematics as a broader domain than did K-6 children, for they included probability, geometry, and measurement in their rationales. However, they did so mostly for obvious situations where an explicit term, symbol or format was used. When the format was not easily recognizable as one which they might have seen in mathematics class, students often indicated that mathematics was not involved and gave reasons such a "There's no problem part" and "there's no way to make it into a problem," or "there's no question." The basic operations were still the major component of mathematics for the junior high students. This appeared in YES and NO rationales alike. Students often identified situations as being mathematical because they "saw" one of the four basic operations present, although not always correctly. Students also made statements like "It's not math because there's no addition, subtraction, multiplication or division." At a more subtle level, several students made the comment "It isn't math because there's no other number," when only one explicit number appeared in a problem, thereby seemingly making an operation impossible.

Junior high students were similar to K-6 children in that they appeared to identify mathematics by what they had seen in mathematics class rather than by recognizing the underlying structure of situation as mathematical. For example, while junior high students were able to identify the mathematics in an item where the word "symmetry" was used, they could not identify the mathematics in similar situations where the concept of symmetry was described but the actual term was not used.

The following additional conclusions were drawn based on the analysis of the junior high students' rationales. Mathematics continues to be an upwardly shifting domain. For example, some of the students gave reasons such as, "That's not math because it's just common sense" or "just logical" or "you just know." Other students echoed what the K-6 children indicated in that mathematics is "work," and also that mathematics requires activity, through statements such as "there's nothing to do" for situations describing a protagonist who was "looking" or "thinking" rather than calculating.

Some junior high students see mathematics as school related in that they identify situations as mathematics because they are what they have done in class. Other perceptions demonstrated by other junior high students included that mathematics is exact and therefore does not involve approximating or estimating, that mathematics is correct, requires calculation, and that it is not done in art, social studies, English or science class.

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LES MATHOPHOBES: UNE EXPERIENCE DE REINSERTION AU NIVEAU COLLEGIAL

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Le problème de la mathophobie fait partie du quotidien des professeurs de mathématiques et, de façon plus gênante, de la vie de certains étudiants. En nous basant sur diverses expériences tentées particulièrement aux Etats-Unis et sur notre propre vécu, nous avons mis sur pied un environnement ayant pour but de réconcilier un certain nombre d'étudiants ayant un vécu négatif face aux mathématiques. Dans notre recherche, nous voulions voir s'il y avait changement d'attitude chez les étudiants qui participaient aux ateliers et nous voulions identifier les raisons qui provoquaient ce changement. Nous espérions trouver une approche de l'enseignement des mathématiques qui minimiserait les situations propices à l'éclosion de la mathophobie.

Notre pratique comme enseignants en mathématiques, à laquelle s'ajoutent les témoignages, commentaires et remarques formulés par d'autres intervenants nous permettent de constater que de très nombreux étudiants refusent de s'inscrire à certains programmes d'étude parce que ceux-ci comportent quelques cours de mathématiques.

A ceux-là, il faut ajouter tous les autres qui s'inscrivent à chacune des sessions mais qui, systématiquement, abandonnent; ou encore, ceux qui retardent, d'une session à l'autre, le moment fatidique où ils devront finalement se résigner à suivre leurs fameux cours de mathématiques. Comme il s'agit souvent d'étudiants qui, par ailleurs, réussissent bien dans d'autres matières, il est difficile d'attribuer cet insuccès à un problème d'ordre intellectuel.

Le problème n'est pas nouveau et plusieurs études en ont déjà été faites: Nimier (1976),Thanh Minh(1981), Aiken (1970), etc. Il ressort de ces études que le côté affectif de l'être a un effet soit stimulant soit perturbateur dans l'apprentissage des mathématiques.

La mathophobie, selon Tobias (1980) est l'état de panique, de paralysie, de désorganisation mentale qu'éprouvent certaines personnes devant un problème de mathématiques. Notre recherche s'articule et se développe sur le postulat voulant que la composante affective de l'apprentissage explique en grande partie les échecs multiples et irrationnels vécus par les sujets identifiés comme mathophobes.

Nous avons mis sur pied, au Cégep du Vieux Montréal, en collaboration avec le service d'aide à l'apprentissage, des ateliers "phobie des maths". Ces ateliers fonctionnent depuis la session d'hiver 1984. À l'automne 1985, nous avions un nouvel objectif : celui d'y faire des observations afin de formuler des hypothèses sur les facteurs affectifs pouvant intervenir dans l'apprentissage des mathématiques.

L'annonce des ateliers se fait au début de session par le service d'aide à l'apprentissage. Au moment de l'inscription, les étudiants prennent rendez-vous avec un psychologue. Les ateliers se déroulent sur une période de cinq semaines à raison d'un soir par semaine.

Les étudiants susceptibles de participer aux ateliers sont ceux qui ont identifié leur insécurité face aux mathématiques et qui se reconnaissent à la lecture du profil proposé par la publicité. Ce profil décrit sommairement les caractéristiques d'un mathophobe tel qu'on peut le rencontrer au Cegep. Pour cette recherche, nous avons travaillé avec un groupe de quatorze étudiants dont l'âge variait entre 17 et 24 ans, pour la plupart. Soulignons que leur participation à ces ateliers est entièrement volontaire.

Les ateliers sont animés par trois personnes : les deux chercheurs qui sont professeurs de mathématiques, et un psychologue engagé à l'occasion de cette recherche. Le premier contact se fait lors une rencontre individuelle entre l'étudiant et le psychologue dans le but de diagnostiquer et de préciser le problème.

La préparation du contenu des ateliers se fait par les deux professeurs de mathématiques. Les ateliers sont animés selon les modalités suivantes: à tout moment, l'étudiant doit se sentir libre de partager ses sentiments avec le groupe, de prendre une pause, de demander de l'aide individuelle, de se joindre à d'autres pour travailler sur un problème. Les animateurs doivent établir un climat de non-compétition dans un environnement soutenant et se montrer particulièrement disponibles pendant les ateliers. Les animateurs écoutent l'étudiant quand il réussit à verbaliser ses problèmes et ses difficultés; ils l'observent et lui font remarquer ses progrès, ses cheminements.

Le premier atelier a une forme un peu particulière car son principal objectif est la prise de contact. Le schéma des autres rencontres est le suivant :

-Retour sur les activités de la semaine précédente.
-Problèmes suggérés sur un thème précis : activités mathématiques.
-Pause.
-Retour sur le processus aux deux niveaux mathématique et affectif (attitudes et comportements face à un problème).
-Fin de l'atelier.
-Tout de suite après la rencontre, les animateurs font un échange d'observations et une brève évaluation de l'atelier.

À partir de la deuxième rencontre, les thèmes présentés sont les suivants : jeux logiques, activités géométriques, algèbre, formules, probabilités.

En plus d'échanger avec les autres participants et les animateurs, l'étudiant a à sa disposition un "journal de bord" dans lequel il est invité à écrire, sous un pseudonyme, toutes ses impressions et ses idées personnelles. Les animateurs tiennent parallèlement un cahier où ils notent leurs impressions.

Nous avons structuré les ateliers avec, à l'esprit, un certain nombre de postulats de base qu'il nous semble important d'expliciter. Nous supposons que le mathophobe ne s'ignore pas et qu'il est capable d'articuler son problème en autant qu'il se sent disposé à le faire. Il faut donc être particulièrement attentif à ce que l'étudiant dit. De plus, nous sommes convaincus que le problème de la mathophobie se règle au coeur de l'activité mathématique. Les problèmes, les activités et le matériel sont choisis à cause de leur richesse et de leur variété. Ils doivent permettre aussi bien l'émergence des réactions mathophobiques que l'occasion de vivre des succès en mathématiques. Nous présumons que le mathophobe révèle ses difficultés à travers son activité.

Le mathophobe peut arriver effectivement à maîtriser la situation, du moment qu'on arrive à clarifier avec lui les dimensions qui sont en jeu. Pour cela, les questions ou les affirmations lancées par les animateurs se regroupent en cinq volets principaux: réflexion sur les activités, confrontation des mythes véhiculés par les mathophobes, partage du vécu mathématique entre les animateurs et les étudiants, partage de l'histoire de la genèse des idées en mathématiques, point de vue du professeur dans son rôle habituel ou stéréotypé.

Les données sont puisées à partir des sources suivantes:

Le résumé des entrevues; les réponses aux questionnaires (pré et post); le journal de bord de chaque participant; les notes des animateurs; les enregistrements des ateliers; les entrevues, rencontres et lectures faites dans le cadre des ateliers et portant sur le même sujet. Ces données ont été analysées en détail.

Les mathophobes en avaient long à nous apprendre. Leur expérience de l'apprentissage mettait en évidence des conditions fondamentales de la démarche mathématique et s'appliquait en fait à quelque chose de beaucoup plus large que le problème de la mathophobie. Nous avons pu observer de très près ce que l'étudiant ressent en faisant des mathématiques, et cette connaissance nous apparaît aussi valable dans le contexte régulier d'une classe que dans le contexte spécifique des ateliers pour mathophobes.

Nos résultats et leur analyse nous ont permis d'explorer différents facteurs sur lesquels les professeurs pourraient intervenir dans une démarche pédagogique régulière.Voici donc les treize hypothèses générées par notre recherche.

H1: Il est admis que l'apprentissage des mathématiques suppose et met en jeu de fortes dimensions affectives. De ce fait, l'apprentissage est souvent facilité par la présence de canaux de communication efficaces. Les étudiants préfèrent se sentir à l'aise dès le début des cours; ils ont besoin qu'on établisse ces canaux de communication au plus tôt.

H2: Il faut, de la part du professeur, s'adresser à la dimension affective de l'apprentissage des mathématiques qui, que le professeur le veuille ou non, est toujours en action; sinon, l'apprentissage est, à la limite, voué à l'échec.

H3: Il faut s'assurer que les étudiants puissent s'exprimer sur leurs perceptions de la matière, du professeur, de leur propre vécu en mathématiques. De la sorte, on peut éviter de perpétuer de fausses impressions, de fausses implications et de fausses dichotomies qui semblent actuellement circuler en grand nombre, au détriment de l'apprentissage des concepts et des méthodes propres aux mathématiques.

H4: Les relations étudiant-étudiant sont très importantes et influencent très positivement l'apprentissage des mathématiques; le professeur doit privilégier les échanges à ce niveau.

H5: L'exploration libre, en groupe, semble un facteur important dans l'apprentissage : les étudiants doivent avoir la possibilité de chercher, d'émettre des hypothèses et de tenter de les vérifier ou d'en tirer des conclusions.

H6: La verbalisation de la démarche poursuivie lors d'une activité mathématique est trop souvent négligée. Face à un pair, l'étudiant forcé de verbaliser sa démarche lui donne une réalité, peut s'en détacher, l'évaluer et la poursuivre.

H7: Le professeur doit transmettre son vécu en mathématiques, c'est-à-dire faire en sorte que l'étudiant puisse s'identifier à la démarche d'interrogation, de recherche et de réflexion que l'enseignant effectue lorsqu'il aborde une problématique mathématique.

H8: Il faut que le professeur ait des occasions de superviser l'apprentissage individuel. De nombreuses séquences ont montré que les animateurs peuvent effectivement guider l'étudiant à mesure qu'il progresse en lui posant des questions judicieuses, en lui faisant remarquer les résultats acquis, en formulant explicitement les hypothèses implicites de l'étudiant, etc...

H9: En relation avec la supervision de l'apprentissage, il semble important de multiplier les moments de prise de conscience des résultats ("Eureka"). On remarque dans quelques séquences que ces moments peuvent mener à la compréhension mais que l'étudiant a aussi tendance à "échapper" ses nouvelles connaissances. Il les conserve du moment qu'on le relance sur la piste.

H10: Le professeur doit favoriser les apports historiques et situer la démarche de l'humanité dans la construction des mathématiques. Ceci permettra à l'étudiant de constater combien de temps et de travail il peut y avoir entre la question et la réponse.

H11: L'étudiant doit pouvoir relier certaines démarches de résolution de problème, de recherche, de vérification à son vécu quotidien.

H12 La valeur des mathématiques doit être transmise mais sans mystification et de façon à ce que l'étudiant puisse les reconnaître comme étant accessibles.

H13: L'environnement mathématique doit être concret, réel, humain, afin d'intéresser l'étudiant. Autant la forme des activités que les contextes choisis doivent être souples, attrayants et faciles d'accès pour piquer la curiosité et stimuler la recherche.

Les résultats de nos observations nous permettent de penser qu'il est possible de remédier à la mathophobie et ce par des moyens que nous pouvons qualifier de pédagogiques : l'enseignant en serait donc le principal facteur. Disons, pour terminer, que cette exploration nous permet d'entrevoir la création d'un modèle d'intervention en classe et d'en envisager l'expérimentation. Il sera ensuite possible d'en évaluer les effets.

INVESTIGATION DES FACTEURS COGNITIFS ET AFFECTIFS DANS LES BLOCAGES EN MATHEMATIQUES

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La revue de la littérature fait apparaître de multiples pistes dans l'explication des difficultés en mathématiques. Nous avons tenté d'y voir plus clair en effectuant une démarche exploratoire auprès de deux groupes de filles de 6e année primaire (âgées d'environ 12 ans), les unes fortes, les autres faibles en mathématiques. Les instruments utilisés ont été des épreuves piagétiennes, des entretiens et deux tests projectifs (Rorschach et T.A.T.). On a trouvé, sur le plan cognitif, une corrélation très élevée entre l'acquisition de l'"opérativité" et le succès en mathématiques. Sur le plan affectif, on a observé une tendance à ce que les mathématiques soient investies d'une valeur phallique et ce, en relation avec le père, ainsi qu'une légère tendance à ce que l'échec en mathématiques soit l'expression d'un refus de plaire à la mère ou de se soumettre à ses exigences. Mais l'analyse individuelle du vécu conscient et inconscient de chacun des sujets a permis de constater que le succès ou l'échec en mathématiques s'inscrivent dans une dynamique propre à chaque élève et qu'on ne saurait en conséquence relier de façon générale le rendement en mathématiques à tel ou tel facteur affectif, de manière privilégiée.

En cherchant dans la littérature, nous avons découvert de multiples pistes concernant l'étiologie des difficultés en mathématiques, allant des troubles neurologiques (Henschen, 1919; Gertsmann, 1964; Hasaerts Van Geertruyden E., 1970; etc.) jusqu'aux fantasmes inconscients et aux problèmes d'ordre psycho-sexuel (Klein, 1923; Baudouin, 1951; Salzi, 1959; Male, 1964; Mauco, 1968; Diatkine, 1973; Nimier, 1976; etc.), en passant par les facteurs psycho-pédagogiques (Mialaret, 1957, 1959; Dienes, 1964; Baruk, 1973, 1977, 1985; Tobias, 1980; Weyl-Kailey, 1985, etc.). Certains auteurs privilégient la composante affective (Klein, 1923; Baudouin, 1951; Salzi, 1959; Nimier, 1976; etc.), d'autres la dimension cognitive (Dodwell, 1961; Hood, 1962; Freyberg, 1967; etc.), d'autres les méthodes pédagogiques (Hug Colette, 1968; etc.), etc. Quelques-uns font référence à une variété de facteurs (Male, 1964;

Jaulin-Mannoni, 1965; Beauvais, 1970; etc.). Plusieurs ne formulent que des hypothèses, certains se sont aventurés dans des recherches assez mal définies, superficielles ou boîteuses sur le plan méthodologique, si bien que les expérimentations à caractère scientifique sont peu nombreuses dans ce domaine. Et dans cette diversité, on trouve des contradictions et des juxtapositions qu'on ne sait comment concilier. Nous avons alors voulu voir un peu plus clair dans tout ce dédale de variables et de facteurs évoqués et, en l'occurrence, nous avons tenté d'effectuer nous-même une recherche.

Notre cheminement personnel ayant orienté davantage notre intérêt vers les facteurs psychologiques liés aux difficultés en mathématiques, nous avons envisagé de ne considérer que les dimensions cognitive et affective qui, d'ailleurs, regroupent à elles seules une multitude de facettes et ont été trop souvent étudiées séparément. Ne voulant privilégier aucune d'entre elles a priori dans notre recherche, nous avons décidé de ne pas nous fixer d'hypothèse de base en particulier. Notre démarche a voulu être essentiellement exploratoire et permettre, à travers l'ampleur de l'investigation, l'émergence par elles-mêmes des composantes majeures, peut-être insoupçonnées, qui peuvent jouer un rôle dans la réussite ou l'échec en mathématiques. Nous avons voulu, plus précisément, recueillir, dans une optique la plus objective possible, un très grand nombre de données sur le vécu conscient et inconscient de chacun des sujets, afin de voir, à travers tout ce matériel, s'il se dégage une dynamique qui a un lien spécifique avec le rendement en mathématiques. Cette étude a donc été réalisée dans une perspective psychanalytique.

Nous avons cependant été contrainte, devant l'ampleur de la tâche, de nous fixer certaines limites quant au nombre d'instruments à utiliser. Des épreuves piagétiennes nous ont semblé être un excellent outil pour déceler le niveau de développement des structures logiques de nos sujets, et, par ailleurs, les entretiens et tout particulièrement deux tests projectifs (le Rorschach et le Thematic Apperception Test) nous ont paru être les meilleurs moyens pour accéder aux niveaux conscient et insconscient, apportant ainsi une vue globale de la dimension affective. En outre, il est apparu fondamental d'augmenter la validité des données en procédant à des comparaisons systématiques entre des élèves fortes et des élèves faibles en mathématiques. Mais, afin d'éviter la prolifération des variables susceptibles d'influer sur les comparaisons, nous avons choisi des sujets de même sexe, de même niveau scolaire et approximativement de même âge et de même niveau socio-culturel. Ici encore, pour des raisons d'ordre pratique, il a fallu nous en tenir à un nombre assez restreint de sujets. Nous avons donc constitué deux groupes de dix sujets chacun: il s'agit plus précisément de filles, âgées de douze ans environ, de classes régulières de 6e année primaire, de la région de Montréal. Les élèves du premier groupe devaient être, depuis au moins trois ans, les meilleures des classes en mathématiques et réussir mieux en mathématiques qu'en français. Celles du second groupe devaient avoir, depuis au moins trois ans, des difficultés spécifiques importantes en mathématiques et avoir nettement plus de facilité en français. Le clivage a été effectué à partir de l'opinion des enseignants et des résultats scolaires des trois dernières années.

Cette approche diffère, semble-t-il, de toutes les recherches effectuées jusqu'ici dans le domaine des échecs en mathématiques du fait qu'elle s'est donné à la fois non seulement un groupe-témoin (on note en effet l'absence fréquente d'un tel groupe dans maintes études concernant les élèves ayant des difficultés en mathématiques), mais aussi deux mesures, l'aspect cognitif et l'aspect affectif, en privilégiant l'emploi de techniques projectives très rarement utilisées pour ce genre d'études.

En ce qui concerne les résultats obtenus, il se dégage de cette investigation une dichotomie très nette entre les deux groupes de sujets, sur le plan cognitif. On observe en effet que, chez les élèves fortes en mathématiques, neuf sur dix sont de niveau nettement opératoire, alors qu'une élève paraît osciller entre les niveaux préopératoire et opératoire. Chez les élèves faibles en mathématiques par ailleurs, aucune d'entre elles n'est franchement opératoire: huit semblent nettement préopératoires, alors que le niveau des deux autres élèves est encore fluctuant entre le préopératoire et l'opératoire. On a constaté, chez toutes les élèves faibles en mathématiques, les nombreuses hésitations et la faible mobilité de la pensée, caractéristiques des sujets qui ne sont pas franchement opératoires. Bref, on trouve une corrélation très élevée entre l'acquisition de la réversibilité ou "opérativité" et le succès en mathématiques.

Sur le plan situationnel, on note, dans les deux groupes de sujets, une sensibilité à l'aspect assimilation des connaissances (règles à retenir). De plus, les mathématiques sont perçues comme un ensemble de lois ayant d'irréductibles exigences, ce qui suscite de la peur, alors que le français est vu comme laissant plus de place à l'imagination et à la créativité. Les élèves faibles en mathématiques réclament un enseignement plus individualisé, moins compliqué, et plus de continuité dans les méthodes d'enseignement. Elles se trouvent beaucoup plus laissées à elles-mêmes.

Sur le plan affectif, nous avons observé une tendance à ce que les mathématiques soient investies d'une valeur phallique et ce, en relation avec le père: plusieurs élèves fortes en mathématiques semblent rechercher dans cette matière une compensation à leur sentiment de castration ou de manque face à leur père, tandis que quelques élèves faibles en mathématiques expriment par leur échec leur dépression sur le plan phallique et leur démission dans leur quête d'un soutien valable de la part de leur père. En d'autres termes, il semble exister un rapport entre recherche active du père et succès en mathématiques, de même qu'entre relation décevante au père et échec en mathématiques. Chez les élèves faibles dans cette matière, nous avons remarqué en outre une tendance à ce que l'échec soit l'expression d'un refus de plaire à la mère ou de se soumettre à ses exigences. Notons cependant que, chez la majorité des sujets concernés, ces facteurs affectifs ne sont pas uniques et primordiaux. L'analyse individuelle du vécu conscient et inconscient de chacun des sujets a permis de constater que le succès ou l'échec en mathématiques s'inscrivent dans toute une dynamique propre à chaque élève et qu'on ne saurait en conséquence relier de façon générale la réussite ou l'échec en mathématiques à tel ou tel facteur affectif, de manière privilégiée. Seul le facteur "niveau de développement des structures logiques" a opéré une différence très marquée entre les deux groupes de sujets.

En conclusion, cette recherche pose le problème des rapports entre le cognitif et l'affectif. Au point de vue diagnostique, elle montre l'importance d'évaluer le développement cognitif et le côté affectif de l'enfant. Elle souligne que le travail rééducatif doit être axé à la fois sur la dimension cognitive et sur la dimension affective: libérer l'enfant des émotions liées à l'échec, réconcilier l'enfant avec l'activité mathématique par un minimum de réussites pour renverser l'engrenage de l'échec et du désintérêt, susciter chez lui une participation active en l'amenant à découvrir par lui-même. Elle invite l'enseignant de classe régulière à dédramatiser l'enseignement et l'apprentissage des mathématiques, tout particulièrement en donnant à l'enfant beaucoup de possibilités de manipulations et cela durant tout le cours primaire. Sur le plan de la prévention, elle incite les enseignants à repérer très tôt les élèves "pré-opératoires" pour leur offrir une pédagogie correspondant à leur "âge cognitif", c'est-à-dire un enseignement qui stimule de plus près l'activité des structures logiques et des méthodes qui les rejoignent plus personnellement.

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CHILDREN'S ATTITUDES TO MATHEMATICS: A PERSONAL CONSTRUCT APPROACH

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Abstract

Much of the work on how children learn mathematics has been based on theories of mental development, particularly Piaget's. While mental development is unquestionably one factor which influences the way in which children learn mathematics, another, it may be hypothesised, consists of the attitudes which they bring to their task. This hypothesis has been investigated on the basis of Kelly's theory of personal constructs. Preliminary results of the study indicate that there are indeed relationships between certain mental constructs and mathematics performance, but that the concept of 'doing mathematics' itself needs refining, in that the relationships appear to be different for routine mathematics and for problem solving. Possible explanations for this finding are discussed.

A group of pupils from an English comprehensive school is taking part in the study. The school was chosen because its catchment area includes a wide range of social backgrounds and because it is the policy of the school to attempt to bring together, in tutor groups, pupils of a wide range of ability. The pupils taking part in the study are members of one such tutor group. They were in their first year, ie aged eleven when the study began, They are now in their third year. They have worked in six ability sets (or tracks in North American English) since the second term of the first year.

The adoption of a personal construct approach reflects the belief that pupils differ from each other in the ways in which they make sense of mathematics lessons, the roles which they, and others, play in those lessons, and even what it means to be 'doing mathematics'. However, a personal construct perspective also embraces the demonstrably obvious view that two or more persons frequently employ similar constructions of events and my study as a whole is concerned with both similar and different constructs to the extent that they affect mathematical performance. For this paper I shall concentrate on the similarities.

The work was in three stages. In the first, each pupil was interviewed for approximately one hour to provide a background picture of their attitudes to, and ideas about school and learning. To avoid the discussion of factors which I, rather than the pupils, might find important, personal constructs were elicited using, as elements, nine of the fourteen school subjects which the pupils were studying at the time. Each pupil selected a combination of elements which would reflect their own likes and dislikes. The constructs, as they were produced, were used as a basis for wider ranging discussion. The interviews were audio-taped, transcribed and analysed. The constructs were analysed using the Focus grid analysis computer program.

The most interesting finding from this very general enquiry was that those in the higher mathematics sets tended to generate constructs which related more to external factors (E), while those in the lower sets tended to generate constructs which related more to personal factors (P).

SETS	E > P	E = P	E < P	TOTAL
1-2	9	1	2	12
3-6	2	3	7	12

For the second stage, I dealt solely with mathematics. I chose a somewhat different approach, because a trial study with another group of pupils pointed up the difficulties of eliciting constructs when mathematical topics were the chosen elements. I, consequently, myself provided three constructs on which pupils then rated eighteen mathematical topics on a scale of 0 to 7.

The constructs were easy/difficult; like/dislike and useful or not in everyday life and work. Pupils were not expected to have objective knowledge about use. It was their ideas that were of interest because, in general, pupils so frequently, and justifiably, complain that they can see no end use in the topic being studied.

For each topic there was a card with one or more examples of the topic, including the answers, drawn or written on it and this was shown to the pupils. This method was used partly because it is well known that people are far better at recognition than at reconstruction and partly because years of experience in teaching mathematics has taught me that pupils tend not to remember the labels given to many of the topics. As a further precaution each topic was discussed before the first rating to ensure that the pupil had some idea of what was being discussed.

Once the topics had been rated the reasons for the placing of each one were discussed and this provided a means for a deeper insight into the pupil's ideas about mathematics.

The aim of the study was to discover pupil's attitudes and beliefs about learning mathematics. Since most formal mathematical learning takes place in school, it seemed expedient to use the topics taught in school as elements. However this meant that the school's or, arguably, the examination board's concept of 'doing mathematics' rather than that of the pupil's was being used as a basis for the enquiry.

Overall, the pupil's ratings correlated quite well with the setting of the pupils. The higher the set, the more likely was the pupil to find the topic easy, enjoy doing it and think it useful and vice versa. This is hardly a surprising finding given the previous argument and it does no more than give confidence in the validity of the method.

As before, the audio-tapes were transcribed and analysed. The method was the same for each set of interviews. First the tapes were transcribed verbatim. Next they were reduced to notes referring to relevant comments. At present these are being used to create a vignette of each individual pupil and they have also been used to find shared attitudes or beliefs in an effort to locate factors which may be of general rather than individual concern. Progress is being made but this in not the subject matter of this paper.

With the aim of obtaining a view of the pupils which was both deeper and more personal the third stage of the study involved the pupils in problem solving sessions in groups of three. The groups were self selected because this seemed to be the most satisfactory way of ensuring that pupils trusted and felt at ease with those with whom they were working. An unlooked for bonus was that the groups were all composed of people from different sets although the two extremes did not come together. Unfortunately, self selection meant that in only one group was there a mixture of boys and girls.

I made clear to the pupils that it was not their success that was of interest but the ways in which they tackled the problems and how they worked together. The sessions were clearly different from the classroom situation but I felt that once the pupils had relaxed, become involved and learned to ignore the camera there would be useful information to be gained from the approach. The fact that the sessions were videorecorded meant that not only could everything the pupils said be carefully analysed but also that silent signs of interest, involvement or enjoyment or alternatively boredom, frustration, disinterest or even anger would be on record.

There were nine problems in all. The pupils were asked to try to consider as many of them as possible, to reach a conclusion through discussion and to move on the the next question only when all were agreed that they either had a solution or wanted to give up trying. A notepad and pen were provided to facilitate the work but only one of each. The intention was to steer the group away from individual work. The problems covered several types, as follows:

Number of problem	Type
1,2	Fairly easy, to overcome nervousness
4,9	Likely to lead to frustration or boredom
6	Open ended, misleading without careful analysis
3	Requiring generalisation
7,8	Related to probability (not taught as yet)
5	Requiring physical manipulation of material

The problems on probability were included because it is a topic about which people learn in everyday life and that on physical manipulation because it could lead to either very close cooperation or totally individual work.

At the end of each session, before the camera was turned off, I had a discussion with the group about what they had felt about taking part in the activity and whether or not it had felt like doing mathematics. Most remarked on how quickly they had forgotten about the camera, how much they enjoyed working in groups and that they did think that the exercise was mathematical. Also, the majority asked to go through the questions to see if they had been successful or to have them explained if they had not understood. This invariably meant them giving up their free time to do so.

As mentioned earlier the aim of these sessions was to gain deeper and more personal insights into the pupils ideas and attitudes. There was no intention of measuring success at either an individual or a comparative level. However, I soon realised that although I was testing affective factors I was in fact in possession of data that tested problem solving abilities. An analysis of the videotapes indicated discrepancies between problem solving performance and setting.

To test this, marks were allotted to each problem and each group as a whole was rated according to their results. Next the protocols of each group were analysed for indications of each pupil's contribution to the success of the solution and the individual results weighted accordingly. Inevitably there is an element of subjectivity involved here but having videotapes to view makes it possible to gain a fairly clear idea of who is contributing what.

The pupils are divided between six ability sets; the members of the lowest three sets are unlikely to gain any mathematical qualifications before leaving school. And yet, in this group, several performed as well as their so-called betters.

SET	NO IN SET	HIGH	MEDIUM	LOW
1	9	6	3	0
2	5	2	1	2
3	6	3	1	2
4-6	9	2	3	4

As a result of the second and third stages of my enquiry, I now had two types of data about attitudes to mathematics, ie one explicit and one implicit. It was not, however, possible to bring them together because they were clearly based on different concepts of mathematics.

The first used an arithmetic and reproductive approach which inevitably involves a large amount of algorithmic learning. The second used the concept of mathematics as problem solving. The first, predictably, correlated quite well, at a general level, with the overall setting of the pupils. The second did not.

These results suggest that the term "Attitudes towards mathematics" is too general. It may be necessary to separate attitudes towards routine work and attitudes towards problem solving. These attitudes might affect pupils' work and there may be lessons to be learned from this. At this stage it is possible only to offer two hypotheses.

The first is that those with a favourable attitude to problem solving who are forced into doing routine work, particularly where this involves learning without understanding, become disillusioned with mathematics and give up trying. The second hypothesis is that abilities at problem solving and abilities at routine work are not highly corralated. It is quite possible that both are valid simultaneously, but for different pupils.

Both hypotheses are relevant to the ongoing debate about how pupils learn and, given either hypothesis, the present attempts in Britain at curriculum reform which are seeking to give a more important place to problem solving may lead to a different ranking of pupils.

To sum up. In the three stages of this work I have shown firstly that the children who, according to the school, are more able at mathematics tend to view school as a whole more on the basis of external than personal constructs, while the opposite is true for the less able. Secondly, the mathematically more able children also tended to find mathematics easier and more useful, and they liked it better. But it was the third stage which cast doubts on these simple interpretations, for it showed that the concept of 'doing mathematics' used in the first two stages related essentially to routine. If the concept referred to problem solving, then no simple correlation with the school's perception of the mathematical ability of the children could be established and I suggested two hypotheses to account for this fact.

One final comment. In the problem solving sessions those in the higher sets tended to show greater confidence than those in the lower sets and boys tended to show more confidence than girls. It was particularly interesting to watch how top set boys failed confidently whilst lower set girls succeeded with great diffidence.

A CONSTRUCTIVIST APPROACH TO RESEARC ON ATTITUDE TOWARD MATHEMATICS

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Abstract

Current research on attitude generally follows a behaviorist or empirical tradition. Recently, however, some psychologists have suggested new approaches that reflect a constructivist position on attitude. The purpose of this paper is to discuss how a constructivist approach could provide a stronger theoretical foundation for research on attitude toward mathematics. The theories of Mandler and Skemp form the basis of the discussion.

In a recent review Leder (in press) presents a state-of-the-art report on research related to attitude toward mathematics. Her review presents the complexities of research on attitude, including the attempts that have been made to provide an adequate theoretical base for this research. Most of the theoretical base has come from behaviorist psychology or social psychology (Ajzen & Fishbein, 1980). Very little of the research reflects the constructivist approach that has become prominent in research on mathematics learning. The purpose of this paper is to suggest how a constructivist approach to attitude could be of substantial help in analyzing how attitudes develop and in making connections between research on attitude and contemporary theories of learning.

RESEARCH ON ATTITUDE TOWARD MATHEMATICS: CURRENT APPROACHES

Research on attitude generally has a foundation in behaviorism, but it often seems to proceed in rather an atheoretical, empirical fashion. A typical approach would be to specify certain factors (e.g., liking, utility, confidence) that are hypothesized to be important in the affective domain, and then devise a questionnaire that measures those factors. The researcher then gathers some data, examines the characteristics of the instrument, and applies the appropriate statistical analysis package. The results are then interpreted and implications drawn for practice, but little thought is ever given to the development of a sound theoretical framework. The driving force in much of this research seems to be the statistical methodology rather than the theory. In following this statistical model, the researcher seems to assume that the domain of interest (attitudes, in this case) can be modeled by a vector space, and that the items on the attitude questionnaire will span the space and produce factors that describe the space adequately.

Although the theoretical foundation for research on attitude has not been strong, a great deal of useful data has been gathered using these empirical methods. Research on attitudes related to the area of gender differences has been particularly successful. For example, a substantial amount of data indicates that females tend to be less confident than males in mathematics (Reyes, 1984). Since confidence is an important predictor of continuing enrollment in secondary mathematics courses, this finding has implications for the underrepresentation of females in more advanced mathematics courses and in mathematical careers. The data on confidence and course selection is quite consistent across different countries and across different measurement techniques (Leder, 1986).

Research on attitude has made progress not only in the consistency of the results, but also in the development of more sophisticated models to guide the research. This line of research has expanded to include investigations of gender differences in attributions of success and failure in mathematics (Reyes, 1984). The connection between research on attitude and on attributions (Weiner, 1979) has been particularly useful in mathematics education, and promises to make further contributions to our understanding of the relationships among attitudes, achievement, and gender (Fennema & Peterson, 1985).

RESEARCH ON ATTITUDE: THE NEED FOR NEW APPROACHES

Although research on attitude has produced useful data in at least some situations, a new approach to the affective domain could yield substantially more progress, especially in developing better theories about attitude and in making connections between research on attitude and contemporary theories of learning. This new approach needs to take into account the view that learners are actively engaged in constructing their knowledge of mathematics, rather than just absorbing it. This new view of the learner is already having a substantial impact on paradigms for research on cognitive issues in mathematics learning and teaching (Romberg & Carpenter, 1986). Now it is time for this new view to influence how we approach research on attitude toward mathematics.

The need for a new view of research on attitude is widely acknowledged among both cognitive psychologists and researchers in mathematics education. Abelson (1976) notes that research on attitude is confusing and contradictory, and suggests that "the present state of attitude theory is frankly a mess" (p. 40). Mandler (1984) observes that research in this area is generally not cumulative, and that researchers have been preoccupied with measurement issues, and neglected the development of theory. In mathematics education, Kulm (1980) has asked for more emphasis on theory development to guide research on attitude toward mathematics, and numerous authors have noted the relatively weak relationship between attitudes and achievement in mathematics (Begle, 1979).

THE DEVELOPMENT OF ATTITUDE: A CONSTRUCTIVIST POSITION

Constructivist views of learning often pay little attention to the affective domain. Recently, however, two leading theorists (Mandler, 1984; Skemp, 1979) have made affect a major part of their constructivist positions.

Mandler (1984), in his analysis of mind and emotion, extends theory and methods of cognitive psychology to the affective domain. His view is that affective responses result mainly from interruptions of plans or planned actions. In the terminology of cognitive psychology, the plans come from the activation of schemas, and the schemas induce actions. If these actions are interrupted, the individual's autonomic nervous system responds with some sign of arousal, such as an increase in heartbeat or a tensing of the muscles. The individual then interprets this reaction of the autonomic nervous system as frustration, surprise, or some other emotion.

Mandler's emphasis on interruptions seems particularly appropriate to student performance in mathematical problem solving. When a student is working on a non-routine problem, interruptions and blockages are inevitable. The student's interpretation of that interruption will depend on the student's knowledge, beliefs, and previous experiences.

Skemp (1979), in his presentation of a theory of learning, pays special attention to the development of emotion. Skemp's framework discusses the importance of goal states (and anti-goal states), and identifies eight major categories of emotion. These include pleasure, which comes from movement toward a goal state, as well as fear (or movement toward an anti-goal state). Skemp also describes emotions that result from the ability (or inability) to control one's movement toward a goal state (or an anti-goal state). For example, he describes confidence as being able to control movement toward a goal state, and anxiety as the inability to direct movement away from an anti-goal state.

Buxton (1981) has carried out a major study that investigates the usefulness of Skemp's ideas on affect. In this study Buxton presents a careful analysis of adults' affective responses to mathematics, and uses the term panic to describe what occurs in the minds of many. This panic is manifested both in chaotic reactions to mathematical tasks, and in the tendency of some people to freeze--to be immobilized when asked to solve a mathematical problem. In Skemp's terms, the affective reaction results from the inability to move away from the anti-goal state of failure on a mathematical task (Skemp, 1979).

Both Mandler and Skemp provide useful frameworks for analyzing affective responses of mathematics learners. Researchers who conduct detailed studies of individual learners should find these frameworks useful. For example, Cobb (1985) discusses the role that affect can play in the development of early number concepts. He compares the learning of two students who differ in their level of confidence and their expressions of anxiety. Confrey (1984) comments on the confusion and frustration that is reported by young women in a special summer program on problem solving. Ginsburg and Allardice (1984) document the intense feelings of sadness and futility that low achievers express in relation to mathematics learning, and call for a renewed emphasis on affective issues in research. Wagner, Rachlin, and Jensen (1984) report how algebra students can get upset and lose control of their solution processes when they are stymied in their attempts to solve problems. Each of these studies provides useful information on how interruptions and blockages can produce negative feelings about mathematics.

It is important to remember that students have positive as well as negative experiences with mathematics, and the theories of both Mandler and Skemp account for the development of these positive feelings about mathematics. Although research has tended to concentrate more on the negative emotions (such as frustration and anxiety) rather than the positive, a number of people have noted the role of positive affective factors in learning. For example, von Glasersfeld (1987) notes the powerful sense of satisfaction that children report when they reach a satisfactory reorganization of their ideas. Lawler (1981) documents the surprise and positive emotions that accompany the moment of insight when a child sees the connection between two previously unconnected schemas. Similarly, Mason, Burton, and Stacey (1982) discuss the importance of savoring the "Aha!" experience when solving problems, and Brown and Walter (1983) discuss the joy of making conjectures.

A FRAMEWORK FOR STUDYING THE DEVELOPMENT OF ATTITUDE

The first task for researchers is to analyze the barriers that children face as they learn mathematics, for it is these barriers that prevent a schema from reaching completion (Mandler, 1984) or that keep a student from reaching a goal (Skemp, 1979). The affective component of the children's reactions to these barriers constitutes the raw material from which attitudes are formed.

The next task for researchers is to describe the affective reactions of students to these barriers. These reactions can be characterized in terms of their direction (positive or negative), intensity, duration, rise time, and consistency (Kagan, 1978; McLeod, in press). When students respond positively (or negatively) on repeated occasions to a series of mathematical tasks, their responses become more and more automatic. The role of automaticity is the same in the affective domain as in the cognitive; human information processing allows certain responses to become more and more automatic, thus freeing the individual's limited processing capacity for action on unfamiliar problems or situations (Resnick & Ford, 1981). As these responses become more automatic, the theory predicts that the affective reactions will be characterized by reduced intensity, increased duration, shorter rise time, and greater consistency from task to task. Once the reactions become consistently positive (or negative), then the student is exhibiting the stable response that is characteristic of the construction of an attitude.

If researchers are to understand the development of attitude toward mathematics, they will need to use the same kinds of methods that are now used to understand cognitive development. For example, research on affect should include the use of individual observations, clinical interviews, and teaching experiments. Since these techniques are standard for constructivist researchers, they should be willing to expand the domain of their interests to include affective as well as cognitive constructions.

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ATTITUDES OF TWELFTH GRADERS TOWARD MATHEMATICS

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Three self-report techniques, the Revised Math Attitude Scale, a researcher developed questionnaire, and an interview schedule, were utilized in gathering data about twelfth grade students' attitudes toward mathematics. In agreement with other studies, the results of this investigation include: as students ascend the academic ladder their attitudes toward mathematics deteriorates; grades 7-8-9 were identified as the period of time most influential in the development of students' attitudes toward mathematics; students perceive mathematics as useful but are hesitant in specifically describing how it is useful; a significantly positive correlation exists between grade point averages and attitudes; and when considering success as a construct of attitude, gender-related differences seem to emerge.

BACKGROUND

The amount of research conducted in the area of students' attitudes toward mathematics has increased appreciably in the last ten years. The increase in research on attitudes toward mathematics may reflect the recognition on the part of mathematics educators that poor attitudes may be behind a decreased enrollment in advanced mathematics classes in high school. Another factor contributing to the increased interest in attitudes is the recognition that certain groups of students have been identified as not achieving to their potential in mathematics. Females, minorities, and students from low-SES families have not particpated in mathematics and mathematics-related activities to the degree that their abilities predict (Reyes, 1984). Affective variables have been found to be related to the underrepresentation of these groups in mathematics classrooms and careers requiring mathematics knowledge.

PURPOSE OF THE STUDY

The primary purpose of this study was twofold: (1) to assess the attitudes of twelfth grade students toward mathematics; and, (2) to identify factors which contributed to the development of their attitudes. Other research efforts have documented that as students ascend the academic ladder their attitudes toward mathematics deteriorates (Aiken, 1970, 1976; Begle, 1979; Carpenter, et al., 1981; Neale, 1969; Reys & Delon, 1968). The intent of this investigation was to initiate work on explaining why attitudes toward mathematics seem to decline as students progress through grades 1-12. Once an explanation is found, research may begin on preventing this decline and, possibly, on reversing the trend.

DESIGN AND METHODOLOGY

Three primary data collection techniques were utilized in addressing the purposes of the study. A self-report attitude scale, the Revised Math Attitude Scale (RMAS) designed by Lewis R. Aiken, Jr., was used to measure the attitudes of 329 twelfth grade students. The RMAS score allowed the investigator to divide the population into three subgroups: Those with a dislike for mathematics (RMAS score range 0-29); those with a neutral attitude towards mathematics (RMAS score range 40-49); and those students who liked mathematics (RMAS score range 60-80). A second self-report instrument, a 4-item questionnaire designed by the researcher, was used to ascertain if the student liked mathematics in elementary school (grades 1-6), junior high (grades 7-8-9), and high school (grades 10-11-12). The fourth item asked students to check the one period of time which they felt contributed most to the development of their attitude towards mathematics: Grades 1-6, Grades 7-8-9, Grades 10-11-12. The students who checked Grades 7-8-9 became candidates for interview. One hundred twenty-six students (38%) identified grades 7-8-9 as the period of time most influential in the development of their attitude towards mathematics. Five males and five females were selected at random from each attitude group. The interview was the third self-report technique utilized to collect data for this study. The same questions were asked of each respondent, but the questionnaire also contained a set of open-ended questions that allowed for probes of the individual's responses. Other information collected on each interview subject included the number and types of mathematics classes they had taken in grades 9-12 and the grades made in these classes.

ANALYSIS OF DATA

The analysis of the data collected with the RMAS and the research questionnaire included means, standard deviations, and tests for significant differences between subgroups. A Pearson Correlation between the RMAS score and the mathematics GPA resulted in a significantly positive correlation ($r = .38$, $p < .05$). The four items from the research questionnaire and the demographic data obtained were analyzed through regression analysis. The reliability coefficient of the RMAS ($r = .97$) was estimated by the coefficient alpha method. Analysis of the interview data consisted of transcribing the interview tapes and studying the responses to search for trends.

RESULTS AND DISCUSSION

Attitudes as measured by the RMAS According to the RMAS mean score (39.6) for the original population, the overall attitude of this particular group of seniors was bordering between neutral and having a tendency to dislike mathematics. The RMAS scores of 159 students (48%) indicated either a strong dislike (n = 50), a dislike (n = 47), or a tendency to dislike (n = 62) mathematics. The RMAS scores of sixty-six students (20%) indicated that their feelings toward mathematics were neutral. One hundred four students (32%) had an RMAS score indicating either a strong liking (n = 16), a liking (n = 32), or a tendency to like (n = 56) mathematics. An item analysis of the RMAS resulted in the identification of several items in which large percentages of the population were responding negatively (Miller, 1986). Mathematics educators might consider focusing on some of these items as change agents in an attempt to alter students general attitudes toward mathematics.

The distribution of the RMAS scores by sex is interesting because almost twice as many females have a strong dislike for mathematics as males (n = 32 vs n = 17). However, three times as many females have a strong liking for mathematics as males (n = 12 vs n = 4). The numerical differences between the sexes for dislike vs like and tendency to dislike vs tendency to like are not as great. More males than females scored within the neutral range (n = 39 vs n = 24, respectively). Are these data supportive of the contingency suggesting gender-related differences in the learning of and attitudes toward mathematics? Possibly; however, one might also surmise that females are not as inhibited as males at expressing the strength of their emotions. The RMAS mean scores for males (40.7) and females (38.5) are not significantly different at the 0.0001 level; thus, refuting any gender-related differences in attitudes for this particular population.

Attitudes as measured by interview Various attributes of attitude were discussed during the interview (Miller, 1986). Of particular interest were the responses when students were asked, "In general, do you think mathematics is useful?" Twenty-nine of them answered with a strong "yes." However, when asked why they said mathematics was useful, their answers were not as immediately forthcoming. The most popular first response to the "why" probe was "Oh well, you know, everybody uses math." Their hesitation in naming a specific reason why mathematics is useful did not coincide with the strength of their initial response. With continued encouragement to specifically explain why they said mathematics was useful, seventeen students expressed some type of involvement with money; balancing a checkbook, making change, and comparative shopping were three specific examples named. Other reasons named included aviation (one student was learning how to fly), construction (three students worked part-time on construction crews), keeping statistics on athletes, using mathematics in other classes like chemistry and computer science, and a few students said that mathematics is useful in some jobs but were not specific. The nature of the students' responses to this question and the follow-up probe suggest that students sense from society, in general, and parents and teachers, more specifically, that mathematics is useful, but they are not exactly sure why or how it is useful.

The interviewer also asked students to describe themselves as being successful or unsuccessful in the mathematics courses they had taken. Ten of the fifteen males interviewed (67%) felt they had been successful in mathematics. Of the fifteen females interviewed, six (40%) described themselves as being successful in mathematics. The responses to this inquiry support the arguments of researchers who contend that sex-related differences in mathematics do exist. Another factor supporting the existence of gender-related differences in mathematics was the comparison of the mean grade point averages (GPA) between the two sexes. The females interviewed seemed to under estimate their success in matnematics when compared to their male counterparts. The mean GPA (based on a 4 point scale; with F = 0 and A = 4) for the 15 female subjects was 2.47; the mean GPA for the 15 males subjects was 2.03. The ten males who considered themselves successful in mathematics had a mean GPA of 2.42. The six females describing themselves as successful in mathematics had a GPA of 2.68. These data lend support to the "fear of success" construct discussed by Leder (1985) and others.

Factors contributing to the development of attitudes Grades 7-8-9 were identified by one hundred twenty-six (38%) of these seniors as the period of time most influential in the development of their attitudes toward mathematics. The data collected in this study, consistent with the results reported in other studies, indicate that as students ascend the academic ladder, their attitudes toward mathematics deteriorate (Miller, 1986). The seventh grade was singled out by the majority of the 30 interview subjects as the one year in which their attitude towards mathematics changed the most. Some students said they started liking mathematics in the seventh grade and others said they started disliking mathematics in the seventh grade. Reasons given for naming the seventh grade as a critical year focused on the changes between the elementary curriculum and junior high school. For example, some students were bored by reviewing in the seventh grade what had been taught in grades 4-5-6. Some students who started algebra in the seventh grade were excited by the challenge of new content. Other students were discouraged by the amount of work required in the seventh grade, unlike grades 4-5-6 when "math was a breeze." Students claiming to have mathematics anxiety, indicated that the seventh grade teachers were not smpathetic. They only spent an hour a day with their seventh grade teacher and that was not enough time for the teacher to get to know them. Other comments included the lack of practical use for the mathematics studied in grade seven and beyond.

Attitude vs GPA Permission was secured to obtain the mathematics grades of the thirty interview subjects. The ten students in the "dislike mathematics" category (RMAS score range 0-29) had a mean grade point average (GPA) of 1.92 (on a 4-point scale; F = 0 and

A = 4). The ten students in the "neutral attitude" category (RMAS score range 40-49) had a mean GPA of 2.18. The subgroup in the "like mathematics" category (RMAS score range 60-80) had a mean GPA of 2.67. The Pearson Correlation coefficient indicated a significantly positive correlation for these two variables ($r = .38$, $p < .05$). The number of students in this sample prohibits the formulation of any strong conclusions from the differences between the means or the correlation coefficient. However, the results clearly indicate that students with a more positive attitude towards mathematics have a higher grade point average.

Predicting attitudes Can a twelfth grade student's attitude towards mathematics be predicted? Using the student's sex and the responses to the four items on the research questionnaire as independent variables, a regression analysis was run with attitude as the dependent variable. The best one variable model found that a student's response to "Have you liked math in high school?" was the best predictor of the student's attitude towards mathematics as measured by the RMAS. This result is not surprising because a person's most recent experiences with an object would probably greatly influence the attitude held towards that object.

Consistency of responses One aspect of the study that was particularly interesting to the investigator was the consistency of the data collected through two different self-report techniques: written questionnaire vs oral report. The research questionnaire asked students to answer the following question: "When would you say that you developed your present attitude towards mathematics? Grades 1-6, Grades 7-8-9, or Grades 10-11-12." Respondents were instructed to check one. All thirty students selected for interview had checked Grades 7-8-9. During the interview, 47% of the students gave a different answer to that question. This result is somewhat disappointing since it questions the validity of the students' responses not only on the research questionnaire but during the interview, too.

SUMMARY

This research has documented that students' attitudes toward mathematics can be measured and analyzed through a variety of data collection techniques. Attitude is a multifaceted construct which develops over a period of time and under the influence of many variables. Conversations with students in this study as well as teachers and other colleagues make this researcher believe that there is much room for improvement in the attitudes people have toward mathematics. Before progress can be made toward reversing the trend of development as students ascend the academic ladder, many investigations must be conducted to ascertain what strategies would be most successful in improving students' attitudes toward mathematics.

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A CRITICAL SURVEY OF STUDIES, DONE IN KENYA, ON THE DEPENDENCE OF ATTITUDES TOWARD MATHEMATICS AND PERFORMANCE IN MATHEMATICS ON SEX DIFFERENCES OF THE SCHOOL PUPILS

By

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ABSTRACT

Investigations that have been carried out in Kenya since 1970's upto the present on the dependence of attitudes toward mathematics and performance in mathematics on sex differences have sought to find out whether there are statistically significant differences between school boys and girls in their attitudes toward mathematics and in their performance in mathematics. Investigators have, as well, tried to find out whether positive attitudes toward mathematics are significantly correlated to better performance in mathematics.

Results indicate that, during secondary school years, boys have more favourable attitudes towards mathematics than girls. Performance in mathematics during primary school years does not depend on the sex of the pupil. Performance, however, depend on the sex of the pupil during middle secondary school years i.e. at 'O' level stage. At the later stage in secondary school, i.e. 'A' level stage, girls perform better in mathematics than boys.

In Kenya positive attitudes toward mathematics are significantly correlated with better performance in mathematics.

A proposal to deal with this situation is suggested.

INTRODUCTION

Studies on the dependence of attitudes toward mathematics and achievement in mathematics on sex differences are numerous. In some of the studies significant correlations between attributes have been found. Explanations as to the causes of the differences have varied from socio-cultural theories to theories that attribute them to biological differences between the sexes.

The view taken by this author is that differences in attitudes toward mathematics and achievement in mathematics, where attributable to sex differences, cannot be explained purely in either environmental or biolcgical terms. They spring from a very complex interaction of these variables and this causes extremely varied behaviour patterns in school pupils. Mathematics educators are thus called upon to search for such instructional strategies and practices for teaching mathematics as are likely to enhance the creation of positive attitudes toward mathematics and raise the level of performance in mathematics.

In thie paper research, done in Kenya during the decades of the '70s and and '80s*, on the dependence of mathematice performance and attitudes toward mathematics on sex differences is reviewed and critiqued. Research questions, methodologies, findings, and further recommendations are examined. The paper concludes with a proposal for the kind of research that could be fruitfully carried out in Kenya, and perhaps elsewhere, in light of the Kenyan experience so far. In 1987, through this paper, Kenyans have had to pause and take stock of investigations on this issue and then map out areas of further research where work could be rewarding if, not more fruitful.

THE QUESTIONS

Investigations - all done in Kenya by Kenyan doctoral and masters' candidates - sought answers to three basic questions.

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*Empirical investigations on the dependence of certain educational attributes on sex differences in school learning situations started in 1974 with Eshiwani(1974) study.

(a) Who performed better in mathematics, if that was the case hool boys or girls ?

(b) Did all school pupils have positive or negative attitudes toward mathematics ? If that was not the case, did the sex of the pupil influence their attitudes toward mathematics ?

(c) Were attitudes toward mathematics and performance in mathematics significantly correlated ?

ON SEX DIFFERENCES AND PERFORMANCE IN MATHEMATICS

Kenyan investigators used some measure of performance in mathematics (scores from self-made achievement test, results from school-constructed examinations,results from Kenya National Examinations Council (KNEC) records and tested statistically if there were significant differences between school boys and girls in their performance in mathematics.

Wamani (1980) tested mathematical abilities of 8 to 12 year olds using self-constructed mathematical ability tests. He found no significant differences in the performance of the pupils of both sexes.

Samumkut (1986) - used scores from school-constructed examinations, Maritim (1985) - used KNEC records, and Kapiyo (1982) - used self-constructed achievement and ability tests. The three investigators used the t-test to seek for significant differences in performance between school boys and girls in the age group 13 to 17 year olds. They found out that in this age group significant difference in favour of boys existed. School boys performed better than school girls.

Maritim (1985) using KNEC results found that for the age group 18 to 19 year olds, i.e. at 'A' level stage, girls performed significantly better than boys in mathematics.

In Kenya there, then, exists a situation where young pupils (8 to 12 years old) of all sexes perform equally well in mathematics. Pupils in the early adolescence (13 to 17 year olds) perform differently in mathematics with the boys performing better than the girls. Pupils in their late addescence (8 to 19 year olds) reverse the earlier situation and the girls perform better in mathematics than the boys in the 'A' level examinations.

These findings are explained in several ways by the investigators. The most persuasive would seem to be the role played by the teacher of mathematics and the availability of adequate resources in the schools. In elementary grades the teacher is seen as friendly to all the children and thus treats them equally irrespective of the sex of the child. In the secondary school grades teachers are seen likely to be more role preseriptive in pupils' task assignment. This could bring out readily different sex role perceptions. These sex role perceptions would include such precepts as "nice girls don't do math". At 'A' level stage learning facilities are uniformly distributed. Girls at this level are motivated to achieve since they have opted to undertake further studies in mathematics. The role of the teacher ceases to be all that predominant.

ON SEX DIFFERENCES AND ATTITUDES TOWARD MATHEMATICS

Investigations centred on the question of whether boys and girls in secondary schools i.e. 13 to 17 year olds differed significantly in their attitudes toward mathematics. Investigators used pupils and teachers quetionnaires and the Likert scale to assess attitudes toward mathematics. The chi-square, percentages and the t-test were used.

Mbuthia (1986), in a well documented study, found that overall boys (97% of the sample) have positive attitudes toward mathematics as opposed to girls (67% of the sample). In further tests he found the difference to be statistically significant at the 0.05 level of significance. The same findings occured in investigations carried out by Opondo(1984), Otieno (1985), Samumkut(1986), and Mbugua (1986).

Mbuthia (1986) as well as Mbugua (1986) found out that all pupils under investigation perceived mathematics as being of value to society. The boys, however, declared that they enjoyed mathematics more (94% of the boys) than the girls (52% of the girls).

Samumkut (1986) found out that girls tended to have positive attitudes toward mathematics and at the same time they performed poorly in mathematics.

Mbugua (1986) found that 57% of the girls surveyed attributed part of the reason for their liking maths to parental encouragement. Otieno (1985) found that 54% of the girls surveyed blamed their teachers for their hatred of mathematics.

It has been suggested in these studies that further studies, on how teachers and parents go about encouraging girls to learn and study mathematics so that positive attitudes toward mathematics are created, be done. The practices identified to be conducive to this enterprise be further reinforced.

ON CORRELATION BETWEEN ATTITUDES TOWARD MATHEMATICS AND ACHIEVEMENT IN MATHEMATICS

Kenyan investigators have sought to find out if positive attitudes toward mathematics by pupils correlates with better performance in mathematics.

Parkar (1974) using 13 - 15 year olds assigned them to control and experimental groups and these were taught using different instructional strategies (programmed work cards versus traditional methods). He found that there was no significant difference in attitudes towards mathematics between treatment groups but that there was positive correlation between achievement in mathematics and attitudes toward mathematics.

Studies by Kibanza (1980), Samumkut (1986), and Patel (1985) use the same age-group and arrived at the same findings. Patel (1985) established that for girls poor attitudes toward mathematics resulted in poor achievement in mathematics.

Discussions under this question seem to arrive at a conceneus that there seems a need to develop instructional strategies and practices in the teaching of mathematics in Kenya that will encourage the creation of positive attitudes toward mathematics for all pupils and especially for girls in secondary schools.

DISCUSSION

Results of investigations done in Kenya in the '70s' and '80s' on performance in mathematics attitudes toward mathematics show that these two important espects of mathematics education partly depend on sex differences.

Findings indicate that for the very young in primary grades performance does not depend on the sex of the pupil. In early adolescence i.e. during secondary school years performance seem to depend on the sex of the pupil such that boys perform better than girls in mathematics. At 'A' level stage, however, girls have been found to perform better in mathematics than boys.

Except in certain specific areas, investigations indicate that the majority of boys, during secondary school years, have favourable attitudes toward mathematics and that it is not so in the case of girls in the same age group.

In Kenya investigations further show that positive attitudes toward mathematics significantly correlate with better performance in mathematics.

The question facing mathematics educators is what can be done about a situation such as this so that half of the school population is not forever disadvantaged. That is assuming that knowledge of mathematics is a worthwhile acquistion for all citizens.

PROPOSAL

It is proposed that Universities in Kenya undertake research, along the following lines, to identify and encourage the use, by teachers of mathematics, of those instructional strategies and practices that have been found successful in the teaching of mathematics for the majority of children in Kenya.

Step One: Mathematics educators, in groups or individually, to identify instructional strategies that have proved, through usage, to be successful in motivating both girls and boys to like mathematics as well as pass mathematics examinations at national level. Make a list of these strategies.

Step Two: Researchers will use the list prepared above to observe actual strategies and practices of successful teachers of mathematics teaching the subject in their classrooms. Researchers will validate their list and improve on it.

Step Three: Researchers will proceed to use Semi-Delphi Technique with a group of teachers in a well selected sample until there is an acceptable degree of agreement about the strategies. It is likely that at the conclusion of this stage some improvement in awareness and use of strategies and practices used by the better teachers of mathematics will be acquired by most of the other teachers of mathematics in the sample.

Step Four: The government or its agencies could then take over and continue the work initiated by the Universities.

The important thing is to take the first step. This author plus two of his graduate students, has done just that. It is hoped to report some results of this experiment in 1988.

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MATHEMATICS TEACHERS' BELIEF SYSTEMS AND TEACHING STYLES: INFLUENCES ON CURRICULUM REFORM

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The Allen Paragraph Completion Instrument and the Schoenfeld Belief Survey were administered to thirteen secondary and middle school teachers to assess their cognitive development level (based on the Perry Scheme) and mathematical belief systems. Results and implications of this study will be discussed.

Research has recently begun to emerge indicating that mathematics teachers' views about the subject matter, teaching, and learning influence their classroom behavior (e.g. Madison-Nason and Lanier, 1986; Carpenter, Fennema, and Peterson, 1986; Thompson, 1984). However, as Thompson noted, the relationship between teachers' belief systems and their instructional practices is far from simple. Thompson (1984), for instance, observed inconsistencies between some teachers' expressed beliefs and their actions in the classroom. A model that has the potential of clarifying these incongruities and providing a theoretical framework for the study of mathematics teachers' belief systems is Perry's (1970) cognitive development scheme.

The Perry scheme, which is an outgrowth of Piaget's theory of cognitive development, is a hierarchial classification of "how people understand or make meaning of their world" (Stonewater, Stonewater, and Perry, 1987). The nine stages of the Perry Scheme are qualitatively different from one another with each representing a more complex order of thinking than the stage previous to it (For further details, see Conceptual Framework). From a theoretical point of view, it is also possible that one's beliefs about mathematics might also vary as a function of cognitive developmental level. Thompson's (1984) study, for instance, appears to support this hypothesis. Although her study did not focus on cognitive development, Thompson's reported data supplied us with sufficient information to make a "best guess" interpretation of her subjects' Perry levels. From this, we found indications of a positive relationship between teachers' cognitive development level and the degree of consistency between their beliefs and instructional behavior. With this in mind, this present study was designed to explore the relationship between secondary mathematics teachers' cognitive development and their belief systems as a first step to discover the extent to which cognitive development level is an intervening variable between beliefs and behavior.

The first phase of this study was the identification and assessment of appropriate paper and pencil instruments to measure mathematics teachers' cognitive development and their beliefs about mathematics and mathematics teaching. These instruments were piloted with thirteen secondary and middle school mathematics teachers. The results and implications of this first phase are the focus of this paper.

The second phase will begin in May 1987 with the administration of these instruments to a control group and participants of the Discrete Mathematics Program (DMP), a project designed to prepare high school mathematics teachers to incorporate discrete mathematics and applications into their existing curriculum (see Note 1). DMP will thus serve as a vehicle to address the following research questions:

1. What is the relationship between secondary mathematics teachers' attitudes, beliefs, cognitive development, and their classroom behavior?
2. What characteristics of the participating teachers are associated with their adoption of the proposed curricular and instructional reform?

In addition to the administration of paper and pencil tests, 6 teachers (3 DMP participants and 3 control teachers) with different Perry pretest ratings, will be interviewed and observed during the 1987-1988 school year. Pretest results of phase two will be presented at the PME-XI conference.

CONCEPTUAL FRAMEWORK

Perry (1970; 1981) describes a sequence of stages that one moves through when seeking to "make meaning" out of experiences. In general, movement is from a "right vs. wrong" dualistic conceptualization of reality to an understanding that all knowledge is embedded in a contextual and relativistic framework. The three major positions of the theory that are relevant here are dualism, multiplicity, and relativism. For a more thorough and complete description of the theory as it relates to mathematics see Copes (1982), Stonewater, Stonewater, and Perry (1987), or Buerk (1982).

Dualism

In this first stage of development, students see the world in right-wrong, black-white dichotomies. Further, the assumption of the dualist is that all knowledge is known, that an authority (usually the teacher) knows it all, and that it is up to this authority to give the student the right answer.

Multiplicity

Movement from dualism into multiplicity represents a significant broadening of the student's understanding. The student begins to realize that there might be more than one "right" answer, procedure, or perspective, but tends to get lost in the muddle of multiple rights since there is no understanding yet of the contextual nature of deciding which right is best. In late multiplicity, this "multiple rights" perspective is seen as license for an anything goes approach. Often multiplistic students will be heard to say, "Everyone is entitled to his or her own opinion on that problem. I don't know why the teacher thinks her answer is right, mine is just as good."

Relativism

As students move into relativism, another major shift in thinking takes place. They finally realize that right answers depend upon context and are now capable of thinking in relativistic or contextual terms. They understand not only that there are multiple perspectives on a given problem or topic in mathematics but that they can reason relativistically about those perspectives. "Truth," as it were, depends upon the mathematical system in which one is working, the assumptions one makes, or the axioms one accepts as true.

METHODS

Subjects

Thirteen teachers from rural and suburban midwestern school districts who are currently involved in two other research projects participated in the pilot study (see Note 1). The 11 females and 2 males had an average of 10.85 years of teaching experience (s.d. = 8.96). Twelve subjects had completed at least some graduate work.

Instruments

Each teacher was asked to complete the Schoenfeld Belief Survey (1985) and the Allen Paragraph Completion Instrument (1983). The

Allen Paragraph Completion Instrument is an assessment of a person's cognitive development level as measured by the Perry scheme. It consists of two essay questions in which subject are asked to evaluate their educational experience and to respond to a situation in which a classmate is disagreeing with a professor about a point in a biology book. For this study, the second item was modified by replacing the work "professor" with "teacher" and the word "biology" with "mathematics." Responses were scored by trained raters who categorized each response as consistent with a position between 2 (dualistic) and 5 (relativistic) on the Perry scheme.

The Schoenfeld Belief Survey consists of 70 closed Likert-type and 10 open questions which assess a person's beliefs about mathematics as well as about teaching and learning mathematics. For this study, the instrument was modified to include only 57 of the closed items and none of the open ones. Furthermore, four items, based on Thompson's (1984) finding, were added.

Many of the Schoenfeld items were designed to distinquish between two possible categories of mathematical belief systems. The first, Mathematics is Closed, asserts that "mathematics is a rigid and closed discipline, inaccessible to discovery by students and best learned by memorizing" while the second, Mathematics is Useful, asserts that "mathematics is useful, enjoyable, and helps me to understand things" (Schoenfeld, 1985, p. 16). It was hypothesized that there would be a positive correlation between the teachers' Perry position and the Mathematics is Useful subtest. Furthermore, there would be a negative correlation between the teachers' Perry position and the Mathematics is Closed subtest.

RESULTS

According to the analysis of the Allen instrument data, 5 teachers were rated as relativistic (5), 5 as late multiplistic (4), and 3 as early multiplistic (3). Table 1 is the distribution of Perry position scores by gender and teaching experience.

Table 1
Distribution of Perry Position Scores
By Gender and Teaching Experience

	Teaching Experience						
	0-10		11-20		> 20		
Perry level	M	F	M	F	M	F	
3	0	3	0	0	0	0	3
4	0	1	0	4	0	0	5
5	0	3	0	0	2	0	3

Means and standard deviations of each item on the Schoenfeld Belief Survey was found for the three subgroups (i.e. relativistic is Group R, late multiplistic is Group LM, and early multiplistic is Group EM). Small sample sizes prevent a discussion of significant differences between the subgroups; however, several significant anomolies were observed. Contrary to the hypotheses, subgroup R did not necessarily rate the Mathematics is Closed items lower (or Mathematics is Useful items higher) than the two multiplistic subgroups. In fact, several items were rated exactly opposite from what had been hypothesized. For example, for the Mathematics is Closed item "math problems can be done correctly in only one way", the means and standard deviations for this item were as follows: Group R, mean = 2.8, s.d. = .75; Group LM, mean = 3.4, s.d. = .45; and Group EM, mean = 3.7, s.d. .47.

Furthermore, the relativistic group tended to have greater variance on items than either multiplistic subgroup. For instance, while the mean was 2 for each group for the item "The validity of mathematical propositions and conclusions is established by the axiomatic methods," the standard deviations were as follows: Group R, s.d. = 1; Group LM, s.d. = 0; and Group EM, s.d. = 0.

DISCUSSION

As the first step in the exploration of the relationship between secondary mathematics teachers' cognitive development and their belief systems, paper and pencil instruments measuring these two variables were piloted and analyzed. We found inconsistencies between our data and the way we expected the two constructs to be related. Closer examination of the Allen instrument leads us to hypothesize that Perry levels might be different with regard to how teachers think about <u>teaching</u> mathematics and how they think about the <u>content</u> of mathematics. Specifically, the second item asked the teachers to react to the following statement: "A classmate disagrees with your mathematics teacher about a point in the text concerning an answer to a problem. They debated off and on for parts of several class periods. Each side of the argument has its supporters in the class." Several teachers responded to this statement from a pedagogical perspective by discussing such issues as power struggle between teacher and students, efficient use of class time, and the need to bring closure to class discussions. However, other teachers responded from a mathematical perspective by discussing the importance of seeing different approaches to a problem. A few teachers addressed the question from both perspectives and the two parts of their answer indicated different Perry levels. Through interviews, Buerk (1982) found similar discrepencies with mathematically anxious women. These observations suggest that the Perry assessment instrument needs to be revised to separate out the potentially confounding mix of pedagogy and content.

The fact that the relativists' means on most of the Schoenfeld Belief Survey items had a higher standard deviation than did the means of the two multiplistic groups was also unexpected. For example, in responding to items about "right answers" in mathematics, we expected the relativists to converge on an understanding that such answers are contextual, not necessarily right and wrong. Yet some relativists agreed with this view while other strongly disagreed. One explanation for this variance is that cognitive development and beliefs (as measured by the Schoenfeld instrument) are in fact not related, resulting in a potentially high variance in responses by Perry level. On the other hand, if the two constructs are related, then this high variance for the relativists can be explained from a cognitive development perspective. Theoretically, a person can use reasoning patterns consistent with his or her current level of thinking or below, but one cannot use reasoning patterns above his or her level. Based on this argument, it is thus entirely consistent with cognitive development theory that the relativists had more levels at which to think (5) than did the pre-relstivist teachers (3 and 4) and could fluctuate between these levels. This could account for the higher degree of variance for the relativistic group.

The second phase of this study will be to investigate the relationship between secondary mathematics teacahers' cognitive development, beliefs, attitudes, and clasroom behavior. This investigation could have profound implications in the area of curricular development and teacher education if they support Carpenter (1986) assertion that "teachers' beliefs ... affect how [they] perceive ... [the inservice] training and new curricula that they receive and the extent to which they implement the training and curricula as intended by the developers" (p. 226).

NOTES

1. The two projects, The Discrete Mathematics Program and the Miami University Teletraining Institute, are supported by a grant from Title II of the Education for Economic Security Act and administered by the Ohio Board of Regents.

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PERSONAL CONSTRUCTS OF MATHEMATICS AND MATHEMATICS TEACHING

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Repertory grid technique and extensive interviews were used to investigate four preservice secondary mathematics teachers' personal constructs of mathematics and mathematics teaching. Kelly's Personal Construct Theory and Perry's developmental scheme provided a framework for the analysis of the experiential, mathematical, and pedagogical perspectives through which the preservice teachers interpreted their undergraduate teacher preparation programs and anticipated their roles as teachers. Constructs related to teaching roles tended to focus on personal, non-intellectual qualities. Constructs relating to mathematics were affected by prior success with pre-college mathematics and anticipated uses of mathematics in teaching roles and were often discordant with the participants' perception of subject-matter preparation at the college level.

Kelly's (1955) Personal Construct Theory and Perry's (1970) developmental scheme provided a framework for investigating the sources and nature of the construction systems employed by four preservice secondary mathematics teachers as they interpreted their prior school experiences and anticipated their future teaching roles.

Kelly's theory represents a constructivist viewpoint, recognizing the learner as an active processor of knowledge--assimilating and organizing experience through an evolving system of bi-polar images, termed constructs, that control the way in which events are perceived. This constantly evolving system is both modified by experience and determines how experiences are perceived by the individual.

An individuals' actions represent choices from alternatives along a flexible and frequently modified network of pathways as the individual seeks to predict, and thus anticipate, future events. However, it is not the pathways themselves, but the constructs that facilitate, or restrict, the choices of paths that constitute the individual's construct system. Kelly developed "repertory grid technique," used in this study, as a method of eliciting and investigating the nature of, and relationships between, various constructs comprising the individuals' conceptual system.

Perry's scheme is used as a complement to Personal Construct Theory, providing a more global framework for describing the participants' developing "worldview" as it relates to teaching and to mathematics. The scheme was designed to describe the intellectual and ethical development of undergraduate college students and is primarily concerned with the relationship of the individual with perceived authority.

Four major stages of growth are posited: Dualism (a dichotomus good/bad, right/wrong, we/others structuring), Multiplism (a plurality of answers is perceived but without internal structure), Relativism (multiple perspectives emerge, allowing for contextual analysis of events), and Commitment (acceptance of personal responsibility for choices in Relativism). Alternatives to growth (Escape, Temporization, and Retreat) are available to the individual at various stages.

DESIGN OF THE STUDY

The study was conducted over a nine-week period during the spring of 1986. Data were collected from each of the seven secondary mathematics education majors enrolled in a post-student-teaching seminar at the University of Georgia. Each completed a series of seven one-hour interviews and a written task in addition to elicitation and ranking instruments characteristic of repertory grid technique. From the six students who had jointly progressed through the mathematics education curriculum four students, representing a range of achievement on college coursework, were chosen for case studies.

Interviews were of three types: open-ended discussions aimed at developing an understanding of the participants' conceptions of mathematics and its teaching, focussed interviews for eliciting participants' reaction to scenarios of hypothetical secondary mathematics classroom situations dealing with student misconceptions, and problem-solving sessions designed to investigate the participants' understanding of major ideas in the secondary mathematics curriculum and the "socially effective symbols" (Kelly, 1955) with which they communicate these understandings. Interview data formed the primary basis for ascertaining the participants' development relative to Perry's scheme and served as a medium for exploration of meanings ascribed to grid items by the participants.

Repertory grids were administered in two stages utilizing construct elicitation and final grid instruments. Two sets of initial

TABLE S-1
SUSAN'S RANK ORDER GRID - TOPICS

	DOING PROOF	GRAPH EQN.	WORD PROB	SOLVE EQN.	WORK FRAC	PROB STAT	
1 invigorating	5	1	3	2	4	6	boring
2 easy	3	4	5	1	2	6	difficult
3 abstract	3	1	2	5	6	4	concrete
4 easiest to learn	6	5	3	2	1	4	hardest to learn
5 essential	6	2	1	4	5	3	unnecessary
6 varied	3	2	1	5	6	4	routine
7 advanced	4	2	1	5	6	3	basic
8 most useful	3	4	1	5	6	2	least useful
9 best at	4	1	5	2	3	6	worst at
10 organized	1	5	2	6	3	4	disorganized
11 conclusive	1	6	2	3	4	5	inconclusive
12 exact	1	5	4	3	2	6	arbitrary
13 creative	2	3	1	4	6	5	standard
14 most liked	6	4	5	1	2	3	least liked
15 easiest to teach	6	2	5	3	1	4	hardest to teach

TABLE S-3
SUSAN'S RANK ORDER GRID - TOPICS
CORRELATION MATRIX - SIGNIFICANT @ P<.05

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
***								+++							1
	***						—								2
		***			+++	+++									3
			***												4
				***		+++									5
					***	+++						+++			6
						***	+++								7
							***								8
								***							9
									***						10
										***					11
											***				12
												***			13
													***		14
														***	15

TABLE S-5
SUSAN'S RANK ORDER GRID - ROLES

	MATH ED. MAJOR	MATH PROF	BEST MATH TEACH	ME	H.S. MATH STUD.	WORST MATH TEACH	
1 intelligent	4	2	1	3	6	5	dumb
2 interesting	5	4	1	2	3	6	dull
3 organized	4	2	1	3	5	6	sloppy
4 serious	1	4	3	5	6	2	fun-loving
5 personable	4	5	1	2	3	6	cold
6 encouraging	4	3	1	2	5	6	intimidating
7 inquisitive	4	3	1	2	5	6	unconcerned
8 aggressive	5	4	1	2	3	6	passive
9 conscientious	2	3	1	4	6	5	lackadaisical
10 flexible	5	4	3	1	2	6	rigid
11 abstract	4	5	3	1	2	6	concrete
12 complex	4	5	3	1	2	6	one-dimensional
13 respected	3	4	1	2	5	6	discreditable
14 authoritative	4	2	1	3	5	6	unreliable
15 motivating	4	3	1	2	5	6	deadening
16 reliable	3	4	1	2	5	6	unreliable
17 people-oriented	3	5	4	1	2	6	content-oriented

TABLE S-7
SUSAN'S RANK ORDER GRID - ROLES
CORRELATION MATRIX - SIGNIFICANT @ p<.05

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	
	+++			+++	+++		+++					+++	+++			1
***			+++	+++	+++	+++		+++					+++			2
	***			+++	+++						+++	+++	+++	+++		3
		***						—								4
			***			+++			+++	+++	+++			+++		5
				***	+++	+++					+++	+++	+++	+++		6
					***	+++					+++	+++	+++	+++		7
						***		+++					+++			8
							***									9
								***	+++	+++					+++	10
									***	+++					+++	11
										***					+++	12
											***	+++	+++	+++		13
												***	+++	+++		14
													***	+++		15
														***		16
															***	17

TABLE 8-13
SUMMARY RANK ORDER GRID - TOPICS
RELATIONSHIP SCORES / RANK

	SUSAN	LAURA	TIM	[illegible]	
1 invigorating	203/15	100/15	549/5	277/12	boring
2 easy	413/8	700/1	539/8	532/3	difficult
3 abstract	475/5	450/13	390/11	294/9	concrete
4 easiest to learn	292/11	678/3	507/9	469/6	hardest to learn
5 essential	327/9	580/6	549/5	178/14	unnecessary
6 varied	550/1	470/11	155/15	533/1	routine
7 advanced	543/2	678/3	392/12	412/7	basic
8 most useful	536/3	580/6	549/5	294/9	least useful
9 best at	264/13	666/5	603/1	373/8	worst at
10 organized	295/10	422/14	603/1	112/15	disorganized
11 conclusive	241/14	552/9	261/13	255/13	inconclusive
12 exact	273/12	552/9	104/14	504/5	arbitrary
13 creative	440/6	678/2	493/10	533/1	standard
14 most liked	478/4	473/12	601/3	292/11	least liked
15 easiest to teach	416/7	571/8	601/3	532/3	hardest to teach

TABLE 8-14
SUMMARY RANK ORDER GRID - ROLES
RELATIONSHIP SCORES / RANK

	SUSAN	LAURA	TIM	[illegible]	
1 intelligent	694/11	438/16	699/5	417/8	dumb
2 interesting	902/6	699/2	324/12	456/4	dull
3 organized	819/9	583/8	542/10	410/7	sloppy
4 serious	296/17	547/11	565/9	507/2	fun-loving
5 personable	852/8	573/9	190/17	507/2	cold
6 encouraging	970/1	699/2	530/11	391/8	intimidating
7 inquisitive	970/1	430/16	692/3	513/1	unconcerned
8 aggressive	902/6	515/13	722/2	194/15	passive
9 conscientious	528/15	543/12	608/4	276/13	lackadaisical
10 flexible	679/12	679/5	253/16	456/4	rigid
11 abstract	652/13	564/10	306/13	142/16	concrete
12 complex	652/13	467/14	293/14	343/9	one-dimensional
13 respected	912/4	701/1	668/5	330/11	discreditable
14 authoritative	819/5	467/14	723/1	323/12	unreliable
15 motivating	970/1	694/4	667/7	223/14	deadening
16 reliable	912/4	615/6	608/8	115/17	unreliable
17 people-oriented	450/16	504/7	291/15	374/10	content-oriented

Tables S-1 and S-5 illustrate one participant's topic and role rankings on the final grids and Tables S-3 and S-7 the corresponding correlation matricies. Tables 8-13 and 8-14 relate the summary relationship scores, by participant, for role and topic constructs. The complete descriptions of the elements, abbreviated above in Table S-1 and S-5, are: for the topic grid - Constructing a proof, Graphing an equation, Solving a word problem, Solving an equation, Working with fractions, and Probability and statistics; for the role grid - A typical secondary mathematics education major, A typical mathematics professor, Your best mathematics teacher, Yourself, A typical high school mathematics student, and Your worst mathematics teacher.

elicitation instruments, one eliciting role constructs and the other topic constructs, were completed by the participants prior to the first interview. Each set involved the presentation of triads of teaching roles or mathematical topics which the participants were instructed to group in the following manner: "Consider the three topics (roles) presented. Describe some way in which you view two of the topics (roles) as similar, yet different from the third." For example, asked to group a favorite high school mathematics teacher, a favorite college mathematics instructor, and a disliked high school mathematics teacher the participant might group the two favorites by describing them as "encouraging" in contrast to the disliked instructor who was perceived as "intimidating".

Descriptors used by the participants to characterize the similarities and differences supplied a range of bi-polar constructs for the resulting final grids. Participants were asked to use these constructs to rank, along a Likert-type scale, a selection of roles (topics) representing teaching (mathematical) elements (TABLE S-1 and S-5). Role and topic elements were chosen from common themes voiced by the participants during the interviews.

Grids were analyzed using procedures suggested by Fransella and Bannister (1977). Correlation matricies (Tables S-1 and S-5), relationship (variance) scores, and cluster graphs (Figures S-1 and S-2) were constructed for each participant's role and topic grid. Summary charts of relationship scores (Tables 8-13 and 8-14) and cluster graphs (Figure 8-1) comparing participants across elements and topics were constructed for cross-case comparison.

Relationship (variance) scores play a pivotal role in personal construct theory. These represent the explained variance from each of the constructs on the final grids and reflect the relative "intensity" with which constructs impact on the participants' interpretation of experience (Fransella and Bannister, 1977). A construct with a higher relationship score is thus posited to represent a more global influence, or control, on how the individual interprets events. Table 8-13 and 8-14 provide the relationship scores and ranking by participant for the role and topic grids.

RESULTS

While constructs in Personal Construct Theory are bi-polar (e.g., "encouraging/intimidating"), only the "likeness" pole (e.g., "encouraging") of the construct is given here for brevity. Comments relating to a participants' positive or negative connotation of a construct refer to the stated pole. For example, if Laura is described as viewing "easy" in a positive sense, this refers to the connotation she attributes to the likeness end ("easy") of the construct "Easy/difficult." Judgments of positive or negative views were based on interview data and correlations with other constructs.

The findings reported here focus on similarities and differences in constructs and worldviews held by two of the participants: Susan and Laura. Each is a twenty-two year old white female with a high (3.37 and 3.80, respectively, on a 4 point scale) grade point average, from a middle-class background, and actively involved in religious and athletic endeavors.

Constructs of Mathematics. On her topic grid, Susan's most "intense" constructs, based on relationship scores, were "varied," "advanced," "most useful," "most liked," and "abstract." Susan viewed each of these constructs in a positive sense. Laura's most intense constructs consisted of "easy," "creative," "advanced" and "easiest to learn" (tie), and "best at." Laura viewed "creative" and "advanced" as negative aspects and the remaining three constructs as positive.

FIGURE 8-1
SUMMARY CLUSTER GRAPHS - TOPICS
EASY/DIFFICULT X MOST USEFUL/LEAST USEFUL

```
                      EASY                                         EASY
 SUSAN               -|-                      LAURA              -|-   9
                     -|-                                         -|-          4
                  12-|-                                          -|-11, 12  15    5
            9 14     -|-                                         -10, 14
            15   4 1-|11 .                                       -|-
 |--|--|--|--|--|--|--|--|--|--|             |--|--|--|--|--1--|--|--|--|--|
 LEAS                -|- 10 13     MOST       LEAST              -|-         MOST
                     -|-   3,5 6              3                  -|-
                     -|-         7                              6|-
                     -|-                      7                  -|-
                     -|-                                     13  -|-
                   DIFFICULT                                DIFFICULT
```

1-invigorating, 2-easy, 3-abstract, 4-easiest to learn, 5-essential, 6-varied, 7-advanced, 8-most useful, 9-best at, 10-organized, 11-conclusive, 12-exact, 13-creative, 14-most liked, 15-easy to teach

Cluster graphs can be used to graphically portray relationships between constructs for an individuals' system. Coordinates represent the signed variance (x 100) between the constructs chosen for the axes and the remaining constructs. A comparison of Susan's and Laura's graphs (Figure 8-1), with "easy/difficult" forming the primary (y) axis and "most useful/least useful" the secondary (x) axis, shows a strikingly different trend for the two participants. To Susan, constructs related to secondary mathematics topics that are viewed as "most useful" also tend to encompass those that are viewed as "difficult." For Laura, the opposite tendency exists.

Susan's constructs suggest a view of mathematics that values complexity, academic achievement, and application, and one in which she considers her own preferences to play a role. Interview data supported these findings, suggesting Susan was in the process of developing a relativistic view of mathematics--characterized by reflection and the use of alternative viewpoints.

These characteristics were evident in her approach to student scenarios and problem-solving situations (she would typically question the context and meanings of the situation/ problem before expressing a view) and in her response to the study itself (each week she would want to discuss ideas that she had developed based on the previous session). Susan easily alternated between expressing her perception of a "teacher view" of a situation and a "student view."

Laura's more "intense" topic-grid constructs suggest a view of mathematics that focuses on simple, straight-forward procedures which she can easily accomplish through easily learned routines. Interview data suggested that Laura conceived of mathematics as a vehicle for "performance," an area in which she had received constant praise but held little meaning outside the classroom context.

Laura's reactions to student scenarios typically involved an attempt to repeat strategies "her (cooperating) teacher" had used. Her approach to problem-solving situations was marked by attempts to apply learned techniques, often replying "I should remember how to do this," and a lack of alternatives when she did not readily recognize a solution strategy.

Laura's position on Perry's scale was deemed to be that of "Escape in Multiplism" where she finds her "identity in carrying out assignments of external authority by performance." From this perspective "creative" and "advanced" mathematics can be threatening to her self-perceived mathematical abilities. "Easy," "easiest to learn," and "best at" suggest constructs supportive of success in accomplishing assigned tasks.

Constructs of Mathematics Teaching. Susan's five highest rated constructs from the role grid were "encouraging" and "motivating" and "inquisitive" (tie), and "respected" and "reliable" (tie). Laura's were "respected," "encouraging" and "interesting" (tie), "motivating," and "flexible." All were viewed as positive aspects by each participant. Each evidenced an "idealized" view of a favorite former mathematics teacher, with Susan rating her "best" mathematics teacher first on 12 of 17 constructs and Laura rating this person highest on 10 of 17. Each demonstrated self rankings that suggested a close identification with their "best" teacher. Both participants described themselves as having a strong social orientation to teaching; scores reflective of more intellectual concerns (e.g., "intelligent," "abstract," "complex") ranked low on both participants' role grids.

Susan's highest rating on "inquisitive" provides a contrast between the two participants' perceptions of teaching; "inquisitive" was the lowest-ranked construct on Laura's role grid. Susan's approach to teaching is an active one in which she sees herself as a decision maker, capable of making judgments on content and methodology. Laura, however, is passive in her approach, deferring to others for decisions on content and methodology. "Flexible," which ranked fourth on her role grid, was interpreted by Laura as an ability to readily follow instructions from those she perceived to be in authority.

CONCLUSIONS

The cases of Susan and Laura, only partially discussed here, highlight the broad differences in perceptions of teaching and of mathematics that can exist between two ostensibly similar participants in a teacher education program. Other case studies (Owens, 1987) suggest that these individuals are not ends of a spectrum but represent part of a complex array of beliefs held by preservice teachers.

The constructs through which preservice teachers view mathematics and mathematics teaching are important determinants of how individuals interpret their undergraduate experiences and anticipate their teaching roles. These constructs are integral to the individuals' developing worldviews which perform an important function in structuring their roles as professionals. Knowledge of preservice teachers constructs and worldviews can provide teacher educators with understandings of how individuals perceive their undergraduate experiences, and should play a central role in the design and conduct of these programs.

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BELIEFS, ATTITUDES, AND EMOTIONS: AFFECTIVE FACTORS IN MATHEMATICS LEARNING

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Abstract

Current research on the role of the affective domain in mathematics learning takes a variety of forms. Some researchers focus on the beliefs about mathematics that are held by students or teachers. Others focus on the attitudes of mathematics learners. A third group is beginning to look at more visceral, emotional responses to mathematics. This paper responds to the research on affective issues that is reported in this volume, and suggests directions for future research.

Research on affective issues is well represented in this volume. There are 13 papers that deal with affect and its relationship to mathematics learning and teaching. This paper will deal first with issues of terminology, and then continue with some comments on each of the papers. Since the papers are required to be brief, and since research on affect is notoriously difficult to communicate accurately, the possibilities for misinterpretation are many. It seems to me that short papers like these may constitute a form of projective test; readers are likely to see in the papers reflections of their own interests. I hope that this reader has not imagined too much of his own interests in the papers. I also hope that the authors of the papers will not find too many errors in my comments.

DESCRIBING THE AFFECTIVE DOMAIN

The difficulties of saying what we mean in the affective domain are well known. In a recent paper, Reyes (1987) outlines the misinterpretations that occur when psychologists and researchers from mathematics education try to discuss affect. Her discussion makes a number of suggestions regarding terminology in the affective domain that I will try to follow here.

The affective domain is used here to refer to a wide range of feelings and moods that are generally regarded as something different from pure cognition. The main terms used to describe the affective domain are beliefs, attitudes, and emotions. These terms vary from "cold" to "hot" in the level of intensity of the feelings that they represent. They also vary in their stability; beliefs and attitudes are generally thought to be relatively stable and resistent to change, but emotional responses to mathematics may change rapidly. For example, students who say they dislike mathematics one day are likely to express the same attitude the next day. However, a student who is frustrated and upset when working on a non-routine problem may express strong positive emotions just a few minutes later when the problem is solved. Finally, although it is impossible to separate student responses into cognitive and affective categories, some of these terms have a larger cognitive component than others. For example, beliefs seem to involve mainly cognitive processes that are typically built up over a long period of time. Emotional responses, however, may involve little cognitive processing, and their rise time can be very short. So the terms beliefs, attitudes, and emotions are listed in order of increasing affective involvement, decreasing cognitive involvement, increasing intensity, and decreasing stability.

Sometimes researchers get involved in arguments about whether cognitive processing can be separated from affective processing. A similar argument exists about whether one dominates the other. In this paper I will assume that affect and cognition are inextricably linked, and that we cannot separate the two. However, the presence of both thought and feelings in mathematics students at all times does not imply that the two domains are always equally powerful. Sometimes we are more influenced by affective factors, sometimes less. Now that we have established some preliminary groundwork, let us try to define the three terms: beliefs, attitudes, and emotions.

Beliefs about mathematics generally involve very little affect, and are frequently based as much on cognitive responses as on feelings or affective responses. Beliefs about self may have more of an affective component, but in general beliefs will be viewed as primarily cognitive in nature. For example, students may have beliefs about the usefulness of mathematics, or about their role as mathematics learners. For further discussion of the role of beliefs in mathematics learning and teaching, see Reyes (1987), Schoenfeld (1985), and Silver (1985).

Attitude toward mathematics is used to refer to feelings about mathematics that are relatively consistent. For example, attitude will be used to refer to how much students like mathematics, and to how confident they feel about doing mathematics. Attitudes may have a component that is a belief, but they are distinguished from beliefs by the feelings that accompany the beliefs. For further discussion of alternate definitions of attitude, see Leder (in press) or Reyes (1987).

Emotion is used to describe affective reactions that are more intense than beliefs or attitudes. Emotions generally involve some physiological arousal (tense muscles, rapid heartbeat) and some redirection of the individual's attention. Typical emotions would include joy, anxiety, frustration, and surprise. For further discussion of emotion, see Mandler (1984).

The papers have been grouped into three categories, depending on whether they deal primarily with beliefs, attitudes, or emotions. Of course, many papers deal with more than one of these, and no doubt alternate classifications are pc ssible. However, I think that this means of categorizing the papers will be useful in finding interesting comparisons among them. We begin with papers that deal with beliefs.

STUDENTS' AND TEACHERS' BELIEFS ABOUT MATHEMATICS

Research on students' beliefs about mathematics has become much more prominent in recent years, especially in research on the teaching of mathematical problem solving (Silver, 1985). Student views c mathematics can often have a major impact on their performance, as Schoenfeld (1985) has noted. Most of this research has focused on secondary school students (15 years and older), but some investigators have begun to look at younger students. Among these investigators are Kouba and McDonald in this volume.

Kouba and McDonald have begun a coordinated research program to determine what students believe is part of the domain of mathematics. This research started with elementary school students and their beliefs about what constitutes mathematics, and has now continued with junior high school students (ages 12 and 13). As one might expect, students at this age describe mathematics in terms of their experiences in mathematics classrooms. These experiences are often limited to typical textbook exercises, so students frequently fail to see the mathematics in a particular setting when that setting is different from what is found in most textbooks. New teachers are often surprised that they have to spend so much time answering questions like "Why do we have to learn this stuff?" Given students' limited conception of the domain of mathematics, perhaps their question is more legitimate than we realize. If they had a more mature understanding of what really constitutes mathematics, they would have a better understanding of why schools require mathematics.

Bliss and Sakonidis report related work on student beliefs about truth in various kinds of domains from mathematics to religion. These students (aged 11 to 16) generally agree that mathematics and science have a lot of truth in them, and are less certain about history and religion. The empirical and logical nature of mathematics seem to be the major sources of these judgments, and it was a relief to see that only a few students attributed the truth of mathematics to the fact that the teacher said it was true.

Both of these studies on student beliefs provide useful data that help explain how student perceptions of mathematics develop. Both studies provide the kind of broad picture of student beliefs that can result from statistical studies of questionnaire data on relatively large numbers of students. I hope that future studies will continue to gather data in this way, and also gather some other kinds of data that will supplement that which is reported here. For example, more detailed case studies of a few students would help make the data presented here more real, especially for a constructivist audience. Another strategy for making the data more meaningful would be to provide more cross-sectional or longitudinal data, thus allowing readers to make the comparisons between different ages that would allow us to see these beliefs develop over time.

Two other studies (by Owens and by Oprea and Stonewater) deal with beliefs about mathematics, but these two focus on the beliefs of teachers. Both studies use the Perry Scheme as a structure for the analysis of teachers' beliefs, and both supplement that scheme with related data from a second theoretical framework. Also, both studies use small sample sizes where the emphasis is on gathering substantial amounts of qualitative data on only a few subjects. Presenting this kind of qualitative data in seven pages is a very difficult task, and both authors have clearly worked hard to do the best job possible under the circumstances.

Owens presents convincing data on how the beliefs of two of his subjects can differ substantially, even when both appear to be very similar on other dimensions. Some of the data were presented in compact and very complex grids that were relatively opaque for me; I suspect that the attempts to quantify the mostly qualitative data require more space to make clear than was available here. Oprea and Stonewater have collected data that should be helpful in revising their instruments and their theoretical framework, even though the results which they obtained were in conflict with their expectations.

A major problem in the work on teacher beliefs is the lack of an appropriate theoretical framework. Although the Perry Scheme has some appeal, I am uneasy with its application to mathematics education. Even the recent revision and expansion of the scheme (Belenky, Clinchy, Goldberger, & Tarule, 1986), which tries to improve the theory's application to women as well as men, seems to miss some of the relevant aspects of mathematical beliefs. It seems to me, for example, that the scheme needs to take into account the specific requirements of the discipline of mathematics, including its logical structure and complex ways of representing concepts. I am also uneasy with an approach that does not take into consideration the subjects' specific knowledge of mathematics or their level of general ability, and how these kinds of knowledge may influence their performance on measures designed to classify people into categories of dualism, multiplicity, or relativism. The efforts to supplement the Perry Scheme with other frameworks (by Kelly and by Schoenfeld) are certainly helpful. However, I am left with the feeling that these frameworks also need more development and refining before they will provide an adequate structure for the analysis of teacher beliefs.

ATTITUDES TOWARD MATHEMATICS

Research on beliefs has been troubled by the lack of adequate theory, and the same may be said for the work on attitudes. But one area of accomplishment has been the research on how attitudes toward mathematics differ when we compare girls and boys (Leder, 1986). The work of Makuni makes a substantial addition to this area. In this paper, which is based on a substantial program of research carried out in Kenya, we find that gender-related differences can be identified in one more country, and that the pattern of these differences is generally quite similar to what has been reported in Australia, North America, the United Kingdom, and other areas. The development of effective ways to address these differences deserves high priority all around the world.

Further research along these lines is reported by Miller, who used three separate techniques in assessing the attitudes of twelfth graders. The use of multiple measures is a strength of this study; however, the scales developed by Fennema and Sherman (1976) would have been useful in making this study more comparable to others in the field. Also, the Fennema-Sherman scales provide ways to measure more of the varied facets of the attitude construct than does the Aiken scale.

A major contribution of the Miller study is its investigation of the genesis of negative attitudes toward mathematics. The data from this study suggest that seventh grade is an important point in the development of attitude, a finding that agrees with other research in this area. This finding should encourage the support of current efforts to focus intervention strategies at the early adolescent age group.

A related paper by Lucock deals with both beliefs and attitudes. This work discusses beliefs about the usefulness of mathematics as well as the attitude of liking mathematics. A strength of this study is that it allows us to differentiate student responses that focus on routine mathematics from those that deal with mathematical problem solving of a non-routine sort. Other aspects of the study range widely over a number of topics that are difficult to summarize briefly here. Although the study lacks the kind of technical features that would provide more assurance of the quality of the data, the results conform in general to other studies about the development of attitudes toward mathematics.

EMOTIONAL FACTORS IN MATHEMATICS LEARNING

The remaining papers on affect deal with somewhat more emotional issues, ranging from mathematics anxiety of some degree of intensity to emotional responses that have a physiological aspect to them. We begin with the papers on mathematics anxiety.

The work by Lacasse and Gattuso provides us with the results of their experience in running workshops on mathematics anxiety. Their analysis of the problem shows good practical knowledge of "mathophobes", and makes a number of useful suggestions for providing instruction that alleviates the fears of the anxious, especially those who are adult students. This research is very much in tune with related work on mathematics anxiety in that it is based in practice, not in theory. One of the nice features of this work is that it makes use of the expertise of both psychologists and mathematicians as they address a truly interdisciplinary problem.

In a related study, Evans reports on adults' anxiety about mathematics, including their scores on the MARS scale. Again, I would be more comfortable with the more extensive measures that are a part of the Fennema-Sherman scales, but the MARS instrument does have its adherents, mainly from counseling psychology. More important than my preferences in instruments is the fact that this study, like several of those discussed above in the attitude section, presents consistent data on gender-related differences in affective responses to mathematics. In general, these differences indicate that women express more anxiety than men, and that this difference persists even when the women tend to be more talented in mathematics than the men. Moreover, the unfortunate underrepresentation of women in mathematical careers seems to be one of the results of these differences in affective responses.

The interviews that were a part of the Evans study provide an important part of the data. The subject who referred to "panic" in describing his affective reaction to doing sums in public reflects exactly the kind of emotional response found by Buxton (1981). In an extensive study of these extreme emotional responses, Buxton provides a model of how to study mathematics anxiety. The work has a strong theoretical foundation, and yet speaks directly to the needs of mathematics education for improved practice in dealing with anxiety in the mathematics classroom.

Coutts and Jackson report a study on personality variables that are related to success avoidance in mathematics. The notion of success avoidance is a useful concept that grew out of work by Horner in the 1960's. Research on success avoidance has been interesting, but not as successful as originally hoped. It seems that investigations of a single variable like success avoidance are unlikely to provide as rich a picture of student behavior as we need. This study uses 22 personality variables to look for relationships between these characteristics and success avoidance. The fact that significant relationships were found with two variables is not surprising, but fortunately the data make sense in terms of our practical experience.

The paper by Ligault takes a psychoanalytic perspective on mathematics anxiety--quite an unusual perspective for research in mathematics education. Although I find the Freudian interpretations of the students' views of the relationship to the father to be quite extreme, I am favorably impressed by several aspects of this paper. For example, the role of the unconscious has received very little attention in research on mathematics education, even though mathematicians like Hadamard suggest that the unconscious plays a central role in mathematical problem solving, Some cognitive psychologists (Mandler, 1984) are also attempting to bring back research in this area, so perhaps the time is right for a more serious look at this topic. Also, Legault reports the use of projective techniques to assess affective factors; previous attempts by Fennema and others to use these kinds of techniques met with little success, but perhaps researchers in mathematics education should give them another try. One final aspect of this study that I liked is that it combines Piaget and Freud in an interesting way. This kind of healthy eclecticism is good for research in mathematics education.

The last paper that I will discuss also deals with mathematics anxiety and also uses interesting and unusual (for mathematics education) theories and measuring techniques. Gentry and Underhill base their work on Bandura's ideas about anxiety, and include measures of muscle tension, as well as attitude scales, in their efforts to assess the emotional side of mathematics learners. Although this work may seem a bit exotic to some, I am encouraged by the results, and I see it as a model for the kind of exploratory research that is needed on affective issues in mathematics learning. The importance of the physiological measures is emphasized by the fact that there was little correlation between a traditional attitude measure and the measure of muscle tension. It seems likely that these two measures are tapping into very different aspects of the affective domain. The instructional interventions that were used in the study were well specified, and directly related to currently prominent theories. The cognitive restructuring strategy is very much like what Meichenbaum (1977) would recommend, and Mandler's (1984) theory would be quite relevant to the use of the modified progressive relaxation intervention. Moreover, both of these intervention strategies are directly related to the techniques used in some of the current workshop efforts on the topic of mathematics anxiety. Further research along the lines presented by Gentry and Underhill seems to me to be a major step forward in research on the affective domain in mathematics, and especially research on mathematics anxiety.

The thirteenth and final paper (by McLeod) deals with a constructivist approach to the development of attitudes toward mathematics. It attempts to use concepts from cognitive science to show how attitudes could develop out of the basic emotional responses that are the foundation of Mandler's (1984) theory of affect. The paper fails to pay sufficient attention to the role of beliefs in the development of attitudes toward mathematics, but otherwise I find myself in general agreement with the author.

DIRECTIONS FOR FUTURE RESEARCH

In the limited space that is left to me, I would like to discuss briefly two major issues for future research, specifically the need for better theory and the need for multiple methodologies in the study of affective issues in mathematics education. I will also suggest some specific problem areas that need further elaboration and more attention than they have received so far.

The major weakness of current research on affective issues in mathematics education continues to be the lack of a strong theoretical foundation for the work. This observation has been made on many occasions by many different people, and I believe that we are now in a position to make some improvements. Mandler (1984) has made a significant effort to bring research on affect into the mainstream of work in cognitive psychology and cognitive science. Since he takes a constructivist point of view, Mandler's views seem particularly appropriate for discussion at the PME conference. Meichenbaum (1977) also presents a theoretical position that is cognizant of and consonant with the currently dominant paradigm of cognitive psychology. Skemp (1979) also does a good job of taking affect into account in the development of his theory of learning; since his work is closely tied to both mathematics and to developmental psychology, his theory has special relevance for the the psychology of mathematics education. Finally, Weiner (1986) has developed a theory of affect that builds on current research in social psychology. Weiner's work has been the basis for the most successful research done on affective issues in mathematics education (Reyes, 1984; Fennema & Peterson, 1985). Most of this work has been done in the context of research on gender-related differences.

There are a number of other theories that are also useful, and many of them are referred to earlier in this volume. However, the four listed above are my first choices. There are many other theories (for an overview, see Strongman, 1978), but these four seem to me to be the ones that are most relevant to mathematics learning and teaching. Of course, each of these theories needs to be tailored and refined to meet the needs of research in mathematics education.

In addition to concerns about theory, we need to develop and refine a variety of research methods that will fit the needs of our theories. Research on affect is still dominated by paper-and-pencil instruments, even though research on the cognitive processes of students has long since moved on to extensive use of more clinical methods. Many researchers have chosen to supplement their questionnaire data with individual interviews, and a few have even chosen to use measures of physiological changes that are indicators of affect. Both of these choices are welcome as we try to provide a better picture of the affective domain and its influence on student performance. I would also suggest that we use some of the techniques of our colleagues in psychology and anthropology, including closer investigation of facial responses. Some researchers rely almost entirely on facial expression as an indicator of emotional response (Mandler, 1984), a position that I do not hold. However, good educational research needs data from a variety of perspectives, and obtaining videotapes of facial expression seems much more suited to educational research than some of the other biomedical methods that may be a standard part of the psychological laboratory.

In the area of research on beliefs about mathematics, we need to learn more about the methods of anthropologists, and how they determine the role of culture in student performance (D'Andrade, 1981). In the study of attitudes, we need more than just questionnaires and statistical analyses of the data. In the study of the emotional side of mathematics learning, we need more and better ways to measure students physiological responses and facial expressions, as well as strategies for measuring other indicators of the more intense affective reactions that many students exhibit in relation to mathematical tasks.

In closing, I would like to suggest three topics that deserve special attention in research on affective issues in mathematics education. First, current research on the teaching of higher-order thinking skills and non-routine problem solving needs to pay more attention to affective issues. These more intense intellectual activities are often accompanied by more intense affective reactions, and we need better data on student responses in this area. Second, we need to pay more attention to the role of affect in the life of working mathematicians. For example, recent research by Silver and Metzger (1987) points out that aesthetic considerations play an important part in the decisions that research mathematicians make in solving non-routine problems. Mathematicians frequently talk about "pretty" problems or "elegant" solutions; we need to investigate teaching strategies that will help students develop these desirable characteristics. Finally, the current emphasis on affective influences and gender-related differences needs to be strengthened and expanded. I suggest that all studies of affect should incorporate gender as a part of their concern. Substantial progress has already been made in building our understanding in this area (Fennema & Peterson, 1985; Reyes, 1984), but more progress is needed if we are to do our best in dealing with educational inequity and with correcting the underrepresentation of women in mathematical careers.

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Algebra
in computer environments

LA PENSÉE ALGORITHMIQUE DANS L'INITIATION À L' ALGEBRE

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RÉSUMÉ

Nous pensons qu'il y a lieu de fonder l'apprentissage de l'algèbre sur des structures cognitives déjà présentes chez les élèves, et qu'une façon d'atteindre ce but est de leur proposer des problèmes externes aux mathématiques mais à "potentiel algébrique" et de les inciter, lors de la formalisation, à l'utilisation de noms significatifs concordant avec la sémantique du problème.
D'autre part nous croyons qu'en général on passe trop vite de la situation à modéliser à la représentation algébrique et aux manipulations syntaxiques nécessaires à la résolution. Pour faciliter cette transition et le développement des structures cognitives appropriées, nous proposons l'utilisation de représentations algorithmiques intermédiaires réalisables à l'intérieur d'un "environnement algébrique" informatisé basé sur un langage de programmation dédié.

L'algèbre occupe généralement une place importante dans les programmes d'enseignement des mathématiques au niveau secondaire; par ailleurs, l'apprentissage de l'algèbre semble causer beaucoup de difficultés aux élèves qui l'entreprennent. Dans ce court article, nous essaierons d'identifier certains problèmes rencontrés par les débutants, d'en discuter les causes possibles, et de proposer certains éléments de solution. Dans ce qui suit, nous désignerons par "algèbre élémentaire" l'algèbre enseignée au niveau secondaire: l'algèbre élémentaire comporte donc minimalement l'algèbre des polynômes en une indéterminée, mais aussi les fonctions linéaires, quadratiques, trigonométriques, exponentielles et logarithmiques.

LES TROIS ASPECTS DE LA DÉMARCHE ALGÉBRIQUE

Il importe tout d'abord de préc.ser le cadre conceptuel que nous avons adopté. Dans la pratique de la démarche algébrique, nous distinguons trois aspects principaux: syntaxique sémantique interne et sémantique externe.

Lorsqu'il s'agit de reconnaître le type d'une formule (exemple: un polynôme de degré trois), d'effectuer des manipulations formelles correctes ou erronées (exemple: remplacer $(a + b)^2$ par $a^2 + 2ab + b^2$ ou par $a^2 + b^2$), ou de choisir d'appliquer une règle de réécriture en fonction d'une stratégie heuristique (exemple: pour résoudre l'équation $3x + 2 = 7$, soustraire d'abord 2 aux deux membres de l'équation), nous nous trouvons en présence de l'aspect syntaxique. Il est facile de constater l'hégémonie des pratiques syntaxiques dans l'enseignement actuel de l'algèbre élémentaire: il s'agit là de méthodes abstraites et puissantes en vertu de leur généralité, mais dont la raison d'être échappe le plus souvent aux élèves débutants.

Il est plus difficile de décrire l'aspect sémantique interne de l'algèbre élémentaire, peut-être parce qu'il n'apparaît clairement qu'en algèbre non élémentaire. Considérons, par exemple, la recherche d'un modèle ensembliste rendant vraie l'identité $a+(b*c) = (a+b)*(a+c)$: on est alors amené à spécifier un domaine où les variables a,b,c prendront leurs valeurs possibles, ainsi qu'une interprétation des symboles + et * via des fonctions binaires sur le domaine déjà spécifié. Dans notre cas, on peut choisir comme domaine l'ensemble des parties d'un ensemble X, et interpréter + (respectivement: *) comme la fonction qui associe à un couple de sous-ensembles de X leur différence symétrique (respectivement: leur intersection).

En algèbre élémentaire, le domaine de variation est toujours un ensemble de nombres (intervalle de nombres naturels, entiers, rationnels ou réels) qui très souvent n'est pas précisé, et les opérations ont une interprétation canonique invariable: en réalité, on est en présence d'une interprétation ensembliste unique (et des diverses sous-structures induites par certaines restrictions du domaine). Ainsi l'aspect sémantique interne de l'algèbre élémentaire se résume-t-il en des choix judicieux du domaine de variation des variables (exemple: l'identité $\log(x*y) = \log(x) + \log(y)$ n'a de sens que si x et y sont positifs, même si le membre de gauche est défini quand x et y sont tous deux négatifs) et au calcul numérique (évaluation). On peut exprimer ceci en disant que la sémantique interne voit les expressions algébriques comme des fonctions (toujours algorithmiquement calculables) définies sur des ensembles de nombres, éventuellement représentées par des tableaux de valeurs, des graphes cartésiens, ou des algorithmes de calcul.

Avec l'aspect sémantique externe, nous quittons le domaine strictement mathématique pour nous intéresser d'une part aux représentations algébriques (mathématisations, modélisations) de situations non mathématiques ou de problèmes algébriques narratifs (c.à.d. énoncés en langue naturelle), et d'autre part aux interprétations d'activités algébriques (exemple: manipulations formelles) en termes des contextes représentés. Cet aspect retiendra particulièrement notre attention car c'est à ce niveau que l'élève est suceptible d'établir des liens entre ses propres structures cognitives et les concepts algébriques qu'on lui propose, de construire une signification pour ses activités algébriques.

Nous pensons que l'apprentissage des rudiments de l'algèbre doit se fonder sur des structures cognitives déjà présentes chez les élèves et dont la "distance" aux concepts à construire n'est pas trop grande. Dans ce contexte, une approche prometteuse consiste à proposer des situations-problèmes externes aux mathématiques mais à "potentiel algébrique" et d'inciter à des généralisations successives, obtenues en donnant des noms significatifs explicitant la nature générale des "objets" en présence, comme dans l'exemple suivant:

3	*	5	->	15
3 objets	*	5 dollars par objet	->	15 dollars
nombre d'objets	*	5 dollars par objet	->	prix payé
nombre d'objets	*	coût par objet	->	prix payé.

Remarquons que les expressions de l'algèbre élémentaire (avec leurs noms de variables "abstraits" tels a, b, c, x, y, z) résident à un niveau d'abstraction plus élevé en ce qu'elles généralisent une classe de situations-problèmes: par exemple, l'équation algébrique x * y = z généralise autant la situation précédente que la situation

longueur de la base * longueur de la hauteur -> aire du rectangle.

LES PROBLEMES NARRATIFS

Comme nous venons de le voir, les problèmes algébriques narratifs nous semblent jouer un rôle très important dans la construction des concepts fondamentaux de l'algèbre chez les élèves débutants. Depuis plusieurs années, des chercheurs tentent de modéliser les processus cognitifs mis en oeuvre pour traduire en équations des problèmes algébriques narratifs. Ce travail a débuté avec Bobrow (1968) qui a développé sur ordinateur un programme appelé STUDENT. Paige et Simon (1966) ont démontré que le processus de traduction directe utilisé par STUDENT approxime le traitement par certains individus des problèmes narratifs algébriques; mais ils ont aussi souligné qu'une traduction directe ne peut rendre compte du processus humain de résolution qui s'appuie sur des connaissances sémantiques propres au language naturel. En nos propres mots, le traitement effectué par STUDENT est exclusivement syntaxique alors que l'humain utilise aussi une sémantique externe. D'autres recherches dans ce domaine ont souligné le rôle des schémas dans la représentation des problèmes algébriques narratifs (Hinsley, Hayes et Simon 1977; Mayer 1980; Schank 1982). En dépit de ces progrès théoriques, les enseignants en mathématiques sont relativement dépourvus quand il s'agit d'aider les élèves à représenter les relations des problèmes algébriques narratifs.

Par ailleurs, les chercheurs ont récemment fait de grands progrès dans la modélisation de la façon utilisée par les jeunes enfants pour représenter et résoudre des problèmes arithmétiques narratifs. Ces modèles, tel celui développé par Kintsch et Greeno (1985), construisent une représentation d'un problème narratif qui incorpore l'information requise pour le résoudre. En d'autres mots, la représentation spécifie l'opération à effectuer, telle l'addition, la soustraction ou le dénombrement d'objets. Cette théorie a été perfectionnée par Larkin (1986) qui a distingué trois phases dans le processus de construction d'une représentation. Dans la première phase, l'enfant lit les mots du problème et en construit une représentation interne de base, qui correspond directement à la situation physique décrite. Dans la phase suivante, il ajoute de nouvelles relations mathématiques (en se basant sur ses connaissances antérieures). La représentation mathématique résultante suggère un calcul particulier qui est effectué lors de la troisième phase. Selon Larkin, la phase de représentation mathématique peut être escamotée quand l'enfant tente de calculer (phase 3) en se fiant directement à la représentation de base (phase 1).

On peut constater une disparition semblable de la phase de représentation algébrique lorsque des élèves tentent d'écrire une équation (phase 3) en se fiant sur une représentation de base apauvrie d'un problème algébrique narratif. Les approches usuelles d'enseignement ne semblent pas doter les élèves de moyens de construire des représentations mathématiques adéquates pour les problèmes algébriques narratifs. Selon une étude récente (Clement, Lockhead et Soloway 1980), l'utilisation d'un environnement informatique mettant l'accent sur une sémantique procédurale active des équations semble fournir une méthode puissante de construction de représentations mathématiques efficaces. Clement a montré que l'apprentissage de la programmation sur ordinateur aide les étudiants à se former des représentations mathématiques de certains types de problèmes, en incitant à une approche algorithmique dans l'établissement d'équations.

UNE INTRODUCTION À L'ALGEBRE

Rappelons que nous voulons partir d'activités significatives et motivantes pour l'élève, et qu'il est par conséquent hors de question d'introduire le symbolisme algébrique autrement que comme un codage de problèmes en rapport avec les expériences antérieures de l'élève. Nous pensons que lors de l'introduction des concepts algébriques, on passe trop vite de la situation à modéliser ou du problème narratif à la représentation algébrique et qu'il y aurait lieu de passer par une suite d'étapes intermédiaires en vue d'éclairer et de faciliter éventuellement ce processus. Nous avons déjà mentionné une première étape, consistant à utiliser des variables dont les noms sont significatifs par rapport à la situation-problème à l'étude. De plus, il semble plus facile pour le débutant d'utiliser le mode impératif (basé sur l'affectation informatique) plutôt que le mode déclaratif (basé sur l'égalité mathématique): ainsi au lieu d'affirmer que le coût total égale le produit du nombre d'objet par le coût unitaire, il semble plus simple de décrire comment calculer le coût total en multipliant le nombre d'objets par le coût unitaire. On est ainsi amené à représenter le problème à résoudre comme un programme constitué d'une suite d'affectations simples (évitant ainsi au débutant les difficultés liées à la composition d'opérations dans une même expression et à l'application des règles de priorité des calcul) portant sur des variables (ayant des noms significatifs) dont les valeurs peuvent être spécifiées au départ par l'utilisateur (variables d'entrée) ou affichées à la fin de l'exécution (variables de sortie). Outre les possibilités d'exécution et de trace donnant une rétroaction à l'élève sur l'adéquation du programme à la situation-problème étudiée, cette représentation informatique est suceptible de faciliter grandement la recherche de la ou des solutions du problème, ce qui est un facteur de motivation non négligeable pour l'élève. Dans ce contexte en effet, la recherche d'une solution peut presque toujours se ramener à cerner des valeurs à donner aux variables d'entrée de sorte que deux ou plusieurs variables de sortie deviennent égales; pour le débutant, ceci est beaucoup plus accessible que les manipulations formelles habituellement employées.

Nous nous proposons donc de créer un "environnement algébrique" basé sur un langage de programmation dédié qui pourrait servir de représentation intermédiaire permettant notamment:

* de mettre en évidence les variables pertinentes du problème
* de désigner ces variables par des noms significatifs (mais aussi de permettre éventuellement des abbréviations)
* de pouvoir faire appel à des variables intermédiaires (présentes en informatique mais ignorées en mathématiques)
* donner un sens dynamique aux variables en fonction des exécutions possibles du programme (demande de valeur en entrée, affectation suite à un calcul)
* de trouver une ou plusieurs solutions sans nécessairement devoir faire appel à des méthodes de manipulations syntaxiques (méthodes de recherche numérique avec heuristiques)

Par des observations et des interventions auprès d'élèves en interaction avec cet environnement informatique, nous nous proposons de vérifier son impact sur le développement des stratégies cognitives de construction des représentations algébriques.

Notons que cette représentation d'un problème narratif par un programme faisant la transition avec l'écriture algébrique usuelle n'est qu'une étape dans notre vision algorithmique de l'apprentissage de l'algèbre. On peut aussi voir une équation (reliant deux expressions) comme l'assertion de l'équivalence de deux programmes, et les manipulations algébriques comme des transformations de programmes préservant l'équivalence. Mais nous touchons ici les aspects sémantique interne et syntaxique de l'apprentissage de l'algèbre: ceci fera éventuellement l'objet de communications subséquentes.

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ON THE DEEP STRUCTURE OF FUNCTIONS

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Abstract

The computer program Green globs was analyzed for effecting a global understanding of transformations on functions resulting from altering the coefficients of its algebraic representation. Particular efforts were made to tie together the graph of a function with its algebraic rule. Two groups of students worked with the software; one group worked in a highly structured environment with the computer being used to illustrate and reinforce specific transformations. The other group discovered the effect of the transformations alone. Statistically significant differences between the two groups were not obtained. Test results and interviews indicated that, overall, transformations in specific cases were understood by both groups, but that a general, global understanding eluded them. The potential of using this sort of microworld type environment is discussed.

OBJECTIVE

One of the major goals of school and college mathematics is to lead students to a sound understanding of the major underlying notions associated with the graphing of functions. In particular, the structured relationships between the graphs of functions arising from each other under simple transformations are important in this connection E.g., students should be able handle the following types of tasks:

- to graph functions such as $|f(x)|$, $f(x-a)$, $f^{-1}(x)$ and similar ones from the graph of a given function $f(x)$ (without algebraic description);
- to graph simple rational functions such as $(x^2-4)/(x+2)$ or $(x^2-2)/(x+1)$ after determining, by inspection, their asymptotes and local discontinuities;
- to discuss the graphs of functions such as $f(x)=x^2-5|x|+6$ by relying on their symmetry properties.

Important as these goals are for understanding the deep structure of functions and their graphs, they are usually not achieved through the curriculum. The purpose of this paper is to analyze, from a cognitive viewpoint, activities which lead students to understand such functional relationships.

THEORETICAL FRAMEWORK

The theoretical background to our work is given by three complementary research strands which have been developed over the past few years.

(i) The theoretical framework for analyzing aspects of the function concept proposed by Dreyfus and Eisenberg [1984];

(ii) Experimental evidence about student misconceptions on function concepts obtained by Vinner [1983],

(iii) The constructivist approach to facilitate abstraction via microworlds as developed by Thompson [1985].

According to (ii), a function is typically considered to be an expression or an equation. Students view graphs per se as peripheral to the function itself, as an additional load; and if they can avoid dealing with them, they will A group of pre-service teachers were asked to present a graphical argument for developing the usual formulae for the coordinates of the vertex of a parabola given by $y=ax^2+bx+c$. The expected response of setting the first derivative equal to zero was obtained. Then graphs of equations of the form $y=a(x-p)^2+q$ were discussed; afterwards the students were again asked for a graphical method for finding the coordinates of the vertex of a parabola. Once again, they returned to the first derivative. The relationship between the graphing activity and the coordinates of the vertex completely eluded them.

METHODOLOGY

The teaching of the function concept should be designed with the above considerations in mind. As teachers, we have a twofold task:

(a) To transmit to the students a more well-rounded concept of what a function is, namely an abstract mathematical object having any of several concrete representations, one of the most useful of which is a graph; and

(b) To teach students to recognize those situations where graphical processing of functional relationships is more efficient than algebraic processing.

Point (iii) above has produced evidence pointing to the potential of mathematical microworlds for promoting abstraction The present work is in keeping with these results. It uses the commercially available software Green Globs [Dugdale, 1982]. This software presents a set of points in the plane and the student is supposed to generate a graph traversing neighborhoods of these points. By focussing on the effect of changing particular parameters in the equations, insights and generalizations on the deep structure of functions is, theoretically, obtained.

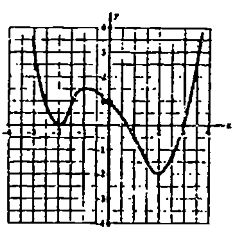
On the basis of the theoretical background and the properties of the software, we arrived at the following hypotheses for this study:

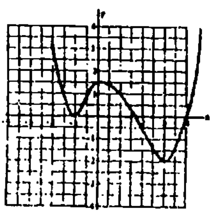
Hypotheses

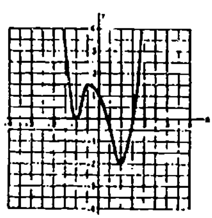
(1) Understanding the relationship between the algebraic and graphical representation of a function is facilitated by the activity of discovering and specifying in algebraic form graphs which traverse given regions.

(2) The influence of changing the parameters of a function on its graph can be understood by structured activities as described above.

5. Given. the graph of $y=f(x)$.
The graphs below were obtained from that of $y=f(x)$. Match each graph with its formula.
(Five different funtions were given in the test; two of them are shown here.)

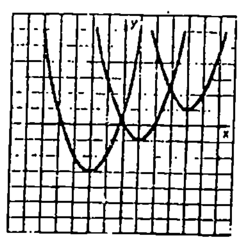






$y=f(x)-1$ $\quad$ $y=f(x)+1$
$y=f(x-1)$ $\quad$ $y=f(x+1)$
$y=2f(x)$ $\quad$ $y=\frac{1}{2}f(x)$
$y=f(2x)$ $\quad$ $y=f(\frac{1}{2}x)$
$y=-f(x)$ $\quad$ $y=f(-x)$
$y=f(|x|)$ $\quad$ $y=f(-|x|)$
$y=|f(x)|$

6. Given: A set of three graphs which fit the formula $y=a(x-d)^2+e$. Write down, which of the parameters a, d, e are identical for the three graphs in the sketch.
(Six different sets of three graphs were given in the test; only one of them is shown here.)



10. Write next to each formula which one of the four graphs in the sketch fits it.

○ $y=x^2-2x+1$
○ $y=1-x^2$
○ $y=x^2-2x$
○ $y=1+x^2$

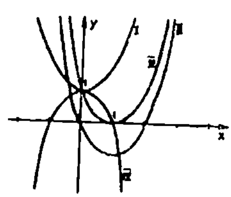


Figure 1: Representative Test Questions

Procedure

Two matched groups of 8 students each were chosen from 45 eleventh and twelfth graders in an academic high school The students were chosen and matched on the basis of their scores on a pretest and teacher recommendations. Within each group of eight, the students were paired according to their performance on the pretest, the two strongest ones together, and so on. The groups used the Green Globs software, in pairs, for six 50 minute sessions spread over four weeks. In these sessions, Group A was free to choose functions to p'ay with, help being provided by the tutors only when requested The activities in Group B were highly structured· In each session, Group B students were directed to use a certain type of function and to investigate the effect of changing the parameters of these functions Such effects were then discussed with the group as a whole. As a consequence, Group B actually used the software somewhat less than Group A.

The actions of the students were followed in detail and recorded by an observer. After the end of the instructional period, a posttest was administered, the posttest was identical to the pretest Representative test questions are listed in Figure 1. Moreover, one student from each pair was interviewed for about 30 minutes; the interviews were semi-structured; while predetermined questions and hints were being used as guide posts, an effort was made to keep the discussion flowing freely. Representative interview questions are listed in Figure 2.

1 Given $f(x)=x^3-3x^2$, let $g(x)=f(x+3)$. Find $g(-2)$.
(This question was accompanied by a graph of $f(x)$).

3.

In the accompanying figure the graph of the function $y=f(x)$ is given. Sketch qualitatively the graph of the function $g(x)=1/f(x)$.

Figure 2: Representative Interview Questions

While the wr tten tests focussed on achievement with respect to the skills under investigation (the influence of parameters on graphs and the effect of transformations on graphs), the interviews were designed to uncover the reasoning processes employed by the students in order to answer the given problems. The analysis of the observer's records, pre- and posttest scores, and the interviews comprise the data for this study

RESULTS

At the outset of the instructional period, students in both groups worked solely with linear functions. Their linearity-boundedness was very strong, in spite of extensive exposure they have had in their studies with other types of functions. Markowits, Eylon and Bruckheimer [1986] noted this gravitation to linearity in junior high school students, but the strength to which it was observed with these advanced students was surprising. Only after being explicitly and repeatedly required to do so, did the students experiment with other types of functions: polynomials and absolute value functions. And even then, they were frequently coming back to the linear, and later quadratic functions. This tendency, amazingly, was least strong among the weakest pair in each group. In fact, while the strongest students tended to spend a lot of time designing expressions according to their needs, the weakest ones tended to proceed on a purely experimental, often somewhat arbitrary fashion, just trying out what happens if they type in a certain formula. As a consequence, the weaker students were more likely than the stronger ones to work with more complicated and more advanced functions; a typical example they used was $f(x)=|x^3-45+x^2-x|$ (note the order of the terms). They would, however, have but the most elusive idea of the graph to be drawn by the program. Overall, the "what if not" sort of thinking as described by Brown and Walter (1969) was not internal to the students: The better ones were too hesitant to experiment with unfamiliar functions while the less able ones experimented with new formulae without any attempt at thinking them through beforehand.

The discussion of the pre- and posttests will focus on those questions which concern our main interest: the effect of changing parameters and the shifting/stretching transformations $f(x\pm a)$, $f(x)\pm a$, $f(\pm ax)$, $\pm af(x)$. These questions constituted 70% of the test and 100% of the interview. Henceforth these questions will be called non-standard. The standard questions included algebraic computations, graph reading and graph identifications such as in Question 10 (see Figure 1). The test results were not statistically different for Groups A and B. Therefore the combined results are listed in Table 1.

Table 1: Mean Percent Subtest Scores

Questions	Pretest	Posttest
Standard	71%	75%
Non-standard	24%	50%

Hypothesis (1) appears to be borne out, even if less strongly than could have been expected. The students did make progress, during the instructional period, on the very difficult topics that were addressed. On the non-standard questions the students doubled their scores; that is, understanding the effect of altering the parameters of a function is facilitated through the Green Globs software. But their overall 50% score on these questions indicates that this link is still not very strong. A more detailed analysis of the test results reveals that on Question 6 they more than trebled their scores with only relatively slight improvement shown on Question 5. The familiarity of the graphs used in Question 6 may be responsible for these differences. In Question 5 the students had to deal with a higher level of abstraction because the functions were given graphically only, and did not correspond to any equations known to them. Hypothesis (2) can not be accepted on the basis of the test results alone. Both groups improved approximately to the same extent. It thus appears that the activities with Green Globs in general caused this improvement rather than the structuring of the activities which was particular to Group B.

The interviews focussed in particular on the extent to which the students had established the connection between the graphical and the algebraic representation of functions. Almost all students did adopt a visual mode of operation; this mode, however, was often on a purely intuitive level; in most cases, an integration between the visual and the analytic mode was achieved by only three out of the eight students interviewed. The others did not fully link the rule of a function (its algebraic representation) to its graph (its geometric representation). Although they could confidently discuss the graphs of specific linear functions, they found it difficult to specify which among several graphs satisfied $y=ax+b$ with $a>1$ and $b>1$. This lack of global understanding manifested itself on all questions except those with quadratic functions: While not a single student missed Question 10 (see Figure 1), it emerged from the interviews that the advantage of working with quadratics in the form $a(x-d)^2+e$ rather than ax^2+bx+c was recognized by more than half the students (evenly divided between Groups A and B).

DISCUSSION

One of the goals of the Green Globs software is to place the emphasis on the geometric representation, subordinating to a lesser role the algebraic rule, rather than vice-versa - which is the way things are usually handled in school. The direct link between the two representations established by the software, has helped the students in the study progress towards establishing an analogous mental link. Overall, the understanding of this link has, however, remained vague for more than half of the students.

This study has to be viewed as one of a series of similar studies, which have all been undertaken within the theoretical framework of mathematical microworlds (see (iii)). In each of these studies, the goal has been one of achieving a process of abstraction on the part of the student; all these studies have met with only partial success [Dreyfus & Thompson, 1985; Dreyfus, 1986]. The question naturally arises, whether the theoretical framework needs to be revised in view of these limited successes. At present, this does not seem to be appropriate; in fact, all three studies referred to were rather short term. Extremely high level activities are required for the processes involved in abstraction in general, and in particular in the conception of a function as an abstract mathematical object, and the establishment of the connection between different representations of this object. It is hoped that longer and more systematic exposure to dual and triple representations of mathematical objects will achieve a clearer effect. But, at present, this is simply speculation.

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BELIEVING IS SEEING:
HOW PRECONCEPTIONS INFLUENCE THE PERCEPTION OF GRAPHS

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Contrary to a common assumption that graphs of function are somehow inherently more accessible to students than are symbolic presentations—after all, students have spent years perceiving and drawing visual forms before they first encounter algebraic symbols—visual presentations have their own conventions and ambiguities. Perceptual strategies that are sufficient for interpreting scale and relative position in real-world scenes are inappropriate when dealing with the infinite and relatively featureless objects in coordinate graphs of simple polynomial functions. Aided by software that dynamically links graphical and symbolic representations of function, our preliminary clinical studies show that perceptual illusions and shifts of attention from one feature to another obscure some of what the educational use of graphs is supposed to illucidate. The paper is illustrated with specific examples of illusions with linear and quadratic graphs.

Software that allows students to probe the nature of function by exploring with linked graphical and algebraic representations has recently been proliferating for three reasons: there is a perceived need to increase the emphasis on graphing in the algebra curriculum; it is theoretically reasonable to suppose that appropriate visual representations help invest meaning in, and thereby promote the learning of, the symbol system with which algebra students must cope; and computer technology lends itself well to this application.

As is often the case, new technological capabilities bring new questions to the fore. While investigating what had initially seemed to be straightforward questions like "Does this kind of software help students learn to make fewer of the canonical errors?" my colleagues and I found ourselves faced with several surprises and some new questions about fundamental issues in how people perceive graphs.

Common-sense supports the notion that the use of more than one representation of function will help learners understand what remains less clear when only one representation is used. Presented thoughtfully, multiple linked representations increase redundancy and thus can reduce ambiguities that might be inherent in any single representation. Algebraic expressions specify the exact relationship, but give neither single examples nor a visual gestalt. Graphs provide a gestalt within the limits of the graph but leave precise details unclear. Tables provide examples of the mapping but do not specify its nature. Said another way, each well-chosen representation views a function from a particular perspective that captures some aspect of the function well, but leaves another less clear: taken together, multiple representations should improve the fidelity of the whole message.

The theoretical arguments presented above are reasonable enough, but they may not be valid. In fact, little is known about the cognitive impact of multiple linked representation in algebra and until recently it has been impractical to examine these suppositions clinically.

Our early experiments have shown that students often misinterpreted what they saw in graphic representations of function. Left alone to experiment, they could induce rules that were wrong. We began to study how students develop strategies for using graphical information to give meaning to symbolic representations of function and vice versa. To what, in fact, do students attend when they look at graphs? What are the misconceptions that they bring with them and how do these misconceptions distort the information that they glean from the graphs? And how, if at all, do these distortions affect a student's ability to use graphical representations of function to inform the understanding and manipulation of the symbolic algebraic representation?

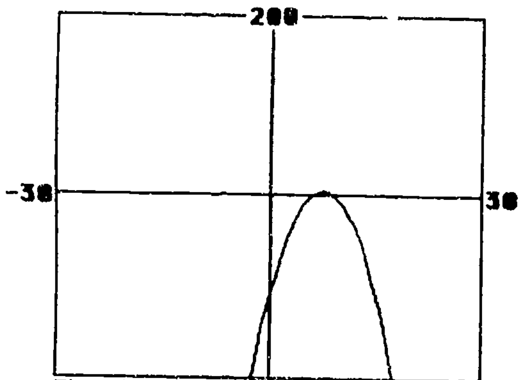

Figure 1

These questions grew initially out of observations with two bright, successful, second year algebra students. They were shown the computer screen illustrated in figure 1, and asked to discover the polynomial $(-2x^2 + 30x - 108)$ that created this graph. They were encouraged to use whatever means they chose, including making computer-supported measurements on the graph and trying out various expressions and observing differences between the graphs they created and the target.

Although the students had not previously tried to match a target graph, they had had some prior experience using the software to explore graphs of this type and had built up some expectations about the effect of the constant term and the coefficient of x^2 in the graph of a parabola. Appropriately, their first analyses made use of these notions.

They believed that something they referred to as "shape" was controlled by the coefficient of x^2 and also knew that if the parabola was "upside down," the coefficient must be negative. After a single experiment trying to match the target parabola they reasoned from its "pointiness" that its x^2 coefficient must be -2. They also had a notion of "height" and believed it was controlled by the constant term. They chose the value of the constant term by estimating where the parabola crossed the y-axis. Seeing that the y-intercept was roughly midway between the origin and the bottom of the graph (at -200), they tried -100. Although they believed that the x-coefficient controlled left-right placement, they said that they had no idea what value to use for it and so they made an arbitrary choice and picked 2.

Figure 2 shows the graph of their function (solid) superimposed on the target parabola (dashed). Their parabola appeared to have the same "shape" as the one they were trying to match, so they felt confident in their choice of the coefficient of the x^2 term. Further, as well as one could see at this scale, the two curves had the same y-intercept, which fit their criterion for the choice of the constant term. Only the x coefficient remained undiscovered.

Yet, despite the confirmation of two of their reasoned choices that *we* may derive from the graph, and, remarkably, despite their expressed awareness that their third choice was the least trusted even from the outset, *they* saw the graph as disconfirming their choice of the constant term. Recall that they *chose* the constant term by examining the y-intercept—a measure intended eventually to insure that the two functions had the same height for identical values of x—but they *coded* what they were doing as determining "height," not y-intercept. When we compare the "height" of two objects, we do so by attending to corresponding features of the two objects—in this case, the vertex.

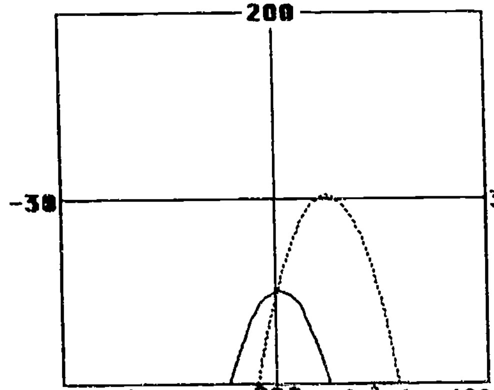

Figure 2

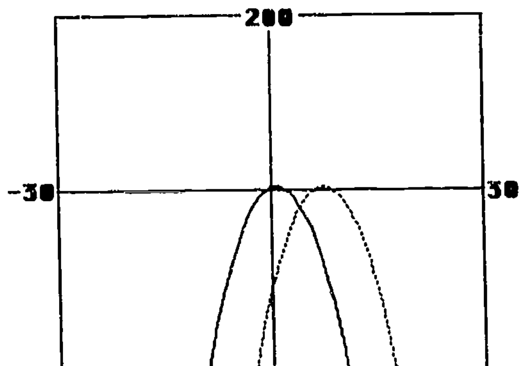

Figure 3

The visual impression so dismayed them that they capitulated and "corrected" the constant term from -100 to 4 to account for their "error." The new expression was farther from the target but was more satisfying because its graph was just as "high" as the graph that they were trying to match (see fig. 3).

It is important to characterize what happened clearly. To *my* eyes, figure 2 showed that they were correct; the parabolas are equally "high" at the y-axis. But in *their* eyes, figure 2 was disconfirming. What we believe influences what we see in the graph.

The way in which the illusion distracted them from their originally correct analysis of the problem is reminiscent of the not-quite-conserver in the Piagetian task in which equal quantities of juice are poured into glasses that differ in width. Initially two identical glasses are filled with juice and the child verifies that they are the same. When the juice from one of these glasses is then poured into a narrower container, its level rises higher than the level in the other original glass. Young children's thinking in this situation seems dominated by the visual impression: the new glass must have more. Older children and adults witnessing this experiment are guided more strongly by their expectation that quantity remains invariant despite appearances. In between, there is an intriguing stage when a child might well *expect* that after pouring from one container to the other the amounts would be the same, but would then give in—even spontaneously expressing surprise as the algebra students did—to the perception. Logical thinking has developed considerably, but is not robust enough to prevail over perception.

The students' confusion in this case appeared to result from a shift of attention from one feature (the y-intercept) to another (the vertex). Not only is the vertex a more salient feature, but in real-world everyday strategies for judging height, it is the feature we would be most likely to use. (More will be said about real-world strategies later.) In other cases, confusion seems to arise from the same mechanisms that give rise to familiar perceptual illusions. Consider, for example, what happens when the students take the expression they have just developed and begin to change the coefficient of x to "move it over." Figure 4 shows how the graphs appear when they have all but the constant term correct. When they looked at graphs like this, they knew that they needed to adjust the constant, but it also appeared to them that the inner parabola was more obtuse than the outer. Here, the target parabola (dashes) appears blunter than the trial parabola above it.

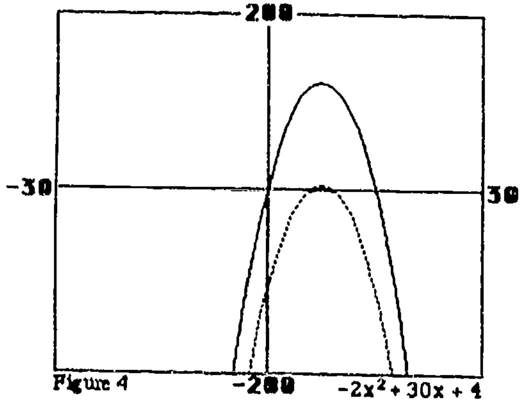

Figure 4

As was true of the confusion regarding the meaning of "height," illusions such as the one illustrated here were sometimes powerful enough to draw their attention back to the already correct coefficient of the x^2 term and cause them to change it.

A GENERALIZED THEORY OF ILLUSIONS

The Cartesian graph spaces with which we are confronted in books and on computer screens are rectangular segments of a plane on which some shape appears. Most commonly—always in the case of algebraic functions that are defined over the entire domain of real numbers—only a portion of the shape appears. Through our experiences with partial views of real objects (e.g., views of things being shifted up or down as viewed through a window) we develop working strategies for interpreting such views. It makes sense that as we first learn to read graphs, we interpret what we see in them according to those strategies that have been successful for us in other realms, and we continue to use such strategies until our new experiences teach us to do otherwise.

In fact, when the object being viewed is infinite in size and relatively poor in discrete identifiable features, our everyday strategies fail: what we experience is often a perceptual or attentional illusion. The student work described above gives examples of both.

Imagine a person slowly descending on a scaffold outside your office window. As the person's feet first appear at the top of your window, you already have a very good idea of the overall shape of the person. Assuming a constant rate of descent, you have a good idea when that person will be fully visible. Aided in part by the availability of readily identifiable, discrete elements in the scene (e.g., shoelaces, buttons), you have no difficulty at all knowing which direction (down) the person is moving. Finally, because people are not too variable in size (among other clues) you can tell that you are seeing a 6 foot person descending immediately outside your window and not a 240 foot person 40 feet away.

By contrast, overall shape, magnitude of a translation (corresponding to the rate of descent of the person in the window), direction of movement, and scale (corresponding to the distance of the person viewed in the window) may all become ambiguous when the object viewed is infinite in size and has the kinds of regularities of shape and lack of readily recognizable sub-elements inherent in lines and parabolas.

TOWARD THE DEVELOPMENT OF A TAXONOMY OF ILLUSIONS

We are currently beginning to classify the sources of confusions that arise in the perception and interpretation of graphs. This paper will illustrate only two: orientation of the graph within its window, and interaction between scale and function.

Consider, for example, what one sees when looking at a family of linear functions Ax+B that differ only in the value of the constant B. If we already know the algebra, we have built up some analytic expectations. What we expect to see is that the graph of a line moves up as B is increased, as in figure 5.

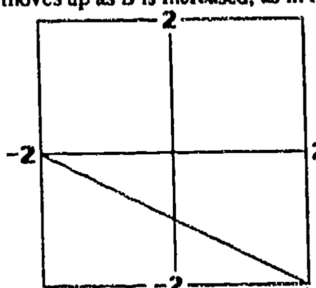

Figure 5a $-.5x - 1$

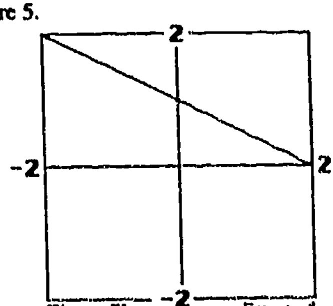

Figure 5b $-.5x + 1$

Because an infinite line presents us with no discrete points to watch, however, it may also appear to be moving from left to right as the constant term increases (figures 6a, 6b) or even from right to left if the line slope is positive. The way the line appears to move depends totally on the angle it makes with the window through which you view it. Though the appearance is a perceptual phenomenon—not one that any amount of algebraic sophistication can change—algebraic sophistication can lead us to ignore appearances. We may even be able to "see" that the segment of line visible in figure 6a has "moved off the top of figure 6b" and a new segment, previously unseen, is now visible.

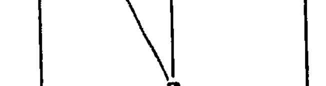

Figure 6a $-2x - 2$

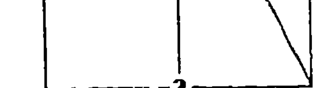

Figure 6b $-2x + 2$

A student who is learning the algebra for the first time, however, has no such analytic expectations. This has important implications for the use of inductive learning experiences with graphing software. Exploration with such software may certainly lead students to "correct" conclusions, but it may also lead to very complex rules like:

There are five cases that describe how the graph of a linear function $Ax+B$ changes as B increases.

$\lvert A\rvert << 1$	the line moves up as B increases
$\lvert A\rvert >> 1,\ A > 0$	the line moves to the left
$\lvert A\rvert >> 1,\ A < 0$	the line moves to the right
$\lvert A\rvert = 1,\ A < 0$	the line moves diagonally to the northeast
$\lvert A\rvert = 1,\ A > 0$	the line moves diagonally to the northwest

This may be an interesting place for students to be at some stage in their mathematical learning, but is certainly not where we want them to arrive as a final destination. Worse yet, student propensity to choose integer values rather than decimals—therefore missing the cases where $\lvert A\rvert < 1$—makes it highly likely that students will choose initial examples that lead them to the left-right theory without even seeing the up-down or diagonal movements that might lead some to expect or want a simplification.

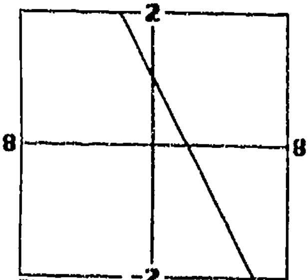

Figure 7 -2 -.5x + 1

Finally, there is an added complication. Even the complex rule given above assumes that x and y are symmetric on the graph. The (visual) angle that a line makes with its "window" depends both on the line's (mathematical) slope and on the relationship of the scales of the two axes. In figures 5 and 6, the x and y axes are represented in the same scale. Figure 7 represents the same function that appears in figure 5b but, because the scale has been changed, its graph resembles the family of functions represented in figure 6.

Scale affects perception in other ways as well. An infinite line viewed close up (figure 8a) or from afar (figure 8b) appears not to change shape, though it moves "closer" to the center of the window. This accords perfectly with our everyday experience with normal objects: as we veiw an (ordinary) object from the same direction but at varying distances, angles in the object are preserved but distances (in this case, the distance from the center of the window) are not. (Of course, the line in figure 8b may equally well be perceived as "higher," suggesting that there has been a change in the constant term.)

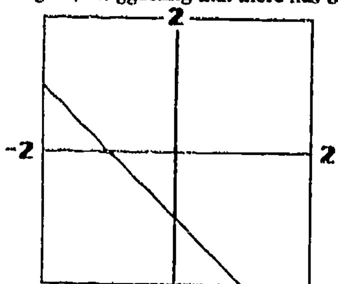

Figure 8a -2 - x - 1

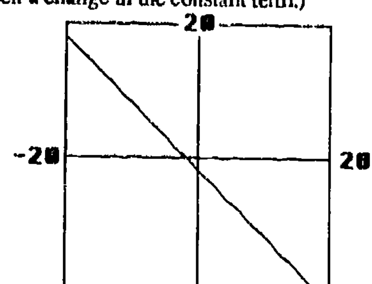

Figure 8b -20 - x - 1

We have a very different experience with the parabola. Figure 9 shows a closeup view. The small box in figure 10 is a reduction of figure 9, one-tenth its linear dimensions. The extended view of that parabola is how it would appear on a scale symmetric in x and y and running from -20 to +20. Unlike a straight line, the parabola does appear to change shape.

Although the portion of the plane that we see in figure 9 corresponds to a fraction of that shown in all of figure 10, we automatically compare the tiny chunk in figure 9 with all of figure 10, not just the segment in the box, fooling our eyes into thinking that angle has not been preserved.

If we compare equivalent portions, we avoid the illusion. The boxed area of figure 10 alone does not appear to be a different shape from that shown in figure 9. Thus, when the scale of the parabola changes in a window of fixed size, the parabola appears to change shape Yet, when the scale of the *window* changes along with the scale of the object in it—that is, when we see the window as well as the parabola from afar—we have no such illusion.

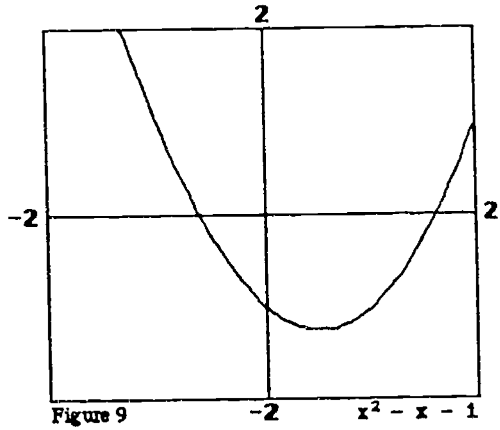

Figure 9

Figure 10

Understanding this interaction between scale and "shape" is important because students typically use "shape" of a parabola (on a constant scale and in a fixed-size window) to determine the A coefficient. Thus, though they learn strategies for solving their problem, the strategies are based on an underlying notion—that parabolas may have different shapes—that is erroneous. The shape that they see is, in part, an artifact of scale

We are now studying the implications of these illusions. As suggested in the metaphor of viewing the window-washer from your office, the unavailability of easily trackable points—like ankles and shoelaces—causes some of the confusion. Perhaps students need more experience with discrete functions—or continuous functions such as A|x-B|+C which have "special" points—as a background for continuous ones. Another problem appears to be due to the infinite range and domain of the functions. What are the implications in this case? The interaction between scale and function is leading us to invite students to graph familiar functions on variously distorted graph papers, to help us learn what they consider the necessary features of a graphing environment, and to help them explore the invariants of graphs under particular transformations.

Understanding feedback systems.

Claude Janvier and Maurice Garançon.

This paper attempts to define what is meant by understanding feedback systems. Understanding is envisaged as a form of coordination of three external representations in view of characterizing the evolution of the system and predicting the effects of changing one variable. Difficulties are described on the basis of this analysis. A computer program that is meant to study further this understanding is described together with the planned experimentations.

Feedback systems are sets of variables interconnected in a special way and whose values evolve with time. A simple ecological system which involves a few populations characterized by eating habits is an example of such a feedback system. When the preys happen to grow in number, the population of predators increases and this has a consequence on the preys themselves. This action of the population of preys on itself is the main feature of feedback systems and is called a **feedback loop**. For similar reasons, the "stock" of an item with the "number of items ordered" at fixed interval of time are two variables which as a system can be regarded as a feedback system. In fact, a change in the order induces a change in the stock which, in turn, as a feedback effect, brings about an adjustment of the next order. The variables of a feedback system and their relations are basically represented by a causal diagram such as the ones we have used to illustre the two examples provided so far.

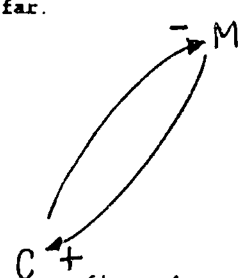

figure 1:
ecological system consisting of a population of cats and one of mice.

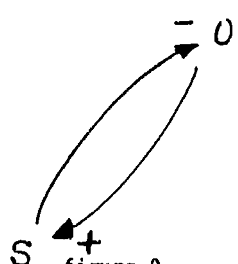

figure 2:
system formed by a stock and the regular order of the items

As the diagrams show (namely the "+" sign towards the end of the arrow), an increase in the amount of mice will induce an increase in the population of cats, while an increase in the amount of cats (predator) will bring about a reduction in the population of mice (see the "-" sign). Similarly, a change in the stock of an item will involve the order to adjuct and consequently will bring about a change in the stock itself.

Modeling and interpretation

One can envisage any feedback system as a set of mathematically defined relations between the variables. They can generally be expressed as a set of differential or difference equations which determine a model. Such a model determines a class of possible interactions between the variables. Mathematically speaking, we can say that there remain some parameters that are left to be fixed and which will define a particular instance of the model. For instance, particular values given to the populations' size at time t_0 will define a particularized model. In other words, a model in which the parameters have been determined defines a precise system. However, a particularized model will never behave like a real world system since it is impossible to take into account all the factors determining the evolution of such a system. A first step into the understanding of feedback systems involves showing mastery of the process of modeling which requires manipulating the abstract concepts of the model while keeping in mind the meaning of the relation provided by the context. This is a form of abstraction quite comparable to the one characterizing the process of interpretation of cartesian graphs as described in Janvier(1980).

Along this line of thought, it must be pointed out that even though concrete objects are sometimes involved in such systems, they remain basically abstract in the sense that the individuals are always considered as parts of a population and, as a consequence, the relations between the populations are essentially statistical in nature. Moreover, the measures that are used are often very complex such birth rates, fluctuations of the inflation rate...

Understanding feedback systems: a first approach.

Even though a set of equations defines totally the relations between the variables, they are meaningless in practical terms because the interactions are more vividly expressed in terms of the solutions of these equations which are explicitly represented by a cartesian graph. We feel well-founded to take the stand that setting up the equations of a system does not ~~necessarily~~ involve understanding it. In other words, we believe that understanding feedback systems goes far beyond establishing the relations on the causal diagram.

Apart from the difficulties related to the process of modeling, it then follows that understanding feedback systems involves using efficiently the different representations of particularized models in order to supply characterizations of the evolution of the system.

In fact, as it has just been mentioned, a particularized system is more adequately described by a set of curves showing the evolution of each variables on the same cartesian graph which we call a **multi-curve diagram** (see figure 3). On the other hand, the relations illustrated by the causal diagram is in no way contained in a multi-curve diagram. As it is shown in Janvier(1978) and Preece(1983), the relations between variables of ecological systems illustrated only by cartesian graph are difficult to interpret. A first attempt to help students' interpretations would consist of providing them with an adequate causal diagram as a sort of support for their reasoning. We are then led to define in a first approximation **understanding feedback systems as a form of coordination between the causal diagram and the corresponding (multi-curve) cartesian graph in view of being able to supply a characterization of the evolution of the system.**

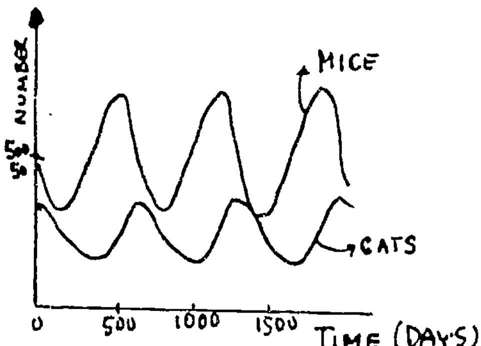

figure 3.

Some difficulties.

With this notion of understanding in mind, we shall examine the structure of such systems in order to determine the difficulties which one meet in dealing efficiently with them.

• **The basic component is the feedback loop** and we shall examine its internal complexity."Didactically speaking",one can distinguish two irreducible kinds of feedback loops.
The first kind is a loop in which one of the elements is introduced in order to control the level of a variable. Such systems are simple in the sense that the value of the controlled variable tends to pre-determined objectives that are attained through well known patterns. Examples are the temperature of a room with its thermostat, the speed of a steam machine together a Watt centrifugal controller. The system shown if figure 3 is another example.
In the second type, two populations interact while they obey some internal growth laws. the system described by figure 2 would then be more adequately represented by figure 2'(next page) which shows clearly why the characterization of the evolution of the system is more difficult to achieved.

• Feedback loops involve **variables** whose changes induce an increase or a decrease in the other variables. Now, René de Cotret (1985), Jao Ponte(1983), Kerslake(1977) and Janvier(1978) have shown that for a single variable students have trouble going beyond a linear relation. They also showed how difficult it is for them to dissociate a rate of change from the actual value of the variable at one instant. For example, the tide is said to rise fast when it is high. René de Cotret appears to have reduced this sort of difficulty with a teaching method based on meaningfulful experimental work. However, feedback loops are more exacting in the sense that they involve two variables and their respective rates of change. **Understanding feedback consequently requires being able to coordinate the interaction of two variables and their rate of change within a certain class of inter-actions**

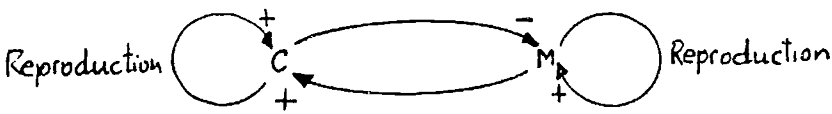

figure 2'

• Another kind of difficulty **an understanding** of feedback systems **must** take into account is **a new kind of relations between variables** due to the fact that feedback loops describe a process which is basically dynamic. More explicitly, the first phenomena that are introduced to students in their science courses are such that one variable determines another one and vice versa. For example, the temperature will determine the length of a rod; the density of a particular liquid, the rate of a certain chemical reaction... The analytical scientific method presupposes that experiments can be carried out in which we examine the effect of one variable on another, while the other ones are being kept fixed. This is not possible for feedback loops. No variable can be assumed to be in a way **controllable** nor can they be considered independent. In feedback systems, **there are no independent variables (except time possibly!)**. This fact constitutes a major obstacle.

Towards the introduction of the phase diagrams.

Since understanding can be regarded as as a form of coordination between two forms of external representations, it seems pedagogically sound that it would be developed through a simulation that would facilitate the coordination of representations when some control of the system can be achieved through inter-active features. There exist on the market to-day several computer programs which simulate a particularized model directly from the definition of the relations established with only the help of the causal diagram. However, we dismiss the fact that this kind of simulation can be beneficial because it does not allow real experimentation with the system. In other words, when the relations between the variables are fixed and the population determined, the simulation is carried out and illustrated by a multi-curve diagram, and no changes can be made when the simulation is under way. An experimentation is needed that would allow the students to modify the conditions or the relation of the system at any time and examine the consequences of such modifications on the system. Clearly, the coordination between cartesian graphs and causal diagrams is not sufficient because the changes induced on the curves are complex and very confusing.

As we look for a more refined representation, a more profound trouble with the coordination of cartesian graph with the causal diagram can be singled out. The relation suggested by the causal diagram contradicts what is revealed on the multi-curve one. When one looks at the behavior of the populations (see figure 3) around day 800, an increase in the number of the mice takes place at the same time as a reduction in the number of cats which contradicts the relation "*the more preys, the more predators*". In order to remove ambiguity, the relation could be reformulated the following way: "an increase in the number of preys will magnify the enviesaged increase of the population of predators or will slow down a reduction of the population of predators which would be underway. However, this makes the resulting causal diagram pretty awkward.

Understanding feedback systems: a more refined approach.

Inspired by several fundamental studies in the field Schaefer (1967) and Braunschweig(1985)we think it is necessary to introduce in our analysis **phase diagrams** that work well for 2 variables. It consists of a curve in a cartesian plane (see figure 4) whose points (two co-ordinates) represent respectively the size of two populations at one time. The evolution of both populations is then represented by this continuous curve in the plane. The main shortcoming of such diagram is that time as a variable is not represented as in the cartesian graph. Temporal reference must be added now and then according to the needs as we have done with the T_1, T_2, T_3,... A system of three variables would require using the space. The evolution of the system would then be a three-dimensional path.

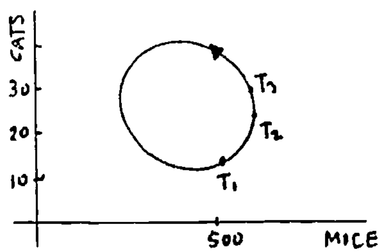

figure 4: Phase diagram corresponding to the cartesian diagram of figure 3.

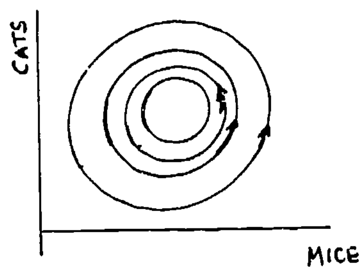

figure 5:

Phase diagrams are very efficient for representing a series of particularized models because their genuine cyclic evolution gives rise to closed curves in the phase diagram. This is shown in figure

5. Needless to say, we simplify things here since in certain models the populations may spiral in or out. In other words, the evolution of systems involving any combination of two populations would be modelized (under rather general conditions) by one of the path shown in figure 5. Changing one population during the evolution of a system would mean going from one curve another one. Figure 6 shows what is the consequence in terms of evolutions to reduce the number of mice.

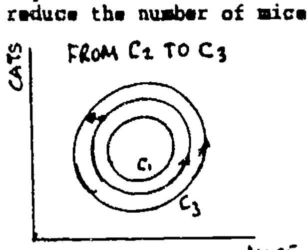

figure 6

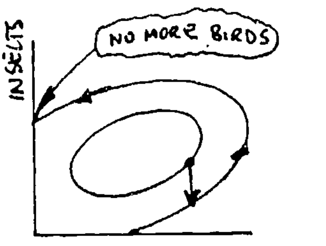

figure 7

Consequently, when understanding feedback systems is associated with coordinating efficiently causal, multi-curve and phase diagrams, it involves then not only being able to supply characterizations of the evolution of a system but also predicting the effect of a change of one population of the system at any moment of its evolution.

Clearly all the previous difficulties stay the same. However more is required. Phase diagrams show clearly the contradiction mentioned and reveals a lot about the dynamical structure of feedback systems. Figure 7 shows the dramatic consequence of killing too many insects. Birds disappear and insects come back *en force*.

Testing understanding.

We plan to conduct three experiments all related in a special manner to the notion of understanding of feedback systems defined above. The three are intimately linked with a computer program that is now in production. It consists of a game in which the student play the role of a pisciculturist who exploits a fishing reserve (pisciculture entreprise). Two kinds of fish are involved : predator (blue) and prey (yellow). They are symbolically mixed in a rectangular "lake" in the center of the screen. They do not appear individually but only in a homogeneous mixture. According to their ratio, the green color of the "lake" may be more bluish or more yellowish. A column on each side of the rectangle represents anyhow the size of each population permanently. The pisciculturist can allow fishing and be paid; or he can stock the lake with blue or yellow fish and pay for it. There is on the screen a "bank account" that varies along with the transactions. The winner is the one who makes the best performance at controlling the system and makes more money with it while "keeping the lake in good shape". The program will be used in another version in which a phase diagram appearing in the corner of the screen will exhibit the state of the system at each instant.

1.- We will test the efficiency of the two versions. This amounts to verifying how effective is the use of a phase diagram for controlling the system. It may happen that the regular change of the green color of the "lake" is sufficient to detect the right strategy allowing a rational exploitation of the pisciculture entreprise. In such a case, we shall have shown that certain understandings of feedback systems do not fit our theoritical framework.

2.- We will verify whether the knowledge derives from playing with the computer program enriched with the phase diagram can be transfered to more complex systems such as the sardine-seal-fishermen ecological system of the St-Lawrence Gulf. In fact, there exists a film describing technically the relations between these variables. We wish to check how the computer program would prepare the student to better interpret the content of the film.

3.- Preece (1986) has created interesting tasks involving ecological systems that are perturbed and come back to their equilibrium position. We envisage testing the influence of the computer program on the students' responses to these tasks.

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DIENES REVISITED: MULTIPLE EMBODIMENTS IN COMPUTER ENVIRONMENTS

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WICAT

ABSTRACT

This paper will describe significant ways that computer-based instruction can encourage teachers and students to make greater use of activities with concrete materials, while at the same time providing a useful context for implementing some of the best instructional strategies associated with mathematics laboratories— including some strategies, which before, have never worked well using concrete materials. There is not enough space in this paper to present research results concerning the success of the computer-based activities used to illustrate Dienes' instructional principles; however, our presentation will focus on these results, particularly as they apply to higher order thinking.

DIENES' MULTIPLE EMBODIMENT PRINCIPLE

Past RN, PR, and AMPS publications (e.g., Lesh, Landau, & Hamilton, 1980; and Behr, Lesh, Post, & Silver, 1984) have identified five distinct representation systems that occur in mathematics learning and problem solving. These are (a) "scripts" in which knowledge is organized around "real world" events that serve as models for interpreting and solving other kinds of problem situations; (b) manipulative models (such as Cuisenaire rods, arithmetic blocks, fraction bars, number lines, etc.) in which the "elements" in the system have little meaning per se, but the "built-in" relationships and operations fit many everyday situations; (c) pictures which, like manipulative models, can be internalized as "images"; (d) spoken languages, including specialized sub-languages (e.g., logic, etc.); and (5) written symbols which, like spoken languages, can involve specialized sentences and phrases, such as: ($x + 3 = 7$, $A' \cup B' = (A \cap B)'$) as well as normal English sentences and phrases.

Not only are the translation processes between models in different representational systems important components of understanding a given idea, they also correspond to some of the most important "modeling" processes needed to use this idea in everyday situations. Essential features of modeling include (1) simplifying the original situation by ignoring irrelevant characteristics in order to focus on more relevant factors, (2) establishing a mapping between the original situation and the "model," (3) investigating the properties of the model in order to generate predictions about the original situation, (4) translating (or mapping) the predictions back into the original situation, and (5) checking to see whether the translated prediction is useful.

Here is an example where the preceding steps are used to solve a standard algebra word problem:

> Al has an after-school job. He earns $6 per hour if he works 15 hours per week. If he works more than 15 hours, he gets paid "time and a half" for overtime. How many hours must Al work to earn $135 during one week?

To solve this problem, students may begin by paraphrasing the given "English sentence" into their own words, perhaps accompanied by a diagram or picture of the situation. Next, the description of the problem can be translated into an "algebraic sentence": $(6 \times 15) + 9(x - 15) = 135$. Then, a series of algebraic transformations can be used to convert this algebraic model into an arithmetic sentence that is sufficient with which to find the answer. The final transformed description is:

$$x = \frac{135 - (6 \times 15)}{9} + 15$$

Finally, by using a series of arithmetic simplifications, this arithmetic sentence can be reduced to: $x = 20$.

So, beyond the paraphrasing and diagramming, the entire solution process involves three significant translations: (1) from an English sentence to an algebraic sentence, (2) from an algebraic sentence to an arithmetic sentence, and (3) from an arithmetic sentence back into the original problem situation.

Notice that the algebraic sentence that most naturally describes the preceding problem situation does not immediately fit an arithmetic computation procedure. This possibility of "first describing, and then calculating" is one of the key features that makes algebra different from arithmetic.

As the preceding problem illustrates, problem solving often occurs by (1) translating from the "given situation" to a mathematical model, (2) translating the model-based result back into the original problem situation to see if it is useful. However, the modeling process usually is not this simple. Instead, in modeling students frequently use several representation systems (or models), in series or in parallel, with each depicting only a portion of the given problem situation.

We found that for realistic textbook word problems, good problem solvers are flexible in their use of various relevant representational systems—they instinctively switch to the most efficient representation at any given point in the solution process.

DIENES' CONSTRUCTIVE PRINCIPLE AND PERCEPTUAL VARIABILITY

Helping students construct a system of mathematical relationships is similar to helping students coordinate systems of overt activities like those involved in playing tennis or riding bicycles; that is, the student begins in situations in which the complexity of the system and the degree of coordination are minimal (e.g., all of the balls come waist high on the forehand side just within arm's reach) and gradually progresses to situations that require more complex and well-coordinated systems (e.g., where "getting in position" is important).

In general, building more complex systems involves: (1) *integration*— e.g., simple systems are linked together to build more complex systems, as when a tennis serve is built up by gradually linking together the toss, the hit, the follow-through, etc.; (2) *differentiation*—e.g., a single system is differentiated to produce two or more distinct variations, as when a forehand volley is varied slightly to produce top spin or backspin.

Poorly integrated mathematical systems are similar to poorly coordinated systems because (1) the student will not "read out" all of the available information—e.g., when first learning to ride a bicycle or hit tennis balls, a great deal of relevant information is not noticed; (2) the student "reads in" interpretations that are not objectively given—e.g., when first learning to ride a bicycle or hit tennis balls, the student's description of an activity is often distorted and biased.

Both of these factors also appear when, for example, an "eye witness" to an accident interprets given information in a way that is biased (because only selected pieces of information are noticed) and distorted (because what "made sense" and what was "expected" influenced the interpretation of what actually happened). Similar biased and distorted interpretations also influence students' mathematical judgements in graphics-related problems like the examples in this section.

Next, an example will be given to show how the basic approach of "taking apart" and "reassembling" mathematical ideas can be extended to basic algebraic concepts. We will focus on "unpacking" the systems of operations, relations, and transformations that underly the basic concepts of linear equations and simple polynomials.

The activities that follow are based on a symbol-manipulator/function-plotter called SAM that WICAT developed to enable students to write, graph, transform, and solve algebraic expressions and equations. In lessons, SAM helps students learn some of the most important basic ideas in algebra or calculus, and the algebra ideas can make SAM more useful for problem-solving situations that students want to address. However, SAM is more than a calculator; it has the following characteristics:

1. SAM can serve as an expression checker. We don't have to wait untill students give final answers to know whether they are proceeding along correct solution paths. We can, for example, assess whether they "set up" the equations correctly.

2. SAM is LISP-based, so it not only generates answers, it can produce solution path "traces" that create many instructional capabilities. For example, it allows us to: (a) generate hints by gradually revealing solution steps one at a time, (b) monitor individual steps in students' solution paths, (c) let students examine processes as well as products of solution attempts, and (d) give students the capability to build/edit/ store equation-solving routines (like the quadratic formula) in a LOGO-like fashion.

3. SAM's symbol manipulation capabilities interact with its function plotter to produce graphic interpretations of transformations leading to solutions. This gives students ways to visualize symbol transformations, and (in yet another way) to focus on processes as well as "answers" during solution attempts.

4. SAM can reduce answer-giving phases of problem solving so that attention can be focused on "nonanswer-giving" phases (e.g., problem formulation, trial solution evaluation, the quantification of qualitative information, the examination of alternative possibilities, etc.) where "second order" (i.e., thinking about thinking) monitoring and assessing functions often are especially important. So, SAM is not simply an answer-giver; it can help students to go beyond *thinking* to *think* about *thinking*.

For polynomials, it is easy for students to use SAM to carry out the following kinds of investigations:

1. Pick a value for n, between -10 and 10, and investigate the changes that this value produces in the graph of the linear expression: nx.

2. Plot the graph of the squared term, x^2; then plot the graph of the linear term, nx (as in step 1 above); and finally, plot the graph of the polynomial, $x^2 + nx$. Notice that the polynomial crosses the x-axis at the points zero and $-n$.

For example, Figure 1 shows the graph of x^2 and $4x$. Figure 2 shows the graph of the polynomial $x^2 + 4x$.

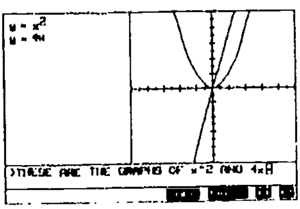

Figure 1

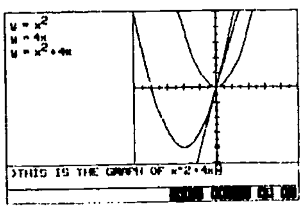

Figure 2

After repeating step 2 for a series of different values for n, it is easy for students to notice that the effect of adding x^2 and nx is to "slide the graph of x^2 downhill along the line nx." Furthermore, it is easy for students to notice that the amount of the slide is just enough to make the polynomial's graph pass through the points zero and $-n$.

3. Polynomials from step 2 can be factored into the form $x(x + b)$, and each of the linear factors can be graphed as shown in Figure 3. Then notice that the two lines pass through the points zero and $-n$.

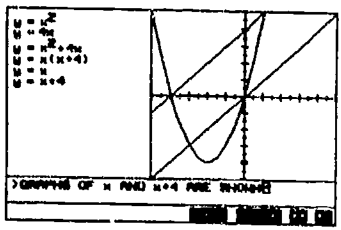

Figure 3

Step 3 shows why we can solve polynomials by factoring, setting each of the linear factors equal to zero, and then solving these linear equations. The linear terms are equal to zero at exactly the same places as the original polynomial.

In this example, the two models involved are (a) written symbols which (although they are on a computer screen) are like those mathematics teachers write on blackboards, and (b) computer graphics, consisting of graphs of equations in a rectangular coordinate system. Nonetheless, the computer-based activities using direct applications of Dienes' instructional principles can be created. For example:

—The *constructive principle* is involved when we "take apart and then reassemble" complex mathematical systems related to polynomials.

—The *multiple embodiment principle* is involved when we focus on mappings between two given models (i.e., written symbols and graphs of equations).

—The *dynamic principle* can be used to show how transformations performed on algebraic equations are reflected in changes in the graphs of the equations at each step. For example, in the next section, we will show how a slight variation on the preceding sequence of activities can be used to show why the "completing the square" process works in the derivation of the quadratic equation.

Even though the "materials" used in the example are computer-based graphics rather than "concrete materials" in the usual sense of this word, the activities can indeed involve overt actions that students can apply to "objects" that they can see and manipulate; and for the first time Dienes' instructional principles can be applied to content areas like "polynomials" which did not seem to lend themselves to a "mathematics laboratory" form of instruction.

DIENES' DYNAMIC PRINCIPLE

Models like coordinate graphs or systems of linear equations can be considered "conceptual amplifiers" because when they are used, they help students use their ideas more effectively. They are not simply inert systems that have no meanings; once students learn to meaningfully embed mathematical systems (ideas and principles) or problem situations within them, students are able to "read out" additional meanings.

A dynamic representation system, once constructed, actually helps students to generate significant new questions and to generate sophisticated solutions related to two of the most fundamental ideas in algebra; that is, our students have used informal language to describe rather deep principles related to (1) invariance under mappings among isomorphic systems, and (2) invariance under transformations within a given system.

The following example illustrates how computer environments are well-suited to Dienes' dynamic principle. Whether we are dealing with linear equations and graphs, fraction diagrams and simple proportional reasoning questions, or with polynomials, computers make it easy for the student to manipulate one model and immediately see corresponding transformations in one or more other models.

This example has to do with the process of "completing the square," which can be used (prior to using the quadratic formula) to find the roots or factors of quadratic equations like $x^2 + 2x - 3 = 0$. Figure 4 shows the graph of $x^2 + 2x - 3 = y$ and $y = 0$. Figure 5 shows the graph of $x^2 + 2x = y$ and $y = 3$. Then, Figure 6 shows the graph of $x^2 + 2x + 1 = y$ and $y = 4$. Notice that the tip of the parabola just touches the x-axis. (Is this significant? Would it happen for other quadratic equations? Which kinds?) Figure 7 shows the graphs of $x + 1 = y$ and $y = \pm 2$. Notice that the diagonal line goes through the x-axis at the same point where the parabola had touched. (Is this significant?) Figure 8 moves the graphs in Figure 7 so that the diagonal line goes through the origin of the graph. (Is this significant?)

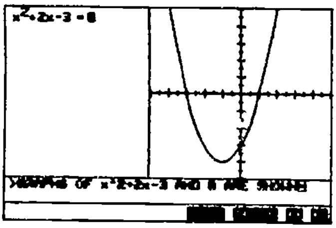

Figure 4

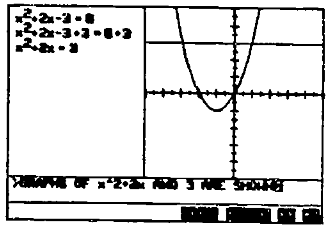

Figure 5

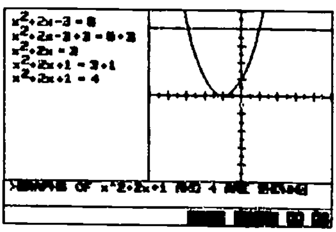

Figure 6

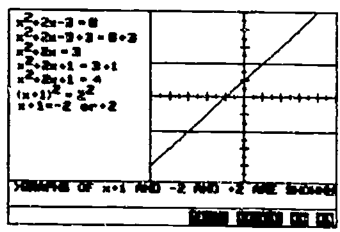

Figure 7

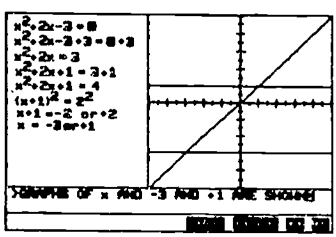

Figure 8

CONCLUSIONS

In general, we are in sympathy with those LEGO BEFORE LOGO proponents who believe that children's mathematical abstractions should be built of a firm foundation of experiences with real manipulable models and realistic problem-solving situations. However, we also know that even real concrete objects often are used only in very abstract ways and that very few teachers successfully use concrete activities as a significant instructional tool. On the other hand, we have seen that when students use the kind of computer-based activities described in this paper (many of which are electronic versions of the kinds of concrete models that we really hope students will have the opportunity to explore), their teachers actually become more likely to use "mathematics laboratory" activities with real concrete materials. This increased use of real concrete activities seems to occur because computer-based simulations of mathematics laboratories tend to minimize the reason why teachers rarely use concrete mathematics laboratory principles.

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USING MICROCOMPUTER-ASSISTED PROBLEM SOLVING TO EXPLORE THE CONCEPT OF LITERAL SYMBOLS- A FOLLOW-UP STUDY

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This paper reports on a follow-up to a 1985 study which used computer-oriented problem solving as a vehicle for investigating the development of the concept of literal symbol. The objectives of this study were: i) to determine ways in which the subjects currently perceive and use literal symbols; ii) to investigate the subjects' concept of literal symbol in light of instructional intervention over the past two years; iii) to determine whether computer-oriented problem solving can have long-term effect on the concept of literal symbols.

In 1985, the author conducted a study using the microcomputer as a tool in investigating concept development (Nelson, 1985; 1986). The study's purpose was to investigate ways that computer-oriented problem-solving activities influenced the learning of the concept of literal symbols and their use in certain noncomputer contexts. A secondary purpose was to investigate the subjects' perceptions of literal symbols in LOGO procedures.

The four subjects were average-ability fourth-grade students with no previous experience in the use of LOGO. They were taught to use LOGO to solve problems involving number sentences, rectangles, and recursion. The following sample procedures, taken from the subjects' work disks, illustrate such uses.

1.
```
TO DT
IF 12 = 5 + :BOX [PR :BOX STOP]
MAKE "BOX :BOX + 1
DT
END
```

2.
```
TO FRED
FD 20
LT 90
FD :X
LT 90
FD 20
LT 90
FD :X
END
```

3.
```
TO DT.
PR :C
IF :C = 1 [STOP]
MAKE "C :C - 1
DT.
END
```

The first procedure uses a recursive technique to solve the sentence 12 = 5 + *x*. The second one draws a rectangle whose width is assigned by the user. The third procedure generates a sequence of numbers from *C* to 1, where the value of *C* is assigned by the user.

Each subject was interviewed before and after the instructional and problem-solving sessions. There were six sets of tasks in the initial interview. Task set 1, which included items such as 8 + 7 = 19, was used to investigate the concept of equivalence. Task sets 2 and 3, which included items such as □ + 9 = 16 and *x* + 8 = 19, respectively, were used to examine the subjects' perceptions of non-literal and literal symbols in number sentences. Items in sets 4, 5, and 6, which all related to rectangles, were designed to explore the subjects' knowledge of rectangles and area. The tasks in the final interview included six sets of tasks similar to those used in the initial interview, as well as tasks which required the use of LOGO.

During the winter of 1987, the author conducted individual interviews with three of the subjects, Alex, Josh, and Dick.

Beth could not be interviewed since she had moved the previous year. Each child was presented with tasks similar to those given during the final interview in 1985. The interviews were videotaped and transcribed for use as data. The objectives of the study were:

1) to determine ways in which the subjects currently perceive and use literal symbols as compared to 1985;
2) to investigate the subjects' concept of literal symbols in light of instructional intervention over the past two years;
3) to determine whether computer-oriented problem solving can have long-term effect on the concept of literal symbols.

THE SUBJECTS

Alex

When presented with the task $14 + x = 20$, Alex indicated that the x was "like a box" and represented a number. He also stated that replacing x with a different letter did not affect the missing value. His perception and use of literal symbols in equations were consistent with his behaviors during the final interview in 1985.

Alex could compute the area of a rectangle when given the length and width and was able to write an expression for the area when one dimension was missing. Given a 7 by x rectangle, Alex indicated that the area was 7 times x. He knew that the area could be computed only when x was given a value. During the final interview in 1985, Alex could write expressions for area, but he always tried to estimate a value for any missing dimension.

He remembered the LOGO commands FORWARD, LEFT, and RIGHT. Alex also remembered that, in the statement FD :D, the letter *D* represented a missing number. However, he had forgotten how to use the MAKE command to assign a value to *D*.

Alex was also shown procedures which found a missing number, or generated sequences of numbers. He was aware that the literal symbols represented numbers, that a symbol could represent any of a set of numbers, and that changing the letter did not affect the output.

Alex had not worked with LOGO since the 1985 study, yet he recalled all of the basic LOGO commands and could interpret some procedures. Through discussions with the interviewer, he demonstrated the ability to analyze procedures which used literal symbols to count or solve simple equations.

Dick

In 1985, Dick initially attempted to solve sentences such as $x + 9 = 24$ by using a one-to-one correspondence between the positive integers and the letters of the alphabet. During the final interview, he correctly solved all equations, indicating that the letters represented a number and that changing the letter did not affect the value. In the follow-up study, Dick's concept of literal symbols appeared unchanged since he still solved sentences correctly and indicated the same understanding of the symbols and their use in the context of equations.

Dick could also write expressions for area using letters, although he attempted to estimate the length when given a 4 by *n* rectangle. He soon corrected himself, stating that the expression "4 times *n*" represented the area and that the *n* stood for a missing number.
Dick recognized the LOGO commands FORWARD, LEFT, and RIGHT; he stated that "RT 90" told the TURTLE to turn right 90 de-

grees. He remembered that letters in LOGO procedures represented numbers in memory, but he was unable to use MAKE to assign values. When prompted by the researcher, Dick could analyze LOGO procedures involving recursion or solutions of equations. He could discuss the role of literal symbols, but did not recall the LOGO commands. This is not surprising, since Dick had not written or used any LOGO procedures in two years.

Josh

At the end of the 1985 study, Josh was able to solve equations correctly, even though he had initially used an alphabetic correspondence to find missing numbers. He was aware that literal symbols could be "anything you want" and that using different letters did not change an answer.

Given the sentence 14 + *x* = 20, Josh found the answer by subtracting 14 from 20. He still knew that the letter was used "to put something in." It appeared that his concept of literal symbols in this context had not changed.

During the discussion of a 4 by *n* rectangle, Josh wrote "4 x *n*" for the area, stating that the *n* stood for the width. He then stated that, instead of representing any number, *n* was the number "that would fit for the length." Josh could interpret and discuss the use of literal symbols in counting procedures and in procedures that solved equations, although he did not recall all of the LOGO commands. Josh, like the others, had not been exposed to any LOGO since the 1985 study.

INSTRUCTIONAL INTERVENTION

The researcher interviewed teachers and examined textbooks to determine the role of instructional intervention in the subjects' perceptions of literal symbols. All subjects attended the same elementary school during the fifth grade, where they were each placed in average-ability groups. They used the text *Heath Mathematics-Level 5* (1979), which did not address literal symbols.

Alex and Dick went to different middle schools, where they were again placed in average-ability groups. The adopted text for the sixth grade is *Growth in Mathematics*, (1978). According to Dick's teacher, letters will be covered at the end of this school year, as an enrichment activity. Alex's class had studied equations in the two weeks immediately prior to this study. The teacher explained that the letters represented "missing numbers," and taught the students to solve simple equations, such as $\underline{y} + 35 = 45$ and $2\underline{p} + 49 = 63$. She recalled that Alex, as well as most of the class, scored well on the unit quiz.

Josh is currently attending a private middle school which uses the text *Arithmetic 6* (Howe, 1981). The unit on equations will be covered in a few weeks; consequently, Josh had not received instruction on literal symbols before the follow-up study.

RESULTS AND CONCLUSIONS

Although none of the subjects had used LOGO since the 1985 study, they were all able to recall and use the basic commands, such as FORWARD, LEFT, and RIGHT. They were also able to interpret literal symbols in LOGO procedures. This suggests that the manipulative nature of LOGO, which allows one to model literal symbols in a semi-concrete manner, contributes to the remembering of both the language and its relationships to literal symbols.

All subjects behaved similarly when responding to tasks which involved literal symbols in equations. This is significant since Alex, who had received instruction in this topic only a week earlier, did not appear to use literal symbols any differently than the other two subjects. Furthermore, all three were able to use literal symbols to represent missing dimensions of rectangles when writing expressions for area.

Based on the above facts, it is the conclusion of the author that, at least for the three subjects and in the given contexts, microcomputer-oriented problem solving has a long-term effect on the concept of literal symbols. The results suggest that the computer can be a powerful tool in the development of mathematical concepts and that it can provide concrete models of literal symbols.

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ACQUISITION D'UN LANGAGE GRAPHIQUE DE CODAGE PAR LA MODELISATION EN TEMPS REEL DES DONNEES D'EXPERIENCES.

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Selon Piaget (1979),"la liaison fondamentale constitutive de toute connaissance n'est pas une simple association entre objets car cette notion néglige de fait l'activité due aux sujets , mais bien l'assimilation des objets à des shèmes de ce sujet". L'acquisition des représentations est vue ici comme étant associée au développement intellectuel, elle permet à l'élève, par la manipulation et l'expérimentation directe de son environnement, de se construire un système de représentations initiales. Cette construction, laissée aux aléas de la vie ou à l'imaginaire sera la base de représentations plus culturelles, plus scientifiques. Le passage entre ces deux types de représentations initiales et scientifiques, n'est pas aisé; c'est l'obstacle épistémologique (Bachelard 1967) qui serait franchi par l'assimilation ou le remplacement des vieilles représentations par des représentations plus scientifiques.

Selon Paivio (1979) nous utiliserions deux systèmes symboliques de codage de l'information un système de représentation verbale qui procede de manière abstraite et un système de représentation imagée qui procède à partir d'expériences concrètes. Si la fonction algébrique du premier degré à cause de son caractère abstrait peut être associée au premier système de codage, la représentation graphique de cette même fonction, à cause de son caractère figuratif, pourrait être associée au second système de codage si l'on est capable de permettre son appréhension à partir d'expériences concrètes et non plus à partir d'une représentation algébrique abstraite. Nous devrions alors partir de la manipulation et de l'expérimentation concrètes dans le but de permettre une construction progressive de la représentation graphique (codage visuel), avant de nous servir de cette représentation, élaborée au contact de la réalité, comme support pour comprendre et assimiler l'interaction entre variables en physique.

OBJECTIFS

Les élèves d'initiation aux sciences ont de la difficulté à se représenter l'interaction entre variables de manière économique et efficace. La représentation graphique de cette interaction est certainement l'outil le plus adéquat pour prendre en compte l'ensemble de ces interactions à la condition que celui-ci soit maitrisé et significatif pour l'élève. Or il semblerait que même lorsque cet outil est maîtrisé en mathématique, il ne devient pas automatiquement disponible à l'élève pour résoudre des problèmes de physique . Les élèves en mathématique sont capables de déduire une valeur de y en fonction de x, à partir du graphique ou de la fonction algébrique. Ils peuvent même être capables de déterminer une fonction algébrique du premier degré à partir de ce graphique en isolant deux points sur celui-ci. Par contre,très peu sont capables de l'utiliser efficacement en dehors des mathématiques pour par exemple prédire ou expliciter une interaction entre variables.

En physique la connection entre le phénomène et sa représentation graphique n'est pas meilleure puisqu'il s'effectue à postériori, lorsque l'expérimentation est terminée. La représentation graphique sert alors à synthétiser les résultats expérimentaux en allant du tableau des mesures au graphique, et l'élève pour comprendre et se représenter l'interaction des variables doit reconstituer mentalement le phénomène physique en même temps qu'il vérifie son évolution sur le graphique. Cette façon de faire apprendre l'intéraction des variables avec un graphique où l'action et la représentation sont temporellement séparées est difficile à appréhender pour l'élève. Nous voulons ici proposer une méthode à caractère technologique qui permettrait de présenter l'action et la représentation de celle-ci en simultanéité. Pour évaluer le bénéfice de cette méthode, nous allons la tester avec des élèves n'ayant pas encore étudié l'algèbre et la fonction du premier degré.

Cette représentation graphique initiale serait alors préalable à l'étude de la fonction algébrique du premier degré et devrait permettre à l'élève de mieux assimiler cette représentation au phénomène physique qu'il étudie. Nous sommes concients que cette pédagogie de la représentation graphique, acquise au seul contact de la réalité, semble utopique sans le support algébrique traditionnellement utilisé. Il n'existe pas à notre connaissance de recherche qui permettrait d'appuyer empiriquement cette idée, aussi allons nous construire un système d'apprentissage de la représentation graphique originale pour permettre à l'étudiant d'acquérir celle-ci au contact direct et sensible de la réalité, par la modélisation en temps réel de données d'expériences en laboratoire.

Ensuite, nous mettrons le prototype de ce système d'apprentissage à l'essai avec le double objectif:

1) d'identifier les bénéfices potentiels de cette idée à partir des interprétations des élèves en situation d'apprentissage

2) de vérifier si cette acquisition de la représentation graphique au contact de la réalité permet à l'élève de prédire à partir de ce graphique les intéractions entre les variables, vitesse, distance et temps en cinématique.

Pour conclure nous présenterons quelques pistes de recherche sous forme d'hypothèse afin de valider cette idée.

METHODOLOGIE

Cette recherche comporte une grande part de développement, la modélisation en temps réel des données d'experiences étant effectuée par un ordinateur qui travaillera simultanément en mode conversationnel, en contrôle de procédé et en mode graphique. Nous avons donc effectué ce développement, concrétisé par un prototype permettant aux élèves de provoquer et conceptualiser des interactions de variables en cinématique par la manipulation et le contrôle via un micro-ordinateur d'un train électrique jouet. Ces élèves peuvent alors planifier des expériences, en commander l'exécution et simultanément au déplacement du train, visualiser la représentation graphique de ce déplacement en fonction du temps. (voir NONNON, 1986).

La source des données

Les données de cette expérience proviennent de deux sources différentes, un ensemble traditionnel de tests comprenant un test de prérequis, un prétest et un postest, (un exemple de question est donné en appendice) et une analyse du cheminement de l'élève effectuée à partir de ses diverses manipulations que nous avons conservées dans un fichier.

Caractéristiques des sujets

Les sujets de cette expérience sont des élèves de 5ième et 6ième années du primaire. Le prétest a été administré en classe sur quarante-trois étudiants.

De ces étudiants, 14 furent sélectionnés pour l'expérimentation selon trois critères: 1) une note au moins égale à la moyenne au test de connaissances préalables (notions de temps, de longueur, de mesure constante, 2) une note égale ou inférieure à la moyenne au prétest, 3) une disponibilité pour se rendre 3 fois de suite au laboratoire à l'Université.

Shème expérimental

Chaque élève recevait la consigne et une démonstration sur l'opération du système expérimental par un élève expert. Il manipulait ensuite durant 20 minutes par séance et réalisait 3 séances à raison d'une par jour, au 3ième jour, il était soumis au post-test pour les 6 premiers élèves, la manipulation était laissée à leur fantaisie alors que pour les 8 derniers, nous leur demandions avant la 2ième séance de faire des prédictions sur les mouvements successifs du train.

LES RESULTATS

Les différences entre le prétest et le postest pour les six premiers élèves n'a montré aucune amélioration passant de 36.4 à 39.4 %. En analysant le cheminement de chaque élève, nous avons compté les manipulations (efficaces) qui impliquaient un nouveau couple de paramètre non encore expérimenté. Nous savons qu'il existe avec les paramètres vitesse, distance et temps. Trois couples possibles (distance en fonction du temps, vitesse en fonction de la distance et vitesse en fonction du temps). Ce qui correspond à une séquence de 6 manipulations simples pour définir entièrement l'interaction de ces trois variables. L'indice que nous avons utilisé sera donc de nombre (6) de manipulations optimums divisé par le mombre de manipulations totales effectuées par l'élève.

Si nous effectuons une corrélation entre les résultats obtenus par chaque élève à cet indice d'efficacité et sa performance telle que mesurée par les différences post-test -pré-test, nous obtenons une corrélation de + 0.814 ($r=2.803$, $dl=4$, $p<0.05$).

Pour les premiers sujets, une corrélation significative entre la performance telle que mesurée au test et l'indice de cheminement de l'élève nous permet d'envisager l'utilisation de cet indice comme critère de performance pour l'élève.

Pour les huit derniers sujets, le fait de les obliger à prédire le résultat avant même de commander leur train semble bénéfique puisque chaque élève a augmenté sa moyenne entre le post-test et le pré-test, la moyenne générale passant de 45 % au pré-test à 84 % au post-test.

Ces mises à l'essai empiriques nous ont permis d'analyser notre prototype et ses conditions d'utilisation. Avec les huit derniers sujets, nous avons pu vérifier que tous les étudiants avaient amélioré leur performance dans la prédiction ou l'interprétation d[illegible] interaction entre les trois variables en jeux; les moyennes passant de 45% au prétest à 84% au postest.

CONCLUSION

Nous avons conçu un système d'apprentissasge de l'abstraction par représentation graphique. Ce système laboratoire qui présente en simultanéité l'action et sa représentation permet à l'élève d'acquérir un langage graphique de codage. Ce langage à composante visuelle, qui constitue une abstraction des interactions de variables au laboratoire est acquis au seul contact de la réalité, sans support verbal. Il s'agit bel et bien d'un langage, d'un outil cognitif à la disposition de l'élève, puisque c'est à travers lui, par une syntaxe implicite (de la correspondance, de l'interpolation, de l'extrapolation, de la variation de pente) que l'élève prédira ou interprétera tous les déplacements de son train en fonction d'une quelconque combinaison des variables. Nos premiers résultats sont encourageants, mais nous avons encore beaucoup à faire pour comprendre et maîtriser ce nouvel outil. Nous allons maintenant, pour terminer vous présenter sous forme d'hypothèses un ensemble de recherches que nous sommes en train de planifier pour en assurer une validation.

HYPOTHESE 1

La présentation concomitante de l'action et de sa représentation graphique favorise l'acquisition d'un langage graphique, disponible pour la résolution de problèmes.

Rationnel: la matrîse de ce langage graphique pourrait se vérifier de différentes façons, par exemple en demandant à l'élève de prédire une interaction non encore expérimentée. Un résultat positif indiquerait ici que l'élève appréhende le mouvement du train à l'aide de la fonction graphique, incluant implicitement les concepts d'interpolation et d'extrapolation, ces concepts n'étant pas encore formalisés verbalement chez lui. Cet apprentissage de l'abstraction serait encore plus significatif, le langage graphique aurait plus de cohérance, si l'on pouvait vérifier l'utilisation spontanée par l'élève de ce langagè pour appréhender un nouveau champ de connaissances. On constaterait alors que l'élève a bien intégré cet instrument conceptuel et qu'il lui est significatif en lui facilitant la production d'hypothèses, la planification des schèmes expérimentaux et l'interprétation des résultats.

HYPOTHESE 2

L'acquisition d'une fonction (graphique) du premier degré, telle que décrite dans notre modèle d'enseignement, est plus efficace et transférable que l'apprentissage de la fonction (algébrique) du premier degré telle qu'enseignée traditionnellement.

Rationnel: mis en présence du phénomène concret et de son substitut graphique, tous les deux en évolution conjuguée, l'élève devrait acquérir l'habileté à opérer des transformations réversibles du concret à l'abstrait.

En outre, l'acquisition de cette fonction graphique du premier degré se faisant dans un contexte de laboratoire, la transférabilité de cet outil cognitif dans d'autres domaines d'application concrète, devrait être supérieure à ce qu'on retrouve pour la notion de fonction enseignée dans une leçon de mathématiques.

HYPOTHESE 3

Un système qui permet de planifier et réaliser deux expériences simultanées est plus efficace pour appréhender l'interaction des variables qu'un système qui impose des expériences de manière successive.

RATIONNEL: la difficulté pour l'élève de planifier et d'exécuter un schème de contrôle des variables est liée au caractère séquentiel de la démarche qui exige au moins deux expériences sucessives pour décrire une interaction. La perception directe et simultanée des résultats de ces deux expériences devrait conduire l'élève à mieux en appréhender les différences essentielles, que s'il avait à reproduire de mémoire les conditions et résultats de la premifère expérience pour les comparer à ceux de la deuxième. Les arguments d'encombrement minimum de la mémoire de travail, d'oubli dans le temps des conditions et résultats passés,ou de leur oubli par interférence avec l'activité présente de la mémoire de travail, vont dans le sens de cette hypothèse.

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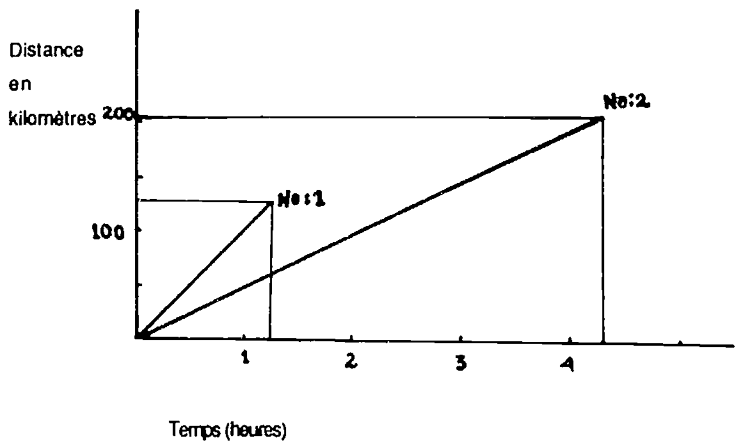
EXEMPLE DE QUESTION

Le train No 1 roule à une vitesse constante pendant 1 heure 25 minutes; il parcourt 123 kilomètres.

Le train No:2 roule à une vitesse constante pendant 4 heures 16 minutes; il parcourt 203 kilomètres.

Quel est le train qui roule le plus vite ?



Encercle la bonne réponse.

A) Le train No:1 roule à la même vitesse que le train No:2
B) Le train No:2 roule plus vite que le train No:1
C) Le train No:1 roule plus vite que le train No:2
D) Le train No:1 roule moins vite que le train No:2

THE REPRESENTATION OF FUNCTION IN THE ALGEBRAIC PROPOSER

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ABSTRACT

Building algebraic functions of adjectival quantity presents the opportunity to represent functions in a variety of ways, at least one of which is both novel and illuminating. The variety of function representations employed by THE ALGEBRAIC PROPOSER, a software environment for algebraic modeling and analysis, is presented and discussed.

If one constructs algebraic functions using quantities that have explicit referents in some external world that one is modeling, then it is possible to represent the functions so generated in some novel and interesting ways. This paper will deal with the variety of representations for functions that are used in THE ALGEBRAIC PROPOSER, a microcomputer based environment for algebraic modeling and problem solving.

Perhaps the easiest way to exhibit the several representations is to work through a sample problem, building and representing functions as we go. Let us consider the following problem.

> Working by himself, person 1 can mow the lawn in 2 hr.
>
> Person 1 and person 2, working together, can mow the lawn in .75 hr.
>
> How long does it take person 2, working by herself, to mow the lawn?

The problem refers to several quantities by value and one quantity by name. These are (1, lawn), (2, hr.), (.75, hr.) and (t, hr.), respectively, where we have used the symbol t to denote the magnitude of the unknown time required for the two persons working together to mow the lawn.

Figure 1 is a prose representation of these quantities (entries A-D) as well as a representation of four other quantities (entries E-H) that are entailed by the original quantities and thus can be thought of as functions of the original quantities. The reader will note that in order to solve the problem, one must constrain the quantity H to be equal to the quantity D.

NOTES

A the job to be done
B mowing time of person 1
C mowing time of person 2
D time for 1 & 2 to mow lawn
E mowing rate of person 1
F mowing rate of person 2
G combined rate of persons 1 & 2
H combined time as function of t
I
J
K
L
M
N

This prose representation is, in some respects, closest to the ways in which people think about situations to be modeled, i.e. in language. While it is richly evocative of the quantities involved, it represents rather poorly, and in some instances, not at all, the relationships among these quantities.

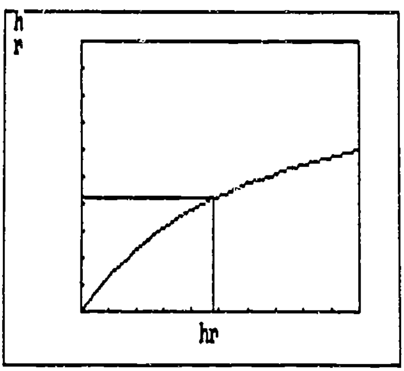
Figure 2 shows the prose representation along with a symbolic representation of the quantities A through H. This representation is, aside from its insistence on the inclusion of the referents of the quantities involved the usual symbolic representation of algebra.

	HOW MANY	WHAT		NOTES
A	1	lawn		A the job to be done
B	2	hr		B mowing time of person 1
C	t	hr		C mowing time of person 2
D	.75	hr		D time for 1 & 2 to mow lawn
E	.5	lawn/hr	A/B	E mowing rate of person 1
F	1/t	lawn/hr	A/C	F mowing rate of person 2
G	.5+[1/t]	lawn/hr	E+F	G combined rate of persons 1 & 2
H	1/[.5+[1/t]]	hr	A/G	H combined time as function of t
I				I
J				J
K				K
L				L
M				M
N				N

The preservation of the referents of the quantities in the representation makes salient the fact that the referent of a function need not be the same as the referents of either the variable or the fixed quantities from which it is composed.

The symbolic representation tends to make the referent quantities less salient while increasing the salience of the relationships among them. Further it represents the relationships with a degree of precision that is totally unavailable to the prose representation.

Figure 3 shows both the graphical and the tabular representations of the quantity H that THE ALGEBRAIC PROPOSER provides. It is in no way remarkable and is presented here only for completeness.



[1/[.5+[1/t]]]

0 < t < 3 hr

0 < VERTICAL < 2 hr

< > ordinates

S change scale

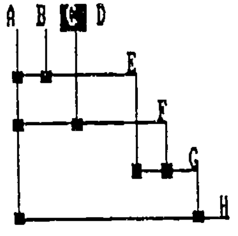
+ - change step size

RETURN

0	undefined
.24	.214
.48	.387
.72	.529
.96	.649
1.2	.75
1.44	.837
t	FUNCTION

Figure 4 shows the prose representation of the both the original and the entailed quantities along with a network representation of the computational dependencies among the quantities. This composite prose and network representation is generated by the software from the user's prose and symbolic representations.

PLAN



NOTES

A the job to be done
B mowing time of person 1
C mowing time of person 2
D time for 1 & 2 to mow lawn
E mowing rate of person 1
F mowing rate of person 2
G combined rate of persons 1 & 2
H combined time as function of t
I
J
K
L
M
N

In particular the representation shows the fact that the quantities A, B, C and D are the original quantities, with A, B and D referred to by value and C referred to by name. It further shows that the quantity E is computed from the quantities A and B, The quantity G is computed from the quantities E and F, etc.

There are several observations to be made about this representation. First of all, it makes salient the ways in which each of the quantities depend on the others. Although it does not represent the binary operations implied by the nodes of the network explicitly, (it could do so at the price of complicating the visual complexity of the network), these may be inferred, often with little difficulty from the semantics of the referents in the associated prose representation.

Second, the network as it stands represents a set of functions. The reader will notice that the network has two "loose ends". These are the quantities H and D. Constraining the network by requiring that these two quantities be equal to one another forces the network to have a solution set. This is a general property of well-posed problems in this representation, i.e. that the equation(s) or inequality(s) that model the problem are formed by constraining the loose ends of the prose-network representation.

Third, the prose-network representation contains no reference to any of the quantities, either original or computed, by value. Thus this representation represents the semantics of the modeled situation without the confounding offered by the particular values. In this sense one may say that the prose-network representation represents an ensemble of problems that have the same structure. This is an attractive notion because it makes possible a discussion of problem types and similarity of problems beyond the surface features normally used to classify problems.

A STUDY OF THE USE AND UNDERSTANDING OF ALGEBRA RELATED CONCEPTS WITHIN A LOGO ENVIRONMENT.

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This paper will present the preliminary results of a three year longitudinal case study of pupils' use and understanding of variable in a Logo programming context. The pupils (aged 11-14) worked in pairs at the computer during their "normal" mathematics lessons. The data consisted predominantly of video recordings of the pupils Logo work and their spoken language. One aim of the research was to relate the pupil's understanding of variable in Logo to their understanding in 'paper and 'pencil algebra, and develop and test out materials designed to help pupils make links between these two contexts. Analysis of the data indicates that most pupils do not naturally choose to use variable in their Logo programming, although with teacher intervention it is possible to find motivating problems which provoke pupils to use variable. Whether or not pupils can make the links between variable in Logo and variable in algebra appears to depend more on the nature and extent of their Logo experience than on any other factor.

Vergnaud has pointed out that "algebra is a detour: students must give up the temptation of calculating the unknown as quickly as possible, they must accept operating on symbols without paying attention to the meaning of these operations in the context referred to" (Vergnaud 1986). He quite rightly says that we must find problems which provoke the use of algebra. This is not an easy task in 'traditional' school mathematics. The computer programming context however does provide problem situations in which variable is a meaningful problem solving tool. It seems appropriate therefore to consider the ways in which the computer can enhance the learning of mathematics, and in particular, as far as this study is concerned, the learning of algebra. We have been investigating, as part of the Logo Maths Project (Sutherland, Hoyles 1987) the hypothesis that certain programming experiences in Logo will provide pupils with a conceptual basis of variable which will enhance their work with 'paper and pencil' algebra.

Ethnographic research methodology was chosen as being the only one possible in an area where technology, pedagogy and the approach to mathematical content were all innovatory. Longitudinal case studies were undertaken for four pairs of pupils (aged 11-14) programming in Logo during their 'normal' mathematics lesson throughout the three years of the project. As researchers we acted as participant observers in the classroom. Pairs where chosen to take into account the spread of mathematical attainment and the teachers' opinions on constructive working partnerships. The data included recordings of the pupils' Logo work, all the language spoken by the pupils (a video recorder was connected between the computer and the monitor), the researchers interventions and a record of all other mathematical work undertaken by the pupils. The video recordings were transcribed and these were combined with researcher observations and teacher and pupil interviews to provide the basis for the research.

At the beginning of the research period, pupils were given the freedom to devise their own goals in order to build up self confidence and autonomy. Our aim was to make interventions related to the idea of variable when appropriate. Analysis of the transcript data at the end of the first year of the research indicated that most pupils did not naturally choose projects for which variable was a functional problem solving tool. It was decided therefore to develop teacher devised tasks which provoked the pupils to use variable. Throughout the second and third year of the research pupils were given a range of teacher devised tasks. One particular task, which used the idea of changing a fixed procedure to a general procedure by scaling distance commands, provided an important starting point.

One aim of the research was to develop materials to help pupils make the links between variable in Logo and variable in 'paper and pencil' algebra. It was decided to base these materials on the similarity between using variable to define a function in Logo and using variable to define a function in algebra (for a fuller discussion of this see Sutherland, 1987). For example the Logo representation:

```
FUNC :x
OUTPUT :x+4
END
```

is equivalent to the algebra representation $FUNC(x) = x+4$ or $x \rightarrow x+4$

The pupils were introduced to these ideas in the form of a game which involved one pupil defining a function and the other pupil predicting the function by trying out a range of inputs. The "guesser" had to define the same function when she was convinced that her prediction was correct. The pupils then had to establish that both functions were identical in structure although the function and variable names used might be different (pupils were encouraged to use a range of variable names including single letter names). It has been reported by Wagner(1981) that in algebra pupils often have the misconception that changing a literal symbol implies changing what the symbol refers to. In this Logo task we were specifically building in the experience that this is not the case.

Categories of variable use were derived from the transcript data and these provided a framework for analysis. (Sutherland, 1987) An overview of the pupils' use of variable analysed according to these categories throughout the three years of the project is presented in Table 1.

Table 1 OVERVIEW OF GENERAL PROCEDURES WRITTEN BY CASE STUDY PUPILS.

CATEGORY OF USE	Pupil 1 SALLY	Pupil 2 ASIM	Pupil 3 GEORGE	Pupil 4 JANET	Pupil 5 JUDE	Pupil 6 RAVI	Pupil 7 LINDA	Pupil 8 SHAHIDUR
(I) One Input	4	2	2	4	3	0	3	1
(S) Input as Scale Factor	3	3	3	3	3	3	6	7
(N) More than One Input	3	2	2	3	0	0	2	0
(O) Input Operated on	6	5	5	6	0	0	0	0
(G) Input to General Superprocedure with Variable Subprocedure	3	3	3	3	2	0	3	0
(F) Input to Mathematical Function	2	2	2	4	1	2	3	2
(C) Input used in Conditional Expression	0	2	2	0	0	0	0	0

All the pupils used input to a Logo function (category F) as part of the Logo/algebra linking materials presented to the pupils in the eighth term of the project. Apart from this Ravi and Shahidur's use of variable was entirely restricted to category S (scaling a distance command). In this context they realised that the variable used affected the size of the object on the screen but they were not necessarily aware of the variable processes within their procedure. Pupils have been ranked (pupil 1 - pupil 8) according to attainment on their school mathematics scheme. Ravi Jude and Shahidur's more limited use of variable was a consequence of them a) being case study pupils for a shorter length of time than the other pupils b) having a higher absence rate than the other pupils and the teacher being consequently more reluctant for them to spend time on Logo work. In choosing to carry out research in a "normal" classroom over a period of three years we had to accept that for reasons beyond our control the pupils were not always available for a "planned" session. However the nature of the transcript data is such that it is possible to reconstruct, for all the case study pupils, the nature of their Logo experience in terms of pupil collaboration, teacher intervention, and computer input and output.

In order to probe the case study pupils' understanding of variable in both Logo and algebra they were all given individual structured interviews at the end of the the three year period of research. They were asked to:

- Make a generalisation and formalise it in an algebra context.
- Make a generalisation and formalise it in a Logo context.
- Answer algebra questions related to the meaning of letters taken from the C.S.M.S[1] project.
- Answer Logo questions related to the meaning of variable names.
- Represent a function in both Logo and algebra.

In addition pupils visited the University laboratories to carry out individually tasks devised to probe their understanding of variable in Logo.

For the purposes of this paper the pupils' understanding of variable in Logo and algebra will be categorised in the following way:

- Acceptance of the 'idea of variable.
- Understanding that a variable name represents a range of numbers.
- Understanding that different variable names could represent the same value.
- Acceptance of "lack of closure" in an expression.
- Ability to establish a second-order relationship between variables.
- Ability to use variable to formalise a generalised method.

Evidence for the pupils understanding of variable in Logo was derived from the structured interview items, the transcript data and the University day tasks, while the understanding of variable in algebra was derived from the structured interview data only.

Acceptance of the idea of variable was deemed present if the pupils were willing to begin to attempt the structured interview questions. All the case study pupils accepted the idea of variable in Logo. None of the pupils had had any experience of algebra before using variable in Logo. Throughout the project the pupils followed their "normal" mathematics curriculum and the type and amount of algebra work carried out by the pupils was not in the control of the researchers. However we know that four of the pupils, Linda, Jude, Shahidur and Ravi, did not carry out any formal algebra work during the period of research. Six out of the eight case study pupils accepted the idea of variable in algebra. The two pupils, Jude and Ravi, who did not respond positively to any of the algebra related categories have both had a very limited experience of variable in Logo (Table 1).

All the pupils accepted the idea that a variable name in Logo represents a range of numbers. Again all except for Ravi and Jude have carried this understanding to the algebra context which contrasts with previous research findings that pupils often regard a letter in algebra as representing an object or a specific unknown. (Booth 1984, Küchemann 1981).

What is the area of this shape?
A =.....................

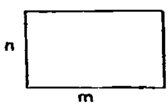
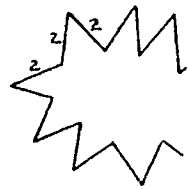


Fig 1

Part of this figure is not drawn. There are n sides altogether all of them length 2.



What is the perimeter of this shape? p=............

Fig 2

Jude attempted to use his Logo understanding of variable in the algebra context when answering a C.S.M.S item (fig 1) but his idea of "any number" soon became confused with "anything" as the following example illustrates.

Jude	*"Does M mean any number miss?"*
Researcher	*"M is any number and N is any number."*
Jude	*"So I can just put anything I want."*

Shahidur had some difficulty with the C.S.M.S item "If John has J marbles and Peter has P marbles what could you write for the number of marbles they have altogether?" and his response indicates the transitional nature of his understanding. Writing down 9 as the solution he gave the explanation:

Shahidur	*"Cos John begins with J and there's four letters in John and Peter begins with P and there's five letters in Peter."*
Researcher	*"Why did you think P stands for 5 ?"*
Shahidur	*"Because I was wondering why they should put J and P."*
Researcher	*"What if they were called Q and R?"*

With this suggestion he immediately wrote down Q + R. Shahidur is not an algebra experienced pupil and his mathematical ttainment is very low. Under these circumstances his responses to the CSMS questions are quite extraordinary. When presented with the perimeter question (fig 2) he wrote down 2 x n's as a solution. When asked to explain his solution he said *"Cos there's the size of them are 2....and there are n's of them ...so 2 times n will be the answer"*.

In order to test the pupils' understanding of whether or not a different variable name can represent the same value they were given the following Logo and algebra questions(not consecutively)(Fig 3).

When is the following true ?

L+M+N=L+P+N

Always. Never. Sometimes,when..................

When do these Logo commands draw the same length line?

```
TO LINES "L "M "N "P
FD :L  FD :M  FD :N
RT 90
FD :L  FD :P  FD :N
END
```

Always. Never. Sometimes,when.......................

Fig 3

Only Sally responded positively to both items but four out of the eight responded positively to the Logo item. When we relate these pupils' understanding to their use of variable in Logo we find that all four have used more than one input to a procedure and have in this context given different variable names the same value. This contrasts with the four pupils who did not respond positively to the Logo item and who had never used more than one input to a procedure.

All of the case study pupils accepted lack of closure in a Logo expression All apart from Jude, Ravi and Linda accepted the idea in algebra. Previous research indicates that this is often a problem for pupils learning algebra (Booth 1984). The case study pupils had used 'unclosed' Logo expressions involving variable in their function machine work.

None of the case study pupils could answer either the C.S.M.S algebra question "Which is the larger 2n or n+2? Explain ...." or the Logo related question correctly. Küchemann maintains that "An important feature of these relationships is that their elements are themselves relationships, so they can be called *'second order'* relationships" (Küchemann 1981). He maintains that it is only when pupils have grasped this notion that they have fully understood the idea of variable. Analysis of the data indicates that none of the pupils had carried out any Logo tasks related to this idea. Although the C.S.M.S question itself can be criticized this result does suggest that further Logo tasks related to this idea need to be devised and that more adequate test items also need to be developed.

Evidence from algebra research suggests that pupils often use use informal methods which cannot easily be generalised and formalised. However in this project pupils were able to interact with the computer and negotiate with their peers so that their intuitive understanding of pattern and structure was developed to the point where they could make a generalisation and formalise this generalisation in Logo. All the case study pupils could formalise a method generalised by them in Logo. However the non-algebra experienced pupils were not able to use algebraic notation to formalise a method generalised by them in the algebra context.

This paper has highlighted the extent to which the pupil's understanding of variable in algebra is related to their use of variable in Logo. The evidence suggests that pupils can use their Logo derived understanding in an algebra context. Possibly one of the most important aspects of the function machine material in helping the pupils to make links was that it provoked the pupils to use a range of variable names, including single letter names.

Footnote 1
As part of the research programme "Concepts in Secondary Mathematics and Science" just under 1000 secondary pupils aged 14+ were tested on their understanding of algebra (generalised arithmetic) (Kuchemann 1981).

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COMPUTER PRESENTATIONS OF STRUCTURE IN ALGEBRA

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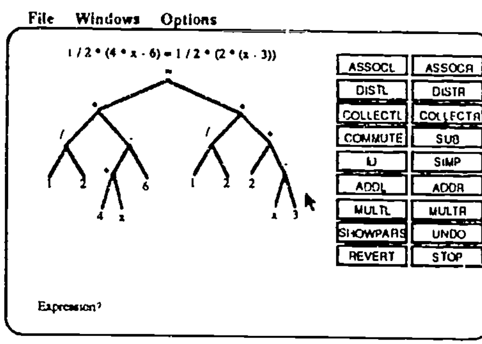
Many errors committed by students of algebra appear to be a result of their long-term inattention to structure of expressions and equations. A special computer program was developed that enabled students to manipulate expressions, but which constrained them to acting on expressions only through their structure. Eight leaving-seventh graders used the program for eight days. An analysis of their actions indicated that errors due to inattention to structure occurred largely while they were first learning a field property or identity, and that afterwards such errors were infrequent.

Typical errors found in previous studies of students' errors in algebra suggest that students studying algebra frequently fail to realize that formulas in mathematical symbol systems have an intrinsic structure (Lewis, 1981; Matz, 1982; Sleeman, 1982, 1984, 1985). In algebra, expressions are structured explicitly by the use of parentheses, and implicitly by assuming conventions for the order in which we perform arithmetic operations. It is hypothesized that many of students' errors in manipulating an algebraic expression are due to their inattention to the expression's structure.

To test this hypothesis, we built a program, called EXPRESSIONS, that presents expressions and equations in two formats: in usual (sentential) form and in the form of an expression tree. The figure to the right shows the screen after having entered the equation 4x–6=2(x–3) and then multiplied both sides by $\frac{1}{2}$. The equation's expressions are shown in sentential notation at the top of the screen. The tree representation of the equation is shown directly below the sentential notation.



To change an expression by the use of a field property or other transformation, students put the mouse pointer on top of one of the buttons along the right side of the screen and then click the mouse to select that action. Then they put the pointer on top of the operation <u>in the tree representation of the expression</u> which defines the expression or subexpression to be transformed, and click the mouse again. The action is performed on the selected expression or subexpression, and the sentential notation and expression tree are changed accordingly.

To transform an expression by the use of an identity, students put the mouse pointer on top of the ID button, click the button, and then click the operation sign within the tree which defines the expression to which the identity is to be applied. The computer will apply one of the identities $a-b = a + {}^-b$, $\frac{x}{y} = x * \frac{1}{y}$, ${}^-x = {}^-1*x$, or $x = 1*x$ to the chosen expression or subexpression, then update the expression tree and sentential display accordingly.

Sample

The sample consisted of eight leaving-seventh graders—six males and two females—from the ISU elementary laboratory school and who volunteered to participate in the study. Their mean age was 13 years 1 month; their mean cumulative mathematics score on the Iowa Test of Basic Skills was 74.6. In the last quarter of seventh grade mathematics, five students received an A, one received a B, and two received a C.

Method

The study took place over nine consecutive weekdays in June of 1986. The first session was devoted to administering a pretest; eight sessions (50 minutes each) were given to direct instruction and practice. The pretest involved assessing students' knowledge of the conventions for order of operations (evaluating numeric expressions), their knowledge of field properties, and their knowledge of variables.

Instruction took place in a classroom at ISU, where the instructor used a Macintosh running EXPRESSIONS. The Macintosh was connected to a projector which created a 6' x 6' image of the screen. All instruction was videotaped.

For practice sessions, students were grouped in pairs by matching their cumulative mathematics score on the Iowa Test of Basic Skills. Practice sessions took place with students in two locations: in a computer room and in the classroom, with two students per computer. Students using the classroom computer were videotaped. Each pair of students was videotaped once. A set of booklets containing examples and practice problems were provided to each student. All students used a version of the program that stored their keystrokes and mouse-clicks in a data file which could be "played back" for later analysis.

Instruction proceeded in this order: order of operations in arithmetical expressions; field properties as transformations of arithmetical expressions; identities and derivations. An outline of the eight days of instruction is given in Table 1.

Day	In Class	In Groups
2	Order of operations; Evaluating expressions	Worksheet 1 (Parts 1 & 2)
3	Parentheses; Expression trees	Worksheet 1 (Part 3); Worksheet 2
4	Discuss Worksheet 2; Commutativity; Associativity; Example from Worksheet 3	————————
5	————————	Worksheet 3 (Part 1)
6	Review commutativity; associativity; introduce distributing and collecting	Worksheet 3 (Part 2)
7	————————	Worksheet 3 (Part 2); Worksheet 4
8	Review field properties; introduce identities	————————
9	————————	Worksheet 5

Table 1. Summary of instruction

The worksheets comprised an integral part of instruction. Table 2 shows the numeric-transformation problems students worked in sessions 6 and 7. Table 3 shows the identity derivation problems students worked in session 9.

	Start With	Change It To
N1.	5*(4+3)	3*5 + 4*5
N2.	5*((4+3)+2)	(5*4)+((2+3)*5)
N3.	(7+3)*(6+5)	(7*6+7*5)+(3*6+3*5)
N4.	(6+$^{-}$5)*(6+5)	(6*6) + ($^{-}$5*5)
N5.	3*(8+4) + 9*(4+8)	(9+3)*(8+4)
N6.	3*(6/9) + (6/9)*7	10*(6/9)
N7.	$^{-}$5*3 + (2+3)*5	0 + 10
N8.	(5+9)*(5+9)	5*5 + 90 + 9*9

Table 2. Numeric transformation problems.

	Start With	Change It To
I1.	(z - q)*u	z*u - q*u
I2.	r*(s/t)	(r*s)/t
I3.	-(p + q)	-p + -q
I4.	(a + b)/c	a/c + b/c
I5.	6x + x	7x
I6.	5x - x	4x
I7.	x + x	2x

Table 3. Identity derivation problems

Results

Pretest

Six of the eight students processed numeric expressions from left to right, ignoring conventions for order of operations (e.g., 8 - 6 + 5 * 3 evaluates to 21), when grouping was not given explicitly. All eight were familiar with commutativity. Seven were familiar with associativity in its simplest form. None was familiar with distributivity. Six differentiated among expressions and equations on the basis of superficial characteristics (e.g. "y+2=5 is different from x-2=5 and x+2=5 because it uses y and the others use x.")

Analysis of computer use

EXPRESSIONS was modified to store all interactions. The stored files were then later rerun for analysis. Students' actions were categorized according to the following scheme:

- A — Appropriate transformation applied at an appropriate place in the expression, given the current and goal expressions.
- IA — Inappropriate transformation, e.g. trying to use the distributive property on $(a*b)+c$.
- AWP — Appropriate action, but applied in a wrong place. This was inferred if a student tried the same transformation twice in a row, first trying it at an inappropriate place in the expression and then applying it appropriately.
- CD — Confused direction. An action was placed in this category if a directional transformation was appropriate (such as using the associative property of multiplication to change the grouping from being on the left to being on the right) but the student chose the wrong direction.

Transforming Numeric Expressions

Table 4 shows the percents of students' actions falling within each category while working the numeric transformation problems (Table 2). Table 5 shows the percents of students' actions falling within each category while working the identity derivation problems (Table 3).

Problem	*A*	*IA*	*AWP*	*CD*
N1	87	0	0	13
N2	75	25	0	0
N3	69	9	17	6
N4	60	31	8	1
N5	82	0	18	0
N6	88	0	0	12
N7	56	39	5	0
N8	89	11	0	0

Table 4. Numeric transformations: Percent per category of all actions. All students completed all problems.

Problem	*A*	*IA*	*AWP*	*CD*
I1	41	48	6	6
I2	*	*	*	*
I3	*	*	*	*
I4	69	23	0	8
I5	56	38	6	0
I6	81	12	6	0
I7	80	20	0	0

Table 5. Identity derivations: Percent per category of all actions; "*" indicates incomplete data .

In many cases, the majority of inappropriate actions occurred early in a problem, suggesting that students were exploring the effects of the available transformations upon expressions. To eliminate the effects of exploratory errors upon the percents in Tables 4 and 5, the data were reanalyzed by the same categorization scheme as previously, but with this exception: All actions prior to two consecutive appropriate actions were discarded. Tables 6 and 7 show the percents of "non-exploratory" actions falling within each of the categories.

The differences between Tables 4 and 6 and between Tables 5 and 7 suggest that students' errors were due to initial play involved in understanding the problems, understanding the available transformations, and making connections between the two. Once students internalized the transformations' structural constraints, they were less likely to commit errors and were more efficient in their solution strategies.

Problem	A	IA	AWP	CD
N1	100	0	0	0
N2	95	5	0	0
N3	85	7	4	4
N4	92	4	0	4
N5	82	0	18	0
N6	88	0	0	12
N7	100	0	0	0
N8	94	6	0	0

Table 6. Numeric transformations: Percent per category of non-exploratory actions.

Problem	A	IA	AWP	CD
I1	58	39	3	0
I2	*	*	*	*
I3	*	*	*	*
I4	86	0	0	14
I5	100	0	0	0
I6	100	0	0	0
I7	100	0	0	0

Table 7. Identity derivations: Percent per category of non-exploratory actions; "*" indicates incomplete data.

Exploratory errors were commonly either irrelevant to the problem being solved (e.g., "what does this button do?") or were attempts at doing something that might take an expression closer to its goal state. For example, one error was to try to use associativity to change $(a+b)*c$ into $a+(b*c)$, to which the computer "responded" by doing nothing. The students wanted b to be multiplied by c, and apparently concluded that the associative property would do that regrouping for them. Also, it was common for students to repeat an errorful action. It appeared that repeating an action supported students in their attempts to reflect on the reasoning they used in first choosing the action, and supported them in understanding the reason that the chosen transformation did not accomplish whatever they had in mind.

Discussion

Previous studies of students' errorful manipulation of expressions and equations proposed that their errors are due to mal-formed rules—perturbations of correct rules. This study asked whether or not such errors were due to students' inattention to structural features of expressions and transformations thereupon. The results suggest that mal-rules need not be a natural occurrence when students operate in an environment that supports explicit attention to expressions' structures, and where structure also imposes constraints on students' actions. We cannot say from the results presented here that errors reported in previous studies were due to students' inattention to structure, but these results indicate that attention to structure is an important consideration.

Students could attempt errorful transformations of expressions while using the computer, but the computer would not carry them out. It appeared that they interpreted this context as one where experimentation became natural and beneficial. We would like to think that students disposition to experimentation was a result of the software and the use made of it. However, it also could have been a result of the instructor's style of instruction, or it could have been that this particular group of students was predisposed to experimentation and reflection.

A limitation of the study is that students were not assessed outside of the computer environment. It is quite conceivable that had these students been left to their own devices, they would have committed errors on paper and pencil that they learned not to make while using the computer. The issue of transfer from computer to noncomputer environments requires extensive research.

Another limitation of the study is that we do not know the depth of commitment that these students had when they "proved" that two expressions were equivalent, or when they derived an identity. Did students think of an identity as a theorem that could be applied in other contexts? We do not know.

A feature of structure which we could not address here with data, but which was addressed explicitly in the study, was that of variable. Many problems (all of those in Tables 2 and 3) were designed so that students would have to treat a subexpression as a unit. When applying field properties and identities to expressions, students regularly needed to substitute a subexpression in an expression for a letter in the canonical statement of a property or identity. They became quite adept at this. Also, students felt no discomfort when letters were first introduced in to-be-transformed expressions. Apparently, by having them transform numerical expressions, they became used to the idea that expressions could be manipulated regardless of their constituent elements. Thus, when letters were introduced, students saw no obstacle in continuing what they had already learned to do with numerical expressions. The approach wherein manipulating algebraic expressions is presented as a natural extension of manipulating numerical expressions deserves further research.

The use of expression trees as one of the representational systems within the computer program proved to be a positive feature of instruction. Students found expression trees to be quite intuitive. When doing Worksheet 1, which focused upon evaluating expressions given in sentential form, students used EXPRESSIONS only to check their answers. They were told only that they needed to click SIMP and then click the top of the tree to evaluate an expression. We found four students who constructed expression trees for complex expressions as an aid to evaluating them, *even though there had been no dis-*

cussion about how expression trees are constructed, and these students had never before seen an expression tree.

Finally, it should be noted that in eight days of instruction these leaving-seventh grade students went from essentially no working knowledge of order of operations to deriving algebraic identities, and did so with some depth of understanding. Even with the limitations stated earlier in this discussion, the fact that such coverage is possible makes us question assumptions that are built into traditional junior high school pre-algebra and algebra curricula about what one can expect of junior high school students in the United States.

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THE EFFECTS OF MICROCOMPUTER SOFTWARE ON INTUITIVE UNDERSTANDING OF GRAPHS OF QUANTITATIVE RELATIONSHIPS

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ABSTRACT

In this paper we describe a software which we designed to help develop intuitive readiness for the encounter with graphs of linear relations. A study was conducted to evaluate the effects of the software on 7th grade pupils (experimental group n=78, control group n=67). Pupils demonstrated intuitive understanding of graphical solutions of linear equations and inequalities. Eight months later a retention follow-up study was applied to the same pupils (now in the 8th grade), just before they started the study of graphs of linear equations. Although the software seems to have been only moderately effective, retention of what was learnt was good.

Whereas most junior high school students successfully read and plot points, they have difficulties in understanding the relations between the two coordinates of points. For example, Hart (1980) found that the relation between straight lines and their equations was understood by only 5-30% of students (depending on age). Some of the difficulties that junior high school students experience in the study of graphs of quantitative relationships, may be due to the necessity for a higher degree of generalization and abstraction than that they have met previously. Butler (1970) maintains that the difficulty may decrease if we teach in such a way that learning activities become intuitive ingredients of future concepts and relations.

The role intuition plays in developing a true understanding of mathematics is emphasized by Kline (1971), Fischbein (1978) and many others. Kieran (1981) investigated how students intuitively extend their existing knowledge in relation to algebraic notions and Dreyfus & Eisenberg assessed the intuitive background of junior high school students as they developed the concept of function. They agree with others that enlarging the base of intuition is a primary goal of education.

The Science Teaching Department at the Weizmann Institute maintains a curriculum project in mathematics for the junior high school. In our program, as in others, we observed students' difficulties with the concepts of graphs of truth sets which are dealt with in Grade 8. It seems that reading and plotting points in Grade 7 gives some familiarity with the coordinate system, but it does not prepare the students for the encounter with the more abstract concepts in Grade 8. The idea was to lead pupils to intuitive understanding of relations between the two coordinates of a point, as a part of the introduction to the coordinate system. We felt that the microcomputer could be more efficient than other media in achieving this aim. Green Globs by Sharon Dugdale (1984) is an exemplary piece of software. In a gaming environment students develop good sense of the relation between the algebraic and graphical representations of functions. In this paper we shall describe another software, Dots and Rules, which we designed to help develop intuitions on graphs of linear relations.

DOTS AND RULES

Dots and Rules offers activities which teach the two-way transfer skill: point ↔ rule. The pupil has to identify which points fit a given rule or which rule fits a given point. All rules are linear and when the student finds among the given points, all those satisfying a particular rule, the picture of the straight line on which these points lie clearly emerges. Visual elements like shapes and colors are used to illustrate the relation between the rules and linearity (see Figure 1, without the colors...).

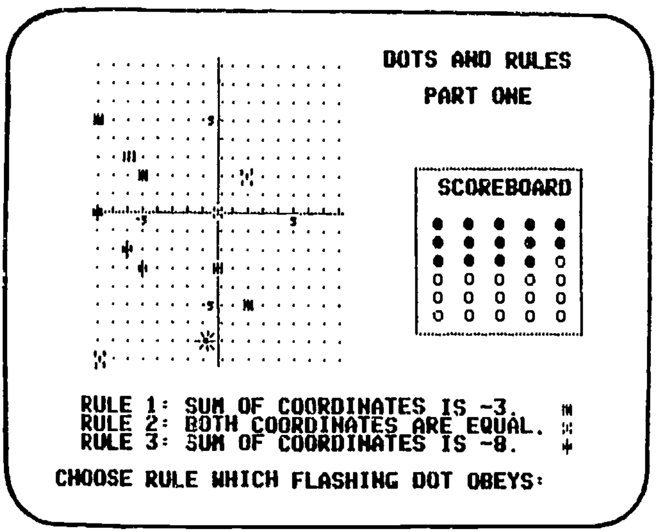


Figure 1

The tasks are imbedded into various activities, from a tutorial on the basic tasks to a competitive game for two players (see Figure 2).

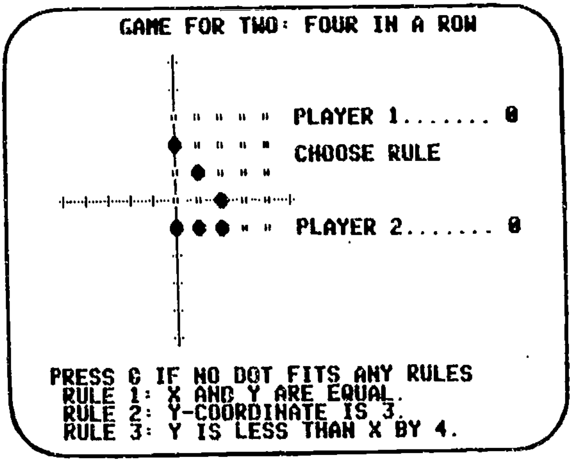


Figure 2

If player 2 chooses rule 3 and identifies the point (3, -1), he completes two "fours" and scores 2 points. While playing, the pupils realize that the point (3, -1), for example, can be caught by other rules as well (e.g. y - coordinate is -1; the sum of the coordinates is 2). The software is not intended to teach the explicit relation between straight lines and their equations, rather its aim is to create some rule-based orientation in the coordinate system, which will provide the intuitive preparation for the introduction of graphs of linear open sentences.

The use of a microcomputer has some didactic advantages; it enables the student to practice different rules with the same pattern, for which the points appear in various parts of the coordinate system. There are, of course, pedagogical advantages like challenge, motivation and feedback in the use of the microcomputer.

In the following we describe the method and results of a study that investigated the effects of Dots and Rules in terms of its aims.

METHOD

Two junior high schools were involved in the study. The schools are located in neighboring urban suburbs having similar socio-economic populations. But schools are of about the same size and use similar criteria for streaming their pupils. Three grade 7 classes with average ability students, from each school participated in the study. The classes in one school formed the experimental group and the others were the control group. A three part questionnaire was prepared, some of the items were in a multiple choice form and others more open.

The first part contained 6 items which test familiarity with the coordinate system. In addition to items on reading and plotting of points, we asked for some generalization; e.g., to identify a property of points on the x or y axes.

The second part contained 7 items and tested the transfer skill: point $\leftrightarrow$ rule, which was explicitly dealt with by the software. For example in item 10 we asked the pupils to identify rules satisfied by the origin (0,0), from the following list of rules:

(a) the coordinates are equal

(b) x is greater than y by 3

(c) the sum of the coordinates is 3

(d) y - coordinate is 4

(e) y is twice x.

The third part contained 13 items which go beyond the scope of the explicit activities of the software and test intuitive rule-based orientation in the coordinate system. For example, in item 20 we asked the pupils to identify all grid points which have the two properties: (a) the coordinates are equal, and (b) the sum of the coordinates is 2. The Kuder-Richardson reliability index for the whole questionnaire is 0.91, for sub1 - 0.67, for sub2 - 0.79 and for sub3 - 0.87.

In March 1986 the software was used by the experimental classes in parallel with the regular introduction to the coordinate system. Treatment of the transfer skill point $\leftrightarrow$ rule was given to the control group without the computer. Then, the questionnaire was given to both experimental and control groups, and the results compared. Eight months later (November 1986) the questionnaire was applied again to the same classes (now in the 8th grade), just before they started the study of graphs of linear equations and inequalities. The results were compared for the experimental and control groups and also with the previous results.

RESULTS AND DISCUSSION

The mean scores (maximum 100) for the whole questionnaire and for the three subquestionnaires are given in Table 1.

	Experimental group		Control group	
	post treatment (n = 78)	retention (n = 67)	post treatment (n = 77)	retention (n = 74)
Total	76	77	71	63
Sub1	82	85	85	76
Sub2	83	82	67	58
Sub3	71	69	66	59

Table 1.

The results are about the same for the two groups on Sub1 at the first application, indicating that average ability is about the same. Although a significant drop in retention occured for the control group we hesitate to draw conclusions since there were only 6 items in Sub1. As anticipated there are significant differences in favor of the experimental group on Sub2. However, the most important finding is in the results of Sub3 which tests the main goal of the software. Although the software seems to have been only moderately effective, retention of what was learnt was good.

To illustrate difficulties which were only partly overcome by the software, we bring the findings for item 10 of Sub2 mentioned above. More than 95% knew that (0, 0) fits the rule "$x=y$", in both groups and both applications. But only 44% of the experimental group realized that it also fits "$y=2x$" in the post-treatment test, and 50% in the retention test. As for the control group, only 8% (!) responded correctly in the first testing and 30% in the second. The "improvement" can be due either to the fact that we used the same test, or to students' experience with graphical presentation of practical situations at the end of Grade 7.

To illustrate the development of some intuitive rule-based orientation, we bring here the results for three items of Sub3.

Item 20 (see above):

A response was considered correct if the point (1, 1) was identified in the the graph.

Correct responses (%):

Experimental group (grade 7) - 64

Control group (grade 7) - 42

At the second application,

Experimental group (grade 8) - 75

Control group (grade 8) - 55

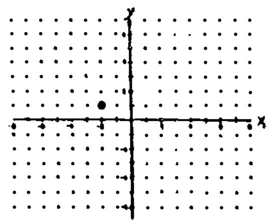
The increase in correct responses from grade 7 to grade 8 is due to the use of some algebra by the latter.

Item 18:

State two properties of the marked point.

(a) ______________

(b) ______________



The results were about the same in the two applications of the test, with a clear advantage to the experimental group. In the control group more than 25% gave only one rule; the common first rule was "$x=-2$" and the second was "$y=1$". In the experimental group less than 10% gave only one rule. The above responses were given by about 30%, and the rest stated more "interesting" rules. The most popular were: "the sum of the coordinates is -1", and "y is greater than x by 3".

Item 25:

In each quadrant in which it is possible, mark a grid point for which the sum of the coordinates is greater than 5.

The given coordinate system was 8x8 (with the origin at the center) and a response was considered correct if correct points were marked in quadrants I, II and IV.

Correct responses (%):

Experimental group (grade 7) - 75

Control group (grade 7) - 61

At the second application,

Experimental group (grade 8) - 73

Control group (grade 8) - 57

In addition, about one fifth of the experimental students attempted, without being asked, to give a full graphical solution to the given inequality.

CONCLUSION

The results indicate that software of this kind can be effective in achieving its main goal - creating intuitive readiness for future concepts. Students related a point to several rules, they "saw" lines and illustrated graphical solutions of quantitative relationships. More software with similar goals for other topics in algebra has been developed. It is likely that the increased use of this media will affect approaches, teaching strategies and the organization of the course.

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Algebra in a Computer Environment

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This reaction has been commissioned by the PME XI organising commitee to consider the contribution of each paper grouped under this heading, to seek common threads, to formulate major questions that still need to be answered and to look for indications in the papers as to how these questions might be tackled. The task is a daunting one. It is rather like attempting to put together a jigsaw puzzle whose pieces were not created to fit together in a master plan, each with a life of its own. It is a problem-solving activity and I shall approach it in a problem-solving spirit. In doing so I should like to acknowledge the help given me by Michael Thomas in formulating this reaction.

1.The contribution of the papers to the research area

The papers grouped under "algebra in a computer environment" range widely from initial ideas in the subject to the graphical representation of algebraic functions, and some expand the domain to more general functions and analytic relationships between variables and their rate of change in "feed-back systems". Although these would not all be classified *mathematically* within algebra, they *cognitively* embrace algebra concepts, beginning with the translation from real world problems to algebraic notation, with its surface syntax and underlying semantic structure, linking with relationships to other representational systems.

The papers also represent very different stages in the research process which are fruitfully considered from a problem-solving viewpoint, passing through various phases after the style discussed by Mason *et al.* (1982). An initial **entry** phase gathers together what is known, what one wants to know, and what tools one might assemble in preparation for the **attack** phase where the empirical work is done. This may result in an impasse or a significant gain, when it becomes appropriate to **review** and refine what has been achieved before either re-entering the problem for a different attack, or **extending** the work in new areas through a new a spiral of entry, attack and review.

Some of the papers have completed a full research cycle, others describe only part of a longer span, for instance, the entry phase to new research, reviewing the literature from earlier phases, proposing theories and setting out plans of attack.

Boileau *et al*, are beginning a new phase of attack in **"La Pensée Algorithmique dans l'initiation à l'algebra"**. They propose to start the study of algebra with activities that are "both significant and motivating to the student", "coding problems ... relating to the students' prior experience" by providing a "tailor-made programming language which will serve as an intermediate representation ... between the problem to be solve and the final coding." They formulate some characteristics of the environment but stop short at giving information as to the state of the development of the system or any empirical testing. Their distinction between the*syntactic*, *internal semantic* and *external semantic* aspects of algebra is one which may prove a useful link with other papers.

In "**Believing is seeing: How preconceptions influence the perception of graphs**", Goldenberg begins an entry phase, based on experiences using computer software and preliminary observations with two "bright, successful, second year algebra students". He leads into a discussion of "how perceptual illusions and shifts of attention from one feature to another obscure some of what educational use of graphs is supposed to illucidate", particularly where the representation lacks familiar perceptual clues, thus raising some concern as to the efficacy of certain aspects of multiple linked represention s.

Thompson & Thompson introduce some significant new software in "**computer representations of structure in algebra**", linking an algebraic expression ot its tree structure allowing free mixing of numbers and letters. They have made an intial empirical attack with a week's instruction/exploration of the software with eight seventh-grade students. They report that the students "felt no discomfort when letters were first introduced in to-be-transformed expressions" and that, after an initial period of experimentation, errors due to inattention to structure were infrequent.

Judah Schwartz also has a reputation for producing innovative software and his paper on "**the representation of function in the algebraic proposer**" is no exception in this respect. The original proposal had hoped to include empirical research with 12 college freshmen, but, in the event, the paper is restricted to a presentation and discussion of the software only, giving a tantalizing glimpse of the possibilities of providing a word problem an algebraic description and interrelating it with graphical and numerical representations.

Dreyfus and Eisenberg present a complete research cycle "**on the deep structure of functions**", entering with a theoretical framework for analysing aspects of the function concept, empirical knowledge of student misconceptions, and a constructivist approach to abstraction using computer microworlds. They hypothesise that the understanding of the relationship between the algebraic and graphical representation of a function is facilitated by using a specific piece of software and that this can be improved by providing structured activities for the students. One group of eight students worked in a highly structured teaching environment whilst a second group were allowed to explore freely. A pre- and post-test revealed a significant improvement by both groups on "non-standard" questions, relating to shifting and stretching transformations on graphs, but the difference between groups was not significant.

In "**Dienes revisited: multiple embodiments in computer environments**", Lesh

& Herre report part of a major on-going project of research and curriculum development which reveals "significant ways that computer-based instruction can encourage teachers and students to make greater use of activities with concrete materials... at the same time ... implementing some of the best instructional strategies associated with mathematics laboratories". They discuss a symbol-manipulator/function plotter called SAM which provides direct links between algebraic manipulation on equations and the graphical representations of the functions on each side of the equals sign. The general questions raised are broad and important but the page restriction regrettably leaves no room to report empirical results.

Zehavi *et al* cover a complete research cycle in "**the effects of microcomputer software on intuitive understanding of graphs and quantitative relationships**". They describes a new piece of software, "Dots and Rules", designed to help intuitions on graphs of linear relationships, tested using pupils of "average ability", in three experimental classes compared with three control classes, selected from similar schools. Tests were given immediately after the treatment and eight months later and showed that "although the software seems to have been only moderately effective, retention of what was learnt was good". "The results indicate that software of this kind can be effective in achieving its main goal - creating intuitive readiness for future concepts."

Two papers look at the role of programming in Logo and its relationship to 'paper and pencil' algebra. Sutherland outlines the preliminary results of a three year case study on "**... the use and understanding of algebra-related concepts within a Logo environment**". She reports that "analysis of the data indicates that most pupils do not naturally choose to use variable in their Logo programming, although with teacher intervention it is possible to find motivating problems which provoke pupils to use variables". Under these circumstances there is evidence that "pupils can use their Logo derived understanding in an algebra context".

In "**using micro-computer assisted problem-solving to explore the concept of literal symbols - a follow-up study**", Nelson interviewed three "average ability students" a year after a study in which they had been "taught to use Logo to solve problems involving number sentences, rectangles and recursion". They remembered most of the Logo commands used a year before, though none recalled the MAKE command for variables and "were able to use literal symbols to represent missing dimensions of rectangles when writing expressions for area". The author concludes that "microcomputer-oriented problem solving has a long-term effect on the concept of literal symbols".

Two other papers beginning new entry phases of research pass beyond algebra into concepts linking variables and their rates of change. In "**Un système d'apprentissage de l'abstraction par représentation graphique**", Nonnon describes software allowing young pupils to control the motion of an electric train, and simultaneously to see its position graphed as a function of time, to enable them to acquire a graphical coding language to predict the interaction between the variables for distance, speed and time. The prototype software has been trialled, using pre-test and post-test to show a significant improvement in predicting and interpreting relationships between the three variables.

Garançon & Janvier report the entry stage into new research in **"the understanding of feedback systems with micro-computer software"**. They formulate the general notion of a feed-back system as "a set of mathematically defined relations between variables" which can "generally be expressed as a set of differential or difference equations". They envisage the understanding of the system as a form of coordination of three representations of the system: an iconic representation of the feed-back loop relating the variables, the superimposition of the cartesian graphs of the variables as functions of time, and the phase plane diagram representing the implicit relationship between the variables. Current mathematical research into dynamical systems shows just how complex these systems can be and one looks forward with interest to the results of research into students' understanding of the specific systems designed for the research program.

2. Common Links in the papers

It will already be apparent that the papers cover a wide range of activities. A closer inspection also shows that *no two papers cite a common reference.* (As a humorous aside, I found it pleasant to see that I am not the only author who refers to my own papers more than anyone else...) Despite the apparent anarchy that this may imply, there are certain underlying trends that can be seen.

2.1 Multiple Linked Representations

More than half the papers use software that links algebraic notation to a graphical representation, one links a real-world situation with a graph, one links the algebraic representation of an expression to its binary tree structure.

Kaput (1987) has suggested four sources of meaning in mathematics:

1. By transformations within, and operations on, a particular representational system,
2. By translation across mathematical representation systems,
3. By translation between mathematical and non-mathematical representations (such as natural language, visual images, etc.),
4. [Reflective abstraction] By the consolidation and reification of actions, procedures and concepts into phenomenological objects which can then serve as the basis of new actions procedures and concepts at a higher level.

It is helpful to review the papers within this framework to see their span over a range of activities. For instance, Nonnon links a graphical interpretation to the real world which

> "permet à l'élève d'acquérir un langage graphique de codage ... acquis au seul contact de la réalité, sans support verbal."

Boileau *et al* also wish to link the pupils' experience with mathematical concepts, this time through programming, whilst other papers concentrate more on translation *between* systems. When one of those systems is graphical, it is often seen as a more "intuitive" system". For example, Zehavi *et al.* comment that the main goal of their software is "creating intuitive readiness for future concepts".

Yet Goldenberg warns of difficulties with multi-representational software:

> "Common-sense supports the notion that the use of more than one representation of a function will help learners understand what remains less clear when only one representation is used. Presented thoughtfully, multiple linked representations increase redundancy and thus can reduce ambiguities that might be inherent in any single representation ... taken together, multiple representations should improve the fidelity of the whole message. The theoretical arguments ... are reasonable enough, but they may not be valid."

His case questioning validity is based on his two subjects' misconceptions of the nature of graphs. Other research supports this concern. For example, Nachmias & Linn (1987) show that a computer-generated graphical representation of a cooling curve of liquid in real-time was misinterpreted by 30% of the children involved, because the large pixels on-screen gave the impression that the liquid remained at a constant temperature for a time and then suddenly dropped a little (to the next pixel level). These students believe in the absolute veracity of the computer. My own observations using computer graphs with older children students suggest that it is possible to discuss such limitations meaningfully, but there are clear indications of conceptual obstacles that need to be researched.

Lesh & Herre suggest that

> "Good problem-solvers are flexible in their use of various representational systems - they instinctively switch to the most efficient representation at any given point in the solution process".

Although preliminary empirical data shows the value of multiple linked representations, more data of how students of differing ability and experience cope will be of great value.

2.2 Microworlds and the Role of the Teacher

The vision of Papert was that, by giving children access to rich microworlds, such as programming in Logo, they would develop "powerful ideas". The reality of this vision is that they may not develop the powerful ideas that may be deemed desirable. For example, the children in the Sutherland study "did not naturally choose to use variables in their Logo programming" and teacher intervention was necessary to provoke suitable activities.

Dreyfus and Eisenberg comment on the "partial success" of several experiments using microworlds in "achieving a process of abstraction on the part of the student" and question "whether the framework needs to be revised". They conclude that "this does not seem to be appropriate" as the studies were "rather short term" and "extremely high level activities are required for the processes involved in abstraction in general". They hope that "longer and more systematic exposure to dual and triple representations of mathematical objects will achieve a clearer effect ... but at present this is simply speculation".

2.3 The Notion of Variable

A noticeable feature of the papers is the variety of different meanings given to a variable. The pupils in Sutherland's study all used (local) variables as inputs to procedures whilst those in Nelson's used global variables with the command MAKE (which they subsequently forgot). Neither paper refers to the difference between a variable in algebra and in programming. (For instance, a Logo variable has a name "X and a value :X.)

Although Boileau *et al* consider elementary algebra as "minimalement l'algèbre des polynômes en une indeterminée, mais aussi les fonctions linéares, quadratiques, trigonométriques, exponentielles et logarithmiques", they later speak of

> "des fonctions (toujours algorithmiquement calculables) definies sur des ensembles de nombres, éventuallement représentées par des tableaux de valeurs, des graphes cartésiens, ou des algorithmes de calcul"

which suggests the possibility of more general procedures. Interestingly, no paper mentions procedural functions even though , when "Al ... earns $6 per hour if he works 15 hours .... [and] gets paid time and a half for overtime" (in Lesh & Herre), his actual wage, for any number of hours, can be calculated in Logo as

```
TO WAGE "HOURS
IF HOURS<15 [OP 6 * :HOURS] [OP (6 * 15 ) + 9 * ( :HOURS - 15) ]
END
```

or in structured BASIC as

```
DEF FNwage(x): IF x < 15 THEN :=6*x   ELSE  := 6*15+9*(x-15).
```

Either of these will easily generate a full table of values for his wage against the number of hours worked (normal and overtime), giving a more interesting and realistic function than the algebraic expression for overtime only.

In Thompson's Expressions Microworld, letters have a more abstract use, standing either for numbers or other expressions, whilst, in some other papers, variables are parts of formulae related to graphical representations. Only Lesh & Herre and Thompson & Thompson concern themselves with the *manipulation* of expressions. Lesh & Herre make the important observation that "the possibility of first describing, then calculating is one of the key features that distinguishes algebra from arithmetic". It is telling to note that

Sutherland's Logo pupils without algebra experience use variables only to *store* numbers, not to *manipulate* them.

2.4 Research Methodology

Of the eleven papers presented, only two have a traditional experimental v. control methodology, two use pre- and post-tests with the experimental students only whilst others used observational techniques or clinical interviews. Sutherland chose ethnographic methodology "as being the only one possible in an area where technology, pedagogy and the approach to mathematical content were all innovatory". Perhaps different techniques are required in different phases of research, with ethnographic methods more suited to the entry phase and a traditional methodology more suited to review, though this division is clearly not hard and fast.

3. Major Questions that Still Need to be Answered

3.1 Algebra in a Computer Environment

First and foremost we must begin to address ourselves to the role of algebra in a future computer-oriented paradigm. Most of the research presented here is concerned with the manner in which *traditional* algebra may be enhanced by the computer with little emphasis on a modern procedural approach. Many interesting functions such as the price of a postage stamp as a function of weight, are given procedurally rather than as a simple formula. Modern computer programs, such as the modelling program Stella (1986), allow functions to be typed in as formulae, as logical expressions, or even as piecewise straight graphs specified using an on-screen pointer under the control of a mouse. The new Hewlett Packard HP 28C symbolic calculator allows variables to have values including complex numbers, vectors, matrices and lists; thus a list of information such as the details required for drawing a graph (ranges, independent variable, number of points etc) can be stored as a variable and recalled when required.

An important global question framing all our research should therefore be

> **How can we direct our use of the computer in mathematics education to the concentrate on the algebra of the future, in addition to the algebra of the past and present?**

In particular we should spend a little time thinking about the role of symbolic manipulators. My own hunch in using them is that they (at present) offer a powerful way of handling the *syntax*, but the user needs to have a coherent understanding of the *semantics*.

It is important also to address ourselves to the question of the needs of different user populations. Several of the research papers talk about pupils of "average ability" (a term which is sometimes a little difficult to interpret). Twenty years ago (in Britain at any rate) *pupils of average ability did not study algebra.* Leitzel & Demana (1987) suggest an *arithmetic* approach to algebra. May we sometimes be wasting our time looking at the difficulties of sections of the population for which formal algebra may be of no relevance? Should all children study the same kind of algebra, or do we need different types of algebra for different populations?

3.2 Multiple Linked Representations

Given the high profile of dynamically linked representations, it is clearly important to obtain far more empirical evidence of their use. In particular we should ask:

> **In what ways do students, of differing ages, abilities and experience, use dynamically linked representations in different curriculum contexts, and how do they conceptualize the relationships between the representations? What cognitive obstacles are likely to occur in their use?**
>
> **What is a suitable theory (or theories) underlying the provision of suitable developmental sequences?**
>
> **In what ways can multiple linked representations be integrated into the curriculum for learning, teaching, problem-solving, and assessment?**

Here we note that the links between representations can take differing forms, for example, Garançon & Janvier view the understanding of feed-back systems as a coordination of three distinct representations, one of which is the *statement* of the problem (the feed-back loop) and others are *solutions*. Other systems simply *translate*,say, symbolic information into graphical form.

> **For a given system, are there simple translations between two representations, or does the relationship involve some kind of solution process?**
>
> **Does the "understanding" of the relationship between two representations involve a direct logical relationship, or is it an intuitive one, or perhaps a combination of the two?**

It would be useful to debate the interplay between syntax and semantics, in terms of the classification proposed by Kaput, the notions of syntax and internal/external semantics of Boileau *et al* and the new evaluation of Dienes' principles as described by Lesh & Herre.

3.3 Programming

Two clearly distinct threads arise in the papers, one proposing specially designed software to enhance learning, the other to encourage constructive acts through programming. These may be seen as totally separate methods of approach, or as being *complementary*, fulfilling two different, but essential, roles. We ask:

In what way are programming and the use of prepared software complementary, and what constitutes an optimum combination of the two in terms of understanding and efficiency (time on task)?

Boileau *et al* speak of a new language for learning algebra, whilst other papers use Logo. It is important to discuss what kind of computer language is appropriate, not just for doing algebra, but also for developing a growing awareness of algebraic structure during the learning process.

3.4 The Role of the Teacher

Lesh & Herre suggest that the use of certain software will encourage teachers to take a "mathematics laboratory" approach to learning and teaching, but Boileau *et al* remark that

> "En dépit de ces progrès théoriques, les enseignants en mathématiques sont relativement dépourvus quand il s'agit d'aider les élèves à se représenter les relations des problèmes algébriques narratifs."

I suggest that *teachers are not convinced by theoretical research, but by ideas and materials that work, for them in the classroom*. The role of the teacher should surely be an explicit part of our theories of mathematics education. With the complexity of the representational systems and the need for teachers to embrace computer technology, we must ask:

How can we encourage teachers to participate actively in our work so that our research is both relevant and suitable for implementation?

3.5 Artificial Intelligence

Few of the papers mention the use of tutoring systems, though the Expressions Microworld and the symbol manipulator/function plotter SAM are both written in Lisp, which gives them the possibility of being used in a more diagnostic/predictive mode. The Expressions Microworld has been explicitly written *to do nothing* if it is given an inappropriate command by the user, thus encouraging users to think about the consequences of their own actions. SAM can produce solution path "traces" to create many instructional capabilities and do other things that are intended to "help students go beyond *thinking* to *think about thinking*". One view is that it is the teacher and the pupil who provide the intelligence, in a way that cannot be provided by the machine, another uses the machine to infer action from a database of knowledge.

Particularly in the case of algebra, which has both a syntactic and a semantic role to play in mathematics, we should ask:

In what ways can computer environments be designed and used to provide intelligent support to the learning process?

3.6 Constructivism

This conference has constructivism as a major theme, and it is implicit in several of the articles, if not always explicit. My own belief is that learning is facilitated by the intelligent action of the pupil, with the teacher acting as a guide and mentor, and I have been struck by the power of the computer to provide a cybernetic environment that acts in a reasonable and predictive way to enable the pupil to build and test new concepts represented dynamically by the software. But do we all share this belief?

Davis (1986) poses the fundamental question:

> **Every educational use of computers is based upon someone's specific philosophy of what, exactly, is to be learned, and upon someone's philosophy of effective pedagogy. These "foundations" are, at present, extremely insecure.**
>
> **In the present case, exactly how do we want our students to think about algebra?**

To this one must add:

> **How can we use computers to encourage students' active participation to develop this algebraic thinking and to *think about thinking* ?**

4. The Way Ahead

I am aware that although some of the questions I have highlighted are phrased as research questions, others are not. Our discussion must include an attempt to focus on specific research hypotheses. It was part of my brief to seek indications from the papers as to how to tackle the highlighted problems. As most of the authors concentrate on putting over their own message in a limited seven page span, it would not be fair to expect the papers to be addressed explicitly to questions formulated after the papers were written, however, I am confident that the collective wisdom and experience of the authors may be brought to bear in the discussion at P.M.E.

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Algebraic thinking

CONCEPTUAL OBSTACLES TO THE DEVELOPMENT OF ALGEBRAIC THINKING

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Children who have not been taught any formal algebra nonetheless bring notions both well-formed and partly-formed to this study from their earlier work in arithmetic. Yet they are often unaware of the conflicts between earlier views and those needed to simplify expressions and solve equations. The initial concepts brought from arithmetic are bound to number experiences and are imprecise because they are explicable in terms of this experience. Algebraic concepts are more abstract, not readily related to experience. They involve notions of the elements being operated upon, the operations that are performed, the way these operations are indicated and carried out, and the way the statements symbolising them are interpreted. Difficulties in making the transition to this abstract view derive from a lack of appreciation of a need for algebraic symbols as much as from procedural difficulties with their manipulation.

The teaching of Algebra may be perceived by students as an initiation into rules and procedures which, though very powerful (and therefore attractive to teachers), are often seen by students as meaningless.

K. Hart (1981)

Children's procedural difficulties with algebra are well known. Most relate to the introduction of symbolic values and the extension of the numbers to which they refer but there are also changes in the meaning of the concepts and operations. In arithmetic the equals sign is predominately used to connect a problem with its numerical result and used in a manner that signifies equivalence or equality interchangeably. In Algebra equivalence and equals have very separate meanings and uses but this may not be apparent when an equals sign is used to signify both. Further, initial work in simplifying expressions and in transforming equations may even lead children to interpret the equals sign as a means of signifying a transformation (Kieran, 1981). The introduction of the equals symbol as a substitute for the verb "is" or "are" also leads to an implicit assumption that equations are always read from left to right. Yet many equations are more readily solved if the unknown term is grouped on the right, leading to right to left working. Attempts to avoid this often introduce negative numbera, creating at least as many difficulties as they solve.

The operations also undergo changes which need clarification. While addition was initially introduced as a binary operation, in algebra the use in essentially unary; the aign is attached to the number or unknown whether its use is as an operation or to indicate a positive value. Subtraction is also used in a unary sense and these new conceptions are essential when numbers and terms are collected together either by adding and subtracting like values or by using the inverse operation in transferring from one side of an equation to the other. Without this realisation, many studenta simply ignore the unknown value and collect the numbers first, using a left-to-right order of operating rather than combining each addition or subtraction symbol with the number or letter it precedes. A further reason for these difficulties is that the teaching of arithmetic has emphasised the notion that subtraction is the inverse of addition far more strongly than the reverse case that addition is the inverse of subtraction. This underlies some of the tendency of children to successfully "change the sign" when transposing values with an attached addition symbol but not doing so with values that have an associated subtraction sign.

Multiplication or division in a number situation almost always involve multiplying or dividing with the number in question; in algebra it invariably means using only one or some of the factors. While it is also possible to multiply and divide numbers by using factors, this aspect is not stressed in arithmetic despite the attention given to writing numbers as products of their prime factors. Multiplying and dividing by factors composed of unknowns and numbers adds further complexities ato algebra. Indeed, what is really happening is that children are called on to suspend the operation in algebra rather than express the result as they did in arithmetic (Matz, 1981). As Collis (1974) has shown, the ability to accept this lack of closure of an operation does not develop until the child is in a concrete operational level in mathematics, a level which may not occur until after this aspect of algebra has been introduced.

However, it is with the use of letters to represent unknowns, variables and general processes that children's difficulties are most apparent, particularly as they attempt to generalise procedures, conventions and use of language from arithmetic to algebra. In many instances letters in algebra behave like numbers; they represent a single value, and the operations to determine them are just the familiar operations of arithmetic operating on the other numbers in an equation. At the same time, the initial use of letters as abbreviations may introduce the thinking that they behave like words rather than numbers, as a placeholder analogous to the use of pronouns in ordinary language. It is this contradictory use that while many different values are possible for a letter in an algebraic expression, when the same letter is used more than once it must have the same value(s) that causes students so much conflict (Wagner , 1983). Further conflicts occur as the notion of variable is extended to include other unknowns when a series of algebraic identities is created to help in factorising expressions.

In many ways our knowledge of children's procedural difficulties in algebra parallels the understanding we have of children's computational difficulties in arithmetic. When computational difficulties were largely viewed as mechanical breakdowns, little progress was made in overcoming them; efforts were made to repair malfunctioning algorithms but these efforts were not particularly productive. In recent times, the analysis of computational difficulties has gone a lot further and brought out the crucial role of children's understanding of number itself in providing for skills and understanding in calculation. While children arrive at school with a fairly well developed sense of number based on counting, this knowledge by itself is insufficient. They need to build a broader understanding in terms of grouping and place value by using concrete materials if the computational procedures are to be mastered.

A similar transformation is needed of the base on which algebraic ideas and procedures is to be built. Usually the use of letters as pronumerals is introduced with little or no context as "letters to stand for unknown numbers". While the use of materials to represent these values appears attractive by analogy with the number situation, in reality the material does not serve as a forerunner to the use of letters; rather letters label the material which is manipulated. It also leaves the question of why these letters should themselves be the object of mathematical manipulation unanswered. Algebra evolved through a need for the concise representation of general relationships and procedures. Such a representation may then enable a wide range of problems to be solved and allow new relationships and procedures to be derived by logical manipulation of the old (Booth, 1986). Excessive attention to this last aspect has led children to view algebra as little more than a set of arbitrary manipulative techniques with little, if any , purpose. Rather than focus on this procedural side of algebra from the outset, it would be more appropriate to build up an awareness of the need for a concise representation of relationships and, indeed , to focus on the determination of these general relationships. Arithmetic has taught children to expect answers and that each problem has its own answer. Algebra involves the extension of general pattern finding activities in mathematics to the identification of classes of problems which have essentially the same result.

A sequence of experiences which lead from concrete arithmetic situations to algebraic generalisations must establish that the use of letters is a useful means to express such results. A first use is simply as labels to identify the objects being examined and thus grows naturally out of words used to describe them in a manner analogous to the use of letters in measurement. When this has been established and accepted, relationships between the objects which have been identified and labelled can also be expressed using the letters that have provided the labels. The use of tables of values to show these relationships can then in turn suggest more concise ways of expressing the results by means of the number which identifies a particular entry. In this way, the use of letters to express relationships occurs somewhat naturally and lays the way for using the letters themselves to find and verify patterns. Only when the development of a generalised arithmetic has established the need for and power of algebraic symbols can algebra be extended to a topic in its own right and meaningful procedures for manipulating the symbols be considered. While the difficulties that students experience with algebra reveal themselves in the use of symbols and the rules that govern their use, it is a lack of acceptance of the symbols as legitimate mathematical entitles in the first place that is the fundamental problem. So much is known about the procedural difficulties that it is possible to provide the means to avoid or overcome them; but since the use of symbols has little or no meaning for the students who have to manipulate them there is no basis for overcoming the difficulties.

The changes that need to be made to student's earlier knowledge from arithmetic are usually overlooked in the development of algebraic procedures. In particular it is the change from the manipulation of numbers to solve for an unknown to the manipulation of the unknowns themselves, labelled a "didactial turning point" by Filloy (1985), that marks the entry into algebra proper. To introduce students to the fuller algebraic meanings of the notions they met and mastered earlier in arithmetic demands the building in of conceptual conflicts when the algebraic extensions are introduced. There is also a need for the broadening of topics traditionally covered in school arithmetic so that all future needs are considered when initial concepts are introduced in arithmetic, when computational rules and procedures are established and when problem situations are constructed.

But most of all, there is a need to establish the usefulness of algebraic symbolism to express relationships and eventually to find and verify the patterns on which these relationships are based. When the need for the objects of algebra is built up, both student and teacher will work together to avoid and overcome the procedural difficulties that are most obviously the problem in mastering algebra, for the need for such manipulations will no longer be in dispute.

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EQUATIONS REVISITED

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Abstract: The idea that children can be guided to construct meaning for formal mathematical procedures from suitably structured concrete experiences underlies much of mathematics teaching. Despite this approach, however, many children do not acquire the desired levels of understanding. Possible reasons for this were investigated within the context of learning to solve linear equations, by interviewing a sample of six 11-12 year old children before, during, immediately after and three months after a concrete-based teaching program aimed at developing a formal equation-solving procedure based on the application of equivalent and inverse operations. Findings suggested that children's lack of prerequisite concepts, and the use of 'concrete' situations which do not appropriately mirror the formal procedure taught, together with the existence of informal 'child-methods', may all contribute to the lack of success of the formal teaching.

In many countries, considerable emphasis is placed upon the development of concrete or experiential approaches to the learning of mathematics. The rationale underlying these approaches is that by providing children with appropriately structured concrete experiences, the children will be guided to develop referential meaning for the formal symbolic procedures or models which are the actual goal of instruction.

However, despite such approaches, it seems that many children do not acquire the desired level of understanding of the taught models and procedures. These children resort either to instrumentally (Skemp, 1976) operating within the formal symbol system of mathematics, often making syntactic errors or inventing 'malrules' (*cf* Matz, 1980), or they adhere to 'child-methods' (Booth, 1981; Hart, 1984) which they construct from their experiences within or outside the mathematics classroom.

This lack of success in helping children develop referentially meaningful symbolic procedures has been suggested to derive from an inadequate relating of concrete and symbolic representations (e.g. Hart, 1987). This in turn may derive from:

(a) insufficient attention to the kinds of methods or models which children may have available prior to instruction (Booth, 1981);

(b) The use of concrete situations which do not appropriately mirror the formal model or procedure (Booth, 1976; Hart, 1986); and

(c) inattention prior to or during teaching to the prerequisite concepts or skills upon which the procedure being taught depends (*cf* Hart, 1986).

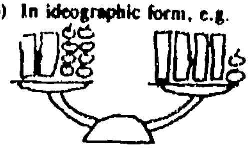
The present study examines these possibilities within the context of solving linear equations in one unknown.[1] The equation-solving procedures which children used were examined prior to, during, and after a 'concrete-based' teaching program designed to help children develop a formal equation-solving procedure based on the application of equivalent and inverse operations. The children's understanding of these latter concepts was therefore also investigated at the same time, together with other notions thought important to an understanding of the formal procedure, such as the meaning of letters and the expression of numerical and algebraic relationships.

METHOD

One class of 11-12 year olds from the 4th year of a middle school in England was involved in the study.[2] The teaching approach adopted was the approach normally used by the class teacher, and involved the use of a 'balance' model and the ideographic representation of equations as states of equilibrium between configurations of 'boxes of apples' and 'loose apples', the unknown being conceptualised as the number of apples contained in a 'box' (see Task 5(b), Table 1). The formal equation-solving procedure to be developed was based on the application of equivalent and inverse operations, and was intended to be directly modelled by, and hence have its meaning derived from, the procedures used to handle the equations as represented ideographically.

The investigation was conducted by interviewing a total of six children, comprising two each identified by the teacher as above average, average, and below average in mathematical attainment. The interview tasks (Table 1) were selected to give information on the children's equation-solving methods, and their understanding of equivalent and inverse operations and the conventions for representing mathematical relationships, including the use of letters to represent unknown values. The children were interviewed immediately before, during, immediately after, and three months after the teaching program in question.

TABLE 1: EXAMPLES OF INTERVIEW TASKS

TASK	INTERVIEWER'S QUESTIONS (Examples)
Mathematical Representation	
1. I think of a number, add 17, and the answer is 31.	1(a) Can you write down the problem some way, to help you work it out? (b) How would you work it out?
2. $x+y=6$	2. What can you say about x and y?
3. Equivalence	
a) $10 + 6 =$	a) Say I choose these cards and place them like this (see diagram). What cards could *you* choose to make the sentence true? Why? Could you choose any others?
b) E.g. $10 + 6 = 16$ ↓ ↓ Remove $10 \quad = 16$	b) Let's use this sentence you have made ($10+6=16$). Now suppose I remove the $\boxed{+}\boxed{6}$ from my side. What have you got to do to your side to make the sentence true again? (Show by choosing and placing the appropriate cards.)
c) $10 + 6 = 16$ $3 \times (10 + 6) = 16$	c) Now suppose I start with your sentence again, only this time I'm going to multiply the *whole* of my side by 3. What have you got to do to your side to make the sentence true again?
4. Inverse Operations	
a) $\square + 3 \; ? = \square$	a) Here I've started with a mystery number ($\square$). You don't know what number it is. But I'll tell you I've added 3 to it. Now it's your turn. What have you got to put in the gap (?) if we want to get back to the *same* mystery number we started with as our answer? (Choose and place appropriate cards to show.)
b) $\square \times 6 \; ? = \square$	b) Now I've started with a mystery number again, but this time I've multiplied it by 6. What should you put in the gap if we want to get back to the *same* mystery number we started with as our answer?
5. Solving Equations Presented: a) Algebraically: i) $18 + a = 47$ ii) $n - 14 = 32$ iii) $3p + 5 = 14$ iv) $2x + 8 = 4x + 2$ v) $7 + 5x = 20$	i) What are we trying to find here? Show me how you would work it out. ii) What does the letter (naming the letter in question) mean? What does '$3p$' (for example) mean?
b) In ideographic form, e.g. 	iii) Can you write an equation to match the diagram? (Note: this ideographic form was the form used in the teaching program.)

FINDINGS

Solutions of equations: After the teaching program, only one of the six students had adopted the formal equation-solving procedure taught. One other adopted a procedure which was part-formal and part-constructive (the inverse operation being used to 'undo' addition/subtraction, but not multiplication which was handled constructively as ?x5=20 (similarly for ?x5=21, which was consequently left unanswered)). The remaining four students used a constructive procedure for equations of the form '$18+a=47$', but trial-and-error for 'harder' equations such as '$12x+8=4x+2$' or '$7+5x=20$'. Both these methods had been demonstrated by these students prior to the teaching.

When presented with an ideographic representation, only the first two students described above spontaneously saw any connection between the algebraic and ideographic forms, and these were the only students able to supply a correct algebraic equation to match a given ideographic example. In both cases, however, the ideographic version was solved using a 'matching' procedure (*cf* Collis, quoted in Galvin & Bell, 1977), and only the 'formal equation solver' was able to match the procedure used on the diagram by a correct sequence of algebraic statements. Of the remaining students, one was unable to proceed with a solution, and the other three used the 'matching' procedure. Two of these latter students wrote algebraic forms for the ideographic representation which used 'b' to represent 'boxes' and introduced 'a'to represent 'apples' thereby producing an equation containing two letters, which the students could not solve. The other two students were unable to write any algebraic representation. All four of these students interpreted '$3b$' as '3 boxes' in this context, and also showed confusion in an 'abstract algebra' context between '$3b$' as '$3b$'s' and '$3+b$'. Prior to the teaching, however, three of these four had interpreted '$3b$' correctly as '3 times b', where b was interpreted as a number.

Equivalence and inverse operations: In the case of addition or subtraction, it is extremely difficult to tell whether children maintain equivalence or apply inverse operations on the grounds of logic or empiricism. In the case of multiplication, however the distinction is clearer. Thus in the equivalence task '3x(10+6)=16?', only two of the six students recognised from the beginning that the equivalence could be maintained by likewise multiplying the RHS by 3. The remaining four students achieved equivalence by evaluating the LHS of the expression, and then *adding* an appropriate amount to the RHS. These respective behaviours were maintained throughout and after the teaching of equation-solving. Similarly, where the inverse operations task for multiplication was concerned (i.e. $\square$x6?=$\square$), none of the students was initially able to solve this, except by assuming a particular value for the unknown and then *subtracting* an appropriate quantity in order to arrive back at the given 'unknown' value. During the teaching, the same two students who had recognised equivalence for the multiplication task, came to recognise that division necessarily undid the effect achieved by multiplying. The remaining four students, however, continued with their empirical subtractive approach, except for one student who came to recognise that subtraction was not suitable, since the amount subtracted would vary according to the value of the 'unknown', but was unable to suggest any alternative. The two students who attained recognition of both 'equivalence' and 'inverse operations' in these tasks were the same two students who were more successful in learning the equation-solving procedures taught.

Mathematical representation: The algebraic representation of equations presented in ideographic form has already been discussed. Of additional interest was students' written representations of the 'I think of a number' task. On the initial interview, none of the children interviewed wrote equations involving placeholders or letters. Instead, each student wrote either a verbal or numerical expression. All the numerical expressions were incorrect, although the students were able to proceed to a correct solution. The common error was to ignore the bidirectional nature of the equals sign, thus producing expressions which, although joined by an equals sign, were not equivalent, but rather represented a procedural statement of how the problem was solved, e.g. '$17+3=20+10=30+1=31$' (*cf* Vergnaud *et al*, quoted in Kieran, 1981). By the third interview (immediately after the teaching program), three of the six students wrote an equation for this task, but interestingly used placeholders rather than letters (the teaching unit had begun with placeholders, but quickly moved to using letters). In working through the equation thus produced, however, only one student (the same one who used the formal equation-solving procedure) maintained equivalence in each successive statement.

Other findings: Also of interest were the findings, supporting results obtained elsewhere, that (a) in the example $x+y=6$, x could not have the same value as y (3 out of 6 children), and only integers formed the replacement set (5 out of 6) (e.g. Küchmann, 1981; Booth, 1984); (b) expressions such as '2:4' and '4:2' were regarded as equivalent (3 out of 6 students) (Booth, ibid; Kerslake, 1986); and (c) students did not necessarily view the same letter as having the same value on two different sides of the same equation (3 out of 6 students) (*cf* Kieran, 1986).

DISCUSSION

The picture emerging from this study is that students' formal (in the sense of logically rather than empirically necessary) recognition of what constitutes equivalent or inverse operations within a numerical context is not to be relied upon (see also Filloy & Rojano, 1986). The students in this study who did not show a formal recognition of equivalent and inverse operations in a numerical context did not show such recognition in an algebraic context, nor did they learn a formal equation-solving procedure based upon these understandings. In addition, 'concrete' or ideographic approaches, though designed to help children gain in understanding of the formal procedures, may be unsuccessful in doing so if children never see the connection between the two. In the present study, the only students who saw and were able to make use of the relationship between the ideographic and algebraic representations were the two 'above average' students who perhaps had least need of the ideographic approach in the first place. In choosing a concrete or other representation of a formal model or procedure, attention needs to be given to the precise nature of the concept or procedure thereby instantiated. Where the particular representation used evokes a concept or method which is not directly analogous to the formal model or procedure at issue, the use of that representation may in fact hinder development of the formal procedure required. Furthermore, unless great care is taken, the use of the concrete model may result in inappropriate 'concrete' interpretations of terms and concepts being made, perhaps resulting in later error. This is not to say that concrete models or alternative representations should not be used in teaching mathematical procedures, but rather that careful thought needs to be given to the kind of model used, to the ways in which the model is related to the formal procedure, and to the limitations and misleading notions that might be inherent in the particular models adopted. Finally, attention is drawn yet again to students' use of informal methods and 'alternative conventions' concerning mathematical representation, and to the fact that important relationships in mathematics which students are assumed to know from arithmetic may either be not recognised by them at all, or alternatively are apprehended only on an empirical (as opposed to formal) basis, with consequent implications for their subsequent mathematical understanding (Booth, in preparation).

Note 1: The work described in this paper was conducted as part of the 'Children's Mathematical Frameworks' (CMF) Project funded by the ESRC and conducted at Chelsea (now KQC) College from 1983 to 1985.

Note 2: The other teaching studies will be described in the report on the CMF project (in preparation).

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MANIPULATING EQUIVALENCES IN THE MARKET AND IN MATHS

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This study analyzes the cognitive model developed through the use of two-plate weighing scales among market vendors. Thirty subjects were observed at work and then asked to solve two sets of transfer tasks, one regarding volumes and the other involving more complex problems with scales. Results suggest that this work experience promotes the acquisition of skills which surpass the work routine. Almost one third of subjects were able to either learn vary quickly or spontaneously develop problem solving methods which allowed for the solution of problems with two-unknown, which do not emerge in their daily activity.

Studies of working intelligence have shown that mathematical concepts and abilities can be developed at work generating efficient problem solving behavior. However, the status of this knowledge is unclear and must be examined in detail. The cognitive model used by the problem solver may be based upon the acquisition of specific work routines or on the understanding of mathematical models.

This study analyzes the abilities underlying the use of two-plate weighing scales. These traditional scales are used in street markets in small towns in the Northeast of Brazil, where the technology of digital scales has not been introduced. One plate holds the weight; the other, the merchandise. Each scale has a set of weights with the values appropriate for the merchandise at hand. For instance, merchandise sold in larger amounts, such as flour and corn, is weighed by comparison to weights of 50, 100, 200, 500, 1,000, 2,000, and 5,000 grams. If a customer asks for 350 grams, three weights, 50, 100, and 200 grams, are placed on one plate and the merchandise in the other; this constitutes an additive solution. An alternative subtractive solution can be obtained by placing 500 grams on one plate and 150 grams on the other plate with the merchandise. This situation affords practice with number operations and an underlying notion of equivalence. Different cognitive skills may develop as a result of such practice. On one hand, subjects may learn a simple routine for weighing because there are few variations in practice. On the other hand, subjects may learn a mathematical model of equivalences, which can be transfered to other situations. Two types of models could underly this knowledge. One would be a simple understanding of equivalences, which can be transfered to other measures, such as litres, and different basic values. The other would be a deeper understanding of equivalences and cancelations, which can be applyed to the solution of equations, such as Filloy & Rojano and Vergnaud have use in teaching situations. If this more powerful understanding is gained, market vendors would be able to either solve problems of some coeplexity with unknowns on their own or learn how to solve these problems through cancelation with relative ease.

METHOD

This study was carried out in Gravata, a town of approximately 70,000 people in the Northeast of Brazil and a commercial center for the surrounding area. Subjects were located in the street fair or markets during working hours. Subjects were approached after the exaeiners observed that they worked with two-plate scales. There were no attempts to select participantes by sex, age, or level of schooling.

The study was carried out in two phases. First, subjects were asked in the natural setting to weigh 400 or 900 grams of any product they sold--a request which was often justified by the experimenters because the quantities are unusual for certain items. All but one vendor (who had just started working at the fair) succeeded in obtaining the desired amount by subtraction. After buying one or more items from the prospective subject, the experimenters introduced themselves as researchers interested in daily mathematics and asked for permission to present new problems. Three refusals were observed; 28 subjects (6 feeales and 22 males with levels of schooling varying between illiterate and secondary school) were willing to participate.

In the second, more formal part of the study, two transfer tasks were administered. The Voluees Task was a simple transfer task, which consisted of changing the variable in the problems from weight to volume and maintaing the overall structure of the problems unchanged. Subjects were asked to obtain five desired total volumes (3, 6.5, 4, 9 and 9.5 litres) by using cups which allowed them to measure exactly 1/2, 1, 2, 5 and 10 litres. Two questions allowed for additive solutions; the problees which required subtraction were parallel to those with subtractive solutions when scales are used (i.e., 40, 400, 90, 900, 95 and 950 grams are all obtained by subtraction). Because there was no reason to expect differences in item difficulty amongst subtractive solutions, a fixed order of presentation was used to allow for the analysis of practice effects on the task.

The Scales Task consisted of presenting the subjects with pictures of two plate scales on which some weights and packages had been placed. The subjects task was to figure out the weight of the indicated packages. Three types of problems were used: (1) two items with one unkown on one plate (e.g., 2x + 800 g = 1,000 g), the purpose of which was to obtain the subject's adaptation to the formal task situation; (2) two items with two unkowns, one of which the subject did not have to solve for and could cancel out, obtaining a simplified problem which could be worked out as a one-unkown problem (e.g., x + y + 900 g = y + 1,000 g); and (3) three items with unkowns on both sides of the equation (e.g., 3x + 250 g = 2x + 500 g), the purpose of which was to test for the development of a more general model used in the manipulation of equivalences. The adaptation items were always presented first. The other items were randomly organized into a list, which was presented to alternated subjects in direct (A to E) or inverse (E to A) order to control for order effects. When subjects had already solved or failed on the third and fourth problems, the experimenters demonstrated the use of a general method (termed below 'manipulating equivalences') in order to test how easily it would be learned by those who did not spontaneously use it. Figure 1 presents a sample problem.

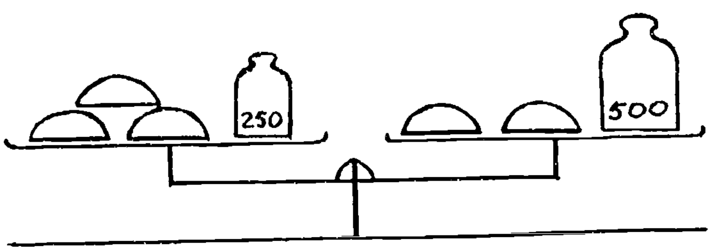

Figure 1

Order of formal tasks was varied across subjects. Some subjects answered both tasks on the same day. Others were tested on different days at most one week apart. Three subjects were not located for the

Scale Task after having solved the Volumes Task; two others were not located for the Volumes Task after having solved the Scales Task.

RESULTS AND DISCUSSION

In the Volumes Task, all addition problems were solved correctly by all subjects. However, observations indicate that a single subject had probably learned from his experience with scales simply a work routine which allowed him to obtain the desired weights on the scale; this subject solved the additive problems but did not solve any of the subtraction problems. Because order of presentation of problems was fixed, different results were obtained for 4, 9 and 9.5 litres. Of the total of 26 subjects, the 4litres question yielded 7.8% wrong answers, 46.1 immediate correct responses and 46.1 correct responses after the experimenters either suggested the analogy with weighing 400 g or that the subject could start with the 5 litres cup. The two other questions, 9 and 9.5 litres showed clear effects of practice and adaptation to the task. For the 9 litres question, 88.5% of the subjects gave immediate correct responses, 7.7 % only produced a correct response after the suggestion of analogy to the previous item and 3.8% did not solve the problem. The 9.5 litres question was correctly solved immediately by 96.4% subjects. While the immediate correct responses in this task could be quite independent of practice with two-plate scales, the fact that a suggestion to solve the 4 litres problem by analogy to weighing 400 g was helpful, can be interpreted as indication of transfer from one task to the other.

The Scales Task showed an easy adaptation of subjects to the first set of items: 100% solved the first item correctly and 96% solved both items correctly. This result can be taken as indicstive that subjects recognized the formal Scales Task as similar to their daily occupation. Questions in which packages of unkown value appeared on both sides of the scale varied in difficulty according to the need to solve for both unkowns (type 3 items) or not (type 2), with the latter type being slightly easier (72.9% of correct answers against 65.3%).

Three basic approsches to these questions could be identified, all of which can be seen as transfer from the working situation but refer to transfering different sorts of conceptions. A higher level conception, which we will term <u>manipulation of equivalences</u>, consisted of treating the situation as one in which equivalences are being manipulated, i.e., subjects were able to understand spontaneously or after suggestion that the equivalences would be preserved if packages with equal even though unknown weights were removed from both sides of the scale. This strategy was used at least in one problem by 8 subjects (4 spontaneously and 4 after suggestion from the examiner) and led to a quick solution of any problem. Subjects who spontaneously worked by manipulating equivalences did so consistently on all items; those who used the method after the experimenters' demonstration tended to generalize it to all following problems. A second type of conception consisted of treating the situation as one in which equivalences must be maintained but no manipulations were performed. These subjects attempted to obtain solution by testing several hypothesis through substitution of the unkown by hypothetical values--a strategy that will be termed here <u>hypotheses testing</u>. This method led to solution on several problems but was slower than the previous one because it involved trial and error. Eleven subjects used this method at least once. The third approach involved attempts to work out the total weights on each side of the scale fitting these values to usual purchases, such as 1 or 1/2 kilo. This strategy will be referred to as <u>fitting values to a total</u>, and often involved a difficulty in accepting the task demand of making all x's equal. Although this method is inappropriate for solving problems in this particular task, it is consistent with demands of the work experience, in which a customer either asks for a total weight or finds goods which must be weighed; when the weight is only approximately measurable, vendors will frequently offer some extra amount 'in order to complete a kilo', for example. This method was used at least once by 10 subjects. Subjects resorting to the last two methods showed less consistency than those who resorted to the manipulation of equivalences; their choice of strategy was strongly affected by the values in the task. Hypothesis testing was more frequently used in those problems which contained two unknowns but the subject only had to solve for one of them. Fitting values to a total, on the other hand, was a more common method when the values in the problem involved a half and a quarter kilo and the total was actually one kilo.

The differences in efficiency between strategies were rather clear despite the possibility of correct solutions through methods inappropriate for the task. When the manipulation of equivalences was used, no errors were observed. Other methods resulted in 25% of incorrect responses.

Performance in the Scales Task improved with higher levels of schooling. However, even illiterate subjects were able to learn the method of manipulating equivalences.

CONCLUSIONS

The main significant findings are summarized below. First, it seems unlikely that subjects working with scales learn only a routine for weighing but appear to learn at least a simple equivalence of measures. While it is not possible to attribute success in the Volumes Task to a transfer from work experience, a simple reminder of the work routine was sufficient to improve performance in this task. Second, transference from the practical setting to a hypothetical one with unknowns on only one side of the scale was observed in all cases. Third, transference to situations with two unknowns is observed less frequently and is not always obtained by means which are equivalent to the mathematical model usually taught in school for solving algebra problems; other methods which avoid the difficulties of two unknowns emerged in this setting. Finally, it must be noted that while this work experience cannot guarantee the understanding of the manipulation of equivalences in this type of problem, the percentage that learns to do so, either spontaneously or after one or two teaching trials, may be considered rather remarkable.

MODELLING AND THE TEACHING OF ALGEBRA

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We present some results of recent Mexican works in the field of Algebra teaching. An analysis is made within the theoretical framework of the experimentation design, which is essentially that of the Pragmatics of Algebraic Language and the Psychology of Information Processing, combined with the acquisitions from Semiotics in the production of codes. The resulting analyses are included as well as analyses of propositions from the beginning of the century.

The study of theoretical problems presented by Algebra Teaching has been gathering force in recent years. The National Council of Teachers of Mathematics of the U.S.A in recognition of this organized, last March, a conference dedicated to analysing the foundations on which theoretical research in this field should be built. The Mexican works mentioned in the bibliography are presented and discussed here indicating the continuous work that has already covered a span of more than five years. In constrast with what happened then with studies in other parts of the world, as much in the data-processing methodology as in the theoretical aspects, these Mexican studies started out with the intention of moving experimental research closer to teaching (planned and executed in the Mexican school system in the medium level schools). These experiments start from the observation of the student's difficulties in learning, given the strategies present in the traditional and innovatory teaching models, used in today's secondary schools.

In a world context, theoretical analysis has been enriched by the problems concentrating on the use of knowledge derived from research on Artificial Intelligence. Also, related more to the psychological processes of the construction of mathematical language signs in general, another group have actively participated with their theoretical works. In Mexico, in attempting establishment of particular mechanism of algebraic language there has arisen the need for a theorical framework that lies half way between the Pragmatics of normal language, the theoretical acquisitions of Semiotics (see 15] and the theory of information and codification. Thus concepts such as semantics syntax, context and reading at one language level etc. have been combined with concepts deriving from the psychology of information processing such as memory, semantics, short term memory, inhibitory mechanisms process unravelling mechanisms, analysis mechanisms, permanency in a semantic field etc. The empirical evidence now accummulated permits us to foresee that an interpretation of the learning processes, practice and communication with the algebraic language (teaching strategies in particular) demands all of these theoretical instruments and that now is the most propitious moment for theoretical reflection on new problems that would put these (theoretical models) to the test. This article concentrates on the description of some experimental data which in the light of an analysis of recently-mentioned concepts, reflect a previously unsuspected depth of interdependence.

CONCRETE MODELS VERSUS DRILL AND PRACTICE.

At the beginning of the twenties, in |16|, Thorndike aimed at including in his work all that seemed pertinent to him for the advance of Algebra teaching in his time. This monumental work still provides an essential programme for any theoretical and experimental approach, setting aside perhaps some emphasis and preoccupations singular to this theoretical perspective. What is still very relevant is his central motivation which had already appeared in an earlier article | 17 |, specifically in the last paragraph: " Algebraic computation, as we recognise it today is, without doubt, an intellectual skill. It is not such an indication of intellect as problem-solving, in part because it demands a lower-grade of abstraction, selection and original thinking, and in part also because it only includes numbers and non-numbers and words. It is nevertheless very much superior what it is claimed to be - a mechanical routine that can be learnt and operated without the use of thought "

In the following sixty something years the research emphasis has varied enormously until about the middle of the century when priority was given not to what Thorndike called problem- solving, but to the structural components of study material: Algebra (in fact all mathematics). We have the case of the medium level French education study programmes where right up to this date traditional teaching situations do not appear in the subject called Algebra, since Algebra is considered a continuation of Arithmetic | see 2 |. As a reaction to this there was a change of direction in the seventies towards the use of teaching models based on situations similar to, but more concrete than, those proposed by Thorndike, mechanizing the handling of algebraic expressions and achieving a speedy use of the syntactic rules.

In | 7 | there can be found examples of situations of concrete modelling which give the following results:

Modelling has two fundamental components: one, that of translation, through which it applies meanings in a more concrete context to the new objects and operations being introduced, the same as appear in more abtstract situations. That is through traslation these objects and operations are related to elements of a "concrete" situation. This is a state of affairs that represents, at the same time, a condition of circumstances in the most abstract situation (in the case of a geometric model for example, the equality between areas or magnitudes corresponds to an equality between algebraic expressions) and from what we already know at the most "concrete" level about the solution of such situations, operations are introduced that, although they are carried out in the "concrete ", also attempt to function on the corresponding objects at the most abstract level. It is thus necessary to have a translation of movement from one context to another to make feasible the identification of each opertion of the most abstract level with the corresponding one in the 'concrete" model.

A second component of modelling is the separation of the new objects and operations with the most "concrete" meaning from that which they were introduced. That is, the modelling attempts to detach itself from the semantics of the "concrete" model. since what is requirred is not to solve a situation already known to be solvable, but to find a means of solving more abstract situations through more abtstract operations. This second component is a motor principle that orientates the function of the model towards the construction of an extra-model syntax.

The study shows that the predominance of the first of these components of the model (traslation) can weaken or inhibit the development of the second. This is the case with subjects that achieve a good handling of the "concrete" model, but that, due to this, also develop a tendency to stay and progress within this context. This anchoring to the model works against the other component that of abstraction of the operations to a syntactic level, which presupposes a breaking with the semantics of the " concrete" model.

The afore-mentioned obstructions constitute a kind of essential insufficiency in the sense that the model (left to the spontaneous development on the child s part) on being strengthened in only one of its components, tends to hide precisely what is intended to be taught, that they are new concepts and operations.

This kind of dialectic bewteen the processes corresponding to the two model components should be taken into account in teaching, which should try to develop harmoniously the two kinds of processes, so that one does not obstruct the other, and viceversa. In fact, from the case analysis performed here, it is clear that this is a teaching task, given that this second aspect of the model, that of breaking with former ideas and operations where the introduction of new skills is encouraged is a process that consists of the negation of parts of the model's semantics. These partial negations take place during the transference of the use of the model, from one situation of a problem to another but, when this generalization in the model's use remains at the expense of the spontaneous development on the part of the student, the partial negations can occur in essential parts of this. It is because of this that it becomes essential to intervene with instruction in the development of these processes of detachment and negation of the model, in order to direct the student towards the instruction of the new notions.

SYNTACTIC MODELS

The idea of the concrete teaching model can be extended to the strategies proposed by Thorndike that will here be called syntactic models in contrast with those of that we shall call semantic, since here they emphasize working with a semantic emphasis in all the signs and operations involved. In the syntactic model, in contrast, the emphasis is placed on the general rule used to construct the habits leading to algegraic operations.

With respect to these models, the empirical evidence (see 12) indicates that apart from generating private semantics (of the subject) that confer meaning on the terms proposed by the general rule and to the algebraic operations involved (they could be called spontaneous connections to use Thorndike terminology) there also appears the phenomenon of reading of the proposed situations, through senses that have been previously conferred on the rules that have to unfold in order to perform the syntactic task. For example, in 3 there appears the case of a subject that, on first confronting equations of the type Ax B · C(A, B, C > 0), always give B a positive value while giving A the negative value, guided by the sense derived from the previous practices that he had carried out with the equations of the type Ax + B · C

In this respect the emphasis placed by Thorndike, not only on drill but also on its preoccupation with practice and the consequences that this has on the times of training that the learning experiences impose, has a new meaning, faced with the need to rectify the spontaneous readings, generated here, not by semantics but by syntax. This is a syntactic context that directs the ('natural') erroneous reading, due to the anticipatory mechanisms of the subject, this theoretical unit is indispensable also in the Pragmatics of Normal Language.

PROBLEM SOLVING AND SYNTAX

In | 9 | and | 18 | there is empirical evidence to show that the analytical process in a typical problematical situation (expressed in the normal language) produces reading phenomena of the situation which inhibit the development of equation solving algorhythms that moments before were performed easily and correctly. Thus, in the presence of a written expression in the normal algebraic language of a first grade equation, the subject is incapable of decoding as such and because of this, he is unable to use brilliant operating skills which moments before he had exhibited with the same equation. In the works mentioned at the beginning of this paragraph it is possible to find more examples illustrating this phenomenon. More illustrative than these however, there occur examples of problematic situations (in the parts where translation of the normal language to algebraic language is made) that reveal the existance of a tension between the interpretation of the algebraic expression, given by a reading that comes from the context of the algebraic language itself, and the use of drill in the operations,inhibiting the necessary reading given by the semantic interpreta tion which confers the concrete situation on the verbal problem. A syntactic reading inhibits the reading of the concrete context where the problem is situated. It does not allow the aplication to this algebraic expression of an interpretation that would permit it to continue with the correct solution strategy that would provide the solution and including as one of its tactics this part of the translation.

It is at this moment of the discussion that some of Thorndike's theoretical pre-occupations and their implication in teaching come into their own since the need to automatize becomes urgent, not only some algebraic operations arising from the decoding of a concrete problematic situation (problems of age, mixtures, alloys, money, work, etc.) neither the sense of the necessary algorhythm nor the semantic interpretation (in terms of the contexts of these algebraic operations were practised) nor the anticipatory mechanisms (especially the inhibitory ones) should obstruct the unfolding of a solution strategy. Besides, it is essential that, when this latter is placed in the short term memory, the time that it will feasibly remain there should not negate the possibility of considering all the necessary intermediary tactics for the proposed solution. This should be the case provided that the concatenation of all the tactics, without making all the steps essential for the obtaining of these partial goals, can be carried out in this part of the memory (the short term) which it would be difficult to maintain alert for such a lenght of time. One could say that the skill of storing important quantities of information, in order to be able to move out from this memory space to bring in new and important information, is not easily found among average students. It demands large intellectual resources not proportioned by normal teaching. Because of this, drill, resulting from intense practise, allows the optimum use of algebraic expressions and the normal operations in algebraic language and this breaks with the anticipatory mechanisms inhibiting the unfolding of necessary solution strategies.

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COMMON DIFFICULTIES IN THE LEARNING OF ALGEBRA AMONG CHILDREN DISPLAYING LOW AND MEDIUM PRE-ALGEBRAIC PROFICIENCY LEVELS

(A clinical study with children 12-13 years old).

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Here we study the pre-algebraic behaviour and the phenomenon of transition from arithmetical thinking to algebraic, in children of low academic achievement, those belonging to the low level and some from the middle level. This student population offers an amplified version of the difficulties confronted by their companions and, in an essential way, questions the process of educational evaluation, indicating that the classification by levels - high, middle and low- depends on the objective of the study and not the subject itself. Through an analysis of video-taped interviews, we show important skills and resources, acquired by these students, that are not reflected in the didactic data. We also see areas of difficulty in algebra-learning that contributes to explain low academic achievement, and to reveal intrinsic problems in the study matter an its teaching.

ANTECEDENTS AND PRESENTATION OF THE STUDY

This work is part of a general project on the "*Evolution of Symbolization in a School Population of 12-18 years of age* ". developed in the Sección de Matemática Educativa del CINVESTAV and the Centro Educativo Hermanos Revueltas in Mexico City, since 1982. The methodology employed in the research project is developed in two directions: 1.- That of the field of historical development of mathematical ideas. 2.- That of the field of educational research. In this latter we look at the research topic '*Operation of the Unknown* '[1] where a transversal study is made of a population of students of 12 to 13 years of age in a controlled teaching system. Previously the antecedents of the student population, in terms of various pre-algebra sub-themes, were determined. A resulting stratification for each sub-theme was developed and this resulted in three levels -high, middle and low. It was discovered that the subjects, who were suited for the study of the phenomena of transition from arithmetical to algebraic thought, were those belonging to the high level. As a result the clinical study described in the research consists fundamentally of an analysis of interviews of this level. The present work, however, studies the video-taped interviews of children of low academic achievement. Their relevance can be seen from two points of view: First, because they give an '*amplified version*' of the difficulties confronted by the rest of the students (6). and secondly, the process of educational evaluation is questioned through the illustration that this classification by levels depends on the objective of the study and not the subject itself. The clinical method illustrates the important skills and resources that these students posses, not visible in didactic data. As a counterpart, it also indicates areas of difficulties in the learning of algebra which partially explain the low academic achievement in these students. The areas, detected by the clinical study are as follows: 1).- Operations, 2).- The nature of numbers. 3).- Primitive methods, the strategy of trial and error. 4).- The interaction between the semantics and the syntax of elemental algebra. 5).- The didactic cut in the study of linear equations.

The analysis presented in these areas is restricted to an attempt to explain the data observed by the clinical method. It does not attempt to be a study in depth, because of the complex problems which each one of these contains at a theoretical level. The important thing to note is that the students' difficulties uncover intrinsic problems contained in the material under study and in its teaching in this case in algebra. Nevertheless, although it is not considered in this work, it is worth indicating the importance of questioning the following: which factors, apart from those intrinsic to the subject matter, have a determining influence on whether a student has a low learning capacity or not. We can not go into this problem here because of the limitations imposed by the methodology applied.

THE CLINICAL INTERVIEW

The basic format of the interviews is of 5 series: The series E of Equations -of the form $x \pm A = B$, $Ax = B$, $A \times (x \pm B) = C$, and $(x \pm A) \times B = C$, where A, B, and C are particular whole numbers distinct from zero. The series C, Cancellation: $x \pm A = B \pm A$, $Ax \pm B = x \pm B$; $x \pm \frac{x}{A} = B \pm \frac{x}{A}$ and $x + x = A + x$. The series I (Operation of the Unknown) that present items such as $Ax \pm B = Cx$ and $Ax \pm B = Cx \pm D$. Finally the series Solution and Invention of Problems, that takes a further look at the equations presented in the previous series.

In the majority of high level cases we introduce a phase of instruction in the Series I (which contains equations for which there has been no class teaching), after having observed the spontaneous replies to the first equations of this series. This instruction did not occur with the children from the low level but in some cases with the middle level child.

AREAS OF DIFFICULTY ANALYSED (Description and Observations)

Next we present the description of the areas in question, with examples of observed difficulties with items taken from clinical interviews.

1. OPERATIONS

1.1 The Duality of the Operation. Letters do not constitute a very intuitive notation for the symbolic values since they do not appear at first instance to refer to numbers. Although the unknowns are frequently used in arithmetic through *'empty spaces'*, numerical sentences such as $3 + \square = 7$, this concept is not generalized in a natural form to one with symbolic value. The *'empty spaces'* are not *"worked"* in the equations nor *"deformed'* by operations that alter their structure, but they have an inherent connotation of *"being filled"*. Since the *"empty space"* is not associated clearly with a letter, the students do not notice, in principle, that the variables can be exemplified with numbers. Thus, the students of this study, when faced with 2x normaly answer, *"You cannot multiply by x because you don't know what x is"*. In fact, the arithmetical idea of *'performing an operation"* such as multiplication is transformed into *"how to write the result"*. On the other hand, there is a tendency to reject an algebraic expression as a result, when the signs +, , ×, ÷ $\sqrt{\ }$, $(\ )^2$ appear in the equations the student immediately works the equation. Here lies the duality of the operation: the permanence of the action when faced with an order of execution.

1.2 The Reading of the Operation. Generally arithmetical operations are performed vertically. In Algebra, horizontal reading of the operation on the equations predominates [2]. These directions in readings, combined with the use of plus and minus considered at the one time binary and unitary, lead to problems. For example, the item $x + 1\,568 = 392$ both readings were present. To solve the previous equation, one student used a calculator and verbalized *"I take 1 568 away from 392. The result is 1 176"*. He read the expression horizontally. Nevertheless, when the interviewer asked him to do the operation on paper he turned to incorrect vertical writing $\begin{array}{r} 392 \\ -1568 \\ \hline 824 \end{array}$.

In the case of the symbolic signs of the values, he cannot be sure whether the sign is explicit or if it is *"contained"* in the symbolic value. Thus in the item $x + 1\,568 = 392$ another interviewee states *"the equation cannot be solved, since x is always positive. So that x could be negative it should be written as -x"*. On the other hand, there is also some confusion between the various operations, interpreting addition as substraction, substraction as division, and taking the square root as making to a power and division at the same time. For example, in the item $x + \sqrt{13} = \sqrt{13}$ the student asks if $\sqrt{13}$ can be a decimal. He is answered in the affirmative. He replies *"In that case, the answer is 6.5 because 13 divided by 2 is 6.5"*. When this same student is presented with the equation $x \cdot \sqrt{3} = 0$ the following occurs:

Student: *"It is 3, no it is 9"*. Interviewer: *"Why?"* Student: *"Take 9 away from 9. The $\sqrt{3}$ is 9"*.

1.3 Inversion of Operations. We can frequently observe that the idea of inverse operation is not consolidated in the student in the transition from arithmetic to algebra On occasion this leads to the inversion of primitive rules which takes him to the correct solution. These rules function in extra-school situations as in the case of the *"reverse"* rule. Thus in the item $13x = 39$ the interviewee states that, in order to find the value of x" one has to divide 13 by 39". He obtains the value 0.333 and confirms that it is wrong. When the interviewer asks him *"What can have happened?"* He answers *"It has to be done in reverse, we have to divide 39 by 13"*. One can note that this procedure is applied to various daily situations and does not necessarily correspond to the inversion of operations.

1.4 The Nature of Equality. In arithmetic, the equal sign is used fundamentally to relate a problem to its numerical answer, in algebra the equal sign has a dual character; as an operator (assymetric character of equality) and as an equivalence (symmetric character of equality). When the idea of operator and not of equivalence is emphasized in the solution of equations mistakes are made. Thus in the case interviews, the *"Quasi-equality"* [3] scheme is present. The student constructs the rule *"it is not important where the operations are performed, as long as they are performed once"*. Thus, $3x + 154 = 475$ is considered equal to $3x = 475 + 154$" since it is the same if you add before or after the equals sign". The preoccupation with operating immediately leads them to ignore the equals sign.

In the series of Cancellation there are present different interpretation of equality [4] at this level: 1).- Arithmetic Equality: $x + A = \underbrace{B + A}$. That is, the student, before giving any reply, "reads" the terms on the right hand side of the equation, as one single number ('close the operation'). 2).- Equality of the two sides $\underbrace{x + A} = \underbrace{B + A}$. Consider each side as a unit. There unfolds a "visual reading" where, at times, the operation involved in the expression is unknown. Thus, in $x + 5 = 5 + 2$, the student replies: x is 2 because they are equivalent.

2. THE NATURE OF NUMBERS.

2.1 Positive Whole Numbers. In the field of positive whole numbers, the zero and one stand out as special numbers [5]. These special numbers appears in the context of the rule of identities, that is $A \cdot 1 = A$ and $A + 0 = A$. Here, we should point out that in $1 \cdot x = x$ they generally interpret 1x as x but do not mention that x is equal to 1.x. On the other hand, when the solution of the equation is zero, there are students that do not accept this solution as valid because they see the zero as *"the absence of value"* and continue to look for another number that might satisfy the equation. Finally, the majority of cases interviewed show a preference for the positive whole numbers to the point that they force the value of the x so that the equation does not contain fractionary expressions.

2.2. Negative Whole Numbers. As far as their teaching is concerned there exists an assymetry between positive numbers and negatives. The positive numbers are more concrete in the sense of their relation with measuring activities, and they can therefore be operated. The negative numbers are secondary, introduced as a result of the operations [7]. In the cases analysed in this work, it is shown through the interviews how difficult it is for the student to understand and accept negatives. On the other hand, we have already mentioned the problem of signs, unitary and binary, in the Area of Operations, and also the lack of link between adding and substracting as inverse operations. For example: (I: Interviewer P: pupil). I: "How do you solve: $x + 1568 = 392$?" P: "By taking 392 from 1568, 1176 (note that he substracts the greater from the lesser number)". I: "Is the answer correct?". P: "Yes". I: "How did you test it?" P: "I added 392 (quasi-equality scheme)". I: "How do you prove it?" P: "By adding 392, but it does not work out because this is greater than this".

2.3 The Polysemy of the Unknown. It is shown in the following way: in an equation, different readings of the same x are made. That is, it is interpreted as an unknown or generalized number (that it has more than one value). Thus, in the items $x + \frac{x}{4} = 6 + \frac{x}{4}$ and $x + 5 = x + x$ of the Cancellation Series, the typical reply is "This x ($x + \frac{x}{4} = 6 + \frac{x}{4}$) is 6 and these ($x + \frac{x}{4} = 6 + \frac{x}{4}$) can be any number".
This x ($x + 5 = x + x$) is 5 and these ($x + 5 = x + x$) can be any value. In the item $2x + 8 = x + 8$ many verbalize "This x ($2x + 8 = x + 8$) is 4 and this x ($2x + 8 = x + 8$) is 8". They state openly that the x in 2x must be half of the x in the second side so that the value is the same on both sides. What the student tries for is that "the quantity is conserved" at all costs. Note that they still have not consolidated the idea of conditioned equality, that of equation.

3. PRIMITIVE METHODS, THE STRATEGY OF TRIAL AND ERROR.

The majority of the students that, for various reasons, do not accept academic knowledge immediately, attack the first algebraic problems with the same methods that have been successful in arithmetic and that are familiar to them [8].

Here we present two case interviews: The first resorts to the strategy of trial and error. The second uses a systematized exploration. They are asked to solve the item

$6x = 37\ 436$. The first student resorts to the calculator for his computations. He tries the numbers, 175, 365, 465, 563, 633. The interviewer then presents him with the previous equation as $6 \times \square = 37\ 636$, At this moment he tries 630 and 620. Note that the values found by the student when multiplying by 6 lead to: $6 \times 175 = 1050$; $6 \times 365 = 2190$; $6 \times 465 = 2790$; $6 \times 563 = 3378$; $6 \times 633 = 3798$; $6 \times 630 = 3780$; $6 \times 620 = 3720$.

It is observed that these numbers do not reach the required order of magnitude. Nevertheless, his computation becomes systematic from 633 on in that the first two numbers of the total on the right hand side, that is 37, tally with the first two numbers of the total on the right hand side of the given equation. The second student sets about solving the item $6x = 37\ 436$ without using a calculator P: "6 by 6 100 this number could be 6 right?". I:"Let's see. Try it".

The student begins to divide on the paper. The interviewer suggests using the calculator and the student arrives at the correct result, 6 239. Note that, in this case, the student suddenly grasps the order of magnitudes of the number he is looking for.

4. SEMANTICS AND SYNTAX OF ELEMENTAL ALGEBRA

In the case studies, the semantic interaction semantics-syntax is analyzed with respect to the invention of a problem from a given equation.

Given the order, "Invent which problem is solved, for example, with the equation $x + 4 = 28$", the student first finds the solution. The most pressing need is to understand the meaning of the sign. That is, to find the unknown before becoming involved in the construction of the problem. (Language obstruction reflex at a purely syntactic level). On the other hand, we mentioned previously, in the area of Operations, the difficulty of conceiving the algebraic equation as a condition of equality. This occurs on inventing a problem to solve the equation, the student ommits the question, that is the thing that converts the description of a situation to a problem.

Sometimes, the problem proposed by the students is foreign to the equation, for example, in the following case: I: "Can you invent a problem solved by this equation $4(x + 11) = 527$?". P: "A problem or just an Operation?" I: "A problem with marbles, for example...". P: "A child had 5 marbles and won 2 and some were lost, but we don t know how many...".

Note that when the student asks if a problem or only an operation is wanted, it can be that he is trying to solve the equation. On the suggestion of the marbles, that is, a semantic situation, he abandons the previous syntax ($4(x + 11) = 52$) and concentrates on posing "another problem". Observe that the new data is foreing to the initial equation.

5. THE DIDACTIC CUT IN THE STUDY OF LINEAR EQUATIONS.

The work "Operation of the Unknown"[4] corroborates the existence and location of a didactic cut in the evolutionary line from arithmetic to algebra. At a theoretical level this cut arises when there is a need to operate the unknown in the solution of linear equations, with an occurrence of x on both sides of the equation. In the clinical study, the didactic cut is perceived only by the children at the high level. One way of noting this is their verbal manifestation when they are faced with new equations that they cannot solve. Some students even imagine the existence of a school-method of attack for these new equations. on the other hand, the students of low academic achievement do not see any difference between arithmetical equations $Ax + B = C$ and the non-arithmetical equations mentioned here. Fundamentally this is because they do not realise the change of concept from arithmetic to algebra, remaining still in a purely arithmetic field They even try to look for mechanisms that allow them to interpret new equations with two occurrences of the unknown as equations where x appears only once. For example, in the item $5x = 2x + 3$ a student answers: " 5x is equal to 2 by 1, 2 plus 3 is 5". Thus, in performing actions on one side only of the equation as in summing $2x + 3$ once the x has a value assigned to it, he reduces the two occurrences of the unknown to one. It is important here to indicate that the explicit non-perception of a didactic cut is not a denial. In this work, the existence of the areas of difficulty here mentioned, indicates that the cut will not flourish at the low level. It will be necessary to solve these difficulties in pre-algebra before studying the first algebraic equations. This information could not be obtained from the high level children where they have automatized the actions that make evident the explanations of the whys and wherefores of the pre-algebraic situations. That is, children with a great academic achievement do not display the need to make explicit the situation procedures which are completely rutinary to them.

CONCLUSIONS

The results of this study display important skills in the students of low academic achievement. Some of these are 1).- Systematized trial and error exploration. 2) - The tendency to generalization and simplification in the methods of equation solution. 3).- Extra-school resources such as the "reverse" rule and the "quasi-equality scheme." 4).- The use of various languages in the invention and solution of problems. On the other hand, the difficulties encountered by the students indicate some key points to be considered in algebra teaching. Thus, we should consider such questions as i) the duality of the operations, ii) the symmetry or anti-symmetry of equality, iii) the non-indentification between one operation and its inverse, iv) the existence of special numbers, v) the extreme difficulty of the negatives.

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THE MYTH ABOUT BINARY REPRESENTATION IN ALGEBRA

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The parse of algebraic expressions is often indicated explictly by the use of parentheses (e.g. $(x + 1)^2$) or else as artifacts of the positional features of notation (e.g. x^{2y} is interpreted as $x^{(2y)}$). In cases where such explicit indicators are absent, syntax defaults to a conventional hierarchy of operations (e.g. $3x^2$ is interpreted as $3(x^2)$). In usual treatments of syntax, expressions are assigned a binary parse. For example, $x + y + z$ is assigned the parse $(x + y) + z$. Because of the associativity of addition, however, we may legitimately ask if the psychological representation is not $x + (y + z)$, or indeed if $x + y + z$ is assigned a parse at all. This paper presents and supports the hypothesis that the syntactic rule which underlies the mental representations for competent symbol users does not provide for a binary parse. Since standard formal mathematical models treat operations as binary, this amounts to an assault upon an implicit but pervasive assumption that formal mathematical theories explain or underlie the rules by which algebraic expressions are manipulated.

Before presenting and evaluating a detailed and somewhat technical hypothesis about the psychological representation of algebraic operations (addition, subtraction, multiplication, etc.) it is useful to consider briefly the place (or rather lack of place) of such a hypothesis within the context of current research in the psychology of algebra. This report is atypical in that it is primarily about the fluent or competent algebraist. The vast majority of studies which have been undertaken to date are about the novice algebraist; the mistakes which he or she makes or the processes by which new algebraic knowledge is acquired. Few reports (Carry, Lewis & Bernard, 1980, being a notable exception) attempt to specify a detailed account of algebraic competence before plunging into the turbulent waters of knowledge acquisition or knowledge deviation.

At least part of the inattention to competent performance can be attributed to an implicit belief that mathematical theory (in some sense) underlies or generates or explains knowldege used to manipulate algebraic expressions. This belief is evidenced, for example, in the distinction which Brown, Burton, Miller *et al* (1975) make between the "abstract logical structure of the [algebraic] knowledge" and the "reorganized, learner oriented structuring of how he is to use the knowledge for solving algebra problems" p. 84. Apparently they believe that some abstract structure (presumably a formal mathematical model) underlies the psychological representations, however, they do not provide a detailed account of the presumed connection. Whatever its direct value for psychological or educational theory, the present paper also takes aim at the presumed mathematical-theory/psychological-theory connection.

The present hypothesis resides within a linguistic theory of algebraic competence (Kirshner, 1987). It is necessary to outline that theory briefly (see Kirshner, 1985, for a more detailed outline) and to describe some parts of it in detail. In the linguistic theory, a distinction is made between the ***surface form*** (SF) of ordinary algebraic notation and a more abstract ***deep form*** (DF) in which the operations and parse of each expression are explicitly displayed. For example the SF, $5(1 - x + r)^{2y}$, would be represented in DF as 5M[[1Sx]Ar]E[2My]] where "M", "S", "A", and "E" abbreviate operations, and brackets display the parse in the usual way.

DF's and SF's are central psychological constructs of the theory. It is postulated that in manipulating an algebraic expression the SF is decoded into its associated DF. It is the DF to which transformational rules are applied. Finally, the transformed DF is encoded back into SF. As an example, $(3x)^2 - y^2 = (3x - y)(3x + y)$ is accounted for as follows. The initial expression $(3x)^2 - y^2$ is translated to its DF, [[3Mx]E2]S[yE2]. A "difference of squares" transformation is used to derive a new DF, [[3Mx]Sy]M[[3Mx]Ay]. Finally this DF is encoded into its associated SF, $(3x - y)(3x + y)$.

Major components of the linguistic theory are a *Transformational Component*, and a *Translation Component*. The Transformational Component provides a list of the transformational rules used in the manipulation of algebraic expressions. (See Kirshner, 1986, for a discussion of the problems encountered in constructing the Transformational

Component.) It will be necessary to refer later to *associative* and *commutative* transformations for addition and multiplication; to the *expansion* transformation used to multiply together two polynomials of the form $(a + b + \ldots + c)(x + y + \ldots + z)$; and to the *arithmetic* transformation which replaces a binary combination of two numbers by the appropriate result.

THE TRANSLATION COMPONENT

The hypothesis which is the topic of the present paper concerns (most directly) the Translation Component. Four stages are postulated in the translation between deep and surface forms. For the present purposes, we will consider translation from SF to DF although, as is clear from the above discussion, a comprehensive treatment of symbol manipulation must account for both the encoding and decoding of SF's. In the original formulation (Kirshner, 1987) translation is directed from DF to SF, so to maintain consistency with that version, the stages here are numbered in reverse.

Stage 4 cleans up such details of surface representation as the insertion of "2" into square root signs, and the replacement of parentheses and braces by brackets. Stage 3 inserts brackets where parsing cues are indicated only through physical artifacts of the representation of operations. As an example, x^{2y} becomes $x^{[2y]}$ because *being in the exponent* is a parsing cue. Stage 2 expresses operations in the capitalized abbreviated notation of DF. Stage 1 effects the insertion of brackets according to a conventional hierarchy where surface cues in Stage 3 have not already dictated the parse. For example, 3MxE2 ($3x^2$) is parsed as 3M[xE2] because of the relative positions of multiplication and exponentiation in this hierarchy. As an illustration of the Translation Component, the SF, $5(1 - x + r)^{2y}$, is translated to its DF, 5M[[[1Sx]Ar]E[2My]], as follows: $5(1 - x + r)^{2y} \xrightarrow{4} 5[1 - x + r]^{2y} \xrightarrow{3} 5[1 - x + r]^{[2y]} \xrightarrow{2}$ 5M[1SxAr]E[2My] $\xrightarrow{1}$ 5M[[[1Sx]Ar]E[2My]].

The conventional hieratachy of operations which governs parenthesis deletion in Stage 1 is given by Schwartzman (1977):

Operation Hierarchy

Level 1 operations are "A" (addition) and "S" (subtraction)
Level 2 operations are "M" (multiplication) and "D" (division)
Level 3 operations are "E" (exponentiation) and "R" (radical)

(In this classification, Level 3 operations are said to be *higher* than Level 2 operations which in turn are *higher* than Level 1 operations.)

The process of parenthesis insertion (Stage 1) entails repeated passes over the expression. At each step, tests are required to choose the most precedent operation from among those remaining for the appropriate insertion of brackets. (Kirshner [1987] argues that the syntactic structure is assigned from least precedent to most precedent by humans engaged in symbol manipulation, however, that hypothesis is irrelevant to the present concerns.) These tests are provided for in the following rule:

Syntactic Rule

(a) Parentheses are inserted around the subexpression with the highest level operation.

(b) If adjacent operations are of equal level, then brackets are inserted about the subexpression on the left.

(A technical definition of adjacency is not provided here.) In the above example Stage 1 is accomplished in two steps: 5M[1SxAr]E[2My] ——> 5M[[1SxAr]E[2My]] because E is a higher level operation than M (part a); and 5M[[1SxAr]E[2My]] ——> 5M[[[1Sx]Ar]E[2My]] because S and A are of equal level, and S is to the left of A (part b).

THE HYPOTHESIS

This version of the Syntactic Rule leads in a straightforward way to the usual binary parse of expressions. For example, x + y + z is assigned the DF [xAy]Az. Due to the associativity of addition, [xAy]Az and xA[yAz] represent equivalent values. Thus it is legitimate to ask whether the DF representation of x + y + z is [xAy]Az or xA[yAz]. More radically, we may question whether x + y + z receives a binary parse at all. Perhaps the two additions are treated as equally precedent.

For associative operations, addition and multiplication, the non-binary hypothesis is relatively straightforward. The suggestion of this paper, however, is more far-reaching. It is proposed that the syntactic rule which underlies the parsing operation for the competent symbol user does not assign a parse for any expression whose operations are of equal level. The technical formulation of this hypothesis is accomplished by the simple deletion of part (b) of the above Syntactic Rule.

This proposal is not as problematic as might appear at first glance. Almost all of the non-associative operations (division, exponentiation and radical) have a binary interpretation imposed at Stage 3 of translation. For example, concerns over the intermediate form xDyMz do not materialize since the division operation would have specified a parse at Stage 3. For each of the potential SF representations of xDyMz, $\frac{x}{yz}$ and $\frac{x}{y}z$, the position of the symbols and extension of the vinculum determines an unambiguous parse at Stage 3. Thus "xDyMz" will have already been assigned a parse before arriving at Stage 1 of translation. Subtraction is the only non-associative operation for which a binary interpretation is not imposed in SF. Thus according to the present hypothesis, xSyAz would remain unparsed even though the binary alternatives, [xSy]Az and xS[yAz] are nonequivalent. This leads, for example, to the possibility of applying the commutative transformation for addition to xSyAz, yielding the nonequivalent DF, xSzAy.

Despite this serious drawback, the hypothesis warrants further consideration. A difficulty encountered in the Transformational Component of the linguistic theory concerns the representation of subtraction/negation. The transformation delineating polynomial multiplication could not be expressed in terms of subtracted terms. Instead it was necessary to express subtraction as the sum of a negative term (xSy ——> xA[Ny]) prior to applying the expansion transformation. While it is unlikely that subtraction is always so represented (for example it would be difficult to ascribe psychological validity to the reinterpretation of the "Difference-of-Squares" transformation as the "Sum-of-a-Square-Plus-the-Negation-of-a-Square" transformation), it appears to be impossible to operate upon subtractions directly in the context of some polynomial transformations.

Some data which seem to inadvertantly bear on this problem were collected by the author in investigating a quite different question (Kirshner, in press). A sample of 137 fourth year engineering students at the University of British Columbia were asked to evaluate each of the following expressions for $x = 2$:

1) $5x + 7 =$	2) $5x^3 =$	3) $4(6 + x) =$	4) $3 + 4x =$
5) $x^3 - 2 =$	6) $2^4 - x + 1 =$	7) $3 + 2x^3 =$	
8) $19 - 4x + 2 =$	9) $3 + (7x - 2) =$	10) $5 - x^2 + 1 =$	

Such problems are very simple for students at this educational level. Indeed only 14 students in the sample *did not* score perfectly (a total of incorrect responses and one ommitted response). Clearly these errors are a marginal phenomena, however, they are not random. Twelve of the 15 errors (including the missing response) occured with the trinomial expressions, #6, #8 and #10 (the lion's share going to #8). In each of these cases the response given (if any) was compatible with the incorrect parse of the expression; e.g. $19 - 4x + 2 = 19 - (4x + 2)$.

There are many explanations possible for these errors. It could be that unlike their peers whose syntactic representations are binary these students construct a non-binary representation. (Of course it would still remain to explain that ten out of eleven of these students got two of three similar questions correct.) Alternatively, it could be that subtraction, which for their peers is represented as addition of a negative, for these students is just subtraction. This, however, would seem to lead to the prediction that these senior engineering undergraduates would be unable to correctly rearrange terms in simple polynomials.

A third possibility does not require postulating such major deviation in the cognitive structures of the erring students. Questions #6, #8, and #10 are nonstandard problems in that usually only one or none of the terms in a polynomial is constant. It could be that the evaluation of polynomials is governed by an *ad hoc* left-to-right procedure. The need for initial focussing on a middle term (for substitution purposes) embedded between two constants may have been just sufficiently distracting to override this *ad hoc* constraint for a small minority of students. This explanation has the advantage of leaving the syntactic structure of expressions and the representation of subtraction homogeneous for the entire sample, entailing only a slight modification of cognitive structures to explain the errant behaviour.

This third explanation is consistent with the hypothesis of non-binary representation. A polynomial expression such as $3x^3 - 4x^2 - 2x + 1$ is not assigned the complex parse, $\{[[3(x^3)] - [4(x^2)]] - (2x)\} + 1$, but the much simpler and more flexible parse, $[3(x^3)] - [4(x^2)] - (2x) + 1$. *Ad hoc* constraints then prevent the **application** of transformational rules (e.g. Commutative and Arithmetic transformations) in ways which would lead to errant results.

Clearly this issue is not finally resolved by the sketchy considerations and evidence presented above, however, a case for the plausability of non-binary representations has been made. Besides recommending the issue for further analysis and research this report is also intended to bring into question the automatic practise of assuming some explanatory link between formal mathematical models (in which operations are binary, for example) and psychological models, and to emphasize the need for detailed and rigorous formulations of the latter.

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THE STATUS AND UNDERSTANDING OF GENERALISED ALGEBRAIC STATEMENTS BY HIGH SCHOOL STUDENTS

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A brief review of the results of the first year of a research project looking at grade 10 students' algebraic concepts is followed by a specific look at their understanding of generalised algebraic statements Attention is focussed on the work of one student across three problems The first problem reveals that students are fairly competent at producing generalised algebraic statements once a usable pattern has been perceived. A lack of flexibility in pattern perception seems to be the main stumbling block. In the second problem we see that once generalised statements are produced most students do not invest them with any meaning or see any use for them other than as a condensation of the problem statement. Only a minority of students seemed to see their use in substantiating a generalisation. That few students use algebra or appreciate its role in justifying a general statement about numbers, is the conclusion of the third problem.

This paper focuses on one aspect of a research project conducted by David Wheeler at Concordia University and funded by the Social Sciences and Humanities Research Council of Canada in 1986. Algebraic Thinking in High School Students Their Conceptions of Generalisation and Justification. A full report of the first year's work is available.

A test instrument of 4 questions was administered to 350 grade 10 students in three Montreal schools at the end of February. Each student responded to one question from each of four question groups involving a bank of 12 questions in all. Twenty-five of the tested students were subsequently interviewed for 30 minutes each while working on similar or the same questions Analysis of the test results and interview protocols supports the following general conclusions:

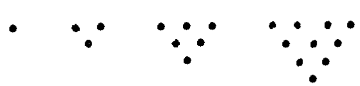
1. A majority of students do not appreciate the implicit generality of algebraic statements involving variables.
2. For most students, numerical instances of generalisation carry more conviction than an algebraic demonstration of the generalisation.
3. Many students do not appreciate that a single numerical counter-example is sufficient to disprove a hypothesised generalisation

4 Students who can competently handle the forms and procedures of algebra rarely turn spontaneously to algebra to solve a problem even when other methods are more lengthy and less sure

The first two findings are compatible with results obtained by other researchers (e g Bell, 1976, Fischbein and Kedem, 1982), though our data is generally richer and covers a greater variety of algebraic situations The last two findings have not, to our knowledge, emerged so clearly before

In a paper presented at PME-NA in Septemper 1986 we looked at students' conception of justification in algebra as revealed through the test and interview performance of one student, Eve This paper is in some sense a complement to that paper in that we will examine the other theme of our research, generalisation,with particular attention to the work of a second student, Yves. Yves is in many ways the complement of Eve Whereas Eve's work was reasonably typical of that of the majority of students tested, Yves' performance was quite unusual Judged by his regular teacher to be one of the weaker students, Yves nevertheless appears resourceful and comfortable with the language of algebra.

Students' ability to produce a generalised algebraic statement was tested using a series of problems involving generalisation of dot and number patterns We will examine here the dot triangle problem which was given to 8 interview students



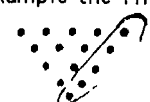
Suppose the above sequence of dot-triangles is continued according to th same rule, how many dots will there be in (i) the 5th triangle (ii) th 100th triangle (iii) the nth triangle?

A similar problem involving dot-rectangles was given to 176 students on the test and another 8 interview students Although the rectangle pattern seemed to be much easier for students, their work there does contribute to our analysis of that done on the dot-triangle.

Yves, who was given the dot-triangle sequence, perceived a whole series of patterns His first perception seemed to be a diagonal one. He drew the fifth triangle from the fourth by adding a diagonal realizing that the number of dots across the top equaled the number in the diagonal as well as the number of the triangle At the same time he established the number pattern for the total number of dots in the first six triangles While thinking about the hundredth triangle he established the formula "$x+x-1$..until $x=1$" which dissatisfied him because "it takes a long time". Asked about the hundredth triangle he wrote $100+99+98+..1$ and said he thought there might be a key on the calculator which would shorten the work At this point Yves switched to an entirely different pattern perception He began studying the dot triangles again and relating them to the total number of dots in each He began to see a pattern in the ratio (number of dots in triangle) ÷ (number of dots along side) "it seems to go down by 5" He explained to the interviewer "I'm just trying to get a constant" Very engrossed in calculations he suddenly wrote $x.((x\ 0{,}5)+0{,}5)$, encircled it and declared "That's it"

Looking more closely at Yves' shifting perceptions of pattern here, we might illustrate them as follows

1 Diagonal pattern each triangle is obtained from the previous one by adding a diagonal of n dots For example the fifth is obtained from the fourth by adding five dots



2 Equality pattern the number position of each triangle in the series equals the number of dots across the top as well as the number of dots along the side Yves' expressed it this way "dots across= # of triangle dots down= # of triangle"

3 Total number of dot pattern. students count the numbers of dots in the first five or six triangles and then proceed to establish the general term of the series 1, 3, 6, 10, 21, .. Most students, like Yves, arrive at the general expression $x+(x-1)+(x-2)$ ... $2+1$.

4. Linking two number series here triangles are ignored and a relationship is sought between the two number series (i) 1, 2, 3, 4, 5, ... representing for Yves the number of dots along the side of the triangles and (ii) 1, 3, 6, 10, 15, ... the total number of dots in the triangles. Dividing the terms of the second series by the first Yves gets the series 1, 1.5, 2, 2.5, 3, .. which he realizes is going up by 0.5 Letting x represent any number in the first series and focussing particularly on the fifth and sixth triangles, Yves creates the expression $x(0.5)+0.5$ which he then multiplies by x to get his final response This can be seen to be another form of the formula for the sum of the first n natural numbers.

Students exhibited many other perceptions of pattern here Some were more useful than others in suggesting a formula The key to success however seemed to be flexibility of pattern perception such as we witnessed in Yves' work Many students who had a single perception and who, like Yves, arrived at the sum of the first n natural numbers blocked there because they were unable to find a formula for this Interviewer interventions were particularly confusing to students in these questions because interviewers were constantly talking to their own pattern perception which in many cases was not that of the student Two pedagogical lessons can be learned here Firstly the importance of teaching students flexibility in pattern perception and secondly the importance as teachers of being aware of our own pattern perceptions and sensitive to other possibilities.

Expressing the perceived pattern in algebraic language did not seem to be a major problem for most students in these problems. Asked what the n th element of the sequence would be, students had no choice but to produce an algebraic generalisation The main stumbling block in producing a generalised algebraic statement was the ability to perceive a usable pattern Only one other student was able to solve the dot-triangle problem and she, like Yves, showed great flexibility in pattern perception

A second question, given to 116 of the test students and 9 interview students, involved both generalisation and justification

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A girl multiplies a number by 5 and then adds 12. She then subtracts her starting number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, "I think that would happen, whatever number I started with."
Is she right? Explain carefully why your answer is right.

This question was dealt with in considerable detail in the PME-NA '86 paper but with the accent on justification rather than generalisation In that paper we looked particularly at the work of Eve whose work was typical of that of a third of the students given this question. "As I read the problem I wrote down the formula That's what I always do." was Eve's explanation of the correct algebraic identity written directly underneath the test question. The status of Eve's generalised algebraic statement became increasingly clear as the interview progressed. Eve used the identity to set up her first numeric example and then abandoned it Successive examples were created from the first example and her conclusion that the girl was not right was based on her four numeric examples Eve's creation of a generalised algebraic statement seemed to be automatic but entirely meaningless for her

Yves did not go into automatic algebra mode as Eve did and his progress through this problem was a constant struggle to understand why the girl is right After trying the particular cases of starting with 3 and with 6, he said he thought she was right Asked how he knew, he referred to the fact that "you can always divide it by 4 for some reason". Asked if it would work starting with 5287, he reexamined his example starting with 6 and eventually said, "I guess whenever you times the number by 5 and subtract what you timesed it by, and you add 12, it's divisible by 4 ... Yeah You always get, whenever you, uh, multiply something by 5 and you subtract by what you multiplied it by, it's always going to be a multiple of 4" Yves checked another example "Why do you think that is?"

He now quite spontaneously generalised his observation and began checking it out "I'm trying it Instead of using 5 I'm using the number below . and it works. Like if you multiply by 4, if you multiply any number by 4 and you subtract what you multiplied by, it's going to be a multiple of the lower number, the one below (i e 3)" Asked why, he said "Maybe it's just the way numbers work", and invited to establish the property without recourse to vague statements he wrote y x - y = mult x-1

On the evidence of this protocol, Yves is able to "see the general in the particular" and to move confidently from particular examples to generalisations and vice-versa When asked if 5287 would work as a starting number, he goes back to his worked example of 6 to find an answer, and he finds without prompting a generalisation of the structure of 5x-x. Krutetskii talks of "seeing the general in the particular" as a characteristic marking off able students from the rest

> " . there is another way, in which able pupils, without comparing the 'similar' .. independently generalise mathematical objects, relations and operations 'on the spot', on the basis of the analysis of just *one* phenomenon They recognise every specific problem at once as the representative of a class of problems of a single type . " (Krutetskii, 1976)

It seems possible that this method of generalisation is not confined to able students but is paradigmatic of everyone's generalising style. This point, however, is not made in the psychological literature on generalisation and may be difficult to establish

The third and final problem we will examine here was one of a series of three problems concerning the justification of statements about consecutive numbers.

The product of two consecutive whole numbers is an even number
Is this statement true? Can you explain how you know?

How does one justify a general statement about numbers? Does algebra have a role in this? In a test question where he was asked to choose the best response to a similar problem concerning the sum of consecutive numbers Yves, like 40% of students chose an algebraic demonstration over a verbal one and a justification based entirely on 3 examples

Given the above problem Yves wrote the numbers from 1 to 10 with the product of each consecutive pair underneath and decided the statement was true He searched for an explanation why and repeated the problem statement, suggesting that ". an odd times . I think it cancels out, or something, and the even wins." Later. "it's a law or something". When asked for "something more mathematical" he produced the algebraic expression x^2+x and used it to demonstrate evenness by considering the cases x odd and x even separately He worked through the case of x even and x odd referring to x=6 and x=7 but in a very different way than most students who introduced numeric examples Looking at his argument we see that the 6 and 7 are used more to illustrate than justify.

> "Every time you square an ev .. an odd number you get an odd square I ... forty-nine. Whenever you add two odd numbers together it's always an even. (Here interviewer says "That's too fast for me") Okay, well x squared is these sevens ... is forty-nine (Here he writes 49+7=56) .. If you use an even number let me think. Five, no, I mean six .. plus six. You get 36 plus 6 which is equal to 42 So it's always even." (Writes $6^2+6=42$)

When the interviewer asked "but how do you know that's going to work for other even numbers?", Yves replied " 'Cause of the formula, it should."

Although Yves was slow to introduce algebra here he did not seem to be satisfied with his explanation until he used the algebra. He was the only interview student who used an algebraic demonstration here On a similar test problem which asked students to explain why the sum of two consecutive numbers is always an odd number and their product even, only

8 of the 118 students used algebra in their justification of the sum and only 2 of these were able to use it in the case of the product 27 students did express consecutive numbers as x and x+1 and showed they were able to write the expressions for their sums and products but the majority of these used their algebra either to create examples by substituting values for x or to set up equations and solve for x (i e 2x+1=7, x=3, x+1=4 which are consecutive numbers) The justification produced in the consecutive numbers questions appeared to be very solidly entrenched in number examples and both the algebra and to a lesser degree the even/odd non-algebraic discussions were more for the form or peripheral to the main work Yves was one of a very small number of students who seemed to appreciate the role of algebra in justifying a general statement about numbers

Our look at student's appreciation of generalised algebraic statements is very incomplete and will need to be the object of much more systematic research We hope to continue our research looking at the influence of the instructional context on students' understanding of generalisation in algebra and undertaking a teaching experiment to determine whether students' understanding of generalisation can be improved by special instruction

To date the research literature has not been extremely helpful Considerable literature exists concerning the theme of generalisation. Some of this literature concerns generalisation as a human activity, some restricts discussion to mathematical generalisation, and some touches on algebraic generalisation All authors seem to presume that everyone knows what generalisation is although no two authors seem to be considering the same activity and many tend to jump about in the meaning they give to generalisation within a single discussion. The confusion surrounding generalisation is compounded by a lack of clarity on what constitutes algebra leaving the definition of algebraic generalisation totally arbitrary. For example, in the Open University text Routes to/Roots of Algebra (1985) we read. "Generality is the lifeblood of mathematics and algebra is the language of generality" (p.8) Later however in the same text we read: "algebraic language provides both one way (there are others) of expressing generality because it is compact and succinct, as well as a tool for manipulating general expressions to reveal new relationships among them." (p.56) What is clear from the literature as well as our own research is that much more work needs to be done in the area of generalisation if it is indeed "the lifeblood of mathematics" and more particularly in the area of algebraic generalisation if algebra is to become "the language of generality" for our students.

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A PSYCHOLINGUISTIC PERSPECTIVE OF ALGEBRAIC LANGUAGE

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This paper includes a description of a psycholinguistic perspective from which we examine relationships among language, cognition, and experiential phenomena. Language influences both thought and perception - this thesis is followed through in an explanation of common syntactical misconceptions in students' interpretations of algebraic structure.

There has been, over the last two decades, an expanding body of research dealing with the learning of algebra. Results of several recent studies include the delineation of students' difficulties with algebra that are associated with algebraic symbolism. Foe example, Matz (1979) and Chalouh & Herscovics (1983) have identified and investigated misconceptions about concatenation of numerals and literal variables; Kieran (1984), among others, has indicated that students sometimes perceive variables in algebraic equations differently than they do in algebraic expressions; Wagner (1981) pointed out difficulties some students had with changing the literal variable in equations and has also described some of the semantic differences between verbal and numeric variables (1983); Wagner, Rachlin & Jensen (1984) and others' have made important contributions to the literature concerning students' interpretations of algebraic language. The variety of syntactic and semantic interpretations which students give to algebraic language suggests that a psycholinguistic perspective of these interpretations may be helpful in elucidating students' understanding of algebra.

PSYCHOLINGUISTIC PERSPECTIVES

Since the turn of the century psycholinguistics has developed into a complex, eclectic field reflecting a variety linguistic perspectives, epistemologies, and theories of cognition and cognitive development (e.g., Hörmann, 1970). This is not the place for a retrospective of the historical development of psycholinguistics; however, a few words may help to illuminate the genesis of the perspective taken in this paper. [A more fully developed and detailed exposition of a psycholinguistic approach to algebra will be found in Norman (in preparation).]

In its simplest formulation the object of psycholinguistics is to describe the processes of language use. The work of Vygotsky (1962) and Luria (1982), particularly in reference to the ontogenesis of language and the role of language in the regulation of thinking, has significantly influenced my general approach to the application of psycholinguistics to algebraic language. Additionally, Whorf (1956) proposed complementary hypotheses of linguistic determinism (language determines the categories in which we think) and linguistic relativity (different languages constrain the development of relatively different cognitive categories) which, in a modified form, underlie some of the assumptions taken here. Although reaction to Whorf's theses has been mostly negative, a recently developed paradigm, cognitive structuralism, has placed on a much firmer theoretical base investigations of the related question, "What are the conditions and constraints on the influence of linguistic constructs in the shaping of thought?". The cognitive structuralist perspective, founded in the work of Piaget and Chomsky, holds, as guiding tenets, that:

1) Cognitive structures (separate from behavior) mediate between perceived phenomena and our reactions (behaviors) to those perceptions;
2) Cognitive structures develop via interaction with external phenomena; and
3) Cognitive structures are distinct from, but influenced and elaborated by, language (Bloom, 1981).

Figure 1 is a simplified schematic representation of the cognitive structuralist view of the associations among perceived phenomena (world), the mediating function of cognitive structures (cognition), and language. The bidirectional arrows represent cognitive mechanisms (such as perception) and non-cognitive processes (such as sociocultural influences) which interactively affect the three constructs of world, cognition, and language.

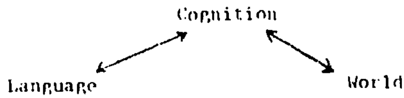

FIGURE 1

The diagram does not adequately reflect the dynamical relationships and interactions among the three constructs. In particular, note that linguistic understanding (and thus its influence) is subject to evolutionary processes, and develops, if not in parallel, at least synchronously with the evolution of cognition and the world, continuously influencing and being influenced by them.

ALGEBRAIC LANGUAGE

The origins of natural verbal and written language are different - verbal language develops sympractically, primarily through social interactions, whereas written language emerges from special learning (Luria, 1982). Nevertheless, there is a direct syntactic isomorphism between natural verbal and natural written language. Note that I do not infer a semantic isomorphism, although for the native speaker of a language the natural association of written language with its verbal image (under the syntactic isomorphism) is semantically rich. A prime feature that distinguishes symbolic algebraic language from symbolic (i.e., written) natural language is the evident fact that algebraic language no longer has a direct, coherent <u>semantic</u> association with verbal language. Any experientially-based frames of reference for algebraic language are too weak to supply students with an adequate semantic support for algebraic understanding.

Mathematical language is envisioned here as a web of symbolic dialects- an arithmetic dialect, an algebraic dialect, a set-theoretic dialect, and so on. Now, keeping in mind the tenets of cognitive structuralism, we consider some linguistic influences on learners' construction of an algebraic grammar (following Chomsky). Two important influences on the construction of an algebraic grammar are related to the depth of knowledge of natural language and knowledge of the arithmetic dialect. Evidence from some of the studies mentioned previously suggests that sometimes students attempt to gain a syntactical understanding of algebraic structure by applying (often inappropriately) syntactic rules from natural language or arithmetic. Thus, both inter- and intralinguistic mechanisms are involved in students' constructions of algebraic grammar.

A simple illustration of the above remarks can be found in students' misconceptions involving concatenation of a numeral with a literal. A student, say, who perceives the algebraic expression 5y as an integer in the 50's has made an inappropriate application of legitimate arithmetic rules of syntax in an algebraic situation (which, of course, has its own, quite different, syntactical structure). The example here shows that the arithmetic dialect influences the perceived structure of the algebraic dialect (see diagram below).

arithmetic syntax ⟶ algebraic syntax

The following example illustrates how the syntax of natural language might influence students' perceptions of algebraic structure. Consider the following two problems given to a class of elementary education majors (they were not presented consecutively):

(1) Larry made two donations to the World Wildlife Fund totaling \$60. One donation was for \$40. How much was the other?

(2) Joan drove a total of 50 miles in one hour. On one part of the trip she drove 35 miles per hour. How fast did she drive during the other?

Although the semantic content, especially the quantity structure, of these two problems is significantly different, the syntactical structure is identical. In each problem we have "quantity 1 combined with quantity 2 results in total quantity." The syntax appears to impose an additive structure in the transition to algebraic language- perhaps, $Q1 + Q2 = T$. In fact, 8 of 22 students apparently found the natural language syntax powerful enough to dominate both the natural language and algebraic semantics of the situation, and arrived at a comfortable 15 miles per hour for the second part of Joan's trip. Thus, we have a clear example of natural language syntax influencing the structure of algebraic syntax.

natural language syntax ⟶ algebraic syntax

Actually, it is not too surprising that students would ignore the semantic content of the natural langu... questions posed in th previous example. After all, students are encouraged to make such translations whenever possible (the "whenever possible" is th often overlooked key). Percentage problems come to mind as a class of statements which, when stated as textbooks usually do, are sy...actically isomorphic with algebraic language.

The powerful influence that both natural language and arithmetic syntactical structures have on the development of an algebraic syntax suggests that there is a linguistic aspect to the cognitive obstacles that arise when a student is experiencing conflict among two or more frames of reference. Herscovics & Chalouh (1985) have described some of these obstacles as they emerge in the transition from an arithmetic to an algebraic frame of reference. It is worthwhile to note here that these researchers attempted to have their students circumvent some of the obstacles by linking the arithmetic and algebraic structures together via a common geometric association. It may be the case that the figural syntax of geometry is an effective mediator between arithmetic and algebraic language.

A third example of how syntax has a structuring influence on the development of an algebraic grammar illustrates more than the previous examples the evolutionary nature of the grammar. In fact, this example is intradialectical, almost self-referential, because it illustrates a syntactical transition within the algebraic dialect itself. Consider the introduction of functional notation (e.g., y = f(x), sin(x), etc.). This new notation is more subtle (syntactically) than we might think (I still have calculus students who write sin(x)/x = sin) because the symbolism is not new, but the interpretation is. Just when students know parenthetical expressions are multiplied (f(x) means f·x and f(x + h) = fx + fh), as are concatenated literals (xxyx3 = $3x^3y$ and similarly csc(2) should be $2c^2s$), they find out otherwise! Whatever grammar the students have when they are first exposed to functional notation, it must be elaborated in order to accommodate the new structural interpretations. (Note that part of this elaboration will include rules which must take into account mathematical conventions. Conventions are essentially a product of social interactions, thus we see that a Vygotskian perspective is relevant.)

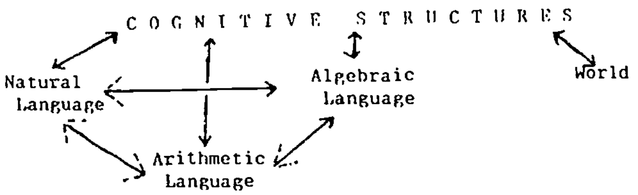

FIGURE 2

Figure 2 summarizes some of the connections which have been discussed so far. The dashed arrowheads indicate directional influences that have not been discussed but which fall within the constraints of the psycholinguistic perspective used here.

An idea that is reinforced by the three examples we discussed is that the development of an algebraic grammar is an evolutionary process. Apropos of syntactics, such a grammar is simply a theory of correct syntax which is constantly undergoing testing and modification until the theory becomes an adequate formulation of rules for the construction ot algebraically grammatical syntax. What has been said so far does not even address the more important question of semantics. Semantic knowledge of algebraic grammar can not be rule driven (as syntax can) any more than can semantic knowledge of natural language. But, as there is no coherent link between algebra and natural language, it is difficult to see how to provide an enriched semantical structure. The ideal situation would be to find a semantically rich representation system and an isomorphism which linked such a system to algebra. A more promising alternative would be to model the development of algebra after the development of verbal language- i.e., sympractically. The implications of such an approach are exciting and could be far reaching.

CONCLUDING REMARKS

This paper represents a first response to Wheeler & Lee (1986) who suggest the importance of opening a dialogue concerning the roles that various aspects of psychology might play in investigations of algebra. I have endeavored here to take a look, albeit through a rather narrow lens, at a few linguistic influences on algebraic language. I suggest that the field of psycholinguistics holds great promise in providing us with tools for examining and eventually understanding students' conceptions of algebra.

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ERROR PATTERNS AND STRATEGIES IN ALGEBRAIC SIMPLIFICATION

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This study examines the underlying strategies utilized in simplifying algebraic expressions and students' rationales for their processes. Written tests were given to 230 grade 10 students and 20 of these students were individually interviewed and asked to simplify expressions involving the product of two monomials. An analysis of audio-taped interviews indicated that students were operating in two distinct, but not disjoint spaces, namely an algebraic and arithmetic space. Incorrect solutions often resulted from the application of inappropriate algorithms or principles within the algebraic space. This was caused by the invalid generalization of an arithmetic algorithm or principle to the algebraic space. Other errors occurred because of the incomplete conceptualization of the algebraic space.

INTRODUCTION

The study of algebra constitutes a major component of high school mathematics. The basis of algebra is the concept of a variable and its associated notations. Comprehending the solution of equations, factorization, polynomials, etc., depends on students' comprehension of algebraic symbolization. Without this comprehension, algebra will be internalized as a set of disjoint and meaningless rules.

Matz (1980) in investigating the nature of algebraic errors indicated that students try to extend and adapt their arithmetic knowledge to algebraic space and this model has been utilized by many researchers in their attempts to analyze and discuss algebraic errors (Kieran, 1984; Herscovics & Chalouh, 1985). The question arises as to the extent the mathematical relationship between arithmetic and algebra parallels students' perceptions.

Researchers such as Kuchemann (1981), Mason and Pimm (1984) and Booth (1984) have investigated students' interpretations of a variable. Their research indicates that students develop different interpretations. Furthermore, many of these interpretations do not appear to derive from an extension of arithmetic space to algebraic space.

Other researchers such as Carry, Lewis and Bernard (1980), and Pereira-Mendoza (1984) found that students do not interpret algebra as generalized arithmetic.

As Byers and Erlwanger (1984) stated:

> It is well known that traditional high school algebra consists largely of repetitive symbolic manipulation and that, pe-haps of necessity, this characteristic of the subject persists to the present day. It is equally well known that by and large students do not understand what they are doing. (pp. 265-6)

If one accepts the reality that students do not understand the algebraic manipulations they undertake, then two questions arise: What underlying rules do students utilize to manipulate algebraic expressions? How do they do they interpret the rules?

SAMPLE AND PROCEDURE

Written tests were given to approximately 230 Grade 10 students randomly selected from various schools in the Province. In the algebraic test the students were asked to simplify the product of monomials such as $3y^3.4y$ $5y^4.6y^2$ etc.

The types of errors were categorized according to the pattern of errors. For example, patterns involving the incorrect combining of exponents, patterns involving the addition of coefficients and patterns involving incorrect signs when multiplying integer coefficients. A subgroup of 20 students was selected for in-depth interviews. Those interviewed included both students with identifiable error patterns (16 students) as well as students who had obtained correct solutions (4 students). The students were individually interviewed by the researcher and the sessions audio-taped. The purpose of the interviews was to determine the rationales for the processes the students were using to simplify the expressions. This report presents part of the results from the interviews.

RESULTS

1. A very common error was simplifying problems involving y.y. For example, many students simplified 3y.4y as 12y. There were three main rationales for this error:

a. Students incorrectly generalize the distributive principle.

e.g. interpreting $3y.4y$ as $(3.4)y$.

In the interviews students would make comments such as "you take out the y". One particular student stated that he was applying the distributive principle. When asked to explain the distributive principle most students would resort to either a general statement such as it means "taking out the common term", or select an arithmetic or algebraic example involving the distributivity of multiplication over addition. When asked to compare what they had written or said with their solution to the problem (which involved only multiplication), only in two cases did this result in a changed view (the students corrected their process). In all other cases they could see no 'difference' since all the expressions have a common term.

b. Students interpret y as meaning just y and hence they conclude that $y.y = y$.

When asked to explain, students would make comments such as "y.y is just y". When probed further it was clear that these students did not have any comprehension of a variable. In fact, students used many of the interpretations of a variable found in the literature. The following is part of an interview with Ann (A) and the investigator (I).

[Ann wrote $3y.4y = 12y$]

I: Can you explain what you did?

A: 3 times 4 is 12 and y times y is y

I: What do you mean by y times y is y?

A: y is just y

I: What is y?

A: y is algebra... we use it all the time. Sometimes we use x or z... any letter will do.

c. Students interpret . to mean 'multiply everything' so they conclude that $3y^4.2y^5 = 6y^{20}$

Comments in the interviews tended to include statements such as . means multiply so you multiply everything.

2. Students do not see algebra as generalized arithmetic. Even students who have obtained correct solutions were unable to clearly articulate the relationship between arithmetic and algebraic simplification; for example,

the relationship between simplifying 3y.y and (3.471).471. This was clear from a variety of responses.

a. Even when asked how they might check their answer, few students suggested substituting a number. Two students who did undertake the substitution concluded that "the answers were different because one is algebra and the other arithmetic." They could see no contradiction between obtaining, $12y^{24}$ for $3y^4.4y^6$ and $12(371)^{10}$ for $3(371)^4.4(371)^6$

b. For those students who did not try a numerical substitution the investigator made the suggestion that they check their answer by substituting a numerical value for y. Four students asked, "What number should I use?" On questioning they were unsure whether the number used would alter the relationship. The following is part of an interview between John (J) and the investigator (I).

I: Could you try a number?

J: What number should I use?

I: Does it matter?

J: I'm not sure. I could try 2 or 3...

I: Try them and see what happens

J: [Student substitutes]

It's not working...[Student checks work]

I: Why not?

J: Don't ...Must have done it wrong... This isn't algebra...

c. Any substitution was a whole number. Even some students who were sure that any whole number would do, were uncertain if a decimal would work.

3. The following student was of particular interest. Joanne solved all the problems by substituting a value for y. She assigned a numerical value from the outset and proceeded to solve the arithmetic problem, concluding by reverting to the corresponding algebraic expression for the solution. When asked to explain she indicated that she "had trouble with letters", but using a number and "going backwards always worked." Her explanations seemed to indicated that she comprehended that the algebraic rules were the same as the arithmetic rules,, although she was not sure why.

DISCUSSION

Overall, the analysis seemed to show that students operate in two distinct, but not disjoint spaces, namely an algebraic and arithmetic space. Students' perceptions of the relationships between these spaces is complex, being influenced by their comprehension of arithmetic and algebraic concepts, principles and rules.

The surface explanation for many errors was the misapplication of a principle, or the invention of an algebraic algorithm for a given situation. For example, the misapplication of the distributive principle resulted in students' concluding that 3y.4y = 12y since y is a 'common term'. Students invented a rule that . meant multiply everything, resulting in students multiplying exponents. Such surface explanations do not explain the cognitive processes underlying the development of invalid algorithms or the invalid generalization of principles. The underlying problem appears to be a combination of the perceived relationship (or lack of a relationship) between the algebraic and arithmetic spaces, together with an incomplete conceptualization of algebraic space.

Students first experience with number is in a physical situation. In the development of arithmetic ideas it is expected that students will progressively and slowly develop an abstract notion of number. Thus, arithmetic algorithms and principles can be attached to abstract situations, because it is assumed that students have had the concrete experiential base on which to build. A parallel experiential base for learning algebraic manipulation does not exist. Students are expected to make the connection between arithmetic space and its generalized form (algebraic space) without the appropriate foundation. This results in the development of an algebraic space that is faulty in terms of its structure and is incompletely conceptualized. Consequently, when arithmetic algorithms, principles, etc., are mapped onto the algebraic space, the resulting transformed algorithms, principles, etc. are invalid and result in incorrect solutions. An example would be the attempt to apply the distributive principle to the expression 3y.4y obtaining the answer 12y. This is caused by both an incorrect mapping of the principle and a miscomprehension of the meaning of a variable. Thus, what on the surface appears to be a invalid application of a principle is, in many cases, the 'correct' application of a principle in an invalid situation. The fault lies in the application of the principle to a faulty algebraic space, not in the principle, per se. Similarly, when students generate invalid algorithms, the basis of the error lies in their view of algebraic space.

In conclusion, it is important to note that even students who could correct simplify the algebraic expressions did not have a well developed cognitive basis for their procedures. Rather, the correct solutions often resulted from a pragmatic application of algorithms. In fact, when pushed to explain a correct procedure, one student got annoyed and finally informed me that "It's the rule when you have letters. Everyone knows that you add the numbers" (referring to the adding of exponents).

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UNDERSTANDING SIGN CHANGE TRANSFORMATIONS

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This study examined students' understanding of sign-change rules in elementary algebra, with a focus on their informal, intuitive understanding of quantities in situations and their ability to link this understanding to formal mathematical expressions. Students from grades 5, 7, and 9 participated in an interview in which they judged the equivalence of formal expressions and of story situations, matched expressions to situations, and modified situations to fit expressions. Students were considerably more successful in judging the equivalence of the situations than of the formal expressions and made few spontaneous links between the two domains. Errors made in modifying situations to match expressions revealed difficulties in applying successive transformations and in interpreting expressions as representing quantities.

An important part of learning elementary algebra is learning to apply various transformations to the symbols in algebraic expressions and equations. Algebra derives its power from the representation of situations (such as those described in word problems) in a formal language in which manipulations can be made independent of the initial situation. Much of the elementary algebra curriculum focuses on the learning of the rules for manipulation of this formal symbolic system--rules for transforming expressions and equations. But students often attempt to learn these rules without linking them to their informal, intuitive understanding of mathematics, reflecting the broader problem of formal school mathematics learning often failing to build upon more informal quantitative knowledge (Ginsburg, 1977; Resnick, in press). Similarly, current theories of algebra learning (Matz, 1983; Sleeman, 1984) account for errors as deformations of symbol manipulation rules; they involve no representation of the quantities or relationships among quantities. Our research examines students' <u>understanding</u> of the manipulations and transformations of algebra, with a focus on situations to which algebra expressions might refer.

In this study we examined students' understanding of a basic set of transformational rules in elementary algebra: sign-changes in addition and subtraction expressions with parentheses (e.g., a-(b+c)= a-b-c). The focus was on (a) children's informal, intuitive understanding of the principles underlying the sign change rules, (b) their formal knowledge of the rules applied to symbols, and (c) their ability to link the two.

Because we were interested in students' intuitive understanding of the principles underlying the sign-change rules before as well as after instruction in algebra, we interviewed students in grades 5, 7, and 9. Our main sample consisted of 28 students from each grade level in urban and suburban parochial schools. In addition we interviewed 8 ninth-grade students from an accelerated algebra class and 14 ninth-grade students from slower paced algebra classes. Each student participated in a three-phase interview, in which he or she judged the equivalence of story situations, judged the equivalence of pairs of expressions, and chose expressions that fit story situations.

EXPRESSIONS AND SITUATIONS USED

Two sets of expressions were used in the interviews. The first set consisted of the expression a-(b+c), its correct transformation, a-b-c, and its frequently made incorrect transformation, a-b+c. We call this set parentheses-plus because of the plus sign inside the parentheses. The second set, parentheses-minus, consisted of the expressions a-(b-c), a-b+c, and a-b-c. The expressions seen by students used numeric values in place of the letters. Story situations that can be described by the expressions were generated for each set of expressions. The story settings involved adding and subtracting money (in a store) or combining and changing sets of discrete objects. Following are two of the story sets:

Parentheses-Plus Situations in Discrete Object Setting (Cupcakes)

1. David took 18 cupcakes to the bake sale. He sold 7 chocolate ones and 2 yellow ones. (This story is described by 18-(7+2))

2. David took 18 cupcakes to the bake sale. At lunch time he sold 7 chocolate ones. After school he sold 2 yellow ones. (This story is described by 18-7-2)

3. David took 18 cupcakes to the bake sale. He sold 7 chocolate ones. Then David's mother brought him 2 more yellow cupcakes. (This story is described by 18-7+2)

Parentheses-Minus Situations in Money Setting (Record Store)

1. Sally went to the record store with $14 and bought a record. The record was usually $8 but was marked $3 off. (This story is described by 14-(8-3))

2. Sally went to the record store with $14. She bought a record for $8. After Sally paid for the record, she remembered she had a $3 gift certificate. So, the clerk gave her $3 in cash for it. (This story is described by 14-8+3)

3. Sally went to the record store with $14. She bought a record for $8. On her way out Sally saw another record she wanted to buy. She bought it for $3. (This story is described by 14-8-3)

THE INTERVIEW: PROCEDURE AND RESULTS

Each child participated in a three-phase interview. Phase 1 assessed the student's informal, implicit understanding of the principles underlying the sign change rules. The student was presented with each of the sets of three stories, asked to say which stories were "about the same," and to justify the choice. Students were generally quite successful in judging the equivalence of the story situations and justifying the equivalence in informal terms. An adequate explanation of the Cupcakes stories 1 and 2 (see stories above) would state that in both stories David sold the same number of cupcakes; it does not matter whether he sold the chocolate and yellow ones at the same time or at different times. The percentages of students choosing the correct story pairs ranged from 77% correct for fifth-graders to 91% for the ninth-graders. Percentages of students who gave adequate explanations of the equivalence of the various stories are presented in Table 1. As can be seen, performance generally improved over the grades. Students were considerably more successful judging and explaining the parentheses-plus situations than the parentheses-minus sets.

Phase 2 assessed knowledge of the sign change rules applied to formal expressions by having the student judge the equivalence of six pairs of expressions to which the sign change transformations had been correctly and incorrectly applied (16-(8+3) compared to 16-8+3, 16+8-3, and 16-8-3; 11-(5-2) compared to 11-5-2, 11-5+2, and 11+5-2).

Table 1

Percentages of Adequate Explanations of Story Equivalence in Phase 1

	Parentheses-Plus		Parentheses-Minus		
Grade	Objects (Cupcakes)	Money (Toys)	Objects (Cookies)	Money (Records)	Row Mean
5	74	86	26	43	57
7	96	93	48	64	75
9	85	89	68	79	80

Students were much less successful with these formal comparisons than they had been in judging the situations in Phase 1. Only 23% of the students correctly judged the expressions 16-(8+3) and 16-8-3 to be equal; 31% of students made correct equivalence judgments of the expressions 11-(5-2) and 11-5+2. Even the ninth-graders, who were taking algebra, did poorly in making formal equivalence judgments. Thus, students' knowledge of the sign-change rule applied to mathematical expressions was weak, even after instruction in algebra. The students did not apply their informal knowledge of the quantitative relationships involved to the formal expressions. Further evidence of this failure to draw upon informal knowledge is offered by the fact that students were <u>less</u> successful in correctly judging the <u>parentheses-plus</u> expression pairs than the <u>parentheses-minus</u> pairs--the opposite of the pattern found in judging the informal situations. In addition, the justifications students gave for their judgments of the formal expressions never involved reference to the structurally equivalent situations in Phase 1. Rather students relied on computation-based justifications, surface-level comparisons, or the application (often incorrect) of rules for operating on the symbols.

Phase 3 examined the student's ability to map the formal expressions and the situations. The student was presented with the story from each set that was best described by the expression with parentheses (e.g., the first story in each of the sets presented above). The student was asked to choose from a set of three expressions the one that best fit the story and to explain why that expression fit. Success on this task paralleled performance in

Phase 1, with approximately 70% to 90% of students choosing an appropriate expression and 50% to 80% giving adequate justifications of why the expressions fit.

For the other expressions in each set (the ones not chosen as describing the story) the student was asked to modify the story to make it fit the expression. This task produced an interesting array of errors. Many of the errors reflected difficulties in conceiving of successive transformations to a quantity. For example, for the expression 18-7+2, one ninth-grader modified the Cupcake story as follows: "David took 18 cupcakes to the bake sale. He sold 7 chocolate ones and, (pause) he didn't sell the yellow ones." This student was unable to incorporate the second transformation (+2) appropriately into the story, resulting in its interpretation as a state (number of cookies left over) instead of a transformation (number of cookies sold, or removed, from the start set).

The largest categories of erroneous modifications appeared to be a result of students treating expressions in a strictly linear and localized fashion, rather than conceiving of the entire expression as representing a quantity. For example, in a typical modification of the Record Store situation to fit the expression 14-8+3, one ninth-grade student said "Sally went to the record (store) with $14 and bought a record; the record was usually $8, but it was marked higher; it was raised, the price was raised $3." This student correctly interpreted the +3 as an increase ("it was raised"), but applied the increase to the wrong quantity--the price of the record instead of the amount of money Sally had. The student did not seem able to construct an adequate representation of the two successive transformations in the expression. Errors like this were made by numerous students on all of the stories and expression types. They had considerable difficulty constructing appropriate situations for expressions, again reflecting difficulties in linking the formal symbols with the reference domains represented in the situations.

Even after matching expressions to and modifying the situations in Phase 3, few students were able to justify the equivalence of the expressions in terms of the stories. The potential power of thinking of the expressions in terms of the situational referents is, however, illustrated by the students who were successful in explaining the equivalence of the expressions in terms of the situations. For example, one seventh-grader in Phase 2 had declared the expressions 16-(8+3) and 16-8-3 to be not equal because in the first "you're saying 16 minus 8 plus 3" and in the second "you're saying 16 minus 8 minus 3." Note that the student seemed to be ignoring the parentheses, focusing only on the fact that there are different operations in the two expressions. After some false starts matching expressions of the same type to the Cupcake story in Phase 3 (Cupcake story 1 above), this student correctly chose and justified the expression 18-(7+2) as best fitting the story, and modified the story appropriately for the expressions 18-7-2 ("he sold 7 yellow cupcakes and the next day he went to another bake sale and sold 2") and 18-7+2 ("He sold 7 cupcakes at the bake sale, and then when he got home, his mother baked him 2 more."). When subsequently shown just the three expressions (18-(7+2), 18-7-2, and 18-7+2) and asked if any of them "would come out to be equal," the student correctly said that 18-(7+2) and 18-7-2 would be equal "because in each story David sold the same amount of cupcakes out of 18, so naturally it's going to come out the same answer." By mapping the symbols in the expressions to quantities in the situations, it had become obvious to this student that the two expressions are equivalent. The student was thus able to link the formal symbols to this more intuitive knowledge about how quantities behave in situations.

SIGNIFICANCE

The ultimate goal of this line of research is to develop ways to improve students' understanding of the symbolic manipulations they learn in algebra. We believe that increasing students' understanding of the referential meaning of algebra's formal symbol system may facilitate the learning of formal rules and the application of algebra to problem solving and learning more advanced mathematics. This study provides important psychological description needed as a base for the development of instructional interventions.

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PME XI ALGEBRA PAPERS: A REPRESENTATIONAL FRAMEWORK

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This set of papers is uniformly excellent in depth of analysis and in importance of issues addressed. These papers and other recent developments in mathematics education research are the basis for real optimism in the future of algebra learning and teaching, especially for long term constructive improvement of the algebra curriculum. Research is beginning to identify specific reasons why algebra is so hard to learn and what appropriate curricular and pedagogical responses might be. This is not an easy task, because algebra is a complex domain, both in the structure and in the multiplicity of its representations.

This paper has two parts. The first is an attempt to draw curricular conclusions from the work presented. The second develops and applies a theoretical framework to this reviewed research. I will not explicitly cite other work published by the authors relating to the papers under discussion despite the fact that some of that other work has frequently influenced what I have written here. Readers can find such references in the bibliographies of the papers under review.

A. SHARED PERSPECTIVES ON CURRICULUM.

Too Much Meaningless Symbol Pushing - Algebra Alienation.

First of all, there is choral unanimity calling for much less curricular emphasis on manipulation of algebraic objects in the absence of meanings for those objects and the actions on them. This unanimity extends far beyond the researchers represented here - virtually everyone who has given a critical look at the standard algebra curriculum criticizes it on these grounds.

This experienced meaninglessness of school mathematics is at the heart of the attendant and devastating problems of lack of motivation and inability to apply mathematics as a tool of personal insight and problem solving. Further, this core problem of school mathematics alienation is compounded by the inherent difficulties in dealing with a formal symbol system isolated from other knowledge that might provide informative feedback regarding the appropriateness of actions taken or a cognitively stabilizing context for those actions.

The traditional curricular response to student difficulty with maneuvering symbols in isolation is to sequence small pieces of activity carefully organized by syntactical features of the symbol system and to isolate this activity from the messiness of "applications" and wider interpretations. The standard pedagogical response is to schedule ever more practice with such symbol maneuvering. One mode of educational research response, feeding from recent skill acquisition work in cognitive science, is to parse the structure of such symbol manipulation skill learning and application in order to design appropriate instruction. (I am happy to note that none of the research in this collection of papers is of this genre.) A recent technological response is to accept the "isolationist" approach, but to augment the skill learning environment with additional features to enrich the experience of symbol manipulation, by providing "history windows," explicit representation of computation or reasoning paths, "inspectable experts," etc., or to add "intelligent advice" on how to manipulate symbols in isolation of wider meanings.

The response on the part of the students has been highly adaptive - to use ever more superficial learning strategies, resulting in even more alienation, which in turn feeds the responses already listed. It is too early to know the effect of the kinds of technological responses mentioned here. My bets for breaking the feedback cycle lie with intelligent use of information technology (not necessarily "artificial intelligence" based, however) that respects the role of teacher and student as sense-making creatures and that capitalizes on student knowledge and skill that have been developed outside the mathematics classroom. Apart from the technology, it appears that most of the authors reviewed here would agree with this view.

Integrate Arithmetic and Algebra - But It Won't Be Easy.

Here the consensus is not as plain, and is much more implicit. Several papers (**Booker's** especially and **Lee's** by implication) call for a better integration of the arithmetic and algebra curricula in view of the difficulties students face in the change of meanings for operations and the equality sign, for example, as they move from arithmetic to algebra (see also Kaput, 1979). After all, if one interpretation of the symbols is learned for years before an abrupt switch to another interpretation, trouble is the *only* possibility. But looking more closely at the key ideas in algebra, those of variable, function, and conditional equality, it is not clear exactly how the integration should be organized. For example, should variable frames be introduced in, say, grades 3 and 4, followed in grades 5 and 6 by variables denoted by letters - as successfully done by the Japanese (Miwa, in press)? And if so, how? Should they be used consistently for writing arithmetic sentences, especially in modeling situations? ("Minnie has some marbles before losing 4 to Zeke, leaving her with 5 marbles. How many did she have at the beginning?" Model this as $\square - 4 = 5$, where the goal is to put a number in the box that makes the equation true.) **Gallardo and Rojano** show that the transition from frames to literals is not trivial - simple replacement of frames by literals is not sufficient. Moreover, **Putnam and** colleagues show that using natural language based story contexts to model the syntax of sign changes requires special care to establish the mapping model. And **Booth and Filloy** show that the use of concrete models, especially those which inherently freeze variable values (see below), can hobble student conceptions of equation. So early introduction of algebra is not easy.

B. A THEORETICAL FRAME OF REFERENCE.

We need a set of languages - representations - with which to communicate and think about the languages of algebra. Given the widespread interest developing in algebra research, and the variegated phenomena being uncovered, this need is more urgent than ever. Natural language has normally been the primary language for this purpose, but for the same reasons that any substantial research domain requires specialized representations that go beyond standard useage in everyday discourse, algebra inquiry needs them. The aims of a comprehensive language and theoretical framework are threefold:

1. to provide means for describing the web of related languages that constitute the languages of algebra (expressions, equations, coordinate graphs, tables of data, hybrid constructions involving natural language fragments, etc.), thereby, in support of **Norman's** main point:
2. to complement with a linguistically/representationally oriented language the traditional cognitively oriented language used to describe student algebra learning and application phenomena, and
3. to provide means for discussing and evaluating the characteristics of new or potential algebra learning and algebra application environments, including environments with cybernetic support features.

Four Sources of Meaning in Mathematics.

Mathematics learning can be regarded as meaning-building. Although the idea of representation system will be illustrated more concretely later, you may assume such systems include the familiar representation systems of coordinate graphs, algebraic equations, and so on, as well as the non-mathematical representation systems of natural language and pictures. In these terms we assert that mathematical meaning can be established in at least four ways:

1. By transformations within a particular representation system without reference to another representation, (these referentially isolated transformations currently dominate the curriculum and lead to the difficulties described by **Pereira-Mendoza**),
2. By translations across mathematical representation systems, say A-->B and, most importantly,
3. By translations between mathematical and non-mathematical representations (such as natural language, visual images, etc.)

Repeated experience with the above three sources leads to a derivative, but essential, fourth source of longer term meaning growth that occurs all across mathematics:

4. The consolidation and reification of actions, procedures and concepts into phenomenological objects which can then serve as the basis of new actions, procedures and concepts at a higher level of organization. (The process by which this is achieved is sometimes called "reflective abstraction.")

To describe adequately the establishing of mathematical meanings, one must necessarily be able to describe in a systematic way the structural features of the representations involved and, especially, how the features interact with one another, since dealing with their differences is what translation is all about.

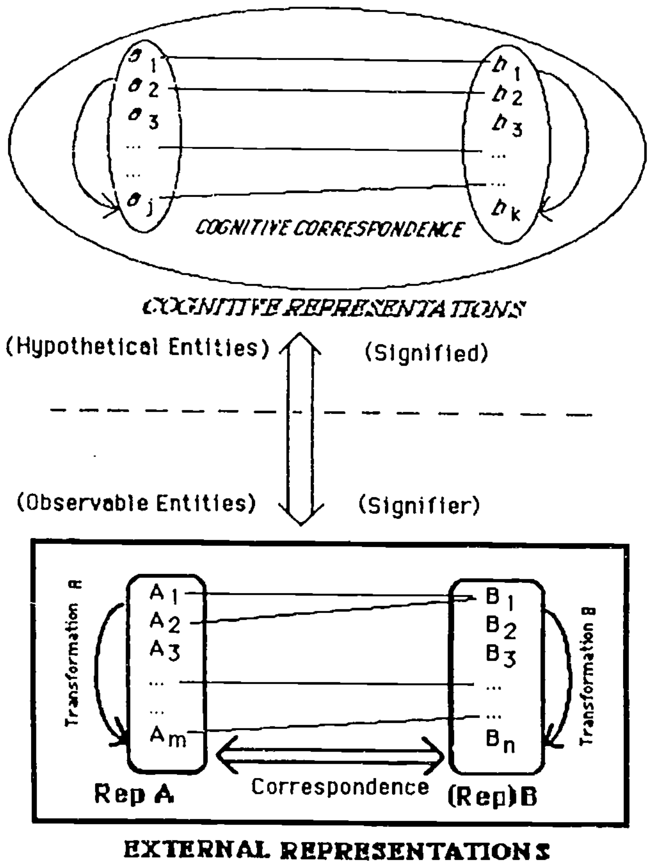
An important consequence of this primarily referential theory of meaning is that we do not assume the existence of absolute meanings, or absolute sources of meaning. Rather, meanings are developed within or relative to particular representations. Thus, for example, there is no absolute meaning for the mathematical word "function" (Platonic or otherwise), but rather a whole web of meanings built out of the many representations of functions and correspondences among them that we have available. Some of these are inherently procedural (function as a transformer of numbers) and some relational (function as a relation between numbers). And each of these families of meanings has its more congenial representations, e.g., the "f(x)=..." as procedural and the "y=..." as relational.

"Mathematical Representation" Unpacked.

I find very helpful an unpacking strategy that explicitly acknowledges the representational aspects of mathematics and hence separates the representing entity from the represented entity. A starting point is provided by the figure below, intended to provide a general and systematic frame for describing representational acts, not to provide some grand formal theory. Later, such systematic descriptions might serve to explain regularities in the representational acts observed. To help understand this point of view we must distinguish:

• the notion of mental representation as the means by which an individual organizes and manages the flow of experience - the upper half of the figure - and
• the notion of representation system as a materially realizable cultural or linguistic artifact shared by a cultural or language community.

"Materially realized" symbols are physical instantiations produced by pen on paper, or keystroke on computer screen, etc. - the lower half of the figure below. Representation systems, when learned, are used by individuals to structure the creation and elaboration of their own mental representations. It is useful to think of mathematical representation systems as functionally corresponding to the grammatical structures of natural language - they are the conventionally defined organizers of the "content" we wish to express. A central goal of algebra research is to determine how those representational forms are learned and applied by individuals to produce useful mental representations - in the figure below, how the vertical arrow comes to be. This picture is intended to depict the major ingredients in representational acts involving two representations, allowing for the possibility that each might be used to represent the other.



Here **A** and **B** can be any representations whatever, mathematical or non-mathematical, and the media in which they are instantiated can likewise be virtually anything, paper-pencil, physical apparatus (e.g., a balance scale), a computer screen, sound, etc. The horizontal correspondence is not assumed to have a particular direction until a particular representational act is specified - then we assume that the arrow "points to" the the thing represented. Note that some, in the Piagetian tradition, refer to the top part of the diagram as the "signified" and the bottom as the "signifier" (Vergnaud, 1987). We would all agree that the constituents of the top are purely hypothetical. A very similar diagram applies when, say **B**, is not a representation system, but rather is a thing or situation being represented by **A** - so the horizontal arrow at the bottom of the diagram points from left to right, from **A** to **B** (although the cognitive version may be bidirectional). Finally, it is often the case that **B** in turn is representing yet something else, in which case we introduce **C**, and so on.

To help clarify things, consider the following simple example involving two familiar mathematical representations: **A** is an alphanumeric representation of a function, **B** is its coördinate graphic representation, and the correspondence is the usual one. It may be the case, as with some software environments that we are familiar with, that one can perform transformations of the algebraic representation of a function and such transformations are reflected by corresponding transformations of its graph. This fact that we frequently use representations in order to act on them is reflected in the existence of the transformational arrows in the picture. Others sometimes refer to the transformed representation as a "new representation," as when one rewrites a function in more convenient form to emphasize a particular feature in the context of problem solving. In this case we would refer to the new representation as a transformed version of the old in order to emphasize that the transformation took place <u>within</u> a particular representation system and did not involve a translation across systems. (Of course, one might then attempt to interpret - i.e., translate - the newly emphasized feature in another representation system.)

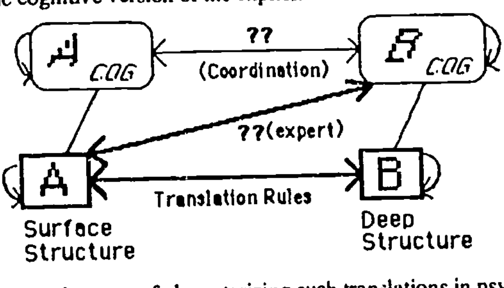
It is clear that while some features of representations correspond to each other, the correspondence is generally imperfect, with features of one not related to features of the other and vice-versa, with several features of one perhaps collapsed into a single feature of another, etc. Furthermore, the correspondence itself may be understood (i.e., cognitively represented) imperfectly. (There are some subtle philosophical points we choose to ignore in this paper. See Kaput, 1987, and in press-b, for details.)

Applying the Theoretical Framework: An Attempt at Synthesis.

We begin by analyzing Kirshner's ideas because we will then be in a position to understand a few of the other papers as well.

Kirshner:

Kirshner's work, as only partly revealed in the paper under review, provides a penetrating analysis of how the alphanumeric algebra symbol system is understood and applied, especially via the relation between the spatially organized features, the "surface structure," and the "deep structure" that they are presumed to represent. In the framework offered above, **A** is the surface structure - the symbols that we see and respond to. We are cued by these to form (through the acts of identification and parsing) a cognitive representation of the symbol strings that we see. Kirshner posits the existence of a deep structure of such symbol strings, which we put in the role of **B**. Here **B** may or may not be representing anything else, so may or may not be serving as a representation system itself. Knowledge of the syntax of algebra then amounts to possessing a reliable and flexible cognitive version of **B** that is well coordinated with one's cognitive version of **A**. The correspondence, from **A** to **B**, is describable in terms of the translation rules that he offers. Much of what he says about the psychological reality of deep structure and the translation rules is therefore about the reality of the cognitive version of the explicit translation rules and the



appropriateness of characterizing such translations in psychological rather than purely mathematical terms (e.g., field properties) or other formal terms. I agree wholeheartedly with the thrust of his remarks, including his assertions regarding the non-binary psychological parsing rules and especially the default to left-right processing (at least among those who read their native language that way).

An important consequence of accepting Kirshner's "parsing" of algebra symbol use acts is the need to interpose the above two-part diagram whenever we talk of a person using the algebra symbol system to represent anything: It is not **A** that is doing the representing, but rather its deep structure **B**, which, by the way, is not to be identified with Kirshner's particular representation of it. His is a fresh and enlightening perspective that may deserve application beyond the symbol system to which it has been applied.

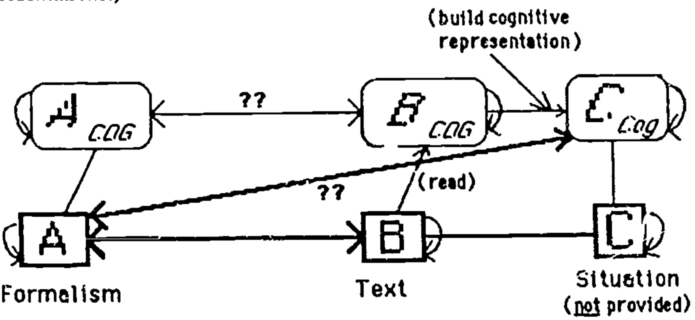
Pereira-Mendoza:

This paper provides some nice examples of student transformation rules based on surface structure which are independent of deep structure - students operating on **A** using <u>its</u> cognitive representation only. After all, the spatial regularities apparent in symbol-string behavior, especially if based on limited experience, can be codified in ways other than those dictated by the conventional rules. And if we are to believe Lee's work that the status of algebraic formulas as generalizations of arithmetic patterns is not established in many students' minds, then the rules are not constrained by those students' arithmetic knowledge. Hence it is entirely reasonable that they will be assimilated into whatever meager spatially-oriented patterns that are available from their limited experience. In particular, the fact that they are willing to replace $3y \cdot 4y$ by $12y$ is an easy superposition of (a) $(3+4) \cdot y = 7y$ with (b) $y \cdot y = y^2$ understood as the rule that allows you to replace two appearances of y by a single one. Note that while the reason given for the first statement is the "distributive law," this "law" has a perfectly consistent <u>surface structure interpretation</u> as "add the numbers and take out one y." Hence a student who provides the distributive law as a reason for (a) above is quite likely to be thinking "multiply the numbers and take out one y." Moreover, if y is merely a character (which might be "modified" by a numerical coefficient-adjective) then again, spatially-based surface structure rules can easily account for replacing $y \cdot y$ by y.

The key to understanding all these sorts of referentially isolated transformation rule phenomena is to regard them as surface structure rules generated out of the immediate character-string experience combined with some natural language patterns and perhaps some arithmetic experience (although not formal generalizations of arithmetic rules). The students are being perfectly reasonable in the limited symbol system context that they are being asked to perform in. A good question: How to engender the usual deep structure rules that are at the heart of algebraic syntax? My suggested answer is first to put the student in the position of using the algebraic statements to represent something that already has an established cognitive referent - either a numerical pattern or some phenomenon that, in a well-understood way, gives rise to a numerical pattern. Then transform the thing being represented in such a way that a transformation of the algebraic representation is required to maintain the correspondence. This brings us to the next paper.

Putnam, Lesgold, Resnick, & Sterrett:

Typical acts of algebraic modeling start with **B** as a natural language, textual representation of some situation **C** embodying some quantifiable relationships. The goal is to construct some algebraic representation **A** of that situation, and perhaps to use that representation to reason about **C**. This particular paper illustrates clearly that connecting and coordinating the cognitions associated with each of these representations is not an easy task. The research examines linkages between student understanding of arithmetic sign change transformation rules (applied to **A**) and their understanding of situations and transformations of those situations (**C**) as represented in text **B**. Here we must distinguish between procedure representing objects, i.e., a phrase such as "16-(8+3)" (which represents a procedure embodied in some story situation - selling cupcakes or buying records) and a transformation of one such object into another: either into an equivalent object such as "16-8-3" or into a nonequivalent object such as "16-8+3." The difference is confusing for students because equivalent objects can be linked to distinct story situations **C** - just as nonequivalent objects could be linked to distinct situations. (See Kaput, in press-a, for more detail on the distinctions between display representations, procedure representing objects, and action representations.)



The researchers found that students had considerable difficulty linking transformations across the representations in such a way that could apply reasoning about the equivalence of situations to reasoning about the formalisms. An interesting issue is the extent that the natural language cues served to identify differences in stories. It seemed to me that often these cues are rather subtle, hence provide relatively weak features to distinguish the formalisms (see, for example, the first two cupcake stories - denoting slightly different action-situations with equivalent, but semantically distinct formal representations).

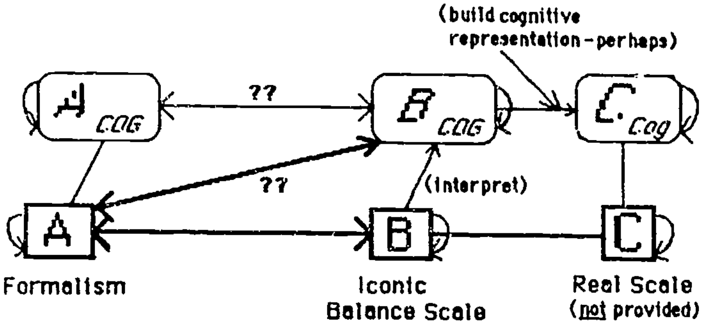
Norman:

I believe the discussion in Norman's paper relating to semantics and syntax can be fruitfully cast in the terms of representation systems as indicated in our first Figure. The semantics of representation **A** are to be found in a reference field for **A**, say **B** - which means, for us, that referential semantics is relative: There is no absolute semantics for **A**. The syntax of **A** consists of the rules that identify and define equivalence of its objects and its allowable transformations. (For more on semantic equivalence, see Kaput, 1987, and in press-a.) Some of Norman's comments on the special role of natural language in interpreting algebraic statements find even stronger illustration in the "Student-Professor Problem" phenomena, e.g., (Clement, 1982; Kaput & Sims-Knight, 1983). As noted above, the whole approach here is in line with Norman's call for a psycholinguistic approach to algebra research.

Booth and Filloy:

In Booth's paper **A** is the representation system of single variable linear equations and **B** is the system of ideographs of balance scales, with parallel transformation rules defined for each. Here **B** is assumed to represent imaginary, or at least invisible, balance scales **C** which are further assumed to have readily available cognitive referents C_{cog} which are assumed rich enough to guide and constrain actions on **B**, which in turn do the same for **A**.

As Booth points out, a critical matter in the design of learning situations based on linking two or more representations is how the features match up, and how the central ideas - in this case variable and equivalence - are represented. A box icon in the paper-pencil medium representing an apple box containing an unknown number of apples does not represent a true



variable, but rather a letter standing for a single, unknown number (see below). And if that unknown is represented as an object (a box), then the translation process, governed by the reference patterns of natural language, yields letter-as-label rather than letter as variable. And how are equivalence and transformations (e.g., inverse operations) represented? For example, if a balance scale is the initial model for equivalence, then the process of transformation to maintain equivalence is likely to be strictly additive, as was observed, rather than multiplicative - because the underlying metaphor for balancing a scale is additive.

Filloy and **Booth** (as well as **Booker**) emphatically point out the potential weakness in using concrete models to represent algebraic statements, including conditional equality, because of the inherent particularity of such models - a particularity which runs entirely opposite to the inherent generality and abstractness of algebraic statements. Here is a very important place where cybernetic models can capture the concreteness that enables the student to use existing cognitive structures without being frozen into particular values of variables. But perhaps even more importantly, such computer-based models can serve not only as display representations, but also as action representations (Kaput, in press-a) that support easy transformations. A fundamental issue is the role of the medium in which models are embedded and the ability of that medium to carry an idea such as variable. Static and dynamic media differ greatly in their ability to support the learning of this central notion. Indeed, I feel that one of the reasons that the idea of variable has been so difficult to learn is the static nature of the media in which we have historically been forced to represent it.

I suggest that the work by **Filloy** and **Booth** using concrete models would have vastly different outcomes (1) if their concrete models had been instantiated in the computer medium, a medium much more congenial to variation and hence conceptual generalization, and, even more importantly, (2) if those models were then actively linked to the associated algebraic formalisms, so that transformations of a concrete model would have salient consequences in its formal counterpart, and vice versa. This then supports the learning of the syntax of representation **A** by providing it a semantics in the model **B**. Further, by appropriately defining the environment, one can traverse the "didactical turning point" identified by **Filloy** and emphasized by **Booker** marking the true entry into algebra as introduced by Vieta - where one acts on variables as well as on numbers.

A larger message in this episode concerns the need to focus research on the possible learning environments of the future rather than those of the past - to take an inherent difficulty such as identified by **Filloy**, and then build and test new teaching and learning environments that respond to that difficulty. (Although I am not an economist, my guess is that the necessary information technology to support such environments will be affordable at least at the level of one computer per teacher in most countries in time for the next generation of students.)

Booker and Lee:

I have already mentioned Booker's arguments, mainly implicit, for doing a better job of integrating the arithmetic and algebra curricula. Lee's results regarding the failure of many students to recognize algebraic statements as general statements of quantitative relationships dovetail extremely well with **Booker's** exhortations to generate situations that require students to use algebra to formalize patterns in numerical data that in turn arises in meaningful contexts. (These students' algebra does not represent anything!) Among the best materials I know of that deliberately do this have been generated by a team initially led by Joan Leitzel at Ohio State University (Leitzel & Osborne, 1985 - other materials are in preparation), although materials with a similar style have been developed by Zalman Usiskin's group at the University of Chicago School Mathematics Project (Usiskin, et al, 1985). Both sets of materials rely heavily on the use of a calculator to generate or elaborate data, and also involve students in plotting the data on coordinate graphs. Hence the algebraic statements are seen not only as formalizations of numerical relationships, but also as ways of describing lines and curves in the plane. Surely, others have done likewise in other countries, e.g., (Miwa, 1987).

Lee's extremely rich paper provides us with a good opportunity to take the representational perspective - see our first Figure - to yet another level of detail, because much of her paper concerns the translation process explicitly. She looks closely at a couple of students translating from the numerosity of arrays of dots **B** to algebraic functions **A** and from natural language based procedures **B** to algebraic equations **A** (which involve putative constraints on the procedures). Her close look at the correspondences used to move from **B** to **A** involves examining exactly how the features of the respective representations are used in such translations. For example, the number of dots on the edge of the (equilateral) triangular dot array provided a feature B_1 that was used as the value of the key variable A_1 in the algebraic representation - so the edge B_2 came to correspond to the variable A_2 itself (and the relationship between B_1 and B_2 as the "numerosity of B_2" gets encoded as the relationship between A_1 and A_2, which is "value of A_2.") She then cites two other approaches to the translation process that are based on different features of **B**. I find fascinating the ways that the differences in "fit" between the various features attended to affect the translation process - somehow they differ in the cognitive structures that they generate, so that the translation-cognitions (where, of course all the action is) are vastly different: first there are ducks and then there are rabbits.

But perhaps even more interesting is the role of the natural language representation system as a mediator in the translation process. Between **A** and **B**, a natural language based **C** was interposed that seems to feed the cognitive versions of both **A** and **B**: Yves wrote natural language statements as an intermediate step in the translation process, which is a clue to the important, perhaps primary role that natural language plays in the interpretation of his mathematical experience.

The second translation process, associated with understanding the results of a numerical procedure described in natural language terms, involves natural language even more directly. We see a strong contrast between Eve and Yves. One of the main differences between these students is the degree to which algebraic statements represent general relationships among quantities, or in our terms here, the extent to which they have a cognitive version of the correspondence between the algebraic and the arithmetic representation systems. The strength and richness of such a correspondence in turn determine the strength and richness of their respective cognitive versions of the algebraic system - because it inherits much of its initial structure from arithmetic experience by means of that correspondence. Hence we see Eve doing what approximates a transliteration from natural language to algebra and then abandoning the result as a support for reasoning about the issue at hand because her cognitive version of the algebraic representation is so impoverished and so isolated from her arithmetic experience. But these students also differ in the richness of their arithmetic knowledge. The richness of Yves' arithmetic structures, interestingly, in this problem do not initially contribute to the building of an algebraic representation in which to reason about the problem, but rather to his natural language based representation of arithmetic procedures. He manages to represent the generality of the procedure in natural language rather than algebra, which for him in this situation seemed sufficient - until prompted to represent it algebraically, which he apparently did as well.

CONCLUSION

Space limitations prevent as full an examination of these valuable papers from a representational perspective as I would wish. Hopefully, time available at the conference will afford that fuller examination - and thereby help strengthen the case that a systematic discussion of the complex phenomena of learning and using algebra can be facilitated by giving explicit attention to the representations involved, especially how their specific features interact in the cognitive realm. Note also that a much fuller discussion of this framework can be found in (Kaput, in press-a).

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Fractions

and

rational numbers

UNIT FRACTIONS OF A CONTINUOUS WHOLE

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The first part of this paper on unit fractions of a continuous whole introduces the theoretical framework. Fractions are defined in terms of "quantification of the part-whole relationship". This leads to a distinction between three levels of the notion of measure: iterative measure, fractional measure, and sub-unitary measure. The experimental work reported in the second part deals with various aspects of unit fractions as observed among 45 elementary school children in grades 3 to 6. The results indicate that the problem of equi-partition still appears in the upper grades, but that by then, the problems of reversibility and invariance have been resolved. Also, the usual vocabulary appears to create a cognitive obstacle for the third graders.

THEORETICAL FRAMEWORK

The general concept of rational number has been investigated extensively throughout the world (Post et al,1985 ; Hart, 1981; Hunting, 1984; Novillis Larson, 1986; Southwell, 1984; Streefland, 1984). Most of these studies have been quite broad and ranged over various related topics such as the different representations of m/n, the notion of equivalence, and the four operations. However, because of their wide scope, these investigations dealt with the primitive notion of unit fraction almost incidentally, without going too deeply into it. Surprisingly few papers focused on the child's acquisition of the fundamental concept of a unit fraction, that is 1/n. And yet, while m/n can be viewed as "1/n of m" or "m x 1/n", both interpretations must rest on a prior construction of the notion of unit fraction. The most important work on this topic dates back to 1948 when Piaget, Inhelder and Szeminska studied how children between the ages of three and eight handle tasks involving the part-whole and part-part relationships when partitioning circles, rectangles, and squares. More recently, Hiebert & Tonnessen (1978) have attempted to extend the above study to discrete sets, while Pothier & Sawada (1983) have investigated the development of the partitioning process.

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In analyzing the concept of fraction, Piaget et al (1948) have pointed out that "initially, a 'part' is simply a piece detached from the whole, and not an element embedded in the whole, that is, remaining mentally linked to it even after having been separated". Their research showed that children master equi-partitioning in the following order: two, four, three, then five and six pieces, and that the subdivision of rectangles seems easier than squares, which in turn seems easier than circles. While the Genevans' work is the finest to date, neither their conceptual analysis nor the tasks they have set, go far enough to claim that they are dealing with fractions in the arithmetical sense, that is, as numbers. The tasks they have designed require the subjects to split up the geometric figures into equal parts, but does success in equi-partitioning imply that the numerical concept of fraction is necessarily present in their mind?

Owens (1985) has reported on the classroom implications of recent research on rational numbers. He pointed out that Kieran (1980) also found that the part-whole paradigm is somehow insufficient to account for the fraction concept. Kieren suggests alternative models for rational number, that of measure, quotient, ratio and operator. While all these models are important in the construction of the general concept of fraction, including both the continuous and discrete case, not all of them prove to be useful in the initial construction of the notion of unit fraction of a continuous whole. In our own conceptual analysis we find that while equi-partitioning results in the production of equal parts, the notion of <u>fraction as a number</u> can only emerge from the <u>quantification of the part-whole relationship</u>. It is not enough for the child to view a piece as part of the whole. The arithmetical concept of unit fraction requires more: that the learner should know <u>what</u> part of the whole is involved. Such quantification requires both <u>a new concept of measure</u> and a <u>primitive sense of ratio</u>.

In reporting Kieren's measure subconstruct of rational number, Owens (1985) indicates that it appears in the context of the quantification of the surface area of a region or the length of a segment. "A suitable unit is chosen and fractional parts are derived by successive partitionings to make the measurement more precise". This notion of measure is very general and quite advanced. But it conceals the fact that it is based on two preliminary stages in its construction. The first one is the well-known concept of <u>iterated measure</u> which involves the iterated use of a measuring unit. This is sufficient if the quantity measured is an exact multiple of the unit of measure. However, the measure of part of a unit requires a new and different notion of measure, that of fractional measure. For example, if asked to measure a certain length which is not an exact multiple of a given unit, young children will provide approximations, stating that "it measures seven and a bit" or "almost eight". But they do not as yet perceive the left over part as being measurable. And this is perfectly normal since they do not at this stage view the unit of measure as being itself divisible.

The initial concept of fractional measure does not require any standard unit of measure. It starts from the perception of a whole as being divisible. Children may have this perception regardless of whether they can perform an equi-partition or not. When presented with a pie subdivided into six equal parts they can recognize the equi-partition even if they cannot produce it themselves. In either case, the next step is crucial in the development of fractional measure. They must now quantify the part-whole relationship: "Since the whole has been subdivided into n equal parts, each part must be an n^{th} of the whole". It is in this sense that fraction is a measure of the part-whole relationship.

While fractional measure results from the equi-partition of the whole, the reverse process, the reconstitution of the whole from one of its parts, requires the appropriate iteration of the given part, and hence is similar to the concept of iterated measure. In this case, the given part is used as the unit of measure in the reconstruction of the whole. The similarity is not quite complete since a fractional part exists only with respect to a whole whereas a unit of measure exists independently and need not be part of a whole.

Of course, fractional measure of a whole is not restricted to unit fractions and these can easily be generalized to multiples of unit fractions of a whole ($m \times 1/n = m/n$). But even then, children are not necessarily ready to handle the more advanced notion of measure involving the use of both units and sub-units. For indeed, they may have acquired the concept of fractional measure without fractions being as yet interpreted as sub-units. For this to occur, the learner has to perceive that the initial unit can be equi-partitioned and that the resulting parts can then in turn be used to obtain a more precise measure. The outcome of this construction is a higher level of the notion of measure that can be called sub-unitary measure.

As mentioned earlier, the quantification of the part-whole relationship also involves a primitive sense of ratio. The general concept of ratio refers to a numerical comparison of two sets, as for example "the elements in set A and set B are in a ratio of 3 to 7". However, in the case of a unit fraction of a continuous whole, the quantification results from a comparison of one part to an equi-partitioned whole. The notion of ratio involved here is primitive in the sense that one of the sets compared is a singleton, resulting in a ratio 1:n.

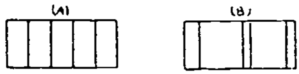
In the light of our conceptual analysis, we have designed an experiment in order to investigate different aspects of elementary school children's knowledge of unit fractions of a continuous whole. In this paper, we will report on a part of this experiment dealing with the learners' awareness of the necessity for equal parts in a partition, of their ability to reconstruct the whole from one of its parts, and of their awareness of the invariance of a fraction relative to the mode of division and the size of the initial figures.

EXPERIMENTATION

To investigate these questions, 45 elementary school children, in 22 different French schools of Greater Montreal, were interviewed (10 from grade 3, 13 from grade 4, 8 from grade 5, 14 from grade 6). The interviews were conducted by 19 teams of prospective elementary school teachers who were in their third and final year of their B.Ed. program, and as such had enrolled in a second course on the teaching of arithmetic at the primary level. The 46 future teachers were grouped into small teams (from 2 to 4). Their training consisted of various simulations, the study of videotaped interviews, and the study of the semi-standardized questionnaire to be used in the assessment. Each interview was handled by two team members, one interviewing, the other one observing and audio-recording. Each recording was then totally transcribed.

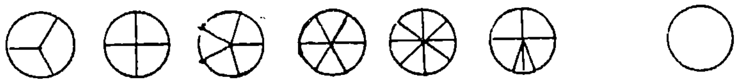
Equi-partition. The children's awareness of the necessity for equal parts in a partition was investigated in two sets of questions. The first set presented the subjects with two rectangles cut up into equal and unequal parts as follows:

Each child was asked:



HERE ARE TWO RECTANGLES, CAN YOU GIVE ME A FIFTH OF A RECTANGLE? ... CAN YOU FIND A FIFTH IN THE OTHER RECTANGLE?

In a second question, children were presented with a row of circles cut up in 3,4,5,6,8 and 6 parts as well as an uncut one



They were asked: HERE ARE SOME CARDBOARD PIES WHICH HAVE BEEN CUT UP. CAN YOU USE A PIECE OF ONE OF THESE PIES TO DRAW A SIXTH OF THIS PIE HERE (indicating the uncut pie)? ... CAN YOU FIND A SIXTH IN ANOTHER PIE?

In interviewing the children, we have found that some did not understand the questions. Among those who did, most felt that the parts had to be equal, but a non-negligible minority accepted unequal parts. On each task, some were classified as transitional because of their mixed responses which focused alternately either on the number of parts or on the necessity of equal parts. When children provided similar responses in both the rectangle and the circle contexts, they were considered to be consistent. The following table provides the distribution for each grade:

Grade	n	RECTANGLE				CIRCLE				Consistency
		Do not understand question	Parts must be equal	Unequal parts accepted	Transitional	Do not understand question	Parts must be equal	Unequal parts accepted	Transitional	
3	10	6		3	1	6		3	1	10
4	12*		8	2	2		9	2	1	11
5	8		5	3		1	5	2		6
6	14		11	3			9	3	2	12

* one subject was eliminated due to nonsensical answers

The very marked change occuring between the third and fourth grades simply reflects the fact that fractions start being taught in grade four. Questioning third graders has proved revealing, especially those six who did not understand the questions,for their answers were quite logical. When asked to find a fifth of a rectangle, some of them gave the question an ordinal interpretation, as the fifth part (counting from the left). To them, finding a sixth of a circle made no sense at all since there was no initial piece. Another interpretation was more of a cardinal nature, with subjects referring to the whole subdivided rectangle or the whole subdivided pie rather than a part of them. Also in this table, one finds evidence that the acceptance of unequal parts is not restricted to the initial learning period but persists to some extent in grades 5 and 6. In all grades the children show a remarkable consistency in their answers regarding the rectangles and the circles.

<u>Reversibility</u>. While the previous tasks dealt with the selection of fractions to be chosen from appropriately partitioned whole units, the next three sets of questions were aimed at the reverse process, that of the reconstitution of the whole from one of its parts. The first task consisted in presenting children with a sector of a circle (1/7) a pencil and paper, while asking them:

HERE IS A PIECE OF PIE. WOULD YOU HAVE A WAY OF FINDING OUT WHAT PART OF A PIE THIS IS ?

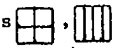
The next task was slightly different in that it investigated if, when given specified parts of a whole, children would anticipate the number of pieces required and reconstitute the whole.

HERE IS A FIFTH OF A PIE . HOW MANY PIECES LIKE THIS DO I NEED TO HAVE A WHOLE PIE ? WOULD YOU LIKE TO DRAW THE WHOLE PIE ?

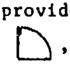
(If unable to do so, the child was asked to trace out the given sector and the question was repeated). The next question was similar except that the children were given a square piece of "cardboard chocolate" and were told that it was a sixth of a chocolate bar. Finally, the first question was repeated using another sector of circle (1/6) in order to verify it for consistency with the initial response or the possible acquisition of new skills which might have been induced by these tasks. The following table provides the distribution of the students' responses:

Grade	n	△ (1/7)		△ (1/5)		□ (1/6)		△ (1/6)	
		can draw whole pie	can name part	can predict no. of parts	can draw whole pie	can predict no. of parts	can draw whole bar	can draw whole pie	can name part
3	10	6	2	4	8	5	7	8	3
4	12	11	11	12	11	12	12	12	12
5	8	8	8	8	8	8	8	8	8
6	14	14	14	14	14	14	14	14	14

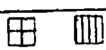
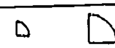
As can be seen from the data, by grade 4, all our subjects have mastered the process of reconstituting the whole from one of its parts. A comparison of the first and last tasks indicates that some learning might have been induced in two third graders and one child in grade four. Again, it is the third graders' responses that have provided us with unexpected insights. In all the above tasks, these children have a greater success rate in drawing the whole pie or chocolate bar than in naming the fraction involved or predicting the number of necessary parts. This brings up the question of whether or not the notion of fraction exists prior to the acquisition of the relevant vocabulary. Evidence of the existence of this notion was provided to us by the two children who, on the first task, found a way of expressing the notion of fraction by drawing the whole pie and then telling us that the original piece was "one piece out of seven" ("un morceau de sept") and "one of seven" ("un de sept") respectively.

Invariance. The last two tasks dealt with the invariance of fractions. The first one aimed at assessing if children viewed a fraction as invariant with respect to different equi-partitions. They were presented with the following two squares . They were asked to verify that they were the same size, to count the number of parts, and to identify what part of the square each piece could be. At that point the interviewer raised the question:

IF I TAKE ONE QUARTER OF THIS BISCUIT (coloring it in front of the child) AND YOU TAKE ONE QUARTER OF THE OTHER BISCUIT (coloring it in front of the child), DO YOU THINK THAT WE WILL HAVE THE SAME AMOUNT OF BISCUIT, OR THAT ONE OF US WILL HAVE MORE THAN THE OTHER, OR LESS THAN THE OTHER?

The next task assessed if children could perceive the invariance of a fraction with respect to the size of the initial figure. They were provided with two quarters of pies of radii 2" and 4" respectively  , and asked if each one of them could be a quarter of a pie. If they thought that the pieces could not both be quarters, they were requested to use each piece in turn to draw a complete pie and identify which part of the pie it was. And then, they were asked a second time: DO YOU THINK THAT THE SMALL PIECE AND THE BIG PIECE CAN BOTH BE QUARTERS OF A PIE?

The following table describes the distribution of the responses:

Grade	n	Invariance wrt equi-partition 		Invariance wrt the size of the initial figure 			
				Initial response		After drawing	
		Quarters perceived as equal	Quarters perceived as unequal	both sectors perceived as quarters	sectors not perceived as both being quarters	both sectors perceived as quarters	sectors not perceived as both being quarters
3	10*	5 (3)**	5	3	6	5	4
4	12	10 (4)**	2	12	0		
5	8	8 (1)**	0	8	0		
6	14	13 (7)**	1	12	2	14	0

* one subject did not understand the word "quarter" ("quart").
** the numbers in parentheses refer to the number of subjects who used visual compensation as a justification of their answer.

Regarding the invariance with respect to different equi-partitions of the square, it seems at first sight that by the fourth grade most of our subjects were aware of it. However, a closer examination of their justifications indicates that this conclusion is premature for many of them (see numbers in parentheses) explained that the rectangular quarter could be split to make up a square quarter. Thus, their rationale has little to do with the invariance of fraction since it merely reflects visual compensation of surfaces. The second task on invariance also indicates that by the fourth grade our subjects seem to be conscious of it. That the drawing of the full circles had a certain impact is evidenced by the two third graders who then found out that both sectors were quarters, and on the two sixth graders who corrected themselves.

CONCLUSION

In our exploratory study of unit fractions of a continuous whole, we have felt the need to provide a clearer definition of this concept. And we have come up with a functional approach, a fraction being defined as "a quantification of the part-whole relationship". This has led us to distinguish between three distinct levels of the notion of measure: iterative measure, fractional measure, and sub-unitary measure. The experimental work reported in this paper has dealt with three aspects of unitary fractions related to fractional measure. We were surprised to find that the problem of equi-partition lingered on among our subjects in the upper grades but we were equally surprised to find that the problems of reversibility and invariance had been so well resolved.

Our study of third graders has revealed that the language used to describe unit fractions created a cognitive obstacle for the children. Either they simply did not understand the words we used or they assigned to them a meaning other than the intended fractional one. For instance, while all subjects understood "moitié" or "demie" for half, they did not necessarily understand "tiers" and "quart" for third and quarter, often preferring "troisième" and "quatrième". But then, as with fifth, sixth, and other unit fractions, many children associated with these words the only meanings they had previously acquired, that is ordinal and cardinal meanings instead of a fractional one. However, we found that even if young children have not yet learned the conventional vocabulary for unit fractions, they can nevertheless find ways to express their quantification of the part-whole relationship using expressions such as "one of n parts". In fact, until pupils become aware of the fractional context, using such expressions in the initial introduction may overcome the cognitive obstacle caused by the use of words having other meanings.

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SOME DIFFICULTIES WHICH OBSCURE THE APPROPIATON OF THE FRACTION CONCEPT

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In the present work we will describe several difficulties identified via analyses of the answers given by students between 11-14 to exercises included on a diagnostic questionnaire. The work we have been carring out has the purpose to further clarify the relationship between the acquisition of the fraction concept and the development of those abilities required to interpret and use the geometric language included in the pictures that frecuently appear in teaching vehicles (specially textbooks) for the contextualisation of the fraction concept.

Theoretical framework and related studies.

Rational numbers appear in the Mexican curriculum [1] from the very first years of elementary school. A curricular analysis of the textbooks [8] shows that the teaching of this topic encompasses various meanings of rational numbers. These meanings are introduced concurrently throughout the six years of elementary school.

The teaching approaches of the different interpretations of rational numbers emphazise different aspects. For example, in the elementary curriculum of our country, we have seen that:

- *Fractions of the unit* are introduced highlighting the importance of the actions that are carried out with a given whole.

- *The meaning of a fraction as a subset of a collection* is approached either within the problem solving context or as numerical computations; the latter tend to have a strong algebraic flavour.

- *Decimals* are introduced through measuring, but very quickly they are immersed in algorithmic processes, where the emphasis is on computational rules.

A careful inspection of the textbooks [2] relfects the use of various types of language in the treatment of the different meanings of rationals. For example,

- With respect to the treatment of *fractions of the unit*, pictures of objects and geometic forms are used. All the actions, such as partitioning in equal parts, exhaustive division of the whole and identification of the fraction, are represented in these pictures (geometric language). Later on, those pictures are related to the numeral associated to the fraction which expresses the results of such actions (arithmetic language).

[1] *In México, there exists one single curriculum for the teaching of elementary school mathematics in the whole country. Curricular principles and syllabi are included in the "Teacher's Guide"* [17]. *Children use the "cost-free textbooks"* [16] *which have been prepared in accordance with that general plan.*

In order to understand, via a particular treatment, a certain meaning together with the relations between the different kinds of language included in this treatment, it is necessary to develop some specific abilities.

- With reference to the treatment of *fractions of the unit, visualization, perception and spatial imagination* are called upon the learner; as well as his ability to translate from one language into another.

We have called *teaching model*, the set formed by *meaning, treatment, languages* and *necessary abilities;* together with the inherent relations that exist among them.

Under a particular teaching model, a pupil constructs a especific *conception of rational number.* Each conception is related to a certain meaning. In the process of constructing different conceptions, links between them get also established. By establishing these links, the pupil is building a new mental image of rational number. The reiterative occurrence of this process results in the pupil's adquisition of the construct rational number.

Lately, we have been trying to detect the difficulties that inhibit the establishing of such links between two or more conceptions. We have also tried to determine the plausible moments where the juxtaposition of the teaching model would favour the transferring of knowledge from one conception to another, as well as to foster the creation of those links which are seldom spontaneously established.

The literature on rational numbers is vast, because it englobes studies related with the different interpretation of rationals. Among those that have focused on the teaching models[3], we can mention Freudenthal, H.[7], Streefland,L.[18], Kieren T.[11] Brousseau, G [3] and the work of our mexican colleagues (see for instance [2]). Some projects, like the English CSMS[4] and SESM[5] and the American RNP[6] were set up with the purpose of understanding the relationships between the teaching models and the acquired conceptions through the process of instruction. Kieren T. et al [12] and the Pothier, Y & Sawada, D. [15] have lately reported their results related to children's uses of geometric language in partitioning tasks.

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[2] *The teaching at the elementary school in our country, due to its own characteristics (in the urban zones, one group have ~50 students; in the rural areas, you can find in the same classroom students that belong to different grades, etc.), fundamentally supports on the textbooks that the Ministry of Education prepares and distributes. Frecuently, children have access solely to such books.*

[3] *We are not trying to make a review of the literature on rationals, our purpose is to mention some of the documents of those researchers that have been working in this area and whose work is more related with the one we are doing.*

[4] *CMSM: "Concepts in Secondary Mathematics and Science", see for instance, Hart. K.* [9].

[5] *SESM: "Strategies and Errors of the Secondary Mathematics", see for instance, Kershke, D.* [10].

[6] *RNP: "Rational Number Project", a brief description can be found in* [1].

Aims, methodology and stages of our study.

One of the main purposes of the research activities to which we have devoted our efforts for the past four years is to try to clarify the relation between the acquisition of the fraction concept, and the development of such abilities as are necessary to interpret and use the simbolic - geometric language involved in the drawings used for the contextualisation of the aforesaid concept.

First Stage.- From the questionnaires used by two studies [*13*], [*14*] and those of the CSMS project [*9*] , we selected those items which included pictures. We then made a comparative analysis of these questions and of the results of the three studies. Once the most significant difficulties were identified, we designed a number of exercises in which the role of drawings was fundamental. In order to explore the pupils responses to these exercises, we conduced and videorecorded several interviews with children of ages between 11 and 13. The analyses of these interviews were the starting point for the next stage of the study; (the most interesting results of these analyses were ruported in [*4*], where a more detailed description of this stage can be found).

Second Stage[7] . We worked out a diagnostic questionnaire structured in a way that it would permit us to examine various aspects of the concept of fraction. This evaluation contains 48 questions and it includes two different meanings of rationals: the one associated with a fraction as a subset of a collection (which we denominated discrete case) and the meaning of fraction of a unit. With respect to the latter interpretation, the questions are referred to geometric plane forms (we called this the concrete case) and the plane representations of tri-dimensional figures.

The questionnaire had been applied during three consecutive years to students of the first grade of secondary school[8], at the begining of the mathematics course (in 1984: one group - 32 students; in 1985: two groups - 43 students; and in 1986: two groups - 36 students).

At this moment where elementary school is over and pupils are initiating their secondary school, we consider that the observation is crucial. For us, this is an important *didactical cut*. During the elementary school, rational numbers have been introduced within various teaching models. The syllabus of the secondary includerationals, but the approach to this topic focuses to the properties of the algebraic structure of these numbers. In other words, the teaching treatment of rationals turns to a formal and abstract approach. This teaching model presupposes that the links between the different conceptions acquired in the elementary school, have been appropiately established.

[*7*] *A partial report of this stage can be found in* [*5*]

[*8*] *Work on this study has been done with students from the "Centro Escolar Hermanos Revueltas" an experimental school in Mexico City where we can control the teaching process.*

With the data obtained in the application of the diagnostic questionnaire to the students of the first generation (1984), we started a cualitative analyses [9]. The purpose of these analyses was the characterization of the strategies used by the students to solve the items of the questionnaire. At first, we classify such procedures in two groups: one of them contains the strategies that lead pupils to a success and the other one englobes those in which we observed difficulties and conduced pupils to a failure.

For each of such groups we endeavored to categorize the strategies developed by children, according to the features they displayed (i.e. considering the resources to which they had resorted, and the meaning emerging in each answer).

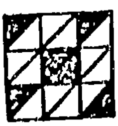
The characterisation of the answers to the items corresponding to the continuous case is completely finished. In this case, we found 14 classes associated to failure and 13 to success. Subcategories of these classes were also assigned. Subsequently, we carried out a comparison with the data obtained with the preceding generations (1985, 1986). The main objective of such comparison was to distinguish those obstructions which appear repeatedly.

In what follows we will describe the more significant categories associated with failure We selected only those classes that are directly related with the fraction concept. These categories are meaningful because of their incidence of appearance in various contexts, as well as for the characterization of the difficulties they encompass. *Such hindrancies inhibit the pupil's appropiation of the aforesaid concept. And efforts should be accentuated in the teaching process so that students are helped to surmount such difficulties.*

The predominance of the cardinality of the part.

In this class we have reunited those strategies where the fraction given in the item is not considered as such. These procedures reveals a disassociation of the numeral and a tendency to centralise the number that corresponds to the numerator. For this problem we found three types of subcategories. We will ilustrate them with answers of the students.

a) The numerator of the fraction imposes and the denominator is displaced (see Figure 1).

PROBLEM	ANSWER
What fraction is shaded? 	The sixth part of eighteen. $\frac{6}{18}$ the sixth part of eighteen.

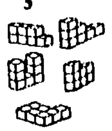
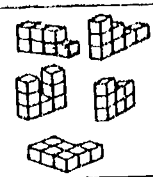
PROBLEM	ANSWER
Color $\frac{2}{5}$ of the cubes. 	

Figure 1: Examples of answers that corresponds to the subcategory a), (continous case at the left and discrete case at the right).

[9] *A detailed description of these analyses can be found in the partial report of our research* [6]

b) The graphical representation of the fraction adapts exclusively to the numerator of the given fraction; the denominator is substituted by another number, (see Figure 2).

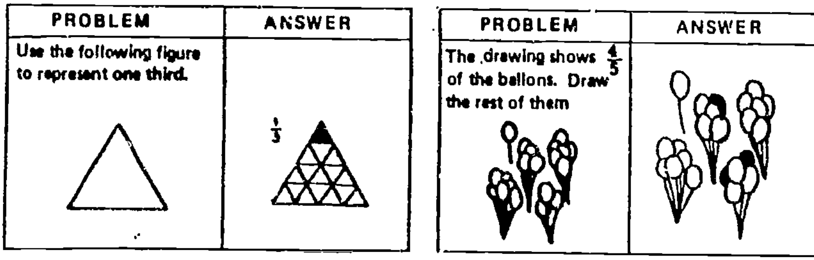


Figure 2: Examples of the answers that corresponds to the subcategory b), (continous case at the left and discrate case at the right).

c) The numerator of the given fractions is separated from the denominator, and in this abscence of the relation constituing the fraction, the first number is treated as a whole number (see Figure 3).

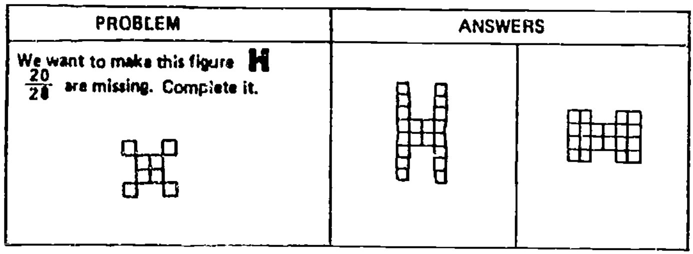


Figure 3: Examples of answers of the subcategory c), for the continous case.

The unequalness of the parts.

The strategies that we have grouped in this class are those in which we identify unequal parts. These difficulties appeared in partitioning tasks of geometric forms[10] (continous case, see Figure 4).

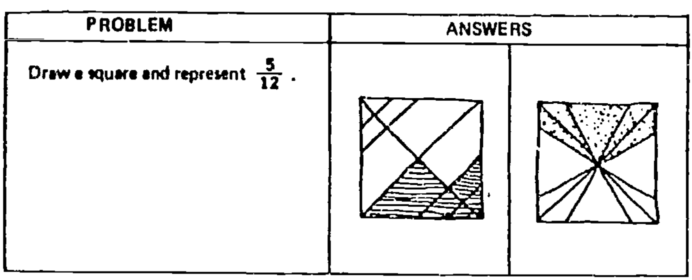


Figure 4: Examples of answers classified as unequalness of parts.

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[10] *The traditional conventions considers as unequal parts, those subdivisions of a plane figure where the resulting parts are not congruent. Our interpretation of this problem is different: we identify as unequal parts those graphic representations where the subdivisions reveal variations of the area of the resulting parts. In [5] there is a discussion of this matter.*

This problem also emerged in the discrete case, where the classification criteria is more evident; the subsets in which the whole is divided have a different cardinality.

Difficulties in the partition

Identified in this category are those problems related to the connection between the sudivision of the whole and the recognition of the fraction.

These difficulties emerged in some partitioning tasks of figures that have a complex structure and whose subdivision adapted to the fraction that appears in the item, imposes the simultaneous use of more than one unit of partition, see figure 5. One of the strategies that lead pupils to a success in the exercise that ilustrates this category, was the considera· tion of an equivalent fraction.

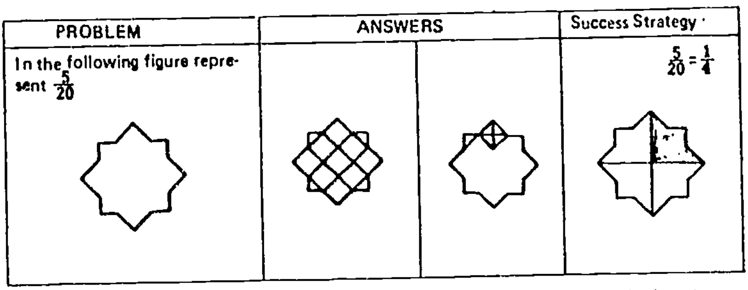

Figure 5: Examples of answers , the middle ones represent procedures that are included in the category difficulties in the partition

The predominance of the cardinality of the denominator.

In this category we have included those readings of the fraction where the value of the denominator takes precedence. Again these procedures reveals a dissasociation of the numeral and a tendency to centralize the number that corresponds to the denominator, assigning to it the meaning of part.

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COMPUTER-AIDED DIAGNOSIS AND REMEDATION IN FRACTIONS

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ABSTRACT

Computer has been introduced for use as an artificial intelligence to analyse data in the area of education. Educationists have found that computer has not only help their work more efficiently but also it can generate information which has not been encountered before. The use of computer is not considered new in education but using it to analyse the cognitive thinking processes of students is quite scarce in mathematics education.

This research study was conducted in line with Ashlock, Brwon, Burton and VanLehn on the analysis of error patterns. The objectives of the present study are to develop an automated computer system for diagnosis and remediation and to construct a conceptual model of remediation in fractions.

The initial stage of the project began with the construction of an instrument to investigate the subjects' performances in fractions. The items were based on a set of 13 objectives on the 4 operations of fractions. The test was administered to 3000 subjects who were classified as below average in fractions. The test was readministered after a week later. The responses of the subjects were analysed and systematic errors were classified.

An automated computer system for diagnosis and remediation in the 4 operations of fractions was developed. It consisted of three sub-systems viz. (a) diagnostic system of errors, (b) tutorial system for remediation and (c) automated generation of text materials for remedation.

To accompany the computer system for diagnosis and remediation a conceptual model in remediation of fractions was developed which was based on the hypothetical remedial activities.

Diagnosis and remediation in the teaching of mathematics have been seen by many teachers as essential for effective teaching. Okey (76) reported that pupils' achievements tended to go up when teachers give diagnostic tests frequently. It seems to indicate that it is beneficial to research further into this area of teachng and learning.

Although the use of diagnosis and remediation in teaching seems to be encouraging, the amount of time required for implementing the test and analysing the data to find out the actual cause of pupils' errors will be tremendous. Unless the amount of this time can be reduced, teachers will normally reluctant to carry out this strategy to help their pupils. Another factor which cannot be ignored is to determine the accuracy of diagnosing pupils' errors in mathematics. It would be futile to conduct remedial classes which are mainly based on erroneous diagnosis. In view of the problems identified above in teaching mathematics, there is a need for mathematics educationist to look into ways which can help teachers to reduce their burden in diagnosis and analysis of data. The topic on fractions is used as an example.

REVIEW OF LITERATURE

"Diagnosis and remediation' is not something new in the mathematics education curriculum. However, research using computer (especially microcomputer) to help diagnosing and remediating pupils with mathematics difficulties is not numerous.

Basically the concept and work done in the area of diagnosis are pursued in two direction. The first group of mathematics diagnosticians concentrated their work on categorising the types of errors according to some major classifications. Robert (68) had classified four error categories viz. wrong operation, obvious computation error, defective algorithm and random response. The work of Engelhardt (77), Cox (75) and Knifong (80) were quite close to Robert's work on errors analysis. This area of research was found to have two limitations. First, researchers tended to emphasise on written responses and there had been few attempts to analyse pupils difficulties by talking to them. Second, emphasis was placed on difficulties related to a type of mathematical task rather than a whole range of difficulties which pupil experience.

The work of Ashlock (76) had indeed given rise to another group of mathematics diagnosticians. His work was concerned with the identifications of error patterns in computation. Methods for correcting pupils' errors in computation were suggested in his book 'Error Patterns in Computation'. Brown and Burton (78) constructed some diagnostic models of basic skills (addition, subtraction, etc.) using a representation technique called 'procedural networks'. Using these diagnostic models, two computer-based systems, BUGGY and DEBUGGY, were developed to teach both students and student teachers about the strategies of diagnosing bugs. Later, Brown and Van-Lehn (81) introduced the Repair Theory in an attempt to explain how the bugs (systematic errors) were acquired by students and how they were held. Travis and Carry (83) and Woerner (80) did similar kind of work to identify students' errors in multiplication and addition of fractions respectively. Travis and Carry concluded in their study that the diagnosis-remediation combinations were effective for remediating students' errors in multiplication. Woerner concluded that the use of computer for diagnosis was effective for probing more information. Bright (84) suggested that further computer-based diagnostic system should incorporate CAI for remediation.

OBJECTIVES

In view of the previous research and suggestions discussed on the previous paragraphs, a research project was initiated to investigate further into this area on fractions. The main objectives of the research study are to

(1) classify a near-exhaustive set of error patterns in fractions.

(2) develop a computer system for

  (a) analysing pupils' erroneous algorithms in fractions.

  (b) generating tutorial questions in remediation.

  (c) generating text materials for remedation.

(3) derive a diagnostic model for remediation in fractions.

(4) test the accuracy of the computer system in diagnosing pupils' errors in fractions.

(5) investigate the effectiveness of this approach as compared with the 'usual method' for remediation adopted in the local context.

At the time of writing this paper, objectives (4) and (5) above have not been realised.

METHODS

Sample

The sample for this study consisted of about 3000 average and below average pupils from 30 schools in Singapore. They were selected from the Primary 5 and 6 of the Normal Stream and the Primary 6, 7 and 8 of the Extended Stream (pupils take 6 years and 8 years to complete the Primary Education in the Normal and Extended Streams respectively).

Instrument

A diagnostic test on the addition of fractions was constructed which was based on the pre-determined objectives. The thirteen objectives identified for the test were

Addition of Fraction (Denominator $\leq$ 12)

(1) Addition of simple fractions with like denominators.

(2) Addition of simple fractions with unlike denominators.

(3) Addition of mixed numbers with like denominators.

(4) Addition of mixed numbers with unlike denominators.

Subtraction of Fractions (Denominator $\doteq$ 12)

(5) Subtraction of simple fractions with like denominators.

(6) Subtraction of simple fractions with unlike denominators.

(7) Subtraction of mixed numbers with like denominators.

(8) Subtraction of mixed numbers with unlike denominators.

Multiplication of Fractions (Denominator $=$ 12)

(9) Multiplication of a simple fraction and a whole number.

(10) Multiplication of a simple fraction with a simple fraction.

Division of Fractions (Denominator $=$ 12)

(11) Division of a simple fraction by a whole number.

(12) Division of a simple fraction by a simple fraction.

(13) Division of a whole number by a simple fraction.

In each objective identified above, 4 parallel items were used to test the subjects' knowledge in the algorithmic skills. This was to ensure that the different types of errors were identified viz. systematic errors and non-systematic errors due to misreading a question or guessing a solution.

Procedure

The above diagnostic test was administered to the 3000 subjects with the help of 60 Certificate in Eduction students of the Institute. The subjects were retested in the following week. In both tests, no time limit was imposed on the subjects. They were told to hand in their papers as soon as they had finished their work. Pupils' responses to each item of the tests were marked. Incorrect responses were carefully analysed to determine the actual error pattern of each mistake. Subjects were also interviewed when their errors made were randomised or they would be asked to think aloud on working a similar problem. The results obtained in the second test were used to check whether the erroneous strategies used by the subjects were systematic.

COMPUTER SYSTEM FOR DIAGNOSIS AND REMEDIATION

Owing to the nature of the topic on fraction, it is not the intention of this study to construct a procedural networks to show a general diagnostic model in fractions. However it was found that, on the average, about 8 error patterns, were identified in each objective. It would not be possible to list all of them here in this short paper.

It can be envisaged that teachers find difficulty to memorise all these error patterns. Besides it is also time consuming to analyse individual's error in performing operations in fractions and other topics. Hence an automated computer system is developed to reduce the burden of teachers who would, presumably, reluctant to perform the above tasks without such a system.

The Automated Computer System developed consists of three sub-systems. They are the

(1) Diagnostic System of Errors in Fractions.
(2) Tutorial System for Remediation in Fractions.
(3) Automated Generation of Text Materials in Fractions for Remediation.

The Diagnostic System of Errors in Fractions is a system that can generate randomised questions which were based on the 13 pre-determined objectives. It can also determine the subjects' errorenous strategies in performing the 4 operations of fractions. The subject is expected, if desired, to work out the problem on a piece of paper. The answer is keyed into the computer and it will logically analyse the subject's work and the probable cause of error is printed out.

The following tables show an examinee's performances printed out from this computer system.

Table 1 : Analysis of a Pupil's Performances in Fractions

Objective (3) : Addition of mixed numbers with like denominators

Item 1 : (a) Time taken : 9 secs

(b) Question : $2\frac{3}{11} + 2\frac{7}{11} = 4\frac{10}{22}$

(c) Error Pattern : Add the whole numbers, the numerators and the denominators correspondingly.

Table 2 : Summary of Results

Objective No.	Item No.	Result	Time Taken (sec)
1	1	Wrong	16
	2	Wrong	6
	3	Wrong	7
	4	Wrong	8
2	1	Wrong	10
	2	Right	7
	3	Wrong	6
	4	Wrong	5

The Tutorial System for Remediation is a system that generate randomised questions for drill and practice. The system is used to provide questions for drill and practice after the subjects have undergone remedial lessons conducted by the remedial teachers. The following table shows an example of the printout which summarises the examinee's performances.

Table 3 : Summary of the Pupil's Performances in Fractions

Objective No.	No. of Question	No. Right	No. Wrong	% Right	Time Taken (sec)
1	10	5	5	50	12
2	10	6	4	60	23
3	10	3	7	30	13
4	10	2	8	20	43
Total	40	16	24		91
Average				40	22

The Automated Generation of Text Materials in Fractions for Remediation is designed to generate additional materials for the subjects to practice at home. The answers are also provided for the subjects to check the accuracy of their work.

MODEL FOR REMEDIATION IN FRACTIONS

An overview of the error patterns made by the subjects in this study shows that most of the errors made are rudimentary. It is possible to use Brown and Van-Lehn's Repair Theory to explain the occurances of the bugs. The examinees tended to apply a simpler strategy to work out the algorithmic operation.

An analysis of each error pattern was carried out and it was found that it has its own error identity. On the basis of its uniqueness, a set of hypothetical remedial activities was suggested that would most likely alleviate the weaknesses of the subjects. The following example shows an error pattern in multiplication of 2 fractions and the possible remedial actions for those subjects who err in this type of problem.

Question : $\frac{N1}{D1} \times \frac{N2}{D2} = \_\_\_$

Error Pattern : $\frac{N1}{D1} \times \frac{N2}{D2} = \frac{N1D2 \times N2D2}{D1D2}$

Treating 'x' as '+'

Weaknesses : Recognition of symbols

Remedial Activities :
(1) Further diagnosis on the recognition of symbols + and x
(2) Concept of Multiplication of 2 fractions
(3) Algorithm in multiplication of 2 fractions
(4) Comparing addition and multiplication algorithms

To illustrate an example of the construction of a remediation model, an analysis of the remedial activities to cater for the subjects who have not mastered the multiplication of a simple fraction with another simple fraction/whole number was carried out. Using these remedial activities, a conceptual model for remediation of multiplication of fractions is constructed as shown in figure 1 on page 10.

Each remedial activity is placed at one of the six levels identified To help teachers identify the exact level at which the subject has not achieved, the computer system may print out the required level for remediation. Based on the conceptual model for remediation of multiplication of fractions, teachers are able to select a set of those remedial activities classified at and below the level identified by the computer system.

CONCLUSION

Two important outcomes are seen to emerge out of this study viz. development of an automated computer system for diagnosis and conceptual model for remediation of fractions. This computer system and the conceptual framework for remediation provide an alternative approach for individualising instruction in mathematics. It serves as a prototype system to cater for other areas of mathematics.

Some features of this system are worth noted for future implementation. It does not only provide with accurate diagnosis of errors but also it helps to reduce the investigator's time to analyse examinees' errors. With the remedial information printed out, investigator may conduct remedial activities immediately without wasting much time in looking for remediation materials. In the process of using the system for diagnosis, the investigator may also be able to collect further information on error patterns as the set of error patterns identified earlier may not be exhaustive. This provides additional information for research.

Two assumptions have been made in this study. The hypothetical remedial activities are assumed to be effective and exhaustive. Further research should concentrate on verifying the remediation model and the accuracy of the diagnostic system.

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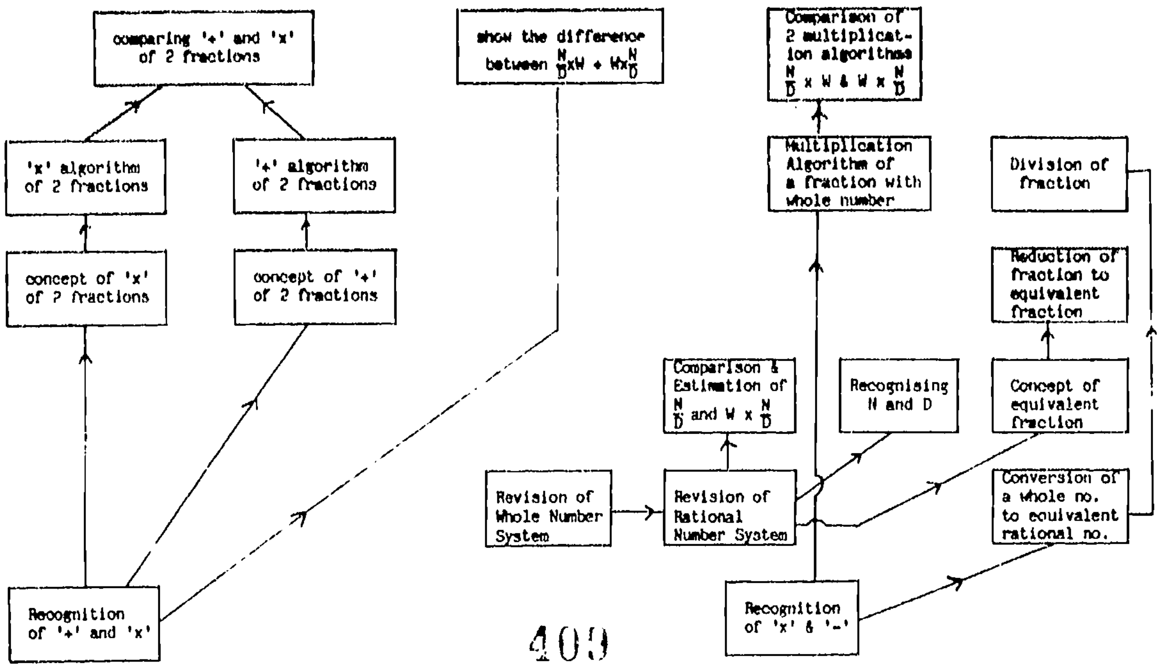
CONCEPTUAL MODEL FOR REMEDIATION OF MULTIPLICATION OF FRACTIONS



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CATEGORIZATION OF FRACTION WORD PROBLEMS

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What information in a word problem does a problem solver use to decide that it should be solved similarly to another problem? Do nonexperts and experts use different types of information? This set of four studies showed nonexpert and expert problem solvers do categorize problems differently, nonexperts relying more on surface feature similarity. However, nonexperts improved judgments of solution similarity when the problem type was constant. The results suggest: 1) the distinction of surface and deep features may not be rich enough for describing categorization of problems, and 2) problem solvers attempt to use all features of similarity that they perceive.

What information in a word problem does a person use in order to decide what operation to perform to solve the problem, and how is that information used? One way in which this question can be approached is to ask whether a problem would be solved similarly to another problem. Studies of problem solvers in physics (Chi, Feltovich and Glaser, 1981; Larkin, McDermott, Simon, and Simon, 1980) suggest that novices and experts attend to different types of information when classifying problems according to solution similarity: novices classify problems mainly with respect to surface similarity, whereas experts classify problems on the basis of principle of solution. Similar observations have been made with good and poor problem solvers in the same grade in school: poor problem solvers are often misled by surface structure or "pseudo" similarities (Silver, 1979, 1981). Although the product of categorization tasks is different for nonexperts than experts, it is not obvious that nonexperts and experts actually approach the task in different manners. The intention of the following set of studies is to determine whether expert and novice categorization behavior can be explained using a single set of principles.

I will begin by proposing the Surface Feature hypothesis and attempt to show that it cannot account for the data. The Surface Feature hypothesis is that in categorizing problems, nonexperts attend to the surface structure, or storyline, while experts consider only the deep structure, principle by which the problem should be solved. One limitation of this hypothesis is that it provides no obvious route for the acquisition of expewrtise: how does the nonexpert progress from using surface features to classifying by principles? The transition might be accomplished if one assumes that the nonexpert is capable of categorizing on the basis of principles and does so when possible. However, later I will argue that the correct and incorrect categorizations of experts and nonexperts can be accounted for by assuming that all perceived similarities are used in the categorization process.

Two types of tasks are employed in this set of studies to infer how the information in the problem statement is used in solving a problem: 1) the choice of operation used to solve a problem, and 2) judgments of the similarity between problems. Arithmetic word problems containing fractions were chosen as the domain of investigation, since the types of problem can be well defined, and since a large proportion of high school students (Carpenter, et al, 1980) and adults (Watson, 1980) are unable to use rational number concepts fluently, even after a considerable period of instruction in school. A secondary goal of these studies was to try to understand why arithmetic word problems that contain fractions are difficult to solve.

One plausible explanation is that all word problems are more difficult to solve than correspondingly similar computational problems. However, the NAEP data (Carpenter et.al., 1980) for 13 year olds, clearly argues against this explanation, as the differences in performance between word and computational problems of similar types are neither constant in size nor always in the same direction.

A second explanation is that since fractions are more complex numbers, being composed of two parts, they add incrementally to the difficulty of a problem. Although such complexity may adversely affect the development of computational skills, the presence of fractions in a word problem does not neccessarily make it harder to understand the problem situation. This is an experimental question, which will be addressed in Study 1.

A third explanation is that number type (whole number or fraction) influences the kinds of units allowed, and hence the possible structures of word problems. Because of such differences, it is not always possible to simply substitute fractions for whole numbers (and vice versa) and retain the structure of a word problem. In cases where there is a lack of parallelism in problem struture, nonexperts would likely have difficulty interpreting a problem statement. For example, consider the following fraction multiplication problem:

> "Margret had 4/5 of a gallon of ice cream. She gave 1/5 of the ice cream to her sister, Anne Marie. How much ice cream did Anne Marie receive?"

The basic solution approach is unaffected if Margret had 4 instead of 4/5 of a gallon of ice cream. However, the 1/5 cannot be replaced with a whole number and have the problem remain a multiplication problem. This is true for all word problems in which the multiplier, or operating number, is a fraction. In general, the structure of addition and subtraction problems makes it possible to substitute fractions and whole numbers without altering the meaning. Measurement division problems are likewise unaffected, implying for all three cases little difference between nonexperts' understanding of the whole number and fraction operations. However, nonexperts will be likely to experience difficulties with fraction multiplication problems because of the lack of parallelism.

The four studies reported here concerned: 1) assessing relative difficulties of problem understanding, 2) judgment of solution similarity by experts and nonexperts, 3) how judgments of solution similarity may be facilitated, and 4) replication of results with a younger population. The subjects were two groups of college students (N=48 and N=57) enrolled psychology classes at UMass, and one group of eighth graders (N=52) from a local junior high school. Preliminary analyses for each study showed no main effects or major interactions involving sex.

In all studies, the problems had two numbers, were solved with one-step, and the result of the operation was unknown. Two types of word problems requiring each operation were used: Active and Passive. Active problems involved an action integral to the problem storyline, while Passive problems described and asked about a relationship between the two problem elements. For example:

> Hansel began the trip with 3/4 of a pound of bread. He used 1/4 of a pound of the bread to mark the trail. How much bread did Hansel have then?
>
> Ernest had 3/5 of a box of typing paper. George had 4/5 of a box of typing paper. How much more paper did George have than Ernest?

STUDY 1: ASSESSMENT OF PROBLEM TYPE DIFFICULTY

The purposes of Study 1 were to: 1) assess relative differences in levels of understanding among the set of fraction problem types, 2) determine whether the complexity of fractional numbers could be ruled out as an explanation for the poor performance on fraction problems, and 3) provide an index of expertise in solving fraction word problems.

The subjects were given eight fraction word problems, one of each of the eight types, and were to indicate which one of the four arithmetic operations should be performed on the two numbers given in the problem in order to solve it. The group 2 adults were also given a set of eight problems that each contained two whole numbers.

Results The mean performances (and patterns of correct answers) for Groups 1 and 2 on the fraction problems were similar: 67% versus 69%. There were considerable differences among the four operations for both groups, $F(3,141) = 45.95$, $p < .0001$, and $F(3,168) = 40.78$, $p < .0001$; the means were addition: 92%, subtraction: 86%, division: 37%, and multiplication: 34% (all pairwise differences were significant with a Bonferrroni test ($p < .008$), except between addition and subtraction).

There were considerable differences within operations as well: subjects did not understand equally well all problems which require the same operation, as indicated by the significant activeness within operation effect, $F(4,188) = 9.62$, $p < .0001$ and $F(4,224) = 8.87$, $p < .0001$. Better performance was generally associated with the active problems, but the size of the effect was quite variable.

If the poor results on this task with multiplication and division problems result from a poor understanding of these operations, this should be reflected in performance on whole number problems. However, Group 2's performance on the whole number problems rules out this explanation, since the mean percent correct was 98% (versus 69%), and ranged from 95% to 100% correct, making the pattern of results quite different.

STUDY 2: JUDGMENT OF SOLUTION SIMILARITY

Study 2 tested the Surface Feature hypothesis: Do nonexperts consistently categorize problems on the basis of surface features, and do experts consistently sort on the basis of deep structure? In this study, subjects were given a standard problem with four alternatives, and were to determine which two of the four alternatives would be solved similarly to the standard. The alternatives were structured so there was a match in: 1) both surface structure and operation (B), 2) only the operation (O), 3) only the surface structure (S). and 4) neither dimension (N). The Surface Feature hypothesis predicts that all subjects should choose the B alternative, since it has a similar storyline and requires the same operation for solution as the standard. However, for the second choice, nonexperts should consistently choose the S alternative, while experts would always choose the O alternative.

The problems varied in operation, activeness of standard, and in the difficulty of performing the computations with the numbers in the problems. Expert subjects were those who made zero or one mistake in identifying the operations for solution in the study 1 task.

Results There was a main effect of error level, $F(3,36) = 4.42$, $p = 0.0096$; the more experts subjects (0-1 errors) performed better overall (84% correct) than the less expert subjects (2, 3, and 4-5 errors, 66% correct), $t(38, \text{onetailed}) = 3.37$, $p < 0.001$. There were no significant differences among the three nonexpert groups.

As predicted, both experts and nonexperts frequently (89%) choose the B alternative. It was chosen more often than the O alternative (53%), $F(1,36) = 94.67$, $p < 0.0001$, indicating that a match in surface structure facilitates the decision that problems are solved similarly. When one of the selections was incorrect, 62% of the time the S alternative was chosen, indicating a tendency to judge solution similarity on the basis of surface features. However, 47% of the time the nonexperts did correctly choose the O alternative. In contrast, experts did not consistently judge similarity on the basis of deep structure, choosing the O alternative only 71% of the time. Together, these results imply that the Surface Feature hypothesis cannot be true.

STUDY 3: FACILITING OF JUDGMENTS OF SOLUTION SIMILARITY

As study 2 has shown, judgments of solution similarity can be facilitated by similarity of storyline. However, it is possible that similarity in the types of words, actions, and situations that occur are sufficient to produce such facilitation. For example, consider the following two active multiplication problems:

> Mary cooked a 3/4 pound steak for dinner. She ate 1/3 of the steak. How much steak did she eat?

> Tom found 1/4 of a bottle of glue. He used 3/4 of the glue building a birdhouse. How much glue did he use?

Both problems concern the size of a fraction of the original quantity that has been consumed. In contrast, the following passive problem involves no consumption:

> 7/8 of the sandwiches the waitress delivered were hamburgers. 1/4 of the hamburgers were cheeseburgers. What fraction of the sandwiches served were cheeseburgers?

Although these problem are similar, in that they require the same operation for solution, the similarity seems harder to recognize than that which occurs when the problem type is the same. In study 3, the hypothesis tested is that a match in problem type facilitates the judgment of solution similarity.

The task was to choose which of four alternatives (requiring addition, subtraction, multiplication, and division for solution) would be solved similarly to a specified standard. The sets of alternatives were structured so that: a) all alternatives were of the same type, active or passive, and b) they had story lines that were as similar as possible. The standard had a different story line and could either match or mismatch the alternatives in problem type. There were 16 items on this task: 4 operations x 2 types of standards x 2 (matching or mismatching) sets of alternatives.

Results The results indicate that similarity of problem structure facilitates the decision that two problems require the same operation for solution. The main effect of match in problem type was highly significant, $F(1, 56) = 28.00$, $p < .0001$: subjects chose the correct alternative more often when the problem types matched. However, this size of this effect differed with operation, $F(3,168) = 9.41$, $p < 0.0001$. A match in problem type provided the most facilitation for subtraction (73% vs 40%) and division (71% vs 44%) items, a smaller facilitation for addition items (88% vs 80%), but no facilitation for multiplication items (54% vs 54%). Thus, facilitation is greater for problems that students have moderate difficulty understanding.

Study 3 indicates problem solvers are able to utilize features of similarity due to problem type to judge solution similarity. Such features might include common patterns of actions, such as "giving-to" or "-from" (see Kintsch and Greeno, 1985 for other types of action patterns), similar questions, and the use of related words or phrases, such as "gave away", "spent", and "lost." Since there was little overlap of key word phrases, the similarity perceived is related to the meaning of the words, not the actual words.

STUDY 4: REPLICABILITY WITH A YOUNGER POPULATION

One objection which could be raised concerning the first three studies is that nonexpert college students should not be considered true novices, because they have had a considerable opportunity to practice and apply inappropriate problem solving stratgies. To determine whether this objection has any justification, the studies were repeated with eighth grade students, who were the youngest students available who had completed all instruction in fractions. It was conceivable that overall levels of performance would be higher for the more experienced subjects, but the trends should correspond.

Results In general, the performance of the eighth graders was quite similar to that of the college students. In the study 1 task, the overall performance of eighth graders was lower than that of the college students, $F(1, 105) = 20.98$, $p < 0.0001$. However, there were no interactions with age, indicating eighth graders had difficulty with the same types of problems. They also tended to err in the same way as adults.

For eighth graders, the study 2 task was modified slightly, so that performance on task 1 could be used to predict when subjects would err on the similarity judgment task. It was predicted that subjects would consistently confuse operations between tasks. In fact, performance on task 1 did correlate with performance on task 2, $r = 0.571$, $t(51) = 4.96$, $p < 0.001$, suggesting subjects tend to make surface feature errors when they do not understand the operation with fractions.

In study 3, the eighth graders performed nearly as well the adults on the matching task (58% vs 62% adult, N.S.). The one significant effect involving age was an interaction of age, match, and operation, $F(3,153) = 4.38$, $p = 0.0055$; this seemed mainly due to eighth graders having more difficulty distinguishing active subtraction and active multiplication problems. In conclusion, the ways in which adults differ from eighth graders are also ways in which thay are better than eighth graders: nonexpert adults do not make different kinds of errors from "true novices".

GENERAL DISCUSSION

The studies reported here imply that the strong form of the Surface Feature hypothesis is false: nonexperts do not consistently use similarity of surface features as a basis for a judgment of solution similarity. They do tend to rely on surface feature similarity if they have difficulty understanding the concepts of the operation, but the eighth graders performance on study 2 showed this tendency is not consistent. Hence, even a weaker form of the Surface Feature hypothesis would not appear to account for the data.

In fact, study 3 suggests that the surface/deep structure distinction may not provide a sufficiently rich scheme for understanding problem classification, since judgments of solution similarity were facilitated by a match in problem type. Problem type must provide the basis for a structural analysis which is intermediate between surface and deep structure. Actually, since the deep structure must be derived from the problem text, it is plausible that a useful level of structure might result without a complete analysis of the deep structure. If this is true, it also provides a reasonable explanation for why experts occassionally err; subjects of all levels of expertise categorize problems on the basis of the features of similarity that they perceive. If the analysis of structure is halted before it is complete, either because of a lack of knowledge or from falsely perceived similarity, the subject is likely to be incorrect.

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COGNITIVE EFFECTS OF INSTRUCTION DESIGNED TO PROMOTE MEANING FOR WRITTEN MATHEMATICAL SYMBOLS

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Students in grades four, five, and six were taught a special two-three week unit designed to assist them in creating appropriate meanings for decimal fraction symbols. Based on theoretical analyses, it was hypothesized that students would acquire appropriate meanings and would use the meanings to solve a variety of decimal problems. Many of the students in the first study, who were instructed in small groups, did acquire and use semantic-based processes to solve decimal problems, including novel transfer problems. Students in the second study, who were instructed in a whole classroom, largely exhibited semantic processes on the instructed problems but they did not transfer the processes to novel situations. Possible explanations for the differences between samples are discussed.

One of the most widespread and persistent complaints about students' behavior on mathematical tasks is that it is overly mechanical and inflexible. Even when students perform well, further analyses often show that their skills are applied in a rigid way and are tied to particular tasks (Carpenter, Matthews, Lindquist, & Silver, 1984; Hiebert & Wearne, 1986).

Currently we are exploring the nature of one potentially fundamental cause for the rigidity of students' mathematical behavior and their coincident low performance on even slightly nonroutine tasks. Our hypothesis is that much of students' mechanical behavior in mathematics results from drill-and-practice of symbol manipulation rules before establishing meaning for the written symbols they manipulate. In other words, conventional instruction is not sufficiently sensitive to the importance of creating sound, rich meanings for written representations at the outset. We are examining this hypothesis by studying the effects of altered instruction on the cognitive processes students use to solve mathematical tasks. The domain of interest is the introduction of decimal fractions to elementary school students.

Theoretical Context

We propose that competence with written mathematical symbols develops through the sequential and cumulative mastery of four distinct cognitive processes (Hiebert, 1987). The first two processes develop the semantics of the particular symbol system and the second two elaborate its syntax. The four processes are: (1) creating meaning for individual symbols by connecting them with familiar or meaningful referents; (2) developing symbol manipulation procedures by reflecting on the meanings of the symbols; (3) elaborating and routinizing the procedures and rules for symbols; and (4) using the symbols and rules as referents for building a more abstract symbol system.

The first two processes are of most interest here because they are the processes that we hypothesize to be crucial for further success in mathematics and, at the same time, are processes that apparently are not cultivated in conventional instructional programs. Our objective was to provide explicit opportunities for students to acquire processes that create meaning for written symbols, and processes that use these meanings to guide the development of simple procedures on symbols. By monitoring students' behavior over the special instructional sequence, it was possible to document changes in cognitive processes and to trace the effects of these changes on performance. In particular, it was possible to examine the role of semantic-based processes in developing initial competence with written symbols.

Methodology

Samples

Two different samples were used to provide different instructional settings. In the first study, the sample consisted of nine students in grade four and ten students in each of grades five and six. All fourth graders and five of the fifth graders had not been instructed previously in decimals. Students represented different achievement, racial, and gender groups. This sample was used to examine the effects of instruction in small group settings.

In the second study, the sample consisted of an entire classroom of 30 fifth graders. Most students in the class scored between the 40th and 60th percentiles on recent standardized achievement tests in mathematics; they represented a mix of gender and racial groups. Most of the students had received brief previous instruction in decimals. This sample was used to investigate the effects of instruction in whole classroom settings.

Instruction

Instruction in the two studies differed slightly so they will be described separately. In study one, students participated in a sequence of nine activities covered in seven to nine 25 minute lessons taught by one of the authors. The first five activities focused on process one--developing explicit connections between the written notation for decimal fractions (through hundredths) and referents that represented quantity in a concrete way. Dienes base ten blocks were used for the referents. The last four activities focused on process two in the context of addition and subtraction. Students were asked to use the block referents and the joining and separating action on blocks to guide their decisions about how to deal with the associated written symbols. They were not shown procedural rules with symbols, such as lining up decimal points before adding or subtracting.

In the second study, students participated in a sequence of eleven 35 minute lessons taught by one of the authors. The first eight lessons were similar to the nine lesson sequence employed in study one. Lesson nine returned to process one using the number line as a referent. The aim here was to enrich the meanings students could connect with written symbols by providing another visual representation of decimal fraction quantities. Lessons ten and eleven focused on process two in the context of ordering decimals and changing between decimal and common fraction form. Again, students were not shown rules for manipulating symbols but were asked to use what they knew about decimal fractions to deal with the symbols appropriately.

Evaluation

Two kinds of measures were used to assess the effects of instruction on students' performance and on the processes responsibl for performance. Direct measures were tasks like those used during instruction; they assessed in a straightforward way whether students had acquired processes one and two. Transfer measures were tasks that had not been introduced or discussed during instruction but could be solved by a flexible application of processes one and two.

The evaluation schedules and specific assessment instruments differed between studies. All students in study one were interviewed individually before instruction and six weeks after instruction.

After completing each task, students were asked to explain how they decided what to do and what they were thinking while they solved the task. Verbal explanations were coded in terms of whether they appealed to quantitative meanings of the written symbols (semantic processes), whether they indicated only a recall and application of previously learned symbol manipulation rules (syntactic processes), or whether they were ambiguous (uncodable). Interrater agreement in coding student protocols was above 90 percent.

Evaluation in study two was structured so that the effects of particular subsets of instructional activities could be traced more precisely. Before instruction, all students received a written test containing both direct and transfer measures. Six target students also received individual interviews similar to those conducted with sample one. Forms of the written test were readministered after lessons four, eight, and eleven, and six weeks later. Interviews were conducted again with the same six students after lessons four and nine, and six weeks after instruction.

Results

Study One

Results from the first study only will be summarized here; some of these data are described in more detail in other manuscripts (Wearne & Hiebert, 1986). Two types of direct measures were included in the interviews--tasks assessing connections between Dienes block and written representations of decimal fractions, and addition problems presented with symbols written in horizontal form. On the representation task, performance improved from 3 of the 29 students correct before instruction to 24 students correct after instruction. On the addition problems, the primary concern was the process used to complete the tasks. Before instruction, 2 of the 29 students used semantic processes (considered the meanings of the symbols, i.e., the quantitative values of the digits) in deciding how to add 2.3 + .62 or 5 + .3. After instruction, 19 students did so. From these results it appears that, given appropriate instruction, most students can acquire the semantic processes, processes one and two in the theoretical description summarized earlier.

A critical question is whether students can use the processes flexibly. Transfer tasks involved ordering decimal fractions (choose the larger of .5 and .42) and changing between common fraction and decimal form (write .7 as a fraction and write 8/100 as a decimal). Of the 19 students who used semantic processes on the addition problems, 11 students transferred these processes to at least two of the three novel tasks.

Transfer of fundamental cognitive reasoning processes by nearly 60 percent of the students who acquired them is notable given the long-standing difficulty of inducing transfer in learning experiments. The result is especially interesting given the relatively brief instructional sequence (not much longer than is ordinarily spent practicing rules for solving these problems) and the fact that many students had never seen these kinds of problems and certainly had not used semantic reasoning processes to solve them.

The hypothesis that overly syntactic behavior results from drill-and-practice before establishing meaning for symbols is tentatively supported by findings which suggest that early routinization of syntactic rules may inhibit students from developing semantic processes. For the 15 students who had received previous instruction, who had been taught rules for adding decimal numbers, 33 percent of the responses to the addition items changed over our instruction from syntactic-like or uncodable to semantic-based responses. In contrast, 64 percent of the responses by the 14 students who had not yet been taught decimals changed to semantic-based responses. The sample of students and tasks is too small to draw definitive conclusions, but it appears that the question of when instruction should focus on the meaning of written symbols is worth pursuing further.

Study Two

Assessments were given more frequently than in study one, and particular measures changed roles (from transfer measures to direct measures) as students received direct instruction on the topic. Tasks like those in the first study were used in the second study, both in the interviews and on the written tests.

After students had received instruction on a particular kind of task, an average of four-five of the six interviewed students used semantic processes to solve the task. None of the students had used such processes before our instruction began. These findings are consistent with those in study one.

However, most of the target students did not use semantic processes on the items until similar items were discussed during instruction. That is, most of the students (five of the six on most tasks) did not extend or transfer the semantic processes that they had discussed early in instruction to novel tasks, tasks that appeared later in the instructional activities. The lack of transfer is at odds with the results of study one.

Assuming the six target students are representative of the class, we can predict that performance on the written tests will be relatively high on the direct measures but quite low on the transfer measures. The prediction assumes that semantic processes yield correct performance and nonsemantic processes do not, at least on transfer measures. Based on other analyses, the assumption is not unreasonable (Wearne & Hiebert, 1986). The percentages from the written tests, as related below, largely confirm the predictions.

Before our instruction began, an average of 9 percent of the responses to decimal tasks were correct. After students had received some instruction, but just prior to instruction on a particular type of task, an average of 18 percent of the responses were correct. This can be considered a measure of transfer because these tasks were administered before students had considered them during instruction, but after students had been provided opportunities to engage in process one (connect symbols with referents). In contrast, after students discussed the use of semantic processes on similar tasks, and had seen such processes modeled, an average of 69 percent of the responses were correct.

Discussion

A finding of particular interest was the difference between the two samples in process use on the transfer items. There are three viable explanations for the differences, all of which we currently are investigating further. The first explanation, and perhaps the most obvious, is that acquiring and applying semantic processes is not an all or none phenomenon, and students in the second study simply did not acquire the processes as completely or as deeply as those in the first study. This certainly is possible given the more difficult instructional setting and the likely accompanying effects---reduced attention, less engaged time, etc.

The second potential explanation for the differences between samples is that most of the fifth graders in the second study had already studied decimals in grade four. Although it is not clear how much they had practiced symbol manipulation rules, a number of the interviewed students persisted in citing rules like "you always have to line up the decimal points" to justify their behavior. The special instructional activities that might have provided a meaningful rationale for the rules failed to do so. It seems that it is difficult for students to penetrate their own routinized procedures with meaningful information (see also Resnick & Omanson, 1987).

The third possible explanation for the low transfer of semantic processes by the students in study two overlaps with the first two but is worth considering separately. It may be that students had acquired the semantic processes but did not recognize their applicability in novel situations, or for some reason chose not to use them. VanLeuvan-Lefevre (1987) reports a related result. Second graders who had developed relatively rich conceptual knowledge of fraction symbols (through process one) did not use the knowledge in novel situations. Using Greeno, Riley, & Gelman's (1984) term, students did not acquire utilization competence. Perhaps instruction needs to attend as carefully to the appropriate use of semantic processes as to their acquisition.

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REGIONS, NUMBER LINES, AND RULERS AS MODELS FOR FRACTIONS

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This study investigated the measure subconstruct of the rational number concept by contrasting students' knowledge of two exemplars of this subconstruct, an eighth-inch ruler and the number line. Students' proficiency with an area model (part-whole subconstruct) was also compared to their proficiency with the two measure models.

The concept of rational numbers has been analyzed by various researchers into a number of various components that are usually referred to as subconstructs in accordance with Kieren's analysis (1976). Two of these agreed upon subconstructs of the rational number concept identified by Kieren (1976) are part-whole and measure. The part-whole subconstruct is acknowledged by Behr, Lesh, Post and Silver (1983) as being fundamental to all other interpretations of rational numbers. The measure subconstruct as represented by associating fractions with points on number lines of length 1 and greater has been shown to be difficult for elementary and junior high school students (Novillis, 1976; Larson, 1980; Behr et al., 1983; Behr and Bright, 1984; Armstrong and Larson, 1985). Identifying the unit on a number line has been mentioned by most of these researchers as being one area of difficulty for students. Students sometimes disregard the scaling and treat the whole number line as the unit. Another variable that increases the difficulty of the number line model is the number of segments in each unit as related to the denominator. When the number of segments is a multiple of the denominator, students seem unable to disregard the extra points in order to associate a reduced equivalent fraction with the correct point on the number line.

The principal purpose of this study was to further investigate intermediate grade students' understanding of the measure subconstruct of the rational number concept. The major question addressed in this study related to this purpose was: Are intermediate grade students equally proficient in relating fractions to the number line as they are in relating fractions to a ruler scaled to the eighth-inch? The measure subconstruct in the past has mostly been associated with the number line model. The ruler is another representation that is very similar to the number line in that both contain a scale and involve the measure of length. The main difference is that the eighth-inch ruler is a common measuring instrument that is found in and out of school. In terms of Lesh's (1976) five representational systems, the number line would be classified as a "picture" (static figural model), but using a ruler would be classified as a "real world situation". Another difference is that fractional parts on an eighth-inch ruler are limited to halves, fourths, and eighths.

Since in all previous rational number research students at this age have been most successful with area models these were also included in the study for purposes of comparison. It was assumed that the students would be more successful on the area model tasks across all types of fractions than on the number line tasks. Of major interest was the difficulty of the ruler model as compared to the area and number line models.

METHOD

Seventy-three fifth-graders and 48 sixth-graders were administered an 84 item Fraction Test in two parts in October, 1985. The students comprised all of the fifth- and sixth-graders in one school who returned parental approval forms. The school is in a lower to middle socio-economic area in Tucson, Arizona, and contains many minority students.

The Fraction Test contained 40 multiple choice and 44 open-ended items. It measured the students' ability to associate proper fractions, improper fractions and mixed numerals with area models (rectangular regions), points on number lines, and line segments measured with an eighth-inch ruler. Equivalence was also tested by partitioning the unit into twice as many parts as the denominator of the related fraction. Each type of fraction associated with each type of model was tested by the following four types of test items: a) given a model, the students selected the appropriate fraction; b) given a fraction, students selected the appropriate model; c) given a model, students wrote the appropriate fraction; and d) given a fraction, students indicated that fractional part of the model.

Another portion of the test contained items at the abstract level to measure students' ability to recognize and to produce equivalent fractions, to reduce fractions to lowest terms, to produce a missing numerator or denominator in a pair of equivalent fractions, and to equate and to produce related improper fractions and mixed numerals.

RESULTS AND DISCUSSION

The Fraction Test can be divided into a number of various subtests by considering the following variables: type of model, type of fraction, and type of test item. The means for the subtests based on the three models investigated and the abstract portion of the test are presented in Table 1.

Table 1

Means for Four Subtests: Area Model, Number Line, Ruler, and Abstract

	Area Model	Number Line	Ruler	Abstract	Total Test
Number of Items	20	24	20	20	84
Grade 5 n=73	24%	8%	23%	18%	18%
Grade 6 n=48	27%	9%	31%	30%	23%
Total	25%	9%	26%	23%	20%

The means for the total test and each subtest were very low. From previous research, it was expected that the scores on the area model subtest would be higher (Novillis, 1976; Armstrong & Larson, 1985). As expected the number line model was more difficult than was the area model. An unexpected result was that students' overall scores on the area model and ruler subtests were approximately the same. In past research, students were more successful with area part-whole models than with measure models. Even though the students had similar mean scores on these two subtests, there was a difference in their success in associating the different types of fractions with these two models. The means for the four types of fractions for each type of model are presented in Table 2.

Table 2

Means for Type of Fraction for Each Type of Model

Type of Fraction	Area Model Gr5	Area Model Gr6	Number Line Gr5	Number Line Gr6	Ruler Gr5	Ruler Gr6	Abstract Gr5	Abstract Gr6
Proper Fractions	63%	56% (4 items)	8%	10% (8 items)	21%	30% (4 items)	N/A	
Equivalent Fractions	5%	10% (8 items)	4%	4% (8 items)	21%	24% (3 items)	25%	43% (12 items)
Mixed Numerals	39%	44% (4 items)	17%	20% (4 items)	30%	38% (8 items)	N/A	
Improper Fractions	7%	14% (4 items)	9%	5% (4 items)	11%	17% (4 items)	N/A	
Relationship of Improper Fractions and Mixed Numerals							6%	11% (8 items)

Students were more successful in associating proper fractions with area models than they were with associating mixed numerals with area models. But when using the eighth-inch ruler, students were more successful with mixed numerals than they were with proper fractions. Even though the scores were lower the same pattern exists for the number line model as for the ruler. In order to be successful on a mixed numeral task, students must first associate the whole number portion of the mixed numeral to these models and then find the fractional part. The second step is all that is required in order to associate a proper fraction to the number line or ruler. Is it more difficult to associate a proper fraction than a mixed numeral to a measure model because the unit is the interval from 0 to 1? Why would the interval from 0 to 1, when partitioned into fractional parts, be more difficult than other partitioned intervals, e.g., 1 to 2 and 2 to 3? Some errors made on the mixed numeral tasks involved selecting the incorrect fractional part after correctly selecting the whole number part. Such errors indicate that some students are familiar with the syntax of mixed numerals but are having problems with the semantics of the fractional part of the mixed numeral.

Another difference to be noted between the area model and the ruler is the level of success with equivalent fractions as compared to proper fractions. When associating these two types of fractions with the ruler students scored at approximately the same level. Whereas, with the area model there was a large difference in their success rate with both types of fractions. When considering these four subtests, students were most successful in associating proper fractions with area models and least successful in associating reduced equivalent fractions with area models. Students' increased success with associating reduced equivalent fractions with the ruler might be due to the fact that halves and fourths are the only equivalent fractions that appear on an eighth-inch ruler. Also, the fractional parts of each inch in the ruler are marked by vertical lines of differing lengths that serve as visual cues to the various equivalent fractions associated with each mark. Both of these task variables probably contribute to students' increased success with equivalent fractions when using a ruler compared to area and number line models.

Most of the students were unable to recognize models for equivalent fractions--the means for associating equivalent fractions with the three models were 7% for fifth-graders and 10% for sixth-graders. Yet the sixth-graders had their third highest score on the subtest that tested equivalent fractions at the abstract level. On this 12 item subtest, students supplied or recognized an equivalent fraction, reduced fractions to lowest terms, and found a missing numerator or denominator in a pair of equivalent fractions. The greatest difference in achievement between the two grade levels occurred on this subtest. The sixth-graders' increase in competence in identifying and generating equivalent fractions at the abstract level without a parallel increase in their ability to associate equivalent fractions with appropriate models could be indicative of mathematics instruction that focuses on symbol manipulation without concept development.

CONCLUDING REMARKS

In textbooks, rulers and other measurement tools that contain fractional parts of units are generally located in the "Measurement Chapter." Students are introduced to a specific fractional part of a unit (e.g., a half-hour, half-cup, half-inch, quarter hour, fourth of a cup, fourth-inch) and are then involved in practice activities to learn to identify that fractional part on the appropriate scale. On these textbook pages there is little reference to the more general fraction concept. In the "Fraction Chapters" area and set models and to a lesser degree number line models are used to develop meaning for the various types of fractions. One usually thinks of the use of fractional sub-units on measurement scales as an application of the general fraction concept but this might not be the case for some students.

Consideration of the above description of the curriculum and the results from this study might explain why some children are more successful with the ruler than with the number line. When using the ruler they might not be applying a generalized fraction concept but instead learned specific sub-units--half-inch, fourth-inch and eighth-inch--in the same way that they identify previously learned units such as, inch and foot. This would explain their increased success with equivalent fractions when using the ruler.

In order to help students recognize and integrate the common aspects of specific measurement scales and the number line model for rational numbers these should be related in instruction. The relationship of rulers calibrated in different sub-units to number lines similarly partitioned might be a key in helping students understand number line models. Also, it could aid students in interpreting measurement scales calibrated to fractional parts other than halves and fourths. Additional research is needed to investigate this proposal.

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FREE PRODUCTION OF FRACTION MONOGRAPHS

L.STREEFLAND

RESEARCHGROUP OW & OC, UTRECHT

Une recherche de développement a été entreprise pendant la période du septembre 1983 au février 1986 en vue de la création d'un cours de tendance 'réaliste' sur les fractions et de la formation d'une théorie d'enseignement et d'apprentissage annexe, suivant le cadre théorique tel qu'il a été développé par Treffers pour l'enseignement mathématique 'réaliste' en général. Notre contribution présentera ce que nous appellons 'les productions libres' des élèves qui ont participé à cette recherche, en particulier par rapport au dressage de 'monographies de fractions'.

1 DEVELOPMENTAL RESEARCH

The research took place from September '83 to February '86 and was guided by two main objectives:

a. developing new course material for fractions aimed at children from 9-12;
b. developing a theory for teaching and learning fractions within the broader framework of a theory for realistic mathematics education.

A teaching experiment was therefore arranged, interspersed regularly by (clinical) interviews, in order to trace long term individual learning processes [1].
A tentative elaboration of the course was based on former development and research [2]. It had to be provisional. For instance, there was no experience with children's own free productions, which is an essential tenet of realistic mathematics education (cf. section 2). With respect to this the research was intended to produce hypotheses.
In order to realize our objectives, a continual shifting from the teaching process to the individual learning processes and vice-versa was necessary.
The researchgroup was small (19 pupils) and came from a 'simulation school', which means that an extra teacher was present because of the socio-economic status of the parents, i.e. untrained labourers or members of ethnic minorities.

2 THEORETICAL FRAMEWORK

The background was formed by the theory of realistic mathematics education composed by Treffers [3]. He distinguished as its dimensions:

2.1 The Van Hiele levels

> 'Thinking is continued acting. ... At the higher level the acting of the lower level becomes an object of analysis' [4].

At the first level, the objects of mathematical thought are embedded in operations with material - for fractions for instance, visible models and figures. At the second level, relations between fractional numbers and figures are explored and, as a result, those numbers and figures become symbols for properties. At the third level, the relations themselves, such as the equivalence of fractions for instance, become objects of mathematical thought; their nature and interrelated properties are ascertained, which makes it possible to derive them from each other.

> 'The organisation now fits into a logical and connected system. The bonds between the various levels across different courses are fairly complex. So the third level of the arithmetic system represents the concrete basis for the first level in algebra instruction and the third level of 'fractions' is the basis for 'probability' according to Van Hiele.' [5]

2.2 Didactical phenomenology

This is the second dimension. This type of analysis considers mathematical concepts, operations and so on as organizational tools for phenomena with an eye to their mental constitution in learning processes. This method is therefore in opposition to concept attainment by concrete embodiments [6]. Its consequences for fractions I have already explained elsewhere. [7]

2.3 Progressive mathematisation

This third dimension is guided by the following instructional principles:

a. the dominating place occupied by context problems, serving both as a source and as a field of application of mathematical concepts;
b. the great amount of attention payed to (the development of) situation models, schemas and symbolising (cf. [7]);
c. the large contribution children make to the course by their own productions and constructions, which lead them from the informal to the formal methods;
d. the interactive character of the learning process;
e. the firm intertwining of (related) learning strands [8].

Tenet (c) is the heart of our matter. It will be considered with respect to:

3 PRODUCING FRACTION MONOGRAPHS

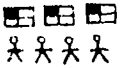
3.1 Analysis

A monograph describes the process of fair sharing and its outcome in situations like 'share 3 chocolate bars among 4 children'. [9]

First level

a Example: Distribution performed and described.

Each will get $\frac{1}{4}$ (of a bar) + $\frac{1}{4}$ (") + $\frac{1}{4}$



$3 \times \frac{1}{4}$

fig.1 $\frac{3}{4}$, or

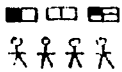


fig.2 $\frac{1}{2} + \frac{1}{4}$ ($\frac{3}{4}$, each.

A child's observation of facts like 'two quarters hide in one half' signals the transition to the next level. In the research the children invented terms like 'hidename', 'alias' or 'pseudonym'. The last term was chosen 'officially' for this phenomenon.

A monograph like '$\frac{3}{4}$' ($\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$, $3 \times \frac{1}{4}$, $\frac{1}{2}+\frac{1}{4}$, ...) is purely descriptive at this stage. As such it replaces descriptions from every day life like 'each one gets one quarter and one quarter ...'

By way of inversion one can ask to write in full the numbersentences as well as to reconstruct the distribution situation. As a consequence the concrete source, every day language of repeated halving and the budding application of numbersymbols for fractions will be connected firmly.

b. Like a. The different ways of sharing (cf. fig.1 and 2) and their outcomes are registered in symbols successively.

E.g. (fig.1) After 1 bar: each $\frac{1}{4}$.

After 2 bars: each $\frac{1}{4}+\frac{1}{4} = 2 \times \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$.

c. Like a., but now a reconstruction of the situation with more divers material.

E.g. Each one got: $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$; $\frac{1}{2}+\frac{1}{4}$; $1 - \frac{1}{4}$. How much is it? Which story of fair sharing fits?

So the mathematical material strongly refers to (imaginary) distribution situations. The methods of sharing (unit by unit, global) are the principles of production behind the monographs.

Second level

Concrete stories of fair sharing as principles of production gradually melt into the background. Other methods of production will supplement these methods and later replace them, e.g. the twosided application of pseudonyms, that is replacing a fraction by a pseudonym and also considering a given fraction as a pseudonym for (an)other fraction(s). The commutative laws for addition and multiplication may also become methods of production.

Instead of describing concrete stories of fair sharing, the activities change into the composition and decomposition of real fractions according to production methods as mentioned above. However, it is preferably not to wipe out all the traces of the concrete foundation. It can be very productive to start with the decomposition in unit - fractions based on the unit by unit division.

$$\text{E.g.}\quad \frac{5}{6} = \underbrace{\frac{1}{6}+\frac{1}{6}+\frac{1}{6}}+\underbrace{\frac{1}{6}+\frac{1}{6}} \qquad \begin{cases} = 5 \times \frac{1}{6} = \frac{1}{6} \times 5 \\ = 1 \times \frac{5}{6} = \frac{5}{6} \times 1 \end{cases}$$

$$\begin{aligned} &= \frac{3}{6} + \frac{2}{6} \\ &= \frac{2}{6} + \frac{3}{6} \\ &= \frac{1}{3} + \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{3} \qquad \text{and so on.} \end{aligned}$$

Based on activities like this, the pupils will become able to prepare and develop their own methods of operating with fractions. The understanding of equivalence can increase considerably in this way. Another striking feature is that laws such as that of the commutativity do not put their seal on the system of fractions afterwards as tools of formalization; on the contrary, these laws are developed from within the system in order to be able to organise it formally with progressive refinement.

Third level

Once again, a change of sight will have to be made in order to reach this level. The rules for the composition and decomposition of fractions will become the objects of mathematical thought which holds, too, for their character and the way they are connected. The methods of production which turn out to be the most efficient might become standard procedures or algorithms.

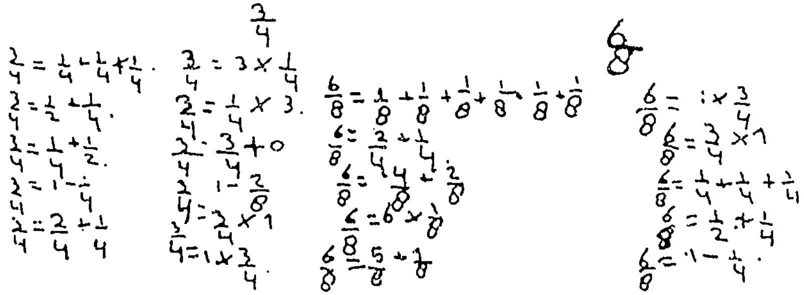
E.g. $\frac{1}{2}+\frac{1}{3}$. To which monograph does this sentence belong? Explain your rule to make a decision about it.

$\frac{1}{2}+\frac{1}{3} \rightarrow$ Rule: both fractions are probably pseudonyms for other fractions. Produce these 'other fractions' like $\frac{1}{2} \rightarrow \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots$ and $\frac{1}{3} \rightarrow \frac{1}{3}, \frac{2}{6}, \ldots$ and aim at the first name in common, or perhaps the second or third ... $\rightarrow \frac{3}{6}+\frac{2}{6}=\frac{5}{6}$ or $\frac{6}{12}+\frac{4}{12}=\frac{10}{12}$ or $\frac{9}{18}+\frac{6}{18}=\frac{15}{18}$ ...

3.2 Research results

13 Pupils followed the complete course. Only their results will be considered. They applied *four* methods of production with increasing consciousness and consistency.

a. *starting with the decomposition in unit fractions*



Example of pupils work.

Four pupils behaved this way consistently and the others incidentally.

b. *varying the operation*

The group differentiatied as follows:

prefered operations	number of pupils
+	1
+, -	2
+, x	2
+, -, x	6
+, -, x, :	2

c. *twosided application of pseudonyms*

11 pupils behaved this way in general. The remaining two confined themselves to the equivalence of the simplest fractions within the system of repeated halving, like $\frac{1}{2}=\frac{2}{4}$ and $\frac{2}{8}=\frac{1}{4}$.

d. *application of laws like commutativity*

Seven pupils applied laws like a + b = b + a, a x b = b x a and a x b = 2a x $\frac{1}{2}$b (a, b, $\in Q^+$) systematically. Moreover, a x b = (a - p) x b + pb, a - b = (a + p) - (b + p), a : b = pa : pb and other forms of composed number sentences were used.

Some striking aspects of the learning processes were:

— the increasing skill of all the pupils in producing equivalent fractions;
— the initial dominance of addition and the gradual increase in the use of other operations; division however, remained somewhat of an exception;
— the general dominance of the elementary laws for operations as methods of production.

The results contain sufficient evidence to assume that giving the opportunity to produce monographs is a means for making progress in vertical mathematisation for all the pupils. That is, provided these productions will be alternated with interactive lessons in which individual methods of production are discussed and interchanged (at the conference an example will be elaborated upon).

4 CONCLUSION

It should be clear that only a corner of the veil could be lifted.
In general free productions possess a double evaluative function.

a. They reflect the teaching process and as such shed light on, for instance, the subordinate role played in it by division. On the other hand, the phenomenon of making N-distractorfailures occurred very rarely. This means that fighting against them was rather successful [10].

b. They also force pupils to reflect on their own learning processes [11].

And - finally - producing monographs seems to be a promising first step into algebra.

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[11] cf. Treffers (1987).

LA NOTION DE GROUPEMENTS CHEZ PIAGET: PROLONGEMENTS FORMELS ET APPLICATION À LA GENÈSE DU CONCEPT DE FRACTION

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La notion de groupement occupe une place centrale dans la théorie structuraliste du développement cognitif chez Piaget. Elle expliquerait, selon ce dernier, l'émergence du concept de nombre comme provenant de la combinaison des groupements liés à la sériation et à la classification, de même que la formation des principales conservations physiques. Elle ne décrit cependant que l'étape finale dans la formation d'un concept. A partir de la formalisation mathématique de E.Wittmann, nous avons élaboré un modèle mathématique qui tente de cerner cette notion, c'est-à-dire sa genèse et ses prolongements. Par la suite, nous avons testé ce modèle à diverses formations de concepts, en particulier à celle de la notion de fraction chez l'enfant de 6 à 16 ans, en nous guidant sur une expérimentation menée par G. Noelting.

1. Présentation générale

1.1 Problématique

Peu de psychologues et de didacticiens savent à quel point la notion de groupement, et plus généralement celle de structure mathématique, jouent un rôle central dans les théories de la connaissance échafaudées par Piaget. Cela s'explique en grande partie par la confusion dans la présentation et l'exploitation de ce concept chez ce dernier. Plusieurs, conscients de son importance, ont tenté de clarifier cette notion en lui donnant une forme mathématique plus intelligible: citons, J.B. Grize (1960), G. Granger (1965), K. Witz (1971), H. Wermus (1972) et enfin E. Wittmann (1973, 1975, 1982).

Bien que la formulation de Wittmann dans le cadre du langage des catégories donne une prise excellente sur cette notion, il n'en reste pas moins que cette dernière ne formalise que l'étape finale de l'acquisition d'un concept. Du point de vue épistémologique, si nous désirons pénétrer à l'intérieur du mécanisme de construction des connaissances, ce sont les étapes conduisant à cette structuration qui demeurent les plus intéressantes, de même que les étapes ultérieures combinant ou perfectionnant ces structurations finales pour accéder à des connaissances plus complexes ou plus profondes. Piaget, dans ses études sur la genèse du nombre, fournit - bien maladroitement et de façon erronée - des pistes de ces étapes préliminaires à la formation de groupements qu'il dénomme "groupements de classe et de sériation" et dont la mystérieuse combinaison expliquerait, selon lui, l'éclosion du concept de nombre.

L'objectif que nous avons poursuivi a été de retracer ces étapes, antérieures et postérieures au groupement, et de les formaliser dans un modèle mathématique général englobant la définition de Wittmann. Chemin faisant, notre problématique s'est élargie à la critique de la théorie structuraliste de Piaget et de l'utilisation abusive et souvent fausse de la mathématique dans ses écrits, concrétisée, entre autres, dans le groupe INRC.

1.2 Réponse à la problématique

Nous avons été ainsi amené à construire et à définir dans le langage des catégories et des graphes des notions préliminaires au groupement, comme celles de schème, de fragment de groupement, de sous- groupement, de groupement engendré par un fragment, etc, et des notions postérieures comme celles d'isomorphisme et d'union de groupements, de groupement libre, de groupement-quotient, etc, et, finalement, la notion centrale de produit amalgamé de groupements. Cet outillage nous a alors servi à mieux cerner et décrire dans un modèle unique les liens implicites établis par Piaget entre les concepts de schème, de groupement, de mécanisme cognitif (assimilation, accomodation, abstraction réfléchissante, etc.) et de structures cognitives associées aux divers stades du développement intellectuel.

1.3 Application du modèle

Nous nous restreindrons ici à une seule application de notre modèle, à savoir celle de la genèse de la notion de fraction dont les étapes ont été si joliment mises en évidence par G. Noelting (1978). Nous montrerons comment les notions développées peuvent servir à décrire les groupements, leur genèse, leur développement ou leur abandon, ainsi que leurs interactions, intervenant dans la résolution du problème consistant à comparer deux fractions quelconques.

2. Description sommaire de l'expérimentation de G. Noelting sur les fractions (cf [1])

Après une démonstration concrète, on présente à l'enfant (ou à la classe) 25 dessins du type suivant:

où les figures pleines représentent des verres de jus d'orange et les figures vides, des verres d'eau.

On demande alors si, après avoir mélangé chaque groupe de verres dans un seul grand verre, le groupe de gauche aura plus ou moins ou autant le goût de jus d'orange que le groupe de droite; en d'autres mots, on demande de comparer (3,1) à (1,4). L'enfant doit de plus justifier sa réponse.

L'analyse des réponses sur un grand nombre d'enfants fait ressortir des stratégies se complexifiant avec l'âge jusqu'à la stratégie finale. Ainsi, à 3 ans et 6 mois (âge moyen), l'enfant fonde sa réponse sur la seule comparaison des numérateurs (le nombre de verres de jus); à 6 ans et 4 mois, à numérateurs égaux, il compare les dénominateurs (la quantité d'eau); à 7 ans, l'idée de rapport apparaît (plus de jus que d'eau, par exemple); à 8 ans et 1 mois, il reconnaît les fractions équivalentes du type (a,a); à 10 ans et 5 mois, il acquiert la notion générale de fraction équivalente; à 12 ans et 2 mois, il sait comparer certaines fractions après avoir ramené l'une d'elles au même dénominateur que l'autre par équivalence; enfin, à 15 ans et 10 mois, il sait résoudre le problème général en modifiant, s'il y a lieu, les deux fractions pour comparer ensuite leurs numérateurs.

3. La notion de groupement

3.1 Définition de Piaget

Selon Piaget, l'enfant construit ses connaissances par le biais d'actions, d'abord concrètes, ensuite intériorisées et finalement organisées en une structure, nommée groupement, qui lui permet de les inverser et de les combiner. Cette capacité d'organisation apparaîtrait vers 7 ou 8 ans et donnerait lieu à un certain nombre de groupements fondamentaux à l'origine, entre autres, du nombre et des principales conservations physiques. Cette structure obéit globalement aux règles suivantes:

- deux actions intériorisées ("opérations") voisines peuvent être combinées pour donner lieu à une opération plus complexe;
- toute opération est inversible;
- la combinaison d'une opération et de son inverse résulte en l'opération nulle;
- la combinaison d'opérations est associative, au sens mathématique du terme.

La formation de ces groupements assurerait à l'enfant un équilibre cognitif ou une adaptation cognitive résultant de l'application sur le plan intellectuel des mécanismes d'assimilation et d'accommodation. On trouvera, par exemple, en [2] une description élaborée de ces idées.

Notre but n'est pas de discuter ici du bien-fondé de ces hypothèses ni de la théorie structuraliste qui en découle. Nous référons pour cela le lecteur à [3]. Mais nous retenons l'idée que la notion de groupement, sous sa forme générale, est un modèle utile pour décrire la formation d'un concept ou d'une habileté intellectuelle.

3.2 Définition de Wittmann (légèrement modifée)

Un groupement est un quadruplet (S,T,C,o) où S est un ensemble d'états, T est un ensemble d'opérations, C est un sous-ensemble de T dont les éléments sont appelés opérations élémentaires et o est une loi de composition partielle sur T.

Ce quadruplet obéit aux lois suivantes:

1. (S,T,o) est une catégorie où tout morphisme est inversible;
2. T est engendré par C au sens suivant:
   pour tout $f \in T$, $F=g_1 o g_2 o \ldots o g_n$ où g_i CUC^{-1}

4. Description sommaire du modèle mathématique

Quelles sont les étapes identifiables précédant la formation d'un groupement? Comment un concept descriptible en terme de groupement peut-il résulter de la combinaison de deux autres groupements? Comment un groupement décrivant un concept peut-il donner lieu à un groupement décrivant un concept plus fin? Dès le moment où l'on accepte, comme le fait Piaget, qu'un concept comme celui de nombre naturel peut se décrire en termes de groupements, les questions précédentes se posent. Ainsi, peut-on décrire à l'aide d'un modèle mathématique cohérent la genèse des composantes numérale et quantitative du nombre, leur combinaison et, par exemple, la conception des nombres pairs?

Les définitions qui suivent proviennent du modèle que nous avons élaboré pour répondre à ces questions. Il s'agit, bien sûr, d'une description très sommaire et partielle; nous référons le lecteur à [3] pour les détails et les autres parties du modèle.

Soient $G_1 = (S_1, T_1, C_1, o_1)$ et $G_2 = (S_2, T_2, C_2, o_2)$ deux groupements.

4.1 Un foncteur de groupements F entre les groupements G_1 et G_2 est un foncteur entre les catégories (S_1, T_1, o_1) et (S_2, T_2, o_2) tel qu'il existe $n \in N$ avec $F(C_1) \subset \bigcup_{r \leq n} C_2^r$.

4.2 Un foncteur de groupements F: G_1 G_2 est dit un isomorphisme de groupements s'il est une double bijection sur les états et les opérations et si $F(C_1) = C_2$.

4.3 Un sous-groupement G'_1 de G_1 est un quadruplet (S'_1, T'_1, C'_1, o'_1) où $S'_1 \subset S_1$, $T'_1 \subset T_1$, $C'_1 \subset C_1$, $o'_1 = o_1$ restreint à T'_1, et qui est lui-même un groupement.

4.4 $F'_1 = (S'_1, T'_1, C'_1, o'_1)$ est dit fragment de groupement de G_1 si $\emptyset \neq S'_1 \subset S_1$, $C'_1 \subset C_1$, $C'_1 \subset T'_1 \subset \overline{C'_1}$ et $o'_1 = o_1$, restreint a T_1, où C'_1 porte sur S'_1 et $\overline{C'_1}$ est la fermeture réflexo-symétro-transitive de C'_1.

4.5 L'union de G_1 et G_2 sera le fragment de groupement:
$G_1 \cup G_2 = (S_1 \cup S_2,\ T_1 \cup T_2,\ C_1 \cup C_2,\ o_1 \cup o_2)$

4.6 Si F_1 est un fragment de G_1, le sous-groupement de G_1 engendré par F_1 sera le "plus petit" sous-groupement contenant F_1 comme fragment (au sens de la relation d'ordre: "est sous-groupement").

4.7 Soit (S,C) un graphe sans point isolé. Le groupement libre engendré par (S,C) noté $G_L(C)$, sera le groupement (S,T,C,o) où (S,T,o) est la catégorie libre engendrée par $C \cup C^{-1}$.

4.8 Soient $G=(S,T,C,o)$ un groupement et R une fonction qui assigne a tout couple d'états (a,b), $a,b \in S$, une relation d'équivalence $\mathcal{R}_{a,b}$ sur l'ensemble des morphismes de a vers b. Nous appellerons groupement quotient, noté G/R le groupement (S,T',C',o') où (S',T',o') est la catégorie-quotient obtenue de (S,T,o) et de R et où C' est l'ensemble des classes d'équivalence de C.
Nous noterons $G/R=(S,T/R,C/R,o/R)$.
Nous noterons R_1 la fonction qui assigne à tout couple (a,b) la relation d'équivalence triviale qui identifie toutes les flèches de a vers b.

4.9 La fusion de G_1 et G_2, notée G_1+G2, sera le fragment:
$G_1+G_2 = (S_1US_2,\ T_1UT_2/R_1,\ C_1UC_2/R_1,\ o_1Uo_2/R_1)$

4.10 Le produit amalgamé de G_1 et G_2, noté $G_1 * G_2$, sera le groupement libre engendré par le graphe $(S_1US_2,\ C_1UC_2)$ quotienté par R_1, c'est-à-dire,

$$G_1 * G_2 = G_{L(C_1UC_2)}/R_1$$

4.11 $G'=(S',T',C',o')$ est dit sous-groupement large de G si $S' \subset S$, $T' \subset T$, $o'=o$ restreint à T'.

4.12 Si $G'=(S',T',C',o')$ est un sous-groupement large de G, il sera dit sous-groupement direct s'il existe $n \in N$ tel que $C' \subset \bigcup_{r \leq n} C^r$.

4.13 Un schème sera représenté par un graphe (S,C) où C est un ensemble de flèches orientées.

5. Application du modèle au concept de fraction

5.1 Schèmes et groupements intervenant dans la comparaison des fractions.

L'analyse, sous l'angle des groupements de l'expérimentation de G. Noelting nous a permis d'identifier divers fragments, sous-groupements et groupements correspondant aux différentes étapes menant à la comparaison générale des fractions; elle nous a aussi permis de caractériser l'étape finale à l'aide d'un produit amalgamé de deux groupements, et, finalement, de décrire l'enchaînement des groupements conduisant à cette dernière étape. Voici très schématiquement et partiellement, pour chacune de ces étapes, les schèmes et groupements tirés des réponses et des explications des enfants (G_i désigne un groupement et C_i le schème qui l'engendre).

1ère étape

G_0 : associé à la comparaison du nombre de verres de jus (sans tenir compte de l'eau).

C_0 : "ajouter un verre de jus pour accentuer le goût en modifiant ou non le nombre de verres d'eau".

G_0 est isomorphe à un sous-groupement du groupement correspondant à la composante numérale du nombre.

G_0 et C_0 se raffinent lors de cette étape pour donner lieu à:

G_1 : associé à la comparaison du nombre de verres de jus en laissant l'eau inchangée.

C_1 : "ajouter un verre de jus pour accentuer le goût en laissant inchangé le nombre de verres d'eau".

G_1 est un sous-groupement de G_0.

2e étape

G_2 : associé à la comparaison du nombre de verres d'eau en laissant le jus inchangé.

C_2 : "enlever un verre d'eau pour accentuer le goût du jus en laissant le jus constant".

G_2 est un groupement isomorphe au groupement dual de G_1.

3e étape

G_3 : associé à la comparaison de couples (a,b) et (c,d) du type $a \leq b$ et $c > d$ (ou $a > b$ et $c \leq d$)

C_3 : "comparer le nombre de verres de jus au nombre de verres d'eau dans le premier couple; s'il est inférieur, relier (pour accentuer le goût) aux cas où c'est supérieur ou égal; s'il est égal, relier aux cas où c'est supérieur".

C_3 résulte de la tentative de combiner C_1 et C_2.

4e étape

G_4 : associé à la <u>conservation</u> des rapports pour les couples du type (a,a).

C_4 : "ajouter un verre de jus et un verre d'eau pour ne pas modifier le rapport et conséquemment le goût".

C_4 est la combinaison de C_1 et de C_2.

<u>5</u>

5e étape

G_5 : associé à la <u>conservation générale</u> des rapports.

C_5 : "multiplier par m le nombre de verres de jus et le nombre de verres d'eau pour conserver le goût".

G_4 est isomorphe à un sous-groupement large de G_5 qui en devient ainsi le prolongement.

6e étape

G_6 : associé à la comparaison de fractions en ne modifiant que l'une d'elles par équivalence.

C_6 : "rendre équivalent à une fraction ayant même numérateur (ou dénominateur) et comparer les dénominateurs (numérateurs)".

C_6 est obtenu de la combinaison de C_5, C_1 et C_2.

7e étape

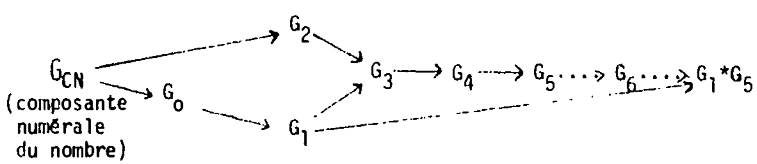
G_7 : associé à la comparaison générale de fraction;

C_7 : "remplacer les deux couples par des couples équivalents ayant même dénominateur et comparer les numérateurs".

C_7 est obtenu de la combinaison de C_5 et de C_1.

$G_7 = G_1 * G_5$; G_1 et G_5 sont des sous-groupements de G_7.

5.2 <u>Enchaînement des groupements</u>



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THE DEVELOPMENT OF CONCEPTUAL UNDERSTANDING OF THE RELATIONSHIPS AND PROCEDURAL KNOWLEDGE OF THE TRANISTION BETWEEN THE SYSTEMS OF COMMON FRACTIONS AND DECIMAL RATIONALS

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Forty seven 6 to 9 graders of average mathematical ability were tested on the systems of common fractions and decimals and their relationships. Development was slight and misconceptions and procedural bugs were found in almost every aspect: the identification of common fraction with their equivalent decimal rationals and vice versa, the translation of common fractions into their decimal equivalents, and the reverse operation when the denominators were not 10 exponents. The observed difficulties were accounted by textbooks that (1) enable rote learning of procedures without prior acquisition of the underlying conceptual knowledge, and (2) use the equivalence between the two systems without establishing the meanings of common and decimal notation and their relationships.

A widespread way in which students are introduced to the system of decimal fractions is via the common fraction system. It is characterized by the establishment of and capitalizing on the representational equivalence between the systems of decimal and common fractions, and by shuffling between them. This instruction attempts to form a conceptual model (Lesh, Landau, and Hamilton, 1983) of the rationals, and it is simultaneously based on and aming at all four parts of a conceptual model. It renders the learning of the system of decimal rationals greatly contingent on the mastery of certain prerequisite conceptual and procedural knowledge of the system of common fractions: At least relational understanding of the common fractions system and mastery of the procedures applicable in it; the availability of procedural knowledge for transitions between the two systems whose development starts with the initial acquisition of procedures and ends with the skill of making these transitions; the understanding of the relationship that is being established between the two systems that consisits of the gradual comprehension of the eqivalence or corespondence between the systems; and the understanding of the new system that is being constructed, namely, its concepts, procedures, relationships between the two and the symbols designated to them.

A variety of deficiencies in the conceptual and procedural knowledge regarding each of the systems of common fractions and decimal rationals was reported over the years but only a small portion of it dealt directly with the conceptual and the procedural knowledge pertaining to the <u>relationship</u> between the two systems. The present study deals with the relationship between the two systems and the procedures employed for the shuffling between them. It has three objectives:

a. To identify the conceptual and procedural knowldege that sixth to ninth graders have in each of the representation systems for fractions and to specify how it is used to make transitions between the systems.

b. To characterize the development of that knowledge from the sixth to ninth grade.

c. To identify possible sources for deficiencies in conceptual or procedural knowledge.

METHOD

The subjects were fourty seven 6 to 9 graders with average mathematical ability that were randomly selected from two schools in Beer-Sheva. The first study involved 6 students in each grade level and the second involved 6 students in grades 6-8 and 5 in grade 9.

The study comprised two parts, each using a different test that were administered individually. In each test the student first solved individually all test items and wrote down his or her answers. Then a clinical interview was conducted in which he or she further explained the answers and were thoroughly interrogated. The interview was also used to test certain interpretations and hypotheses regarding the answers. The tests related to each of the systems of common and decimal fractions and their relationships and in some problems the student was required to make transitions between these systems.

The first test comprised 13 items that are described later on. The second test comprised 12 problems that centered around 3 topics: The understanding of common fraction and decimal notations and mutual transformations between the two systems, expressing by decimal a given proportion of a continuous quantity and vice versa, and locating common and decimal numbers on the number line. To attain the major goal of this test, most problems demanded (a) that the same task be proformed in both types of numbers, and (b) to trasnform fractions from one representation to the other one.

RESULTS AND DISCUSSION

The results of the first test show that most students did not acquire mastery in most of the topics, only slight development occurred, and most deficiencies and misconceptions persisted from grade 6 to 9. Failure rates were between 50% and 100% in grade 6, and between 33% and 50% in grade 9, pointing at serious deficiencies in both declarative and procedural knowledge within each of the systems and their coordination. Three topics were not acquired by most students and did not develop from grade 6 to 9: Ordering decimals and common fractions, determining a proportion (different from the well known 0.25 and 0.5) of one quantity out another (both continuous and discrete), and increasing and decreasing decimal numbers by 10 exponents. In seven topics slight development was observed, but they were not acquired by all students: Marking common and decimal numbers on the number line; expressing proportions (different from 0.25 and 0.5) between continuous quantities (line . ents) by common fraction and decimal notation; comparison of lengths (in meters and kilometers), and of weihgts (in grams and kilograms); number of tenths, hundredths, and thousandths in decimals; comparing common fraction and percent operators; continuing common and decimal series; and using distance, time and speed relationship with whole, decimals, and common fractions values. Only one topic was fully mastered at all grade levels: ordering times expressed as common fractions hours and whole numbers of minutes. Lack of deveopmental effects were reprted for other topics in that domain as well (e.g., Lesh et al., 1983; Post, Behr, Lesh, and Wachsmuth, 1985).

The identified deficiencies in declarative and procedural knowledge regarding the two systems are as follows:

1. Ordering and comparison of decimals were performed on the basis of their standard notation with no reference to common fractions or to concrete models, and two opposite well documented (Nesher and Peled, 1985; Hoz and Gorodetsky, in press) incorrect procedures were used. (a) "The longer the part to the right of the decimal period the larger the number", and (b) "the longer the part to the right of the decimal period the smaller the number". Three incorrect procedures were used to order common fractions. (a) "The larger the denominator the smaller the fraction, but the size of the fraction is not influenced by the size of numerator." (b) "If in one fraction the complement to 1 is larger than that of a second fraction then the first fraction is larger than the second." This misunderstanding of the concept "complement" points to the lack of the part-whole schema with regard to the 1 and the difference between it and the given fraction. (c)

"One fraction is larger than another if its numerator and denominator are larger, respectively, from those of the other fraction." This rule resembles a combination of two common fraction comparison rules, each of which pertains to either the numerator or the denominator. This combination works in some but not all cases.

2. Misinterpretation of the decimal and common fraction notations and lack of ability to judge equivalence between these representations were revealed in the performance on several tasks. (i) A decimal fraction was characterized as a string of numbers with a decimal period or as a common fraction with a 10 exponent. (ii) The zeros on both sides of the period were ignored and only nonzero digits were considered. (iii) 0.8 and 0.10 were written as the next terms in the series 0.2, 0.4,... A whole numbers rule was employed only to the mantissa and the meaning of the period was distorted. (iv) Confusion arose regarding the direction in which to move the period and the number of places to increase or decrease decimals by 10 exponents. (It should be noted that the need to choose the operation was clearly a hinderance in the test problems since the students were not told whether to multiply or divide by the 10 exponent.) (v) The numerator and denominator of a common fraction considered separately when these were to be conceived of as one unit. Similar difficulties were also reported by Post et al. (1985). (vi) The decimal and common fraction notations could not be used to designate proportions. (vii) Translations were made between the decimal and common fraction systems employing several incorrect transformations (e.g., 4/12 is 4.12, 12.4, 4.8, or 8.4) apparently without resorting to any concrete model. Moreover, the translations were made without any attempt to check the obtained fraction (e.g., applying a known transformation procedure).

Knowledge of the two systems was rather poor, but that of the common fractions was mastered more fully than that of the decimal system, and the former system was clearly insufficiently learned to base the latter system on it.

The location of common fractions and decimals on the number line with only whole numbers indicated on it was done satisfactorily with numbers smaller than 1 but was very difficult with numbers larger than 1 (e.g., 1.75).

The only procedure used (regardless of its success) to transform a common fraction into a decimal was to extend the fraction so to make its denominator a 10 exponent and then to place the period in its position in numerator. The reverse transition was achieved by writing the mantissa as numerator and a 10 exponent as denominator, with only few students reducing the obtained common fraction. These procedures were employed very seldom so that conclusions regarding the understanding of the relationship among the systems could not be reached. The second test was designed to help determine additional deficiencies regarding the relationship between the systems and provide more precise descriptions of procedural bugs in the transition between them.

The results of this test replicate and strengthen those of the first one. (a) Development was observed in very few topics and deficiencies in conceptual knowledge and bugs in procedural knowledge persisted from grade 6 to 9. (b) About 65% of the students failed to acquire the meaning of common and decimal notations, to identify the common fraction names of decimal rationals and vice versa, and to transform into decimal rationals common fractions whose denominators were not 10 exponents. The only procedure used to translate a common fraction into a decimal one was to transform the denominator into a 10 exponent. Most failures occurred when this could not be achieved (e.g., 4/12). (c) About 80% of the stu[illegible] failed to locate poi[illegible]ts on the number line and to create the decimal and common fraction names of these points.

These results emphasize the necessity of having a[illegible]lable knowledge of each system in order to be able to more fully understand the other system, a fact that may explain why the instruction did not achieve its objectives regarding the four components of the conceptual model of rational numbers. Examination of the relevant Israeli textbooks points that (1) the majority of them devote relatively little time to the establishment of the system of decimal rationals, (2) on one hand they utilize the equivalence between the two systems but on the other hand deemphasize the meanings of common and decimal notations and their relationships, and (3) they enable students to rote learn procedures without first establishing the underlying conceptual knowledge.

These results may be important for mathematics educators who attempt to base the instruction of the system of decimal rationals on that of the common fractions or to draw parallels between these systems.

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LANGUAGE: ACTION: FRACTION
Reflections on Fractional Number Research Papers

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To be useful personal knowledge in mathematics must serve to organize mathematically oriented phenomena for its knower. For young persons such knowledge manifests itself in LANGUAGE which a person can validate in the world of intuition or everyday actions. Using concepts developed by Frye, language can be seen to be used at four levels. The first two uses heiroglyphic and metaphoric are immanent. Heiroglyphic language use calls one's attention to action in the presence of the action itself, while metaphoric language is "put for" action. While this latter use can occur without the concomnitant action, it is driven by the action. Metonymic language use is at a higher level reflecting language and the related thinking as analogs of action. Here thought is independent of action but can be referenced to it. The language is analytic or dematic in nature; it is neither dependent on action nor does it even presuppose existing related action.

As children build rational number knowledge what kinds of "things" are indicated by the language which they use? The fractional number research studies above, although very diverse, test for or show children (and the researchers) using language in a variety of ways. These reflections attempt to inter-relate these reports by attending to the role of the LANGUAGE used, the role of MATHEMATICAL ACTIONS, and the attendent fractional number knowledge.

Bergeron and Herscovics study the quantification of the part whole relationships considering a set of related action/language pertaining to unit fractions of a continuous whole. Although their research tasks are not school tasks, they relate to school learned language in that children's responses are based on pre-given partitions or results of partitions (pieces called fifths, sevenths, and quarters) which might well be named in a heiroglyphic or metaphoric sense by children. In general, Bergeron and Herscovics observe that after grade four children can in this context identify parts from a partitioned whole and reconstitute wholes from given parts. In the latter, it is interesting that the researchers give a part and then ask "how many?" parts make a whole, thus orienting the student to a learned counting relationship. The invariance items are done in quarters (seen in other research to be well known by children), but equipartition results are different from those seen elsewhere. Here again the researchers use language "quarters" for the piece in each trial case. Does this language provoke the children to make a response which goes beyond the perceptual?

The above results on unit fractions of a whole are not generally at variance with other studies (eg: Noelting) which suggest an early control of even more general notions of unit fractions. What is of interest is that the researchers attended to this notion through a set of related tasks which can be seen as related to the groupment notion of Therien. Although not given in the brief paper it would appear that the children's language/action exhibited a unit fraction groupment in the domain of part-whole (as opposed to a more dynamic unit notion).

The Figueras et al research also looked at the relationship of spatial perception and fraction knowing but with older children and more sophisticated tasks. Their students were not as uniformally successful as those in the Bergeron/Herscovics study. Some of the behaviors, for example in finding one third of a triangle, indicated that some students faced with these tasks illustrated Therien's Stage 1 behaviours of fixing on one of the two elements in the fractional task. Other results vividly show the role of language use. One task asked students to show of 5/20 of a starlike figure. If one metaphorically put 5/20 for 5 of 20 pieces this task would be difficult (as it is for many). If one thought 5/20 of a figure as a quantitative part (1/4) then the task was easy.

Several of the studies reflected on the fact that traditional instruction makes an early and probably unwarranted emphasis on symbolic manipulation and computation with common or decimal fractions. To the extent that this is true children are probably forced to treat these symbols as concrete objects in and of themselves and hence build knowledge based inappropriately on patterns in the symbols ("count decimal places"", "like denominators") or generate purely nominal knowledge.

The presence of largely nominal knowledge is vividly illustrated in the paper by Tomer, Wolf, and Hoz. With the exception of a half and a quarter (language which controls actions usually well known to children), their sample of 11-14 year olds exhibited many instances of a focus on decimal symbols as patterns in and of themselves. Unless these symbols are tied to a personal conceptual base prior to instructions on procedural manipulations, it appears that there are serious limitations in the useful fractional knowledge of these children and young adults.

Even when pictoral representations are used in fraction number instruction, the conseqences are not always positive. From Novillis Larson's study, it is evident that while measurement models would be useful particularily in work with fractions greater than one and equivalent fractions, students could handle the fractions on a ruler much better than they could perform with a number line measurement representation. It would appear that children could use fraction language somewhat accurately as metaphors for measures indicated on a ruler. The static number line seemed to present a conflict with whole number knowledge. If instruction is to use the number line as an intuitive model for fractions, attention must be paid in performing fractional actions in this context and tying language to them.

The paper by Hiebert and Wearne and that of Streefland provide evidence that programs of instruction based on the premise of tying language to significant mathematical object/actions are both possible and have the desired effect. The Hiebert and Wearne paper tests a carefully developed theory of symbolic knowledge building. In this model meaning of individual symbols is built on meaningful referents (in this case base 10 blocks used to illustrate some decimal fractions) Although the authors do not mention it in their theory it seems critical to note that the reference base and exercises allow decimal language to be used in a manner which illustrates the quotient relationship between the unit and a decimal fraction at least in a limited sense. Referring to the Therien theory the referent system used in the Hiebert/Wearne instruction admits the structure of a groupment at least up to the level of common denominators. In both studies reported by Hiebert and Wearne students used appropriate semantic processes after instruction. However, in the full classroom group, the level of transfer to related decomal tasks before direct instruction was low. Hiebert and Wearne suggest that learning in the more complex setting was less complete and hence less transferable. In terms of language use this might mean that the children's use of language in the second group was more immanent and thus tied to the particular objects in the particular setting. While for at least some students in the small group decimal language reflected their ideas about actions, that is language called an abstraction to attention, in the large group student language still referred directly to objects.

In the Streefland study of fraction monograph production, he makes explicit reference to a levelling in which language, image, and action form a base for fraction language development where mathematical fraction language first simply replaces everyday language. This is followed by levels of language and rule "play" where the language can be validated back in an object-world familiar to the child.

The Kheong paper presents a model of fractional knowledge based on symbolic computations with fractions. The resultant computer diagnostic system would appear effective in dealing with nominal fractional number knowledge. It would appear that the analysis is based on exhausting categorization of symbolc error and direct error correction. What it does not do is examine the meaning of the language used. Thus the remedial system takes no cognizance of the semantic processes underlying operations on fractional numbers.

Finally the paper by Hardiman presented some very interesting results on simple word problems with fractions. Two things stand out in the findings. If fractions can be interpreted in a who whole number problems. Secondly, language use surrounding the multiplicative aspects of fractional numbers is confounding both to the subjects and the basic thrust of the surface structure/deep structure research.

SUMMARY

What seems clear form the above cited studies is that a fraction curriculum in which symbols are not tied to meaningful object actions has inhibiting effects. While object image based instruction can lead to some systematic fraction knowledge, it is important to examine these environments to see if they actually can support a full dymnamic fractional number knowledge system. It would appear that the richer environments and systematic attention to growth in language development lead to robust fractional number knowledge. Therien reminds that such knowledge has a systematic structure itself, a chain of groupments. However, because in some settings fractional numbers can refer directly to quantities, the kinds of component groupments needed to explain actual child behavior may well be very different than those posed by Therien.

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PROCEEDINGS OF THE
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PME-XI

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Geometry in computer environments

SIMILARITY:
UNRAVELING A CONCEPTUAL KNOT WITH THE AID OF TECHNOLOGY

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Educational Technology Center, Harvard Graduate School of Education

Previous work has identified three areas of difficulty that students seem to have with the topic of similarity: proportional reasoning, dimensional growth relationships, and correspondences in right triangle similarity. A unit addressing these three difficulties was constructed for use with the GEOMETRIC SUPPOSER. Students were observed as they learned similarity with this unit. From these observations, clarification of the three part characterization of student difficulties will be sought. The use of technology, specifically the GEOMETRIC SUPPOSER, provides two benefits. First, it supports a pedagogy which seeks to directly attack the students difficulties in understanding similarity. Second, the lab setting allows researchers as well as teachers to directly examine student thought processes.

INTRODUCTION

Teachers in the Educational Technology Center's 1985-86 geometry study (reported at the eighth PME-NA) considered similarity the most difficult topic in the curriculum for their students to understand. From experiences in these classrooms, students' initial conceptions about similarity were characterized, as well as the difficulties they experienced in learning about the topic. Based on these insights and on a review of the pertinent literature, three areas of difficulty with the topic were identified. A unit of problems for use with the GEOMETRIC SUPPOSER was then designed. The unit asks students to explore

*The research was conducted at Education Development Center under a subcontract from the Educational Technology Center of the Harvard Graduate School of Education. It was supported by the Office of Educational Research and Improvement contract #400-83-0041.

constructions involving similarity and, where possible, tries to force a confrontation between their naive theories and contrary evidence.

THREE DIFFICULTIES IN LEARNING ABOUT SIMILARITY

The first and foremost difficulty that students experienced in the 1985-86 study involves a crucial part of the definition of similarity, ratio and proportion. For example, students seemed to think that by extending the sides of a triangle by equal lengths they would always get a similar triangle. This conception was very resistant to change and appears to be related to similar additive strategies exhibited by students on ratio tasks (Hart, 1984).

The literature on similarity also indicates that an understanding of the relationship of area growth in similar plane figures and the generalized problem of dimensional growth in higher dimensional similar objects is difficult for students to acquire (Friedlander et al., unpublished). Similarity is confounded with dimension; students are suprised to see that area does not grow in the same ratio as sides do. The majority of research on this difficulty focuses on the relationship between linear and area growth and was done with junior high school students.

A third difficulty is associated with the mean proportional relationships found in right triangles with an altitude drawn from the right angle vertex. The students experienced difficulties solving problems that demanded the use of these ratios and the correct identification of the correspondences and proportions among the segments involved. This difficulty seemed to have two major components beyond bookkeeping difficulties. First, in order to develop the correct proportions, students must be able to match the corresponding sides of the three triangles in the construction. Such a matching is difficult since one of the triangles must be mentally flipped (not just rotated) in order to make the correspondence. It is not clear that students understand that flipping two-dimensional figures does not change the figures. Second, in these proportions, the same segment plays different roles in three triangles. For example, in the mean proportion involving the altitude, the altitude is the short leg of one right triangle and the long leg of another, as well as the altitude of a third.

PURPOSE

The purpose of this paper is to further examine the typology of the three areas of difficulty in the learning of similarity. Preliminary results related to this purpose are reported below.

Specifically, there are three sets of questions which will be examined:

1. Is students' preference for additive versus multiplicative strategies to create similarity a replicable phenomenon? If so, does it seem to be a difficulty in working with ratios or is it a geometric difficulty?

2. At the end of their studies about similarity, can students recognize linear, area and volume growth relationships in solids? If so, how do they manage the conflict between the descriptions?

3. With right triangles, does the students' difficulty stem from the fact that the similar triangles are rotated or flipped? Alternatively, does the difficulty stem from tha fact that the same segment, the altitude, is both the small leg of one triangle and the large leg of another triangle as well as the altitude of a third?

METHODS

Intervention

The unit consisted of eight computer tasks that students did in the computer lab during class time. Students also learned about similarity in their ragular classroom. The unit was used by four geometry classes, two in one location and two in another. The classes in the first location had no prior experiance with the SUPPOSER; the classes in the second location had used the SUPPOSER from September until March prior to beginning the unit.

Data Collection

Obsarvers visited the experimental and comparison classes while the topic of similarity was taught. Students' computer assignments were collectad from the experimental classes. Final tests on similarity were collected from all classes.

The performance of students in these four experimental classes was compared with the performance of students in two comparison classes. The measures used were a pretest of fraction ability and ratio and proportion skills, and a posttest of ratio and proportion skills and understanding of similarity.

Four students from each of the experimental and comparison classes were also interviewed. The interviews focused on the three areas of difficulty outlined above.

The majority of the results reported below are from the data collected at the site where students did not have previous experience with the Supposer. The data from the other site have not yet been completely analyzed; they will be reported at the PME conference.

PRELIMINARY RESULTS

Additive versus Multiplicative Strategies

Early in their work last year with the SUPPOSER, a group of students in one of the classes developed the notion of "rescaled" triangles, triangles that have the same angles but are different sizes. The name derives from the "Scale Change" option that is present in the SUPPOSER. The students were convinced that in order to get the rescaled version from the smaller version, one should add a set amount to the length of each side. They stuck to this opinion in the face of counter evidence.

This phenomenon was not recreated in any of the experimental classes this year. When asked to explain how one could create a rescaled version of a triangle, students used proportions in their explanations. However, one of the computer tasks in the unit brought to light evidence that suggests that the students preference for multiplicative strategies was not strongly rooted.

In this task, students were asked to create a triangle similar to a given triangle, by extending two of its sides. In two of the four experimental classes, students were first given the option of choosing additive or multiplicative strategies. Their choices were very different in the two classes. In one class, where this task was given the same day that students investigated figures whose sides were proportionate, eight out of eleven working groups chose a multiplicative strategy. In the other class, where a weekend intervened between the Side-Side-Side activity and the task of extending the two sides, eight out of eleven groups tried additive strategies. The two groups had scored similarly on the ratio and proportion pretest. The "multipliers" had 5 out of 16 students at Hart's fourth level of ratio understanding; 12 students used additive strategies on her test. The "adders" had 3 out of 16 students at level four; 12 students used additive strategies in the ratio and proportion pretest.

On the next day, the teacher followed up the two-option task with a task that did not allow choice. In this version, a student is asked to investigate an additive strategy and then a multiplicative strategy. The class that previously had used multiplicative strategies now expressed surprise that an additive strategy would not work. It took the students half of the period to connect this phenomenon to their previous assignment. In the "additive" class, the students immediately explained that the activity was the same as the activity that they had done earlier.

In the other two experimental classes, students had done much better on Hart's test of ratio understanding. In one class 14 out of 23 scored at level 4 with 6 using additive strategies, while in the other class 15 out of 22 scored at that level with only 2 using additive strategies. In these classes, students were not given choices; they were told to first construct additively and then multiplicatively. In both classes, many students didn't bother measuring to check their conjectures. They were positive that an additive strategy would yield similar figures. Only when they tried a multiplicative strategy on the same figure did they feel the need to go back and make measurements. They expressed much consternation when they found that extending by equal lengths did not necessarily yield parallel lines. Thus, even those students who show an understanding of ratio, may not exhibit that understanding in certain geometric contexts.

Side, Area and Volume Growth Relationships in Similar Solids

At the end of the similarity unit intervention, we interviewed twelve students at one of the sites--four from each of the two experimental classes and two from each comparison class. One of the tasks on the interviews involved two similar rectangular solids. The dimensions of one of the solids were twice the dimensions of the other. Students were asked to describe how much larger the large solid was than the small solid.

No student responded with an additive strategy. All twelve students interviewed responded with at least one multiplicative strategy (either comparison of sides or of volumes), while five students suggested exactly two strategies (usually sides and volume). Four students also noted a third possibility (comparison of areas).

Onl. .e of the students who suggested more than one strategy expressed surprise. This student thought that the area relationship should be the same as the sides relationship. He counted and then revised his opinion to 2:1 for sides and 4:1 for area.

When asked to explain the possibility of different answers or when asked to choose between two answers they had given, eight of nine students responded that both or all three descriptions were true. Only one of the students was disconcerted by the fact that two different descriptions seemed to hold. He settled on the volume relationship as truer than the relationship between sides. The more articulate students suggested that "the answer depends on the question" or that "the quantities being measured are different."

Only two of the students explicitly mentioned that the solids were similar. Only one student had an explanation for why the volume relationship is 8:1. She explained that since it is a cube, one cubes the ratio of the sides. No student was moved to make any generalization about dimensions.

Corresponding Ratios in Right Triangles

During these same interviews, two of the other tasks were structured to test two hypotheses about students' difficulties with proportions in right triangles. First, we tested the notion that students do not identify the similarity of the triangles because of the flipping involved. This was accomplished as part of a task that asked students to identify similar shapes. Some of the figures needed to be flipped in order to recognize that they were indeed similar. Nine out of the twelve students had no difficulties performing the necessary flips and recognizing the similar figures. Three students missed one or two of a total of five necessary flips.

On the second task, students were given 6 metal pieces (two of equal size) that were cut into lengths that could be joined end to end to make two similar right triangles. These two triangles can then be joined to make a third similar right triangle. Students were first given the six pieces and asked to create two similar right triangles. After they had done so, one of the two equal-sized pieces was removed and students were asked to create two or three similar right triangles with the remaining five pieces by connecting them end to end.

The students' performance was very hard to analyze, but in general students did not seem to have a problem with the notion that one metal piece would have to function as a member of two or three triangles.

PRELIMINARY DISCUSSION

The phenomenon of using additive strategies to construct similar triangles was replicated. Whether this is a difficulty with ratio reasoning or with geometric reasoning is a difficult question. However, it does seem that the context of extending two sides of a triangle to make a similar triangle encourages additive understandings of proportions even among those students who have already come to a multiplicative understanding of proportions.

Students in general were able to recognize the different relationships, area being the most elusive. They also tended not to experience conflict between the different descriptions, although they were not able to explicitly coordinate the descriptions. They did not generalize the relationships between sides, areas and volumes.

The evidence from the interviews does not strongly support either of the two hypotheses. Few students had difficulties flipping geometric shapes. The students did not see this as an inappropriate action. Students also did not have difficulties with the notion that one rod could be a part of two or three triangles. No one was stymied by the task of making similar right triangles with the five remaining pieces.

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THÉORÈME DE THALÈS ET MICRO-ORDINATEUR

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ABSTRACT. — The theorem of Thales enables mathematicians to calculate large lengths by means of computations on small lenghts and realises in this way a change of space. The use of the theorem of Thales changes a geometric approach of a problem into a numeric one where proportionality corresponds to parallelism.

The object of this research is to investigate the difficulties for the pupils when using the theorem of Thales in the solution of a problem. It was carried out in two environments :

– a computer environment
– a paper-pencil environment.

The modification of the pupil's strategies due to the computer is also investigated.

Introduction. — Le théorème de Thalès est une notion charnière des mathématiques et de leur enseignement en France en fin de collège (14-15 ans). Il permet un passage du cadre géométrique au cadre numérique où le parallélisme est mis en relation avec la proportionnalité [DOUADY 1985]. Les difficultés à son sujet sont de différents types. Elles sont cependant beaucoup plus importantes quand le résultat demandé est non pas numérique, cas pour lequel le changement de cadre apparaît d'emblée nécessaire, mais une propriété géométrique à démontrer.

Problématique du théorème de Thalès. — Historiquement à l'époque de Thalès (mesure des pyramides) actuellement encore pour des problèmes concrets (par exemple le cubage d'un sapin sur pied) le théorème de Thalès (*) permet de calculer une longueur que l'on ne peut pas mesurer directement à cause de sa taille importante et de difficultés techniques à partir d'autres longueurs.

On peut distinguer trois espaces dans lesquels les problèmes ne se posent pas de la même façon parce qu'ils ne mettent pas en jeu les mêmes possibilités de contrôle [BROUSSEAU 1983]. Ce sont :

- le micro-espace : l'espace des objets qu'on peut déplacer sur une table;
- le meso-espace : entre 0.5 et 50 fois la taille du sujet;
- le macro-espace qui met en jeu des problèmes de repérage et d'orientation [LABORDE 1986].

Thalès permet, pour les mesures de longueurs, un passage de l'un quelconque à l'autre de ces trois espaces.

Quand on cherche à conférer une structure d'espace vectoriel à R^2 ou R^3, on utilise Thalès pour définir la multiplication par un scalaire dans l'ensemble des classes d'équivalence de bipoints du plan ou de l'espace.

Thalès, aussi bien historiquement, que dans sa présentation dans l'enseignement ou son emploi dans des problèmes apparaît donc comme un outil [DOUADY 1985]. En particulier Thalès permet de transformer une classe de problèmes où intervient le parallélisme dans un cadre géométrique en une classe de problèmes où intervient la proportionnalité dans un cadre numérique.

(*) à partir de maintenant nous dirons Thalès pour le théorème de Thalès

Objectif de l'étude. — Nous avons choisi d'étudier les difficultés pour les élèves concernant les problèmes de la classe de troisième de collège (14-15 ans) nécessitant l'utilisation de Thalès ou de sa réciproque. Dans un tel problème se produisent différentes phases :

1. — décision d'utiliser Thalès ou sa réciproque comme élément de la solution;
2. — reconnaissance des configurations où le théorème a lieu d'être appliqué;
3. — détermination des rapports utilisés que j'appellerai choix de la forme de Thalès ou de sa réciproque;
4. — maniement des rapports;
5. — calcul algébrique.

Choix des variables des problèmes posés aux élèves. — Nous appellerons variables les éléments du problème dont un changement de valeur entraîne un changement de stratégie des élèves. L'étude des différentes phases de résolution de problème et des observations faites avant l'expérimentation nous permettent d'établir la liste des variables concernant le texte de la façon suivante :

La première variable concerne *le type de réponse demandé*. Soit l'élève doit réaliser un calcul (*calcul d'une longueur, d'un rapport de mesures algébriques*) soit l'élève doit réaliser une démonstration de propriétés géométriques.

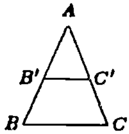
La seconde variable est celle de *configuration*. Elle se décompose en une série de sous-variables qui sont les suivantes :

* *présence ou non de figures reconnues* et nombre de ces figures. Celles-ci peuvent être des parallélogrammes, des trapèzes, des triangles;

* *utilisation de Thalès ou de sa réciproque dans le triangle*. Il y a alors deux cas particuliers qui sont les suivants :

1) le théorème employé quand on applique Thalès est en fait le théorème du milieu : dans un triangle une parallèle à un côté passant par le milieu d'un autre côté passe par le milieu du troisième côté;

2) un des rapports utilisés est le rapport d'un côté sur la longueur du segment découpé sur une parallèle par les deux autres côtés.



(utilisation du rapport $\frac{\overline{B'C'}}{\overline{BC}}$
$\frac{\overline{B'C'}}{\overline{BC}} = \frac{\overline{AB'}}{\overline{AB}}$)

On dira alors que l'on utilise la deuxième forme de Thalès dans le triangle.

* *direction des parallèles.*
* *nombre de segments* composant la figure.

La variable suivante concerne *l'utilisation de Thalès*. Elle se décompose de la façon suivante

* nombre de fois où Thalès ou sa réciproque est appliqué;
* nombre de différentes formes de Thalès ou de sa réciproque utilisées;
* sous-variable directe ou réciproque suivant que le théorème appliqué est Thalès ou sa réciproque.

Nous avons une *variable mesure algébrique* et une *variable calcul algébrique* prenant la valeur vraie si un calcul autre que la simple écriture des rapports doit être fait.

Enfin, nous définissons la *variable changement d'espace* qui prend la valeur vraie si Thalès permet le passage d'un macro-espace à un meso-espace ou d'un meso-espace à un micro-espace.

On pourrait penser qu'il faut définir une variable vecteur qui prendrait la valeur vraie si des vecteurs figuraient dans l'énoncé. Nous ne le ferons pas car alors ce n'est pas Thalès qui est utilisé mais la propriété de l'espace qui est de posséder une structure d'espace affine associé à un espace vectoriel.

Il nous est apparu ensuite que les modalités de la situation de résolution de problème avaient un rôle important à jouer.

Hypothèse de travail et choix d'un dispositif. — Le travail qui suit est fondé sur l'hypothèse qu'une des difficultés les plus importantes dans la mise en œuvre de Thalès est la décomposition de la figure en groupes d'éléments où il peut être appliqué. Habituellement, pour résoudre un problème de géométrie, les élèves sont placés dans des conditions dites *papier crayon*. Nous avons cherché à voir si l'introduction d'un logiciel de tracé avec certaines facilités de modification de la figure que ne présentent pas les conditions *papier-crayon* faciliterait la résolution des phases un, deux et trois.

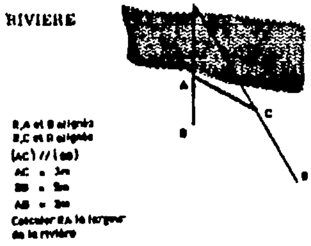
Le logiciel choisi est Mac Draw sur Macintosh. Les actions de tracés indiquées aux élèves pour modifier éventuellement la figure sont :

1 – le tracé d'un segment d'un point à un autre
2 – la translation d'un segment
3 – le changement d'épaisseur des traits d'un segment
4 – la suppression d'un segment
5 – la mise en place d'un quadrillage.

Les actions de tracé 2 et 3 permettent d'isoler rapidement des groupes d'éléments de figure et de la rendre ainsi dynamique. Nous définissons des variables de modalités qui sont les suivantes :

- la variable de dispositif qui prendra les valeurs suivantes :
  - Macintosh avec Mac Draw
  - *papier-crayon* avec règle, compas, équerre, rapporteur.
- la variable de tracé de la figure.

Expérimentation. — Un choix de neuf énoncés présentant un échantillonnage des valeurs des variables a été fait. Les énoncés ont été couplés par 2. Un énoncé a été pris deux fois, parcequ'il semble présenter le plus de difficultés pour les élèves. On a associé quand cela était possible des énoncés présentant des types de réponses différentes et sinon des formes différentes de Thalès. Nous donnons ici un exemple d'énoncé :

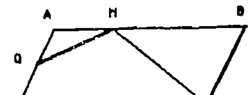


Valeur des variables. Le type de réponse demandé est numérique. La configuration présente 2 triangles. La figure actuelle compte 7 segments. Les parallèles sont obliques. Il y a une seule utilisation du théorème de Thalès sous sa deuxième forme dans le triangle. Les variables mesure algébrique et calcul algébrique prennent la valeur vraie. La figure présente des segments manquants et les traits sont d'épaisseurs différentes

Les élèves travaillaient par deux pendant une heure. La consigne était de rédiger par écrit une solution en utilisant uniquement, pour les élèves travaillant sur Macintosh, la figure de l'écran. Celle-ci était susceptible d'être modifiée grâce aux actions de tracé indiquées précédemment. L'expérimentation a eu lieu dans une classe de troisième technique d'un collège de la banlieue grenobloise (19 élèves de 15 à 17 ans) et dans une classe de troisième clasique du même collège (24 élèves de 14-15 ans). 10 paires d'élèves provenant de différentes classes de troisième classique d'autres collèges de l'agglomération grenobloise ont également participé aux expérimentations. Par suite, chaque groupe de deux textes a été donné (dans ce type de conditions) à 6 paires d'élèves différentes. Un seul texte a été cherché douze fois. Une seule élève, provenant de la troisième technique, a travaillé individuellement. Chaque problème a été réalisé également par deux paires d'élèves de 3ème classique travaillant dans les conditions *papier-crayon* .

Nbre d'élèves / Classe	total	ayant abordé 1 problème	ayant abordé 2 problèmes	ayant abordé 3 problèmes
3ème technique	19	4	12	3
3ème classique	24	4	20	
élèves d'autres 3ème classiques	20	6	14	
élèves dans les conditions papier-crayon	20	8	12	

Rôle du dispositif dans le processus de résolution. —



Soit un parallélogramme (ABCD),
M un point de (AB).
On mène par M la parallèle à (BD),
elle coupe (AD) en Q.
On mène par M la parallèle à (AC),
elle coupe (BC) en N.
On mène par N la parallèle à (BD),
elle coupe (DC) en P
Que peut-on dire du quadrilatère MNPQ

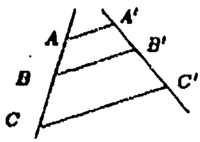
La présence de traits d'épaisseurs différentes apparaît essentielle chez des élèves ayant reconnu que la résolution du problème nécessitait l'utilisation de Thalès. Pour les élèves travaillant dans le type de condition Macintosh avec Mac Draw pour certaines figures comme PARALLELOGRAMME.B où Thalès ou sa réciproque sont employés un grand nombre de fois, c'est la décomposition de la figure réalisée en changeant l'épaisseur de certains traits qui est systématiquement utilisée. Pour d'autres figures on voit également des mises en évidence de configurations obtenues par la translation des segments inutiles.

Les élèves commencent systématiquement par lire le texte en repérant les tracés sur l'écran quand ils y sont déjà en ajoutant éventuellement les segments manquants. Les segments superflus ne sont jamais effacés. Cette tâche remplace celle de tracé de la figure dans les conditions *papier-crayon* et semble jouer un rôle analogue. Ensuite, ils recherchent une stratégie globale. La décision d'appliquer Thalès ou d'utiliser des projections est déclenchée par la vision de parallèles. Alors seulement les élèves isolent un groupe d'éléments de la figure. Par rapport aux conditions *papier-crayon* les élèves passent alors plus facilement en revue les différentes formes de Thalès ou de sa réciproque pour décider laquelle utiliser et font ensuite sans difficulté les autres décompositions éventuelles de la figure pour les applications successives de Thalès ou de sa réciproque.

L'effet des variables concernant le nombre de fois où Thalès est appliqué et le nombre de formes de Thalès est atténué par rapport à ce qui se passe dans les conditions *papier-crayon*. Comme Mac Draw facilite en pratique la décomposition des figures, les procédures de résolution comportant plusieurs emplois différents éventuellement de Thalès ne sont pas mises à l'écart comme dans les conditions *papier-crayon*. La difficulté provenant de l'intersection des sécantes entre les parallèles est réduite par la possibilité avec Mac Draw de translater une sécante. Par contre la difficulté concernant l'utilisation de parallèles horizontales subsiste. Cette difficulté est en rapport avec l'absence de présentation de ce type de figure dans les manuels.

Rôle des variables dans le processus de résolution. —

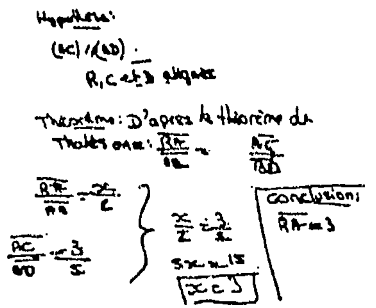
Variable concernant la forme de Thalès. — Nous constatons dans tous les cas un enfermement dans une forme de Thalès qui est la suivante :



rapports utilisés $\frac{\overline{AB}}{\overline{BC}} = \frac{\overline{A'B'}}{\overline{B'C'}}$

La forme de Thalès qui présente la plus grande difficulté est la deuxième forme de Thalès dans le triangle

Solution d'une paire d'élèves pour RIVIERE

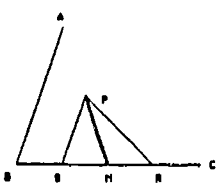


Nous constatons dans cette production d'élève l'introduction du rapport $\frac{RA}{AB}$ qui correspond à la forme usuelle d'utilisation de Thalès avec une tentative d'utilisation de sa deuxième forme dans le triangle qui devrait donner $\frac{AC}{BD} = \frac{RA}{RB}$.

C'est dans la classe de troisième technologique que l'utilisation de la deuxième forme de Thalès dans le triangle s'est produite avec le temps de recherche le plus faible et le moins d'erreurs de rapport.

Théorème de Thalès et théorème du milieu. —

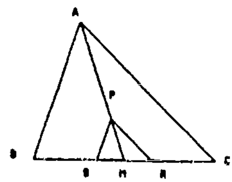
TRIANGLE.B



Soit un triangle (ABC).
M le milieu de [BC].
P le milieu de [AM].
On trace la parallèle à (AB) passant par P
Elle coupe (BC) en S
On trace la parallèle à (AC) passant par P
Elle coupe (BC) en R.

Montrez que R est le symétrique de S par rapport à M.

TRIANGLE.C



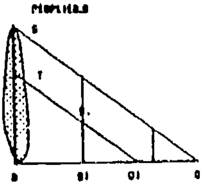
Soit un triangle (ABC)
M le milieu de [BC].
P un point de (AM)
On trace la parallèle à (AB) passant par P
Elle coupe (BC) en S
On trace la parallèle à (AC) passant par P
Elle coupe (BC) en R

Montrez que R est le symétrique de S par rapport à M.

Les deux problèmes précédents ne diffèrent que par le fait que dans TRIANGLE.B, P soit le milieu de AM. Cette propriété qui n'est pas nécessaire dans les hypothèses pour résoudre le problème, permet de remplacer dans la résolution du problème le théorème de Thalès par le théorème du milieu. Ce dernier théorème a été appris l'année précédente en classe de quatrième de collège. Il évite par ailleurs, d'utiliser les rapports de mesures algébriques. Un peu plus de la moitié seulement des élèves appliquent le théorème du milieu pour TRIANGLE.B. Le reste des élèves, ce qui ne constitue pas une proportion négligeable, applique le théorème de Thalès deux fois en écrivant $\frac{\overline{SM}}{\overline{BM}} = \frac{\overline{PM}}{\overline{AM}} = \frac{\overline{RM}}{\overline{CM}}$.

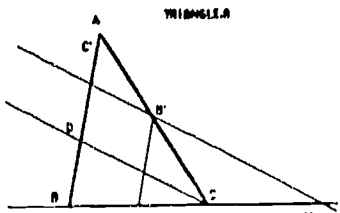
Variable mesure algébrique et variable calcul algébrique. — Nous remarquons que plus de la moitié des élèves remplacent les mesures algébriques par des longueurs. Pour les élèves ayant conservés les mesures algébriques nous voyons fréquemment dans la résolution de TRIANGLE.C : M milieu de $[BC]$ entraîne $\overline{MB} = \overline{MC}$. Pour PARALLELOGRAMME.B 2 paires d'élèves ont utilisé des sommes au lieu de rapports et ont écrit ainsi la solution : en appliquant Thalès $\overline{AQ} + \overline{AD} = \overline{AM} + \overline{AB} = \overline{CN} + \overline{CB} = \overline{CP} + \overline{CD}$; en appliquant la réciproque de Thalès on obtient (QP) parallèle à (AC).

Variable type de réponse et variable dimension. —



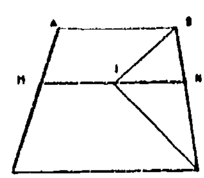
Soit un peuplier de base B de sommet S
Soit BO l'ombre du peuplier
On place un bâton de base B1 de sommet S1 à côté de l'ombre du peuplier
Soit B1O1 son ombre
On mesure BO, B1O1, B1S1
Calculez la hauteur du peuplier
(On suppose que les rayons du soleil arrivent parallèles)

TRIANGLE.A



Soit un triangle ABC
Une droite Δ coupe (BC),(CA) et (AB) respectivement en A',B' et C'
La parallèle à Δ passant par C coupe (AB) en D.
Montrez que

$$\frac{\overline{A'B}}{\overline{A'C}} \times \frac{\overline{B'C}}{\overline{B'A}} \times \frac{\overline{C'A}}{\overline{C'B}} = 1$$



Soit un trapèze ABCD
Les diagonales (AC) et (BD) se coupent en I
Par I on trace la parallèle à (AB).
Elle coupe (AD) en M et (BC) en N.

Montrez que I est le milieu de [MN].

Dans les deux types de condition Macintosh avec Mac Draw et *papier-crayon* ces variables paraissent fondamentales. Les élèves ont une réticence à réaliser un passage du cadre géométrique au cadre numérique quand le type de la réponse comme les hypothèses appartiennent au cadre géométrique ce qui nécessite un double passage. La présence de changement d'espace semble une condition favorisante pour la décision d'emploi de Thalès.

Dans le type de condition Macintosh avec Mac Draw, pour PEUPLIER.A et TRIANGLE.A les élèves prennent rapidement la décision d'utiliser Thalès et recherchent les formes d'utilisation. Par contre TRAPEZE.A est la figure qui a été placée deux fois dans les énoncés parce qu'elle semble présenter le plus de difficultés pour les élèves. Pourtant PEUPLIER.A et TRIANGLE.A sont les problèmes dont les résolutions comportent le plus grand nombre d'utilisation de Thalès sous différentes formes et dont la figure comporte le plus grand nombre de segments et de configurations possibles dans lesquelles Thalès peut être appliqué. Nous remarquerons que les élèves progressent plus rapidement dans la résolution de PEUPLIER.A qui présente en cas de changement d'espace que dans celle de TRIANGLE.A.

La présence de parallèles associée à la demande d'une mesure de longueur semble être un critère pour la décision d'utiliser Thalès. Son efficacité est renforcée par la présence d'un changement d'espace.

Conclusion. — Le théorème de Thalès apparaît comme une notion possédant un vaste champ conceptuel. A l'élève en situation de résolution de problème cela impose d'une part

- de disposer de critères portant sur la figure et le texte permettant de déterminer si le théorème de Thalès ou sa réciproque ont lieu d'être appliqués et sous quelle forme;
- d'autre part de savoir mettre en œuvre des méthodes de résolution de problèmes, algébriques et géométriques.

L'utilisation du logiciel Mac Draw sur Macintosh permet de favoriser les procédures comportant différentes décompositions de la figure. Cela a permis une plus grande facilité de résolution quand Thalès ou sa réciproque devait être employé plusieurs fois ou quand Thalès n'apparaissait pas sous sa forme usuelle. Les difficultés restantes sont celles concernant la direction des parallèles, l'utilisation de la deuxième forme de Thalès dans le triangle et les difficultés liées au calcul algébrique.

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(23 mars 1987)

SEEING WHAT MATTERS: DEVELOPING AN UNDERSTANDING OF THE CONCEPT OF PARALLELOGRAM THROUGH A LOGO MICROWORLD

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Abstract: We report the results of a study in which pupils engaged in mathematical activity through interaction in a Logo microworld based on the concept of a parallelogram. The objective was to identify ways in which the pupils progressively became aware of and generalised the embedded relationships within parallelograms. The data was analysed from the perspective of a general model for learning mathematics within functional and meaningful situations. The analysis of the data has provided insight into the ways in which a structured Logo environment can provide a context in which concepts can be first used and later understood, based on the interaction between symbolic and visual modes of thinking, the partial layers of discrimination which are constructed, and the way in which the computer acts as cognitive scaffolding for the learner.

Our starting point is that pupils should learn mathematics by engaging in functional mathematical activity. We suggest that the major difficulties which confront school mathematics are :

i. the separation of mathematics from any sort of meaningful activity, and

ii. the separation of pupils' conceptions from their formalisation (for a further elaboration of these points see Hoyles 1985; Noss, in press).

We aim to construct learning environments in which mathematical ideas and operations are applied as tools. In such circumstances the learner's attention is focussed on the use or outcome, and it is evident that she may not be aware in any explicit sense of the mathematical concepts and relationships embedded in the activity. The problem therefore is to raise these implicit mathematical structures to conscious awareness.

With this background in mind, a model for learning mathematics has been proposed which involves the dynamically related components of using, discriminating, generalising and synthesising (abbreviated to UDGS). The components of the model (see Hoyles 1986) are as follows:

Using: where a concept is used as a tool for functional purposes to achieve particular goals;

Discriminating: where the different parts of the structure of a concept used as a tool are progressively made explicit;

<u>Generalising</u>: where the range of application of the concept used as a tool is consciously extended from a particular to a more general case;

<u>Synthesising</u>: where the range of application of the concept used as a tool is consciously integrated with other contexts of application -- that is, where multiple representations of the same knowledge in different symbolic forms derived from different domains, are reformulated into an integral whole.

A fundamental criterion of the UDGS model's applicability lies in the extent to which exploration and experimentation are provided for within the learning situation. It therefore fits rather well in the context of interactive computer environments, at least those in which the learner is both engaged in the construction of executable symbolic representations and is provided with informative feedback. We would claim that computer-based environments of this kind can provide a special learning situation which can aid in the restructuring of the pupil's knowledge from its initial basis within 'theorems in action' (Vergnaud 1982), to more abstract cognitive structures. Our image of the computer is as an intellectual resource for provoking the child to use mathematical ideas, explore situations, and to pose and solve problems. Viewed in this light, the computer may be seen to act as a collaborator which can stimulate changes in the representation of a problem and thus make possible their solution. This perspective owes much to Vygotsky who acknowledged the discrepancy between solitary and collaborative problem solving. Vygotsky argued that cognitive development occurs as the inter-psychological processes found in social interactions become internalised as intra-psychological functions. Here we investigate whether this process might occur during interaction with the computer and how the ontogenesis of individual representations might be a function of this interaction.

OBJECTIVES

We set out to provide a small number of children with a Logo-based parallelogram microworld consisting of a structured set of tasks, some of which were to be attempted off the computer and some on the computer. Our hope was that these activies would enable us to gain insight into the development of the children's understanding of the 'essence' of a parallelogram, and in particular how the computer served as a catalyst in this development -- by affecting both the way the concept was represented and the range of methods of task solution available. In terms of the theoretical framework of the UDGS model, we planned to investigate the cycle in which children:

i. used a given Logo program for a parallelogram and learned to see its outcome as a totality;

ii. progressively discriminated the function of its component parts;

iii. separated these component parts into cognitive units: that is perceived the semantic meaning of the units and connected these with visual outcomes;

iv. generalised both the program and their understanding of the mathematical concepts embedded within it;

v. synthesised conceptions developed in the Logo microworld with those developed in other contexts.

In this paper, we report findings only relating to the first three issues. (for a fuller discussion see Hoyles and Noss, in press). As well as describing the pupils' mathematical activity within each particular phase of the cycle, we also hoped to gain insight into how transitions came about between the components of the UDGS model; that is, how and why change occurred.

METHODOLOGY

We have been working with a group of seven children aged between 13 and 14 years with considerable Logo experience (around 120 hours over four years). During the year 1985-6, we collected together a series of case studies of their Logo activities within their mathematics classroom, which provided the background for the formulation of the set of tasks which are the basis of this paper. These tasks were undertaken during a two-hour session in our computing laboratory. Data was obtained by reference to the dribble files (which record all pupil-computer interactions automatically) of the pupils work, the researchers' notes (both researchers were present during the activity but no interventions were made by them), and the written work of the pupils. This background case study material was also used to aid in the interpretation of the data.

Pupils were given the following procedure:

```
TO SHAPE :SIDE1 :SIDE2
FD :SIDE1  RT 40
FD :SIDE2  RT 140
FD :SIDE1  RT 40
FD :SIDE2  RT 140
END
```

The pupils were initially encouraged to play with the procedure by using it in the construction of a tiling pattern. This activity was intended to reveal pupils' initial conceptions. By constraints built into subsequent tasks, it was planned that the pupils' attention would be progressively drawn to the meanings and implicit relationships in the procedure's code which were necessary for the output to be a parallelogram i.e.:

a) that the inputs to the procedure (SIDE1 and SIDE2) represented the lengths of the two sides of the figure.

b) that SIDE1 and SIDE2 were each called twice and alternately;

c) that the inputs to RT represented the turtle turn between the drawing of the sides;

d) that the sum of the two turtle turns (if both in the same direction) between adjacent sides, was a constant which was equal to 180.

We were interested in investigating whether pupils were able to identify the parts of the program that might be changed to produce a given figure and the laws determining 'correct' changes. Specifically we wanted to know whether by finding and using pairs of turns that 'worked' (that is produced a required parallelogram) pupils were able to abstract the relationship between their sizes, and whether they were provoked to make this relationship explicit in a Logo program (i.e. articulate it in symbolic form). We also wished to explore how the pupils made links between the Logo program and its graphical output; that is between the values of SIDE1 and SIDE2 and the inputs to RT in one mode of representation, and the lengths of the sides of the parallelogram and the angles between them in the other.

RESULTS

Competing frames: Logo versus pencil-and-paper mathematics

The results illustrate the ways in which the pupils switched, or failed to switch, between two competing frames -- namely Logo and pencil & paper 'maths'. This confusion was illustrated in the pupils' responses to the first question, where they were asked to predict on paper what SHAPE 150 200 would produce, and label the sizes of the sides and angles of their figure. Four of the seven children used what we have termed 'Logo labels' for angles as illustrated in Figure 1.

Figure 1
An example of a 'Logo label'

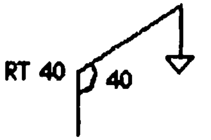

Figure 2
Noel's 'combined labels

This type of labelling indicated that the children were identifying with the turtle and thinking about turtle turn. They were not however, making any attempt to relate this notion of turtle turn to the angles of their drawing. Two of the remaining children (probably the most able of the group) labelled the interior angle of their drawings as 40 and 140 respectively. The remaining pupil, Noel, did both -- that is, he labelled 40 as the interior angle of his parallelogram and used a Logo label (see Figure 2).

Using procedures to generate implicit generalisations

Our results show how pupils come to use mathematical relationships quite spontaneously in order to fulfill the requirements of a task; how these relationships are at first merely routines and implicit parts of the activity, but subsequently can form the basis of more conscious reflection. For example, in answer to a question which invited the construction of a variety of parallelograms, Nicola worked in direct mode. She found, by trial and error, three pairs of turtle turns that could be used to produce the shapes -- 90 90, 45 135, and 70 110 -- and she was able to generate more pairs by adding or subtracting equal amounts to each input. Nicola's direct drive solution allowed her to succeed, at least at a product-oriented level. It seems that building up a figure in action in this way maintains a close relationship between the figure and the symbolic code, but does not guarantee an awareness of the relationships at a conscious level (in this case that RT :A .... RT 180 - :A). It is our feeling that such an awareness will not necessarily happen without intervention although the possibility of the pupil developing fragmented pieces of knowledge which can be subsequently used cannot be ruled out. We should recognise that even this relatively unsophisticated 'add and subtract' strategy does imply a feeling for invariance; it captures the idea of a theorem if not the theorem itself (after all, the fact that the sum is actually 180 is only a matter of convention!).

Discrimination is not 'all or nothing'

Our analysis of the pupils' responses within the parallelogram microworld leads us to suggest that there are various layers in the discrimination process. For example, it had seemed evident from Mathew's early responses that he 'knew' that the first two turns to form a parallelogram must add up to 180; that is he had indentified the meaning of parts of the procedure and used a quantitative relationship between these parts. However, his later work indicated that he had not discriminated the limits of applicability of this invariant sum relationship, in that he applied his rule even when the first turn merely served to orient the parallelogram correctly. While concentrating solely on the symbolic representation he inappropriately applied his relationship without thinking through its meaning in terms of its visual output.

This and other examples lead us to conclude that there are three identifiable stages of discrimination in which the following are present:

i. discrimination of the features of the figure without regard to its available symbolic represention(s);

ii. discrimination within the symbolic representation; that is perceiving its structure and pattern without regard to the visual outcome.

iii. discriminations which can be transferred between contexts, and relationships isolated when represented in different forms. This would constitute a first step towards a more explicit synthesis.

Using the computer as scaffolding

Our data indicates that the symbolic representation of a computer program can act as a form of scaffolding which allows the learner to sketch out the whole problem as she sees it, and then attend to the elements of the concept on which work still needs to be done. Viewed in this light, the issue of scaffolding is intimately bound up with the idea of the synthesis between visual and symbolic modes. As an example, Nicola wanted to make a more general procedure to use for her tiling than SHAPE. She wrote the procedure SHAPE1 (i.e. TO SHAPE1 :SIDE1 :SIDE2 :MOVE), introducing one new input (MOVE) for all the turns so that all her turns were RT :MOVE. This procedure showed discrimination of the 'opposite lengths are equal' rule. A casual observer might assume that Nicola believed that all the turtle turns to form a parallelogram must be equal. However she immediately edited her procedure (without running it) to add an extra input MOVE1, which was then used as a distinct input to the appropriate (second and fourth) RT commands. Our interpretation of Nicola's work is that initially she did not want to focus on the turns but on the overall program structure; she simply wanted to put down a marker that they should vary. The computer allowed her to do this and then to return and 'correct the details'.

CONCLUSION

Our analysis has provided a key for understanding the processes by which children make their way around the UDGS model while working in a Logo environment; that is the extent to which the pupil, through interaction in a Logo microworld, is able:

i. to synthesise the symbolic descriptions in terms of programs (or fragments of programs) with the geometric image on paper or on the screen, and

ii. to use the computer as scaffolding for the construction of generalisations.

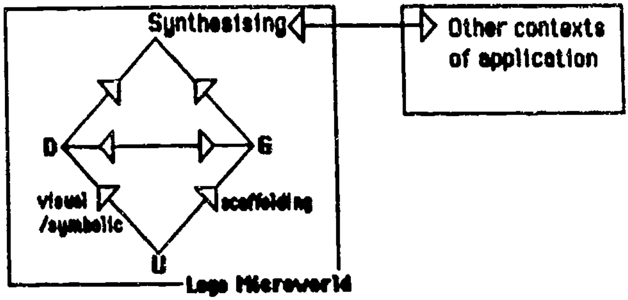

Figure 3. The UDGS model

From this perspective, the model can be characterised diagramatically as in Figure 3. We would maintain that the need for formalisation implied by the programming environment underpins progression from both U to D and from U to G. More specifically, discrimination involves a synthesis between the geometric and symbolic representations of some part of the concept facilitated by the Logo environment, while generalisation is aided by the scaffolding role of the computer in the way described above.

The study indicates that there is considerably more that we need to learn about the ways in which the presence of the computer can both influence children's mathematical conceptions, and provide a context in which concepts can be first used and later understood. We think that we have begun to isolate some of the components of these transitions: the interaction between symbolic and visual modes of thinking, the partial layers of discrimination which are constructed, and way in which the computer acts as cognitive scaffolding for the learner. Before we can synthesise these into a coherent theory, more attention will need to be focussed on children's existing conceptions and on their levels of understanding within the UDGS framework, in both Logo and non-Logo contexts.

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THE EFFECTS OF LOGO-BASED LEARNING EXPERIENCES ON STUDENTS' NON-VERBAL COGNITIVE ABILITIES

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This paper reports on specific cognitive abilities which may be influenced by a LOGO Learning Environment designed for exploration of geometric relationships. When comparing LOGO students to comparison groups, results suggest that the LOGO experience may very well enhance the abilities assessed. This impact seems to be dependent, in part, on the improved training and skills of teachers who are well versed in the philosophy of a LOGO educational culture.

INTRODUCTION

This research study is part of a larger project investigating the teaching and understanding of geometric relationships through LOGO. The focus of the larger project is the development of LOGO learning experiences designed to give ninth grade students opportunities to explore various geometric relationships in order to enhance their study of formal geometry in tenth grade. This report examines specific non-verbal cognitive abilities believed to be related to the goals of the LOGO class and the subsequent understanding of geometric relationships. These non-verbal abilities were assessed using subtests of the Cognitive Abilities Test (CogAT) published by Riverside (1985).

General Project Description

During a two year period (1984-1986) several mathematics teachers from two urban high schools were given training in using LOGO to teach geometric relationships. In the 1985-86 school year two LOGO classes (one per school) were taught each semester and became the main focus for project research. The same teacher at each school taught both semesters. Consistent with the developmental focus of the Project, feedback from the first semester LOGO classes was used to improve both the learning experiences and instructional strategies for the second semester classes. It was predicted that in each semester at each school the LOGO class would demonstrate higher gains than the comparison group on project measures. This difference was expected to be greater in the second semester.

RESEARCH METHODOLOGY

Research Focus

The overall concern of the project was to explore geometric understanding and subsequently, through the use of LOGO, enhance students' ability to succeed in their high school geometry course. An aspect of this interest focuses on the question, "What impact might a LOGO Learning Experience have on certain non-verbal cognitive abilities which are believed to be related to success in formal geometry coursework?"

Design

In each semester at each school, a Pre/Post-test Comparison Groups design was used to address the above questions regarding the effects of the LOGO Learning Experience. In order to form the main sample for the study, an attempt was made to identify all ninth grade students enrolled in Algebra 1 (or its equivalent) at the two participating high schools. Non-random LOGO classes and comparison groups were formed from this sample. Selection for the LOGO classes was based on students' scheduling requirements and willingness to take the LOGO course. All other students in the main sample were considered part of the comparison groups.

Brief Description of the LOGO Classes

The Project classes were taught at two city high schools, one with an all black population and the other with an equal distribution of black and white students. The LOGO classes consisted of between 12 and 21 ninth grade algebra students who were on track for tenth grade geometry. The LOGO course was an elective carrying graduation credit in computer science. It was taught once each semester for approximately 16 weeks, with classes meeting every day for a 50 minute period. Project equipment for each classroom included 14 microcomputers, at least one printer, one graphic plotter and a modem. The equipment at each school was housed in computer laboratories.

Students generally worked in pairs on the computers. There were opportunities, however, for individuals to work alone. Student pairs were generally self-chosen and of the same sex. Students were encouraged to discuss their work within pairs and to rotate the role of computer operator. Pairs of students were also encouraged to look at the work of other pairs and to share ideas. There was frequently a background level of student verbal interaction within and across pairs.

The teacher at School 1 (the all black school) was a black male with 16 years teaching experience at School 1 in grades 9 - 12. He had taught all mathematics courses for those grade levels in addition to computer science courses. The teacher at School 2 was a white female with only four years teaching experience. She had previously worked as a computer programmer in industry. She had taught most mathematics courses (except geometry) and BASIC programming courses. The role of both LOGO teachers was at various times that of motivator, instructor, coach, co-explorer, and evaluator.

Curriculum Sequence

Although a general curriculum outline was developed, it was not the intention of the Project to standardize curriculum but rather to offer guidelines and materials which the teacher could use in his or her own way to best meet the needs of specific students. The curriculum at both schools was more focused on geometric relationships during the second semester than during the first. It should be noted, however, that the sequence of topics taught in each classroom was not identical. Classes at both schools attempted to start from students' intuitive knowledge of motion, direction, angles, and geometric shapes. They then used LOGO to build on these intuitive notions - to explore the relationships within and among regular polygons, rectangles, parallelograms, and circles. They explored geometric transformations (slides, rotations and reflections). A more detailed description of the LOGO classes and activities can be found in the Project Interim Report (Olive, Lankenau & Scally, 1986).

Assessment of Cognitive Abilities

Four subtests from the Cognitive Abilities Test were administered to all project students (N=124) in the fall of 1985 prior to the first LOGO class. The same four subtests were administered at the end of each semester (January and May, 1986) as post-tests for the designated groups (see Table 1). The four subtests were chosen because they appeared to be assessing abilities which were believed to be enhanced by the LOGO experiences. The Figure Classification subtest deals with the ability to identify the common element of several geometric figures and to select another figure that shares the same property.

Similarly, the LOGO activities involved the classification of geometric properties. The Figure Analogies subtest assesses the ability to determine a relationship between two figures and to then identify the missing figure of a second pair which would demonstrate the same relationship. Working with commonalities between figures was an integral part of the LOGO experience. The Figure Analysis subtest assesses the ability to make mental transformations of a figure. Many of the LOGO activities involved transformations of geometric figures and the ability to anticipate results of such transformations. The Equation Building subtest deals with the ability to construct relationships between numbers and arithmetic operations; the LOGO curriculum also emphasized the construction of relationships (both arithmetic and geometric) in the programming tasks.

TABLE 1: TESTING SCHEDULE
Cognitive Abilities Test

Pre 1st Sem. 9th Grade	Post 1st Sem. 9th Grade	Post 2nd Sem. 9th Grade	Post 2nd Sem. 10th Grade
C1	C1	--	C1
L1	L1	--	L1
C2	--	C2	C2
L2	--	L2	L2

C = COMPARISON GROUP 1 = 1ST SEMESTER GROUP
L = LOGO CLASS GROUP 2 = 2ND SEMESTER GROUP

Additional Assessments

Data on pre-treatment characteristics have also been collected and will be used to assess comparability of groups. These include standard achievement measures, a learning styles inventory, surveys of attitudes towards mathematics and computers. Post-treatment data have also been collected from a variety of other sources, including algebra grades, geometry tests and grades. Final post-testing of all students will take place at the end of their tenth grade year (May, 1987). Documentation of the LOGO experiences came from interviews, observations and dribble files of LOGO students' work.

RESEARCH RESULTS

Analyses of covariance were performed on the data on each of the four subtests. The post-test scores were the dependent variables with pre-test scores used as covariates. Separate analyses between the LOGO and Comparison Groups were computed for each semester for each school. See Table 2 for the statistical results discussed below.

TABLE 2: COGNITIVE ABILITIES TEST PRE AND POST TESTING RESULTS 1985-86 NINTH GRADE STUDENTS

	SCHOOL 1			SCHOOL 2			
GROUPS: ==>	L1	C2	L2	C1	L1	C2	L2
SUBTESTS:							
QUANTITATIVE (Total Score = 15)							
N	14	7	19	1	17	1	2
PRE MEAN	7.00	7.57	6.58	-	7.18	-	-
(SD)	(2.99)	(1.40)	(1.77)	-	(1.94)	-	-
POST MEAN	7.07	6.43	6.84	-	8.06	-	-
(SD)	(3.03)	(2.76)	(2.46)	-	(2.36)	-	-
r	.60	.27	.51	-	.30	-	-
2-Tailed p =	.03	.56	.03	-	.13	-	-
ANCOVA:							
ADJ. POST MEANS	-	5.94	7.02	-		-	
MAIN EFFECT	-	F(1,23) =	1.04	-		-	
p =	-		.32	-		-	
FIGURE CLASSIFICATION (Total Score = 25)							
N	15	10	21	21	14	26	11
PRE MEAN	8.07	7.50	7.05	9.05	10.14	9.96	10.27
(SD)	(4.54)	(2.42)	(2.73)	(3.06)	(3.70)	(3.76)	(3.00)
POST MEAN	9.27	8.80	10.43	10.19	13.14	12.58	15.27
(SD)	(4.62)	(4.61)	(3.80)	(3.43)	(4.22)	(4.68)	(4.10)
r	.71	.68	.38	.63	.39	.70	.66
2-Tailed p =	.00	.03	.09	.00	.17	.00	.03
ANCOVA:							
ADJ. POST MEANS	-	8.57	10.53	10.44	12.76	12.66	15.08
MAIN EFFECT	-	F(1,28) =	1.94	F(1,32) =	4.08	F(1,34) =	4.07
p =	-		.17		.05		.05
FIGURE ANALOGIES (Total Score = 25)							
N	15	10	21	21	14	26	11
PRE MEAN	10.67	8.40	10.71	13.76	15.07	14.58	14.46
(SD)	(3.83)	(3.78)	(4.04)	(2.63)	(2.84)	(4.40)	(4.57)
POST MEAN	12.60	11.50	13.00	14.62	15.64	16.15	18.36
(SD)	(4.07)	(4.45)	(4.16)	(3.72)	(4.13)	(4.63)	(3.11)
r	.75	.53	.80	.59	.20	.87	.76
2-Tailed p =	.00	.12	.00	.01	.50	.00	.01
ANCOVA:							
ADJ. POST MEANS	-	12.70	12.43	14.93	15.17	16.12	18.43
MAIN EFFECT	-	F(1,28) =	.05	F(1,32) =	.04	F(1,34) =	7.19
p =	-		.83		.85		.01
FIGURE ANALYSIS (Total Score = 15)							
N	15	8	18	12	14	16	9
PRE MEAN	4.00	5.63	3.83	4.33	5.50	4.94	6.11
(SD)	(2.00)	(2.62)	(2.26)	(2.96)	(3.01)	(3.02)	(4.14)
POST MEAN	5.67	6.00	5.67	6.58	7.21	7.50	8.00
(SD)	(2.13)	(2.14)	(3.38)	(3.68)	(2.89)	(3.72)	(3.78)
r	.34	.31	.59	.45	.61	.13	.50
2-Tailed p =	.22	.46	.01	.15	.02	.64	.17
ANCOVA:							
ADJ. POST MEANS	-	5.18	6.03	6.94	6.90	7.52	7.97
MAIN EFFECT	-	F(1,23) =	.49	F(1,23) =	.00	F(1,22) =	.09
p =	-		.49		.98		.77

KEY: C1 = 1ST SEMESTER COMPARISON GROUP L1 = 1ST SEMESTER LOGO CLASS
C2 = 2ND SEMESTER COMPARISON GROUP L2 = 2ND SEMESTER LOGO CLASS

School 1

First Semester: Because there were insufficient numbers of ninth grade algebra students at School 1 to have a comparison group for each semester, all non-LOGO students were assigned to the second semester comparison group. Consequently between group analyses are not available for the first semester.

Second Semester: The LOGO group made slightly larger gains than the comparison group on all subtests except Figure Analogies. This subtest was the one instance where adjusting for the covariate actually reversed the raw score post-test results, showing the comparison group as having made a slightly higher gain.

School 2

First Semester: Analyses indicate that the LOGO group at School 2 made slightly larger gains between pre and post-tests than the comparison group on both Figure Classification and Figure Analogies. Although the LOGO group did score higher than the comparison group on the post-test for Figure Analysis, adjustment for the pre-test difference equalized the two groups. An analysis of the Equation Building subtest was not possible at School 2 as only one comparison student took both the pre and post for this subtest.

Second Semester: Again, no analysis was possible for the Equation Building subtest as only three students were present for both the pre and post administrations at School 2. There were significant main effects for groups for both the Figure Classification and Figure Analogies subtests. This effect was in favor of the LOGO group. Although not statistically significant, the LOGO group made greater gains on the Figure Analysis, as well.

SUMMARY DISCUSSION

Although statistical significance (alpha = .05) for differences between groups was usually not obtained for all of the analyses, the trend clearly shows that the LOGO groups scored higher across the non-verbal cognitive subtests. The gains were particularly noteworthy for the Figure Classification and Analogies subtests at School 2. The second semester LOGO group at School 1 also appears to have maintained the trend of larger gains on the quantitative Equation Building subtest.

Implications and New Questions

The first implication of these results is that the LOGO experiences may well be enhancing students' non-verbal cognitive abilities. The second implication is that the second semester LOGO classes may have had more effect on those abilities than the first semester classes. Thus, the improved activites and teacher expertise evidenced in the second semester LOGO classes appear to be important factors. This supports evidence from other project investigations that the potential value of LOGO may lie in the effectiveness of teacher development relative to classroom climate, instructional expertise and educational philosophy, as well as the degree to which student-student interaction is encouraged.

The comparison of first semester and second semester results for both LOGO and Comparison groups at School 2 also suggests the possibility of a 'developmental effect.' Both second semester comparison and LOGO groups scored higher on the post-tests of all three spacial subtests than their first semester counterparts at School 2. This develomental influence could explain the apparent lack of difference between the first semester LOGO group at School 1 and the second semester comparison group which was post-tested more than four months later. In addition to general maturation, school experiences common to both LOGO and Comparison groups, such as the Algebra course, could also have influenced the second semester post-test scores.

An obvious question arising from these results is whether or not this apparent gain in spacial abilities evidenced by the LOGO groups will help with their subsequent studies in geometry. Initial data on enrollment in geometry courses indicates that a much larger proportion of the LOGO students have enrolled in geometry this year than the Comparison students and that the difference is greater for the second semester groups. The final testing of students' non-verbal cognitive abilities will take place at the end of their tenth grade year (May, 1987). With this additional data we will explore the relationships between students' LOGO experiences, their geometry experiences and non-verbal cognitive abilities.

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L'OUTIL INFORMATIQUE ET L'ENSEIGNEMENT DE LA GEOMETRIE DANS L'ESPACE

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(conference will be in English)
L'outil informatique peut-il aider à surmonter les problèmes posés par la représentation plane d'objets spaciaux, dans le cadre de l'enseignement de la géométrie de l'espace? peut-il contribuer à l'évolution et au contrôle de l' "activité perceptive", lors de la lecture ou de la production de dessins illustrant un problème de géométrie dans l'espace?
Cette recherche essaie d'utiliser les possibilités d'action qu'offre l'informatique pour créer une situation d'enseignement où le traitement dynamique de dessins est le moyen de résolution d'un problème, qui fasse évoluer certaines connaissances géométriques.

La recherche exposée ici se veut comme un essai d'exploration et, jusqu'à certaines limites, de remédiation à certains problèmes posés par la perception, lors de la lecture et de la production de représentations graphiques, dans le cadre de l'enseignement de la géométrie de l'espace. En effet, une des difficultés de la géométrie dans l'espace résulte du fait que l'accès aux situations spatiales se fait à travers des représentations planes. Certaines caractéristiques de la configuration représentée sont alors absentes ou modifiées, d'où la nécessité d'un code de représentation. Bien qu'il soit nécessaire, ce code ne peut être suffisant pour surmonter les difficultés de coordination et de construction de rapports entre espace graphique et espace physique. Un contrôle de la perception est indispensable à une telle fin; or la perception ne consiste pas, selon PIAGET, en une simple lecture des données sensorielles, mais comporte une organisation active, de plus en plus influencée par le développement de l'intelligence; cette "activité perceptive", au sens de VURPILLOT, doit être contrôlée par des connaissances géométriques antérieures, et sera conduite de façon de plus en plus élaborée, en fonction de l'élaboration de la connaissance des objets impliqués, et des relations spatiales qui les régissent.

Ce travail n'aurait pas pu être réalisé sans l'aide précieuse de Annie BESSOT et Madeleine EBERHARD, de l'équipe de Grenoble, qui ont bien voulu participer aux expérimentations, et dont les conseils étaient toujours pertinents. Je leur exprime ici ma reconnaissance.

En partant de l'hypothèse d'une interaction entre la maitrise des représentations planes de certains objets et la connaissance des propriétés géométriques de ces objets, notre recherche essaie de développer une problématique de la représentation graphique qui rende nécessaires certaines connaissances de la géométrie pour la construction de dessins, et qui fasse du dessin un problème dont la résolution contribue à l'évolution de ces connaissances.

Deux séquences d'enseignement ont été construites et réalisées, dans le cadre d'un atelier d'informatique, avec une classe de 3e et une classe de CPPN (élèves en difficulté, 14-16 ans). Les élèves trvaillent par binômes, ce qui a permis d'extérioriser leurs démarches, et de créer des conflits, qui les obligent à formuler des arguments pour soutenir un point de vue. Au cours de cette expérience, on a enregistré les différentes étapes du travail des élèves, ainsi que tous leurs propos, en vue d'une analyse clinique de l'évolution de leurs stratégies.

Notre démarche est de construire des situations-problèmes où, d'une part, l'ordinateur joue un rôle important comme outil d'aide, par la puissance d'action qu'il amène au traitement de représentations graphiques, et où, d'autre part, on tente de multiplier de plus en plus les ambiguités perceptives, pour inciter les élèves à mettre en oeuvre des moyens de contrôle qui se basent sur des règles géométriques; le but principal est de surmonter le problème de la vision dans l'espace, en déstabilisant la confiance en la perception, pour orienter vers des stratégies opératives, basées sur la mise en rapport et la coordination des constituants de la configuration spatiale, par opposition au procédé figuratif, basé sur la perception intuitive et statique. Pour ce faire, on a adopté un type particulier de représentation:

TYPE DE REPRESENTATION ADOPTE

Etant par excellence le moyen de représentation adopté dans les manuels scolaires et la pratique de l'enseignement, et dans les ateliers de formation technique et professionnelle, le modèle de la perspective cavalière s'est imposé. D'autres raisons ont aussi guidé ce choix: supposant un point de vue imaginaire, rejeté à l'infini, la perspective cavalière implique, par opposition à la perspective conique, une limitation des modifications inférées par les déplacements du point de vue: à une direction de vue donnée, on a une seule représentation en perspective cavalière d'un objet donné.

D'autre part, le critère de notre choix étant le degré d'ambiguités perceptives introduites, la perspective cavalière répond à cette exigence; en effet, la perspective cavalière "transparente" est une représentation polysémique: un dessin peut évoquer, soit plusieurs objets, soit le même objet associé à plusieurs directions de vue.

En nous appuyant sur une typologie des perspectives cavalières selon les degrés d'ambiguité qu'elles induisent, nous avons adopté un type particulier de perspective, ayant les caractéristiques suivantes: (fig.1)

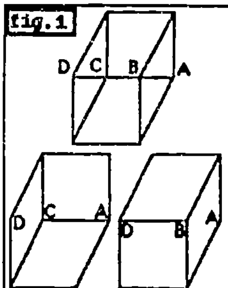
* perspective cavalière frontale,

* ambigüe par l'alignement des segments AC et BD, représentant deux arêtes situées dans deux plans frontaux différents, et à des niveaux de hauteur différents,

* les quatre points A, B, C et D, qui représentent quatre sommets situés, deux à deux, à deux niveaux différents de profondeur, sont alignés, et partagent le segment DA en trois segments de même longueur. Ceci implique des coïncidences et des alignements qui multiplient l'ambiguité, dans les cas où on aurait plusieurs cubes juxtaposés.

* le segment BC joue un rôle bivalent : il fait partie des deux arêtes AC et BD; par conséquent, il n'est effacé dans aucune des deux représentations de cube opaque.

fig.1



Cube en perspective cavalière adoptée
en haut : représentation du cube transparent
en bas : 2 représentations possibles de cube opaque qui en résultent

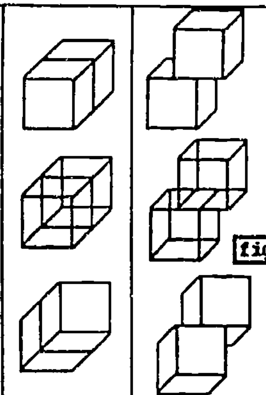
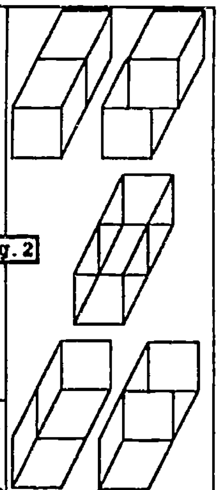


fig.2

Comparaison des degrés d'ambiguité induits par la représentation de 2 cubes : en perspective adoptée et en perspective cavalière normalisée



Persp. adoptée

* comme pour toute perspective cavalière, la représentation d'un cube transparent peut évoquer, selon les arêtes qu'on considère comme cachées, deux cubes opaques. Une des particularités de la perspective adoptée réside dans le fait que cette non-univocité est multipliée à partir du cas où on a une représentation de deux cubes, comme le montre la comparaison dans la fig.2: d'après une représentation de deux cubes transparents avec la perspective adoptée, on peut déduire quatre configurations de deux cubes opaques qui, avec la perspective cavalière normalisée, nécessiteraient deux représentations "transparentes" différentes.

CHOIX DES OBJETS REPRESENTES

De par sa place comme une unité constitutive de l'espace physique, la configuration "cube" est la base du monde d'objets impliqués dans notre situation. Ces objets, qui sont des assemblages de cubes (réels ou virtuels) de même taille, relèvent du micro-espace, et constituent une première mise en ordre de micro-espace: organisation selon la structure du trièdre euclidien trirectangle. Notons que, dans une représentation graphique, les ambiguités perceptives résultant de la juxtaposition de plusieurs cubes ne se réduisent pas à la somme de celles impliquées par chacun des cubes; elles la dépassent pour mettre en jeu des problèmes provenant de coïncidences et d'alignements entre les représentations des différents constituants de la configuration (voir fig.2).

PLACE DE L'OUTIL INFORMATIQUE DANS LA SITUATION

L'ordinateur intervient dans cette situation comme outil d'aide à l'enseignement; l'informatique n'est donc pas notre objet d'enseignement, pas plus que l'entrainement à l'utilisation de logiciels. Notre recherche s'intéresse, dans le cadre de la résolution des problèmes posés, aux processus d'adaptation ou d'évolution des stratégies des élèves, dans le contexte de leur confrontation aux logiciels utilisés, basés sur d'autres systèmes de connaissance et de traitement de l'information, qui ne suivent pas nécessairement la même logique de fonctionnement que l'élève.

Deux logiciels de dessin graphique sont utilisés: Mac Paint et Mac Space.

le premier permet un traitement de figures planes, à l'aide d'outils graphiques assez proches de ceux qu'on utilise dans une situation habituelle de dessin: un crayon, une gomme, une règle,...

le deuxième permet d'obtenir, dans une fenêtre de contrôle non accessible au traitement, la perspective cavalière d'une configuration, à partir de ses trois vues qu'on peut construire dans trois autres fenêtres (normes du dessin technique).

Dans ce cadre, une catégorie des difficultés liées au graphisme (tracés corrects de lignes, parallélisme, angles droits,...) sont écartées, par la donnée des outils disponibles dans le contexte de chacun des deux logiciels. D'autre part, l'informatique permet un traitement dynamique des informations: on peut agir sur le dessin, le modifier, le corriger, tout en sachant qu'à tout moment on peut récupérer le dessin de départ, ou des dessins intermédiaires qu'on aurait enregistrés. Le dessin n'est plus, comme c'est souvent le cas, un support statique de représentation de l'objet, il est au coeur du problème, et c'est à travers lui que se manifestent et évoluent les conceptions des élèves.

Finalement, l'outil informatique intervient dans cette situation par deux ensembles de contraintes, imposées par deux conceptions différentes des objets: une conception des représentations fondée, dans Mac Space, sur une géométrie particulière, qu'on pourrait appeler "géométrie des facettes", et qu'on espère développer dans un travail ultérieur, et, dans Mac Paint, sur le traitement direct de la représentation en perspective. L'importance de cet outil se révèle surtout au niveau du transfert de dessins: de l'environnement du logiciel Mac Space, où les dessins sont reconnus comme des représentations d'objets spatiaux, à travers un lien permanent entre espace physique et espace graphique, vers l'environnement du logiciel Mac Paint, où ce lien rompu doit être restauré par les élèves (fig.3). C'est donc un transfert d'un environnement où les transformations affectent l'objet lui-même, donc, où le dessin est opératif (Mac Space), vers un autre, où les modifications ne peuvent se traduire en transformations spatiales, donc, où le dessin n'est que figuratif (Mac Paint).

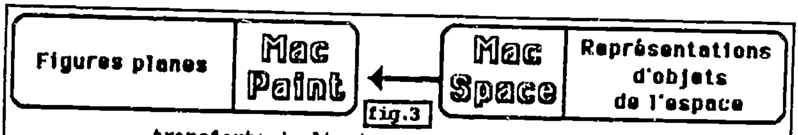


fig.3

transfert: de l'univers des représentations de l'espace physique vers l'espace graphique

EXEMPLE ET ANALYSE D'UNE SITUATION-PROBLEME

En nous situant dans ce cadre théorique, nous analyserons une situation-problème, dans le cours de notre séquence.

Au début de cette activité, les élèves disposent d'un type de repérage lié à l'objet, où la configuration "cube" est l'unité, et où un cube de l'assemblage considéré joue le rôle de "cube-origine". La tâche présente a pour objectif le passage à un autre type de repérage, qui approche le repérage sous-jacent à Mac Space, qu'ils utiliseront dans une activité ultérieure. Ce passage se fera à travers des stratégies de dépassement du conflit entre perception et connaissances géométriques (voir la consigne dans la fig.4).

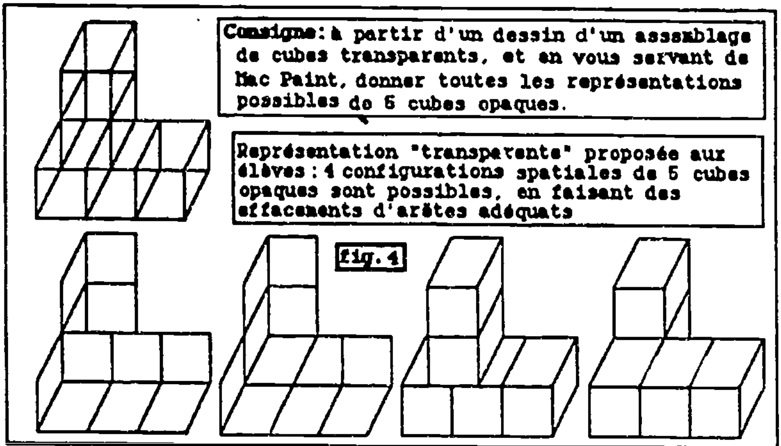


fig. 4

La tâche présente met les élèves dans une situation de: décodage d'un dessin en perspective "transparente" polysémique, et transformation de ce dessin pour montrer à autrui le plus grand nombre possible d'assemblages de cubes qu'il peut représenter. La tâche exige une stabilité des représentations, qui sera fragilisée par les ambiguités perceptives, résultant d'une part de la "transparence", et d'autre part de coïncidences, alignements et superpositions trompeurs liés au type de perspective. Ce n'est que par l'élaboration de relations géométriques entre les éléments de la configuration que les élèves pourront surmonter cette instabilité. Ils seront obligés à mettre en oeuvre des:

* relations d'incidence, d'adjacence, d'alignement, de coplanarité et d'interaection entre les différents éléments de la configuration représentée,
* relations d'ordre spatial: devant, derrière, dessus, dessous,...

* relations d'appartenance (de points à des arêtes ou des faces) et de contenance (de segments dans des faces),

* rapports permanents entre le dessin et la configuration représentée.

l'analyse des stratégies des élèves, à partir de leurs productions (fig.5), le long de la résolution de ce problème, a montré qu'elles suivent un cours d'évolution général qui va de l'analyse locale vers l'analyse globale:

1-les premières stratégies consistent à traiter les segments séparément. Se révélant très coûteuses, et aboutissant à des productions fausses, ces stratégies seront remplacées par d'autres:

2-l'élève essaie d' "opacifier" localement un ou plusieurs cubes, et d'intégrer ces parties par rapport au dessin global, ce qui n'est pas toujours possible.

3-reconnaissance de portions de plans comme entités unifiées du dessin: des référentiels locaux, par rapport auxquels seront repérés des ensembles d'éléments de la configuration, et testés, par suite, d'un seul coup, des ensembles de traits.

Ces stratégies ont été évaluées par rapport à une stratégie optimale que nous considèrerons comme l'aboutissement théorique de leur évolution; elle est basée sur la découverte de référentiels globaux: *les plans équidistants parallèles au plan horizontal. Dans cette classe, un plan particulier: la face dessus de l'objet,

*les plans équidistants parallèles au plan vertical frontal. Dans cette classe, un plan particulier: la face devant de l'objet,

*les plans équidistants parallèles au plan vertical, perpendiculaire au plan frontal. Dans cette classe, un plan particulier: la face de droite de l'objet.

Ces trois plans particuliers constituent, en fait, un trièdre trirectangle, qui est la base de l'espace euclidien; il aura un rôle fonctionnel dans la stratégie optimale qui consiste, pour le pavé par exemple (fig.6), à: repérer le plan représentant la face devant d'une configuration, et nettoyer les carrés qui le constituent; répéter la même opération pour la face dessus et la face de droite de l'objet. Particulier au pavé, cet algorithme de fonctionnement peut être adapté à d'autres configurations.

Ces fonctions de contrôle développées par les élèves à des niveaux plus ou moins élaborés peuvent être analysées du point de vue d'un repérage dans l'espace, où l'entité "cube" est détruite, pour laisser la place à des objets géométriques plus abstraits, qui quadrillant l'espace par des plans structurés en trois classes. Ce repérage fonctionne comme un moyen de contrôle et de dépassement de la perception.

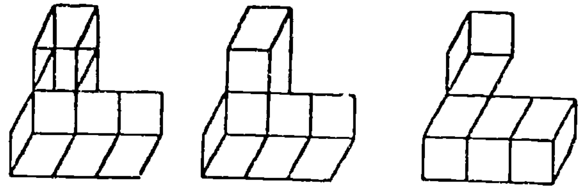
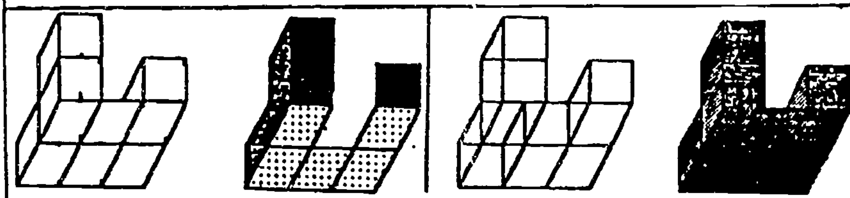
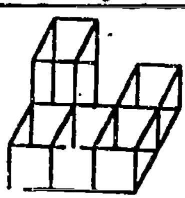


fig.5

Stratégies d'analyse locale, donnant des dessins de configurations impossibles selon la perspective cavalière (deux codes différents de la perspective régissent les parties du dessin)

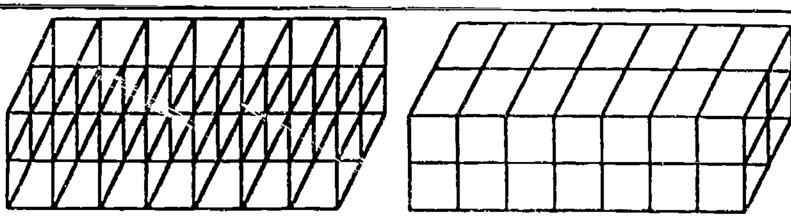


Exemples de représentations de configurations spatiales, correctes du point de vue du code de la perspective, mais qui ne font pas partie de l'univers des objets concernés: assemblages de cubes non-accrochables.

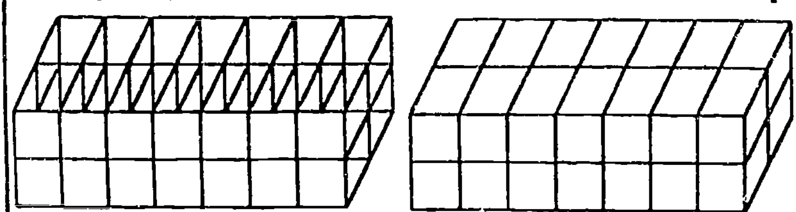


Débuts d'applications locales

de la stratégie optimale



Stratégie optimale, basée sur la structuration de l'espace



en trièdre trirectangle

fig. 6

LES HABILETÉS PERCEPTIVES D'OBJETS POLYÉDRIQUES

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Richard ALLAIRE
U.Q.A.M.

RÉSUMÉ

L'objectif général de la recherche était d'étudier comment un environnement ordinateur de type LOGO peut permettre de développer les habiletés perceptives d'"observation", d'"abstraction" et de "communication", correspondant aux opérations respectives de "visualisation", de "structuration" et de "transfiguration", définies dans le cadre d'une typologie globale de la perception structurale d'objets géométriques, que nous avons limité aux polyèdres, pour l'instant.

Le rapport qui suit présente des éléments de la problématique, la méthodologie suivie, ainsi que quelques résultats obtenus en 1985-86. L'expérience se poursuit au niveau de deux autres habiletés définies dans la typologie.

PROBLÉMATIQUE

L'environnement ordinateur

Nous avons cherché un environnement ordinateur de type LOGO afin de retrouver un certain contexte d'apprentissage propre aux enfants-programmeurs. Cette contrainte nous a obligés à initier les élèves du groupe expérimental aux rudiments du langage LOGO: primitives graphiques, définition de procédures et modularisation de celles-ci, notion de variables, répétition .. Le progiciel retenu fut d'abord EXPERLOGO, puis MAC LOGO, qui roulent sur un appareil Macintosh et qui contiennent déjà les fonctions de base simulant l'espace tridimensionnel au moyen d'une projection centrale et permettant conséquemment les rotations autour des axes x, y et z

Le système LOGO fournit un langage de base qui, augmenté de fonctions pertinentes, fournit un excellent moyen pour pallier à des déficiences linguistiques d'abord, mais aussi manipulatoires, car intervenir physiquement sur des objets tridimensionnels et leur transformation n'est pas chose aisée.

La typologie des habiletés perceptives

Depuis plusieurs années, nous affinons une typologie d'opérations et d'habiletés correspondantes permettant de définir d'un point de vue opératoire la perception spatiale [1]. La présente recherche s'est concentrée sur les trois premières opérations de la typologie, à savoir la visualisation, la structuration et la transfiguration, correspondant aux habiletés respectives d'observation, d'abstraction et de communication. Ces trois premières étapes se situent au niveau de la compréhension de l'objet plutôt que de sa transformation, au niveau de l'imagerie mentale que l'objet suggère plutôt que de son invention, au niveau cognitif plutôt que créateur et enfin au niveau analytique plutôt que synthétique, par opposition aux trois dernières opérations de la typologie.

Le développement d'habiletés perceptives d'objets géométriques à l'aide d'un environnement ordinateur

L'arrivée des micro-ordinateurs dans les écoles permet d'entrevoir la possibilité de se servir de cet instrument d'apprentissage pour que l'enfant puisse construire ses propres structures cognitives au moment de sa vie où il est peut-être le plus interrogé par le réel sur le plan des relations spatiales et des propriétés géométriques afférentes. En se basant sur la typologie des habiletés en cause dans la perception structurale d'un espace géométrique où intervient entre autres le primat piagétien du topologique [2], et sur l'apport d'un langage informatique évolué, interactif et modulaire, permettant l'ajout de primitives fonctionnelles et leur utilisation dans les programmes de jeunes élèves, la problématique se situe donc essentiellement au plan de l'étude des interactions qui facilitent la structuration progressive du monde des images, au centre de la triade dynamique de Caleb Gattegno: Pensée - Image - Action.

Bien qu'exploratoire, notre recherche visait à établir le critère de vérité de l'hypothèse centrale suivante: l'environnement ordinateur de type LOGO, incluant les logiciels pertinents permettant d'extensionner les primitives originales du langage, permet de développer les premières habiletés perceptives de l'espace.

MÉTHODOLOGIE

Les logiciels

Pour développer les habiletés déjà citées d'observation, d'abstraction et de communication, nous nous sommes situés dans un contexte radical, celui d'un environnement ordinateur, excluant totalement les formes tridimensionnelles elles-mêmes. Nous avons donc dû élaborer un certain nombre de logiciels permettant de simuler des polyèdres à l'écran, en les dessinant dans une projection centrale et en permettant la rotation de l'objet représenté, comme s'il tournait vraiment devant nos yeux. Les logiciels devaient donc permettre aux sujets de modifier la présentation des objets dessinés, selon des rotations autour des axes x, y et z, les trois axes orthogonaux dans R^3.

LOG1 vise à développer l'habileté d'**observation** en se servant d'activités de **visualisation**. La fenêtre centrale du bas va donc faire apparaître au hasard une des trois formes du haut et la représenter dans une projection centrale fixe, qui ne sera jamais identique à celles que l'élève peut obtenir par la rotation des formes dont il peut changer la perspective. Chaque forme peut d'ailleurs être représentée par 46 656 positions différentes, i.e 36^3 positions possibles. Le problème consiste donc à trouver lequel des trois dessins du haut représente la même forme que celui d'en bas.

LOG2 vise à developper l'habileté d'**abstraction** par l'intermédiaire d'activités de **structuration**. Le problème à résoudre va consister à identifier lequel des dessins du haut représente une forme qui n'entre pas dans la construction de la forme représentée par le dessin du centre en bas. Avant de faire son choix, l'élève peut de nouveau simuler à volonté la rotation des trois formes représentées par les dessins du haut.

LOG3 implique l'habileté de **communication**, mise de l'avant par des opérations de **transfiguration**. Deux sortes de transferts ont été retenus, en concordance avec la nature de l'outil didactique utilisé: le dessin topologique (ou diagramme de Schlegel) et la définition littéraire. Dans les deux cas, il s'agit pour l'élève d'identifier le dessin représentant la forme décrite, soit par une définition, soit par son graphe topologique.

Les groupes de sujets

Les 15 enfants du **groupe expérimental** étaient de niveau 5è année (9 et 10 ans) et provenaient de deux écoles de la Rive-Sud de Montréal. Ils ont d'abord été initiés au langage LOGO, à raison de deux heures par semaine, à partir d'octobre jusqu'à la mi-décembre 1985. A la fin, les sujets étaient en mesure de réaliser de petites procédures indicées, de les appeler entre elles, d'utiliser la répétition d'un certain nombre de commandes, d'utiliser les changements angulaires facilement et, bien sûr, de gérer leurs fichiers. En janvier 1986, ils furent initiés à l'appareil Macintosh, ainsi qu'au langage EXPERLOGO, selon le même rythme horaire.

Le 3 février, les sujets du groupe expérimental passèrent le pré-test. Ensuite, au rythme d'un par mois, les logiciels d'entraînement LOG1, LOG2 et LOG3 furent présentés aux enfants, sur lesquels ils purent travailler suffisamment de temps. En moyenne, chaque enfant a travaillé une heure par semaine sur les logiciels d'entraînement. Des grilles d'observation ont permis également aux animateurs de relever certains faits significatifs. Le 12 mai, les sujets passaient le test de relance.

Tant qu'à eux, les 15 élèves du **groupe contrôlé**, du même âge et de la même région, n'avaient pas accès à l'entraînement décrit précédemment. Seuls leurs apprentissages scolaires normaux, en géométrie par exemple, ainsi que la période de temps entre le pré-test (31 janvier) et le test de relance (20 mai), impliquant une certaine maturation naturelle, pouvaient agir sur leur développement perceptif.

Le test-entrevue

Le pré-test et le test de relance furent identiques. Les cent jours et plus qui les séparaient, justifiaient cette décision. La forme retenue fut celle du test-entrevue. Pour démontrer l'efficacité de l'environnement ordinateur, il s'avérait nécessaire de construire un test en utilisant de véritables objets, et non seulement leurs représentations sur papier. Le test collectif était donc exclu. Chaque entrevue durait de 20 à 25 minutes.

COMPORTEMENTS OBSERVÉS

Un des buts de cette recherche était de comprendre comment les élèves arrivent à faire des choix et quelle est l'évolution de leurs comportements dans le développement des habiletés perceptives concernées. L'analyse qualitative a porté sur des observations recueillies lors de l'appropriation par les élèves des trois logiciels: LOG1, LOG2 et LOG3. Les observations recueillies furent consignées dans un journal de bord. Les élèves ont pratiquement toujours travaillé en équipe de deux, ce qui a permis des échanges et des discussions qui se sont avérés fructueux pour l'acquisition des habiletés perceptives impliquées dans ces trois logiciels.

Lors des activités d'apprentissage, nous avons observé un certain nombre de comportements développés par les élèves. C'est l'évolution de ces comportements qui leur ont permis de faire des choix de plus en plus rationnels. Les stratégies de décision se sont élaborées au fur et à mesure des acquis et des expériences vécus avec les logiciels. Nous relatons, à partir de chacun des logiciels, l'évolution de cette prise de décision faite par les élèves.

LOG1 (habileté d'observation)

Dans un premier temps les élèves choisissaient au **hasard** afin de finir les premiers. Nous sommes revenus à la charge à deux reprises en insistant sur le fait qu'ils devaient faire un choix à partir de critères qu'ils devaient se donner.

Le premier critère choisi fut l'**apparence générale du solide**. Etait-il petit ou grand par rapport aux autres, contenait-il une forme particulière telle un cube, une pyramide ou un prisme triangulaire, etc? Ce choix porté sur l'une ou l'autre des figures permettait une élimination. Le raisonnement suivant prenait place: si ce n'est pas celui-ci, donc c'est l'un de ces deux-là. Ainsi, par déductions successives, ils arrivaient à faire un choix plus judicieux. Il demeure que ce critère était très primitif et qu'il fut la cause de plusieurs choix infructueux.

LOG2 (habileté d'abstraction)

Lorsque les élèves ont commencé à travailler avec le deuxième logiciel, ils ont eu tendance à utiliser le hasard à nouveau, mais ils se sont vite rendus compte des difficultés encourues, à cause de la fréquence des mauvais choix. Suite à des discussions entre eux, et plus intéressés à faire des choix corrects, ils se sont mis à analyser les polyèdres plus en détail. Ils se sont fixés de nouveaux critères. La **forme des faces** des solides représentés a pris de l'importance. Qu'il s'agisse de triangles, de carrés, d'hexagones ou d'autres figures. Le **nombre de faces** fut aussi choisi comme critère additionnel. Les erreurs dans le choix du solide approprié se sont avérées plus rares et le temps de décision plus long, laissant plus de place à l'étude de ces solides.

LOG3 (habileté de communication)

A ce niveau le choix des élèves a été, de façon générale, beaucoup plus déductif qu'aléatoire ou inductif comme au début des activités. Pour chaque problème, ils ont discriminé à partir des propriétés des polyèdres qui leur étaient connues. Le **nombre de sommets** et le **nombre d'arêtes** furent des critères importants dans leur prise de décision. Certaines régularités dans la structure de l'objet telle que pour le rhombicuboctaèdre ont aussi servi de critère. Dans ce logiciel, il y avait deux média utilisés: la définition caractérisant l'un des solides et la représentation de l'objet à l'aide du diagramme de Schlegel.

A ce stade, on a observé une étude plus fine, de la part des élèves. Les critères de discrimination s'additionnent et donnent à la prise de décision une meilleur justification. L'analyse qu'ils font des objets en regard des premières activités où le hasard jouait un rôle de premier rang, dénote une plus grande compréhension de la structure des objet étudiés. Ces types de comportement sont les résultats d'une plus grande appropriation des habiletés perceptives. Il est à noter que cette typologie des habiletés perceptives est hiéarchisée.

Ces critéres développés par les élèves eux mêmes, sont perçus, lors d'un choix à faire, comme étant cumulatifs. C'est la conjonction d'un ensemble de propriétés qui permet de prendre une décision plus juste. Ces observations nous amènent ainsi au coeur du triplet pensée - image - action.

QUELQUES RÉSULTATS

Les performances de l'ensemble des sujets au pré-test ont confirmé la hiérarchie des habiletés impliquées dans la perception des formes géométriques, à savoir l'observation, l'abstraction et la communication, correspondant respectivement aux activités de visualisation, de structuration et de transfiguration.

Alors qu'au pré-test, c'est uniquement au niveau de la visualisation que les sujets du groupe contrôlé performaient significativement moins que ceux du groupe expérimental, c'est précisément de nouveau à ce seul même niveau que ceux-là ont augmenté leur performance moyenne entre les deux tests, de telle sorte que sans aucun entraînement, ils se replacent au même niveau de performance que les sujets du groupe expérimental, lors du pré-test.

Tant qu'au groupe expérimental, ses performances ont augmenté significativement partout ($p < 0{,}02$) et se distancent significativement de celles du groupe contrôlé également partout ($p < 0{,}06$). Une analyse selon le sexe révèle que ce sont surtout les garçons du groupe expérimental qui ont contribué à l'amélioration des performances de ce groupe entre les deux tests.

Enfin, l'analyse des correspondances a révélé l'importance des trois (3) premières activités, quant à leur capacité de discriminer les sujets entre eux, eu égard à leur sexe (axe 1: filles et observation tactile versus garçons et observation visuelle), ou à leur performance en structuration (axe 2: dessin en perspective et faible structuration et axe 3: dessin topologique et forte structuration).

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THE EFFECTS OF LEARNING LOGO ON NINTH GRADE STUDENTS' UNDERSTANDING OF GEOMETRIC RELATIONS

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The literature suggests that students do not have the necessary prerequisite experiences at the third van Hiele level - that of geometric relationships - to succeed in their formal axiomatic study of tenth grade geometry. In this project a LOGO learning environment was proposed as a means of providing these experiences. The environment was established and its effects analyzed in a variety of ways. In this study, students' van Hiele levels of geometric thought were assessed by analyzing transcripts of clinical interviews using a pre- post- comparison groups design. The paper describes the angle tasks and van Hiele level descriptors which were developed. Results indicated that there were few between groups levels differences on either the pre- or post-interviews. When the interviews were analyzed for within subjects levels differences, though, the LOGO students evidenced more gains than their counterparts.

RATIONALE

The van Hiele model of geometric thought development suggests that prior informal explorations of relationships between or among geometric properties and relations are necessary for students to be able to work with the formal deductive geometry encountered in high school. The standard elementary and middle school geometry curricula in the United States do not provide adequate experiences that foster students' thinking at the third van Hiele level - that of relationships.

The Turtle geometry of LOGO provides a learning environment that encourages students to explore and thereby acquire understanding of certain geometric relationships. The study reported here is part of a larger project whose overall purpose was to investigate the effects of a LOGO learning environment on ninth grade students' understanding of geometric relationships.

A course in Turtle geometry was taught to one ninth grade class each semester (sixteen weeks per class) in two urban high schools during the past academic year. Approximately 20 students enrolled in each class. (A more complete description of teacher and student characteristics, and of the LOGO curriculum may be found in Olive & Lankenau, elsewhere in these proceedings.) In this part of the project, the effects of the course on students' van Hiele levels of reasoning were assessed using a clinical interview technique.

DESIGN OF THE STUDY

Interview items were developed for the topic of angles, modeled on those developed by Burger and Shaughnessy (1986) for triangles and quadrilaterals. At the beginning and end of each semester, one-to-one interviews of approximately 50 minutes length were administered to a sample of ninth grade LOGO students and to a sample of similar students who had not had the LOGO experience. Student responses were audiotaped, and the audiotapes were used to analyze the effects of the LOGO course on students' van Hiele levels of geometric reasoning. Follow-up interviews will be conducted when the students complete their tenth grade geometry courses.

Angle Tasks

On the first task students are asked to draw an angle, then another that is different, and another that is still different in an attempt to discover which properties of the angle the student identifies as significant and to explore whether the student believes the number of possible angles to be limitless. In the next task they are asked to identify angles in two contexts (the first a "real life" drawing of a building, and the second a page of "textbook" drawings of angles), to justify their identifications, and to tell what they would have a friend look for to make similar identifications. This task explores definitions and class inclusions. In the third task students sort cut out models of angles and explain how they are alike or different to determine the properties the students identify for the angles. Next the students estimate the number of degrees and in which direction to turn an arrow in order to aim it at a target ball, then are asked to predict the degree of turn in the opposite direction as well. This task explores angle relations. Assessment of students' understanding of angle relations continues in the fifth task which explores complementary, supplementary, and congruent angles. The final task, which asks students to justify the interior sum of the angles in a triangle, provides an opportunity for the students to generate an informal proof.

Angle Descriptors

Operationalizing the levels for the angle topic was a necessary step in analyzing the interviews. That endeavor involved making decisions both about the nature of the tasks in which the students were engaged and about the characteristics of the levels themselves. Students in this sample were not expected to perform at the fourth or fifth van Hiele levels, therefore only the first three were considered. The following descriptors of student behaviors regarding angles are a synthesis of the Oregon State (Burger 1982) and Brooklyn College (Fuys & Geddes, 1985) descriptors of level indicators and of observations of student behaviors on the angle tasks. Where the Oregon State and Brooklyn College descriptore differed slightly, the investigator chose to match angle descriptors to the Brooklyn College descriptors because they were referenced so closely to translations of the van Hieles' writings.

First Level

In general, the student identifies, characterizes, and operates on angles according to their appearance. Specifically the student:

1) Draws angles independently.

2) Identifies angles in a simple drawing or more complex figure.

3) Names or labels angles and uses standard and/or non-standard names appropriately.

4) Includes irrelevant attributes when describing angles.

5) Excludes relevant attributes when characterizing angles, such as straightness.

6) Compares and sorts angles on the basis of their appearance as a whole, specifically not having the 90 degree referent for the sortings, making inconsistent sortings, or sorting by an inappropriate attribute.

7) Pairs congruent, complimentary, and supplementary angles on a looks-like basis.

8) Does not conceive of an infinite number of angles.

9) Does not think of properties as characterizing angles.

10) Does not make generalizations about angles.

Second Level

In general, the student establishes properties of angles and uses properties to solve problems. Specifically, the student:

1) Analyzes and compares angles in terms of their properties.

2) Identifies and tests relationships among angles within figures.

3) Recalls and uses appropriate vocabulary for relationships, such as corresponding angles are congruent.

4) States a litany of properties rather than determining necessary and sufficient properties when describing angles.

5) Is able to decentrate when trying to decide whether to turn a spinner to the left or right to aim at a target ball in a task to estimate turning angle. (Whether a student can orient turning relative to a spinner's position or whether the student orients turning relative to his or her own body's position is called the ability to decentrate.)

6) Accurately estimates angle measure by using known properties (such as right angles measure 90 degrees) or by insightful approaches.

7) Formulates and uses generalizations about properties of angles in problem solving situations and uses related language (all, every, none) but a) does not explain how properties are interrelated, b) does not use formal textbook definitions, c) does not explain subclass relations, d) does not see a need for logical explanations of such generalizations and does not use language related to explanations (if-then).

Third Level

In general the student formulates and uses definitions, gives informal arguments that order previously discovered properties, and follows and gives deductive arguments. Specifically the student:

1) Identifies necessary and sufficient properties that characterize angles.

2) Formulates and uses complete definitions, a) explicitly referring to them, b) accepting equivalent forms of definitions, and c) immediately accepting definitions of new concepts.

3) Explicitly describes relationships between properties, including sub-class relations.

4) Presents an informal argument or informal proof, justifying the conclusion using logical relationships, a) orders properties, b) interrelates several properties, and discovers new properties by deduction.

5) Recognizes the role of deductive argument and approaches problems in a deductive manner.

6) Explicitly uses "if", "then" statements.

7) Forms correct informal deductive arguments (implicitly using such logical forms as the chain rule and modus ponens), but a) does not grasp the meaning of deduction in an axiomatic sense, b) does not formally distinguish between a statement and its converse, and c) confuses the roles of axiom and theorem.

Analyses

For each individual's pre- and post-interview a matrix was completed indicating incidence of occurrence of each van Hiele level descriptor on each angle task. Student movement, either within or between levels was then documented by comparing the analyses of the two interviews. Two types of movement were documented: gain, and moderate gain.

Movement was judged to be a gain from pre- to post-interview when progress was documented between levels, or within levels when the student provided more correct answers on several questions within a given task. Gain also occurred within levels when a student employed a new strategy successfully in a problem solving situation, or provided several instances of increased accurate use of vocabulary.

Moderate gain was noted when a student engaged a task, perhaps with limited success, that s/he was unable to engage on the first interview, or when s/he employed a previously used strategy more successfully on the post-interview. Gain was also considered moderate when the student provided more correct answers on only a few questions within a given task, or when answers to questions were not substantively different from the pre-interview, but were provided more efficiently and confidently on the post-interview. Occasional instances of increased vocabulary were considered moderate gains.

As will happen in field research, __all__ of the first semester comparison students who were interviewed enrolled in LOGO during the second semester. For the purposes of the longitudinal study, then, there are no first semester comparison students. Results of the analyses of the second semester students' interviews are reported below.

RESULTS

Incidence of gain on van Hiele assessment of angle tasks by student.

Note: < symbol means angle, L refers to LOGO, and C to comparison.

Student	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6	Totals
	Draw <	Identify <	Sort <	< Measure	< Relation	< Deduction	
L 1				Gain		Moderate	
L 2	Gain		Gain	Gain	Gain	Moderate	
L 3	Gain		Gain	Moderate	Moderate	Moderate	
L 4	Gain	Moderate		Gain	Moderate		
L 5		Moderate		Moderate	Moderate		
L 6	Moderate			Gain		Moderate	
Subtotal	4 (1 mod)	2 (2 mod)	2	6 (2 mod)	4 (3 mod)	4 (4 mod)	22 (12 mod)
C 1		Moderate			Gain	Moderate	
C 2							
C 3					Gain		
C 4	Moderate		Moderate				
C 5	Gain	Moderate	Moderate	Moderate	Moderate		
C 6	Moderate						
Subtotal	3 (2 mod)	2 (2 mod)	2 (2 mod)	1 (1 mod)	3 (1 mod)	1 (1 mod)	12 (9 mod)

__Discussion__

The results of this study are descriptive. When the interviews were analyzed for between groups performance on van Hiele levels, there appeared to be no differences in level performance on either measure for the LOGO and Comparison students. The vast majority of student responses on both pre- and post-interviews were at the first and second van Hiele levels. Only two students, one LOGO and one Comparison, evidenced third level behaviors. Within subjects changes in level performance were then analyzed for evidence of treatment effects (see table). Across tasks the LOGO students evidenced 22 gains, 12 of them moderate. The Comparison students evidenced 12 gains, 9 of them moderate. Both qualitatively and quantitatively, then, the LOGO students evidenced more gains overall. Within task differences in favor of the LOGO students were noted for task 4, angle measure, task 5, angle relations, and task 6, angle deduction. These tasks are closely related to topics in the LOGO curriculum, which focused on geometric relationships. The purpose of the overall Project is to assess whether experience in a LOGO learning environment enhances students' understanding of geometric relations. In this study, qualitative analyses of van Hiele levels for this sample of students indicates that it very well may. Further analyses are planned, along with follow-up interviews when the students complete their tenth grade geometry studies.

By characterizing the van Hiele levels for the topic of angles the work has served to further operationalize the levels. Given that there is a student need to understand geometric relationships prior to undertaking high school geometry courses, the van Hiele model and interview instrument developed in this Project may assist educators in assessing such understanding. The application of the model to assess the impact of a LOGO experience should not only enhance understanding of students' thought processes in geometry but also provide necessary information about improving educational methods of promoting such understanding.

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EFFECTIVE PROBLEM POSING IN AN INQUIRY ENVIRONMENT: A CASE STUDY USING THE GEOMETRIC SUPPOSER

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The paper closely examines three considerations which must be taken into account in posing problems for use with the GEOMETRIC SUPPOSER in high school geometry classrooms: the kind and size of task, the amount of process instructions and the nature of the specification of the construction. The discussion is based on the work of three high school classes during 1985-86 where geometry was taught in a guided inquiry approach aided by technology. In the conclusion, the issue of students' geometric knowledge is also discussed.

INTRODUCTION

For the past four years, members of our group have taught high school geometry courses with the aid of the GEOMETRIC SUPPOSER, a microcomputer program. We characterize our approach to teaching geometry with the SUPPOSER as "guided inquiry" to distinguish it both from traditional instruction that relies on lectures and from "discovery-oriented" teaching.

As in traditional teaching, our approach includes class sessions during which a teacher introduces new material. However, we also use lab periods and class discussion periods where students must take a more active role in their learning. During a lab period students are given a task to work on, usually in pairs. Discussion periods are devoted to sharing data, conjectures, and supporting arguments that students generated in response to the lab task. Unlike a "discovery"

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approach, students in our classes are not expected to generate all of the theorems that they will use during the year. They are given theorems and gain understanding by applying their knowledge of these theorems to the lab tasks.

We quickly realized that the tasks posed to students in the lab are the main engine of the learning process in this type of approach. Therefore, we spent a lot of time designing "good" problems. In the course of writing these problems, we developed a set of principles which are delicately poised between guidance and student inquiry. For example, we concluded that the task should be rich and should yield more than a single, easily defined answer. It should not aks a student to rephrase or prove a given assertion. We also concluded that the instructions should be explicit without providing step-by-step directions and should suggest mechanisms for organizing and summarizing results.

To write this paper, we stood back from creating problems, analysed the activities that we used, and drew conclusions about the types of problems that have been most useful and effective in conjunction with the SUPPOSER. This analysis was based on problems used by three high school geometry teachers during 1985-86, collected student work and classroom observations.

Since the goal of analyzing our experiences with posing problems was very large, in the spirit of induction, we first limited our analysis to one type of SUPPOSER problem--construction problems or problems whose endproduct is a construction. As a result of the analysis of construction problems, we identified three important issues in posing problems: kind and size of task, the nature of the process instructions, and the way in which the construction is specified. We then sharpened our understanding of these three issues in light of our experiences with non-construction problems. In this paper, we use these three issues as a framework to present observations about students' responses to both construction and non-construction problems.

OBSERVATIONS

1. KIND AND "SIZE" OF TASK

There are different types of problems that can be posed in a SUPPOSER environment. One way to categorize problems is to group them by their endproducts. There are problems that ask for a specific construction, others request a solution to a construction problem that will work in many cases, still others request conjectures, w. a fourth type of problem focuses on the presentation of an argument or a proof.

Some of these kinds of tasks are more difficult than others. To illustrate, let us consider two types of construction problems: those that require an example of a construction and those that demand a general method.

A construction problem that requires only the construction of one example is much easier than a problem that requires a construction that must work on a whole set of figures. No conjecture is necessary. After doing the construction and checking that it works, one can be sure that the problem is solved. One doesn't have to try it on other figures or make an hypothesis. By contrast, a problem that calls for a general method requires a leap of faith or a proof in order to be convincing.

Another illustration of the differences between tasks comes from comparing construction problems and conjecture problems. Students seemed to enjoy the construction problems more than other types of problems. They worked hard and generated many good methods of solution. In addition, they used both inductive (e.g. trial and error) and deductive techniques (e.g. making use of knowledge of geometric theorems in deciding what constructions to try) more often when doing construction problems.

A second way to categorize tasks is to examine the role that the teacher assigns to the problem in the classroom. There are at least three different roles that a problem can play.

First, a teacher may want to have the students discover the theorems, postulates and definitions of geometry. Second, a problem can be used to allow students to become familiar with a given field of inquiry--a set of definitions or a particular construction--before the teacher teaches the material. A third way to use a problem is to pose problems that require students to apply concepts they have already learned to a new construction. These three approaches differ in the emphasis that is attached to specific conjectures. In some cases, the conjecture is important in and of itself, while in others it is not.

In addition to type of task, the size of the task may also effect student performance. First, some clarification of "size" is in order. A problem is a small one if it cannot be explored in a variety of directions or approaches, does not have many parts, and does not demand a lot of work from students.

One type of large problem requires the full range of inquiry activities. This type of problem calls for a student to derive general ideas from empirical experience and then to construct formal proofs of those ideas. In general, students did not have difficulties with this kind of large problem.

However, there was another type of large problem that did present an obstacle. Small problems may become large ones if after examining the small problem one changes features of the problem: replaces an altitude with an angle bisector or a median, generalizes from one type of triangle to other types of triangles, generalizes from triangles to other polygons, or generalizes from two dimensional shapes to three, four, or n dimensional shapes. In our inquiry approach with the SUPPOSER, it is important to change the nature of the problem, to ask "What if not?" questions. However, changing a problem in this way is not usually considered appropriate in school settings. Not suprisingly, students did not usually change the given problem.

2. PROCESS INSTRUCTIONS

On occasion, the teachers we worked with this year felt that the problems we had developed were both too large and too vague. They used process instructions to clarify the tasks and to break the problem into manageable parts.

We distinguish between two types of process instructions, those which define the task at hand and those which break down the inquiry process into separate activities. This second type of instruction provides guidance for students whose inquiry skills are not polished.

Different types of tasks require different kinds of clarification and specification. In a discovery task, where the teacher wants students to discover particular relationships, it is important to indicate which relationships are under scrutiny. In exploration type tasks, this kind of specification of relationships may be less necessary. The curriculum does not hinge on the discovery of a particular relationship.

Teachers also wanted to aid students whose inquiry skills were not strong. We had assumed that this type of instructions would be unnecessary because they would become "second nature" to students. However, this year's work with low ability students indicates that these kinds of instructions are important throughout the year. Explicit process instructions seem to be especially important when students must isolate one variable at a time.

It is difficult to know how much assistance is appropriate; too much aid can be constricting. One student commented in a derogatory tone:

> On the worksheets the problem is all mapped out for you. The problem that they just gave us, you have to find the solution and find the work. They give you step by step, number one, number two, number three.

We tried many strategies for giving aid. One strategy used by some teachers this year that we would not recommend is to include tables and charts as part of the problem. Teachers used the column titles as the instructions to guide the discovery process and to indicate the subject of the expected conjectures. We do not argue that tables should never be used in the inquiry process. They are appropriate tools for students to use after they have determined what measurements are to be made or organized in the chart.

Below is an example of a set of instructions that tries to strike the correct balance. These instructions accompanied a large problem.

> ...Make a brief restatement of what the problem asks for.
> Make an outline of the steps you think necessary to explore and solve your problem.
> You will collect, examine and study the data you think will help you to make conjectures about the relationships in the problem...
> A conjecture sheet will be due...
> The last segment of your project is to prove "formally" as many of your conjectures as possible, but no fewer than three...

These instructions seem general enough that they could be used with many problems. They get students to step through a model of the kind of inquiry desired by the teacher. Such directions do not restrict students who no longer need this type of organization, yet they are helpful to others who may need them.

3. SPECIFYING THE CONSTRUCTION

In some problems, a construction is the endproduct of the inquiry; in others, it determines the situation which will be examined. In either case, a construction must be specified.

Analysis of this year's work suggests that these specifications should be couched in a general form. It is an interesting finding in light of the traditional focus in geometry courses on conceptual, general terminology (e.g. exterior angles) as opposed to labels which are specific to a diagram (angle BCD). The expectation in a traditional class is that students will translate from the conceptual terminology into properties of a given figure in terms of its labels and back again.

Using the SUPPOSER, we have the same expectations.

Since the specifications of the construction should be general ones, diagrams can be problematic. In most cases, a diagram's role in a problem is to simplify, minimize, or clarify written information. The figure is a form of shorthand, providing information succinctly. In such cases, the diagram acts as a model, not as a specific entity. It has a reference field that is larger than itself. However, it is not possible to produce a 'global' or universally valid picture. There is always the risk that a certain facet of the infinitely many characteristics of the picture will be identified by the solver as a special property.

Part of the rationale behind the SUPPOSER was to provide a tool that could help students understand that a picture is a special case and that examining one picture is part of a larger process that includes viewing many special cases and not one static example. Thus, since the SUPPOSER makes examples easy to create, work with the program should stress the importance of examples and at the same time should minimize the importance of any single example.

It is important to note that the diagrams that we provide with a problem are very different in character from those that students make on their papers after copying from their screens. Our diagrams are single instances of a set of possible diagrams which represent the whole set. By contrast, students' diagrams are specific cases which only represent themselves. A different word for each kind of diagram is appropriate.

In our classroom observations, we noted that students had difficulty developing an understanding of the representative nature of diagrams. One suggestion for aiding this understanding is to qualify every diagram that accompanies a problem by saying: "Your construction should look something like this."

A second observation about the specification of the construction is that it varies in explictness according to the type of problem. Construction problems are different from conjecture problems. Since constructions are the "given" in a conjecture problem and the task is to generate assertions about the construction, the construction can be specified in a direct manner without compromising the task.

In contrast, the construction in a construction problem must be described in a general way, in formal language that will appropriately describe all possible solutions. Giving the construction in a step-by-step manner will provide the students with the solution to the problem.

CONCLUSION

In addition to the three considerations that we have already examined, there is at least one other factor that should be taken into account. The amount of geometric knowledge students have is crucial to understanding their response to a problem. We have observed students growth over the course of the year both in geometric knowledge and in comfort with geometric inquiry using the SUPPOSER. This comfort has both a technical aspect (i.e. how do I use the program?) and a more abstract aspect (i.e. how does inquiry work in this field?). This is an important issue, but it is difficult to investigate since one cannot usually give the same problems to beginning students and to students who have had instruction in geometry.

This evolution on the part of students can be illustrated by the change in students' relationship to diagrams. In the beginning of the year, students treat diagrams that are supposed to serve as models as if they are examples. This mistake disappears later in the year. Progress is also evident in students' approach to a problem. We believe that students attack a problem by developing an initial conjecture and by refining that conjecture in the light of further experience. How does one develop an initial conjecture? When students have little geometric knowledge, conjectures are likely to be a hit or miss affair. As the year progreeses, students derive their first conjectures from their geometric knowledge and then use the SUPPOSER to elaborate and verify their conjectures.

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GEOMETRY AND THE COMPUTER ENVIRONMENT

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In acting as respondent to a grouped set of papers on Geometry and Computer Environment, I will try to bring the reported research into perspective and suggest directions for future work. I wish however at the outset to acknowledge the difficulties I have faced in undertaking this task. Firstly, several of the papers are in French and, although my competence in this language is reasonable, I have had difficulty in understanding some of the more complex arguments. Secondly, some of the software forming the basis of the research projects is not familiar to me. I have therefore been forced to rely solely on the written descriptions as a means of grasping its structure which is not altogether satisfactory. Thirdly, I have had very little time to both absorb and reflect upon the findings.

In order to begin to think about the area of geometry and computer environment it seemed sensible to first consider what geometry actually is. Bishop, 1983, proposed the following definition:

> "Geometry is the mathematics of space, and mathematicians approach space differently to artists, designers, geographers, or architects. They search for mathematical interpretations of space. Mathematics educators, therefore, are concerned with helping pupils gain knowledge and skills in the mathematical interpretations of space. Depending on many factors, such as one's philosophy of mathematics education, geometry education can range from learning well-established geometries, such as Euclidean or more modern transformation geometry, to developing the pupil's own geometrical ideas" (Bishop 1983 p.175).

Bishop goes on to contrast two approaches to geometry; one w emphasises "other peoples investigations" (for example learning theorems) and the other which starts from "actual mathematising of space by pupils involving --- experimentation with materials and representations, classifying, defining, and analysing why certain relationships occur" (Bishop 1983 p.175).

There would appear to be many research avenues still to explore. We are still relatively ignorant about the learning of spatial and geometric ideas in general, as pointed out by Bishop, and even more ignorant as to the influence the computer might have on such learning. There is enormous potential here for mathematics education research - the computer, perhaps uniquely, allows children to construct graphical representations, see the result of their constructions, manipulate and change them and use them in more sophisticated designs. The research described gives us valuable insights into this potential. I would like to distinguish three broad research themes:-

I. Research which explores the development of children's understandings of geometrical and spatial meanings and how progression (for example, from globality to increased differentiation) might be affected by computer "treatments".

II. Research which investigates the "training" influence of computer environments on different spatial abilities for example, on the two ability constructs distinguished by Bishop (1983):-

> "1. The ability for interpreting figural information (IFI). This ability involves understanding the visual representations and spatial vocabulary used in geometric work, graphs, charts, and diagrams of all types. Mathematics abounds with such forms and it concerns the reading, understanding, and interpreting of such information. It is an ability of content and of context, and relates particularly to the form of the stimulus material.
> 2. The ability for visual processing (VP). This ability involves visualization and the translation of abstract relationships and nonfigural information into visual terms. It also includes the manipulation and transformation of visual representations and visual imagery. It is an ability of process, and does not relate to the form of the stimulus material presented."(p184).

III. Research which takes as a starting point the design of geometric computer-based situations which confront the students with specific "obstacles" and seeks to identify student/computer strategies, the meanings students construct and how these meanings relate to the representations made available by the computer "tools". Such research more or less explicitly uses the computer to create didactical tools to facilitate the acquisition of specific mathematical conceptions or understandings.

The papers grouped within the topic area Geometry and Computer Environment appear to fall within the research themes II and III above and it would seem appropriate therefore to consider together the papers within each of these separate themes. In theme II, I would suggest there are three papers: J. Olive and C. A. Lankenau, The Effects of Logo-Based Learning Experiences on Students' Non-Verbal Cognitive Abilities; S. Scally, The Effects of Learning Logo on Ninth Grade Students' Understanding of Geometric Relations; and R. Pallascio and R. Allaire, Les Habiletés Perceptives d'Objets Polyédriques. I will briefly summarise them below:

Olive and Lankenau investigated the effects of a Logo environment on four subtests from the Cognitive Abilities Test - the Figure Classification Subtest, the Figure Analogies Subtest, the Figure Analysis Subtest and the Equation Building Subtest. They found after using analysis of covariance to compare performance between Logo and Comparison groups that the Logo groups tended to achieve higher scores on these tests, a finding which was more marked in the first two tests mentioned above. They also pointed to a developmental effect and the importance of the teacher's "instructional expertise and educational philosophy". Scally, using a clinical approach with pre and post tests investigated how the Logo environment might provide experiences at the third van Hiele level - that of geometric relationships and deductive structure. Her work involved the operationalising of van Hiele levels for the topic of angle as a basis for the analysis of student interviews which took place before and after a Logo experience lasting over a semester. Her work suggests that both qualitatively and quantitatively the Logo students evidenced more overall gains. Pallascio and Allaire developed software in order to represent and manipulate 3-D objects on the screen. The software used was Logo, the rationale being the extensibility of the Logo language. The research aimed to develop the abilities of observation, abstraction and communication by means of visualisation activities, structuration activities and transfiguration activities respectively. Pre and post tests were used and the results indicated that there was an improvement in visualisation in the experimental group but that it was the boys in this group who seemed to account for the differences observed.

In theme III, I would suggest the following papers can be grouped: D. Chazan, Similarity: Unraveling a Conceptual Knot with the Aid of Technology; E. Gallou-Dumiel, Théorème de Thalès et Micro-Ordinateur; C. Hoyles and R. Noss, Seeing What Matters: Developing an Understanding of the Concept of Parallelogram through a Logo Microworld; C. Janvier and M. Garançon, Understanding Feedback Systems; I. Osta, L'Outil Informatique et l'Enseignement de al Geometrie dans l'espace; and M. Yerushalmy and D. Chazan, Effective Problem Posing in an Inquiry Environment: A Case Study using the Geometric Supposer. Each paper identified specific pupil strategies. which relate closely to the software used and the mathematics modelled so cannot be summarised in any general way. I have however attempted below to capture some significant points in each of the papers.

Chazan used the Geometric Supposer to provoke a confrontation with some naive theories that students hold of the mathematical concepts - similarity, proportional reasoning, dimensional growth relationships and correspondences in right triangular similarity. Gallou-Dumiel used the software MacDraw, to investigate the difficulties students might have in using the theorem of Thalès. She compared the strategies used and the problems thrown up in this computer environment with those observed in a paper and pencil environment. Hoyles and Noss used a Logo based parallelogram microworld to investigate how students came to understand the "essence of a parallelogram" through modifications of the formalism of the given program. They identified ways in which students progressively became aware of and generalised the relationships embedded within the parallelogram procedure. Janvier and Garançon presented a detailed analysis of difficulties in understanding feedback systems and how software could be designed specifically to assist in overcoming these difficulties. Osta set out to identify ways the computer might help in overcoming problems of representing 3-D objects in 2-D. The research described teaching situations where there was a dynamic interaction with the 2-D designs and where the aim was to solve a range of problems constructed in order to develop specific geometric knowledge. Two types of software tools were used, MacSpace, where operations can actually be done on the 3-D object itself and MacPaint, where operations can only be undertaken on the figurative design. Student strategies in these two environments were compared and in particular a development from local to global activity described. Yerushalmy and Chazan examined some issues arising from their work using the Geometric Supposer. The paper makes important points about the didactical conditions which facilitate student learning and distinguishes the important influences of: the kind and size of task, the amount of process instructions and the nature of the specification of the construction.

DISCUSSION QUESTIONS

I would like to raise a series of questions some or all of which might be useful as starting points for discussion of the research papers to be presented:

1. What type of software is used and what is the rationale for its choice and the design of the computer environment?

2. What is the purpose of the research:- to assess the influence of the computer environment on specific mathematical conceptions or on more general spatial abilities? If the former how is the nature of the conceptions or misconceptions identified and if the latter what is the theoretical framework adopted?

3. What is the geometric knowledge investigated and how is it modelled in the computer environment?

4. What is the influence of the computer on the learning environment? It is possible to approach such a question in two ways - how do the computer tools affect the representation of the knowledge; and how does the computer environment effect the process of learning?

5. What is the relationship of the understandings developed in the computer context with other contexts? It is plausible to suggest that in unfamiliar simulated environments on the computer embedded relationships are more easily discerned but how does the recognition of these relationships transfer across contexts?

6. What is the role of the teacher and the nature of the teacher's intervention in the learning process?

7. What is the student reaction to the software - for example, were they motivated to experiment? Could they cope with the technical aspect? Were any individual differences or differences between groups observed?

RESEARCH METHODOLOGIES

In considering the papers in this topic it is possible to discern a wide range of methodologies in terms of the age of the pupils, sample size, time scale of experiment, research design (case study, use of comparison groups) and context (research laboratory or school based). Data sources also vary: within theme area II, results tend to be based upon the administration of tests of spatial abilities or the identification of these from interview protocols; within theme II, the research data tends to be the identification of student strategies whilst attempting the tasks presented.

COMMON THREADS IN THE PAPERS

1. An important common thread in the papers is that all the computer environments described focus on the development of student understandings or abilities as a result of student/computer interaction. No computer environment exhibited characteristics of CAI, that is a concern for the explicit transmission of facts and skills. Thus the computer is not viewed as a teacher but as a tool for the manipulation of graphical representations which has the power to provide informative feedback. The software is thus seen as an environment for mathematising space with instructions are added by printed material or teacher intervention. The software is therefore not content specific but adaptable for different instructional purposes or different groups of students. The software environments also embody formalisms allowing the user to represent the effect of his/her actions.

2. Leading on from point 1, there seems to be a common rationale for the computer use; that is that perceptual ambiguities/conflicts which inevitably arise from student activity in an interactive computer environment will provoke students to be suspicious of perceptual cues, rethink their original intuitions and try to make explicit their geometric knowledge in relation to the graphical feedback.

3. Most of the papers described students working in pairs at the computer. This would seem to imply a role for the computer as a tool to provoke discussion.

4. Several researchers identified the importance of the computer as a means of "seeing the general in the particular", that is provoking a move away from the manipulation of specific objects on the screen to seeing them as representative of a class of objects or as generic examples. Thus the formalism required in computer use is seen as a way of assisting the user in distinguishing the central mathematics from the peripheral and identifying the invariants which define this mathematics.

FURTHER QUESTIONS

1. Many papers pointed to the importance of the role of the teacher. There is a need for more explicit research on the nature of teacher intervention and how this affects learning in a computer environment. How for example is it possible to guide students so they do not work randomly (which often is the quickest way to the 'result') and therefore do not grapple with the mathematical meanings embedded in the activity? How can the pupils be moved to adopt a more reflective/deductive approach which takes into account both global and local features of the task? How can and should guidance vary in terms of level of abstraction and generalisation with respect to different pupils?

2. More comparative research would be helpful; that is comparisons of outcomes on tasks embedding specific geometric knowledge constructed within different computer environments and in non-computer environments.

3. Research might usefully consider how robust are the misconceptions identified in different contexts and how far do they arise from particular types of software environments or pedagogic practices? A detailed analysis of the basis of misconceptions, visual or symbolic, for example would also be of considerable interest.

4. More specific attention in this research area could be focussed on differences in approach or outcome which might reflect group (for example gender, ethnicity, class) as opposed to individual characteristics.

5. Since most of the research describes students working in pairs, it would seem important at some point to attempt to distinguish the influence on learning outcome of the student/student discussion from the computer interaction.

6. There would seem to be a need for more research within a developmental framework involving longitudinal study and evaluation of children working in a computer environment.

CONCLUSION

There remains an enormous amount of research to be done on both the effect of interaction in a computer environment on general spatial abilities and the influence on students' construction of geometrical knowledge of activity with software tools which allow the manipulation of graphical objects on the screen. All the studies presented in this group acknowledge the power of the computer for learning geometry. This power was very evident in my own attempts at grappling with the research papers. I was largely working outside a computer environment and was forced therefore to visualise the effects of the manipulations described without the chance to experiment and receive feedback. The difficulties I experienced in doing this gave me much food for thought!

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In-service teacher training

TEACHERS' VIEWS AND ATTITUDES ABOUT CLASSROOM COMPUTER USE

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Computers may bring deep changes in the nature and contents of mathematics teaching and learning. This study was designed to yield information on the nature of the issues that teachers face when they try to use computers in their classrooms. Fifteen teachers were involved in this project, covering the grade level range from 6th to 12th grade. The software most used were demonstrations and practice programs. Teachers tended to use the programs that they knew best and perceived as akin to their own teaching styles. It is concluded that the teachers who are willing to use the computer in the classroom should be provided with the opportunity to reflect on the nature of the learning activities intended for the students.

Computer science and computer technology may have the potential to bring about deep changes to mathematics education. Computers can be used for individualized learning, as "electronic blackboards", as supercalculating machines, to promote project work, to conduct investigations or problem solving activities, or simply as motivational devices. According to Taylor (1980), the computer can be seen either as a tutor, a tutee, or a tool. Most of the research concerning the impact of the computer in mathematics education has focused on the presumable effects on the student. However, if we regard learning as a continued social process, we must also pay attention to the ways teachers regard and react to the new technologies (Phillips et al., 1984).

In our country there is already some experience of using the computer in club activities, usually with high involvement and enjoyment from pupils and teachers. Interested students, very often not the brighest ones on the discipline of mathematics, sometimes develop complex programs related or not to curricular topics, which means a significant effort of independent study. However, one may ask if the fundamental effects of the computer in mathematics learning will come from this kind of activities or from what will happen within classrooms.

THE STUDY

Objectives. Throughout 1986, some mathematics teachers were encouraged to use computers in their classrooms. We were interested in studying the eventual new educational phenomena that could develop in such a situation, namely new kinds of interactions both among students and among students and teacher. We were also concerned with teachers' beliefs and views about the use of computers regarding their classroom activity.

More specifically, the main objectives of this investigation concerned the ways teachers react to the different problems they face when they take the computers into the classroom: (a) what are their purposes in the use of computers; (b) what kind of software do they tend to choose; (c) how do they use it?; and (d) what kind of new educational issues have the teachers to face and how do they react to them?

The teachers. Fifteen teachers were involved in this project. They were teaching in middle (grades 5-6) or secondary (grades 7-12) schools. Some of them had previously learned something about programing (generally using BASIC), although attaining very different levels of expertise. A small number of them contributed to the conception or the construction of educational programs. However, most had, at least at the beginning, only a very short experience of work with computers.

During the first semester of 1986, these teachers participated in a seminar at our Department, in the University of Lisbon, dedicated to the educational use of computers. The seminar was organized in weekly sessions of about two hours. The work included sometimes solving and discussing problems through programing activities, but it was generally dedicated to discussing the educational value of different programs and the possibilities of their use in the classroom. In some occasions, the experience of one teacher that had used a particular program was reported to the group.

During the seminar, there was no unanimity concerning teachers' involvement, interests or views about computers and their use in the classroom. On the contrary, while some teachers were always present and knew all the available programs, others missed a variable number of sessions and their contact with some programs was limited to short demonstrations.

Available programs. About twenty programs were selected to be used whenever necessary. Obviously, the possibilities of using computers in the classroom was not strictly limited to these programs or even to the existence of previously prepared software. Short programs written by the teacher in the classroom or student programing activities were also educational possibilities to be considered and encouraged.

The suggested programs can be classified in different groups according to their prevailing characteristics (Hatfield, 1984):

(1) Demonstration - dedicated to help teachers to introduce or explore new content topics, generally in a pre-established way, although eventually with some interactive characteristics;

(2) Educational games - providing challenge to strategy and, in most cases, involving some mathematical idea;

(3) Practice - very often with gaming features, but clearly intending to help students to become proficient in a given kind of operation, rule or skill;

(4) Problem-solving - demanding for a way to solve a problem or a situation affected by random factors;

(5) Simulation - representing an aspect of a real life situation and allowing students to change values for the variables or parameters and study the effects of their choices.

(6) Tool - utilitarian programs, for example for drawing graphs of any given functions.

None of the suggested programs was a tutorial. Obviously, most of them presented characteristics belonging to more than one of the categories mentioned above.

Data recording. Two kinds of report sheets were developed to record the relevant information after every lesson in which computers had been used. One of those report sheets focused on functional aspects about hardware and software, objectives of the computer use in that particular lesson, how students' activities were organized, and main positive and negative features of the program in relation to the mentioned objectives. The second report sheet asked for the most relevant issues that occured from cognitive, affective, social, and global (educational) points of view. Specific questions were included to help the record (for example: learning difficulties about the involved concepts, students initiative in developing strategies, interest or enthusiasm of the students, cooperation, participation of boys and girls, learning atmosphere, relations among students and among students and teacher).

The report sheets were completed by the teacher at the end of the lesson. In some cases, lessons were observed by one of the researchers and short discussions were conducted with the teacher to talk about what happened in the lesson and discuss the most salient occurrences.

RESULTS

Considering the report sheets that were returned, we notice that only seven different programs were used by thirteen different teachers. Table 1 describes the situation, indicating the names of those programs and, for each of them, some conditions about the way it was used (grade level, number of different classes, number of computers in the classroom), and the classification type according to the categories stated above -- with (x) we refer to the prevailing characteristic and with (*) other significant ones, whenever this is the case.

Table 1 shows that most classes (17/21) belong to grade levels 7 and 8 (the two first levels of secondary schools). This cannot be related to the classes of these particular teachers -- in general, secondary school teachers are responsible for classes in both levels 7-9 and 10-12. This aspect may partially be attributed to the nature of the existing software and perhaps also to the fact that teachers appear to be more comfortable in using the computer with younger rather than with older students. Another aspect, maybe even more significant, is that simulations were virtually ignored and that demonstration and practice programs were preferred.

TABLE 1

Programs Used by the Teachers

Program	Level	Classes	Computers	Characteristics (1)	(2)	(3)	(4)	(5)	(6)
Number Facts	5/7	2/2	1	x		*			
Grandprix	7	3	1	*		x			
Relative Numbers	7	4	1	x					
Sistems	8	1	3		x	x			
Equations	8	3	1		x				
Estimates	8	3/1 (a)	2,5,3/1 (a)			*	x		
Function Graphs	11	1/1 (b)	1/1 (b)						x

Note: (a) and (b) indicate situations that, for their special interest, will be referred in detail later in the text.

However, a deeper analysis needs to take into account the written reports in the context of our knowledge about the particular teachers, their involvement and their views about the use of computers and also their usual ways of organizing classroom activities. From this point of view, some interesting observations seem to be:

1. Teachers made generally a quite positive evaluation of the use of computers, considering that the choosen program had corresponded to the stated objectives and referring as especially positive the involvement of the students.

2. Teachers tended to select, for their own use in the classroom, programs with which they were very familier. In some cases, a program was only used by its own author (or co--author) and, at most, by colleagues of the same school. Moreover, some teachers, whose involvement in the project was not very strong, limited their use of computers to the few programs about which they had participated in detailed demonstrations and discussions about their educational possibilities.

3. Teachers seemed to avoid programs that can disturb their usual way of managing classroom activities. That is probably

why demonstrations and practice programs were preferred and why, in several cases, one single computer was used. Educational games were used in a similar way, in teacher-centered lessons. Or, alternatively, they served to support some kind of "extra-lessons". Programs of problem-solving type were used by the teachers who regularly organize in their classes practical team work as students' activities, generally associated to the simultaneous use of several computers.

4. Most lessons did not include any tasks for students but those provided by the computer program. On the other hand, activities suggested by the program were not, in most cases, retaken in any subsequent lessons. It appears that there was an identification between the objectives of the lesson and the specific objectives of the software.

However, this way of using the computers in the classroom, largely dominating at the beginning, gave partially place in the last period to some more original and creative ways of organizing students' activities. The two following examples correspond to notes (a) and (b) from Table 1:

(a) In an 8th grade class, the teacher organized her students in five different groups and prepared six lessons about related subject matters involving errors, aproximate values, and racional numbers. During each lesson one of the groups worked with the program "Estimates" which deals with estimations to a given decimal place, demanding the solution of a random situation in the least number of trials. It intends to promote the devolopment of estimation ability and strategic ideas, and to provide some practice. At the same time, the other groups worked in different tasks, not involving the computer, on related subjects. The sixth lesson was used to organize a contest among the groups using the program. Another teacher, using the same program in a single class period with several computers, decided for the subsequent lesson to choose some situations as if they were generated by the computer, wrote them on the blackboard, and discussed with the class adequate strategies and methods to manage those situations.

(b) In an 11th grade class, the teacher prepared a worksheet suggesting the exploration of different situations involving graphs of trigonometric functions and relations among them, and motivated her students to study those situations with the help of the computer. The students worked in small groups and used a graphic drawing program. He the computer was used as a tool and was not, in any sense, the focus of the lesson.

IMPLICATIONS

The introduction of the computer in the classroom is charged with pedagogical implications. The computer can be used to reinforce traditional teaching practices or as an instrument of change. A teacher who is willing to use the computer in the classroom should reflect on the nature of the learning activities that he or she intends to proportionate to the students.

Computers provide an opportunity for change (Fey, 1984). However, one should not expect that teachers will modify their styles from one day to the other. Therefore, it is necessary to have a diversified set of programs and computer related materials, that can be used in a flexible way allowing teachers to choose according to their particular inclinations.

The most important and urgent changes in mathematics teaching and learning are not likely to occur just by a spontaneous process. To foster change, to stimulate teachers to perceive the potentialities and limitations of computers and of courseware packages in relation to students' difficulties and progresses, it is necessary a continued process of reflection and exchange of experiences. That needs to be taken into account by those charged with the development of software, support documentation, and the organization of teacher training.

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CHANGES IN MATHEMATICS TEACHERS VIA IN-SCHOOL IN-SERVICE EDUCATION

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A model for a teacher in-service program concentrating on changing teacher attitudes and behaviors is discussed. This model was implemented in 7 socially disadvantaged schools over a two-year period. Difficulties of implementation and initial teacher reaction to the project are discussed. Both subjective and objective measures assessing teacher change and resultant pupil achievement are presented.

Introduction

The attempt to understand and solve the problem of "why Johnny can't add" has generated great activity among mathematics educators both in the realm of educational psychological research and in the area of curriculum development. Despite the resultant increased understanding of learning difficulties in mathematics and the abundance of approaches and materials, many of our schools are still severely plagued by low mathematics achievement.

The reasons for this situation are complex: sometimes there is a concentration of "problematic" children in a school, such as in the case of schools in disadvantaged areas, which makes instruction particularly difficult; sometimes the teachers teaching mathematics have not been trained for their task, having been recruited from other fields; sometimes qualified mathematics teachers have not kept themselves abreast of changes in their area, and sometimes all, or some combination, of these factors exist simultaneously. It is certainly a shock to specialists in mathematics education to visit a classroom and discover that advances in their field have simply not reached the mathematics teacher.

Usually in-service education is proposed to ameliorate such circumstances. While the short-term effects of in-service education have been cited, the long-term effects have not been so evident. As pointed out by Richard Skemp (1985;1986), in-service has tended to fail as a long-term remedy essentially because the focus has been on individual teachers in artificial contexts who find it difficult to apply what they learn in schools where the old ways dominate. It has been suggested that in-service must deal with teachers in their own environment in order to be really effective.

The purpose of this paper is to describe the operation, and evaluation of a comprehensive in-school in-service approach for improving mathematics teaching which operates under assumptions similar to Skemp's. At this point in time, the Department of Science Teaching at the Weizmann Institute of Science has applied this approach for two years in two junior high schools and for one year in five more schools. All mathematics teachers in each school have been involved (n=40), a condition for participation in the project. The duration of the total intervention is currently scheduled for a three-year period.

The Problem

All of the junior high schools participating in the program were identified as low achieving in mathematics. Initial testing of the children at all grade levels (7 to 9) indicated that knowledge of prerequisite topics (simple fractions, decimal fractions, and elementary geometry) was very low, the average score by school being about 57% correct on minimal competency tests. Achievement tests measuring knowledge of the junior high school curriculum itself yielded scores ranging by school from 20% correct to 55%. (Average "advantaged" schools tend to yield scores of about 85 on the prerequisite tests and 73 on the curriculum-based tests.)

In general, the schools were characterized by lack of a clearly defined mathematics program suitable to their pupils. Moreover, classroom instruction was found to be in need of serious improvement. Mathematics lessons tended to be of the conventional "chalk and talk" variety, and reflected little forethought or planning on the part of the teacher. Class exercises, homework assignments, and explanations were often unrelated; lack of prerequisite knowledge received little systematic treatment; lower and higher level cognitive skills were dealt with inadequately; and mathematical errors occasionally crept into teachers' explanations.

The In-Service Program

The major goal of the program undertaken to alter the above situation was to make mathematics teaching in these schools more effective and more varied. The main focus of the program was to be the mathematics teacher under the assumption that lasting change could only be accomplished in this way. Program planners were definitely of the opinion, so aptly expressed by Lee Shulman (1979), that "...any changes in curriculum and instruction must be mediated through the minds, motives and activities of teachers."

In this belief, teachers were provided with both individual and group assistance. Throughout the school year the teachers were counselled individually by "master" teachers, who observed their lessons at regular intervals, discussed instructional topics with them and offerred advice along the way. In addition, group activities were provided for the teachers through the observation and analysis of demonstration lessons given by colleagues or "master" teachers and, more importantly, in the form of workshops aimed at enriching the teachers' knowledge of mathematics and exposing them to various strategies and teaching aids.

The major aim of this intervention was to alter the teachers' behaviors in the classroom by encouraging them to be more reflective about what was occurring during a lesson and by offering them alternative instructional approaches. Teachers were expected to pass through a number of phases in becoming better teachers. First, their consciousness would be raised as to the need for change and as to the various alternatives available. Next, their motivation to change would increase and under the friendly coaxing of a "master" teacher they would become more willing to try new ways. Finally, they would actually try out new strategies and gradually learn to put them to use effectively in the classroom.

In addition to the program's efforts to effect instructional change on the personal level, intervention was done on an organizational level. Such intervention was carried out with the intent of establishing a unified plan for mathematics instruction suitable to the specific pupils at each grade level in each school. Decisions were made at the start of the project by staff from the Weizmann Institute in consultation with the teachers themselves, as to which curricular topics would receive greatest emphasis, which textbooks were most appropriate, and how remedial help would be given for lack of student prerequisite knowledge.

The operational model developed for this in-school in-service program is depicted in Figure 1. As indicated, change in teacher classroom behaviors together with the use of appropriate curricular materials are expected to produce an increase in pupil mathematics achievement.

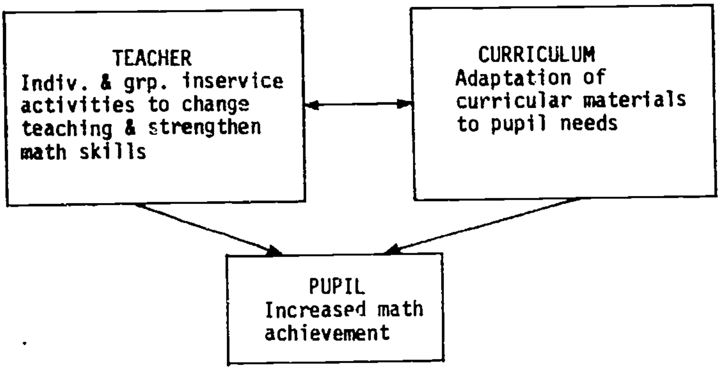

Figure 1: Model of in-school intervention and effects.

Implementation Difficulties

There were many hitches to the smooth implementation of the in-school intervention program which should be mentioned before presenting evaluation findings. Essentially, difficulties were of two kinds: technical difficulties and psychological barriers. On the technical side, problems were encountered in the scheduling of group activities and in selecting a suitable and convenient location for them. Various solutions were adopted, ranging from group meetings held in school during school hours, to meetings held in school after hours, to meeting held at the Weizmann Institute on released-time.

A more serious technical problem which still persists today is that master teachers often find themselves with insufficient time for serious discussion and planning with individual teachers, and for developing a close on-going working relationship with them.

Psychological and attitudinal barriers were, and still are in part, the most critical impediments to project implementation. Teacher participation in the program was mandatory in two schools, being dictated from above. In the other five schools, the majority of teachers accepted the project which obligated the remaining teachers to participate. While local school authorities and school principals were strongly supportive of the program, the teachers themselves were only mildly interested. As indicated by results of a needs assessment questionnaire completed by the teachers at the outset of the project, teacher attitude was generally positive towards the proposed group activities and generally negative towards receiving individual assistance or towards any activity which might be construed as personally threatening (such as observation of their classes or preparation of a lesson to be viewed by fellow teachers). Clearly, the first year of the in-school intervention pr[illegible]am in all of the schools was greatly devoted to overcoming suspicions and to creating a relaxed, cooperative environment in which teachers and master teachers could work together to accomplish common goals.

Evaluation

Evaluation of the in-school in-service program focused first and foremost upon the perceptions and behavior of the teachers, and concentrated only secondly upon their pupils' achievement and attitudes. Measures included classroom observations, interviews with teachers, questionnaires for both teachers and pupils, and finally achievement tests administered to the pupils. Some of the measures were adapted from already existing ones, while others were constructed specifically for this project.

After one year of the program, some positive change in teacher attitudes and behavior could be discerned, mainly from classroom observations and interviews with the teachers. For example, teachers were more willing to admit to both teaching deficiencies and even to their lack of in-depth knowledge of the mathematics content of the curriculum. Teachers concurred among themselves that even if their actual teaching had not undergone great alteration, their perception of mathematics teaching had been transformed. Some teachers were able to translate such cognitive changes into behavioral changes. Towards the end of the first year, their lessons were better planned, learning goals were more clearly defined, homework assignments were more carefully selected, and math games, worksheets, group activities and investigative learning were gradually entering their repertoire of instructional approaches. This is a start in the righ. direction, however it is not enough. Since behavioral change is a slow process, it can only be hoped, that after several years of such a program, stable and lasting teacher change will occur.

Results also indicate that mathematics learning has already been positively affected as a consequence of this program. At this stage it would be difficult to claim that change in pupil learning is a direct result of teacher change, since teachers are, for the most part, only at the initial stages of altering their instructional approaches. Increased pupil achievement at this point, as measured by standardized achievement tests, must be attributed to the organizational intervention of the project staff into the curricular content, pacing of topics and choice of basic learning materials. There is some evidence, however, that these early gains will help convince the teachers that change is in order and indeed possible.

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ELEMENTARY SCHOOL TEACHERS' PERCEPTION OF MATHEMATICS AND MATHEMATICS TEACHING AND LEARNING: TWELVE CASE STUDIES

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Abstract

Four problems affected in the past and still affect mathematics teaching and learning. These are the insistence put on symbolism, the great influence of formalism, the heavy presence of behaviorist learning theories and the focus that many teachers put on their pupils'answers instead of on their reasoning. Any solution of these problems requires a change in the perceptions that many teachers share about mathematics and mathematics teaching and learning. To that end, a new approach for teacher training, approach based on concept analysis within the framework of a model of understanding, was elaborated and tested. This paper presents the part of the rechearch which is concerned with twelve case studies led within a control and an experimental group. The results show that the aforesaid approach led teachers to a more constructivist perception of mathematics and of mathematics teaching and learning as well.

PROBLEM AND HYPOTHESIS

Four problems, tied to the nature of mathematics, to the philosophy of mathematics, to epistemology and to learning theories, plagued mathematics education for years. And they still do. These problems are:

- the insistence put on symbolism and notation (Ginsburg, 1977),
- the great influence of formalism (Davis et Hersh, 1980),
- the heavy presence of behaviorist learning theories,
- the exagerated focus that many teachers put on their pupil's answers instead of on their reasoning.

As these problems persisted through both time and reforms, we must look elsewhere than in curriculum transformations for their solution. As we have already said, these problems are tied to philosophic perceptions of mathematics and to conceptions of learning. Thus we should address the latter if we want to remedy the situation and improve mathematics teaching. And in this undertaking, teachers are the targets we should favour: firstly, because they are the true workers of mathematics teaching and thus, they are the pivoting factor of any genuine renewal of that teaching (Colmez, 1979). But also because they have been neglected in all past reforms in the senses that they seldom received the necessary profiency courses and that their pre-service training was often barely adequate ((Freudenthal, 1977; Beltzner et al., 1977; Colmez, 1979; Robitaille and Dirks, 1982). A new approach of teaching training, whether pre or in-service, seems to be a sine qua non condition to any possible solution of the problems at hand.

Such a new approach was test in the research. As described in Bergeron et al. (1981) and Herscovics et al. (1981), this approach is based on the initiation of teachers to the analysis of mathematics concepts within the framework of a model of understanding (Herscovics and Bergeron, 1983). This approach of links the psychological, epistemological and pedagogical aspects od the teaching of mathematics to the mathematical content itself. Mathematical concepts are examined in the context of the model of understanding; the focus thus put on the cognitive aspects leads the teacher to reflect on the mental processes involved in the elaboration of a particular notion. This, in turn, may also lead the teacher to restructure his teaching strategies in order to conform more closely with what he has understood about the learning of the notion. According to our hypothesis, this approach should lead the teachers to a more constructivist perception of mathematics teaching and learning.

EXPERIMENTATION AND TOOLS

We tested our hypothesis on an experimental group (n=18) of teachers enrolled in an in-service program at the "Faculté d'Education Permanente" (F.E.P.) at the Université de Montréal. These teachers took a 45 hour mathematics education course in which our approach was used. Simoultaneously, we appealed to a control group (n=16) of teachers also enrolled in the same F.E.P. program but taking a course in a field other than mathematics or mathematics education. Immediatly before and after the 45 hour courses followed by both groups, the participating teachers were subjected to a test and a questionnaire. Furthermore, six participants from each group accepted to answer twice the same long interview, once before and once after the course. Thus, it became possible to evaluate the perceptions shown by the teachers and to evaluate as well the evolution of these perceptions. Therefore, three tools were developed, a test, a questionnaire and an interview: these three tools are described in the following paragraphs.

The first one, called the "correction test" (Dionne, 1985) aimed at evaluating the relative importance the teacher places on the students' answer and on their reasoning. He or she was asked to grade fictive students' answers to two elementary problems: one in arithmetic and the other in geometry. Furthermore, he or she was asked to justify his or her grading.

The second tool was a written questionnaire on which teachers were asked to specify their perception of school mathematics (Dionne, 1984). These different percpetions - traditional, formalist and constructivist - were each caracterized by a set of statements. Teachers were first asked to order these sets according to the importance they assigned to each one and then to give a mark to each of these sets.

The third tool was a long interview in which were explored the teachers'perceptions of mathematics teaching and learning. Four themes were addressed: the function and place of intuition, the role and the place of understanding and of skills, the part of discovery and of definitions and, finally, the role and importance given to errors. Within each part of the interview, teachers were asked to specify the meaning they gave to key-words used, to illustrate that meaning with examples and to articulate the role they attribute to each of these elements in their class practice.

THE MODES OF ANALYSIS

The first two tools, test and questionnaire, were used with each group, experimental and control, individually taken as a whole. Results concerning procedures and conclusions were presented in Dionne (1985 and 1984). In this paper, our intention is to concentrate on the twelve case studies related to the interviews. But in each of these cases, the response to all three instruments (test, questionnaire and interviews) were considered and examined as a whole. The purpose of this approach was to clearly isolate the manifest perceptions and their evolution. Next, the answers to "pre" and "post" interviews were analysed. These analyses were afterwards compared to those conducted by two independant experts. The only conclusions retained from this exercise were those on which agreement was unanimous. What should be added here is that these analyses were more qualitative than quantitative. They were based on the search for converging indications: this appears to be more interesting since we do get a real description of the perceptions instead of mere reductionist statistics. And the search for convergent indications - in which we tried to validate the answer to the test, to the questionnaire and to the different parts of the interview - allows for clearer conclusions on a topic which in itself remains difficult to define and delimit.

THE SIX TEACHERS IN THE CONTROL GROUP

Of the six subjects involved with the control group, five showed constructivist perceptions and tendancies while the sixth one, Lise, appeared much more traditional. As all the perceptions did not change much between the pre and post tests, the descriptions that follow in the next paragraphs hold for the period that precedes as well as for the period that follows the proficiency course followed by these subjects.

In their reactions to the correction test, the six subjects proved to be much more concerned with the reasoning of the children than with their answers. There were only few variations in the marks given when, for each category of children's answers, we compare the geometry problem to the arithmetic problem or the pre to the post-test.

It is much more hazardous to try to conclude from the individual answers to the questionnaire: there was often a contradiction between the first classification given by some teachers and the marks they gave afterwards. Global analysis based on the groups' means showed interresting indications (Dionne, 1984). However, it is much more difficult, if not impossible, to characterize in a few general statements the individual reactions of the subjects.

Intuition, the first theme of the interview, was first described in general terms ("to guess", "to anticipate") but four of the six teachers succeed in linking this concept of intuition to learning. For example, they spoke of instinct, trial and error discovery, use of concrete materials. Later, while addressing the theme of learning by discovery, three teachers stood up in defense of very intuitive approaches based on the manipulation of concrete objects, but they did not use the words "intuition" or "intuitive" to characterize these approaches.

Most of the teachers acknowledged the importance of understanding. They described this notion in terms of explanations given by the child, two of them adding the idea of transfer. Lise distinguished herself as she linked the idea of understanding with the ability to memorize and to apply or to use. The idea of skill, also dealt with in this second part of the interview, remained vague for most of the subjects. Only two of them were able to define it in terms of "ease at doing something".

Five out of the six subjects agreed on the effectiveness of learning by discovery. They prefered this approach to what is known as the traditional magisterial approach in which the teacher defines the notions to be learned. Lise, even if she thought of discovery us an "ideal" way of learning, admitted that she confined herself to traditional, authoritative lectures on mathematical concepts and operations.

Finally, mistakes were perceived as normal and even useful by all the six participants. They did not all believe that a good answer can stem out of bad reasoning. However, if such an event were to occur, they thought that the student's explorations would allow them to realize what had happened. Only one teacher claimed that she systematically asked the children for explanations...

With these people from the control group, as we have already pointed out, changes were rare and isolated phenomena, so that, as such, they never indicated to any substantial evolution in these subjects' perceptions. Hence our conclusion that our tools did not initiate modifications of these perceptions.

THE SIX TEACHERS IN THE EXPERIMENTAL GROUP

In both the pre-test and the pre-interview, our six experimental subjects proved to be very much similar to those of the control group:

five of them appeared to be rather constructivists, showing reactions analoguous to the ones already described. Furthermore, Jacques, like Lise, proved to be more traditional. However, changes observed between the pre and post-tests or between the pre and post-interviews were much more numerous and coherent. The next paragraphs give a description of these.

In the correction post-test, our first subject pays a much more nuanced attention to the child's thought processes as she distinguishes between a computation mistake and a reasoning error. And this is confirmed, in the post-interview, by the clear distinction she establishes between poor understanding and a lack of skill. Our subject then describes what she considers her "new" reactions when facing students' mistakes, explicitly asserting here the influence of the experimental course. She also claims this influence when she explains, still in the post-interview, that she now distinguishes levels of understanding and can describe their manifestations.

In the correction post-test, our second subject deprived her students of the benefit of the doubt she had given them in the pre-test. This increased severity is explained in the post-interview when our subject relates how much surprised she was when, asking questions to "good" students, she discovered that they simply could not explain their solutions to particular problems. Still in the post-interview, she presented intuition as a kind of spontaneous and primitive knowledge and described different levels of understanding. For instance, she mentioned that these levels allowed the teacher to overcome the simplistic "you have it or you don't" diagnosis. Finally, she insisted on the ability to explain as an important criterion of understanding, a criterion replacing the ability of problem-solving. This attitudinal change is coherent with the increased severity mentioned above.

Of all the interviewed persons, our third experimental subject was the most constructivist from the outset. Thus, she did not change much but even then, she deepened and structured her beliefs. The course allowed her, notably, to clarify her perceptions of intuition, now seen as the first step in the understanding process and to describe that intuition in terms of cognition and not only in terms of representation. She now could also establish a classification of errors distinguishing between a technical mistake, a wrong choice of process and a lack of understanding.

Our fourth subject was also a very constructivist person but this did not prevent her from renewing her vision of intuition which she described, in the pre-interview, as a vague feeling and, in the post-interview, as primitive knowledge, an important step in the learning process. She also asserted, in the post-interview, that she now believed more than ever in the importance of the process of discovery. And she associated the latter with the manipulation of concrete materials, which, as we believe, constitutes a rather intuitive approach.

Our fifth subject also showed signs of change in her perception . For instance, she explained how the course helped her to get a better understanding of basic arithmetic and, thus, a better understanding of the roots of her students' difficulties. And as she acknowledged this, she was led to a more general use of concrete materials and manipulations, And for her, this was added proof of the importance of intuition and discovery in the learning process.

Jacques, our last subject, was at the beginning what he remained throughout the experiment: a very traditional teacher. For instance, in the correction test, he remained, above all, interrested in answers and did not see any place for intuition in the learning process. He did not find "realistics" the use of discovery learning nor did he consider error as something rather difficult to analyse. His conviction were, for sure, a little more "nuancées" than what this brief summary makes then to be, but as it may, they remained unaffected by the experimental course.

CONCLUSION

There is a clear convergence of indications: five out of the six control subjects proved to be rather constructivist. For instance, their reflections on errors were coherent with their reactions in the correction test; the attention they said, in that test, to the student's reasoning process was confirmed by the importance they attached to understanding. Similarly, the importance they gave to intuition was confirmed by their marked preference for learning by discovery. Lise, alone, remained clearly traditional. She could'nt make room for the child's intuition or for a discovery process in her teaching. Also she gave a rather behaviorist description of understanding.

The preceeding conclusions also apply to the subjects of the experimental group: simply replace Lise by Jacques... For Jacques did not change much. In fact, our constructivist beliefs led us to say that it will always be sheer utopianism to try to change someone's convictions againts its own will. The initial move or impetus has to be initiated by the person himself.

However, if the changes observed within the control group were small and isolated, the presence and coherence of those noted in the perceptions of five out of six of the experimental subjects reveal a real influence of the experimental course on these perceptions. For these individuals, the course was an opportunity to discover something new and/or to deepen and structure beliefs already present: concerning the function of intuition in the learning process, for instance, or the importance of analyzing children's errors. All this confirms and strengthens the indications obtained from the global analysis of answers to the correction test (Dionne, 1985) and to the questionnaire on school mathematics (Dionne, 1984). However, the evolution observed here are not radical. And this is due to the fact that constructivist perceptions were more operative than expected at the beginning, but especially and above all because real change in perceptions is more a continuous process than an event.

Finally, what should be retained is that this research brings novelty in two ways.

- First, by its method. Because of this study, new tools to evaluate teachers' perceptions were built and tested. More importantly, the simultaneous use of these tools, each being very different from the other, allowed a search for converging indications which mutually validated themselves.
- Secondly, by its conclusions. In fact, the results of this study could lead to better teachers training programs where mathematics concepts would be considered simultaneously in an epistemological and psycho-pedagogical perspective. This particular kind of integration of different aspects of teaching

and learning could help to solve the problems mentioned at the beginning of this paper because it would put the focus on the way concepts are known and understood.

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USE OF CLASSROOM VIDEO FOR TEACHER INSERVICE EDUCATION

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Abstract

The first part of this report refers to an informal study in which questions were addressed about the effect on mathematics teachers of their viewing of classroom video tape. Classroom excerpts were viewed by groups of teachers, who were then invited to reconstruct what they had seen, relate it to their own experience, and discuss issues which were raised. The second part concerns continuing work in which beliefs, which resulted from the earlier study are being tested. Lessons of a number of teachers in a secondary school have been filmed and viewed by the teachers both individually and in groups. Audio recordings of the teachers comments, both at the time of the viewing and after subsequent classroom action provide data for analysis.

PART 1

BACKGROUND

The Centre for Mathematics Education at the Open University (U.K.) has filmed many hours of mathematics lessons and has published compilations of excerpts from these lessons. The members of the Centre have subsequently been engaged with issues concerning the use of such video footage for teacher inservice education.

THE INFORMAL STUDY

The Informal Study concerned questions to do with the value of viewing video recordings of mathematics lessons in contributing to the professional development of mathematics teachers. It has been pursued through a series of workshops with teachers and advisory teachers. In some cases the workshop has been based on particular themes or questions. Two examples of such themes were, *the role of practical work in the mathematics classroom*, and *teacher intervention*.

The work has been based on the belief that viewing a video tape creates resonance [1] in the viewer which can stimulate her to reflect on aspects of her own classroom experience and practice (Schon 1983). Moreover, if the viewing is a group activity, it provides a shared experience of the recorded classroom. As a result of this the group have a common basis from which to explore issues raised due to individual resonance.

An important aspect of viewing videotape seems to be what happens when the television is switched off (Mason 1985). We have worked on the principle that this should be the most valuable part of the activity. The videotape itself is no more than an incomplete record of one classroom. The value in viewing this merely as a classroom example or demonstration seems limited, since the viewer is not actively engaged in it. We suggest that it is only when the viewer relates what she has seen to her own experience and starts to work on the issues raised, that she starts to profit from the experience. It is by construing the video images in terms of her own experience that she constructs her own reality (von Glasersfeld 1984).

The length of videotape viewed at any time has usually been quite short, perhaps 3 to 7 minutes, so that viewers could be expected to remember what they saw in some detail. Viewing has been followed by moments of individual silent reconstruction to allow each person to fix their own images before sharing them with others. Then viewers have been invited to tell a neighbour briefly what they saw, without interpretation. Discussion has then been opened to the larger group, to allow sharing of what was seen, and of related personal experience. We hope to include an example of this way of working at our presentation.

One of the first things that we observed was that although everyone watched the same piece of video, different people actually saw different things. Sometimes by negotiation people were able to agree on what they had seen. At other times only a reshowing of the video would convince. As discussion progressed, description turned subtly into interpretation, and the teacher group were no longer discussing what they had seen, but were putting themselves into the situution and responding to it with their professional experience.

Although it was never claimed that excerpts illustrated good practice, this was often assumed by the viewers. In many cases it triggered a response of overt criticism of the teacher concerned. When this happened we felt it important to suggest that little value could result from criticising the teacher filmed, and that perhaps most could be gained from relating what was seen to one's own experience and trying to learn from it by reflecting (Kilpatrick 1986).

At all of the workshops, issues to do with teaching mathematics were raised which involved most of the participants in discussion. Often a considerable debate resulted. We suggest that the video was the catalyst in the creation of this debate, and that as a result the teachers confronted issues that they might not otherwise have articulated.

Many events could be reported. There are no formal records, and the teachers were not formally interviewed after the sessions. However, comments from many teachers indicated that they found the sessions stimulating and thought provoking. A valuable further study could be to follow up some of the teachers and explore what effect the experience had on their classroom practice.

A number of questions were raised as a result of this work:

1) What happens to the teacher's classroom work as a result of engaging in issues provoked by the video?

2) How does the selection of the excerpt for viewing affect issues which are raised?

3) How does the length of tape viewed affect the resulting discussion?

PART II

These questions and others are being considered in a further study which concerns exploration of the use of video tape by the members of the mathematics department in a secondary school.

THE CONTINUING STUDY

Our intention is to study in more detail, the effects on teachers of viewing classroom video tape. The video tape used in this case is filmed in the classrooms of the teachers who are taking part. The teachers are interested in the use of video recording to inform their own classroom practice. Several lessons have been filmed, and the following use has been made of the resulting video tape:

1) Teachers have individually viewed video of their own classrooms and their subsequent comments have been recorded on audio tape.

2) The teachers, as a group, have viewed excerpts from each other's lessons and the discussions which followed have been recorded on audio-tape.

Current analysis consists of working on transcripts of the audio-tapes, and talking with the teachers about the issues raised and the role of the video-tape in raising the issues.

A result which is already noticeable concerns teachers working on ideas together and sharing classroom experiences in a way that did not occur before. For example, the group viewed a video excerpt of the Head of Department (HoD), presenting an activity to his class in a particularly open-ended form. The activity was one on which the other teachers were also working with their own classes, and so the lesson viewed had relevance for them all. The HoD had chosen the excerpt, and he had some questions which he wanted to share with the group.

He had presented the activity in a open-ended form in which pupils had been invited to pose and explore their own questions. He felt that the resulting exploration had produced few mathematical insights because pupils had worked with too many variables. He had tried hard to get pupils to constrain the situation themselves. so that it became possible to relate some of the variables, but felt that he had not succeeded. His questions were to do with the value of an open-ended approach, the desire on the part of the teacher for recognisable mathematical outcomes, and the training of pupils in problem solving strategies.

The other members of the department were quickly drawn into the debate, and a number of issues emerged:

* The different modes of presentation of an activity; (witnessed by the different experiences of the teachers in presenting the activity themselves).

* The value of open-ended activities - losses and gains.

* The time spent by a teacher 'up front' as opposed to working with individuals or groups within the class

One teacher felt that, despite the HoD's worries, his pupils could gain from being given more freedom to explore for themselves, and so he decided to try out a more open-ended approach to his next classroom activity.

The HoD believed that he had spent too much time at the front of the class, and decided to work on reducing this in subsequent lessons.

After viewing tapes individually, one of the teachers commented on the value of the video-tape in providing glimpses of the working of pupil groups to which she had not had access because she was elsewhere. She said,

> *I found it very useful to see what they did, and hear what they said after I'd left them. Normally you wouldn't know, and I was very pleased. They were smiling and seemed confident.*

Another teacher remarked that he had found it interesting to observe his intervention with pupil groups and he wanted to think about how this effected the work of the pupils and his perception of their understanding.

As a result of one video excerpt of a group of four pupils working on a problem, a meeting took place between the HoD and the pupils who had been filmed. The pupils were encouraged to talk about how they had felt when they were working, to recall different stages in their working, and to relate them to their mathematical progress. The HoD felt that this had value in at least two ways:

1) He gained insight into the students' perceptions of what he had asked them to do.

2) He felt that a better understanding might result between the students and himself to the advantage of future work.

Video-tape has contributed to the raising of issues in mathematics teaching amongst the teachers involved in this study, for the following reasons:

* Its replay facility enables sharing of events and concerns with colleaguee and pupils, and viewing of events where the teacher was not present.

* Resonance with what is viewed brings out issues and concerns for group discussion and possibly subsequent classroom action.

Questions still being pursued:

1) What changes in classroom action and outcome can be related directly to video viewing?

2) Can and should excerpts for viewing be chosen with specific issues in mind? How does what is offered for group viewing influence resulting discussion and action?

3) Is there any efficient or economical way in which group viewing of video recordings can take place, perhaps to maximise benefits from this group activity?

4) What do the teachers regard as the benefits?

CONCLUSION

The video is having an effect on the teachers' thinking. The methods of its use and aspects of classroom action and outcome need further exploration.

We realise that there are issues to do with the making of the video - e.g. technical limitations and implicit editing by the person holding the camera - which might affect viewing. We are aware of these as potential issues but are not explicitly addressing them here.

NOTES

1. St Augustine wrote in *De Magister*, in the late fourth century, the following passage which captures beautifully what we mean by resonance:

   If anyone hears me speak of them (images of things once perceived), provided he has seen them himself, he does not learn from my words, but recognises the truth of what I say by the images which he has in his own memory. But if he has not had these sensations, obviously he believes my words rather than learns from them. When we have to do with things which we behold with the mind, ... we speak of things which we look upon directly in the inner light of truth ... .

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THE MIDDLE GRADES MATHEMATICS PROJECT: COACHING AS A STRATEGY FOR CHANGING TEACHER PRACTICE

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The Middle Grades Mathematics Project represents a two year effort to improve the teaching and learning of middle school mathematics. The goal of the project is to ascertain the amount of assistance middle school teachers need to implement effectively an instructional model that would promote the development of mathematical understanding through conceptually oriented instruction. Twelve middle school mathematics teachers were separated into three groups: One group, four "uncoached" teachers, received mathematics materials using the instructional model and a workshop focused on using these materials; a second group, four "coached" teachers, received the materials and workshop training and were also coached in their classrooms; the third group, four "lead" teachers, received the materials, workshop training, and coaching and were also expected to coach a colleague the second year. Interim results from the first year of the study indicated the "lead" teachers made greater changes in their thinking about instruction than did their "coached" or "uncoached" counterparts. There is evidence that 1) coaching does make a difference in changing teacher practice and 2) the length of support time needed to help teachers learn a new instructional mode is at least two years.

Fullan and Pomfret (1977) provided a comprehensive review of research on curriculum and instruction implementation. They suggested that there are five dimensions of implementation in practice: changes in materials, structure, role/behavior, knowledge and understanding, and value internalization. The goals of implementation must also include changing teachers' behaviors or roles in the classroom in such a way as to encourage the acquisition of student process goals. Teachers can implement new materials in the classroom, at a surface level, so students learn the skills and "algorithms" of the content without developing deeper understanding of the concepts and processes inherent in the mathematics. According to the Rand Study (McLaughlin & Marsh, 1978), professional development aimed at changing experienced teachers' practices may need a level of personal involvement with the teachers in addition to providing exemplary mathematics materials.

Howson (1979) says, "If new materials are to be handled with understanding, then training is insufficient -- one can train teachers to handle a new learning system, yet to cope with difficulties which arise in its use, the teachers must be reeducated." Joyce & Showers (1981) hypothesize that a fully elaborated training system including theory, demonstration and practice, and feedback generally will ensure skill acquisition on the part of the teachers. However, if transfer is to occur they suggest that further help is needed which could be provided by "coaching".

Showers (1983) reports a very promising study in which the notion of coaching is elaborated. Coaching was conceived in this study as a combination of several elements: the provision of companionship, the giving of technical feedback, and the analysis of application.

Good & Brophy (1974) demonstrated the power of intensive observations and feedback for assisting teachers to alter certain kinds of behavior. Lanier (1983) used an intensive advisor strategy to change teacher behaviors in general mathematics classrooms. There are many similarities between the characteristics of an advisor's work and that of a "coach". Andreae (1972) emphasizes that the advisor's role is to provide assistance in terms of teacher's needs. Apelman (1981) says that "stimulating and extending teachers' thinking about their goals raises advising above merely technical aid." An advisor's ultimate task is to elicit in the teacher a problem-solving and reflective attitude, that will enable him/her to overcome successfully future challenges. Incorporating the strengths of the advisory role with the more behavioral coaching role seems to be a very profitable direction in teacher inservice.

PROJECT GOALS

The Middle Grades Mathematics Project (MGMP) represents a wide-ranging effort to improve the teaching and learning of mathematics by teachers and students in grades six, seven, and eight. The early phase of the project focused on the development of exemplary units of important mathematics ideas appropriate for students in these grade levels. The instructional model (Shroyer, 1984) around which the units are built (Launch-Explore-Summarize) provides teachers with exemplars of conceptually oriented mathematics instruction.

Good pointed out that his research with Grouws on mathematics instruction showed that many teachers tend to emphasize computation, memorization, and mechanics. However, the students of teachers who emphasize conceptual understanding received higher achievement scores in mathematics. (Notes & News, 1984)

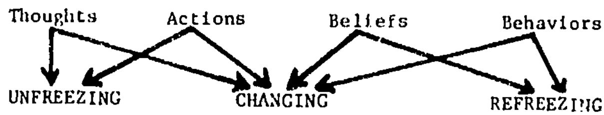
The MGMP staff identified changes in teacher practices that need to occur in order to implement effectively the Launch-Explore-Summarize (LES) model. These changes in teachers' beliefs and behaviors included the following major areas:

1) Patterns of Communication;
2) Planning for Instruction;
3) Quality of Direct Instruction; and,
4) Instructional Thoughts and Actions.

Previous work with the implementation of these exemplary materials had shown, as the literature predicts, that the materials alone did not produce the desired changes in teachers' instructional beliefs and classroom practices. This study examines the impact of classroom consultation (referred to as coaching) on producing the desired changes. The major question is, "How effective is coaching as a strategy in changing teacher's instructional emphasis from a computational to a conceptual orientation as reflected in the exemplary mathematical materials (MGMP units)?"

THE THEORETICAL FRAME

The model of the nature of teacher change that the staff theorized would be found is based on Lewin's general model for the process of change. As Blanchard (1981) explained, the Lewin model consists of three phases: The first phase, unfreezing, prepared or motivated people for change; the second, changing phase, took place when people learned new patterns of behavior; the third phase, refreezing, was the process by which the newly acquired behavior was adapted or integrated into the individuals repertoire. We imposed a series of change states on this model that we conjectured the teachers would move through in varying degrees during the change process.



Eight of the twelve project teachers were coached by the staff. Of the eight, four were identified as "lead" teachers with the expectation from the beginning that they would coach a colleague in their school during the second year of the project. The four "coached" teachers were not expected to coach a colleague. The remaining four teachers were "uncoached" - they received the same summer workshop training and the exemplary mathematical units as did the "lead" and the "coached" teachers but did not have any coaching follow-up in their classrooms during the school year.

INTERIM RESULTS OF THE TEACHER STYLE INVENTORY
AND STUDENT SURVEY OF THE CLASSROOM

In an effort to capture the changes in teacher's thoughts, actions, beliefs, and behaviors surveys were administered to the project teachers and the students in their classes. The surveys used a Likert-scale requiring a response from 1 to 5. Students completed the Student Survey twice during the school year - once in the Fall and Spring. The teachers completed the Teaching Style Inventory at the start of the project, after the first year, and will be given the inventory at the end of the second year. It was believed the results of the Teaching Style Inventory would provide evidence of a teacher's changed thoughts and beliefs about instruction and classroom practice. In addition, the results of the Student Survey would reflect the teacher's changed actions and behaviors in the classroom.

The Teacher and Student Surveys from the Spring of 1985 (pre-project) and the Spring of 1986 (interim) were analyzed. A method of analysis was employed that captured the amount and degree of change the teachers made in their thinking across the first year. This analysis involved making a comparison between an "actual" response on an item with the "ideal" response for that item. For example, if a teacher's or students' "actual" response to an item was 2 and the "ideal" response for that item was 5, then a 3 was recorded. This value of 3 signified that the "actual" response was 3 levels away from the "ideal". A sum of all the items distances from the "ideal" was calculated for each pre- and interim survey for the teachers and their classes. The difference between the sums on the pre- and interim surveys represented the Index of Change - or the amount of change which occurred from the pre- to the interim survey. (Figure 1)

The results suggest that each of the project's teachers had made some changes in their thinking about instruction and classroom practice across the first year. Interestingly, three of the four "lead" teachers ranked first, third, and fourth in the interim Teacher Survey results. This would indicate these teachers had changed more in their thinking than did their project counterparts. The results from the Student Surveys indicated the students in the teachers' classes noticed some change in the classroom practices of their teachers. Although the "lead" teachers showed more change in their perceptions, this change was not reflected in classroom practice. The "coached" teachers showed less change in their perceptions (from the Teacher Survey results), but their students noticed more change in their classes (from the Student Survey results).

PRE- TO INTERIM RESULTS ON THE MGMP

TEACHER STYLE INVENTORY AND STUDENT SURVEY OF THE CLASSROOM

TEACHING STYLE INVENTORY

	Tchr Code	Spring '85	Spring '86	Chng	Rank '85	Rank '86
Lead	A-1	53	36	+17	B-2	A-4
	A-2	61	36	+25	B-3	B-2
	A-3	62	53	+9	A-4	A-2
	A-4	53	22	+31	A-1	A-1
Coached	B-1	65	51	+14	B-4	C-2
	B-2	48	32	+16	C-2	B-3
	B-3	51	43	+8	A-2	B-4
	B-4	54	46	+8	A-3	B-1
Un-Coached	C-1	67	43	+14	B-1	A-3
	C-2	55	38	+17	C-1	C-1
	C-3	80	71	+9	C-3	C-3
	$\overline{X}$	59	43			
	SD	9.2	10.2			

STUDENT CLASSROOM SURVEY

Tchr Code	Spring '85	Spring '86	Chng	Rank '85	Rank '86
A-1	28.8	27.4	+1.4	C-2	B-3
A-2	30.2	27.6	+2.6	C-1	C-1
A-3	26.8	25.5	+1.3	A-4	C-2
A-4	26.4	26.4	0.0	A-3	B-1
B-1	27.7	23.1	+4.6	B-3	B-2
B-2	27.0	23.4	+3.6	B-2	B-4
B-3	27.0	21.4	+5.6	B-1	A-3
B-4	28.0	24.3	+3.7	C-3	A-4
C-1	26.1	22.1	+4.0	B-4	C-3
C-2	24.1	22.5	+1.6	A-1	A-1
C-3	27.8	26.8	+1.0	A-2	A-2
$\overline{X}$	27.3	24.6			
SD	0.51	2.13			

Figure 1

The summary results show changes in how teachers thought about instruction and some small changes in their classroom practice as perceived by their students. We believe that the results from the second year will likely show more dramatic changes in the "lead" and "coached" teachers and less change in the "uncoached" teachers. The "lead" and "coached" teachers during the second year seem to be transforming their thoughts and actions into beliefs and behaviors about practice - this has not been accomplished by the "uncoached" teachers.

DISCUSSION AND EMERGING CONCLUSIONS

None of the project teachers reached a state of changing their beliefs or behaviors in a consistant habituated way by the end of the first year. Although most of the "lead" and "coached" teachers had moved into Lewin's Change Phase by the end of the first year they were inconsistent in their classroom practices. While one lesson would be very good, the following one might show a return to a previous instructional mode. For example, a return to questioning such as, "Tell me what you do to find the area of a rectangle." (requiring a computational response rather than a conceptual one). If there was a surprise in the data from this first year it was in the length of time we found teachers needed support in order to make substantial changes in their instruction.

IMPLICATIONS

At this point in the project there is clear evidence that coaching does make a difference and that the length of support time needed to help teachers learn a new instructional strategy - to make real change - is at least two years. The process of changing teachers' instructional modes involves moving them through phases of unfreezing, changing and refreezing by changing their thinking and acting in the classroom first, followed by the more comprehensive changes in their instructional beliefs and behaviors in regards to the teaching of mathematics.

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ANALYZING THE PROBLEM SOLVING BEHAVIOR OF TEACHERS AS LEARNERS

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Twenty-six educators (including elementary teachers, administrators, and graduate students) working in small groups participated in a mathematics problem-solving activity as part of a teacher development project. Participants were asked to use geoboards to construct solutions to a series of multi-step problems involving fractions. Analysis of their problem-solving behavior in an exploratory investigation indicated that generally the participants first constructed a physical model which they evaluated and refined using knowledge of numbers, rather than first constructing a symbolic solution and then modeling it physically.

The model providing the basis for this study has as its goal the professional development of mathematics teachers. Described elsewhere (Maher, 1986; Maher, in press), the model proposes a program, based on a constructivist view of learning and teaching, that enables teachers to facilitate mathematics instruction by creating learning environments for children. One component of the program is that the teachers experience the construction of the mathematics they are required to teach by working in small groups to solve problems using particular physical embodiments to represent the mathematical concepts. The approach is based on the view that such experience can better prepare teachers to observe and guide children's mathematical learning and consequently better equip them to design, implement, and evaluate problem-solving activities for children. The model is currently being implemented in a three year project at the Harding School in Kenilworth, N.J.[1] (Maher, Alston, & Landis, 1986; Maher & Alston, in press; Maher & Landis, in press).

Another view of teaching which advocates both understanding of mathematical content and of the development of childrens' knowledge is offered by Carpenter, Fennema, and Peterson (1986). They hold that the complexity of the processes of both learning and teaching causes prescriptive teaching to be ineffective and propose an alternative which would require that teachers make instructional decisions based on an "understanding of the general stages that children pass through

1 The Project is sponsored by the Rutgers University Center for Mathematics, Science, and Computer Education, the Kenilworth Public Schools and the New Jersey Department of Higher Education.

in acquiring the concepts and procedures in the domain, the processes that children use to solve different problems at each stage, and the nature of the knowledge that underlies these processes" (pp. 227-228).

In order to respond to the present crisis in education, Mary Futrell, President of the National Education Association, also expressed a need to restructure teaching both from the perspective of children's learning and the professional status of teachers. She called for continued research by educators on learning and teaching to be implemented as a basis for developing programs that would challenge and prepare teachers to create environments in which students could be actively engaged in explorations that facilitate learning (Futrell, 1986).

Components of the knowledge domain, identified by researchers as critical for mathematics teachers if they are to make informed instructional decisions, are knowledge of the content as well as both knowledge of how the content is learned and how problems are solved by children at various ages. Shulman (1986) has indicated the need to consider knowledge of content in research on teaching. Romberg and Carpenter (1986) have indicated that research on teaching and research on learning in mathematics are conducted as if these were separate areas and call for an integration of research in the two disciplines.

Understanding teachers as learners requires studying the ways teachers construct solutions to mathematical problems, reflect on their own strategies as well as possible alternatives, and consider how children of varying ages and mathematical experience may represent their solutions. This report is focused on that part of the model that considers the teacher as learner. The approach is based on the view that teachers, in preparation for their working with children, profit from experiences doing mathematics that call upon their engagement in constructing solutions to problems and modeling them with the use of physical objects.

OBJECTIVES

Specific objectives of this investigation were to describe how groups and individuals within groups solved mathematical problems. The particular problem-solving activity involved concepts and operations with fractions to be modeled using geoboards. The following behaviors were studied: (1) construction of solutions based on the representation of the physical model(s) and/or monitored and revised based on an interaction of their conceptual/procedural knowledge of fractions with their physical model(s); and/or (2) construction of solutions based on conceptual/procedural knowledge without prior reference to a physical model(s).

METHODS AND PROCEDURES

The study took place in a K-8 elementary school with participants of the Rutgers-Kenilworth Project. The population consisted of a mixed group of 26 educators: 18 elementary teachers (from grades 1-8), 8 graduate students (including 2 elementary and 4 secondary mathematics teachers, and 2 teachers with no prior teaching experience), the school principal and the curriculum coordinator. Five groups naturally formed by self-selection, each including both teachers and graduate students and two including also an administrator.

In this two-hour session participants were asked to use a geo-board to construct solutions to four multi-step word problems involving equivalency and operations with fractions. Previous sessions, similarly organized, included activities that provided a variety of physical objects with which the participants were to construct solutions to problems including activities with fractions using geoboards. The third of the four problems given in this session was used for this report. On average, 20 minutes was required for the task. It was presented in a context of the school environment. Each of the five groups was videotaped and transcripts of these tapes and the written work of the participants provide the data for this paper.

The statement of the problem is as follows:

> The first, second, and third grades have been given a garden plot to develop as a project. The third grade is to have 1/2 of the plot, the second grade is to have 2/3 of the remainder of the plot and the first grade is to have what is left. Show (1) a possible model of the garden by grades, (2) how much of the garden the second grade will have, (3) how much the first grade will have and (4) how much of the total plot Linda's class will have if the three first grade sections share their part of the garden equally.

RESULTS

In each of the five groups, the initial attempt to construct a solution was based on the physical embodiment provided. One or more of the group members began by enclosing a rectangle on the board and then looking to see if it could be partitioned according to problem specifications. For instance in Group 1, composed of a 5th grade teacher, a 7th grade teacher, 2 graduate students, and one administrator, the administrator began with:

> "It's 6 - or 12 - or 16. (Enclosing each of these areas in turn and counting squares inside.) You can't do 16 because half of 16 is 8 and you need 2/3 of that."

Four of the five groups completed their solutions to the problem based on these models. For each of these groups, individuals' procedural knowledge of fractions was used to accept, reject, or refine a particular construction.

Two strategies seemed most prevalent in the construction of these solutions. In the first, successive models were built, the areas of which were consecutive multiples of 6. Members of each of the five groups began with a rectangle of 6 square units which met the requirements of the problem that half of the plot could be divided into two parts: one, 2/3 of the area of that half and the other, 1/3. In order to fulfill the final requirement that the smallest part be shared among 3 classes, the rectangle was doubled and 12 square units enclosed. In each case it was discovered that this construction would not work.

As an illustration, consider the dialogue between a 3rd grade teacher (T3) and a graduate student (G) in Group 2.

> T3: "Here's the half. (Points to half of the 12 square units.) Then 3rds. This is what the 2nd grade gets." G: "You divide it into 6ths - so the 2nd grade gets 1 - 2, 2/6 of the total. The 1st grade gets 1/6 - " T3: "Now just divide this into 3rds - How? It's not dividing!" G: "I see what you're saying."

Members of three of the groups then combined two geoboards in order to enlarge the rectangle. Within Group 4, a 1st grade teacher (T1) and a 5th grade teacher (T5) provide an example of a number of attempts to construct a solution based on 24 square units before successfully constructing a model of 36.

> T1: "Let's put our boards together. We want 24 squares. (Encloses a 6 by 4 rectangle.) - Any way you look at it, it's not going to work - You have this little triangle left." T5: "You need 36." T1: "(Constructing a 36 unit square.) That's going to work!"

A second strategy was employed by members of two of the groups based on a 9 unit square. In each of these cases an individual working parallel to the rest of the group developed the solution while the other members were constructing models based on the first strategy. After completing the solution, that person shared it with the others.

Group 3, including an administrator (A), a 5th grade teacher (T5), 3 2nd grade teachers, and a graduate student (G), exemplify this approach.

All of the members of the group constructed models of rectangles, first with 6 square units and then with 12.

> A: "This is what's left - 2 square units - You have to divide it by 3." T2: "That's 2/12 (Constructing her model)." A: "I don't see a way." G: "9. Here is half. (Pointing to her square which is divided by a rubber band along the diagonal.) Now we have 3 (parts). - 1 and 1/2, 1 and 1/2, 1 and 1/2 - see, we can divide it. There it is! A: "You're right - I understand." A then explains to T5 and other members of the group. All 6 construct the model and T5 continues the explanation to the 2nd grade teachers. T5: "This is my 6 halves. Here are 3 halves for the 1st grade - this unit and 1/2."

Group 5 provided the only exception to constructing a solution from the model. This group began with a 5th grade teacher (T5) and 2 graduate students, both of whom teach college mathematics, (G1 and G2). They were joined midway by a 3rd graduate student, also a college mathematics teacher (G3). The 3 began, as the other groups had done, by constructing first a 6 unit rectangle and then one having 12 square units. They had just proposed building a 36 unit model when G3 joined the group and they paused to explain their strategy to him.

> G3: "Why not use 6?" G2: "Because how do you take - You have 1 unit left - How do you divide that into 3rds? - (Reads problem)." G1: "The 24ths aren't going to give us help - " T5: "36 would give it." G3: "18 would do it better - It would be a perfect fit." G2: "Do you know what bothers me when you came up with 18 - Isn't it kind of underhanded? In other words, we knew basically what we wanted to find in this thing."

CONCLUSIONS AND IMPLICATIONS

A goal of this model is that teachers show certain changes in perspective and practice in their classroom mathematics instruction. Specifically for this investigation, the question was whether the activity that called for participants working in groups to construct physical models to represent their solutions in mathematical problem solving was appropriate. Qualitative observations of the five groups indicated that there was a high level of involvement among the participants. Each person was active in constructing models of the solution and all but one of the participants were involved to some degree in dialogue with group members concerning particular strategies. The roles assumed by various members of each group seemed to be important in considering the process of constructing problem solutions, but space limitations preclude their being reported here.

It is interesting to observe that in the case of solutions using a nine unit square, the solution seemed to be based on the constructed model rather than the knowledge that 18 sections could be found using the least common multiple of 2, 3, and 6, although the area had been divided into 18 pieces. Also, those individuals and groups who used the strategy of doubling the area never attempted to construct a model of 18 square units, which is the lowest common denominator for 1/2, 1/3, and 1/6.

Implications of this investigation suggest that teachers working as partners with administrators and graduate students can be engaged in constructing solutions using physical objects to represent concepts that are a part of elementary school mathematics. Continued research is needed to observe the effect, if any, that participation in these activities has on the teaching behavior of the participants and on their ability to construct problem solving activities appropriate for the children whom they teach. Also to be explored are the effects, both cognitive and affective, of natural heterogeneous grouping such as those formed in this project.

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PROJECT WORK WITH TEACHERS INVOLVED IN A PROGRAM FOR THE USE OF COMPUTERS IN EDUCATION

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This research describes the evaluation of a teacher training program set up to implement the use of computers as a support for project work. The teachers, who entered in the program mainly with the motivation of learning how to use computers, recognized changes in their attitudes and pedagogical practices.

The national Project Minerva was established in 1985 to promote the introduction of computers in Portuguese elementary, middle and secondary schools. This project pretends to contribute to the technological updating of school curricula and methods, and has a concern for pedagogical transformation. Universities and Superior Schools of Education are charged with the training of teachers, curriculum development and its evaluation, as well as with the necessary support to the work carried out at the schools.

Teachers of several disciplines are involved in this project. However, many of those who are most interested and become leaders within the schools are mathematics teachers. So far, most of work involving students has been realized through extra-curricular activities. Some of the teachers tried to use the computer in the classroom, but this has been difficult given the scarcity of appropriate software and insufficient quantity of existing hardware.

To use the computer in the classroom does not imply necessarily a change of pedagogical attitudes, student/teacher relationships, and learning processes. The computer can just be used to reinforce a tradicional style of teaching. In project work, students have the possibility to participate in the choice of the problems that they want to deal with and to define the corresponding strategies, methods, and forms of presentation of the results. This pedagogical approach, which remotes to Dewey and Kilpatrick, intends to assign the students a responsible and independent role in their own learning process. Dewey (1959) wanted to give a livelier content to education, in opposition to teaching just from listening and from books, by following the principles of motivation, dedication and organized work in order to achieve a learning goal. In his views about the use of computers in education, Papert (1980) also stresses the importance of the deep involvement of the students in the learning process through their personnal project.

Many contemporary teachers are in a way or another sensitive to these proposals. However, project work is not easy to implement in a long term basis, and most for the actual activities carried out in todays schools still draw from the traditions of straightforward transmission of ready made knowledge, memorization, and passive learning. To make teachers aware of the possibilities, difficulties, and conditions of sucess of project work and to invite them to start using this methodology with their students, it seems reasonable to involve them in a set of activities of a similiar format.

THE STUDY

This research intended to evaluate the effects of a training program in project work on teachers, attitudes concerning this kind of pedagogical strategy and to evaluate its effectiveness in developing their ability to conduct students in project work oriented activities, using computers. This evaluation was also intendend to provide informa tion to improve the design of the training program.

Specific objectives of this program on project work were to make teachers: (a) develop skills of organisation and cooperation in group work, (b) develop research skills and the ability to organize and present information, (c) be aware of different aspects of verbal and nonverbal communication, (d) view knowledge in a interdisciplinary perspective, (e) recognize the importance of intrinsic motivation, and (f) stimulate their iniciative and selfconfidence.

THE PROGRAM

Involved in this study were 22 teachers, all participants in the Project Minerva. Of these, 13 were teachers of mathematics and 7 teachers of other subjects.

The training program was developd in two phases. The first phase consisted of a four day workshop which main objective was to give an overview of project work methodology. The second phase concerned the implementation and evaluation of project activities in the schools.

In the first phase, a general problem was selected in big group discussion and then subdivided in smaller questions which were taken on by different subgroups. Each subgroup selected its own methodological strategies, including labor organization, data collection methods, data analysis, and forms of presentation of the results. After the presentation of each subgroup there was a discussion period in which the different contribuitions were confronted with the general problem initially defined. Finally, there was a general discussion to evaluate all the activity.

The second phase, included the introduction to the use of computer tools such as spreadsheets, data bases, word processing, drawing applications, an initiation to the LOGO language, and monthly seminars for discussing pedagogical themes related to the use of computers and project work and for exchange of experiences and reflection on the ongoing activities in the schools.

EVALUATION

There were two main periods of evaluation: the first took place at the end of the initial project work workschop. Since this activity represented for most of the teachers a first formal contact with this methodology, it seemed important to evaluate it just after the end of the workshop.

There was some discussion to decide if the problem to be selected should concern or not directly computers. The teachers decided that the computer ought to be regarded as just an instrument among others and picked up as their question "what can be done to improve the school?". This question was taken with enthusiasm by the participants, who assumed their role in the study of the subquestion in which it was subsequently divided. Some teachers felt inconfortable in doing actual field work, but the pressures of the needs of the group overcome this difficulty. Most of the final presentations were quite creative and original. In a short Likert type questionnaire they reported to have enjoyed the workshop and some indicated to have aquired new pedagogical perspectives to use in actual practice.

In the second evaluation, carried out six months after the initial workshop, the teachers were asked to respond to a more detailed questionnaire. One group of questions concerned the self-evaluation of change of attitudes by the teachers themselves. Another group concerned the different activities undertaken. Three other open question asked for comments on the difficulties and the potencial of project work. Twenty teachers answered this questionnaire.

The responses to the first group of questions are summarized in Table 1.

A global analysis of the responses to the questionnaire showed that in this phase of the work most of the teachers considered that the activities carried out contributed to improve, either highly or moderately, the qualitity of their work in the mentioned areas. For the whole set of questions, 39% of the responses indicated a high contribution, 46% a moderate contribution, 12% a low contribution, and 3% were "don't know" responses. The item that had most

TABLE 1

Contribution of the teaching training program to specific areas-teachers' responses

	Contributions High	Moderate	Low	Don't Know
A. Development of an attitude of permanet learning	95%	5%	-	-
B. Development of capacities of organization and technics of group work	15%	60%	25%	-
C. Awareness of the importance of the affective aspects in the learning process	25%	45%	20%	10%
D. To view knowledge in an interdisciplinary way	40%	40%	20%	-
E. Awareness of the problems of communication in the school context	35%	60%	5%	-
F. Development of new perspectives concerning the role of the teacher in the school	40%	55%	5%	-
G. Development of a new relationship with students	40%	40%	20%	-
H. Development of the ability to stimulate and to support the project of the students	35%	60%	-	5%
I. Development of a more positive perception of their function as educatores	30%	50%	10%	10%

positive responses concerned the development of an attitude of permanent learning. Following were the itens D, F, and G. The items B,E,H and I received mostly moderate respon ses. The answers of the mathematics teachers and of the tea chers of other topics were similar for all items, except for items E, G and I, in which mathematics teachers were eager than the others in recognizing a high contribution of the training program.

A second set of question concerned the specific contri butions of the several moments of the teacher training program. Some of this moments had a pedagogical emphasis and others concerned the use of specific computer tools. The most valued of the pedagogical activities was the initial workshop (63% high). For the remaining, the teachers tended to rate higher the activities that were mostly related to their actual experiences. The sessions concerning the specific computer tools were in general quite highly rated (all with more than 42% high).

At last, in open questions, we asked for the opinion of the teachers about the difficulties related to project work as well as for suggestions for the improvement of the program. Most of the teachers indicated several difficulties that they face in trying to use the computer as a support for project work in their schools. A content analysis of the 51 answers indicated that the scarcity of available time and the insufficience of material conditions, namely, computers and appropriate working spaces, were mostly refered (29%). Pressures from programs and the negative attitudes of their colleagues were also mentioned several times (10%and 18% of the answers, respectively).

Concerning the role that should be given to the reflection on the pedagogical aspects of the use of computers in education, 89% of the teachers agreed that this aspect should continue to have a strong emphasis in the program. However, they suggested a shift towards more practical issues and more exchange of experiences.

The projects developed in the schools by these teachers may be grouped in two kinds: projects with teachers and projects with students. In the first case, were offered courses related to the use of computer tools, such as drawing applications, word processing, and LOGO. There were also sessions for all school to show the educational potencial of computers. In the second case, there were experiences with LOGO and other computer tools in extra-curricular activities, as well as one experience on teaching Geometry in a 5th grade classroom. Projects such as the school journal and other interdisciplinary activities were also implemented in most of the schools.

A more detailed evaluation of all these projects will be performed on the end schools year in order to improve the working methods and to divulge and extend this kind of activities to other schools.

CONCLUSIONS AND IMPLICATIONS

Overall, we tend to believe that this training program was quite succefull. It seemed to have a reasonable mix of "pedagogical" and "technical" components, which reiforced each other and promoted teachers willigness to change some of their attitudes and practices.

The introduction to the use of computer tools and LOGO constituted for most of the teachers the main motivation. The discussion of pedagogical themes was appreciated and the teachers recognized it as important for the acquision of skills in developing work projects with the computer in their schools. However, we feel that this pedagogical discussions should be more deeply rooted in teachers practical experiences.

It would be unreasonable to expect outstanding results in a rather limited period of time. The inservice training of teachers should be viewed as a long term process. Particularly in this case, the teachers need to learn many new things about a new medium, the computer. However, these teachers are becoming leaders in their schools by introducing the computer as an instrument of pedagogical change. These teachers will participate in the training of their colleagues. So, this involvement may rather be an important part of the training program next year.

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EXPERIMENTATION OF THE MINI-INTERVIEW BY PRIMARY SCHOOLTEACHERS

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The MINI-INTERVIEW (Nantais et al, 1983) is a new tool for evaluation of understanding of mathematics at the primary level and is characterized by a short individual questioning sequence. Concerning the experimentation of this tool, we raised two questions: first, a question of feasability, in which we tried to determine the conditions under which the teacher can use the mini-interview with each of her pupils in the classroom. The second question concerns the use of the mini-interview to determine if the teacher uses it as a tool for formative evaluation, in the sense of providing her with some feedback on her teaching and assessing the pupil's understanding in the construction of the concept. This paper presents some results of the experimentation of the mini-interview by three first grade schoolteachers with all their pupils in their classroom.

The mini-interview has been devised to inform the teacher of the child's thinking and reasoning; for the teacher who wishes to involve her students in the construction of their knowledge must be able to follow their cognitive evolution. That kind of information can only be obtained from individual questioning and this suggests a form of clinical interview as the tool to be used. To achieve its full value, this form of evaluation must be integrated in the teaching task and hence should be used by the teacher herself within the classroom. It is in answer to the needs of the teachers and also to stay within the restrictions of the classroom that we have designed a new tool for the evaluation of the child's understanding of mathematics, a tool we have called the MINI-INTERVIEW. The mini-interview consists of an individual interview of the pupil and is characterized by a short timespan (5 to 10 minutes) and a sequence of questions prepared systematically and rationally. This evaluation aims at the student's understanding in the construction of conceptual schemas, and hence deals only with key notions in the mathematics curriculum. This is why the use of the mini-interview by the teachers requires a serious training in conceptual analysis as well as a good grasp of the methodological guidelines concerning the steering of an interview. This preparation enables them to determine criteria by which they can evaluate the understanding of a given concept, and it also helps them in running rigorous and efficient mini-interviews. The mini-interview used in this experimentation deals with the strategy of "counting on from one of the terms" in the addition of small numbers (Nantais et al.,1983); the sequence of questions in this interview has been carefully studied with many subjects.

To determine the criteria by which to analyze such an experimentation. we must realize that to carry out mini-interviews is no easy task for it does not simply consist in using a new instrument by following precise rules. Indeed, to carry out each interview, teachers must take into account several factors simultaneously such as class organization, the questioning prepared in advance, and abiding by the allotted time (5 to 10 minutes). Moreover, they would need to interpret the child's answers in order to continue questioning adequately and handle all this while respecting the child's rythm and behavior. For it is important to spell out that it is not a test but an attempt to ascertain the pupil's reasoning in the addition problems set. For experimental purposes, with each pupil, the teacher had to audio-register each interview and complete a checklist, as well as write up an assessment report.

CLASSROOM USE OF THE MINI-INTERVIEW

Regarding the question of feasability, we have kept in mind the following elements: the handling of the interviews, the class organization, the type of activities suggested to the students, and the attention paid to the provided time. Thus our first question, about feasability, has been answered positively, for the three teachers involved in the experimentation Louise, Denise, and Rejeanne, succeeded in using the mini-interview with all their students within the regular time frame of their class,over a period of two to three weeks, at a rythm averaging two interviews a day.

Concerning class organization, it was our intention not to suggest any predetermined model, for we wished to examine how each teacher would handle it in carrying out the mini-interviews within the classroom. The three teachers did not make any radical modification but simply adjusted their normal mode of functioning. Thus one teacher, Rejeanne, took advantage of the time allotted to her students for individual work, in order to carry out her mini-interviews. Louise had to initiate her pupils in working on their own more often without asking for help during the

the mini-interviews with individual students. On the other hand, Denise's pupils were already used to work regularly in teams or in workshops on a wide variety of activities; moreover, Denise had already started individual assessments right at the beginning of the school year. In each classroom, a corner had been set up to facilitate audio-registration of the interviews, but even more to enable both student and teacher to concentrate on the task at hand.

Class organization went beyond just the material aspect, for it also involved planning the activities for the rest of the students, activities which were to be used not simply to fill time, but needed to be integrated into the regular learning activies. Activities such as silent reading, research work, and workcards on social sciences, were handed out rather freely at the beginning. But quite soon, the teachers felt the need to plan ahead and spell out,as rigorously as possible, the work intended for the students so that all of them would understand clearly the instructions and the methods to be used in the tasks they were set, thereby reducing the possibility of being interrupted during the interview. Concerning the feasability question of the mini-interview, we had at the beginning questions about the possibility of letting such young children work on their own or in teams. The experimentation shows us that not only is it possible but that it is even desirable; indeed, the three teachers reported that it proved to be very positive,for their pupils learned to organize themselves, to manage without their constant presence and support, and thus became more autonomous. However, this required that the activities had to be well prepared and that all students understood the work they had been set. In a few instances, the teachers have reported that they were able to do two consecutive interviews, their students being so involved and focused on their work.

One of the characteristics of the mini-interview is its short timespan which must not exceed ten minutes. Respecting this time requirement is quite important for the brevity of the mini-interview is a specific element used in determining the classroom feasability question. The time spent by the teachers was as follows: Louise stayed within the allotted time in 21 of her 25 interviews, Denise for 12 out of 19, and Rejeanne for 17 out of 24, providing us with the following percentages, 0.84 , 0.63 , and 0.71 respectively. Considering that the teachers were trying out the mini-interview for the first time and that they were being held to rigorous experimental constraints, these results are very positive and show that in normal conditions, the mini-interviews can be used in a relatively short time since with 70% of the 68 students interviewed, the prescribed time had been observed.

Our analysis enables us to bring out some of the reasons which might explain why several interviews went beyond the allocated time. It seems evident that this can be attributed to the fact that the teachers were experimenting this new tool for the first time but perhaps, it is also because some of the experimental instructions had not been grasped adequately. In a few instances, the teachers tended to transform the interview into a teaching session or to strongly suggest some hints which might have brought the student to provide a desired response; this occurred mostly with students having some problems. In other cases, the pupil's nervousness made it necessary to repeat or reword questions that had not been understood. The lengthening of some interviews can also be attributed to one of the questions which was aimed at assessing the child's memorization of number facts, a question for which the teacher should have provided only a few seconds for an answer. One teacher in particular, Denise, did not understand the objective of this question since each of her students was allowed time to look for an answer and to explain the reasoning behind it. The lack of adherence to the prepared questions also had an impact on the length of the interviews, especially when useless questions were asked or when unintended questions were added. For one of our teachers, Réjeanne, her difficulty in interpreting immediately the student's procedures and answers led her to repeat some questions or to raise additional ones even when the pupil's procedure was quite evident.

THE MINI-INTERVIEW AS A TOOL FOR FORMATIVE EVALUATION

Relative to the second question, we might add that having trained the teachers to analyze concepts, we were convinced that they had learned to view the acquisition of knowledge as a constructivist process. It was thus natural to expect that they would be more aware of the value of formative evaluation. From our analysis, some evidence can be found to the effect that the evaluations performed by these teachers using the mini-interview, have been carried out in a formative perspective.

We first examined if they could identify the procedures used by each pupil in the addition problems. We then compared the teacher's evaluation with ours. In each case of agreement between our two evaluations one point was attributed for each of the four addition procedures correctly identified, and of course, for each one of the pupils. The correspondence between the two evaluations has proved to be excellent as indicated by the index of agreement worked out for each teacher: 0.92 for Louise, 0.91 for Denise, and 0.86 for Réjeanne. This shows that our subjects, the teachers, have correctly interpreted the understanding of their students regarding the identification of procedures used in the addition of small numbers.

The mini-interview has enabled the teachers to determine the level of understanding of their pupils and has led them to modify their perception of some of the students. For example in Réjeanne's own words, "Most of my pupils were more advanced than I tought"; and for Louise, speaking about one of her students, "Contrary to what I thought, she is not ready to move to the level of abstraction". Furthermore, to her great surprise, she found a pupil who did not understand the meaning of addition, despite the fact that they were well into the school year. The other teacher, Denise, identified a few pupils who did not conserve number; this led her to understand why they were not succeeding in counting on while doing addition.

Since the mini-interview aims at a very precise aspect of the construction of a concept, it can be used beyond the assessment of a student's understanding, to uncover quite accurately at which point there might be a problem. Thus, the teachers were able to identify not only those students who needed special attention, but especially, what kind of help was needed; and this is one of the formative elements of the mini-interview. For instance, some students could not solve problems of addition without concrete material: 8 in Louise's class, 6 in Denise's, and 5 in the class of Réjeanne. This information proved to be quite important, for indeed, it led Réjeanne and Louise to perceive the need for more work at the intuitive level, the latter teacher specifying that she is now aware of the need "for some pupils to work more with manipulatives" and that she will have to "drop several textbook exercises which are dealing solely with the symbolic aspect of addition".For other pupils, the counting on procedure was almost mastered; this allowed the teachers to clarify the type of intervention to be used with them by planning supplementary exercises to consolidate the concept.

At the end of each interview, the pupil was asked a question in order to find out his perception of this new type of individual intervention. The children felt that being alone with the teacher had put them in a privileged situation. As Réjeanne remarked: "This is an opportunity for the pupil to speak to the teacher, especially for those who do not express themselves very much in class". For some pupils, the mini- interview was used to strengthen a newly acquired procedure. For example, one of Louise's pupil's who during the interview discovered that counting on from one of the terms "it always works" and continued using the counting on procedure for the remainder of the interview. In such a case, the mini-interview goes beyond its evaluative objective, but where a few additional seconds are sufficient to initiate or reinforce a desired procedure, the opportunity should not be missed. Nevertheless, care must be taken not to transform the mini-interview into a period devoted to teaching or to remediation.

As a last observation, let us recall that the main objective of the mini-interview was to get at the child's thinking, thus to de-emphasize the answer in order to focus on how the pupil solves a problem. Our experimentation has shown that our three subjects were really aiming at finfing indices, either in the children's actions or in their verbal answers, which would provide a better grasp of their reasoning. And this sentence of Denise's indicates definitely a formative effect of the mini-interview when she writes about what she learned from it is that in the future she will be more inclined "to ask the child how the answer was found".

CONCLUSION

These few results from our analysis lead us to believe that the mini-interview is a tool that can be used by teachers themselves in their classroom with each and every one of their pupils. Moreover, the mini-interview can play a formative role in evaluation. As described in this paper, many indications show that in addition to enabling them to gather information on their students' understanding, the teachers' use of the mini-interview provided them with feedback on their teaching.

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CHANGING TEACHING STYLES

The development of a model for effective in-service courses

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This paper discusses a two-year research project on helping teachers to change their classroom styles. It focuses on two interlocking aims. The first was to devise a model for in-service courses which would enable participants to genuinely alter their perceptions of the ways in which they could enable their pupils to learn mathematics. The second aim was concerned with a particular medium through which this change could be effected. There has been growing evidence that children, and indeed adults, achieve greater enjoyment and success if they are encouraged to become involved in mathematical thinking rather than merely receive mathematical thought. Changes in the English public examinations at 16+ have in effect forced this doctrine into the classroom by including investigations, extended projects and problem solving in the syllabus to be assessed. The research project selected investigations as the basis from which to work and aimed to enable reluctant and even hostile teachers to effect the necessary change to a more investigative way of teaching

It was taken for granted that no person can change another. Only the teacher herself can effect personal change and those who offer in-service courses have therefore to provide her with the optimum environment within which this can take place. It is relatively easy for attitudinal changes to occur on a surface level: the zeal of the presenters can be infectious, but "it is not sufficient to take the ordinary teachers...and by a short course of lectures arouse in them a temporary enthusiasm for new methods" (Perry, 1902). For there to be a lasting change, able to withstand the pressures and assumptions back in the world of school, there must be personal motivation, conviction and involvement in the proposed new ways of thinking or behaving. A teacher must take away within herself a belief in the new set of values, a belief that they are relevant to her, a belief that change is necessary and a firm belief that such personal change is possible. To be effective, in-service courses must not provide merely a 'bolt-on package', but a way of altering one's philosophy of education and understanding of learning. There are criteria around which the course model was built, but equally fundamental to the design was acceptance of the premise that change can be very threatening. In this instance it challenged both the perception which teachers had of their knowledge of the content of the subject and their skills of class management.

Confrey (1984) contends that "Implementing instructional approaches that encourage a focus of process and more independence, persistence and flexibility in the students requires changes in students' conceptions of mathematics". True, but it first requires such changes in the teachers. Without these, 'new' curricula can rapidly become moulded to fit the structure of the old.

The major element of the course model was the creation of a supportive environment within which teachers could work and think with confidence. The intention was to make teachers aware of their own strengths, enable them to achieve mathematical success and pleasure themselves through a new medium, support them in their initial attempts to change their classroom practices and create a non-threatening environment in which to discuss their progress. The structure of the course was therefore built around four key elements:

a) each participant's personal experience of doing an investigation;
b) small group reflection on this experience;
c) each participant's personal experience of taking the same investigation into their own classroom;
d) small group reflection on this experience.

The course was spread over a teaching term and took place out of school hours. The first session ran from 5.30 on a Friday evening through to 8.30 on the Saturday evening with the aim of welding the group together as a supportive unit. Subsequent sessions took place fortnightly and were of two to two-and-a-half hours duration. There was a further whole day long meeting in the middle of the course. Time was provided at each session for the teachers to work on a suitably flexible mathematical investigation at their own level of understanding. Small groups of five or six teachers who had been playing with the same problem then spent time together discussing their workings and trying to take their explorations of the problem further. A tutor joined each group with the passive role of listener and covert task of ensuring that more mathematically able or dogmatic teachers did not pose a threat to the less secure and confident. At some stage each evening the participants returned to the whole group for some relevant input by the course leader. This included analysis of processes, elements of classroom planning, approaches to assessment and topic-based work. The final part of the evening was spent back in the small groups planning, with the help of the tutor, how to take their particular investigation into their own classroom - this 'homework' to be done during the two weeks before the next meeting. Careful attention was paid to the composition of these small groups, which were changed for each new set of investigations. It was there that mutual support was vital, but at the same time no teacher was allowed to become, even subconsciously, dependent on any one other teacher. Each had to make the change in teaching style their own. The start of each of the evening sessions was a small group discussion of "how the classroom experience went".

The course model was trialled on a cascade principle. Initially I ran the course at Oxford University. The participants were aware of its experimental nature and the final session was devoted to evaluation. Participants were asked, through an anonymous questionnaire and subsequent informal discussion over refreshments, about their initial expectations of the course, their reasons for not already using an investigative teaching style, their emotional reactions during the course and their current assessments of the effect of the course on their future teaching. Specific detailed suggestions for improvements to the individual sessions were also sought. The presentation and content were modified and the series of sessions written up in such a way that two advisory teachers and two heads of departments in local schools could run a second course at which I was present. This time, teachers were encouraged to come with a colleague from their own department if possible, but were deliberately prevented from working together in the same small group. Again the final session was devoted to evaluation. The heads of department were then able to take an adapted model back to their staff at school and the advisors had the materials to run similar courses within their local education authorities. One of these latter courses was also evaluated. A follow-up study was then initiated to look at the participants of the first two courses. The purpose was to assess the long-term influence on their teaching styles.

The method adopted in the evaluation session - that of open-ended questions such as "What were your expectations of the course?", "Were there occasions when you felt 'good'?" - meant that teachers responded in their own words, but it was possible to group the replies under certain headings. The absence of response, however, did not necessarily imply a negative reaction. Some of the questions were posed to shed light on the course model, and some to examine the area of content, namely investigations. Only those affecting the construction of the course model are presented here. Tables of results are presented in percentages followed by brief comments. General conclusions are suggested after the follow-up study has been considered.

RESULTS OF EVALUATIONS

From Questionnaire	Evaluation at end of: 1st	2nd	3rd course
<u>Expectations of the course?</u>			
Encouragement and know-how	69	53	50
A collection of materials	63	38	40
To meet and discuss with other teachers	38	16	16
To learn how to assess investigations	19(6)*	28(9)*	40(23)*°
Ideas on classroom management	56	34	60
To understand the value of investigations	19(6)*	3	16

*The percentages given in parenthesis indicate those whose expectations were not fulfilled by the course.

°The rising dissatisfaction in the session on assessment can be directly related to the rising panic among teachers, following the government's announcement of the new national examination. Neither the course leaders nor anyone else at that time had 'the answers' on what marking schemes would be used for this examination.

<u>Emotional reactions during the course?</u>				
'Bad'	Mathematically insecure	31(19)	31(25)	37(23)*
	Tired	31	28	10°
	Overwhelmed	19	6	20
	'Know-alls' spoiled it			13**
	Cold workroom			10°°
'Good'	Enjoyment	69	34	20
	'Eureka' when working	63	56	50***
	Working in a group	19	25	30

*The percentages given in parenthesis indicate those who <u>commented</u> that they had overcome these feelings during the course.

°The fall in complaints of tiredness may be attributed to the fact that the third course was held near the participants' schools, therefore involving much less travelling.

**This comment is interesting as it only occurs in the third course questionnaire responses. The original course model provided for monitoring of the small groups to prevent this happening and this feature was not present in the third course.

**Ten weeks later, the teachers still remember the one very cold evening. Comfortable environment is important.

***The high number of spontaneous responses in this category underlines the value of the philosophy behind the course model, which was to allow participants to work at their own mathematics before considering their craft as teachers.

<u>Did more than the set homework?</u>			
No	13	25	37
Same investigation, different class	38	59	37
Same class, more than one investigation	50	31	30
<u>Normal teaching style affected?</u>			
No (and no response)	38	9	33
Yes	62	91	67
These were either: More investigative	44	72	57
and/or: More confident	25	25	20
<u>Intended doing investigations next term?</u>			
No-one suggested that they might not do investigations.			
Regularly	75	56	57
Integrated on the syllabus	56	47	43
One-off events	13	38	37
<u>Have involved colleagues back at school?</u>			
No	50	37	67*
Several	44	50	20
One other	6	13	13

*This figure includes many teachers all of whose departments had been on one of the three courses.

The follow-up study was conducted by Robin Grayson (1986), who interviewed the participants during the year following the courses.

Comments relevant to the course model are discussed here. With hindsight all the participants considered the courses enjoyable and beneficial, with the exception of two teachers who were already well experienced in the use of investigations and confessed to having come on the course merely to get new materials. The use of Saturdays was found to be acceptable, giving time for greater depth of study, and the requirement to "try out and report back" was appreciated. It forged a link between personal work, course theory and classroom practice. Attending with a school colleague was seen as valuable as it provided continued support away from the course but many were also pleased that they had been separated during the sessions as a consequence of the philosophy of the course model, as this provided a wider pool of experiences to draw on. The original intention of this policy was to protect the more junior teachers from the fear of assessment by their heads of department, but this was obviously never made explicit. The difficulties course members encountered when trying to continue investigative work after the course ended were discussed during the interview. One unexpected reaction was to the timing of the course within the school year. The first course was held during the last term of the year, and, not knowing their new timetables, teachers were unable to commit themselves to doing investigations in the future with any specific group of pupils. By contrast, the second and third course participants were able to make statements such as "With classes 2A and 3B...." with a certainty that they would not be inhibited from carrying this intention out by changes in teaching schedules. The factor of pupil resistance was frequently raised, but the teachers felt that pupils became less resistant over a period of time, and the courses had provided the confidence to persist with their new style of teaching. Problems of seemingly unsuitable school sites, teaching schemes and lack of resources were seen as diminishing obstacles to a change in teaching styles, given the personal confidence and motivation to change acquired on the course. Confirmation of the course model came through the finding that, as already to some extent indicated by the above statistics, the major single benefit reported by teachers in the follow-up study was increased personal confidence. It was this which gave them the willingness and impetus to experiment in their teaching.

Several recommendations for those with the immediate problem of designing in-service courses can be culled from these results, but what are the implications which can be drawn for the more general psychological aspects of effecting changes in teaching style? The increase in confidence came, not from instant classroom success as they returned from the course, but out of the personal belief that the new style of working which they themselves had experienced and enjoyed would facilitate the pupils' understanding and enjoyment of their mathematics. This personal conviction, through personal involvement enabled them to comprehend, and not merely to 'hear', the value of the change.

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THE LEFT AND RIGHT HEMISPHERES OF THE BRAIN AS A MODEL FOR IN-SERVICE TEACHER TRAINING.

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Abstract: This report discusses the left brain/right brain model as a method of evaluating didactic teaching methods in response to teachers questions. A right brain bias is implied throughout.

There is an increasing amount of evidence to show that two different aspects of learning mathematics, i.e. language functions vs. mainly spatial processing, are linked to the activities of different halves of the brain.(1)

In the P.M.E. Israel 1983 two papers discussing the influence on learning and teaching of the two hemispheres of the Brain were presented. Fidelman(2) in relation to learning and higher mathematics and Yeshurun(3) in relation to learning and elementary school mathematics. However, Yeshurun, ibid, remarks that a right brain approach with elementary school mathematics is possible since most of the topics can be presented both sequentially and globally but that this mode of presentation is not possible in secondary school mathematics as each branch of mathematics has its own mode of presentation. In other words the topic determines the approach.

Many educationalists appear to dismiss the left/right hemisphere model because they feel that a child's left/right orientation is determined at birth or at least during the formative years and there is not very much one can do to change it. Others maintain that some attempt should be made to determine the degree of left/right brain bias for each individual child and then use appropriate teaching techniques. Still others maintain that mathematics is a predominantly left brain activity and should therefore be taught as such (standard text book approach).

All the above approaches (no matter what the terminology used to relate to right or left brain) appear to use a linear scale such as:

DIAGRAM 1.

as a model of the child's thought (or topic requirement) with the child (or topic) sited somewhere on the scale.

It is my contention that both left and right brain approaches are possible with most topics in mathematics and it is important for the teacher to at least be aware of this and to help me identify various teaching approaches, I use, not a linear scale, but a 2-dimensional approach, viz:

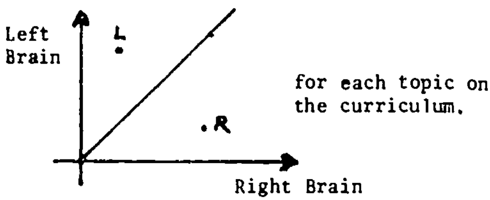


DIAGRAM 2.

In every school maths curriculum and in every age range there is at least one topic (or concept) which presents itself as a barrier to further advancement in maths. It can be as simple as multiplication by 10, or more complicated such as long-division or addition of fractions, with the barriers becoming greater as the pupil climbs higher up the ladder. Eisenberg and Dreyfus[(4)] refer to this; "one continues to take Mathematics courses until one doesn't succeed any more". In the primary to early secondary school situation there is no opportunity to "drop out" entirely, it is usually only possible to "drop down" a stream, so these barriers have to be either "overcome", "bypassed" or may be even just "ignored" or else they remain barriers.

One way to cope is to "learn the rules" and "play by them" or have a machine, like a calculator, that "knows the rules" and so all one has to do is to have confidence in the machine and feed it the information and if the rules for handling the machine are simpler or in the eyes of the user more consistent than those of, say, subtraction of negative numbers, so that the right answer can be achieved most of the time, then everything is rosy. Thus we are presented with a picture of the pupil who succeeds in maths being the one who is able to play by the rules, whether he understands them or not and the successful teacher is one who can get his pupil to play by the rules, whereas a pupil who has grasped the concept "sees" what to do and the barrier disappears.

This "getting the pupils to play by the rules" very often becomes an obsession going in some cases to extremes, for example, I've seen some teachers spending the best part of a lesson on SETS, with instructing the pupils on the correct way to draw the curly brackets $\{\ \}$ or defining an order for removing brackets from an expression like:

$$6\left\{23 - [4.(7 - 5) + 3]\right\}$$

and woe betide any pupil who deigns to write the expression:

$$6[23 - (4.\{7 - 5\} + 3)]$$

and it certainly won't permit a pupil to "see" a solution to a problem such as "the sum of two numbers is 13 and their product is 36, what are the two numbers?".

The "SEE" used is a right brain process. Let us consider some specific requests from middle school teachers during in-service workshops.

<u>Teacher 1)</u>. The class has been occupied with inserting the inequality sign between pairs of directed numbers. As expected some were carrying out the task successfully, others sometimes with the correct answer and at others not and, of course, others practically always wrong. The pupils were using the following working definition:

If the two numbers are positive then the larger number is furthest from zero, and if the two numbers are negative then the number nearest the zero is the larger. (Of course if one number is positive and the other negative then the positive number is the larger.)

The teacher had two objectives:

i) to enable the pupils to see that the larger the number the further right it is on the number line, (a unification and hence right brain approach)

ii) to enable the pupils to give the right answer instinctively (again right brain).

Teacher 2). The class is working well with the addition of directed numbers and is also beginning to handle open sentences such as:

$(+3) + \square = (-1)$ (Right brain)

but for all that, I know that we will not be able to make a successful transfer to subtraction. Eventually I will have to make the statement that to subtract we add the (additive) inverse and the pupils will learn it without understanding the rule in the same way as they learnt how to divide fractions. Working with this rule (left brain) interferes also with the addition that the class had mastery of (or so I thought). Is there no way to unify the two operations? (right brain)

Another Teacher (same topic) 3): my class seems to be handling addition and subtraction of directed numbers without too much difficulty but we don't seem to be convinced that adding the (additive) inverse is equivalent to subtraction.

Teacher 4). When teaching the solution of two simultaneous equations is it better to equate coefficients and subtract or get to the situation where the coefficients are additive inverses and then add?

The above examples are with concepts or topics that the teachers feel must be taught. Let us now consider some topics which the teachers would prefer not to teach.

Teacher 5). Is it in order if I do not teach the area of a circle to my 8th graders? They are better off doing algebra anyway.

Teacher 6). I don't want to teach Pythagoras. It doesn't fit into the syllabus and it just takes up the class time and anyway the children never understand it.

Teacher 7). The book says that we have to teach inequalities of the type: $|x - 3| + |x + 4| < 11$, how is it possible to do so successfully with an 8th grade class in the little time that is available?

I deliberately chose these examples of Teachers' requests for two reasons:

i) they require a reply which appeals to the right brain (left brain solutions didn't satisfy the teachers)

ii) Didactical questions that could be dealt with using a predominantly left brain approach were practically non existent and in any case solutions were usually given by other teachers at the workshop.

One other point, a paper of this type, is by its very nature predominantly left brain orientated so it will be very difficult to demonstrate a right brain approach and therefore apart from one attempt which will follow shortly a full discussion of possible right brain approaches to the teachers questions will be left until this paper is presented when perhaps other suggestions will be forthcoming.

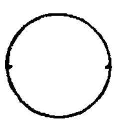
Betty Edwards[5] classifies Left-Mode and Right Mode characteristics as follows:

L - Mode	R - Mode
Verbal	Non verbal
Analytic	Concrete
Symbolic	Analogic
Rational	Spatial
Logical	Intuitive
Linear, etc.	Holistic, etc.

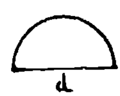
If we are to appeal to the Right brain with our solutions we must be able to describe the approach with more of the R-mode terms than those of the L-mode and the approach will be at some point R (see diagram (2)).

I now present the suggestions accepted by teacher 5,

Take a circle



fold in two



What is this bit (the part labelled d) called?

Fold in two again

What is this part (the part labelled r) called?

Work with your neighbour and make different shapes from the two folded parts of the circle, give names where appropriate. (Encourage different shapes)

semi circle

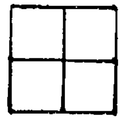
butterfly

Square - Anything special about the square?

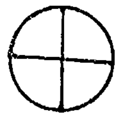
Yes it is the square whose side is the radius of the circle or

"square on the radius"

use the square as a template and draw 4 squares thus



Take your circle back



How many squares will your circle cover?

A left brain approach using the formula πr^2 results in many children (taking π to be 3.14 and r to be 5) using a calculator and keying in 3.14 x 5 =

x

=

i.e. $(3.14 \times 5)^2$ and not 3.14×5^2

Whereas using the right brain approach with the emphasis on the square of the radius the child will generally key in 5 x 5 =

x

3.14

=

Judicial use of a right brain approach even simplifies the algebra from 2 dimensional to uni-dimension(6).

Now see if you can think of right brain orientated suggestions for the other questions listed.

In conclusion, neither the left brain or the right brain can work in isolation, there must be interaction between the two and the teacher must be able to stimulate both hemispheres of his students' brain otherwise no interaction can take place.

It is the task of the teacher to devise (or master) activities or approaches which will (a) encourage the pupil to strengthen his weaker hemisphere and (b) to use his stronger hemisphere to maximum effect in exactly the same way as a soccer coach would devise activities to encourage his player (a) to strengthen the weaker leg and (b) to increase the range of skills on the stronger leg knowing full well that even though the player prefers one leg over the other, both need to be developed.

In order to encourage practising teachers to develop both hemispheres they themselves have to be made aware of the model and in many situations their own appropriate hemispheres have to be strengthened.

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CHILDREN'S AND TEACHERS' MATHEMATICAL THINKING: HELPING MAKE THE CONNECTIONS

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This paper reports on the varying effectiveness of workshop techniques that facilitate reflective awareness in teachers who possess different beliefs about learning, teaching and mathematics. Three prototypes of belief systems were developed to make sense of participant responses toworkshop techniques and were designed to capture the range of beliefs expressed in the workshop. Techniques included the use of videotaped clinical interviews of children solving mathematical problems, autobiography, triadic problem-solving, and a projective test. Workshop techniques encouraged different kinds of teachers to explore their own learning and beliefs and to make connections between their own learning, children's learning and their teaching practices.

OBJECTIVES

The objectives of the research reported in this paper are twofold: (1) To assess the effectiveness of workshop techniques for facilitating reflective awareness in elementary school mathematics teachers and (2) To examine, with the aid of three prototypes of teacher belief systems, how these techniques affected different kinds of teachers.

REVIEW OF THE LITERATURE

The theories and research which inform the design of the workshop and the research on the workshop derive from two sources. The first source is the literature which describes and analyzes children's mathematical thinking and which has resulted in a richer understanding of children's cognitive processes in the mathematical domain (e.g., Ginsburg, 1983). Recent work on the development of mathematical thinking has, in Schoenfeld's (1982) phrase, moved "beyond the purely cognitive" to consider the affective and reflective aspects of mathematical learning. The consensus of this research is that students' awareness of what they are learning, how they are learning, and what they are feeling as they learn, are crucial components of the learning process. Teachers who can facilitate self-awareness in children are likely to improve the learning process. The purpose of the workshop was to facilitate self-awareness in teachers so that they would be able to do so for their students.

The second literature source is on teacher belief systems. Brown and Cooney (1982) suggest that teacher beliefs, which are strongly held and acquired through enculturation and education may not be debatable or open for consideration by teachers. This would help explain the varying degrees to which teachers accept, internalize, and practice what they learn in methods courses (Mundy, Waxman & Confrey, 1984). Teacher belief systems are not only important in understanding teachers' behavior in the classroom; they are also crucial when considering teachers as learners. Furthermore, teachers themselves need to understand the relationship of beliefs to practice.

DESCRIPTION OF WORKSHOP

Innovative teacher education workshops (e.g., "SummerMath for Teachers," Mundy, Waxman & Confrey, 1984) have encouraged teachers to be constructivist in their approach to teaching mathematics. Other recent workshops have used videotapes of students performing mathematical tasks as springboards for teachers' understanding of students' mathematical processes (e.g., Ginsburg's video workshops for teachers, 1985). The workshop examined here incorporates both of these approaches in an attempt to provide teachers with the knowledge, skills and perceptions needed to facilitate self-awareness in children. The objectives of the present workshop "Understanding Teachers' and Children's Mathematical Thinking" were to have teachers examine: (1) The development of children's mathematical thinking, (2) Their own beliefs about mathematics, learning and how they learn, and 3) The connections between their beliefs about themselves as learners and teachers, and how children learn mathematics.

Workshop techniques served the dual purpose of facilitating teacher self-awareness and data collection on teachers' perceptions of self-learning, teaching, the nature of mathematics and children's mathematical thinking. Techniques included: Videotaped clinical interviews of children solving mathematical problems (at different age levels); journal assignments; a mathematical autobiography; triadic problem-solving; exploration of unusual algorithms for arithmetic operations; development of curriculum materials which encourage self-reflection in children; administration and discussion of a projective test (Mueller & Ginsburg, 1986) to tap unconscious attitudes with regard to learning and teaching; and pre, post and follow-up questionnaires.

Subjects who participated in the workshop consisted of nine pre-service and six in-service teachers from a cross-section of backgrounds. Their ages ranged from 24 to 50 and they taught in urban and suburban school systems.

FORMULATION OF PROTOTYPES

Examination of the fifteen workshop participants' responses to the above exercises and techniques allowed us to formulate three hypothetical types of pre and in-service teachers. These prototypes are not meant to be used as unequivocal categories. They were designed to capture the broad range of beliefs expressed by the workshop participants and are used for heuristic purposes. The three types are: The "Traditionalist," the "Constructivist," and the "Reflective Math-Phobic." We characterized these three types on the basis of their beliefs about learning, mathematics, the development of children's mathematical thinking, and how to teach mathematics. We will consider each type according to these belief dimensions.

Beliefs about Learning. The Traditionalist saw learning as a function of rote memory, of mastering one aspect at a time, and of being externally rewarded. Learning was also seen as a function of innate intelligence. The Constructivist saw learning as a function of exploration, discovery and effort. Learning was seen as being its own reward, and discovery was equated with ownership of the material. The Reflective Math-Phobic showed conflict in her beliefs: On the one hand, she believed that learning was a process, however, she also believed that perhaps success in learning was a function of intelligence.

Beliefs about Mathematics. The Traditionalist saw mathematics as the mastery of rote algorithms. The Constructivist saw mathematics as heuristics, problem-solving and part of everyday life. The Reflective Math-Phobic saw mathematics as aesthetic, profound and with important connections to the world of science and art, but felt cut-off from those connections through her lack of understanding of the structure of mathematics.

Beliefs about Teaching. The Traditionalist believes in teaching children by modeling correct algorithmic procedures, drill and practice, flashcards, workbooks and rewards for success. The

Constructivist teaches the underlying reasoning behind algorithms through probing questions to children and provides children the opportunity to work with manipulatives. The Reflective Math-Phobic takes a laissez-faire approach by allowing the children to go at their own rate and does not like to intervene with questions.

INTERACTION OF TYPES WITH TECHNIQUES

1. Autobiography. Prior to coming to the workshop the students were asked to writetheimathematicaautobiography. The Traditionalist did well at mastering algorithms in elementary school but had problems later on with algebra and geometry. The Constructivist did not like arithmetic, but did well with math in high school or with teachers who were more flexible in their approach. The Reflective Math-Phobic did not like math at any level but enjoyed an intellectual approach to history and literature.

2. Analysis of videotaped clinical interviews. The Traditionalist interpreted the children's errors as lack of knowledge or ability. The Constructivist saw errors as a clues to the child's mathematical thinking. The Reflective Math-Phobic saw errors as a function of fear and anxiety.

3. Triadic Problem-Solving. The Traditionalist was anxious about their problem-solving performance, intruded her own solutions when acting as a prober, and did not know what to observe when in the observer role. The Constructivist was comfortable in the problem-solving role even when they could not get the answer, did not impose their own thinking as a prober, and recorded strategies the problem-solver used when in the role of observer. The Reflective Math-Phobic was afraid that her problem-solving performance would reveal their mathematical ability, found it difficult not to impose their own thinking when in the prober role, and had highly detailed observations on all aspects of the task.

4. The Projective Test. Four pictures of learning scenes, one with parents and three in a traditional school setting, were presented to each participant who was then asked to write about each scene. The Traditionalist perceived parents and teachers as telling the child

what to do and the child as not knowing the answer. The Constructivist perceived that the parents and teachers were helping the child or merely being available if the child wished to be helped, and perceived that the child as patiently struggling with the problems at hand. The Reflective Math-Phobic perceived that the parents and teachers were ignoring the child and perceived that the child was daydreaming or too anxious to perform.

SUMMARY AND CONCLUSION

The "Traditionalist" was uncomfortable with many of the workshop exercises but was jarred into rethinking their teaching approaches. The "Constructivist" already possessed a process approach to learning mathematics, both in children and in herself. For this participant, the workshop techniques proved most valuable as a chance to incorporate reflective processes into teaching practices. The "Reflective Math-Phobic" began to feel better about her own learning history and to become more interested and curious about the processes children use to learn mathematics. The discrepancy between her views and her behavior as a teacher came to the fore.

Data from this workshop suggest that it is crucial to take into account teachers' belief systems about learning, mathematics and teaching when formulating teacher education curricula. The teachers' exploration of their own learning and beliefs vivified the connections between their learning, their students learning and educational practice.

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Research Reports on Projects with Inservice Teachers: A Reaction

by

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For the purpose of discussing these papers on inservice teaching, I find it convenient to group some of them together. The papers of Dionne, Waxman&Zelman, and Jaworski&Gates deal with assessing teacher's beliefs or influencing teacher belief systems. The papers of Ben-Chaim, Fresko, &Eisenberg; Pirie, and Nason&Lappan propose intervention via training and workshop activities to bring about desired changes in teachers' styles, beliefs and teaching methodologies.

The two papers from Portugal, (Monteiro&Ponte and Abrantes&Ponte) are concerned with involving teachers in computer use in mathematics classrooms. Finally, we have papers by Nantais on the feasibility of teachers conducting mini-interviews with their students, by Maher and Alston on an inservice project in which teachers solved problems in groups, and by Rosen on the feasibility of left-right brain analysis for didactical questions raised by teachers.

I will not attempt the nearly impossible task of uniting this wide spectrum of papers under any one roof, but I will try to react to them in groups when it is feasible.

Teacher Beliefs

I wish to start with this group of papers, because it makes sense to me to explore the belief systems of the teachers you are interested in changing before you attempt any change activities.

The themes of "reflective activity" and "constructivism" echo throughout the three papers on teacher beliefs. These researchers wish to induce metacognitive activity among in-service teachers for the purpose of developing a more constructivist perception of mathematics. Dionne attempted to lead teachers to a more constructivist view in an experimental inservice course. Beliefs were assessed by having teachers explain how they graded student's answers to a test items, by a questionnaire, and by pre-post interviews with the teachers. Waxman encouraged reflective awareness of mathematical processes among teachers by means of interviews of children solving problems, by teacher autobiographies, and by group problem solving by teachers. Jaworski stimulated reflective activity by having groups of teachers watch videotaped segments of actual mathematics classes, and subsequently relate what they saw to their own classroom experiences.

In Dionne's paper we find evidence that the experimental inservice course had effects on how teachers perceived children's mistakes. After the course subjects made a distinction between computation and reasoning errors among children, and mentioned that the old "children have it or they don't" response is far too simplistic. The beliefs of these predominantly constructivist teachers in Dionne's study appeared to be strengthened by the course. This prompts several questions. First, what exactly was the material covered in this experimental course, and how was it covered, that is, what mathematics was taught, and how was it taught? Information about the course seems crucial to other researchers in this area. Second, what would happen if the volunteer sample had not had a predominantly constructivist orientation to begin with? Is it possible that volunteers for such a study are more likely to be constructivists? Would it be likely that "non-constructivists" teachers would show little change of perception on the post interviews? It seemed that those teachers who came to the study ready to explore and change their beliefs did so, while Jacques did not. More case studies may be needed to see if the expermental course has any hope of moving the Jacques-type teachers away from their fixed perceptions of mathematics teaching and learning.

One of the real strengths of the Dionne study is the three different methods of collecting information about teacher beliefs, the "correcting test", the questionnaire, and the interview. In this way a profile for each teacher's beliefs can be reliably synthesized by analyzing their responses in several different contexts. This same strength is also apparent in the Waxman paper in which three prototype belief systems were distilled from teacher responses to four distinct data collection techniques. The prototypes "Constructivist", "Traditionalist", and "Math-Phobic" may be very useful to other researchers who are attempting to identify teacher beliefs. The paper operationalizes each of these terms by describing specific behaviors and response patterns for each prototype. According to Waxman, Traditionalists believe in rote memory, inate intelligence, drill&practice. Constructionalists believe in exploration&discovery, underlying reasonin processes rather than mere answers, and heuristics and problem solving. Math-Phobics seem to be caught between the other two types, mouthing constructivism while behaving traditionally. The interaction of each prototype with the techniques--videotapes, autobiography, problem solving activity, projective test--suggests that Waxman has devised an excellent way to assess some aspects of teacher's beliefs. For example, traditionalists interpreted children's errors on the videotapes as lack of knowledge, but Constructivists saw clues to the children's thinking processes rather than errors. The Math-Phobic, predictably, felt that errors occurred because the children were anxious.

There are several issues that are raised by the Waxman paper that those of us involved in in-service "change" projects should consider. It may be well to first find out "who" we have in our workshops, that is, what our teachers' beliefs are, and then work on providing a basis of change for those who have a chance of changing. There are teachers who are not going to budge, no matter what kind of inservice we offer them. Another possiblity is to sprinkle a few Traditionalist type teachers among a group of Constructivists so that the majority is already ripe for change, in hopes that some of the Traditionalists can be made more flexible. It would be possible to asses the beliefs ofr teachers ahead of time using the techniques in the studies of Dionne and Waxman. A second issue that comes to mind is the role of mathematical content. Is it possible that the Math-Phobic prototype is on shaky gounds mathematically, and therefore does not have the confidence to merge voiced beliefs with actual practice? In a broader sense, what is the role of mathematical content knowledge in determining teachers' beliefs?

The Jaworski paper also used videotapes, but with the purpose of stimulating group discussion and reflective activity. Teachers viewed taped excerpts both from their own classroom, and from other teachers' classrooms, and their comments during and after the viewing were audiotaped. The paper reports that different teachers, viewing the exact same segment of videotape, reported seeing entirely different things going on. This finding makes sense in light of the three different belief prototypes reported by Waxman, and corresponds to the different ways that teachers analyzed errors on her test items. While Jaworski did not report any systematic investigation of teacher's beliefs, the videotape segments show a great deal of promise for ferreting out beliefs and biases. Thus, Jaworski has provided researchers with yet another creative methodology for assessing beliefs. Jaworski may benefit from using some of the other methods for assessing teacher beliefs, such as those reported in the Dionne and Waxman papers, in conjunction with her videotape reporting sessions. In this way any claims that are made about teacher beliefs can be substantiated by a second or third information source.

Jaworski poses some interesting questions for future research. What is the effect, if any, on the classroom activity of teachers who participate in these videotape sessions? How important is the selected tape excerpt itself to the quality of discussion (one would think very important)? Jaworski's method also has the potential to allow comparison of teacher perspectives with student perspectives on a segment of classroom interaction, since viewers can simulate the role of students, and examine the lesson from their perspective.

In summary, here are some questions that may deserve attention from those researchers who are investigating beliefs of inservice teachers.

1. What is the interaction between mathematical content knowledge and beliefs about teaching and learning mathematics? Is there a relationship between a teacher's preservice content and methods background, and the prototype belief system she(he) espouses (i.e., Constructivist, Traditionalist, etc.)? Beliefs do not form in a vacuum.

2. For that matter, are there any relationships between classroom management techniques, class environment, and teacher belief systems?

3. Are there any relationships between the professional activity of inservice teachers and their beliefs? That is, are teachers who are involved in inservice activities such as workshops, attendance at local and regional meetings, memberships in NCTM and local organizations, more likely to profess certain types of belief systems?

Indeed, these three questions suggest that studies investigating correlates of teacher belief systems are highly desirable. The papers at this conference indicate that techniques for assessing teacher beliefs are already in hand. A natural next step is to ask about relationships of beliefs to other teacher variables.

4. Are the prototypes "Constructivist", "Traditionalist", and "Math Phobic" adequate and useful in describing most teachers beliefs about mathematics? Are there other prototypes that may be discovered if one were to investigate beliefs of secondary level or college level mathematics teachers?

5. What is the next step? That is, once you know something about the beliefs of the teachers you are working with, what do you do about it? What types of inservice experiences do you plan in order to enhance, or even change, those beliefs?

This last question brings us to the second group of papers, research on long term inservice programs.

Changing A Teacher's Styles and Practices

The papers of Pirie, Ben-Chaim, and Madsen-Nason have several common themes. First, the teachers in these studies were actively involved in doing some mathematics in groups. The intent of all three researchers was to model both content and processes in such a way that they could be carried directly into the classroom by the participant teachers. Second, there was a provision for reflective activity and discussion in groups. You will recall that this type of reflective activity on the part of the teachers was also important in the set of papers on beliefs. Third, these researchers are attempting to evoke definitive changes in teachers' perceptions of mathematics, and to induce corresponding changes in classroom teaching behaviors. Finally, there is deep personal involvement and commitment to the teachers over an extended period of time (several years!) on the part of the researchers. The establishment of a "group identity" among the teachers is crucial in the studies of Pirie and Madsen-Nason.

Pirie's paper provides a nice transition from our discussion of teacher beliefs. She is primarily concerned with altering teacher's perceptions of mathematics by involving them in "investigations", reflective activity, and group discussion. Pirie states from the outset that change cannot take place unless the teacher her(him)self effects the change. This is a crucial reminder for those of us who would like to bring about change through inservice activities. Teachers, like any other people, change from within. Since change takes place inside, no amount of "top down" dictating will effect permanent change if a teacher is not ripe for it from within. Thus, Pirie suggests that it is the job of inservice to provide an "optimum environment" within which change can take place. Pirie's environment involves a tightly welded group identity, a comfortable peer group within which teachers feel free to reflect on mathematical investigations, and to discuss and share personal experiences. Many of the elements of a counseling support group are evident in Pirie's group of teachers--peer support, freedom to express without judgment, networking with others (teachers) who have similar concerns (bring a friend to counseling session). Each group session started off with "how things went" this past week. The strength of Pirie's methodology is the establishment of this very strong teacher support group.

Once such a support group has been established, the potential for effecting a change of perception about mathematics is heightened because teachers are not isolated in their attempts to change. This type of support may be particularly important for effecting change among teachers who are "reluctant and even hostile", as Pirie says, to pursue a more investigative way of teaching mathematics. Such group support also is crucial for the teacher who has to go back to a building where perhaps he(she) is the only one whose perceptions of mathematics are changing. It is very difficult to "fight the good fight" all by yourself when your colleagues are criticizing you for doing something different. "Why do you want to do that, we've always done it this way, and it's worked just fine."

It is not, then, surprising that Pirie reports success in altering her teachers' perceptions of mathematics. Several other issues come to mind while reflecting on this paper. Any attempts to reliably assess long term influence of such a course on teaching styles may involve more than just a questionnaire to substantiate the changes. Documentation of increased inservice activity, and classroom visitations and observations over a period of several years, will provide further evidence that the desired changes are actually taking place. It is just too easy for teachers to slip back into old ways when they are out on their own away from that su~port group.

A second issue involves the actual "investigations" of the course. This is another case (as in Dionne above) where other researchers could benifit from knowing exactly what mathematics was "done" by the groups. The selection of appropriate problems for "reluctant and hostile" teachers may be crucial to the successful formation of a mathematical support group. I believe that the actual mathematics is at least as important as the process of investigation.

The key word in Ben-Chaim's paper may be "intervention." This three year project includes not only classroom intervention, but intervention on an administrative and organizational level. The hope is to establish unified plans for mathematics instuction at each grade level in each of the seven project schools. This is a much broader and ambitious task than changing individual teachers' styles and practices. Yet, changing the teachers is at the core of Ben-Chaims' efforts. Without the horses, the cart will not move. Ben-Chaim utilized "master interveners", teachers who observed the lessons of the participating teachers, offered advice, and gave demonstration lessons. These master teachers encouraged reflection on what was occurring in lessons, and offered alternative instructional approaches to the teachers they were working with.

Ben-Chaim reports that while the administrators and principals were very supportive of the program, the teachers themselves were "only mildly interested." The paper reports only the first year's activity, so perhaps more teacher enthusiasm will be forthcoming. On the other hand, the approach in the Ben-Chaim project is top-down. Participation in the program is, or might as well be, mandatory in each school. It is not surprising that administrators like the program, since it appears to be "their baby". One wonders what stake the teachers themselves really have in the program so far. From their point of view, this project may be "happening to them," outside their locus of control. They have administrators telling them that they will participate, and master teachers telling them how to do it right. Perhaps a missing ingredient in this project is the establishment of a peer group among the participating teachers, similar to the Pirie project. It may be possible to rectify this situation, since the project is still in its early stages.

As in some of the other studies from this session, it is not clear in Ben-Chaim's report just what mathematics, or what alternative methods, were proffered to the participants by the masters. Specific examples would be of great benifit to other inservice researchers. Another question concerns the characterization of the existing mathematics programs in the schools. How was the information about the state of mathematics in these schools obtained? Math lessons were found to reflect little forethought and planning, homework assignments and explanations were unrelated, lack of prerequisite mathematical knowledge was overlooked. How did Ben-Chaim assess this situation? Were these schools visited by the researchers prior to the project? Or were the classrooms visited after the project had already been mandated?

Who were the master teachers, and how were they selected? Did they come from the same school, or were they from the outside? Were they classroom teachers, or were they university people? Answers to some of these questions may help explain why the teachers were willing to admit to teaching deficiences and knowledge gaps, while their actual teaching did not manifest any great alterations. If teachers perceive that change is being dictated to them, rather than coming from them, researchers may be hard pressed to effect change in their classrooms. Finally, the problem of "not enough time" is a perennial excuse in schools for why change hasn't taken place. Here are two ways to circumvent the time excuse. First, teachers need to perceive that they have a real stake in any changes that take place so that they will want to try and make time to initiate change, and second, administrative support in the form of release time for planning, reflection, and peer interaction is essential for maintaining the growth of change. Thus, the administrators who are so supportive of this project should perhaps be called upon to actually demonstrate their support to the teachers by providing release time. Some schools accomplish this by providing a "roving substitute" for a day each week.

The Madsen-Nason study includes the major foci present in each of the other two studies. A support group of teachers is established. There is ongoing personal involvement of the researchers with the teachers, and constant encouragement to reflect on what is happening in the classroom. In this case, the intervention that occurs is in the form of a prescriptive teaching model, namely the Launch-Explore-Summarize (LES) model, in conjunction with specific mathematical units created by the Middle Grades Mathematics Project. The purpose of the intervention is to get teachers to transfer the elements of the LES model to topics that they teach in their own classroom. In order to accomplish this, changes in both beliefs and behaviors need to occur. This study seems to be a delicate marriage of the "bottom up" approach to change used by Pirie with the "top down" approach to change used by Ben-Chaim.

All the teachers, "coached", "uncoached", and "lead", were exposed to the same mathematics units for the middle grades. In contrast with the two previous studies, the actual mathematics that actively engaged the teachers has been clearly defined, the five MGMP units (Probability, Spatial Visualization, Factors&Multiples, Similarity, Area&Volume). Subsequently, teachers were coached (or not) in the use of these materials in their own classrooms. The coaching in this study is done by people who have been personally involved with the support group of teachers from the beginning, thus the marriage of top down and bottom up elements of change. One gets the sense that these teachers may have viewed their coaches more as friends than as interveners.

Like the Ben-Chaim project, this project reports the first year of activity. Also, like the Ben-Chaim project, this project reports that changes in teachers' perceptions of mathematics were not necessarily accompanied by corresponding changes in classroom practice. Teachers tended to slip from the LES mode to their previous teaching habits. The study concludes that more support and more coaching time is needed to make a real difference in the classroom.

In summary, the papers on changing teachers' styles and perceptions prompt one to consider the following issues.

1. It may be very useful to assess teacher beliefs in the population we are trying to change before attempting to implement an inservice project. The studies from this session on teacher beliefs provide a multitude of ways to asssess beliefs. Perhaps, then, we should select those teachers we wish to change more carefully, with an eye towards flexibility and leadership. These teachers could then be used as "change agents" to affect change in their own schools or districts.

2. The importance of a base support group for promoting internal change is clear in the studies of Pirie and Madsen-Nason. Our inservice projects should be the soil for change, and we the farmer's who plant the seeds. Thus, grass-roots bottom up approaches probably have more potential for lasting change than do mandated top down changes.

3. "Where's the Beef?" If we are talking about changing teachers' practices through inservice courses it is important for us to attend to and to share both the mathematics content and the methodolgy that we feel represents our desired outcomes.

4. The role of administrative support and the importance of time for reflective discussion is evident in all three of these papers.

5. More attention needs to be paid to the ways we "measure" the outcomes of change in the classrooms of our participants. Videotaped segments, audiotaped interviews, and classroom observation can enhance written data obtained from surveys or questionnaires.

6. Change is a long term process, teachers need lots of time, and lots of support. We should not expect dramatic changes in teachers' styles even after a year. In our initial attempts to assess change, perhaps some of the more subtle cues of change need to be elicited, lest we mistakenly conclude

we are having little impact on our teachers. It is a strength of each of these projects that they will be able to assess and influence change over a period of several years.

Other studies on Teacher Inservice

The papers of Maher, Monteiro, and Abrantes deal with the effects of particular content and/or methods instruction on teachers' views of mathematics. Maher models a constructivist approach to mathematics, embodying fraction concepts on geoboards in a small group problem solving setting. Both the mathematics taught and the need to consider the teacher as a learner received considerable attention in this study. A real strength of this study is the inclusion of school administrators and graduate students in the small group problem solving sessions. Getting teachers and administrators to experience mathematics together is a crucial step in establishing communications lines within a school, and appears to have been quite successful in this project.

The videotapes of the small group sessions that have been recorded by Maher could be used to assess beliefs of the group members. They could also be used as a basis for discussion and reaction, as in the Jaworski study. Thus, Maher has data that can be useful to researchers who are asessing teacher beliefs. However, it appears that the main use for the videotapes in this study was to investigate and analyze the problem solving processes of the groups. Maher states that a goal of this problem solving model is that teachers will show changes in perspective and practice in their classrooms. Unfortunately no evidence is presented either for what teachers believed, or what they did in their classrooms, as a result of the problem solving experience. These issues may be addressed as the project continues.

Monteiro attempts to show teachers the possiblities that computers give them to go beyond mere "transmission of knowledge" in mathematics classrooms, and to investigate the affect of computer exposure on their attitudes and teaching strategies. Co-operative group efforts and interdisciplinary perspectives were dominant aspects of this study. Both teacher and student computer projects were part of the inservice activity. Teachers received courses on computers in LOGO, and applications packages such as word processors and drawing programs.

It would be helpful to know more about the exact experiences of both the teachers and students. What types of projects were undertaken to anwer the question "How can we improve our school?", and how was the the computer used to investigate this question? What were some or the experiences the students had besides some exposure to LOGO? What mathematics was discussed in conjunction with the computer packages? Teacher attitudes were assessed by several questionnaires. It would be helpful to obtain evidence beyond questionnaires, such as from classroom visits, to see how the teachers used the computer. In order to address the teachers' concern about keeping the computer rooted to practical experiences, it may help to attempt to tie the computer more closely to the mathematical content itself. For example, packages such as Algebra Arcade, or The Geometric Supposer are closely tied to specific mathematical content.

The study of Abrantes on implementing computer packages in classrooms contains a familiar theme from the other papers. "One should not expect that teachers will modify their styles from one day to the other." Once again, we have evidence that change is slow in classrooms, and requires a long period of time to implement. Of the twenty programs that Abrantes selected, teachers tended to use the programs that concentrated on demonstration and practice, rather than those involving problem solving, simulations, or educational games, and to use only programs with which they were very familiar. Abrantes attributes this to a reluctance on the part of the teachers to disturb their classroom management policies. An alternative explanation is that these types of programs require the least amount of teacher involvement in the mathematics. Teachers can continue to teach as they always have, and just use the computer as another tool to evoke drill and practice methodologies. The constructivist perspective, that seems so dear to most of the presenters in this session, can be avoided even with the introduction of a computer. Thus, attention to teacher belief systems may need to accompany any inservice activities that involve computer uses for simulations and problem solving. Perhaps we should carefully select our initial group of teachers to be "changed", and then let them affect the rest of the school.

It is a strength of Abrantes' approach that he had teachers evaluate how the computer lessons were organized, and then subsequently asked them to assess the cognitive and affective aspects of the programs on their classrooms. Classroom observations would provide additional evidence for implementation of the computer programs, in addition to the teacher self-reports. Although there is some indication of the types of programs used in Abrantes' paper--number facts, estimates, function&graphs--a complete list would be beneficial to other researchers.

The final two studies, those of Rosen and Nantais, deal with entirely different issues than any of the other papers. Rosen claims that it is important that teachers be aware of both left and right brain approaches to most mathematics topics. A number of questions about the "way" to teach particular mathematics topics are raised by teachers, and shared by Rosen. The impression that I get from the paper is that these teachers are not adequately prepared to deal with so-called right brain approaches to mathematics problems. It is my understanding of right-left brain research that right brain involves visual solutions to mathematics problems, and involves visual thinking about mathematics concepts. In that regard, some of the questions raised by Rosen for right-left brain categorization do not seem appropriate. For example, why should a missing addend problem be considered right brain, and is it even useful to do so? How would one visually represent this type of problem for young learners? Rosen does not offer any hints as to how this could be accomplished in the paper. Similarly, it is difficult to defend how one sequence of keying in the formula for the area of a circle is right brain, while another sequence of key strokes is left brain. No evidence for this contention is supplied in the paper. The impression one gets from Rosen's paper is that teachers are well trained in mathematics as a left brain activity, but that mathematical experiences in visual thinking may be necessary to elicit a right brain orientation to mathematics problems. In this regard, Rosen may find the work of Meyer and Nelson (Math and the Mind's Eye Project, Portland State University) and the units on visual thinking that they have developed quite helpful.

Nantais has investigated the feasibility of having teachers conduct short interviews with their students in order to obtain feedback as to how the student's learning and thinking are progressing. Results indicate that teachers can obtain valuable information from this type of process, although the interviews themselves need to be limited in time lest they be too time consuming for teachers to conduct with all the students in their classes. It would be interesting for Nantais to devise a "script" for such an interview, so that some outside person could conduct interviews with the children as well as the teacher. Such a script would guarantee that any interviewer would ask at least the same basic set of questions in the interview. Then, comparisons could be made between the types of responses that the teacher gets, and those obtained by an outside interviewer. The reason I raise this issue is that it may be that the teacher her(him)self gets information from the children that the children think the teacher wants to hear, rather than completely reliable responses. The work of Nantais is reminiscent of work done by Ed Labinowicz (Cal State Northridge) on teaching teachers to conduct mini-interviews with their students. Labinowicz has written a book on the subject which may be helpful for further work by Nantais.

In summary there are two overall impressions that I draw from all eleven of these papers that need serious consideration.

1. It is important for those of us doing inservice projects to attend to the mathematical content that we wish to model, and to communicate that content to our fellow researchers. This is the "Where's the Beef" question.

2. When we select participants for our inservice activities, if we really want to maximize the potential for change, we should carefully choose our initial perspective change agents after we have assessed their prototype beliefs.

Mathematical
problem solving

TOWARDS A TAXONOMY OF WORD PROBLEMS

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Abstract. In this paper a set of interconnected word problems are presented. To prepare the problems selected task variables were incorporated into a taxonomy or matrix, resulting in a series of interrelated word problems. In the resultant set of word problems the following variables were manipulated: level of vocabulary, type of question, type of extraneous information, type of operation, and computational level. The final outcome was approximately 1200 prototypes for word problems.

In a recent paper, Bachor, Steacy, and Freeze (1986) argued that in previous efforts to construct word problem typologies two fundamental conceptual problems have emerged. First, specific problem solving strategies have been identified and then they have been incorporated into the defining characteristics of the resultant word problem typology. For example, conceptual knowledge or semantic relationships have been assumed to represent learner strategies, which in turn, have been used to define problems (e.g. Carpenter & Moser, 1982; Riley, Greeno & Heller, 1983). Second, generalized learner characteristics, such as a theoretical sub-set of a developmental or cognitive theory, have been selected as the defining features of word problems. For example, Caldwell & Goldin (1979) used Piagetian stages to mark learner characteristics and to distinguish word problems. Some other limitations associated with both of these approaches to theory building already have been noted (e.g. Garofalo & Lester, 1985; Riley, Greeno, & Heller, 1983). Carpenter and Moser (1982) argue that they have not been able to use their framework, which represents the first construction technique, to characterize unambiguously all addition and subtraction problems. Garofalo and Lester (1985) have suggested that one problem associated with the second approach to word problem generation is that cognitive theories are too ill-defined to translate directly into instruction.

An alternative to word problem design has been suggested by Cawley, Fitzmaurice, Shaw, Kahn, and Bates (1979). They have argued that any set of task characteristics can be incorporated into a typology or matrix. The selected task characteristics would be used to provide both the structure and stricture for the constructed word problems. Creating a typology likely will not lead to a resolution of the inconclusive framework problem referred to in the last paragraph. Instead any suggested typology may be best constructed around a set of research hypotheses. Similarly, if Cawley's suggested approach to word problem construction is to be adopted the importance of learner characteristics will need to be addressed independently. Such investigations will depend on at least two factors: a) the degree to which selected learner characteristics can be validated empirically, and b) the degree to which those characteristics can be related to an instructional theory to facilitate optimal decision-making for learners of varying problem solving abilities. This approach to the construction and study of word problems is described briefly in the paragraphs that follow. However, a discussion of learner characteristics will not be undertaken due to limitations of space.

COMPONENTS OF THE WORD PROBLEM MATRIX

In selecting the components to be included in the word problem matrix a number of choices were faced. For example, Caldwell and Goldin (1979) point out that up to seventy-three task factors had been included in a single study. Thus, the following principle was formulated based on an examination of Canadian word problem curricula and on a review of previous research (Bachor, Steacy, & Freeze, 1986) to provide a rationale for word problem design: To unravel the enigma of word problems, task variables must be incorporated in a typology in which the included problems are to be considered simultaneously as requiring language manipulation, logical analysis, and mathematical computation.

Along with this general principle, five concomitant sub-principles have been delineated. These sub-principles provided the basis for word problem preparation and will serve as hypotheses for future research.

Sub-principle 1: Modifying the phrasing of the word problems was hypothesized to be a significant determinant of problem difficulty. Two variations on a basic problem set were prepared, resulting in three levels.

The basic set of problems were written at the third-fourth grade level. To establish grade level, every word used in the problems was judged against both a graded Canadian spelling list (Thomas, 1979) and a graded vocabulary compilation (Dale & O'Rourke, 1976, 1981). Two further problem sets varying in language level then were constructed: a) by inserting adjectives into the basic problem set, or b) by modifying nouns used in the original problems. Before any adjective addition or noun substitution was made, two criteria had to be met. One, both the adjectives and nouns had to be rated in the Dale & O'Rourke list as falling between the sixth and twelfth grade levels. Two, the selected adjectives had to be logically consistant with the nouns incorporated into the basic problem set; and the new nouns had to fall into a logical superordinate category. In addition, some mathematical terms used in the problems were changed to increase their difficulty level, using the same criteria described above. Verbs were held constant at the grade three-four level of difficulty across all three language levels. Some variations in verb selection occurred so that specific word problems would read better and follow logically. Sample problems are provided in Table 1.

Vocabulary Level	Sample Problem
grade 3-4 level	The stranger counted a few chickens. The stranger counted 3 geese. Chickens and geese are birds. The stranger counted 8 birds altogether. How many chickens did the stranger count?
grade 6-12 level, adjective insertion	The determined stranger counted a limited number of savory chickens. The determined stranger counted 5 succulent geese. Chickens and geese are birds. Considered collectively the stranger counted 8 birds. How many chickens did the stranger count?
grade 6-12 level, noun substitution	The conservationist counted a limited number of lynx. The conservationist counted 5 cougars. Lynx and cougars are felines. Considered collectively the conservationist counted 9 felines. How many lynx did the conservationist count?

Table 1: Indirect Problems to Demonstrate Changes in Vocabulary

Sub-principle 2: The type of question incorporated into the problem is hypothesized to affect difficulty (Bachor, 1985). The three types of questions suggested by Pearson and Johnson (1978) were incorporated into the matrix of word problems: a) text explicit (TE), b) text implicit (TI), and c) script implicit (SI).

When incorporated into mathematical word problems, the three types of questions can be seen to vary in the number of assumptions made about the potential problem solver. With TE problems, the assumptions made are that the responder can read the text and locate the required answer;

neither selecting an operation nor completing any computation are required. With TI problems, the latter requirements are assumed. The responder may be required, depending on problem construction, to integrate information from more than one statement and then to complete at least one operation. While problem solvers may need to integrate information, the semantic categories required to complete the problems as given are summarized in the classification statements in the actual problem. With SI problems, a third requirement is added in that any required prior learner knowledge of the semantic relationships between the subordinate and superordinate categories of nouns also is assumed. It should be noted that as a result of this latter assumption, SI problems contain less text than TI problems. Examples of each type of question are given in Table 2.

Type of Question	Sample Problem
Text Explicit	The cook ordered 69 pies. The buyer bought 76 cookies altogether (SIR). The cook prepared 78 more pies (SR). How many pies did the cook order?
Text Implicit	The king photographed 67 tigers. The prince photographed 83 bears. The queen saw 70 zebra (SIR). The duke chased 59 elephants (SR). Kings, princes and dukes are men. Tigers, bears, zebra, and elephants are animals. How many animals did the men photograph?
Script Implicit	The man rented 12 houses. The woman rented 78 houses. The bank sold 86 houses (SIR). The worker fixed 63 houses (SR). How many houses did the people rent?

Table 2: Direct Problems to Demonstrate Question Type and Types of Extraneous Information

Sub-principle 3: Including extraneous information is hypothesized to affect word problem difficulty (Bachor, Steacy, & Freeze, 1986). Two types of extraneous information may be found in the problems. The extraneous information contained in the first type has been termed "set irrelevant extraneous information" (SIR). Two different cues are provided to mark the SIR case: (a) a shift in noun, either in the subject or the object or in both the subject and the object, and (b) a change in the verb. The second type of extraneous information has been called "set relevant information" (SR). Only one cue, a change in the verb used, is provided to the reader in the SR case. A third case of extraneous information is found in some problems. It is termed "set relevant and set irrelevant" (SRSIR) because both types of extraneous information are combined in one problem. Sample problems containing these three variations in extraneous information are illustrated in Table 2.

The specific determination of SIR is more complex as the wording of the problem statements vary as a function of the set complexity of the problem, which is described next. In simple subject, simple object (SSSO) problems, the exclusion occurs in both the subject and the object and in the verb. In the case of simple subject, complex object (SSCO) problems the exclusion for SIR occurs in the complex object and in the verb. In complex subject, simple object (CSSO) problems the exclusion occurs by modifying the noun in the complex subject and by changing the verb. In the final variation of complex subject, complex object (CSCO), the exclusion occurs by changing either the noun in the complex subject or the complex object but not both concurrently and by changing the verb. The result is two variations of the CSCO case. Sample problems for all the variations of the SIR case are given in Table 3.

Sub-principle 4: The need to categorize or classify information into logical sets, referred to as set complexity, is hypothesized to influence problem difficulty (Cawley et al., 1979). Four types of set complexity are found across the word problem matrix: a) simple subject, simple object; b) simple subject, complex object; c) complex subject, simple object; and d) complex subject, complex object. In SSSO problems, responders are not required to make classification decisions. Thus, in Table 3 in the first example, there is no need to classify the basic problem statements as only "wolves" and "chickens" need to be considered. Category inclusion must be determined in the case of the question since this is a script implicit example. In the last example in Table 3, the responder must determine if the statements given belong to the superordinate category in both the subject and the object cases. In the two middle examples, categorization only is necessary for either the object (SSCO) or the subject (CSSO).

Sub-principle 5: The type and number of operations, and to a lesser the computational level, are hypothesized to affect problem difficulty (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980). Two types of problems (Cawley et al., 1979) are used to define the number and type of operation contained in the problems: a) direct and b) indirect. Computational level is controlled by manipulating number. All other elements of the problems, for example, the type of extraneous information, were held constant.

Set Complexity	Sample Problem
Simple subject, simple object	At first the wolf caught 41 chickens. The wolf caught 69 more chickens. The fox picked 38 ducks altogether (SIR). How many birds did the animals catch?
Simple subject, complex object	Tom had 78 cats. Tom had 47 dogs. Tom helped 32 seals. How many pets did Tom have?
Complex subject, simple object	Mary desired 57 horses. Ellen desired 46 horses. Jack led 76 horses. How many horses did the girls desire?
Complex subject, complex object, a) subject exclusion	Mrs. Green caught 23 sharks. Mrs. Brown caught 14 trout. Mr. Smith ordered 13 goldfish. How many fish did the women catch?
b) object exclusion	Mrs. Green caught 35 sharks. Mrs. Brown caught 65 trout. Mrs. Smith saw 87 frogs. How many fish did the women catch?

Table 3: Sample Problems to Demonstrate SIR Exclusion Conditions and Types of Set Complexity

The operations in direct problems are either addition, or multiplication, or both addition and multiplication. A direct question is found when definite quantifiers only are used in the word problems; examples are found in Table 2. Indirect questions have indefinite quantifiers incorporated into the first statement. This change results in a shift in the required operation to subtraction, division, or both subtraction and division. Samples of indirect problems at the three levels of vocabulary are given in Table 1.

All word problems were restricted to whole numbers as the referents in them always are intact objects. Three variations in computational level are found: single digit, double digit, and double digit with regrouping. For the single digit problems, the digits 0 and 1 were omitted (see Bachor, Steacy & Freeze, 1986 for a rationale). All digits were used in the other two types of computational levels. The difference between the last two cases was that regrouping was required at least once during the calculation of the final answer in the second case and was never necessary in the first instance.

CONCLUSION

The result of constructing word problems using the above guidelines is a set of approximately 1200 variations in word problems. It has been argued that the advantage of constructing problems using a matrix is that research hypotheses can be formulated around selected task characteristics. Further, it has been suggested that another advantage of generating problems in this manner is the separation of task and learner characteristics. Finally, it is intended that one use of these specific word problems will be that learners of varying problem solving efficiency can be compared systematically.

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SOLVING WORD PROBLEMS: A DETAILED ANALYSIS USING THINK ALOUD DATA.

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The use of think-aloud data from concurrent verbalisation should not be neglected as a source of information in the analysis of mathematical problem solving. A framework for the use of think-aloud data in the analysis of performance on word-problems is described and an example of its application, at a general level, to one student's think aloud protocol is discussed.

In this paper we present a framework for analysing think aloud data from performance on certain mathematics word problems. We describe the framework and details of its use and then provide an example of its application.

Several different approaches have been taken to the analysis of think-aloud problem-solving performance in mathematics.

Wallas (1926), Polya (1957), Krutetskii (1976), and Luria (1973) proposed models of problem-solving that have been used to analyse performance in studies such as those of Kilpatrick (1967), Rowe (1980), and Schoenfeld (1985). Why present another think aloud analysis? First, this type of analysis provides a rich body of information that is qualitatively different to that derived from non-interactive methods used to develop process descriptions such as error analysis, and chronometric analysis. A second reason for proposing another framework is that such frameworks need to be revised to make use of current thinking about the nature of problem-solving. The Kilpatrick and Rowe frameworks noted above, while sound in basic structure, need revision for this reason. The recent approach of Schoenfeld, was not used because we believe that consideration of problem-solving events in sequence provides both a better representation of the dynamic nature of problem-solving and an account that is more adaptable for instructional purposes than does use of Schoenfeld's time-based episode.

Our approach to the analysis of problem-solving performance has a further affinity with previous studies, in that we too focus upon process or strategy events. In this respect our concern is primarily, though not exclusively, with procedural, rather than with declarative, knowledge. The Analysis, Strategy, Checking, Planning, Review, and

Metacognitive Knowledge categories are derived from recent work on metacognition (Lawson, 1984). Declarative knowledge is considered at a general level through use of the Representation code. The remaining codes are operational codes.

Uncertainty about the validity of think-aloud data (Nisbett & Wilson, 1977) has resulted in its relative neglect. We believe that the rejection of all verbal report data is not justified (Ericsson & Simon, 1984; Shavelson, Webb, & Burstein, 1986).

The most developed framework for considering verbal reports is that of Ericsson and Simon (1984). The key assumptions of that framework for the present study are: (1) information held in STM is available directly to the subject, while information in LTM will not be available until it is retrieved into STM: (2) concurrent verbal reports, which do not require the generation of new information, are reports based on information that is the focus of attention in STM; (3) concurrent verbalisation, under the proper instructions, may add time to a performance but need not change the structure of thought processes; and (4) concurrent verbal reports are reports of deliberate processes.

In this report we first set out a framework for analysing think aloud data from concurrent verbalisation, and then apply this to a student's protocol.

METHOD

Subject and procedure.

The think aloud protocol we will discuss here was provided by a 15 year old student (WA) of average ability in a regular High School who was having some difficulty with mathematics.

After several familiarisation meetings two sessions involved training WA to talk as he solved problems. The training procedure followed that used by Ericsson and Simon (1984, pp.377-379).

The problem given to WA was:

> A fireman stood on the middle rung of his ladder spraying water into a burning building. As the blaze lessened he climbed up 5 rungs. A sudden burst of flames sent him down 10 rungs. When it died down he moved back up 12 rungs. When the fire was out he climbed the remaining 10 rungs to the top of the ladder and entered the building. How many rungs did the ladder have?

For the purposes of analysis the problem was regarded as consisting of a number of dimensions. Briefly, these dimensions are parts of the problem about which a correct encoding, or a wrong encoding, can be made. For the above problem we identified the following dimensions:

1.	Beginning = middle rung.	7.	Movement is up.
2.	Beginning = mid point.	8.	Value of move = 12
3.	Movement is up.	9.	Movement is up.
4.	Value of move = 5.	10.	Value of move = 10
5.	Movement is down.	11.	Top rung reached.
6.	Value of move = 10	12.	Question = How many rungs?

Coding the Protocal

The coded think aloud protocol is given in Table 1. The protocol was coded using the coding schedule set out in Table 2. The schedule has 11 major categories that represent our analysis of the events involved in problem-solving.

WA's protocol was divided into a series of events so that each event was described by one code. A new division of the protocol was recorded when a different event was identified. In Table 1 slashes (/) are used to separate the coded events. Reliability checks on use of the coding system were carried out using a series of raters and these yielded acceptable levels of consistency for identification of the number of events in a transcipt, for assigning of codes to events, and for rating of the one transcript over two occasions.

This coding system allows for the recording of the sequence of events, although sequence will not be addressed here. Rather we will focus on a more general form of analysis in which the codes and their sequences are used to develop higher-level groupings of the events that occurred in the protocol.

RESULTS AND DISCUSSION

Inspection of the events coded under each of the major codes allowed the development of a commentary on each of these aspects of WA's performance.

His analysis is marred by failure to encode dimension 2, failure to realise that the starting point is the mid-point of the ladder. This error of analysis limits the rest of his attempt at solving this problem.

Two representations of the problem are established. The first, which is quite promising, is numerical. WA initially treats the problem as a series of addition and subtraction problems. This approach would, if it had been coupled with the encoding of dimension 2, have led to a correct solution. WA's second definition of the problem, the 'approximate' representation, with the ladder having 'about 20 rungs', could also have been a fruitful approach, if his use of the diagram had not been careless. WA chose his strategies well. He applied the specific strategies correctly, and showed that he can handle negative

Table 1. Transcript

V: OK. Let's have a look at this problem now. I'll just pin this to your jacket here (mini microphone) so that we can hear what you're saying. Now would you like to read that problem out aloud so that you can start.

WA: /The fireman stood on the middle rung of his ladder spraying water into a burning building. As the blaze lessened he climbed up 5 rungs. The sudden burst of flames sent him down 10 rungs. When it died down he moved back up 12 rungs. When the fire was out, he climbed the remaining 10 rungs to the top of the ladder and entered the building. How many rungs did the ladder have?/

V: /Uh huh. Now remember to keep talking out loud about whatever you're doing to try and solve that problem./

WA: Alright. He was on the middle rung/and he went up/5 and/went down/ 10,/so take 10 away from 5./ [REPRESENTATION]

V: Mm/

WA: Negative 5,/then add/12/Is..a.s. 7/

V: Mm/

WA: Then when the fire died out he went up/another 10/so 7 add 10/is 17./

V: Keep talking out aloud whatever you are thinking as you're looking at that problem again./

WA: Got another...got a problem here./ How you supposed to work out how many rungs it have..had..if all you know is he's on the middle rung and/(pause) you haven't been given any that are below it..how many lower?/

V: That's right. You haven't been given how many below it/ So what are you thinking about that,/

WA: It's weird. 'Cos it's hard to solve 'cos/.um..I don't know (pause)/

V: So./

WA: I'm tired, (pause) Um (pause)/

V: Can you tell me what you're thinking now because you arrived at an answer before of 17 which you got by adding up the steps and then taking away and then adding up and I'm just wondering (WA Yeh) what you're thinking now about what you did there./

WA: Er. So he's..(start's drawing steps on a ladder and whispers numbers to himself./ [REPRESENTATION] /

V: Can you talk aloud about what you're doing so that it's recording/

WA: Er. I'm just drawing up a picture of a ladder sort of thing/
So he's in the middle/so go up 10 rungs/1,2,3,4,5,6,7,8,9,10./
So that's about where he is./ (puts X) /Then he goes up/another 5/.1,2,3,4,5/ (puts X_2)/ Then he went up/ then/down / 10,/5,6,7,8,9,10/(puts X_3)/ Then he goes up/another 12/.10,11,12/ (puts X_4) So on that one/ And he's got another 10/to climb/ so add another 10 on here/ 1,2,3,4,5,6,7,8,9,10/(puts X_5)/
So I reckon he probably had about/(counts from/X_1)/1,2,3,4,5,6,7, 8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23/rungs/..approximately./

V: So you think that there are 23 (WA 23) rungs/

WA: Yeh.

numbers. The decision to draw a diagram was also a good one. However he did not apply this general strategy effectively. Without a view of the ladder as one with two halves and a mid-point his use of the labels on his diagram (X1-X5) was careless, resulting in the selection of a wrong starting point (X3) for his final calculation. All computations were carried out correctly.

WA showed one sign that he was monitoring the course of his problem-solving. He was aware that he had a problem in calculating the size of the bottom half of the ladder. He did not, however, persist with this monitoring or go on to attempt to resolve the problem, apart from asking for assistance from the interviewer. Most importantly there was no consideration of either of the solutions he reached. He also failed to check use of his diagram in the final counting of rungs.

WA did comment on both his own state and the nature of the problem. However these did not amount to positive use of knowledge about his own capacity as problem-solver or about the class of tasks of which this problem was a member. The comments he did make suggested a tendency to give in rather easily when confronted by a difficulty.

There were no events coded in the Planning, Review, and Off-Task behaviour categories.

There were errors of both commission and omission in WA's protocol. He did not analyse the problem statement thoroughly and did not, in the latter part of the protocol, represent the problem effectively in setting out his diagram. He also made an error in selecting the starting point for his final count.

Use of this framework has enabled us to build up a detailed picture of the events of problem-solving for this student. To this extent it does fulfil its purpose of proving a means of increasing our understanding of the operation of processes involved in the solution of this type of problem. We can identify types of difficulties met by this student, and where these occur, and recur. The analysis suggests areas that are likely to be profitable ones for extra work. The framework could be used in the mathematics lesson, so that the events of problem-solving become more public. This approach also provides for the further analysis of the sequence of events in problem-solving.

Table 2. Think aloud coding schedule.

00 READING.
The problem statement is read for the first time, or is re-read in whole or in part. Errors of both omission and commission are coded. Errors which do not change the sense of the statement or the nature of the dimensions of the problem are not coded.
001 The reading is error free.
002 Error in reading.

01 PLANNING
A sequence of actions is anticipated. The problem-solver predicts actions which will be taken in solving the problem.
011 Planning of course of solution
012 Error in planning

02 ANALYSIS
The problem statement is interpreted beyond the point of READING. Typically this involves identification of the dimensions of the problem, either initially or at any time during the performance. ANALYSIS is not coded while the problem statement is being read.
021 A specific dimension of the problem is identified or inferred.
022 Error in analysis.
The number of the dimension encoded is noted, e.g. 021-6.

03 REPRESENTATION
A REPRESENTATION code is assigned to the protocol whenever the solver establishes a representation, whether explicit or not. It is assumed that the solver uses a representation while solving the problem. Having done some analysis the problem-solver reformulates the problem into a more-or-less cohesive structure which is used as the basis for further work on the problem.
031 The representation is appropriate.
032 Error in representation. The representation is incorrect in whole or part.

04 STRATEGY
A method of solving the problem is applied.
041 A general strategy, a 'rule of thumb' is applied. These strategies are applicable across a wide range of problems. Heuristics would be coded here.
042 A strategy specific to a part of the mathematics content of the problem is applied. Use of an algorithm would be coded here.
043 Error in strategy application.

05 COMPUTATION
Symbols are manipulated to generate a product. These are the occasions when the solver is working out values not given in the problem statement.
051 Computation
052 Error in computation

06 CHECKING
The course of problem-solving is interrupted to check a previous action.

060 Checks reading.	064 Checks computation.
061 Checks analysis.	065 Requests input
062 Checks representation.	066 Error in checking
063 Checks strategy application.	067 Explains performance

07 GUIDANCE
The interviewer assists the problem-solver.

070 Guidance for reading.	075 Guidance for checking
071 Guidance for analysis.	076 Offers encouragement
072 Guidance for representation.	077 Guidance for planning
073 Guidance for strategy.	078 Guidance for review
074 Guidance for computation.	079 Neutral prompt

08 METACOGNITIVE KNOWLEDGE
The problem-solver makes statements about the task, about the setting for the problem-solving, or about himself/herself as a problem-solver. The statements about the task are distinct from those coded in the REPRESENTATION category.
081 Statement about self
082 Statement about task

09 SOLUTION
A solution, or a solution to one stage of the problem, is given.
091 Error in stage solution.
092 Error in final solution
093 Problem-solver gives up.
094 Correct solution or stage solution is given.

10 REVIEW
The solution, or a part of the solution, is evaluated in relation to the problem statement.

101 Reviews/evaluates product of solution process
102 Error in review

11 OFF-TASK BEHAVIOUR
Any statement that is not classified within this system as being relevant to the problem-solving.
111 Off-task behaviour

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TOWARD A CONSTRUCTIVIST DIRECTION IN MATHEMATICAL PROBLEM SOLVING

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While there has long existed a professional consensus regarding the importance of understanding and teaching mathematical problem solving, few notable gains have been forthcoming. From a general theoretical framework of constructivism provided by Dewey and Piaget and certain more particular research findings of Lesh and his associates, we argue that knowledge development in the classroom suffers when purpose, action and knowledge are not linked. Purposeful real world problem solving situations may provide a framework for just such a development.

> 'Understanding and solving problems are considered the chief goals of mathematical study.' This significant statement is found in the new handbook for mathematics issued by New York State Department of Education. It is typical of the renewed emphasis being placed upon this phase of the work by schools all over the nation (Gilmartin, Kentropp, Dundon,1939).

The above exerpt is from the preface of a middle school mathematics textbook. In that preface the author's extol the virtues of problem solving and then procede to offer thousands of "real life" applications, such as, "Richard said the interest on $372 at 5% for 2 years 7 months and 23 days was $48.24. How great an error did he make?" (p.69).

Even a casual inspection of modern textbooks indicates that notions of mathematical problem solving have not fundamentally changed. Innumerable problems, usually simpler, but otherwise like the one above, can be found in any of a plethora of texts at appropriate levels.

Perhaps more disturbing than the virtual stagnation of our school mathematics textbooks with respect to mathematical problem solving are the often cited National Assessment of Educational Progress (NAEP) reports of what are dismal results of young peoples' performance on mathematical applications problems. (Carpenter, Corbitt, Kepner, Lundquist & Reys,1980).

Yet another detractor to what is clearly not an optimistic picture is the fact that much of the educational and psychological research done in mathematical problem solving has not been very helpful. According to Lester (1983), "Kilpatrick (1969) has characterized the body of mathematical problem-solving research as atheoretical, unsystematic and uncoordinated, dealing primarily with standard textbook word problems (e.g. problems involving one-step translations from words to mathematical sentences) and interested exclusively in quantitative measures of behavior" (p.233).

While recent studies of problem solving seem to be generally more theoretically serious, nonetheless with respect to real world type applications, there continues to exist an alarming research void. Lester (1983) claims, "...there has been an appalling lack of research related to applied problem solving" (p.251).

Let's summarize our characterization of mathematical problem solving. Problem solving has long been considered an important educational endevor; one in which students often evidence poor performance. As yet our related research efforts have not been particularly revealing and consequently changes in actual classroom practice are not imminent.

CONSTRUCTIVISM

If we see the development of knowledge in an organism as a construction (Papert,1980) of reality by that organism, then, "Jean Piaget's work on genetic epistemology teaches us that from the first days of life a child is engaged in an enterprise of extracting mathematical knowledge from the intersection of body with environment" (p.206).

Thus, from the outset, humans are, in some sense, organically interacting with their world in such a manner so as to produce mathematical knowledge of that world. These organic interactions, or constructions of knowledge, are, to say the least, very different from those kinds of experiences involved in the typically nonoperative learning of even the most well managed of school mathematics classrooms. Further these exchanges with the environs are what most closely resemble "real world" applications of mathematics. They are, in point of fact, the real world that our classroom "real world" applications would simulate. Now, if we consider (Furth, 1981) that, "One of the results of Piaget's 'radical constructivism' is his refusal to take objectivity in any but a constructivist sense. *A thing in the world is not an object of knowledge until the knowing organism interacts with it and constitutes it as an object*" (p.19) [emphasis added].

Therefore, from the above perspective, it is apparent why students do so poorly on tests that measure performance in applied problem solving. Since the student has come to "know" mathematical ideas in a figurative rather than an operative or constructivist sense, problems that ask for an understanding which embodies knowing in a Piagetian constructivist sense cannot, in general, be successfully solved. From another perspective, there presently exists a pedagogical separation between school and real life that is manifested in the distinction between classroom exercises that the student may (or may not) know how to accomplish, and applied problems, which essentially embody what are new and unknown interactions; interactions which to the student represent an as yet unconstructed reality. In this light, as we teach our students to walk and then ask them to float, is it any wonder that they sink? Need this be the case?

WHAT IS TO BE DONE ?

"...connection of an object and a topic with the promotion of an activity having a purpose is the first and the last word of a genuine theory of interest in education" (Dewey,1916, p.130).

From our interpretation of certain of the research we argue that curricula designed to encourage conscious purposeful activity of students is effectively a necessary condition for a greater possibility of the development of operative knowledge.

For a constructivist perspective we turn again to Dewey and Piaget. That both of these leading thinkers are of sufficient philosophic proximity that each may be considered a constructivist is not in any way a novel idea (Giarelli,1977). In fact, it has been argued that except for differing emphasis, both approached genetic epistemology from what are nearly equivalent "dialectical" or interactional conceptions of knowledge (p.130). From Dewey (1944), we find the following, "Thinking, in other words, is the intentional endevor to discover specific connections between something which we do and the consequences which result, so that the two become continuous" (p.145).

The above statement, representative of a radical constructivist perspective (Von Glasersfeld, 1981), posits thinking as an aspect of experience; that is, action is a necessary condition for genuine reflective thought. Again from Dewey (1944),

"Thinking is thus equivalent to an explicit rendering of the intelligent element in our experience....The starting point of any process of thinking is something going on, something which just as it stands is incomplete or unfulfilled" (p.146).

From Dewey's comments, we suggest that it is (purposeful) actions which give rise to thinking, and not that thinking is some encapsulated mental process, separated from action. The implications of such statements are far-reaching. For example, from such a view there is no beneficial distinction between a "procedural knowledge" and a "deliberative knowledge" (Merlyn Behr,1985). Such a dualistic view sees "knowledge" as a storehouse of information, and explicitly segregates thought from action. On the other hand, Dewey (1944) believes that, "A separation of the active doing phase from the passive undergoing phase destroys the vital meaning of an experience" (p.151).

And that knowledge is both a process and an end: "While the content of knowledge is what has happened what is taken as finished and hence settled and sure, the reference of knowledge is future or prospective. For knowledge furnishes the means of understanding or giving meaning to what is still going on and what is to be done" (p.341).

In the work of Piaget (1969), we find a similar theme regarding the basic necessity of action for the development of intelligence. Be it basic sensorimotor or more complex coordinations, the equilibration of an organism in conjunction with its environment begins, *a priori* with actions.

"...knowledge is derived from action, not in the sense of simple associative responses, but in the much deeper sense of the assimilation of reality into the necessary and general coordinations of action. To know an object is to act upon it and to transform it, in order to grasp the mechanisms of that transformation as they function in connection with the transformative actions ....intelligence constructs as a direct extension of our actions " (p.29).

Now, a brief summary. Our essential case is that student outcomes in mathematical problem solving are in the relatively dismal state in which we find them because too few of the kinds of meaningful school activities that would make matters otherwise are being undertaken. Piaget and Dewey notwithstanding, in their general everyday operation, the schools deal neither with knowledge as primarily a constructing process that requires action for its development, nor with the goal of conscious understanding of purpose by students which would enable the development of the continuity of those actions.

From our perspective we now turn to recent research on mathematical problem solving by Lesh and his associates. In their work on applied mathematical problem solving (Lesh, Landau, Hamilton, 1983), the researchers concentrated not on detailed analyses of "trick" problems; i.e. cleverly novel exercises usually of intrinsic interest to certain mathematics researchers and cognitive psychologists, but not very appealing to the bulk of all other people, but rather on what they considered, "*realistic problem-solving situations*", involving, "easy to identify *substantive mathematical content*"(p.263). The researchers claim,

"...there is a dynamic interaction between the content of mathematical ideas and the processes used to solve problems based on those ideas. ... Applications and problem solving ... play an important role in the *acquisition* of basic mathematical ideas. We believe that applications and problem solving should not be reserved for consideration only *after* learning has occurred; they can and should be used as a context within which the learning of mathematical ideas takes place" (p.266).

In an even stronger statement (Lesh and Akerstrom, 1981), it is advised that mathematical problem solving researchers redirect their energys toward investigations of subjects' "powerful content-related processes", and away from "general (and weaker) content-independent heuristic techniques". The researchers offer their advice because they, "...reject the dichotomy between content-independent processes and process-independent content...", and that, "...content-independent heuristics have proven to be basically unteachable and of dubious value...", while, "...content-dependent processes ...seem to be not only imminently teachable, but also surfacing time and time again as ...processes of critical importance in the solution of real-world problems "(p.128).

From our perspective, the conclusions and analyses of Lesh and his associates seem clear. Content-related processes speak to the dialectical relation between ideas and the actions necessary to construct knowledge. For constructivists, it may be upon just this interactional focus where research should train its attention. On the other hand, we suggest that content-independent processes seek some kind of grand or universal approach to problem solving in the abstract. As such, this kind of approach is based upon a metaphysical notion of a presumed ability to isolate particulars from universals (Dewey,1944) to which they are interactionally related. Such an assumption almost certainly leads in the direction of a mind-body dualism that separates mental operations from actions.

FRAMEWORK FOR A MODEL

Lesh and his associates, while supporting content-dependent process oriented research in mathematical problem solving, are clearly not supporting the traditional classroom approach to content. How then is this content to be considered? To this formidable question, our short paper offers, at best, some guidelines.

From our perspective, what Dewey and Piaget have said generally, Lesh and his associates have more particularly demonstrated. Our major conclusion from the work of all three is that there is a significant link between purpose, action and knowledge. Purpose, we suggest, informs the action necessary for the development of operative knowledge. Knowledge is not a collection of facts. Rather it is the interactional monitoring process through which action is made increasingly coherent with purpose, and it is concomittantly the state produced by this process. Since purpose, action and knowing are linked, it therefore follows that genuine real world problem situations, where purpose is clear and student interest is encouraged is one scenario in which operative knowledge may have a credible possibility of developing. These situations, broadly considered, would present problems that have mathematical content to students in a class or small group format. The mathematics would typically appear as one aspect within a more complex (hence, more lifelike) situation that must include a conscious consideration of purpose. Research designed in concert with what has been suggested might involve an extension of the Applied Problem Solving Project (Lesh, 1981) to include a concious consideration of values and interests reflected in the choice of the problems. Such research remains to be done.

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BUILDING SEMANTIC COMPUTER MODELS FOR TEACHING NUMBER SYSTEMS AND WORD PROBLEMS

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ABSTRACT

This paper presents issues concerning the construction of models for teaching mathematical concepts and problem solving. As an example of this decision making process it suggests a computer-based model for teaching natural numbers. This model represents natural numbers together with the operations of addition and subtraction, and is aimed at facilitating the solution of word problems as well. We will show how research on children's informal knowledge of numbers and algorithms together with research on how children solve word problems is taken into account.

INTRODUCTION

Because mathematical concepts are abstract, models are needed to communicate mathematical definitions and meanings to children. Models have been used in schools to present mathematical concepts, and yet much current research shows that children are developing incomplete ideas about concepts. It is possible that some of the models, used over the years, have been powerful, but the difficulty of handling them in the classroom has prevented their effective use. It is also possible that the models have not effectively drawn attention to those features of the concepts that are supposed to be taught. The graphics capabilities of some of the newer computers provide a facility for addressing both of these questions. Models on the computer are easy for children to handle using natural movements of touching, dragging, and placing (Hutchins et al., 1985). Computers enable many operations on the model's elements which have not been possible before. For example, it is possible to undo a series of actions which has changed the form of the elements (e.g., cutting and putting back together). It is possible to link symbols to elements of the model in graphically compelling ways. The computer can also be made a helpful tool in telling us what the child has learned from the model (Sleeman & Brown, 1982).

We are constructing such computer-based models for elementary arithmetic. Using terminology borrowed from research on analogies (Gentner, 1983), we can say that the models we are developing will be the "base" for communicating information about the "target," in this case--the mathematical concepts. Learning from this base, we expect the child to develop a valid mental model of the target. We will describe the construction of a computer-based model for natural numbers. The goals of the a model are to represent physically the elements of mathematical concepts, to enable direct manipulation of the elements, to facilitate understanding of the concept to a degree that various situations can be mapped to it, and to enable us to tell what the child knows, through the observation of her manipulations.

THE PROCESS OF CONSTRUCTING A MODEL

We discuss here several of the most important issues that must be addressed in designing the base model.

1. A structural approach vs. a natural-environmental approach:

The first issue in constructing a model is a philosophical one: What is the nature of the model's entities? Are they mathematical objects or real-world objects? Nesher (1987) describes two main opinions on the nature of the exemplifications used to introduce a mathematical concept. The natural-environmental approach suggests that understanding of a concept emerges from dealing with real-world situations; therefore the exemplifications should be the situations themselves, rather than a representation of the abstract mathematical entities. The structural approach, on the other hand, treats the abstract mathematical entities and their mathematical senses as the reference of the exemplifications. Real-world situations, according to the structural approach, should be introduced instructionally only after the formal system has been established. In building a model for natural numbers we have adapted the structural approach. Thus the model will represent natural numbers and the various senses of addition and subtraction.

2. Mathematical structures and psychological structures:

Addition and subtraction have two mathematical meanings of "senses"--the unary and the binary. These two mathematical definitions correspond to psychological structures that have been identified in cognitive research on word problems (Carpenter & Moser [1982], Nesher, Greeno, & Riley [1982], Riley, Greeno, & Heller [1983], Vergnaud [1982], and others). Specifically, unary maps to "change" and binary to "combine" classes of problems. By including both of these mathematical senses in a model we hope to enable children to connect their implicit psychological structures with the mathematical structures that are being developed.

In addition to representing the unary and binary structures of addition/subtraction, we also want to introduce from the start the idea of a mathematical "sentence" (an equation). The structure of a mathematical equation maps to the "compare" psychological structure.

This close connection between the mathematical structures and psychological structures enables us to begin instruction by introducing the number set together with mathematical definitions of addition and subtraction. The child learns to map between an additive structure of the elements of the model and a symbolic number expression. The operations have the senses which are suggested by the model's structures. At a later stage, children can learn to solve word problems using the same model, having available several structures from which to choose.

3. One mode vs. separate modes for the different senses:

When several senses of a concept are to be acquired, we have to choose different representations for them. This could call for building a set of models, each model representing a different sense of the concept. Such an approach helps to highlight the different senses, but may negatively affect the child's ability to integrate all represented meanings and attach them to the single set of numbers. To avoid this possibility we have chosen an alternative in which a single set of elements can be manipulated according to different structural rules in different "zones" of the screen. This permits us to represent several meanings without having to introduce different kinds of elements. The model that has been chosen to represent natural numbers is called the "Trains World." In this world each number is represented by a train of a given length. The world has various zones (see Figure 1) corresponding to the structures the child is to understand.

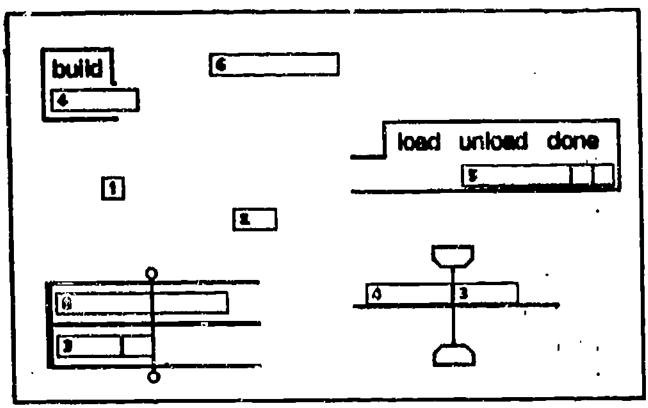

Figure 1: The Trains World. From Top left clockwise: building zone, loading zone, fixing zone, copying zone.

Trains are first constructed in the "Building Zone." The child inputs a number and the zone "outputs" a train the length of which matches the request. The number is also indicated by a matching number appearing on the train. The written numbers facilitate conversation--i.e., if the child inputs "6," the resulting train is referred to as a "6-train."

The "Loading Zone" serves as a representation of the unary definition of addition, operating on one number. In this zone cars are attached, one at a time, to a given train, making it a bigger train. Cars can also be disconnected from a given train, making it a smaller train. The loading process is similar to the counting strategy that children use early in their number development (Fuson, 1982; Resnick, 1983). This zone corresponds to the "change" story structure, in which a given set undergoes a change resulting in a new final set.

The "Fixing Zone" serves as a representation of the binary definition of addition, operating on two numbers. In this zone two existing trains can be "glued" to become one long train. Alternatively, one train can be "cut" into two trains, the size of the smaller trains being determined by the placing of the initial train on the cutting machine. The fixing zone embodies a part/part/whole structure, but one sees either the part/part or the whole (not both) at one point in time. It corresponds to the "combine" story structure, in which two sets are combined to form a union set.

The "Copying Zone" serves as a representation of an equation, having a part/part/whole structure where the parts and the whole can be observed simultaneously. In this zone two trains are arranged in parallel and aligned on one end. Then a "copying machine" runs over them, creating a new train which fills in the difference between the initial trains--thus "equalizing" them. This zone corresponds to the "compare" story structure.

Any object created in one zone can be moved to another zone, where it can be operated on. Operations done in one zone can be undone (in a different way) in another zone. This mobility is intended to promote the creation of an integrated mental model of number.

4. Map between the model and symbolic notation:

We do not want the child's knowledge to remain tied indefinitely to the pictorial representation on the screen. Instead we intend that the child will create an abstract mental model of the numbers together with the operations defined on them. There is reason to believe that formation of this abstraction will be aided if the child constructs a mental "mapping" between the pictorial model and corresponding symbolic expressions (Resnick & Omanson, 1986). This process is aided by referring to the trains as if they were the numbers themselves, e.g., "a 6-train" may be called simply "the 6." The manipulation of trains then becomes a direct manipulation of the numbers themselves, making it more salient that the facts discovered about trains are actually facts about numbers.

Another way to support the connection to the symbolic world is by creating an area on the screen in which symbolic expressions are written. (The symbol area is not shown in Figure 1.) It is possible to "yoke" an operational zone with the symbolic zone so that any action on a train automatically produces a corresponding change in the symbolic expression, or vice versa. With this yoking, children can work to produce desired changes in the trains by manipulating symbols, or vice versa. This serves as a potentially powerful form of "mapping instruction."

5. Choice of Images:

Once the general structure of the model has been established, there still is a whole logical range of possible images for displaying the elements. The images can vary on at least the following dimensions: a. Level of abstractness. b. Continuity.

On the dimension of abstractness the elements can look more or less like trains. they can, for example, have some train characteristics, such as wheels or a smokestack, or they can simply look like rods (See Figure 2). A train which looks more like a real one might be more attractive. Yet there is a possibility that a strong train resemblance might inhibit mapping of other objects onto the model when it is eventually used for solving word problems. Our experience with children indicates that they usually work well with abstract objects. We plan to use the abstract version, and resort to a more realistic version only if we find that certain children cannot handle the abstract model.

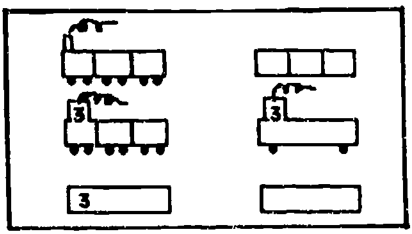

Figure 2: Examples of train alternatives.

The continuity dimension involves several issues. Discrete elements encourage counting, but we want to avoid counting, except for the loading zone, where cars are loaded one by one. With continuous trains the emphasis on a number as an entity is stronger, and in the part/part/whole structure the relations between the sets are more obvious. Another issue in favor of a continuous element will be discussed below when we consider the integration of models.

6. Integration with other models:

A key issue to be considered in designing models is their mathematical scope--i.e., how much of the number system and how many operations should a given model represent? Models with greater scope tend to be complex and are usually avoided by instructional designers. Thus, rods may be used for natural numbers, pie diagrams for fractions, and yet other models for negative numbers and decimals.

In order to make the characteristics of a given set of numbers salient, it may well be necessary to choose models of a limited scope. Yet there should be a way to discover the relations between numbers represented by different models. We are currently addressing this problem as we design models for other parts of the number system. In our train model, the order relation is represented by the length of a train, i.e., the longer the train is, the bigger is the number. This choice of comparison by length together with the choice of continuous elements enables a connection between this model and some existing models of fractions (e.g., strips). However, in building continuous trains we have lost the discrete meaning of integers, which actually differentiates them from other numbers. Further, it is not clear whether we can find a good link from trains to a negative-number model. We will report on this aspect of our work at the meetings.

ON INSTRUCTION

We have built a tool for representing a mathematical concept having taken into account principles which determine some aspects of the instructional sequence. We have, for example, decided to start with defining numbers and operations and only later introduce real world situations. There are still, though, many open questions. For example, we do not know how much time should be spent using the model before story problems are introduced. Further, we don't know how much structure we should put into the child's activities and how much free exploration to allow. There are other open issues with which we also intend to deal--for example, the use of intelligent coaching. In regard to this issue we will need to consider the trade-off between the advantages of adaptive coaching and the constraints limiting the child's activities that are often necessary to allow a computer (even a high-powered and "intelligent" one) to interpret the child's actions (Resnick & Johnson, 1988). Having built the tool, and having shown in pilot work that children are able to handle the basic manipulations of the objects, we are now developing answers to these and related questions in studies that closely observe children while they are manipulating the model's objects and conversing with them about their actions.

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THE MIAMI UNIVERSITY TELETRAINING INSTITUTE FOR MATHEMATICS INSERVICE IN PROBLEM SOLVING: PRELIMINARY RESULTS

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ABSTRACT

This study examines the problem solving performance of 361 middle school students whose teachers were enrolled in a special "teletrained" course on problem solving. Each student completed 5 problems that covered Guess and Check, Working Backwards, Make a Model, Pattern Recognition and Elimination. Correct answers ranged from 39% for Patterns to 77% for Working Backwards. Process analysis indicated that students who attempted using a specific strategy answered more problems correctly than students who did not use a specific strategy. No sex differences were found, but in general, number of correct answers increased by grade level. Differances in process by grade level are dicussed.

In 1986 the Department of Mathematics and Statistics at Miami University, in conjunction with area school districts, WCET educational television in Cincinnati, and the American Telephone and Telegraph Company's Training and Development Division in Cincinnati received funding for a joint venture to establish the Miami University Teletraining Institute (MUTI). The Institute provides "high-tech" based inservice training in mathematics and the teaching of mathematics to teachers throughout the state. Specifically, the Institute utilizes two-way audio, two-way micro-computer communication, and video-tape between the University and locations throughout the state to disseminate instruction and course-work to teachers[1] (Stonewater & Kullman, 1985).

This teletraining system is currently being used to develop and disseminate training for middle school teachers in mathematical problem solving strategies and applications to teaching. It was expected that the participating teachers would subsequently teach the

[1] This study was supported by a grant from Title II of the Education for Economic Security Act and administered by The Ohio Board of Regents.

problem solving material they learned in the course to their own students.

The purpose of the research reported here is to describe the effect on teachers and their students of thirty hours of teletrained instruction in mathematical problem solving and applications to teaching. The intent of this research is not to compare teletraining instruction to other means of delivery, but to describe outcomes relative to a teletraining environment. The research questions address teachers' improvement in using various problem solving strategies, the effect of training on teachers' cognitive development, and students' improvement in problem solving ability.

Since the teletraining problem solving project is currently underway, pre-post data are not available at the time of this writing. Thus, while the presentation at the meeting will cover all three of the above questions, the remainder of this paper will focus only on pretest data on the middle school students' use of five problem solving strategies: Guess and Check, Working Backwards, Make a Model, Patterns, and Elimination. In the following the problems associated with each of the five strategies and the methodology of scoring the problems are discussed. Then, student performance will be analyzed to determine the percent of students who answered each problem correctly and the degree to which students actually used the various strategies to solve the problems. Also, results will be analyzed to determine if there were differences in performance by sex and by grade level.

METHODOLOGY

To assess middle school students' problem solving ability, five problems were developed, each corresponding to one of five problem solving strategies the teachers would learn and subsequently teach in their own classes: Guess and Check (The sum of two numbers is 25 and their difference is 7. What are the two numbers?); Working Backwards (Billy played two games of marbles, but he forgot how many he started with at the beginning. After winning his first game, he had twice as many marbles as he started with. After the next game, he won ten more marbles, which gave him a total of 26 marbles at the end of both games. How many marbles did Billy have before he started to play?); Look for a Pattern (For the following patterns, tell what the next number is: a) 1,2,4,7,11, 16,22,29; b) 1,1/2,1/3,1/4; c) 1,4,9,16,25.); Make a Model (Suppose you have 5 points as shown below. You can connect points A and B to form a line. You can also connect points A and C to form a different line. In all, how many different lines can you form with these five points?); and Solve by Elimination (Find the number described by the clues below. Circle the correct number: a) It is divisible by 4; b) It is larger than 8641; c) It is an even number; d) The sum of the digits is 21; e) It is less than 9756. A list of 17 four digit numbers follows). All of the problems were selected based on procedures recommended by Charles, Lester and O'Daffer's (1984) An Assessment Model for Problem Solving and as such were process problems, varied the mathematical content (geometry, computation, etc.), and varied the types of numbers students worked with (integer, fraction).

Each problem solution was scored for correct answer; Guess and Check and Working Backwards were scored for the extent to which an appropriate strategy was used in obtaining the solution: 0 points-no evidence of applying a Guess & Check or Working Backwards strategy; 1-limited attempt at trying the strategy, but incomplete; 2-evidence of correct application of strategy that should lead to correct answer; 3-correct application of strategy and correct answer; 4-used algebra as a solution strategy.

SAMPLE DESCRIPTION

Seventeen teachers were involved in the teletraining problem solving project. Of the 14 teachers who administered the five problems to their students, three had a bachelors degree, four had some graduate training, and 7 had a masters degree. All but two were female; average years of teaching experience was ten.

A total of 361 students completed all five problems. Of these students, 30 were in grade 4, 60 in grade 6, 196 in grade 7, 42 in grade 8, and 16 in grade 9. Additionally, 17 of these students were in a special class for the academically talented or "gifted". Of the total sample, 155 were male, 169 female, with 37 not indicating sex.

RESULTS

1. Correct Answers by Problem--Correct answers for each of the five problems ranged from a low of 39% of the students who answered all three Pattern problems correctly, to a high of 77% of the students who answered the Working Backwards problem correctly (see Table 1). Other results were: Guess and check, 70% correct; Model, 50%, correct; and Elimination, 54% correct. The remaining Pattern answers were 5.8% answered none correctly, 22.7% answered one of the three correctly, and 32.7% answered two of the three correctly.

TABLE 1

Percent Correct Answer by Grade Level

	Grade 4	6	7	8	9	Gifted	Overall	χ^2	DF	p
G & C	53.3	76.7	62.8	90.5	87.5	94.1	70.1	25.604	5	.0001
W. Back	43.3	80.0	80.0	73.8	81.2	100.0	77.3	26.266	5	.0001
Model	26.7	53.3	48.0	57.1	87.5	58.8	50.4	17.489	5	.0037
Pattern*	13.3	40.0	36.7	45.2	62.5	64.7	38.8	34.586	5	.0028
Elimin.	30.0	45.0	51.5	71.4	93.8	76.5	54.0	28.165	5	.0001

*represents all 3 correct

2. Correct application of Problem Solving Process or Strategy--Only slightly more than one-third of the students (33.5%) showed solid evidence of actually using the Guess and Check strategy and obtained the correct answer (coded "3"), while an additional 10% of the students used an algebraic solution method, and another 12.2% made an incomplete attempt at using the strategy (coded "1" or "2"). Thus, an appropriate strategy or strategy attempt was used by just over one-half of the students in the sample (55.7%). Of these students (n= 201), all but 33 answered the problem correctly; three of

TABLE 2

Ability to use Strategy (Percent)

	no evidence	Improved ability to apply strategy			Algebra
	0	1	2	3	4
Guess & Check	44.3	8.9	3.3	33.5	10.0
Working Backward	16.1	15.0	8.9	55.7	4.4

the errors came from the 36 students who attempted an algebraic solution and 30 came from the 44 students who attempted using Guess and Check (code "1" or "2"). Thus, of the students who showed some evidence of using a problem solving strategy (55.7% of the sample), 86% were able to obtain a correct solution to the problem. It should be pointed out however, that one-third of the students coded "0" (unrecognizeable strategy or no strategy) answered the problem correctly, but provided no evidence of how they obtained their solution. Thus, while it appears that only slightly more than half of the students used the Guess and Check strategy, there may be a much higher percentage of students who used this method in their head but did not give any indication of the process on paper.

A closer examination of how students used the Guess and Check strategy indicates that many seemed to randomly select numbers for their guesses and did not use a systematic search process in finding the correct solution. The students who used more systematic approaches either found pairs of numbers that differed by seven and they "checked" them in order (for example (1,8), (2,9), (3,10), etc.) until finding (9,16) worked. Others systematically subtracted each successive integer from 25 until the correct combination was found (e.g., 25-1, 25-2, 25-3, etc.). Students who obtained incorrect solutions would often confuse the "difference between the two numbers is 7" to mean that one of the numbers was 7, the other 25-7, or 18, resulting in an incorrect answer of (7,18).

In the Working Backwards strategy, 84% of the sample showed some ability to use the strategy (codes 1-3), or used an algebraic solution method. Of this group of 303 students, 257 (85%) obtained the correct answer. Five of the 16 algebraic solutions were incorrect and 41 of the 46 code "1" or "2" were incorrect. Although the number of students who attempted an algebraic solution is small, the relatively high percentage of incorrect algebra solutions (almost 1/3) is noteworthy. Non-algebraic attempts accounted for only 29% of the incorrect solutions.

Incorrect Working Backwards answers were often the result of performing the correct "reverse" operations, but in incorrect order (dividing first instead of subtracting). Other incorrect solutions would use only one of the operations (e.g. 26-10=16) and stop there. Others would fail to reverse an operation at all (multiplying 26 by 2 instead of dividing). A few students attempted to use Guess and Check to determine the beginning number of marbles and a particularly innovative fourth grade female correctly attempted to combine Guess and Check with Working Backwards.

3. Sex and Grade level Differences in Answers and Process-- Chi-Square analyses indicated no differences between males and females in distribution of correct and incorrect answers on any of the five problems (see Table 3).

TABLE 3

Number of Males an Females with Correct/Incorrect Answers

Strategy	Male C	Male INC	Female C	Female INC	χ^2	DF	p
Guess & Check	106	49	120	49	0.263	1	.6082
Working Backwards	126	29	123	46	3.291	1	.0697
Model	77	78	84	85	0.000	1	.9962
Pattern	52,52,39,	12*	73,53,34,9		3.911	1	.2945
Elimination	76	79	95	74	1.673	1	.1959

N of males = 155 N of females = 169
*represents 3, 2, 1 and 0 correct

As might be expected, there were differences is grade level performance. The percent of students who answered each of the five questions correctly generally increased with grade level (see Table 1). However, when the percent of students who correctly applied the strategies is compared across grade level, some interesting trends emerge (see table 4). Here we are examining students who were coded "3" in strategy application, i.e. only those students who applied the strategy completely correctly.

Table 4

Percent Correct* application of Strategy by Grade level

Strategy	4	6	7	8	9	Gifted	χ^2	DF	p
Guess & Check	36.7	38.3	32.1	83.3	93.8	58.8	234.44	20	.0001
W. Backwards	16.7	55.0	62.8	71.4	62.5	94.1	113.712	20	.0001

*"Correct" = code "3" or "4", algebraic solution

For Guess and Check, percent of correct application of the strategy (but not necessarily correct answer) remains fairly stable in 4th through 7th grades (37%, 38%, 32% respectively), but during 8th grade the percent increases to 83%-40% coded "3" plus 43% who solved the problem algebraically. In 9th grade, 94.5% of the students solved the problem algebraically (not all correctly, either). For gifted studenta 41% applied Guess and Check correctly and an additional 18% used algebra. Thus, there appears not only to be a jump in algebra applications during 8th grade as one might expect, but also a slight jump in correct applications of the Guess and Check strategy.

The analysis of those who applied Working Backwards correctly indicates that only 17.5% of the 4th graders could do so, 55% of the 6th graders could, and 63% of the 7th graders could. For 8th graders, 71% used a correct strategy, as did 88% of the 9th graders. For gifted students 82% used Working Backwards, and an additional 12% used algebra, a surprisingly high 94% correct application.

For the Make a Model problem, two categories of classifying how students solved the problem stood out. One common method was to draw all possible lines connecting all combinations of points, and the other was to list all combinations of points that determined a line (i.e., AB, AC, AD, AE, BC, etc.). The selection of the more concrete drawing method decreased with grade level, and the use of the more abstract listing method increased with grade level. Surprisingly, no gifted students selected the listing method. Percentage-wise, almost all 4th graders drew the lines, while only 50% of the 9th graders drew lines. None of the 4th graders listed the combinations, while slightly less than 20% of the 9th graders did so. So, the more abstract method of listing was used less often, but its use increased with grade level, while the more concrete method of drawing lines decreased with grade level.

CONCLUSIONS--While these results represent only a small sample of middle school children and their problem solving performance and while difficulties remain in analyzing problem solving strategies from written solutions, some major trends emerged in this study.

1. Particularly in the higher grades, students were better problem solvers, both with respect to correct answers and to application of strategies, than we expected. While the problems may have been too easy to adequately challenge the students, they did demonstrate facility at using both Guess and Check and Working Backwards, without prior specific training.

2. On the other hand, much "fine tuning" of students' ability to use the strategies is needed. Fourth graders had fairly low scores for correct answers (range 13.3% to 53.3% correct). Much work is needed at this level. Across the board Guess and Check was often done haphazardly, without regard for a systematic solution nor a

thorough analysis of patterns within the chosen guesses to direct future guesses. In short there was little evidence of a "best-guess" approach to the problems. Likewise with Working Backwards. Error patterns reveal that students need practice with subskills embedded in the strategy such as reversing operations and reversing the order of the operations. Instruction focusing on such skills is indicated. This may also be true for algebraic solutions approaches, for which errors were high.

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ASSESSMENT AND EQUITY IN MATHEMATICAL PROBLEM SOLVING IN MIDDLE GRADES

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In an attempt to develop measures to assess the effectiveness of problem solving interventions two paper and pencil problem solving instruments that measured aspects of understanding a problem (e.g., interprating the vocabulary and diagrams, and recognizing what is given in a problem) and problem solving strategies (e.g., using patterns, lists, and guess and check) are described. The tests had moderate reliabilities (Alpha = .78 for the test for fourth through sixth graders; Alpha = .76 for the test for seventh and eighth graders). There were no significant gender differences on the tests but blacks scored lower than whites at all grade levels.

The growing crisis in the effectiveness of mathematics education, most severe among females and minorities, has recently been brought to painful public awareness. Recent studies document American students' poor performance in the area of mathematical problem solving (e.g., Carpenter, Lindquist, Matthews, Silver, 1983; Miller, 1985). Furthermore, there is a substantial body of research which indicates that, by the high school years, males perform significantly better than females on most measures of mathematical achievement (e.g., Lee and Ware, 1986; Peterson and Fennema, 1985) and, at all grade levels, Asian and white students tend to outperform Hispanics and Blacks (Lockheed, Thorpe, Brooks-Gunn, Carpenter et al., 1983).

In an attempt to alleviate these problems, a number of intervention programs which focus on problem solving and/or mathematical equity have been developed. These intervention programs are important not only as a means of improving students' problem solving performance and eliminating race and gender differences but also further our understanding of the underlying theoretical issues.

An essential, but often neglected, component of any intervention program is summative evaluation. Unfortunately, a continuing hinderance to the effective evaluation of problem solving intervention programs has been assessment. Traditional multiple choice mathematics tests focus on computational performance and not on important aspects of problem solving such as understanding the problem, recognizing extraneous information, and identifying and using appropriate strategies. These aspects of problem solving have been identified and studied using interview techniques (e.g., Lester, 1982, Schoenfeld, 1983). While this research method is providing important new insights, it is exceedingly difficult to use with the number of pupils involved in a typical intervention program.

We were recently faced with this predicament when designing a problem solving assessment for the evaluation of EQUALS, an intensive inservice program designed to teach teachers to increase the confidence and competence in mathematical problem solving of their students and to relate the usefulness of mathematics to future career choices (Sutton and Fleming, 1987). A special focus of this program is to address the needs of the traditionally underserved, girls and minorities. A large sample involving 27 urban and suburban school districts led to our decision to use a paper and pencil problem solving instrument. The best existing instrument appeared to be one developed by the Wisconsin Department of Public Instruction (n.d.).

Unlike most traditional multiple choice mathematics tests, The Wisconsin Mathematical Problem Solving Test attempts to measure aspects of problem solving performance in addition to computational proficiency. This instrument is consistent with an assessment model proposed by Charles, Lester, and O'Daffer (1984) which divides problem solving into two components: (1) the ability to perform the thinking processes involved in solving a problem and (2) the ability to get the correct answer to a problem. Each of these components can be further divided into measurable problem solving behaviors. Understanding the question in the problem and recognizing extraneous or missing information, for instance, are two thinking processes categorized under the first component while problem solving strategies, such as making a list and drawing a diagram are categorized under component two.

For the evaluation of this project, two multiple choice tests were developed to assess the problem solving abilities of middle grade students. One test, designed for grades seven and eight, was a 30-item subset of the Wisconsin instrument while the second, intended for grades four through six, was a 20 item simplification of the first. The purposes of this paper are to describe the psychometric characteristics of these two problem solving instruments, and to assess ethnic, gender, and grade differences of these instruments.

METHODS

Subjects

There were 483 seventh and eighth graders (mean age = 14.2 years) and 470 fourth through sixth graders (mean age = 12.0). These subjects were either the students of teacher participants in the EQUALS training program or the students of teachers matched by grade level and district with these EQUALS teachers. (These matched teachers were used as a comparison group in the overall evaluation). The seventh and eighth graders, representing 24 classrooms, were 53% female, 23% Black, and 76% Caucasian. The fourth through sixth graders, representing 19 classrooms, were 51% female, 47% Black, and 48% Caucasian.

Instruments

The Wisconsin Mathematical Problem Solving Test for eighth graders consisted of two forms and students completed only one form. Approximately half the items on each part were designed to measure a student's ability to understand a problem (e.g., interpreting the vocabulary and diagrams, recognizing what is given in the problem, and recognizing extraneous or missing information); the remaining items assessed problem solving strategies (e.g., using patterns, using lists, tables and graphs, and using guess and check). For the EQUALS evaluation, a 30 item test of items selected from both parts of the Wisconsin instrument was created for the seventh and eighth graders. This test is called Test 7/8. Criteria for selection included level of difficulty (easier items were chosen because of the inclusion of seventh grade students) and maintaining an equal proportion of items on understanding the problem (14 items) and problem solving strategies (16 items). Three of the 30 items were modified slightly. On one item, the question was clarified. On two items, changes were made that were consistent with our equity concerns. Specifically, the name "Jack" was changed to "Juanita" and the sport "football" was changed to "softball".

For Test 4/6 (grades four, five, and six), 18 items from Test 7/8 were modified to make them simpler (e.g., elimination of decimals and using smaller numbers) and 2 new items were generated. Again, half the items assessed understanding the problem and the other half assessed problem solving strategies.

Procedure

The teachers administered the problem solving instrument to their own classes in Fall, 1985. Reliable data on test conditions is unavailable; however, the vast majority of the teachers have over 10 years of teaching experience and are accustomed to administering standardised tests.

RESULTS

Reliability and Measurement Analysis

The thirty items of Test 7/8 were subjected to item analysis (SPSSX RELIABILITY) and the Alpha coefficient of internal consistency was .74 (n = 513). Four items with the lowest item-total correlations were eliminated which yielded an Alpha of .76, a test mean of 12.2 and a standard deviation of 4.63. The Alpha for Test 4/6 (20 items) was .77 (n = 416). Two items with negative item-total correlations were eliminated and the this yielded an Alpha of .78 , a test mean of 9.6 and a standard deviation of 3.86. A complete item analysis (means, standard deviations, and item-total correlations of each item) will be presented at the conference.

To assess how well the two instruments were able to distinguish between the two problem solving components proposed in the Charles et al. model (1984), two factors were specified in the factor analysis. The factor analysis of Test 7/8, while not clear-cut, generally supported the a priori categorization of the items as assessing either understanding the problem (Understanding) or problem solving strategies (Strategies). Specifically, 10 of the 16 Strategies items loaded on Factor One (loadings > .3), 8 of the 14 Understanding items loaded on Factor Two, 7 items did not load on either factor, and one Understanding item loaded on Factor One instead of Factor Two. The factor analysis of Test 4/6 did not support a two factor model.

Grade, Ethnic, and Gender Differences

Grade, ethnic, and gender differences were assessed using separate factorial ANOVAs on the total problem solving test, the Understanding subtest, and the Strategies subtest. Any differences with $p \leq .01$ were considered significant.

On Test 4/6, there were significant grade differences on the total test, and the two subtests (Total test, $F = 45.35$, $df = 2,376$, $p < .001$; Understanding subtest, $F = 32.62$, $df = 2,375$, $P < .001$; Strategies subtest, $F = 35.38$, $df = 2,374$, $p < .001$). The Scheffé post hoc test indicated that the fourth graders scored significantly lower than the fifth and sixth graders on the total test and subtests ($p < .001$), but there was no significant difference between the fifth and sixth grade means (Total test means: fourth grade = 7.1, fifth = 10.1, sixth grade = 11.3). On Test 7/8, there were no significant grade differences on the total test and two subtests. There were no significant gender differences for either group of students on the total test and on the subtests.

Whites scored significantly higher than Blacks on the total test and both subtests for both the younger students (Total test 4/6, $F = 22.97$, $df = 1,376$, $p < .001$; Understanding subtest, $F = 12.26$, $df = 1,375$, p<.001; Strategies subtest, F=25.27, df=1,374, p<.001) and the older students (Total Test 7/8, F = 26.46, df = 1,435, p < .001; Understanding subtest, F=21.41, df=1,436, p<.001; Strategies subtest, F=11.61, df=1,426, p=.001). On Test 4/6, the total test mean score for Blacks was 7.9 compared to 11.1 for Whites. On Test 7/8, the total test mean score for Blacks was 9.9 compared to 12.8 for Whites. Information on the socioeconomic status of the students was not available.

There was one significant grade by race interaction for Test 4/6 on the Understanding Subtest (F=7.71, df=2,375, p=.001). Black females scored higher than Black males whereas White females scored lower than White males.

CONCLUSIONS

The present effort enabled us to identify problem solving instruments that go beyond the assessment of computational skills and appear to lend themselves to the evaluation of large scale math problem solving interventions. These instruments have moderate reliabilities with an ethnically diverse population. These tests also seem to be appropriate for programs which focus on ethnic and gender equity since our findings are consistent with current research on ethnic and gender differences (Lockheed, 1986). For example, minority differences consistently appeared but gender differences did not occur in grades four to eight. In addition, our intervention evaluation indicated that Test 7/8 is sensitive to change over a school year (Sutton and Fleming, 1987).

While these instruments can be used in their present forms, we intend major revisions in an effort to improve reliability. Our findings were limited by uncontrolled test conditions and many items appeared to be overly difficult. Thus, we plan to develop a large item pool retaining the same conceptual framework, to administer these items under controlled conditions to a diverse population, and to create new tests containing only those items with good psychometric properties.

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PERUSING THE PROBLEM-SOLVING PANORAMA: COMMENTS ON SIX PAPERS ON MATHEMATICAL PROBLEM SOLVING

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This paper presents a review of six papers (by Bachor; Lawson & Rice; Pace; Peled & Resnick; Stonewater, and Sutton, Oprea, & Fleming). The remarks concerning these papers are embedded in a broader commentary on the current state of research on mathematical problem solving as compared with the situation Kilpatrick described nearly two decades ago. Beyond specific comments on the six research reports, this review argues that these papers, although quite varied in their theoretical foundations and methodological approaches, share some commonalities, in that they represent instantiations of trends in contemporary research on mathematical problem solving.

Almost 20 years ago, Kilpatrick (1969) reviewed the literature on mathematical problem solving and concluded that "problem solving is not being systematically investigated by mathematics educators" (p. 523). In the past two decades, the situation has changed dramatically. During this time, there has accumulated a considerable corpus of research dealing with the nature of mathematical problem-solving performance. Most of the research has been conducted by cognitive psychologists, seeking to develop or validate theories of human learning and problem solving, and by mathematics educators, seeking to understand the nature of the interaction between students and the mathematical subject matter that they study. Moreover, the research tends to be much more systematic than that reviewed by Kilpatrick.

Theoretical Emphases

In his review, Kilpatrick (1969) decried the fact that few studies had an explicit theoretical rationale or built on previous research, but he noted signs of increased interest on the part of mathematics educators in psychological theory and research related to higher-order cognitive processes. Kilpatrick's perception of an emerging trend was apparently correct, for the current situation is quite different from the one he reviewed in 1969. The influence of modern cognitive psychology on current problem-solving research has been substantial. In the United States, most of the current research on mathematical problem solving is based to a great extent on theoretical formulations provided by cognitive psychology. Special treatments of cognitive theory relevant to problem solving and mathematics can be found in Frederickson (1985), Schoenfeld (1985), and Silver (1987).

In the set of papers under consideration in this review, one sees the clear influence of cognitive theory in the work of Lawson and Rice and also in the work of Peled and Resnick. The protocol analysis scheme proposed by Lawson and Rice is heavily influenced by considerations of available information in working memory, and it relies extensively on the cognitive framework developed by Ericsson and Simon (1984). The work of Peled and Resnick concerns itself with mental models and representations of abstract mathematical concepts.

Another contemporary theoretical thrust is "constructivism" in one or another of its many forms. One of the fundamental assumptions of recent research on mathematics learning and problem solving is that new knowledge is in large part constructed by the learner. According to this view, learners do not simply add new information to their store of knowledge; instead they integrate new information into already established knowledge structures and build new relationships among those structures. This process of building new relationships is essential to learning. Recent versions of constructivism are largely compatible with earlier versions, such as Piaget's theories of human learning, although the terminology is somewhat different.

Constructivism is most central to the paper provided by Pace, but the influence of a constructivist perspective is also clearly evident in the work of Peled and Resnick. In the latter case, a constructivist perspective on learning guided the researchers in their construction of appropriate computer models that could form the basis for the mental models that the child would eventually construct for the concepts. The notion of representation is central to the constructivist perspective. In Pace's paper, he takes a constructivist perspective to argue that increased research and instructional attention should be paid to applied problem solving in school mathematics. Although I quarrel with neither his basic constructivist premises nor his conclusion, I find his argument to be less than compelling. There is neither a logical nor a psychological necessity linking the constructivist perspective on learning and the call for the extensive instructional use of applied problems.

Research Foci

Kilpatrick (1969) organized his discussion around five categories: problem-solving ability, problem-solving tasks, problem-solving processes, instructional programs, and teacher influences. These categories vary in the extent to which they are addressed in current research. For example, classical research on problem-solving tasks and the characteristics that contribute to task difficulty - with its emphasis on linear regression models for predicting task difficulty - has generally given way to a detailed consideration of the ways in which task characteristics interact with individual cognitive functioning. The paper by Bachor hints at a consideration of individual cognitive functioning, but we are only presented with data concerning the problem taxonomy. It is clear that Bachor's taxonomy might prove to be a useful tool in studying children's problem solving, especially as it is affected by linguistic and mathematical complexity. One hopes that the taxonomy will be used in conjunction with modern cognitive theories to help provide a rich account of children's problem solving and to suggest powerful instructional sequences.

In contrast to the research on problem-solving ability reviewed by Kilpatrick in 1969, the widespread use of factor analytic approaches and the treatment of problem-solving ability as a (nearly) unitary phenomenon have been replaced by detailed studies of problem-solving processes, often involving extensive clinical interviews. The paper by Lawson and Rice emerges from this tradition. Protocol analysis schemes such as the one they present are fundamental tools both for data reduction and for data synthesis in detailed studies involving individual clinical interviews with problem solvers.

Concern for problem-solving process is also evident in the papers prepared by Stonewater and by Sutton, Oprea, and Fleming. Stonewater's coding of students' problem-solving performance included not only the correctness of their answer but also the strategy used. The process data that were collected in the study are quite limited, however, and it would have strengthened the study greatly if clinical interviews were conducted so that both a richer description and a better estimate of strategy use in the student population could have been established. Nevertheless, the overall aim of studying the effects on students of problem-solving instruction aimed at their teachers is laudable, and the final report of this project should be of considerable interest.

The paper prepared by Sutton, Oprea, and Fleming gives evidence of concern not only for process but also for assessment. As instructional programs to improve teachers' and students' mathematical problem solving have proliferated, it has become evident that there is serious need for adequate assessment tools to measure the effectiveness of such programs. More generally, there is a need for relatively easy to administer tests that measure important problem-solving processes. Several states (e.g., California, Wisconsin) have incorporated these mathematics process items into their state assessment tests. Sutton, et al. analyzed data obtained from the Wisconsin test to examine gender and race differences in performance. Since these were multiple-choice items, I would urge that further analyses involving distractors be undertaken. Marshall (1983) has demonstrated the power of distractor analysis to identify important gender differences on these kinds of process items from the California test.

Methodological Approaches

In 1969, Kilpatrick noted that most studies of problem solving were either "one-shot comparisons of ill-defined 'methods'" or "laboratory studies of arbitrary, highly artificial problems." Methods-comparison studies have almost completely disappeared in contemporary problem-solving research. Moreover, highly artificial problems have been almost completely eliminated from the serious research literature on mathematical problem solving. Recent research has typically involved problem tasks that are drawn from actual textbooks, or realistic problems from students' lives, or problems that are nonstandard but appropriately related to the mathematics that students have studied.

Pace argues in his paper that insufficient research attention has been given to realistic application problems. Nevertheless, in recent years there has been a noticeable increase in research aimed at examining the relationship between one's informal knowledge of the world and one's formal mathematical knowledge. For example, Carpenter (1985) and his colleagues have reported that young children have the ability to solve many types of arithmetic story problems before they receive formal school instruction. They found that many children were able to use their semantic knowledge of the real world and their skill in counting to solve many addition and subtraction story problems. Another strand of research examining the relationship between school mathematics and real world applications is found in the work of anthropologists (e.g., Carraher, Carraher, & Schliemann, 1985; Scribner, 1984) who have found that the mathematics one uses in everyday situational problem solving can bear little relationship to the formal mathematics one learns in school. Persons with very limited and flawed formal mathematical knowledge can be quite skillful in solving quantitative problems encountered in occupational settings, and they often use invented procedures rather than those taught in school.

In 1969, Kilpatrick argued that, given the limited state of our knowledge about mathematical problem solving, researchers might be well-advised to consider clinical studies of individual subjects. His advice was apparently heeded, because contemporary research has heavily emphasized clinical approaches and, to a somewhat lesser extent, case studies of individuals. In the typical clinical study, explicit attention is given not only to the knowledge that a person would need to know in order to solve the problem but also the processes used by the solver.

The most popular technique for studying the processes used in mathematical problem solving has been the "talk aloud" clinical interview. This technique, pioneered by gestalt psychologists, has been widely used to study both the cognitive and metacognitive aspects of problem-solving episodes. In their paper, Lawson and Rice provide an example of a protocol analysis scheme that attempts to capture the problem-solving process of an individual solver in the concurrent verbalization accompanying solution behavior. Unfortunately, the evidence provided in their brief paper is too meager to determine the usefulness of their scheme and its relative merits compared with others that have been proposed (e.g., Schoenfeld, 1985).

Some other popular current approaches have utilized computer technology. For example, some researchers have designed simulations of human problem solving in complex domains like physics and mathematics. These simulations are often based on and/or validated with actual protocols obtained from problem solvers. A somewhat different computer-based approach has involved the construction of "intelligent tutors" that provide problem-solving instruction. The interaction between the learner and the computer tutor provides a rich data source for research into the nature of and requirements for mathematical problem solving.

In the papers for this meeting, the work of Peled and Resnick relates most closely to this trend. Their paper presents an example of some ways in which research on children's cognition can inform the developers of computer software. In their paper, they illustrate that it is the case not only that research results can provide answers to development questions but also that research results, methods, and theoretical constructs can help us to formulate more appropriate development questions and approaches.

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Metacognition
and
problem solving

AWARENESS OF METACOGNITIVE PROCESSES DURING MATHEMATICAL PROBLEM SOLVING

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Awareness of metacognitive processes during mathematical problem solving is an intriguing but as yet poorly understood phenomenon. This study attempts to capture the developement of awareness in a subject, rather than to directly measure its current level. Data were collected in a semester-long problem-solving course and included journal entries, written problem solutions with explicit "metacognitive revelries," optional videotapes of talking aloud while solving problems, and general observation of the students. The argument for the development of awareness or lack of it rests in the conflux of the evidence from the several points of view. The article presents 2 case studies from the data set--Duke, who showed an increasing level of awareness in the course, and Chad, who did not seem to increase her level of awareness.

Metacognitive processes during mathematical problem solving and their relationships to success in such problem solving is an intriguing topic that has been discussed in mathematics education considerably in recent years (e.g., Garofalo & Lester, 1985; Schoenfeld, 1983a). Metacognitive processes include the control and monitoring of one's cognitive processes, aspects that bring awareness and consciousness into the discussion (Brown, 1978; Flavell, 1976, 1979). Schoenfeld (1983b) included the degree of awareness in his dimension matrix of cognitive activity. Lack of awareness of mental operations during problem solving was hypothesized by Gurova (1959/1969) as a "substantial reason for low achievement in the mathematical disciplines" (p. 97). Gurova's report does not contain enough detail about the methodology to replicate the study. Further, it seems plausible that a teacher's level of awareness of his/her own cognitive and metacognitive processes during problem solving would contribute significantly to her/his effectiveness as a teacher of problem solving. However, neither of these hypotheses can be investigated until the phenomenon of awareness is studied. Direct measurement of awareness (i.e., measurement of the current state of the phenomenon in a person) may be possible at some future date but no such methodology currently exists. For now, I have taken a lesson from studies of metacognition in other areas (e. g., in memory, Brown, 1978) and have tried to capture the development of such awareness over time. The present paper presents two case studies from data collected to study awareness of metacognitive processes during problem solving.

Method

A variety of techniques--journal entries, written problem solutions with explicit "metacognitive reveiries," optional videotapes of talking aloud while solving problems, and general observation of the subjects--were used to try to capture the development of awareness from several viewpoints over time. The data were gathered during a semester-long course (15 weeks) on the teaching of problem solving in the mathematics classroom. The course began with an introduction to several problem-solving strategies. During the rest of the course, sessions were devoted to problem-solving episodes (usually quite rich and complex) in groups and as a class, to discussions of problem solving (including metacognition) and the teaching of it, and to problem-solving episodes led by the subjects themselves.

During the course, subjects were given 6 Problem Sets of 2 to 5 problems each to be turned in for evaluation. The written solution was to include all work on the problem, including blind alleys, and was to include a separate column for "metacognitive reveiries." The evaluation gave more weight to the solution processes and metacognitive reveiries than to the correctness of the solution. Subjects also wrote a journal entry each week, using a code name so that the information would not influence their grades in the course. The topics of the journal entries were chosen to encourage reflection upon their own problem solving processes and their own development of confidence, strategies, and awareness during problem solving. Some subjects also volunteered (as part of their course project) to be videotaped while thinking aloud during problem solving. There were three videotape sessions per volunteer--one immediately after the introductory phase of the course, one about midway through the course, and one at the end of the course. The problems used in the videotaping sessions were the following:

Videotape Session 1: Harry's Hamburger Heaven sells milkshakes for 80¢ each. It sells 320 of them a week. Harry figures for each 5¢ increase in price, he would sell 10 fewer milkshakes per week. What price would maximize income?

Videotape Session 2: I have 50 cinder blocks 8" by 8" by 32" to make a 10-step staircase. The end view of the staircase will look like the picture below. Do I have enough blocks?

Videotape Session 3: How many different rectangles are on an 8-by-8 checkerboard? Note, rectangles are considered different if they are different in position or size. So, a 2-by-1 rectangle is considered different than a 1-by-2 rectangle.

The subjects in the data set were 18 graduate and undergraduate students, all inservice and preservice teachers of mathematics, mostly on the middle-school level (grades 6-9, ages 11-15 years), but with some teachers on the intermediate level (grades 4-6) and some on the secondary level (grades 9-12). To develop the case studies presented here, I began by reflecting upon my personal observations of the subjects and my knowledge of their work and metacognitive reveiries on the problem sets throughout the course. I then read the journal entries and the transcriptions of the videotaping sessions. For the journal entries, one must assume that, to a certain extent, the subjects wrote what they felt I wanted to read. In the videotape transcriptions, I looked for explicit monitoring statements. There is no way to know to what extent the subjects may have been aware of metacognitive processes without making explicit verbal references to them. However, the argument for awareness or lack of it rests in the conflux of the evidence from the several points of view, not with the evidence from any one of them.

The case studies presented here were chosen to contrast the development of awareness (Duke) and lack of such development (Chad). I chose subjects with similar mathematics and teaching backgrounds. Both Chad and Duke were undergraduate students with no teaching experience and no methods courses for teaching mathematics. Both had completed 3 college-level mathematics course with grades of A or B. Both were preparing to teach mathematics on the middle-school level. Also, the lengths of their journal entries and videotape transcriptions were similar, so that merely the amount of verbiage would not contribute significantly to differences in explicit references to cognitive monitoring. Throughout the following descriptions, direct quotes are from the subject's Problem Set solutions, journal entries, and videotape transcripts.

Duke

At the beginning of the course, Duke felt a "bit apprehensive," stating that "word problems have never been my favorite." She described her initial level of problem-solving ability as "enough. . .to flounder myself through if necessary" but that she frequently got "bogged down by words." In her first Problem Set, her metacognitive reveiries were limited to statements such as "set up an equation," "draw picture to see where circles are," and "dead end." This last comment indicates some awareness of monitoring but was not followed by any attempt to salvage the positive aspects of the solution path. Her first videotape session included only 4 monitoring statements, of which 3 were almost identical.

By the middle of the course, Duke's confidence in problem solving was increasing. "I can now look at a problem without panic." Her awareness of her own progress (or lack of it) on a problem was definitely growing. In response to a problem-solving session in class, she described herself as reaching "a high level of frustration. . . .I was having trouble changing my perspective of the problem. I had the general idea of what was going on, but couldn't see how I needed to break the problem down into parts." After another class session, she wrote that "I think if I had had enough time I could have figured it out though because I knew I was on the right track." In her Problem Sets, Duke was beginning to include searches for more than one solution and generalizations of patterns. The tendency to generalize a pattern did not carry over to her second videotape session but she did solve the problem in 2 different ways. She still included only 4 monitoring statements, but they were all different, apparently monitoring different things (e.g., one on the difficulty of the solution path, "I think I'm making this too hard."; vs. one on the compatibility of the 2 solutions, "Hope this gives me the same answer as before.")

By the end of the course, Duke's awareness of and confidence in her problem-solving abilities had not kept pace with the development of her abilities. Her solutions in her Problem Sets were quite sophisticated and rich in strategies. Yet, in approaching the take-home exam, Duke said she felt "frazzled" and didn't know "how I will ever be able to do. . .those problems. They look tough." (It is true that the exam contained some very complex problems.) After she turned in the exam and we went over the problems in class, she described her confidence level as high and bemoaned the fact that she had not turned in some of her solutions "because I didn't at that point have enough confidence in what I had done." Apparently, her awareness had not effectively monitored the correctness of her solutions. In her third videotape session, Duke's transcript was literally peppered with monitoring statements. She seemed to simultaneously monitor her progress toward a solution, her progress with a chosen solution path, and the correctness of her local procedures. She was aware that her last solution method (i.e., counting the kinds of rectangles and the number of each kind) would eventually give her a solution but also that she might not have executed it correctly. "If I didn't leave any of them out, then 1092 should be the answer. . .I hope!"

When Duke was asked to reflect upon whether she had developed more awareness of her metacognitive processes during the course or whether she had merely developed a vocabulary with which to express her processes, she felt she had developed more awareness. She admitted that part of what she learned was vocabulary, but added that "on the other hand, I think that I am also more aware that something is going on. When I sit down now to do a problem, I realize that my brain goes into action and that it has all of these avenues it explores, and I can kind of watch it explore and direct it sometimes." Such a statement is in stark contrast to her earlier statements of "floundering" through problems.

Chad

Chad's story begins very similarly to Duke's but does not progress in the same way. Chad also felt "very apprehensive" about the course. She said that her previous experience in problem solving was limited to word problems that "have a certain method of obtaining a set solution." As with Duke, Chad's first Problem Sets contained very limited descriptions of processes. Her first videotape transcript contained several monitoring statements but they were all related to whether or not the income had peaked yet and to correcting her many procedural and arithmetic errors.

Throughout the course, I had several discussions with Chad about elaborating on her cognitive processes in her solutions for the Problem Sets. She frequently said, "I don't know how I do it. I just do it. . . . I don't know why I did it that way. . . . I don't know why I quit doing it that way and tried another way." She seemed unable to comprehend the possibility of monitoring her own processes.

By midway through the course, we were emphasizing alternate solution paths, multiple solutions, and generalizations. Chad wrote in her journal, "Some solutions may come easy to me yet I only see one way to get to this solution. . . .On problems that I cannot figure out the correct solution, I become very frustrated." In the Problem Sets, Chad rarely went beyond a single solution and rarely generalized to an n-th case. In her transcript for the second videotape session, there are no monitoring statements at all. She merely drew the 10-stair picture and counted. She made no attempt to do the problem another way or to relate it to the generalization of triangular numbers that we had used frequently in class and in the Problem Sets.

At the end of the course, Chad's situation is opposite to Duke's. Chad felt "nervous" before the exam but "confident with my problem solving abilities. . .much better about the looking back experiences and I really think that I am ready to do well." However, her exam responses were vague and her solutions, though sometimes correct, were not rich in strategies and processes. In her third videotape session, she did exhibit a certain level of awareness with several monitoring statements. But the statements were related to her confusion in the attempted solution, e.g., "That's not right," "I don't know how else you would go about doing this," "I can't figure out how you would make that chart," and "It's a big difference between those two numbers and I can't justify where either one of them actually came from." She seems to be aware of her confusion. Yet, she forges on to a solution anyway. "So, there would be 848 different rectangles. But I can't actually justify why." Though she does not exhibit great confidence in her solution, she also does not seem aware that it is incorrect or that there does not seem to be much that could have been salvaged from her solution path.

When asked to reflect on whether she had developed awareness during the course or only learned vocabulary to use in expressing her processes, Chad seemed to miss the point of the distinction. She responded in her journal, "At first it was just vocabulary, but now it is used and is quite helpful. . . .In the problems given, they require thought and these ideas (vocabulary) have shown us [ways] through these problems. So, it is not only vocabulary but useful ideas." Another journal entry summarizes Chad's final "state." "I know and understand the sophistication of problem solving, yet I find myself still strictly thinking mathematically and looking just for an answer."

Conclusion

The two subjects described here show clear differences in the development of awareness during mathematical problem solving. I suggest that the differences are shown not only in the number of monitoring statements but in the substance of those statements. Chad was clearly monitoring herself in the last videotape transcript, but the substance of the monitoring was primarily her confusion. In contrast, Duke's monitoring statements indicated that she was monitoring several things simultaneously. The substance of Duke's statements referred to progress toward a solution, progress within the current solution path, and correctness of local procedures.

It is unlikely that significant progress in understanding and measuring the phenomenon of awareness of metacognitive processes during problem solving will come quickly or easily. My own monitoring of our progress towards that goal suggests we have a very long ways to go yet. I believe the study of the development of awareness will be useful in eventually understanding and measuring awareness.

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METACOGNITION AND MATHEMATICAL PROBLEM-SOLVING: PRELIMINARY RESEARCH FINDINGS

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The purpose of this paper is to present preliminary findings from our current National Science Foundation project investigating the role of metacognition in mathematical problem solving. Specifically, the objectives of our research are: (1) to assess the metacognitive behaviors of grade seven children engaged in mathematical problem-solving activity and (2) to explore the extent to which these students can be taught to be more strategic and more self-aware of their problem-solving behaviors.

Metacognition refers to the knowledge and control one has of one's cognitive functioning; that is, what one knows about one's cognitive performance and how one regulates one's cognitive actions during performance. Metacognitive knowledge about one's mathematical performance includes knowing about one's strengths, weaknesses, and processes, together with an awareness of one's repertoire of tactics and strategies and how these can enhance performance. Knowledge or beliefa about mathematics that can effect one's performance are also considered metacognitive in nature. Examples of metacognitive knowledge include knowing that one is sloppy at computation, and knowing that the "key word" strategy cannot be used to solve every word problem. The control and regulation aspect of metacognition has to do with the decisions one makes concerning when, why, and how one should explore a problem, plan a course of action, monitor one's actions, and evaluate one's progress, plans, actions, and results. This self-regulation is influenced by one's metacognitive knowledge. For example,

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if one recognizes a given problem as being complex, then she is more likely to take time to explore the conditions of the problem and plan methods of solution before attempting to determine an answer.

PROJECT DESCRIPTION

Our metacognition project consists of three phases: pre-treatment assessment of mathematical performance and metacognitive behaviors, instructional treatment, and post-treatment assessment. The pre- and post-treatment assessments involve performance on a written test of routine and non-routine mathematics problems, performance of selected students on routine and non-routine problems observed through individual interviews, and performance of selected students observed as they work in pairs on non-routine problems. The individual and pair interviews are being video-taped and performance is being analyzed in terms of behaviors exhibited, strategies used, and decisions made.

The instructional treatment is being presented three days per week for a period of 12 weeks and consists of three metacognitive components: the teacher as external monitor, the teacher as facilitator of students' metacognitive development, and the teacher as model. The "teacher as monitor" component consists of a set of teaching actions for the teacher to engage in: (1) to direct whole-class discussions about a problem to be solved; (2) to observe, question, and guide students as they solve problems; and (3) to lead whole-class discussions about solution attempts. The "teacher as facilitator" component involves the teacher: (1) asking questions and devising assignments that require students to analyze their mathematical performance; (2) pointing out aspects of mathematics that have a bearing on performance; and (3) helping students build a repertoire of heuristics and control strategies, along with knowledge of their usefulness. The "teacher as model" component involves the teacher explicitly demonstrating regulatory decisions and actions while solving problems for the students in the classroom.

The subjects for this study are students in two seventh-grade mathematics classes in Monroe County, Indiana. One class is a regular-level class, the other is an advanced-level class. The instructional treatment is being presented by one of the investigators (FKL).

SOME PRELIMINARY FINDINGS

The preliminary findings reported here consist of: (1) a very abbreviated comparison of the performance of an advanced-level student (AA) with that of a regular-level student (WC) based on observations made during individual interviews with each, and (2) reports of classroom observations made during instruction.

Observations of AA and WC

These observations are based on two 45-minute individual interview sessions with each student. During the interview sessions, the students worked on the four problems given below:

> Kennedy collected 225 tape cassettes and 4 old shoe boxes to put them in. If he puts the same number of cassettes in each box, how many extra cassettes will there be?
>
> The six grade math teacher did an experiment with her students. One at a time, students were to give her change for a 50¢ piece without using pennies. No student could use the same set of coins as someone else. How many students will be able to give her change?
>
> Atlas Steel makes 4 different types of steel. From a shipment of 300 tons of raw steel, the factory produced 60 tons of type I, which sold for $60 a ton; 75 tons of type II, which sold for $65 a ton; 120 tons of type III, which sold for $72 a ton; and 45 tons of type IV, which sold for $95 a ton. Raw steel costs $40 a ton. It costs the factory $2,500 to convert every 300 tons of raw steel into the 4 types. How much profit did Atlas make on this shipment?
>
> There are 10 people at a party. If everyone shakes hands with everyone else, how many handshakes will there be?

On working the "Kennedy" problem WC read the problem several times before dividing 4 into 225. She then read the problem again, mentally estimated 4 times 56, and multiplied 4 times 56. She read the problem again and wrote out "56 in each box" and "1 left over." She claimed she read it over and over "just to make sure it's division." When asked how she usually decides which operation(s) to use, she replied "It's the one that makes sense, and when you check it, you get the numbers in the problem back." She did go on to explain that when unsure, she tries all four of the operations and then picks the one that makes the most sense. AA on the other hand, read the problem over several times then divided 4 into 225, saying under her breath "4 into 225 goes..." She did not reread the problem again, nor did she check her division, nor did she write out her answer.

On the "change" problem, WC read the problem several times then divided 5 into 50, 10 into 50, and 25 into 50. She labelled her quotients 5¢, 10¢, and 25¢ respectively. When it was explained that the change did not have to be in all of the same coin she immediately began adding combinations of 5's, 10's, and 25's, first in a vertical running total then in a horizontal series of single additions (she had trouble tallying the coins using a running total). Her plan was to list as many combinations as she could using at least one of each coin, add these up as she went along, and then go back at the end to check for repeats. She checked periodically for repeats as she went along, but so poorly that she did not spot the ones she had. This may have been because she didn't attend to it very much since her plan was to go back later or because she was overloaded trying to find new combinations and check old ones at the same time. In this manner, she listed and totaled 7 combinations of coins, 5 of which were repeats, which she noticed only when she checked at the end. She checked and rechecked her calculations using a lot of pencil movements, and tallied and labelled each combination of coins. AA's approach was very different. After reading the problem, she began listing combinations of coins, not by their values, but by using q's, d's, and n's. Her plan was to list all the combinations she could. Although the specifics of her plan evolved as she went along, her work was fairly organized. She did a lot of tallying under her breath as she worked and used her pencil to keep track of what she was tallying. She realized that her strategy was not as organized as it could be and referred to it as "a little awkward for making sure I got them all."

On the "Atlas Steel" problem, WC read the problem many times and mainly "looked at the numbers to try to figure out what to do." She started adding weights, then costs, then gave up on addition. She then began to multiply weights by costs. She used her pencil to locate the numbers in the problem to "get the right numbers." After she added up the resulting products she didn't know what to do. She then began dividing the products by 300. She realized this "didn't make sense" so she gave up. As in the previous problem, she used a lot of pencil movements in her calculations to "keep on the right columns." AA, on the other hand, read the problem several times and focussed on the whole problem, not just the numbers. She realized she would have to compute income and outlay and then subtract. She worked with a "profit schema" in mind. Again she talked to herself while calculating and used some pencil movements. She didn't "waste time" checking because she "probably" calculated correctly. To her, the problem is solved "after you know what to do--the rest is just unraveling it."

On the "handshake" problem, WC read the problem, asked if they "shake with both hands," then multiplied 10 times 1 and 10 times 10 (and got 1000). When asked to explain the 10 times 1 and 10 times 10, she realized that "they ain't gonna shake with themselves." She quickly multiplied 9 times 1, and 10 times 9, then checked her calculations and said "ninety." After discussing her work further, she thought "it would only be nine handshakes." She thought about it some more and went back to 90. When the situation was modelled with three people, she shouted "Oh my God, I just thought of something...you're not gonna shake twice." She then divided 2 into 90, initially making a mistake because she "worked too fast." AA also started with 10 times 10, but was unsure. She talked about it under her breath and used her pencil to do the multiplication in the air and not on the paper. She then wrote down ten 9's and said "ninety." After being informed that her answer was incorrect she thought about the problem some more and wrote a 9 and an 8. She then asked if her first solution was indeed incorrect and after being told that it was, she listed 9 through 1, said "plus zero," and added up the numbers.

In summary, WC began working on problems even before she was satisfied that she had a complete understanding of it. She did not spend enough time analyzing the data, exploring her understanding, and planning a solution. This is obvious in her approach to all of the problems. In the "Kennedy" problem she constantly went back to the problem to "make sure it's division;" in the "change" problem she initially misinterpreted the conditions of the problem, and after being informed of this, immediately grasped on to another misinterpretation and began to work, again without evaluating this new interpretation; in the "Atlas Steel" problem she moved from addition, to multiplication, to division hoping that one of these would "make sense;" and in the "handshake" problem, she had to be prompted to realize that people don't shake their own hands and that people don't shake hands twice. WC spent a lot of time calculating and checking her calculations, even when unnecessary (e.g. multiplication by one...). She used a lot of pencil movements to keep on track and was able to make on-line adjustments of some local actions. Also, she was very careful to tally and label quantities when appropriate.

AA, on the other hand, was more concerned with understanding a problem before putting work down on paper than she was with calculating correctly and labelling results. She started working on paper only after she felt she had a good understanding of the problem and had "figured out what to do." This was particularly obvious in the "handshake" problem when her first calculations were close in the air. She didn't use as many pencil movements as WC to keep her on track when calculating, but instead talked to herself when working. She claimed this keeps her attention focussed on what she's doing. She was content that she knew how to solve the problems, and was confident enough her calculations were "probably right" that she felt no need to check them.

Observations During Instruction

The observations reported here are in reference to the three aspects of the instructional treatment.

The teacher as external monitor. In many cases, especially in the regular-level class, the students were weak in the basic skills necessary to complete the problems. Thus, much of the "external monitoring" time was spent explaining how to do calculations or how to reason logically, rather than discussing metacognitive-level considerations. Our observation at this point is that teachers must expect to provide instruction in basic skills and problem-solving strategies simultaneously with instruction in metacognitive awareness.

The teacher as facilitator. One way in which students were directed to reflect upon their own cognition was by completing self-inventory sheets on which they listed their own strengths and weaknesses in problem solving. In the regular-level class many students could think of nothing to record. The advanced-level class had much less difficulty. Similarly, when assigned to solve a problem and then to write a narrative describing their thought processes as they had solved it, the advanced-level students complied with about a paragraph each. The regular students, almost without exception, failed to turn in a narrative. We suspect a combination of factors contributed to this outcome: the weaker mathematics students may also suffer from weaker writing skills to the extent that writing such a narrative was beyond them; the weaker students are not as conscientious about completing homework assignments; and the weaker students may have more difficulty reflecting on their own thought processes.

The teacher as model. Of course, whenever the teacher explained the solution of a problem, he pointed out the importance of rereading to clarify understanding, he discussed why a particular strategy was chosen, he openly checked his calculations, and he always pointed out the importance of comparing the final answer with the conditions of the problem. But these actions are no more than an elaboration on a typical teacher's explanation. Our efforts to go a step further, and have the students observe the monitoring strategies used by an "expert" problem solver as he solved a problem he had not solved before were less than successful at first. The teacher served as "expert," solving, on the spot, a problem posed by a research assistant. However, the teacher found it very difficult to maintain a role of "expert" problem solver. He fell quickly into the role of teacher, and soon was explaining, rather than modelling. Not surprisingly, the students found it hard to focus on the teacher as an expert model, rather than as a teacher-explainer. Some of their notes on what the "expert" did well, and not so well, concerned the effectiveness of what they apparently considered a teaching demonstration rather than a demonstration of expert problem solving. They wrote such observations as "talked too fast," "wrote big and neat on the board," didn't explain clearly," etc. A modification of the modelling procedure, in which the students viewed a video-tape of a research assistant solving a problem at a desk worked much better as a model for expert problem solving. The students had no expectations that the assistant should "write neatly" or "explain clearly," since she was obviously just writing while talking to herself, not to them. Thus, they were better able to concentrate on observing the monitoring strategies that she used.

The above project description and data presentation are admittedly very brief. Space considerations allow us to give only a flavor of our current work. We also refrained from drawing any definite conclusions from this work because both the data analysis and the instructional treatment are still in progress.

SUCCESS AND UNDERSTANDING WHILE SOLVING GEOMETRICAL PROBLEMS IN LOGO.

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An analysis of the planning and debugging activities of 12-year old children solving geometrical problems in LOGO leads to the description of four important general beliefs these children have about successful problem solving, sequentiality, locality, progressive improvement and acceptability.

Their performance on tasks related to previously solved problems (figures produced on the screen), reveals three different levels of awareness of both the solutions found and the geometrical characteristics of the objects that they have worked on, outcome awareness, solution path awareness and implicative awareness, corresponding to fragmented, integral but unidirectional and finally general conceptualization of the object.

RESEARCH OBJECTIVES

The gap between success and understanding is wide and everyone has experienced situations where one has been successful without knowing why, sometimes without even noticing it. Success obtained this way cannot however be regarde as the most fruitful for transfer and further success. As Simon noted, transfer will be secured only if the learner is made or becomes aware of his skills (Simon, 1980). In this perspective, performance analysis should be regarded as useful as much in cases of success as in cases of failure, but solvers seldom take this point of view. The attempt to solve a problem successfully may require so many of the solver's resources that looking for understanding at the same time may be perceived as an obstacle. As Norman put it : "there is no better way to ruin a performance than to think simultaneously about the details of its execution" (Norman, 1982, p.69).

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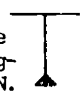
The play between success and understanding is particularly sharp in LOGO where educators' objectives pull towards understanding while children mainly look for success.

In this paper we will report on two aspects of children's solutions in a LOGO environment. We will first show some beliefs, in Schoenfeld's sense (Schoenfeld, 1985), that underlie children's conception of how to be successful while solving geometrical problems in LOGO. We will then show different levels of awareness of both the solutions found and of the characteristics of the tasks.

THE LOGO ENVIRONMENT

The subjects in this study were six 12-year old children (4 boys and 2 girls) of average Grade 6 level in math. They came as volunteers, after school hours, one hour a week for 24 sessions.

Their LOGO environment consisted of a set of six predefined procedures :

TRT :A, TLT :A	Rotates the turtle slowly either to the right or the left by A degrees.
MOVE :N	Moves the turtle N steps in a fixed direction without leaving a trace.
BASELINE :N	Produces a horizontal segment N steps in lenght.
TEE :N	Produces the figure  where both line segments have lenght N.
RECT :X :Y	Produces a rectangle of dimensions X and Y.

The last three procedures are state transparent (i.e. turtle's initial and final position and orientation are the same).

TASK AND DATA DESCRIPTION

For this study we used two kinds of tasks : on- and off-computer tasks.

In the <u>on-computer tasks</u>, subjects had to write programs that would produce geometrical pictures on the screen like the ones shown in Fig.1. Figure 1A was given during sessions 4 and 5 and Figure 1B in session 6. Special requirements were conveyed orally : for Figure 1A, the baseline had to be drawn first and the two big tees had to just meet (no overlap and no gap). For Figure 1B, the outside shape had to be a square, and inside that square, the stem (i.e. the vertical rectangle B in Fig. 1B had to be centered and of the same width as the bar (i.e. the rectangle A in Fig 1B.

Fig. 1 : Two of the figures to be drawn using the commands listed above.

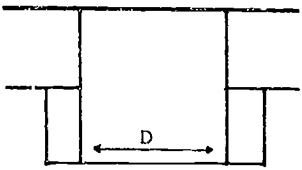


Fig. 1A : The 4-Tees figure

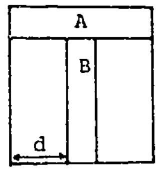


Fig. 1B : The Rectangles figure

These tasks were especially chosen because of the high degree of interdependence among the different components. For example, in the 4-Tees figure, the choice of input for BASELINE completely determines all the subsequent inputs to the TEE and MOVE procedures. This means that a successful solution can only be achieved by analytical means rather than through a strategy of trial and adjustment.

The description of the beliefs underlying LOGO problem solving activities is taken from the careful analysis of the dribble files produced by our subjects, of the changes progressively made in the text of their programs, and of the comments made during the activity of solving twelve such geometrical problems.

The off-computer tasks were given at a later date to assess children's level of awareness of both the characteristics of the figures they had to draw and the solutions they had found. These tasks were :

The Measure Attribution Task (2 weeks after the corresponding production task for Figure 1B; not given for Figure 1A). Given the print-out of the figure, subjects had to express the dimensions of the figure numerically. No conditions were made on the order in which the dimensions should be indicated.

The Input Insertion Task (4 weeks after the corresponding production task for Figure 1A; not given for Figure 1B). Subjects were given the print-out of the figure and the corresponding program without the inputs to the commands BASELINE, TEE and MOVE, and were asked to fill them in. As in the Production Task, the program had BASELINE as its first instruction.

In these two tasks, subjects were encouraged to check their solutions afterwards.

The Description Task (4 months after, for Figure 1A and Figure 1B). Subjects were given print-outs of the figures and were asked to describe verbally, among other things, any difficulties they might have had in producing the figures and how they handled these.

The Procedure Evaluation Task (4 months after, for Figure 1A; not given for Figure 1B). Subjects were given four programs related to the figure, that, for the same BASELINE, presented different inputs for the central interval (distance D in Fig 1A). Subjects were asked to decide on their correctness and to give reasons for their evaluation.

DISCUSSION

BELIEFS UNDERLYING THE SOLUTION OF TURTLE GEOMETRY PROBLEMS.

The debugging strategies used by our subjects can be seen to be directed by the fundamental assumption that the way towards a solution is generally by progressive matching between the current production and the target object.

This assumption underlies at least the four following beliefs about the sources of mistakes and the ways to success, beliefs that in their turn focus the solver's attention to specified parts of the program.

SEQUENTIALITY: Mismatches between the current production and the target figure can be dealt with sequentially rather than in parallel (unless some components of the figure are related in a very obvious way). Most of the time our subjects changed the size of the tees in Figure 1A in pairs and without changing the corresponding intervals.

LOCALITY: A mismatch between a component of the production and that of the target figure is attributable to the input chosen for the component itself rather than to the effect of any other input. The reasons that led to the choice of the input, whether based on visual informations, verbal constraints or determined by previous inputs are judged irrelevant. For instance in the Rectangles figure, Karen's first attempt produced an uncentered but correctly sized stem. The mismatch was nevertheless attributed to the procedure that draw the stem and so she modified its width (which was indeed constrained by that of the bar to meet the problem's condition) rather than its placement (distance d in Figure 1B).

PROGRESSIVE IMPROVEMENT: If a debugging attempt produces a mismatch between previously matched components then go back to the previous state and abandon the debugging strategy. Seeing that the big tees were not touching as they should, (Fig 2A), Rosemary decided to reduce the size of the Baseline. Changing one thing after the other, she left the input for the central interval unchanged and got the production shown in Fig 2B. To fix this gap she reduced the size of the central interval, hence bringing the two big tees closer together (Fig 2C). Rather than continuing this strategy, she then decided to reduce the gap between the two big tees by modifying their size (Fig 2D). Of course, this strategy, which she perhaps considered safer, produced many new mismatches. We see this abandoning of a winning strategy for a "wild goose chase" as having been dictated by the apparent regression. It is significant that, two weeks later, she considered the state shown in Fig 2B as having appeared earlier in her work than that shown in Fig 2A.

Fig. 2 : Effects of four successive changes of inputs carried out by Rosemary in the 4-Tees figure.

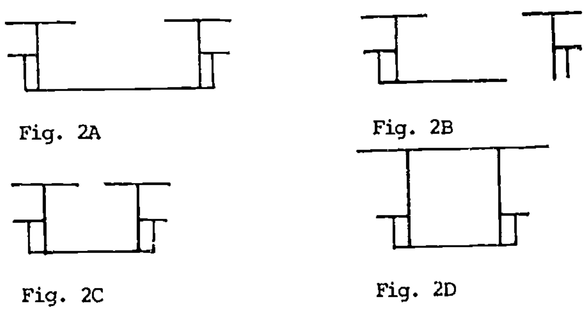


Fig. 2A Fig. 2B Fig. 2C Fig. 2D

ACCEPTABILITY: Any program that leads to a production that has all of the target's characteristics, except for the proportions between its components, can be regarded as successful.

LEVELS OF AWARENESS OF ONE'S SOLUTION AND OBJECT'S CHARACTERISTICS.

According to the subjects's behavior when asked to reproduce their solutions with new initial conditions (e.g. starting with a different input for BASELINE in the 4-Tees figure or for the outer square in the

Rectangles figure) in the Production Task, and to their performance on later off-computer tasks, we propose to distinguish three levels of awareness of the solution and of the object's characteristics. Each level of solution awareness corresponds to a level in the awareness of the object's characteristics. These levels are neither developmental nor according to increasing expertise, since a subject may prove to have different levels of awareness for two sub-problems solved within the same task.

Level 1 : OUTCOME AWARENESS. In the Production Task, subjects are able to note whether they have been successful or not on local problems independently of their performance on the global problem, but no explanation is available of the reasons for this success. This kind of awareness is short-lived and the whole solution process has to be restarted as soon as the specific inputs are forgotten. Later still, even the outcome can be forgotten. In the Description Task, Ben was still aware that he had trouble with the input for the central interval in the 4-Tees figure, but not whether he got it correct or not.

This level corresponds to a FRAGMENTED CONCEPTUALIZATION of the object. In the Measure Attribution Task, for instance, most of the dimensions are estimated visually, leading to some inconsistencies.

Level 2 : SOLUTION PATH AWARENESS. Subjects are able to reconstruct, in the Description Task, a correct solution, by retaining the necessary steps from their previous attempts and dropping the rest. Awareness of success is accompanied by awareness of the reasons for it. Working with relations, subjects may solve the problem starting with different inputs. But the solution path is not fundamentally changed. No solutions for sub-problems emerge from the global solution. For instance, none of our subjects gave, in the Input Insertion Task, inputs to the TEE procedures that were appropriate to the chosen BASELINE, even if they had found the input to give to the BASELINE in the Production Task by using the sizes of the TEEs.

Such a level of awareness of one's solution corresponds to an INTEGRAL but UNIDIRECTIONAL CONCEPTUALIZATION of the object. Dimensions in the Measure Attribution Task will be consistent because all the dimensions can be computed, in some convenient order, as functions of others, unlike in the Input Insertion Task where the order is imposed.

Level 3 : IMPLICATIVE AWARENESS. A successful strategy can not only be reconstructed but the whole solution may be re-planned to be more general. Changes in the way of handling local problems may be derived from the global solution. Anticipation may lead to computation of the sizes of the tees rather than to guesses.

This level of awareness of a problem's solution corresponds to a GENERAL CONCEPTUALIZATION of the object and of its characteristics.

CONCLUSION

This study shows that 12-year old children show some spontaneous beliefs about where their programs are faulty when evaluating the outcomes of such programs. They also have their own criteria for success.

Delayed investigations about their understanding of solutions and of the problems they worked on makes clear that success with LOGO often leads to less understanding than what was hoped for by researchers. Three different levels of awareness have been described, and good parallels appear between the awareness of solution and the conceptualization of the problem, confirming the piagetian hypothesis about the nature of awareness (Piaget, 1976).

Mathematics educators should be alerted to the fact that correct-looking productions do not necessary mean that the student have solved the expected problems. The many ways that lead to a particular target may allow the intrinsic problem to be circumvented.

Special efforts should also be taken to develop original ways to reinforce in children's minds an interest in understanding and in being aware of one's own work, not only on the computational but also on the mathematical level.

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INTERACTIVE DOMAINS, METACOGNITION AND PROBLEM SOLVING

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The Problem Solving and Thinking Project is documenting evidence of metacognitive activity of individuals during mathematical problem-solving and identifying additional phenomena that may affect problem-solving performance. This paper presents preliminary results for Jill, an inservice teacher who is participating in the Project. Although Jill was not successful in solving either of two nonroutine problems attempted, regular and frequent evidence of monitoring and regulation of cognitions was found. Additional analysis produced domains that may be interferring with her problem-solving performance. While it is premature to make any conclusions based on this work, it lays the groundwork for further analysis in the Project.

This paper presents results of identifying and analyzing individual metacognitive activity during mathematical problem-solving sessions and additional phenomena that may affect problem-solving performance. The research is part of the Problem Solving and Thinking Project sponsored by the National Science Foundation, currently completing the first year of a three year study. We are conducting a naturalistic inquiry driven by the assumptions (basic beliefs) of the naturalistic paradigm. These assumptions have shaped and influenced the focus of our inquiry, our choice of methodology, and our analysis techniques.

FOCUS OF THE INQUIRY

The most recent results of the National Assessment of Educational Progress (Lindquist, Carpenter, Silver, & Matthews, 1983) indicate that children in the United States are generally competent in mathematical computation skills, however, they are unable to problem solve. We assume a relationship between mathematical problem-solving performance and metacognitive activity (Garofalo & Lester, 1985; Hart & Schultz, 1985, 1986; Schoenfeld, 1983; and Silver, 1985). Specifically, the monitoring and regulation of one's knowledge, beliefs, and strategies may impact problem-solving performance. From preliminary work the assumption seems substantive and persuasive but we cannot refute or support it until we systematically study metacognition in the context of mathematical problem solving.

We have not accumulated sufficient knowledge on the nature of metacognitive activity to know whether the right questions are being asked, essentially we are "begging the question." The "questions" presented below, therefore, are only given to establish boundaries for our study; they define "the terrain, as it were, that is to be considered the proper territory of the inquiry" (Lincoln & Guba, 1985, p. 227).

1. What categories (domains) of phenomena associated with an individual's problem-solving performance can we identify? While we have assumed the substantive theory of metacognition, we are not implying determinancy. The simultaneous influencing of factors makes it impossible to sort out a single causality. Metacognition is only one of many factors that simultaneously influence problem-solving performance. It is important then to find the range of domains that influence problem-solving performance.

2. What evidence of metacognitive activity can we find? This question focuses on the assumed domain of metacognition. If an individual is in a problem-solving situation, what can we look for that indicates monitoring and/or regulation of cognitions?

THEORETICAL PERSPECTIVE

While naturalistic inquiry generally rejects the use of an a priori theory, we are assuming a general theory of metacognition (Flavell, 1976; Garofalo & Lester, 1985) as representing the state of the art in mathematics education. Although the assumptions of the theory are not apparent, we do not find it inconsistent with our inquiry paradigm. For a more thorough discussion of our perspective see Schultz (1987, these Proceedings).

RESEARCH DESIGN

Due to the unpredictable interaction expected and found in the environment, the design was allowed to emerge rather than to be completely constructed preordinately.

The subject. The subject for this paper is Jill, one of 15 inservice teachers in our Problem Solving and Thinking Project. She is a young, high school mathematics teacher completing her first year of teaching mathematics.

The setting. Jill participated in The Institute on Problem Solving and Thinking, a tuition-free, master's level mathematics education course that we developed and taught as part of our project. It served as a context for the initial stage of our data collection. In addition, Jill has continued on with the project as a volunteer, long-term subject.

The methods. Two methods for collecting qualitative data were observations and interviewing. Observations of Jill were of her solving textbook word problems familiar to her before one of her classes, solving an unfamiliar nonroutine problem in a small group, and solving an unfamiliar nonroutine problem alone. A structured interview was conducted before and after she solved the problem alone. Combining the two methods helps to make distinctions between what Jill does (observation of problem-solving behavior in several contexts) and what she says she knows or believes (interviewing). This triangulation provides validity for our results.

The data. Three videotapes were made of the observations and interview. Data are transcriptions of the three tapes. The first protocol is of Jill teaching her class. The second protocol is of Jill with one other inservice teacher solving an unfamiliar nonroutine mathematical problem. The third protocol is of Jill solving an unfamiliar nonroutine problem in a think-aloud session with the interviewer. This last transcription includes the guided interview.

ANALYSIS

Constant comparison. This qualitative strategy was used to begin developing an answer to our first research question: What categories of phenomena can we identify that are associated with an individual's problem-solving perfomance? It involves scanning the data for categories that emerge, i.e, looking for patterns and identifying domains. While this process begins initially on a "feels right" basis that relies on tacit knowledge, by constantly comparing incidents in the same and different categories we eventually begin to develop properties of the categories and rules for membership.

Parsing the protocols. The parsing technique was used to assist us in answering our second research question: What evidence of metacognitive activity can we find? The protocol of Jill working alone and the protocol of Jill in a group were parsed into periods of time when a single set of like actions (episodes) occurred. The time between episodes (transitions) when the decision is made (or not made) to continue on with a process or to start in a new direction are critical points where evidence of monitoring and regulation may be apparent. Details of this process are described in Hart and Schultz (1985, 1986) and Schoenfeld (1983).

RESULTS

Although our second research question is subsumed in our first in the actual analysis results for question 2 emerged first. The discussion is more appropriate then in that order.

Question 2: Evidence of metacognitive activity. The monitoring and regulation components of metacognition were coded separately in the protocols. Both verbal and inferred evidence of these were found in the data. Verbalized monitoring occasionally occurred without verbalized regulation and vice versa. In those cases, inferences could usually be drawn fram the transcript about non-verbalized activity. The following are examples from Jill's transcript while working in a small group trying to solve the following problem: "I'm a proper fraction in simplest form. The product of my numerator and denominator is a multiple of seven. Their sum is a perfect square. Who am I?"

1. After initial reading of the problem Jill says: "Do you know anything about this?" (evidence of MONITORING); and "Maybe we could talk about what the terms mean?" (evidence of REGULATION).

2. In the third transition there is no verbal evidence of monitoring. From her verbalized regulation we infer evidence of monitoring because she abruptly stops discussing terms and says: "So, let's take a fraction like X over Y," (evidence of REGULATION).

In the transcript of Jill working alone on the problem "Can you find two fractions in simplest terms with different denominators whose difference is 2/13?" we found evidence such as the following:

3. After working a few moments on a problem Jill asks: "Can I have two fractions whose difference eventually in reduced form is 2/13? (evidence of MONITORING); and "O.K. let me see if I can start with a common denominator of 26." (evidence of REGULATION).

4. Again after working for a few more minutes she says: "Oh, do these two fractions have to have a different denominator?" (evidence of MONITORING); and "O.K., what if I have 13 as a denominator and 2 as a denominator" (evidence of REGULATION).

Question 1: Categories of phenomena. After the protocols were analyzed for evidence of metacognitive activity, they were analyzed for patterns. Regular and frequent monitoring was observed, from which two categories emerged: monitoring of her knowledge and monitoring of her understanding. Knowledge in this use was defined as information and beliefs she brought to the problem-solving session. Monitoring of understanding became defined as assessment of the progress of the problem process. An example of monitoring her knowledge is when Jill comments while going over the vocabulary in the problem that "there aren't a heck of a lot of perfect squares." She changes her direction and begins listing perfect squares. An example of monitoring her understanding occurred when she says "This isn't getting us anywhere." At this early stage of our research the properties of these categories are still loosely defined. They do, however, serve as a beginning for our thinking.

While no obvious categories emerged for regulation of cognitions it was observed that all Jill's regulation was done as a result of her internal monitor rather than an external monitor such as another group member or an interviewer.

Four major categories emerged from searching the protocols for patterns and identifying other domains that influence Jill's problem-solving process. They have been loosely labeled Time, Ease, Embarrassment, and Other. After indicating locations of these on the transcripts, Jill was given the opportunity to respond to our observations. Some of the groupings are based on her responses. The following are typical statments that fell into each domain. They are direct quotes.

Time. 1. We don't have a time limit on this do we? 2. It will take me longer to write it down. 3. I could figure it out if I had more time.

Ease. 1. It would be easy if my first 2 fractions had a denominator of 13. 2. It would be easy if I didn't have to have different denominators. 3. It would be easier if I could read it.

Embarrassment. 1. I'm embarrassed. 2. I'm sounding foolish talking out loud. 3. You're enjoying this aren't you? (This comment was placed into this category as a result of Jill's reaction. When asked to elaborate she said she was embarassed and wanted to stop and regroup her thoughts).

Other. Other items of interest were identified and as yet do not fit into a category. 1. I want to know the answer. Tell me what I did wrong. 2. Is this what you wanted? 3. It would be helpful if you had a standard way to do these problems.

CONCLUSIONS

What can we say about Jill's metacognitive behavior? What information does this give us on understanding the relationship of problem solving, metacognition, and other factors? In both problem-solving sessions Jill demonstrated regular and frequent monitoring and regulation of her knowledge and her understandings. Also, an evaluation of her monitoring indicated productive awareness. Yet she was unable to solve either of the problems. We hypothesize other factors are interacting and influencing her level of performance. There appears to be a powerful belief that it is necessary to solve problems quickly in order to be a "good" problem solver and not be "embarrassed." If these related beliefs are driving her problem-solving behavior they may be interferring with her ability to perform successfully on the problems. Her beliefs may have incapacitated her to such a degree that she is unable to live up to the picture she has of good problem-solving performance.

It is expected that these characterizations and interpretations of Jill will continue to evolve as data from other subjects are reviewed. This initial conceptualization will be filtered, organized and interpreted through several more phases of refinement. Our "problem" is to make sense out of this, not just as observers but as participators as well, for after all, we became a part of Jill.

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METACOGNITION: THE ROLE OF THE "INNER TEACHER"(2)

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ABSTRACT

The nature of metacognition and its implication to mathematics education is our ultimate concern to investigate through a series of our researches. We argued in the last paper that metacognition is given by another self or ego which is a substitute of one's teacher and we referred to it as "inner teacher". In this paper we will show a more concrete description of metacognition of children through their recorded responses. Especially we will prove that there is a close correlation between pupils' metacognition and their performance in mathematics.

INTRODUCTION

In our last paper, we presented the concept and the roles of *inner teacher* in the research of metacognition. There we paid attention to children's two ego: one is the acting ego and the other is the executive ego which monitors, assesses, and controls the former. This executive ego is really a substitute or a copy of the teacher from whom the pupil learns. In the teaching-learning context, we refer to the executive ego or the subject of metacognition as the *inner teacher*.

In this paper, we will investigate more clearly the concept of the metacognition through pupils' recorded responses of their solving mathematical problems.

Roughly speaking, we could regard "metacognition" as the knowledges and skills which make the objective knowledges active in one's thinking activities.

There are a few proposals on the categorization of "metacognition" in general but here we will follow to the suggestion of Flavell's and adopt the next three divisions:

1. on the *self*
2. on the *task*
3. on the *strategy*.

Our unique conception is that these metacognitions are thought to be originated from the teacher him/herself. Teacher cannot teach any knowledge *per se* directly to pupils but teach it inevitably through his/her personality with the metacognitions whatsoever.

To express more clearly, teachers give any knowledge or skill always with its monitor who is to manage the pupil as executor. In another word teaching is the activities to settle another *self* (or *ego*) in pupils' mind as the substitute of their teacher.

It is well known that teacher and their method of teaching are the most important component in education especially in mathematics teaching. Our intention is to clarify the implication of this old educational common sense and give it a scientific analysis and develop a more effective method of teaching mathematics.

The most crucial point of our research is to investigate what part of teachers' activities is introduced as their pupils' cognitive elements. And as such, our final aims of research would be as follows:

1. to make a list of teachers' activities in teaching
2. to make a list of pupils' thinking activities in their learning
3. to compare both lists and make a list of metacognition
4. to establish the relation between pupils' mathematical performances and their commanding metacognitions
5. on the basis of these reflections, to develop more effective method of teaching in mathematics.

In this paper, we hope to contribute to attain 3.,4. of the above aims.

THEORETICAL FRAMEWORK OF THE RESEARCH

Logical Model of Metacognition

At first we propose a logical model of metacognition, which will be also available to understand the meaning of several technical terms such as "metacognition", "metaskill", "metaknowledge", "monitor", "assessment", "control", etc..

As an example we wish to observe the case of a pupil who, being given a verbal problem and asked to solve it, thought that

"It is a long problem, it is difficult and I should read it carefully."

This process of the pupil's thought is paraphrased as follows:

M_1: it is a long problem.

K_1: if it is a long problem, it is difficult.

C_1: it is difficult.

$M_2=C_1$

K_2: if it is difficult, we should read it carefully.

C_2: we should read it carefully.

We should like to formalize this cognitive process in a chain of syllogisms like in logic as follows:

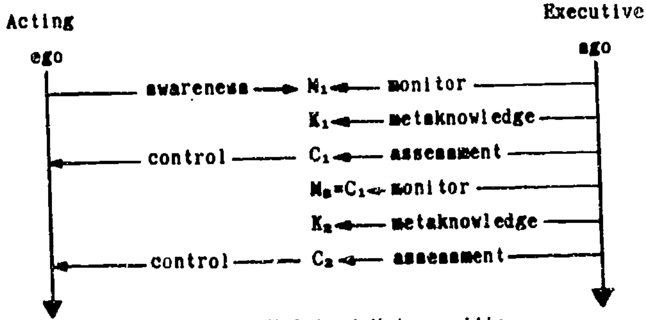

Fig. 1 A Model of Metacognition

1) M_1 is the fact that the acting ego was aware of from the confronted problem situation, and ways of awareness and their adequacy are monitored by another ego that is the executive ego.

2) K_1 is a proposition which comes from the executive ego. We identify this with the metaknowledges which is what we wish to analyse about its nature and its origin.

3) C_1 is a conclusion of *modus ponens* from two premises M_1 and K_1, but it is also assessed by the executive ego on its adequacy in this situation.

On this model we could have a psychological interpretation of each step.

(1) Minor premise M_i comes from the problem confronted; it is an information given from the problem through the monitoring of the executive ego.

(2) Major premise K_i comes from inside of the learner through his/her reflection.

(3) Conclusion C_i is not a mere logical consequence but it is also a result of one's assessment upon the urgent problem situation, and this conclusion will control the direction of the thinking and often becomes the minor premise of the following syllogism as is shown in the diagram.

(4) Furthermore we could suppose some mechanism which guides the whole process, choosing appropriate information, remembering a suitable knowledge and not losing the way. We should like to call this mental mechanism "metaskill", which is not the specific skill such as the calculation or drawing, but seems to have a very general nature, and analysis of this metaskill will be our another theme that we wish to

pursue hereafter.

(5) Finally we would like to refer "metacognition" to the whole chain of these syllogisms including metaskills.

METHODOLOGY OF THE RESEARCH

Giving a verbal problem to children, we asked them to solve it and at the same time to write what they thought in solving it. They are sixth grade pupils, already learned multiplication and division by common fraction.

An example of the problem:

There is an elastic string. When we pull it out by $\frac{3}{5}$ of its length, it becomes 64 cm long.
How long it was before pulling?

We decompose pupils' recorded responses sentence by sentence and classify these sentences into several categories. Here are some protocols:

(1) In case of a pupil A (boy, average level)

1. This problem is the same as that I did last.
2. I feel I have forgotten half of it.
3. I think I may do it, if I show it on the line.
   (and drew a diagram, which is omitted here.)
4. Ah, it's hard.
5. (omitted)
6. (omitted)
7. I did, I had the answer.
8. But I wonder it is correct.
9. Teacher said it is an easy problem but it was hard, perhaps because I was about to forget.
10. There may be another way to think but it is all that I can do.
11. Then so much for this problem.

(2) In case of a pupil B (boy, below average)

1. It's a long problem.
2. I hope teacher's help.

3. I don't like a difficult problem.
4. I like science better than math.
5. I can make mistake: it's not the examination.
6. I may well understand if I draw a diagram.

We divide pupils into three groups according to their previous performance in mathematics, and examine the characteristic features of their recorded responses from the view points of metaknowledges.

RESULTS AND DISCUSSION

1) In these short pupils' self-descriptions we can clearly discern three kinds of cognitions, the typical example of which is shown incidentally in the first three sentences of the pupil A.

In 1. This boy is trying to organize the given problem into his existing system of knowledges. (task)

In 2. He has enough ability to reflect himself in solving the problem. (self)

In 3. He has the knowledges of strategies to apply to this specific problem situation. (strategy)

2) The most remarkable fact was that there was a great distinction among characteristic features of each performance group. This is exemplified by responses of the sixth grade pupils to the next problem.

problem: It takes 12 minutes to go from A-station to B-station by bicycle with speed 240 m per minute. But Yamada wishes to go through this way on foot. His speed is 60 m per minute. How many minutes does it take for him?

The above average group: Most of this group will not express themselves so much, but only describing abbreviated process of solving.

* How long does it go in 12 min., if it goes 240 m in a minute?
* I had the distance from A to B.

The average group: They seems to like chattering, but rather have a negative attitude.

* It's trouble some for me.
* I can understand.
* It's not the examination, so I can do mistake.

The below average group: The most impressive was their badly negative attitude near to hostility against mathematics.

* It's my weak point.
* It's a disagreeable problem.

3) In the above average group, pupils seem to understand the problem very easily, whether or not they can solve it. To know or to understand something means to place it in one's existing system of knowledges and give it an appropriate position that it should deserve. These able pupils can do this very smoothly, almost unconsciously of their reflection efforts, and only describing the process they really followed. But less able pupils seem to be suffering from many varieties of prejudices toward problem and mathematics in general before they choose some believable devices of challenging against the problem.

4) There are negative expression in case of below average pupils, as we showed some of them. We think these expression are originated from the teacher's attitude of teaching until this time.

According to the statistics out of curiosity of someone in our country, the most frequent utterance of mathematics teacher during the lesson is "Understand?"; often it amounts to several times in a minute. This kind of utterance would be the most harmful to foster the healthy metaknowledges, because,

1. it sounds something like authoritarianism to compel pupil to understand anything that teacher says as infallible.
2. it generate a fear or an inferiority complex toward mathematics and it's learning.
3. it generate the belief that mathematics is absolute and the only way to learn it is to learn by heart.

In stead of "Understand?", we recommend the often utterance of "What do you want to do next?". This asking would stimulate pupils' autonomous thinking and urge their active participation in the classroom activity.

CONCLUSION

1) We could have a more clear idea about concept of metacognition that we called "inner teacher". It is another self or ego who is watching, controlling, criticizing the original self like the teacher they have learned from.

2) We highly appreciate Flavell's categorization of metaknowledge. In addition to this, we noticed from the pupils' responses that we could divide these metaknowledges in some other categories: one is *positive-negative* and another is *general-specific*, though these distinction are not absolute but relative to the problem situation and the person. Positive and general metacognitions play a powerful role in the cognitive activities and they should be favored in the teaching of mathematics.

3) Though we have not yet closely examined, we had an insight that there is an intimate correlation between metaknowledge of *self* and performance in mathematics. Less able pupils are apt to have negative views toward themselves, but we could not yet decide of which is cause or result of the other. We also believe that the teacher is responsible for cultivating their positive metaknowledge of themselves.

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THEORETICAL PERSPECTIVES ON INQUIRY INTO PROBLEM SOLVING, KNOWLEDGE ACQUISITION, METACOGNITION, AND TEACHING

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This paper gives the theoretical perspectives for research in progress on metacognition and problem solving. First, a world view of learning, teaching, and teacher training of mathematical problem solving is described as supported by the "new paradigm" of Schwartz and Ogilvy. This is followed by the assumptions made in our research, which are described according to the assumptions of naturalistic inquiry of Lincoln and Guba. Finally, the focus of our inquiry is developed which essentially aims at identifying what our research questions should be.

The purpose of this paper is to present the theoretical perspective of the Schultz and Hart research on metacognition and problem solving currently supported by the National Science Foundation. (See Hart, 1987, in these Proceedings for project description.) In order to do this, it is first necessary to explain our world view of learning, teaching, and teacher training of mathematical problem solving. This is outlined according to Schwartz and Ogilvy's (1979, 1980) seven characteristics of the "new paradigm," which come from disciplinary world views such as those from chemistry, psychology, philosophy, mathematics, etc. Second, our assumptions for researching learning, teaching, and teacher training of mathematical problem solving are explained according to a naturalistic paradigm of inquiry outlined by Lincoln and Guba (1985). There are five assumptions of an inquiry guide called the "naturalistic paradigm" whose tenets are subsumed by the same world view as Schwartz and Ogilvy. It is the method of choice, especially when humans are the objects of the research. This section of the paper is outlined according to these five assumptions. Finally, it will be possible to explain the focus of our inquiry. Discipline state of the art literature and theory will be considered throughout. Judgement calls are made as to the relevance and applicability of the literature and theory to the world view and paradigm presented.

LEARNING, TEACHING, TEACHER TRAINING, AND PROBLEM SOLVING

Complexity. Diversity and interactivity are characteristics of our reality. Schwartz and Ogilvy (1979) say that it is "in principle impossible to separate a thing from its interactive environment" (p. 10). In terms of this characteristic, there are at times arbitrary boundaries between learning, teaching, and teacher training. There is undeniable interdependence between the ecosystems of student and teacher, and teacher and teacher trainer. Teaching and learning for knowledge acquisition through the development of productive problem-solving behaviors (Lesh, Landau, & Hamilton, 1983) requires a reciprocol engagement of systems resulting in such roles as learner as teacher and teacher as learner. Kilpatrick's (1985) metaphors of reflection and recursion suggest that one steps outside the system in order to give meaning to experience. Current theory on teaching and learning mathematics as a constructivist activity supports this notion that neither process nor content is an absolute (Cobb & Steffe, 1983; Confrey, 1985; Maher & Alston, 1986; Steffe, 1986).

Heterarchies. It is believed that if there is order in our "realities," many elements exist side by side. Which element is predominant at any moment depends on "interacting variables" often imposed by human thinking for the sake of discussion rather than as determined by nature. An example is discussion of accommodation and assimilation which describe the equilibration process (Piaget, 1973). These are useful terms for us since disequilibrium is a state of one's cognitions when problem solving. Which comes first, however, is often determined by way of rhetoric in scholarly communication. The application of heterarchy to the teaching and learning of mathematical problem solving includes the belief that the orders of teaching then learning, knowledge acquisition then problem solving, or cognition then metacognition is a function of a scholar's mind not of laws of nature. Facilitation of learning or problem solving rather than teaching by command is viewed as the most useful process for effecting change. Authority of teacher over student or teacher trainer over teacher gives way to voluntary mutual association. This, too, supports the idea of "teacher as learner" developed above.

Holography. This term is used metaphorically to describe the models that we use to describe the nature of the world. Images one has of a system or an organism are "projected" so that complete information from the whole is found in any of its parts. This may be seen in cognitive psychology by the shift from use of mechanistic to organismic models to characterize how children develop cognitively (Carpenter, 1980). The shift was from a stimulus-response concept (machine) to a concept of a multidimensional, developmental, dynamic, interactive concept (a living organism). Another example of this characteristic is found in explanations of mathematical knowledge acquisition, which is no longer viewed through a curriculum of concepts, properties, algorithms, skills, and problem solving tending to promote disjoint knowledge. Rather, it is "seen" as a total process of evolving within-concept systems, between-concept systems, systems of representations, and systems of modeling processes brought on by real problems (Lesh, Landau, Hamilton, 1983). For example, one could look at a learner's understanding of fraction to determine his or her understanding of partition (within concept), or understanding of the relationship between whole numbers and fractions (between concepts), or what written, verbal, or concrete systems are used to communicate fractional ideas (representation and modeling). Let's put this in the context of problem solving. To effectively accomplish teacher training in problem solving, it is our view that this is facilitated by teachers reflecting on not only themselves in the process of learning, but the teacher trainers in the process of teaching, and their students in the process of learning. Observing and reflecting on outcomes of each party involved as it relates to the whole "picture" is a potentially useful strategy to impact change.

Indeterminancy. It is no longer considered completely possible to determine a future state of the world. Schwartz and Ogilvy (1979) say that ambiguity about the future is a condition of nature. For us, precise outcomes of teaching mathematics cannot be predicted. For both teaching and mathematical problem solving are complex systems and teachers and students are complex organisms. Moreover, the very nature of the processes used for determining a learner's state of knowledge, problem-solving ability, or teaching ability affects the outcomes! Formal or informal measurements of these things are determined by the relationship between learner and teacher, or teacher and teacher trainer. Even these relationships, however, are an indeterminant.

Mutual Causality. The synergistic relationship of these seven beliefs is particularly noticable here, where mutual casuality and heterachy cannot stand alone. Mutual causality means "simultaneous influencing of factors over time in such a way that it is no longer relevant to ask which caused which" (Lincoln & Guba, 1985, p. 54). This supports our interpretation of Polya's (1957) stages of problem solving not as linear, but rather as both cyclic and mutually responsible for actions throughout the problem solution space. Even though a problem solver may say he or she really understands a given problem, it might not be until several strategies later before true meaning of the problem statement is absorbed. To move on to another example within this fifth characteristic, even though a teacher models a certain set of beliefs and teaching behaviors, learner outcomes are influenced by yet a complex of mutually interacting "causes." The idea of influencing or facilitating replaces those of teacher as authority. There is a shared authority among knower, learner, and known.

Morphogenesis. The traditional perspective was that change was a result of assembling separate components according to some plan. Morphogenetic thinking supports the notion that change evolves from chaos when the "system is open to external inputs" (Schwartz & Ogilvy, 1979, p. 14). Catastrophy theory is mathematics' way of explaining this. What this means is that learning outcomes are not attributed to controlled treatments, but rather to very complex systems and organisms arising through very dense and complex teaching/learning interactions. Learning to teach mathematical problem solving or learning problem solving itself as a student, teacher, or teacher trainer is viewed as change that is not only continuous and quantitative but discontiuous and qualitative as well. The ability to solve a problem has been studied macroscopically (Hart & Schultz, 1985, 1986; Schoenfeld, 1983) through episodic parsings, which identify locations where significant changes in the solution space occur. Our interpretation of what starts change is supported by morphogenesis--that it wasn't the singular event of a reflective moment by a problem solver, but rather an indeterminant networking of thoughts and behaviors that eventually led to a major shift in the solution space. It helps to go back to the conceptual models construct as an adaptive structure to describe knowledge acquisition as evidence of metacognitive (reflective) activity. Within- and between-concept systems and systems of representation and modeling processes are labels attempting to give an interpretation of the evidence of metacognitive behaviors and explain what happens as a result of managerial decision making (Schoenfeld, 1983). New understandings are made possible by the monitoring and regulation of cognitions--a kind of monitoring and regulation of one's own chaos. Who else is better equipped to do this but one's self? We believe that conceptual models evolve this way from less stable to more stable forms and from being less integrated to more integrated with other models through a plausable complex of influences that one eventually "makes sense" of.

Perspective. Objectivity is now seen as an illusion. However, subjectivity, as the other extreme, is not the suggested atlernative. Rather, it is perspective--"a view at a distance from a particular focus" (Schwartz & Ogilvy, 1979, p. 15). In fact, multiple perspectives are needed to get a more complete picture of the phenomenon. What is believed about mathematics, problem solving, teaching mathematics, or learning mathematics influences what is seen (believing is seeing) (Garofalo & Lester, 1985). To facilitate learning, the teacher takes the learner's perspective. To facilitate teaching, the learner takes the teacher's perspective. Rachlin's (1982) process approach to algebra instruction supports taking different perspectives of an algebraic concept or procedure. He encourages moving from the generalization perspective of an idea or procedure, to flexibility of approach where the student is to switch from one mental operation to another. Finally, the student is to engage in reversibility of mental processes as a switching from a direct to a reverse train of thought. That is, he believes that changing perspective of algebraic ideas and procedures enhances the depth of understanding.

BASIC RESEARCH ASSUMPTIONS OF THE NATURALISTIC PARADIGM

Reality. There are multiple constructed realities (Do we create what we research?) that can only be studied holistically (Lincoln & Guba, 1985). Therefore it is no longer considered useful to hold certain variables constant in order to study the influence of one or two other variables. Given this assumption, we have read the literature and learned the theories. We have experienced, observed and meditated on learning, teaching, and teacher training of mathematical problem solving, but still are unsure of what questions to ask. The stated problem is the inability of students in the United States to be successful in problem solving. We (Schultz and Hart) go on tacit knowledge that the milieu of our investigation should be teacher training in problem solving.

Knower and Known. The researcher and those being researched have an interactive relationship. To study a person means in part to study an interfacing of one's self with that person. This assumption means that if we are to study teachers during inservice training, we are ourselves subjects of study as well. Data collected on the investigators is subject to analysis in order to better interpret that of the teachers. Moreover, to enhance and maximize the expansion of teacher data, teachers continue in a participant/observer role with the researchers, where the activity is interpretation of data and taking opportunities to obtain more data.

Generalization. The research goal is to generate idiographic statements on differences, primarily in the form of working hypotheses that describe the subject of study. Our intent is to generate hypotheses, to decide what questions should be asked. The question of parmount importance, then, is "What are the questions?"

Causality. There are plausible influenc s, as on a web, making it impossible to distinguish causes from effect. Given this assumption, we need to study the whole web. Therefore, we have taken as our objects of study ourselves as teacher trainer/inquirers, teachers, and their students. We are all learning problem solving. In a constructivist format, we are all teachers and learners.

Values. Lincoln and Guba identify at least five corollaries in which inquiry is value-bound. Inquiries are influenced by inquirer values, by choice of paradigm, by choice of substantive theory applied, by content under study, and by whether the first four corollaries are congruent or conflicting with each other.

THE INQUIRY

The goal of the Problem Solving and Thinking Project is to document evidence of metacognitive activity of individuals during mathematical problem solving and identify additional phenomena that may interact during problem-solving performance. This documentation is being used to investigate the complex and interactive environment influencing problem solving. The "questions" we hope to answer are: (a) What categories (domains) of phenomena associated with an individual's problem-solving performance can we identify? (b) What evidence of metacognitive activity can we find? (See Hart (1987) in these Proceedings for elaboration.)

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WHY IS PROBLEM SOLVING SUCH A PROBLEM?
Reactions to a Set of Research Papers

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Countless generations of teachers have voiced concern about the inability of many of their students to solve any but the most routine verbal problems despite the fact that they seem to have "mastered" all of the requisite computational skills and algorithmic procedures. Unfortunately, until rather recently researchers have seemed content to attribute problem-solving difficulties almost exclusively to cognitive aspects of the problem solver's performance. Today there is growing sentiment for the notion that a much broader conception is needed of what mathematical problem solving involves and what factors influence problem-solving performance.

I begin my comments with a discussion of the categories of factors that play prominent roles in an individual's success or failure in solving mathematics problems. I do this in order to establish a basis for my reactions to the set of papers on metacognition and problem solving.

What Influences Problem-solving Performance?

An individual's failure to solve a problem successfully when the individual possesses the necessary knowledge stems from the presence not only of cognitive factors, but also of non-cognitive and metacognitive factors that inhibit the correct utilization of this knowledge. These factors can be placed into five broad, interdependent categories: knowledge, control, affects, beliefs, and socio-cultural conditions. It is not my intention to discuss these categories in detail, but it seems appropriate to make a few comments about each of them.

Knowledge

It is safe to say that the overwhelming majority of research in mathematics education has been devoted to the study of how mathematical knowledge is acquired and utilized. Included in this category are a wide range of resources that can assist the individual's mathematical performance (cf., Schoenfeld, 1985). Especially important types of resources are the following: facts and definitions (e.g., 7 is a prime number, a square is a rectangle having 4 congruent sides), algorithms (e.g., long division), heuristics (e.g., looking for patterns, working backwards), and the host of routine, but not algorithmic, procedures that an individual can bring to bear on a mathematical task (e.g., procedures for solving equations, general techniques of integration). Of particular significance to this discourse is the way individuals organize, represent, and ultimately utilize their knowledge. There is no doubt but that many problem-solving deficiencies can be attributed to the existence of "unstable conceptual systems" (Lesh, 1985). That is, when individuals are engaged in solving a problem it is likely that at least some of the relevant mathematical concepts are at intermediate stages of development. In such cases, to the extent that problem solvers can adapt their concepts to fit the problem situation they are successful in solving the problem. Furthermore, since mathematical concepts evolve from situations, it is natural to suggest that there is a close link between conceptual knowledge and problem-solving performance (Vergnaud, 1982).

Control

Control refers to the marshalling and subsequent allocation of available resources to deal successfully with mathematical situations. More specifically, it includes executive decisions about planning, evaluating, monitoring, and regulating. One aspect of control processes that has become increasingly popular as an object of research in recent years is regulation of cognition. This aspect of control together with knowledge about cognition and beliefs constitute what is currently referred to as metacognition.

I might point out that metacognition, as defined and discussed in several of the preceding papers, is not really a new construct. Indeed, metacognition seems closely related to Skemp's (1979) notion of "reflective intelligence" and to Piaget's (1976) construct "reflexive abstraction." Furthermore, metacognitive behaviors correspond very closely to the executive components in some information-processing models of cognition (e.g., Sternberg, 1980) and to conceptions of problem solving (e.g., Polya, 1957).

Affects

This domain includes individual attitudes and emotions. Mathematics education research in this area often has been limited to examinations of the correlation between attitudes and performance in mathematics. Not surprisingly, attitudes that have been shown to be related to performance include: motivation, interest, confidence, perseverance, willingness to take risks, tolerance of ambiguity, and resistance to premature closure.

Typically, emotions are subjective reactions to specific situations. Of course, emotions can have either a facilitating or debilitating effect on the individual, but negative emotions (e.g., frustration) are not necessarily debilitating nor are positive emotions (e.g., joy) necessarily facilitating. There is a growing body of research to support the notion that emotions and cognitive actions interact in important ways (Mandler, 1986). Most of this research has been restricted to the study of the conditions under which certain emotions occur or to the nature of an individual's behavior when in a particular emotional state.

To distinguish between attitudes and emotions I choose to regard attitudes as traits, albeit perhaps transient ones, of the individual, whereas emotions are situation-specific states. An individual may have developed a particular attitude toward some aspect of mathematics which affects her or his performance (e.g., a student may greatly dislike problems involving percents). At the same time, a particular mathematics task may give rise to an unanticipated emotion (e.g., frustration when little progress is made toward solving a problem after working diligently on it for a considerable amount of time). The point is that an individual's performance on a mathematics task is very much influenced by a host of affective factors, at times to the point of dominating the individual's thinking and actions.

Beliefs

Schoenfeld (1985) refers to beliefs, or "belief systems" to use his term, as the individual's mathematical world view; that is, "the perspective with which one approaches mathematics and mathematical tasks" (p. 45). Beliefs constitute the individual's subjective knowledge (i.e., not necessarily objectively true) about self, mathematics, the environment, and the topics dealt with in particular mathematical tasks. For example, my colleagues and I have found that many elementary school children believe that <u>all</u> mathematics story problems can be solved by direct application of one or more arithmetic operation and which operation to use is determined by the "key words" in the problem. Naturally, this belief has a very powerful effect on the way they solve problems.

It seems apparent that beliefs shape attitudes and emotions and direct the decisions made during mathematical activity. In my own research I have been particularly interested in students' beliefs about the nature of problem solving as well as about their own capabilities and limitations.

Socio-cultural Conditions

In recent years, the point has been raised within the cognitive psychology community that human intellectual behavior must be studied in the context in which it takes place. That is to say, since human beings are immersed in a reality that both affects and is affected by human behavior, it is essential to consider the ways in which socio-cultural factors influence cognition. In particular, the development, understanding, and use of mathematical ideas and techniques grow out of social and cultural situations. D'Ambrosio (1985) argues that children bring to school their own mathematics which has developed within their own socio-cultural environment. This mathematics, which he calls "ethnomathematics," provides the individual with a wealth of intuitions and informal procedures for dealing with mathematical phenomena. The point then is that the wealth of socio-cultural conditions which make up an individual's reality plays a prominent role in determining the individual's potential for success in doing mathematics both in and out of school.

Relationship Among the Categories

As I mentioned earlier, these five categories are interdependent. In particular, socio-cultural conditions directly influence the formation of attitudes, emotions, and beliefs, as well as control and knowledge utilization. Affects influence both control and knowledge but not beliefs or socio-cultural conditions, whereas beliefs directly influence all the other domains except socio-cultural conditions and control directs only the ways in which knowledge is utilized. In my view, it is vital that future research gives serious attention to investigating the nature of the interrelationship among these sets of factors.

Some Thoughts about Current Problem-solving Research

Given the foregoing perspective as a backdrop, my approach to analyzing these reports was to look for underlying themes and issues rather than to enumerate flaws and shortcomings. More specifically, I tried to identify fundamental concerns or questions which seemed to be a motivation for these researchers' efforts, and in a larger sense for many of the types of research studies presently being conducted by mathematics educators on the topic of metacognition and problem solving, and to relate these concerns to one or more of the categories of factors considered earlier in this paper. Consequently, although my remarks are restricted to these seven studies, they apply in a general way to all of the work in this field of inquiry.

What Good Is Inert Knowledge? (K. M. Hart)

In his classic essay, "The Aims of Education," Whitehead insists that the central problem of education is one of "keeping knowledge alive, of preventing it from becoming inert ..." (Whitehead, 1957, p. 5). K. Hart's primary concern is that students often fail to adopt generalizable, formal, school-taught procedures and instead persist in using their own "naive" (her term) methods. As a result they have great difficulty not only with symbolic-level computations, but also with verbal problems that are remote from their direct experiences. That is, these students have a warehouse of inert knowledge that is useful only under a very restricted set of circumstances. Her conjecture, which she attempts to back up with very disheartening descriptions of lessons taught by three volunteer teachers, is that teachers do not make appropriate links between the child's concrete, familiar, informal knowledge and the symbolic, unfamiliar, formal world of mathematics. I think this is a quite reasonable conclusion to draw but it does not go far enough. That is, it is simply not sufficient to establish a bridge between concrete models and symbols. At least one other bridge is needed as well; a bridge from the child's reality to the concrete models of that reality. Without this transitional link it is likely that the child will learn how to manipulate the models in a rote manner only, thereby developing inert knowledge. My point is that if only rote knowledge is acquired through experiences with concrete models, then the teacher will have a very difficult time indeed creating a meaningful link between this knowledge and a more symbolic, formal form.

Finally, K. Hart has focused her attention exclusively on the knowledge category of problem solving. Children may choose to continue to use their own "ungeneralizable" procedures because as they learned these procedures they also learned how to monitor and evaluate their use. Can the same be said of the school-taught procedures? It seems plausible that in addition to failing to establish links between concrete and symbolic procedures, teachers tend not to help their students learn how to take charge of their own mathematical behavior; that is, the students are not learning how to control the new knowledge they are acquiring.

What Does It Mean to Be Successful at Solving a Problem? (Gurtner)

Some years ago a colleague of mine, who had just heard Seymour Papert speak about LOGO, informed me that when children worked in a LOGO environment it was inevitable that they would learn with (relational) "understanding." As incredible as this statement seemed to me at the time, my unfamiliarity with LOGO forced me to withhold comment. Since that time I have read many articles and listened to many testimonials about the merits of LOGO. Only recently has its effectiveness in mathematics learning begun to be subjected to systematic, scientific scrutiny. Gurtner's research is an example of the best work in this area. In particular, his work demonstrates that success in solving a LOGO problem does not necessarily mean that it was solved with understanding. But the merit of his research does not rest solely on this unsurprising result. Rather, his results forcefully illustrate that students' beliefs and their level of awareness of their solution efforts play very influential roles in their problem-solving behavior.

I have two cautions to add about the implications of the results of this research. First, Gurtner observed that the students seemed to be directed by an assumption that a solution could be obtained by progressively matching the current production and the target object. I have no concern about the legitimacy of this assumption, but this sort of "means-end analysis" seems somewhat natural given the nature of the problems posed (i.e., the goal state is given; the problem is one of finding a means of attaining that goal). It seems unlikely that students would make such an assumption if other types of mathematics problems were used. Since this assumption underlies the four beliefs about the nature of their solutions, research of this type with different kinds of problems would likely yield a very different set of beliefs. Second, since metacognition training (i.e., training students to be more aware of their thinking and to be more reflective about it) is rare in mathematics instruction, the students' lack of "planfulness" in their solutions may simply reflect a belief that such behavior is unnecessary when solving mathematics problems.

How Important Is It to Have a Good Sense of What You Know? (DeGuire)

There are three general approaches to problem-solving instruction: teaching for problem solving, teaching about problem solving, and teaching via problem solving. A teacher following the first approach would focus instruction on the acquisition of those mathematical concepts, skills, and processes that are useful for solving problems. The second approach involves the teacher modeling good problem-solving behavior or directing students' attention to salient procedures and strategies. The third approach, the method advocated by Polya, involves teaching mathematics with a problem-solving perspective. DeGuire has undertaken the study of students' awareness of their metacognitive processes in the context of a course on problem solving that seems to be a combination of the first two approaches. Because her study is not simply about the development of awareness of metacognitive processes, but is also about an approach to metacognition training, I think it likely that there was some interaction between the students' development of awareness and the nature of the training they received. Perhaps if the course had been structured differently, very different outcomes would have resulted.

Under the best of circumstances metacognitive processes are extremely difficult to get at and I was struck by the care with which she designed the study. DeGuire correctly points out that the methodology for studying metacognitive behavior does not presently exist. In particular, I am concerned, as she no doubt is as well, about the validity of self reports (journal entries) as a source of data about metacognitive awareness. Does the fact that a student is unable to write cogently about her thinking mean that she is unaware of her thinking? At the same time, is a nicely worded statement evidence of good awareness, or might it simply indicate glibness? A similiar difficulty exists in attempting to analyze students' written work. Consider, for example, the case in which a student works on a problem but does not appear to have used a particular skill or strategy. What can be concluded about this student's behavior? That he or she does not know how to use the strategy? Or did not recognize that the strategy could be used? Or simply chose not to use the strategy? To complicate matters further, if the written work on a paper indicates that a particular strategy was begun but abandoned in favor of another, is it reasonable to claim that the student had decided that the first approach would lead nowhere (a metacognitive decision) and so gave up on it in order to pursue a different strategy?

Finally, I was a bit disappointed by DeGuire's unwillingness to tell the reader just what she thinks she has learned from her efforts to understand awareness. However, despite my reservations about her data sources and my disappointment with her reticence to share her insights, I think she has made a good start toward understanding the role awareness plays in problem solving.

Do We Have an "Inner Teacher" Who Directs Our Learning? (Hirabayashi & Shigematsu)

Hirabayashi and Shigematsu suggest that each of us has an "inner teacher" which functions as a cognitive traffic cop, directing the flow of all the mental traffic that is present during mathematical activity. This in itself is not a novel notion. What is new is that these researchers posit that this homunculus is a copy of the teacher from whom the individual learns. Furthermore, the (physical) teacher influences student learning only indirectly since ideas transmitted by the teacher reach the student only by passing through the inner teacher. Thus, the teacher's primary role in the learning process should be to assist students in the establishment of this inner teacher. The aim of their research then is to investigate how a teacher's actions become a part of the student's inner self. I am struck by the very close parallel between this research and that of Garofalo, Kroll and Lester (discussed later). Both groups are interested in the relation between metacognition and mathematical activity and both seem to believe that the teacher plays a central role in the formation of metacognitive skills. Hirabayashi and Shigematsu are doing very important work and I intend to follow their progress closely.

I was especially intrigued by Hirabayashi and Shigematsu's method for classifying students' responses. They describe a "logical model of metacognition" that serves as a means for analyzing students' responses during problem solving. If my understanding of their model is correct, it appears that it is too general to be of much value in identifying the dynamics of the very close link between metacognitive decisions and cognitive actions. I urge them to consider the framework for the macroscopic analysis of problem-solving protocols that Schoenfeld (1985) has devised.

Are There Any Driving Forces in Problem Solving? (Schultz & L. Hart)

Karen Schultz and Lynn Hart have adopted a perspective toward the study of mathematical problem solving that is essentially the same as the one I articulated at the beginning of this paper. That is, they believe that mathematical activity, in particular problem solving, is influenced by a host of interdependent factors, one of which is metacognition. Thus, they are conjecturing that problem solving has several driving forces and their efforts have been directed toward identifying just what these forces are through the use of a research paradigm that allows the forces to emerge as the study progresses. I applaud their willingness to engage in what they call "naturalistic" inquiry, but I am concerned that their desire to be free from the constraints imposed by adopting a preexisting theoretical stance may cause them to flounder unnecessarily, and I do not understand how interviews can be regarded as "natural." Also, it behooves any researchers to attempt to make clear what they believe, however tentatively, about the phenomena they are studying. To say that one does not want to be shackled by the use of an a priori theory is well and good perhaps, but this does not obviate the importance of specifying as clearly as possible the biases and underlying beliefs one has about the phenomena under investigation. It is unclear to me from reading their papers just what their biases are.

The four categories that emerged from their protocol analysis fit nicely into the categories I discussed in the first part of this paper. Furthermore, the fact that the individual failed to solve either of the problems she attempted despite engaging in a variety of appropriate metacognitive behaviors provides a nice argument for the presence of other "driving" forces for her performance.

Why Is Metacognition So Difficult to Study? (Garofalo, Kroll & Lester)

Some years ago, a manuscript I had submitted for publication was rejected, as one reviewer put it, for reasons cited by the author. I vowed at that time never again to point out to potential readers the weaknesses and limitations of my research. Thus, I will refrain from making evaluative comments about the study currrently being conducted by me and my colleagues, Joe Garofalo and Diana Kroll. Rather, I will make a few general observations about our work in the hope of enlightening others as to the difficulty of doing research on metacognition.

(1) Inter-task variability with respect to metacognitive processes is very high. When problems are chosen it is imperative that consideration be given to their potential for eliciting behaviors associated with the aspects of metacognition that is of interest. For example, problems with superfluous information might be included for their potential for requiring metacognitive behaviors associated with the identification of important information (an aspect of developing an adequate representation of the problem).

(2) Inter-person variability with respect to metacognition is also very high. The differences between the students discussed in DeGuire's report amply illustrates this point. Also, the differences in metacognitive processes between our regular-level and advanced-level students suggests that metacognitive skills may be closely tied to mathematical ability. It is important that researchers describe the characteristics of their subjects (e.g., instructional history, previous mathematics achievement, beliefs, attitudes) as completely as possible.

(3) Asking problem solvers to "think aloud," keep written records of their thinking, or work cooperatively with a partner have typically

proved to be less successful than we had hoped. Thinking aloud during problem solving is often unnatural and sometimes has a debilitating effect on performance. Written retrospective accounts of one's thinking have provided very little information for us. This may be due in part to the students' inexperience with this sort of activity. Cooperative work in small groups has been cited as a natural way to get students to talk aloud and to share their ideas openly. Unfortunately, our experience has been that most students find it quite difficult to do this. We suspect that this reticence is due to the students' beliefs as to what is appropriate classroom behavior and to an atmosphere of competition that is fostered by teachers (here is an example of a social condition that results in a belief that in turn affects performance). I should add that our difficulties may be due partly to the ages of the children we have been working with during the past six years (6-13 years).

(4) As we delve more deeply into the nature of metacognition we are becoming more convinced that it cannot be studied in isolation from cognition and other factors that affect mathematical performance. Indeed, it is possible that metacognition is not really distinct from cognition; rather, it is a fundamental part of it.

A Final Comment

While an enormous amount of work lies ahead, progress is beginning to be made toward bringing into sharper focus the kinds of factors that influence problem-solving behavior. As a result, the prospect seems good that research will one day be able to provide mathematics teachers with specific guidance as to how to make problem solving a more integral part of mathematics instruction.

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Ratio and proportion

THEORETICAL ANALYSIS: STRUCTURE AND HIERARCY, MISSING VALUE PROPORTION PROBLEMS

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A theoretical analysis of problem structure for missing value proportion problems is presented. Three variables: order (location) of the unknown, unit of measure, and divisibility identify 512 problem structures. A problem solving model based on problem presentation, problem representation, problem operators, and solution strategies is presented. There are three classes of representations: understanding, intermediate and procedural. A mathematical group of eight transformations which solvers use to transform problem presentations or representations to other representations are defined. Two problem operators which instantiate on procedural representations give a set of 14 solution strategies. A preference hierarchy for using these strategies is hypothesized. These considerations lead to a partial difficulty hierarchy on the 512 missing value proportion problems.

A missing value proportion (MVP) problem involves three given values and an unknown to be found under the constraint of maintaining the proportion relation. Variables which affect performance on MVP problems have been classified by Tourniaire and Pulos (1985) into student- and task-centered, and the latter subclassified as structural and contextual. This paper presents a theoretical analysis of selected structural variables to give a detailed desciption of MVP problem structures. This leads to hypotheses for hierarchies of problem complexity, and of preference for instantiating problem operators as solution strategies, and to a partial hierarcy of problem difficulty. This analysis will make a substantial contribution to this research area by providing a theoretical structure to guide the manipulation of these structural variables.

The conceptual process of problem solving is described in terms of a problem presentation, problem representations, problem operators, knowledge of the problem domain, and solution strategies (i.e. instances of the problem operators). Investigations in problem solving found that the problem representation formed by the solver is dependent on the problem presentation on the solver's knowledge of the problem domain (Greeno, 1978; Chi, Fetovich, Glaser, 1981). Problem difficulty is also a function of the problem representation.

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PROBLEM PRESENTATION

Any verbal MVP problem involves two statements which express a ratio between two quantities, we refer to these as per statements. One of these, the closed per statement, relates two known quantities; the other relates one known and one unknown quantity -- the open per statement. The structural variable of order is defined in terms of the location of the unknown (left or right) within a per statement, and in terms of the location of the open per statement (top or bottom) with respect to the closed per statement. This defines four problem categories as BL (bottom left), BR, TL, and TR.

A second structural variable is unit of measure. On the subvariable of measure space, MVP problems vary according to whether one or two are involved, and for those with two measure spaces, whether the quantities within, or between, per statements are from the same or different spaces. Here we identify three categories of MVP problems -- one-by-one: one measure space; two-by-one: same measure space within per statements, but different between; two-by-one: different measure spaces within per statements, but the same between. The subvariable dimension distinguishes between different units within a measure space, minutes and hours, for example. Dimension yields four subcategories of the one-by-one and two subcategories for each of two-by-one and one-by-two. (See Harel and Behr, in preparation.)

A third structural variable, left to a subsequent analysis, is the partitionability of the quanitity to which the unit of measure refers. Example: Money vs. child. Our observation is that this may affect the choice of operation -- multiplication or division -- which a child makes in formulating a solution procedure for a MVP problem.

The divisibility variable (or integer ratio), which we have extended to pairs of rational numbers, has been shown in prior research to affect problem difficulty. When an integer ratio exists the direction of the divisibility will affect problem difficulty. The variable of common divisor is related to divisibility. It appears that children have an order of preference to operate with a pair of numbers which (a) have an integer ratio, (b) have a common divisor other than 1, and (c) are relatively prime (Behr et al, in preparation).

The four levels of the problem variable order, eight of unit of measure, sixteen of divisibility, identify (4x8x16) 512 MVP problem structures. Of these, the 256 with dimension 1 form the subclass of ratio problems and the remaining 256, those with dimension 2, the subclass of rate problems.

PROBLEM REPRESENTATIONS AND PROBLEM OPERATORS

Our MVP problem solving model is consistent with that proposed by Simon and Hayes (1974). In order to solve a MVP problem we assume the solver has a set of knowledge structures about the problem. This includes knowledge of: the initial and goal states of the problem, problem operators (information processes) that transform a given problem state into another, and constraints under which the operators can be applied. We refer to problem states formed by the solver to determine a solution path from the initial problem state (the problem presentation) to the goal state as problem representations. Our conceptualization of the solution process is that a solver uses one or more problem representations for understanding the problem, and intermediate representations for exploring the relationships among problem components in order to choose the appropriate problem opeators, and finally, the solver will arrive at a choice for a procedural representation on which an operator will be instatiated as a solution strategy. The procedural representation that a solver forms will reflect both salience of the structural variables and the interaction of these with the solver's preference for how to instantiate a problem operator.

We distinguish three types of problem operators for MVP problems. The first class consists of structural transformations by which the solver, responding to certain structural variables, changes the problem structure. Changing the direction of the divisibility, or the order of the unknown, are possibilities. The second class consists of the unit rate operator. This transforms a closed per statement of the form "a per b" to an equivalent one of the form "1 per b/a" or "b/a per 1." The third class consists of procedural operators. These are applied to a procedural representation to solve the MVP problem. The first class of operators transform intermediate representations to other representations with the ultimate goal of achieving a procedural representation. The unit rate operator transforms a procedural representation into another procedural representation. The procedural operators act on components of the problem structure in a procedural representation to arrive at a value for the unknown.

To illustrate the structural transformations we consider an MVP problem of the form "a per b, c per x." The total of eight structural transformations, which are of three types, form a group of order eight under ordinary composition of transformations. The first type, rate reciprocation, changes the problem structure to "b per a, x per c." This transformation gives the reciprocal of each within-per-statement rate pair. The second, per statement reciprocation, interchanges the order of the two per statements to get the problem structure "c per x, a per b." The third, measure space reciprocation, changes the structure by interchanging b and c or a and x. For a MVP problem with two measure spaces this transformation changes the quantities in the per statements so each per statement is changed from a within measure space comparison to a between space comparison or vica versa. It changes the structure "a per b, c per x" to "a per c, b per x" or to "x per b, c per a."

Applying the procedural operators requires additional knowledge, called procedural knowledge, of how to instantiate these operators for a particular MVP problem. Involved is a sequence of finding the relationship between the quantities in the closed per statement (the RCQ operator) and transposing that to an operation on quantities of the open per statement (the ROQ operator) to find the unknown. Consistent instantiations of the RCQ and ROQ operators results in an identifiable strategy. A strategy is valid if it observes appropriate constraints and invalid if any problem constraint is violated. The frequently observed addition strategy is invalid because it violates the constraint of maintaining the proportion relation. Among the valid strategies we consider several instances of a multiplicative stragegy.

A multiplicative strategy begins by instantiating the RCQ operator by determining the relationship between the two known quantities in the closed per statement and expressing this relation in the form of a multiplication or division equation. This equation has an unknown value u. Let v denote the computed truth value for u. Next the ROQ operator is instantiated by using v and the known quantity of the open per statement in a multiplication or division equation to find the value of the unknown. Thus a multiplication strategy involves sequential instantiations of the RCQ and ROQ operators with some combination of multiplication and division equations. We classify a multiplicative strategy as: a division strategy (DS) or a multiplication strategy (MS), when the RCQ operator is instantiated with a division or multiplication equation, respectively. We thus have the strategies PMS (positive division), and NDS (negative division). Moreover, the equations which are used to instantiate these operators can be formed so that the value needed to be found in either case appears in a missing value equation or as the answer to be found in a direct computation equation. In the first case the equation is indirect (I), and in the second direct (D). Further analysis leads to the following list of 14 strategies for solving MVP problems: PMS-ID, PMS-II, PDS-DD, PDS-DI, PDS-ID, PDS-DI*, PDS-II, PDS-II*, NMS-ID, NMS-II, NMS-II*, NDS-DI, NDS-II, and NDS-ID, where the * means that v appears as the result, rather than operator or operand, in the equation for ROQ.

PREFERENCES FOR INSTANTIATING THE RCQ AND ROQ OPERATORS

Our next objective is to hypothesize a hierarchy of children's preference for these strategies. To instantiate the RCQ operator children must consider the direction of the operation (left or right) and also its type (multiplication or division). Instantiation of the RCQ operator involves two given quantities a and b and an unknown quantity u. The equation that is formed to give the relation between a and b can involve a, b, and u in one of two types of operations and in one of three roles of operator, operand, or answer. Considerations, too lengthy to discuss in this brief report, lead us to six ways to instantiate the RCQ operator (See Harel and Behr, in preparation). Under the constraint of an integer ratio (noninteger ratio involves other considerations) these six ways, listed in the assumed order of children's preference, are as follows:

1. Computing left to right or 2. computing right to left, and finding u

to multiply to the smaller of a and b to given the larger.

3. Computing left to right or 4. computing right to left, and finding u to divide into the larger of a and b to give the smaller.

5. Computing left to right or 6. computing right to left, and dividing the smaller of a and b into the larger to give u.

Once the RCQ operator has been instantiated, to instantiate the ROQ operator judgments need to be made about four variables: preservation of operation direction, preservation of operation type, the role (operator, operand, or answer) in which to use v, the computed value of u, in the known-to-unknown relationship and the level of structural equivalence between the RCQ equation and the ROQ equation. (See Harel and Behr, in preparation, for information on structural equivalence.) All possible equation pairs by which RCQ and ROQ operators can be instantiated for a MVP problem of the form "a per b, c per x" are given in the Table. Similar information for a problem of the form "a per b, x per d" is given in Harel and Behr (in preparation).

<u>Analysis of Strategies and Preference Heirarchy for Instantiating the RCQ Operators for the Procedural Representation: a per b, c per x</u>

	RCQ Eqn	ROQ Eqn	Strategy	RCQ R ROQ[1] Direction Preserved	Operation Preserved	Structural Equivalence[2]	R(RCQ)[3]	Preference Index
1		cXv=x	PMS-ID	Yes	Yes	C	KU	1
2	aXu=b	x÷v=c	NMS-II	No	No	P	UK	5
3		x÷c=v	NMS-II*	No	No	N	0	9
4		xXv=c	PMS-II	Yes	Yes	C	UK	2
5	bXu=a	c÷v=X	NMS-ID	No	No	P	KU	4
6		c÷x=v	NMS-II*	No	No	N	0	9
7		c÷v=x	PDS-ID	Yes	Yes	C	KU	1
8	a÷u=b	xXv=c	NDS-II	No	No	P	UK	5
9		c÷x=v	PDS-II*	Yes	Yes	N	0	7
10		x÷v=c	PDS-II	Yes	Yes	C	UK	2
11	b÷u=a	cXv=x	NDS-ID	No	No	P	KU	4
12		x÷c=v	PDS-II*	Yes	Yes	N	0	7
13		c÷x=v	PDS-DI*	Yes	Yes	C	0	3
14	a÷b=u	c÷v=x	PDS-DD	Yes	Yes	N	KU	6
15		xXv=c	NDS-DI	No	No	N	UK	8
16		x÷c=v	PDS-DI*	Yes	Yes	C	0	3
17	b÷a=u	xXv=c	PDS-DI	Yes	No	N	UK	6
18		cXv=x	PDS-DD	No	Yes	N	KU	10

[1]RCQ R ROQ denotes the relationship between RCQ and ROQ. [2]C, P, N denote respectively, complete, partial, or no equivalence. [3]R(RCQ) denotes the relationship between the known and unknown quantities in the open per sentence according as the equation for the RCQ operator relates them left to right in the order known to unknown (KU), unknown to known (UK), or relates them as operator and operand (0).

PARTIAL DIFICULTY HIERARCHY OF PROBLEM REPRESENTATIONS

An analysis for which we refer the reader to Harel and Behr (in preparation) leads to the identification of exactly 18 procedural representation structures (P_i, $i=1,2,\ldots,18$). Further analysis based on the preference hierarchy for instantiating the RCQ and ROQ operators and further assumptions (see Harel and Behr, in preparation) allows us to order these from most complex to least complex; we assume the notation to be such that this order is $P_1, P_2, \ldots, P_{16}$. The total set S, of the 512 problem representations is partitioned by the group of eight transformations into subsets, $S_{i,j}$. The subset $S_{i,j}$ is the set of all problems prepresentations which are mapped by a transformation T_j unto the procedural representation P_i. For $m \neq n$, the intersection of $S_{m,j}$ and $S_{n,j}$ is empty; that is, there are no two problem representations which get mapped by different transformations unto the same procedural representation. We next consider the sets $S_{1,j}, S_{2,j}, S_{3,j}, \ldots, S_{18,j}$, the sets of problem representations which are mapped by T_j unto P_1, P_2, $P_3, \ldots, P_{18}$, respectively. We take the previously established order of complexity on the procedural representations P_1 to P_{18} as an imposed order on the set of preimages, $S_{1,j}$ to $S_{18,j}$. We define this imposed order to be the order of problem difficulty on this collection of problem representations. This leads to the following partial (in the sense of partial order) difficulty hierarchy on the set of 512 MVP problem structures (Let > denote greater in difficulty).

$S_{1,1} > S_{2,1} > S_{3,1} > \ldots > S_{18,1}$ (Level 1)
$S_{1,2} > S_{2,2} > S_{3,2} > \ldots > S_{18,1}$ (Level 2)
......
$S_{1,8} > S_{2,8} > S_{3,8} > \ldots > S_{18,8}$ (Level 8)

Finally our analysis shows that it is not possible to order the levels (Level 1 through Level 8) according to difficulty.

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MULTIPLICATIVE STRUCTURES:
DEVELOPMENT OF THE UNDERSTANDING OF RATES (INTENSIVE QUANTITIES)

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Patterns of error in verbal problems requiring selection of an operation show that there is considerable confusion concerning the numerator/denominator roles of the two quantities comprising a rate. This conceptual obstacle shows itself in difficulties with the inverse relationship between, for example, grams per penny and pence per gram, a tendency to choose additive complements ($\frac{1}{4}$ and $\frac{3}{4}$) rather than multiplicative ones; the more speed, less time relation for fixed distance; and the general inability to perceive the relations among the three quantities in such cases as constituting an integrated whole. This study used large group testing and interviews; it was followed by a teaching experiment.

Extensive research now exists on the understanding of certain aspects of multiplicative structures, in particular on the beginnings of multiplication, on proportion, and on single-operation problems in contexts. Our own work in this field has mainly been on the last topic, single-operation problems, and has had an emphasis on the design of teaching which embodies the results of research on understanding. The work to be reported here concerns the concept of rate, that is of an "intensive quantity" which is the quotient of two extensive quantities. Examples are speed, unit price, miles per gallon, density, map scales, measure conversion factors, proportions in mixtures; such quantities are very common.

It has been shown that the recognition of the appropriate operation in problems involving such quantities is subject to distractions arising from the well-known numerical misconceptions. Sensitivity to the distinction between partition and quotition in rate problems is revealed, not in general by difference in facility, but by a sharp difference in the pattern of errors; partition structures give rise to more reversal errors, quotition structures to more multiplications. These dominant errors are equivalent in both cases to a confusion about the numerator/denominator roles of the two quantities constituting the rate. (Bell, Fischbein and Greer, 1984; Bell, Greer and Grimison, in preparation). This observation was the starting point of the present work, which consists of a cross-sectional study of the understanding of the rate concept in a sample of 598 students aged 11-14 in British secondary schools. Interviews and group tests were used. A teaching experiment followed.

The test began with a question probing the acceptance of a small/big operation in two problems, one concerning the price per gram of a 250g pack of butter costing 44 pence, the other asking for the length of each piece of ribbon, if 150 pieces were cut from a length of 27 metres. In the latter case, 28% chose the correct operation, in the former only 16%. Thus, although for many pupils small/big is unacceptable, it is more so in the two-quantity problem than in the simple partition question.

In another interview, a pair of girls argued with themselves for several minutes to decide whether 88 grams for 44 pence would be 2 pence per gram or half a penny per gram. The confusion appears to be of the roles of different quantities; the expression 'kilometres per minute', to a person who is not attuned to the crucial significance of the word 'per', does not display clearly whether it is like kilometres or like minutes. In a direct test with two matched questions, appearing, spaced, within a test, in different order for two split halves of the group, a sharp difference in difficulty was observed; there was also a tendency for the 'per' question to be answered correctly more frequently by pupils who had met the 'in one minute' version first. See Table 1.

	% correct A before B	 B before A
A. A rowing crew covered a 3 kilometre course in 7.2 minutes. What was their speed in kilomtres per minute?	28	39
B. A rowing crew covered a 3 kilometre course in 7.2 mintues. How far did they row in one minute?	54	61

(N=114 middle and upper ability 13-15 year old pupils).

Table 1

The potential confusion here was shown starkly in another question in the main test which offered a direct comparison.

Q2. In France two local shops sell raisins at 20 grams per franc.

a) Raisins are cheaper at Pierre's.
b) Raisins are cheapter at Claude's.
c) Both shops are selling raisins at the same price.
d) It is impossible to tell who is selling raisins at a cheaper price.

Which is correct? ______________ Why? ________________________

Correct answer and explanation	55
Correct answer, incomplete explanation	13
Both shops are the same	17
Impossible to tell	5
Omit or unclassifiable	10

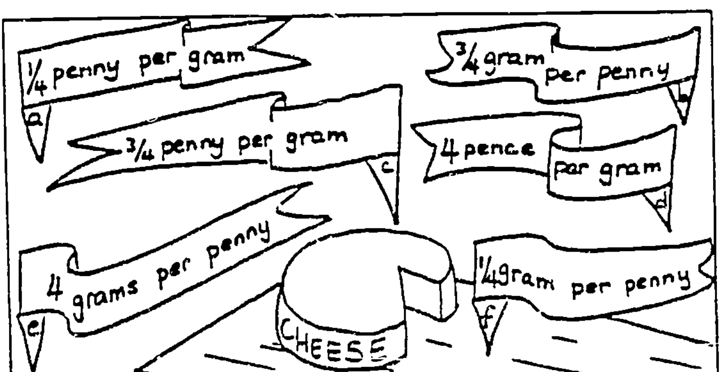
Figures in this and subsequent tables are all % ages, N=598

Table 2

We have 17% +5% of pupils who do not distinguish these prices, even with the choice offered as clearly as this. The quality of explanations varied from one-dimensioned remarks such as: "1 franc is cheaper than 20 francs"; to correct ideas but without calculations: "Because you get 180 more grams at Claude's; to the 3% which mentioned that one price was 400 times the other.

A somewhat more complex question on the same point produced a much lower level of response.

Q3. Prices can be written in more than one way. Find any pairs of flags which show the same price written in different ways.



One or both correct pairs a + e or d + f	21
Ignore rate, choose same numbers	44
There are none	9
Pairing ¼ with ¾	6
Omit, unclassified	19

Table 3

Interviews confirmed that the responses as shown in the table were firm convictions. 4 grams per penny and 4 pence per gram were "the same, just swapped around"; and a and e couldn't ever be the same, because "that's 4 grams per penny and that's ¼ penny per gram". The pairing of ¼ with ¾ suggests a possible awareness of connection, but a feel for values more closely linked, less dramatically different than ¼ and 4.

A question which tested the firmness of grasp of the inverse relationship with more difficult fractional values and in a context somewhat less familiar than price is shown below.

Q4. A cyclist travels 2/5 kilometre per minute.

a) 2/5 minute per kilometre
b) 3/5 minute per kilometre
c) 2½ minutes per kilometre
d) 2 3/5 minutes per kilometre
e) It is impossible to tell the number of minutes per kilomtre.

Which is correct? ____________ Why? ________________

Correct choice with adequate or partial explanation	8 + 8
Ignore rate, choose same number (a)	27
Impossible to tell	29
3/5 or 2 3/5	5
Omit, unclassified	23

Table 4

There were, in the written tasks and the interviews, a number of occasions on which aspects of understanding of the general network of relationships among speed, distance and time or unit price, weight and cost were revealed. For example, Roger, aged 13:

R: Distance + time = speed
Int: Why?
R: Km + hours = km/hr

Two questions tested further awareness of the inverse relationship, this time in the context of speed and time.

Q5. It takes Peter 8 minutes to run a mile. Harry runs twice as fast How long does Harry take to run a mile?

77% responded correctly, and 15% gave the directly proportional answer of 16 minutes.

Q6. Caroline travels 100 kilometres at 34 kilometres per hour. Francis travels 100 kilometres at 44 kilomtres per hour.

a) Caroline takes longer to complete the journey.
b) Francis takes longer to complete the journey.
c) They take the same amount of time.
d) Cannot tell.

Which is correct?_______________ Why? ______________________

In this case, 62% were correct, while 22% chose the more speed, more time response (b). Typical wrong explanations were: "the lower the kph the faster"; "Caroline has a lesser time number than Francis"; "Caroline is going to get there quicker cos she's only got to do 34 kilometres".

18 pupils who had made this type of response were interviewed; 16 of them maintained the same point of view even in discussion.

A written test question asked directly for the selection of correct alternative forms of the relation speed = distance x time.

Q7. I can calculate the speed of my car using the following rule:
Speed = Distance ÷ Time

Circle the other rule(s), written below, which are also true. If there are no other rules which are true, put a tick in this box

a) Time = Speed x Distance
b) Distance = Speed ÷ Time
c) Speed = Distance x Time
d) Time = Distance ÷ Speed
e) Distance = Speed x Time
f) Distance = Time ÷ Speed

Correct, d and e	6
Choosing only one correct form	6
Choosing correct and incorrect forms	17
Choosing all the division relations	4
No other rules true	39
Omit, unclassified	22

Table 5

The choice of incorrect forms is less surprising than the large percentage selecting 'no other rules'. For them, it seems, the given formula is a fixed rule. They are rejecting the possibility that what is essentially the same relation between the three quantities can have different aspects.

The choice of incorrect forms is less surprising than the large percentage selecting 'no other rules'. For them, it seems, the given formula is a fixed rule. They are rejecting the possibility that what is essentially the same relation between the three quantities can have different aspects.

Further questions tested the ability to maintain correct choices of operation in questions with excess or insufficient data.

In the teaching experiment which followed this diagnostic testing, problems similar to those on the test, but somewhat richer, were used to provoke conflict and lead to discussion. Concept-focussed games, and exercises in making up questions were also included. For details see the references.

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QUALITATIVE DIFFERENCES AMONG 7-TH GRADE CHILDREN IN SOLVING A NON NUMERICAL PROPORTIONAL REASONING BLOCKS TASK

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This study aimed to examine differences in the problem representations and strategies employed by low performance and high performance 7-grade children in solving a non numerical proportional reasoning task. The non numerical task, which was developed especially for this aim, involves weight and number relationships between blocks -- a derivative of the density concept. Three categories of representations and four categories of operators were used by the children. The differences between these representations and strategies are like those characterized in research of expert-novice differences among adult solvers in other domains of problem solving, such as physics problems.

The process of solving a problem starts with the formation of a problem representation which happens in two stages: (1) converting the problem presentation into internal mental entities and relations among them; (2) selecting operators to produce new states of knowledge from existing states. The problem solving process proceeds by applying a solution strategy, which means searching the goal state by instantiating the operators previously selected. In certain problem domains, such as physics problems (e.g. Chi, Glaser, and Ress, 1981; Larkin, 1977), the role of problem representation in the solution process is well documented in the research of problem solving. In the domain of proportional reasoning research, on the other hand, the matter of problem representations has been apparently ignored. The efforts have

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been to investigate the effect of context and structural variables on children's performance in solving proportion problems without considering the mental processes that account for this effect. This is a second study in our attempt to investigate the cognitive aspects of proportional reasoning. In the first study (reported in this volume by Behr, Harel, Post and Lesh) we analyzed missing value proportion problems and suggested a problem solving model which takes into account problem presentation, problem representations and operators.

The study reported here investigates differences between low performance and high performance grade-7 children in forming initial representations and selecting operators in the process of solving a non numerical comparison proportionality task. Data reported suggest differences in children's problem solving stages which are like the novice-expert differences characterized by other studies with adults in physics problems (e.g. Chi, Glaser and Rees, 1981).

PROCEDURE

The research paradigm used was that of a modified teaching experiment which was replicated at the two experimental sites, DeKalb, Ill. and Minneapolis, Minn over a period of about 17 weeks. A total of 18 grade-7 children, nine at each site, participated in the research. Each site involved three children judged to be of low mathematics ability and achievement, three of middle ability and achievement and three of high. The major assessment consisted of four one-on-one interviews. The tasks in this report were given as nine problems, among more, in the third interview.

The task involves two pairs of blocks (A, B) and (C, D). Blocks A, C and B, D were constructed from the same kind of unit-blocks, A' and B', respectively; the unit-blocks in A were larger in size than the unit-blocks in B. The number of unit-blocks in A was smaller than the number of unit-blocks in C, and these numbers remained constant across tasks. Three different instances of blocks B and D were used, B_{-1}, B_0, and B_1, and D_{-1}, D_0, and D_1, respectively; the number of unit-blocks within each of the pairs (B_{-1}, D_{-1}), (B_0, D_0), and (B_1, D_1) was one less, the same, or one more compared to the number of unit-blocks in A and C, respectively. The subjects were asked to judge the weight relationship between C and an instance of D based on one of three given weight relationships, less than, greater than, or equals, between A and an instance of B.

The three pairs (A, B_i) reflect 3 different number relationships. This crossed with the three possible weight relationships, <, >, and =, results in nine possible given weight and number relationships. Each of these relationships can be associated with a requirement to find the weight relationship between C and one of the three instances of D. This results in 27 possible problem situations. The nine problems selected and presented to the children are described in detail in following table.

Item	Pair (A, B_i) presented	Given weight relationship	Pair (C, D_i) presented	Correct weight relationship to be found
1	(A, B_0)	=	(C, D_0)	=
2	(A, B_0)	=	(C, D_{-1})	>
3	(A, B_{-1})	=	(C, D_0)	<
4	(A, B_0)	>	(C, D_{-1})	>
5	(A, B_1)	>	(C, D_1)	>
6	(A, B_{-1})	>	(C, D_0)	Undetermined
7	(A, B_0)	<	(C, D_{-1})	Undetermined
8	(A, B_1)	<	(C, D_0)	Undetermined
9	(A, B_{-1})	<	(C, D_{-1})	<

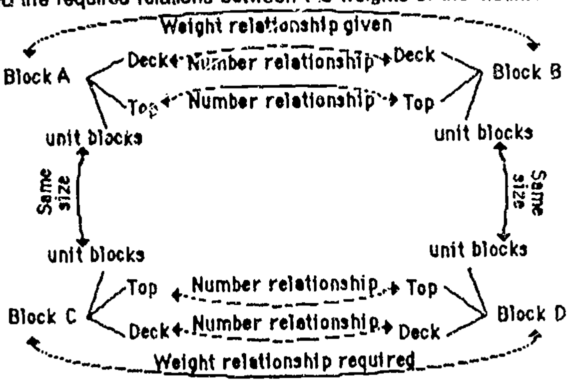
Complexity of the blocks task. Traditional tasks used in research on proportional reasoning involve the requirement to decide which of the relations equal to, less than, or greater than holds between multiplicative relationships a/b and c/d. The nonnumeric proportion task used in this study is more complex than these standard proportion tasks because several relationships must be inferred and coordinated before the final relational judgment can be made in the criterion component of the task. Qualitative proportional reasoning is involved in two episodes in the solution of the blocks task: one is the coordination of the number and weight relationship between A and B to determine the weight relationship, if possible, between A' and B'; the other is in the coordination of the weight relationship between A' and B' and number relationship between the added amounts, uA' and vB', to determine the weight relationship between C and D.

RESULTS

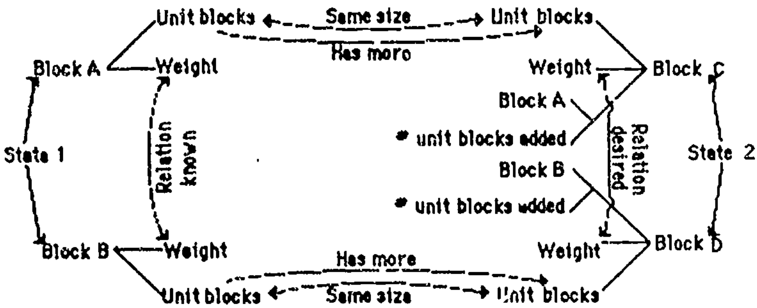
Two processes in the children's responses were examined: one process, reflecting the initial representation of the problem, relates to how they initially viewed the structure of each block and how they interpreted the relationships between these structures; the other process, reflecting the strategies children used to solve the problems, relates to the inferences they made to find the final answer. These representations and strategies were separately classified into 3 and 5 different categories, respectively.

We identified three different initial problem representations of the blocks task, Structure, Complement, and Atomic; listed in order from most to least sophisticated.

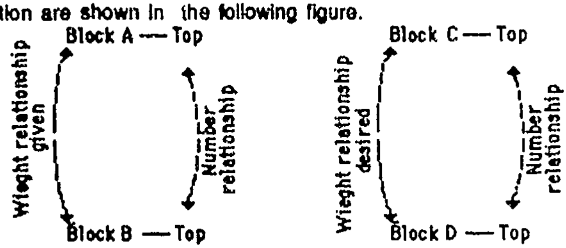
Structure representation. If the Structure Representation was used, each block was envisioned as consisting of two parts, deck and top, and the observable number relationships between the tops and between the decks within the pairs (A, B) and (C, D) were identified. This representation also included the property that each of the pairs (A, C) and (B, D) was constructed with the same size unit-blocks, A' and B', respectively. The following figure describes elements in this representation including the given and the required relations between the weights of the blocks.



Complement representation. If the Complement Representation was used, the children attended to the fact that the number of units in C (and D) was greater than the number of units in A (and B). They also attended to the property that the corresponding blocks were constructed with the same size unit-blocks. Due to these noticed qualities of the blocks, their representation focused on blocks C and D, where C was viewed as an addition of units on A (and D as an addition of units on B). The following figure describes this representation as a network of two states, 1 and 2. In state 1, blocks A and B and the relation between their weights are given; state 2 is a result of changing state 1 by adding units blocks to A and B and getting C and D, respectively.



Atom representation. If the Atom Representation was used, the children viewed each block, separately, as consisting of individual unit-blocks, and the number relationships between the tops of the blocks was considered. Elements of this representation are shown in the following figure.



Five main categories of distinct strategies were identified; listed in order from most sophisticated to least sophisticated, they are as follows: Matching, Balance with three distinct instantiations (Complete, Incomplete, and Deficient), and Counting.

Matching strategy. If the Matching Strategy was used, the child would begin by looking at the relationships between pairs of blocks (A, B) and (C, D). The child would first notice that the number of unit-blocks within the decks of these pairs is equal. Then, he or she would determine the number relationship, μ, between the number of unit-blocks in the tops of A and B, respectively, and the number relationship, μ^*, between the number of unit-blocks in the tops of C and D, respectively. The next step would be to acknowledge the given weight relationships, W, between blocks A and B and the required weight relationship, W^*, between blocks C and D. Children would then observe one of two relations between relationships: One was that the number and weight relationships between A and B are the same relation (<, =, or >), i.e. $\mu = W$; the other was that the number relationship between A and B and between C and D are the same relation (<, =, or >), i.e., $\mu = \mu^*$. Depending which relationship was determined by a child, one of the following two rules, or operators, was instantiated:

(1) $\mu = W \longrightarrow W^* = \mu^*$; (2) $\mu = \mu^* \longrightarrow W = W^*$.

Imposed matching strategy. Problems 3, 4, and 7 (see the above table) can not be solved by the Matching Strategy because neither one of the sufficient conditions $\mu = \mu^*$ or $\mu = W$ in the above rules holds; this posed a problem to those children who depended on this strategy. After finding they were unable to solve a problem using the Matching Strategy one of two avenues were taken. Either the children would use a fall back strategy (i.e., fall back to a less sophisticated strategy) or would use a derivative of the Matching Strategy, which we call the Imposed Matching Strategy. When using this strategy, the child would suppose an equals number relationship between the tops of blocks C and D, so the sufficient condition $\mu = \mu^*$ would hold and rule (1) could be applied to conclude $W^* = W$. Based on the latter relationship he or she would conclude the required relationship between C and D.

Balance strategy. Within the Balance Strategy category, there are three different instantiations: Complete, Incomplete, and Deficient.

Complete balance strategy. If the child solved a task using the Complete Balance Strategy, three relationships were considered. First the children considered the relationship between the weights of A and B. This can be visualized as blocks A and B on a pan balance. The children went on to determine the number of blocks added to A and B which created blocks C and D. At this point blocks C and D are on the pan balance. In order to determine the relationsnip between the weights of C and D, the children used the weight relationship between the unit-blocks A' and B'.

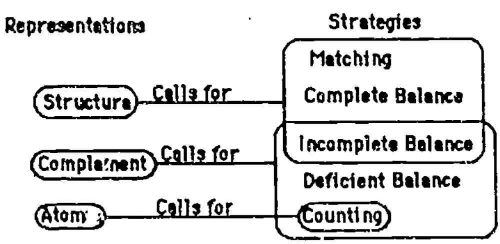
Incomplete balance strategy. The Incomplete Balance Strategy is similar to the Complete Balance Strategy. First the Children considered the relationship between the weights of A and B and then determined the relationship between the number of units added to A and B to solve the problem. Thus, this strategy ignores the relationship between the weights of the unit-blocks.

Deficient balance strategy. In the deficient Balance Strategy only the relationship between the number of units added to A and B was considered to solve the problem; the other two relationships, the number and weight relationship between A and B, were ignored.

Counting strategy. The most simplistic strategy was the Counting Strategy in which the answer to the task was determined by comparing the number of unit-blocks in C and D.

RELATIONSHIPS BETWEEN REPRESENTATIONS AND STRATEGIES

An analysis of the relative sophistication of the representations and strategies described earlier and the relationships among them will be discussed in Harel and Behr (in preparation). A concise description of these relationships is shown in the diagram below. The diagram shows that the most sophisticated representation -- Structure Representation -- calls for Matching Strategy, Complete Balance Strategy, or Incomplete Balance Strategy in this order of frequency; the less sophisticated representation -- Complement Representation -- calls for Incomplete Balance Strategy and Deficient Balance Strategy and Counting Strategy in this order of frequency; and Atom Representation calls only for Counting Strategy. This result is consistent with current theories in problem solving which attribute qualitative differences between a novice and an expert to variability in the quality of their problem representations, especially, in the initial stage of problem analysis. An expert's reasoning about a problem leads to a problem representation that contains structural features of the problem. This representation is superior to that of a novice whose reasoning leads to a representation which incorporates only the surface features of the problem. The sophisticated problem representations of the expert lead to successful solution strategies, while the more primitive representations of the novice lead to unsuccessful solution attempts (see, for example, Chi, Glasnr and Rees, 1981).



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MULTIPLE REPRESENTATIONS AND REASONING WITH DISCRETE INTENSIVE QUANTITIES IN A COMPUTER-BASED ENVIRONMENT.

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We describe several linked-representation learning environments that support learning about and reasoning with the multiplicative structure of discrete intensive quantities. We also describe an icon-based calculation environment that supports the fundamental cognitive actions underlying these reasoning processes. The environments described are part of a larger set that spans the various subconstructs of intensive quantity and which is being extended to include formal algebraic representations as well as representations of continuous intensive quantities. By explicitly linking actions on concrete iconic representations to their consequences in more abstract and formal numerical and graphical representations, we expect to enrich and make more flexible student cognitive models of intensive quantites and the operations on them.

INTRODUCTION

For the past several years our research group examined student reasoning with intensive quantity. This complex web of concepts involves three different aspects, each with corresponding task types: (1) quantitative multiplicative structure (missing value problems), (2) homogeneity-intensivity (sampling tasks), and (3) order (comparison tasks). We concentrate here on the first aspect, describing the different software learning environments developed to support concretely-based strategies for traditional missing value tasks. A fuller description of all parts of our software is available in Kaput & Gordon (1987). Our motivations, detailed rationales for the learning environments, and descriptions of empirical work associated with their development and refinement are available in Kaput (1985), Schwartz, et al (1985), and Kaput, et al (1986), so will not be repeated here. The learning environments will described in the order that we developed them rather than in the order a student would typically encounter them.

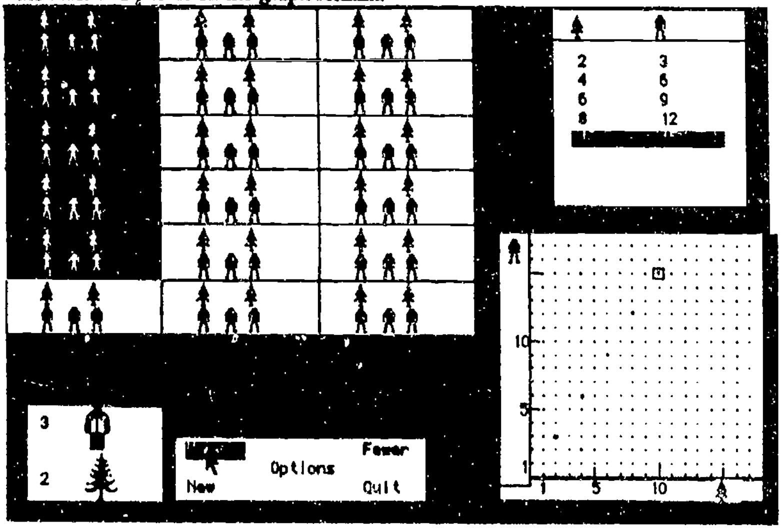
A central objective of our learning environments is to ramp students from their concrete, situation-bound representations of intensive quantities to more abstract and flexible ones. The model for introducing discrete intensive quantities at the concrete level involves regular arrays of icons that the students can manipulate as if they were actual objects. To provide long term curricular coherence, we decided that tables of data and coordinate graphs would be the primary abstract representations for the concept of intensive quantity, to be connected later to formal algebraic representations. Our technique for linking these representations cognitively is to provide experience with them linked cybernetically in an appropriate learning environment.

THE STARTING POINT

We initially constructed the following three-representation environment. One window each is reserved for iconic, numerical, and coordinate graphical representations of an intensive quantity that the student enters into the computer. The intensive quantity is usually associated with a story - for our purposes let us assume that we are planting trees in a park so that every two trees will shade three people. Hence the student selects and then distributes copies of appropriate icons in a model cell that will come to specify the intensive quantity, e.g., 3 "person" icons per 2 tree icons. (The two trees can even be situated so as to "shade" the three people.) This model cell is then replicated to fill an icon window which eventually will contain identical rectangular cells. After the preliminary actions setting up the situation, the screen provides a table of data window labeled by the appropriate icons, and a coordinate graph window, whose axes are similarly labeled. (See the Figure below.)

As the student clicks the MORE button, the cells in the icon window are highlighted, corresponding number pairs are entered in the table of data, and the corresponding points are plotted on the coordinate graph. (Thus the intensive quantity is modeled in the coordinate graph as the slope of a line of discrete points.)

With each click, the latest number pair and the latest point are highlighted to correspond to the number of icons of each type that are highlighted in the icon window. In the figure below, we see the result of 5 clicks of the MORE button. By clicking on FEWER the highlighting process proceeds in reverse; however, the previously deposited number pairs in the table and points on the graph remain.



By clicking on the boundary of any of the windows, the student can turn off that particular representation, so that prediction tasks are possible, e.g., with the table of data turned off one could ask "What number pair will be highlighted if we now clicked on MORE 3 times." Turning off particular representations also helps to control both the novelty when introducing new representations as well as the amount of information on the screen at any particular time.

THE BOXES STRATEGY

The main objective of the Linking Environment is to introduce the two abstract representations and to link them with the icon representation and to one another. But we initially used it with various representations turned off to pose missing value problems, problems such as: *How many people would be shaded by 14 trees in our park?*

Sixth and seventh graders had experience with the rectangular cell layout of the iconic representation, in paper-pencil as well as computer based activities. A surprising number of them generated the following "boxes strategy" to solve the missing value problem:

"Let's see, there are 2 trees per box and so there are 7 boxes of trees. There will be 7 times 3, that's 21 people."

This is a "divide and then multiply" strategy based on an intermediate decomposition of the sets of icons into subsets describable using the intensive quantities

2 trees/box and 3 people/box.

These subsets then support a particular sequence of computations. The first is a quotative division - a division of an extensive quantity by an intensive quantity - divide the given number of trees by the number of trees/box to get the number of boxes. Then multiply this number of boxes by the number of people/box to get the required number of people - the product of an intensive and an extensive quantity.

The divide and multiply strategy amounts to the stepwise solution of the following (scalar) algebraic proportion whose left side involves trees, whose right side involves people, and which overall is a statement about equality of numbers of boxes: $14/2 = x/3$.

Described more fully, the left side is

$$\frac{14 \text{ trees}}{2 \text{ trees/box}}$$

and the right side is

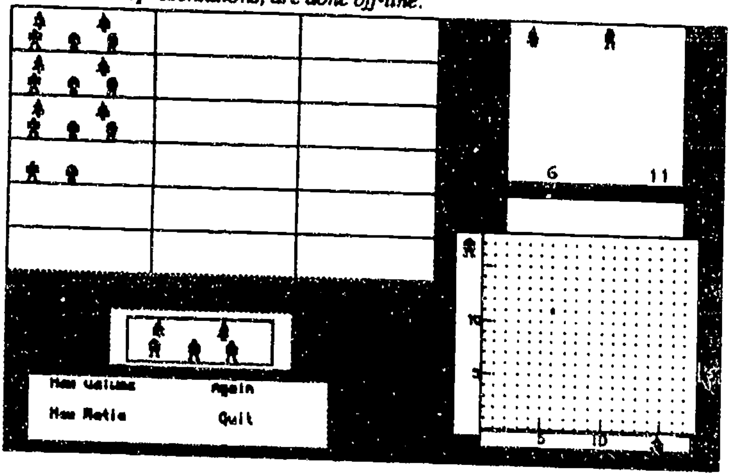
$$\frac{x \text{ people}}{3 \text{ people/box}}$$

Since its algebraic solution is isomorphic to the boxes strategy, the rectangular icon array has produced a very concrete realization of the solution process.

MISSING VALUES PROBLEMS WITH CONCRETE ICONIC FEEDBACK

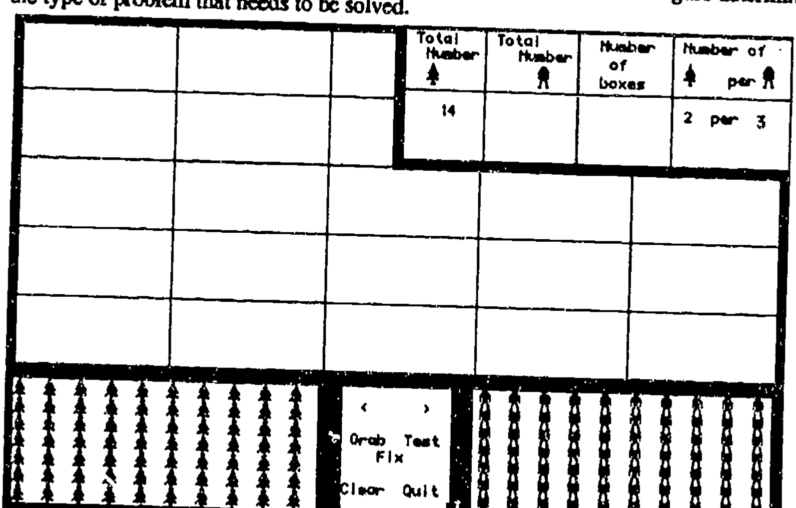
In our next environment, designed specifically for solving missing values problems, the overall appearance and initial actions taken by the student to specify an intensive quantity are similar to that of the Linking Environment. But instead of driving it from representation-independent MORE and FEWER buttons, the student either (1) enters numbers in the table of data or (2) specifies points (by pointing and clicking) in the coordinate graph. In particular, the student provides the corresponding number of an ordered pair when given the other number and the underlying ratio.

By clicking on the boundary to activate the window, the student can view the consequences of his/her input in any other window, most importantly in the icon window. There the computer fills in as many cells as possible identical to the model cell, so an inappropriate input results in cells that do not match the model as in the Figure on the next page. In this case the student can try another input. Correct inputs are preserved as pairs in the table of data and points in the coordinate graph to serve as guides for later inputs. Note, however, *the calculations and reasoning, while guided by the structure of* *the available representations, are done off-line.*



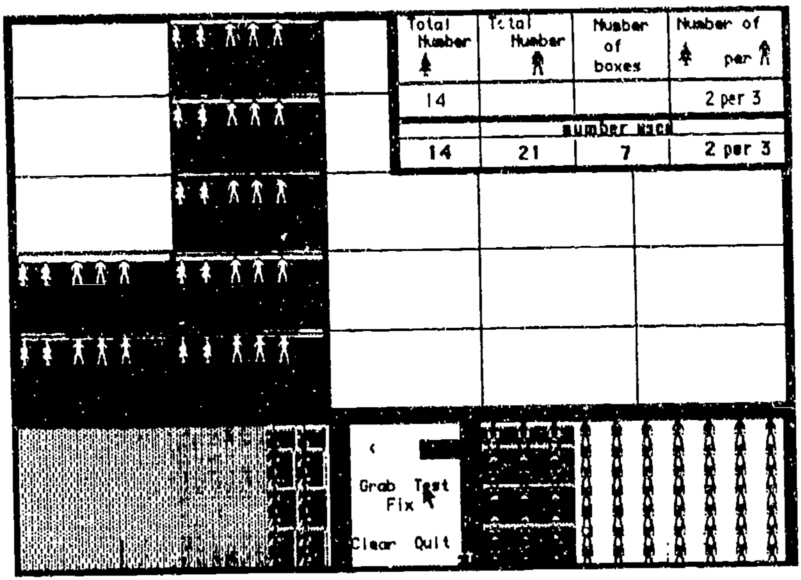
NEXT - THE ICON-BASED CALCULATION ENVIRONMENT

We therefore decided to build an icon-based calculation environment engaging the student in the grouping, matching and counting acts which are at the heart of the primitive concrete strategies we previously observed, and which, in effect, the computer does when providing the the above iconic representation of the solution to a missing value problem. Here, two of the four pieces of information in the data table of the next figure determine the type of problem that needs to be solved.



Clearly, the task of determining the number of the other icons is a traditional missing value problem. To solve it, the student groups icons in the two "warehouses" and then sends them into the cells, involving only grouping (to "grab" a set of icons), matching (the respective contents of the boxes) and counting to complete such a missing value problem.

Although more than one distribution strategy is possible (depending in part on the conditions under which the problem is solved), the "boxes" strategy has a perfectly concrete embodiment in this context: distribute 14 trees into 7 cells (a concretely executed quotative division), and then match each of the groups of 2 trees with a group of 3 people, yielding the layout below. Using the Test command, one can get a progress update on one's progress at any time. In the Figure below, the Test command was requested to confirm the successful completion of the task.

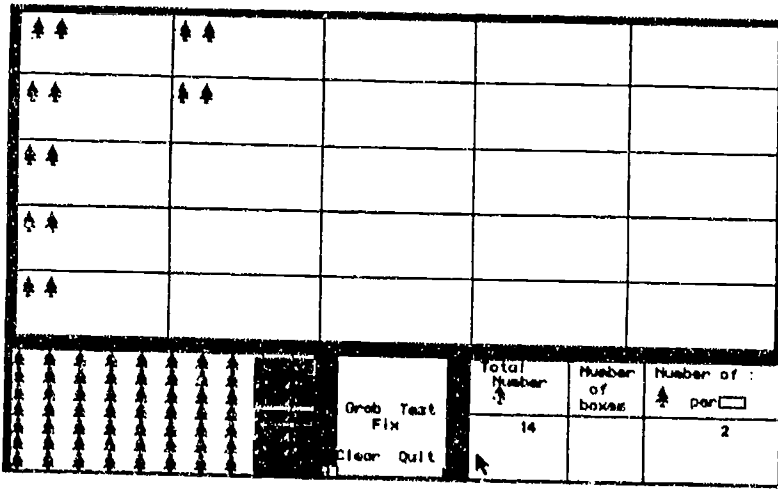


This environment embodies a wide range of actions and problem types, too varied to be described in the space available. For example, when given the respective totals, determining the underlying ratio in simplified form is a challenging distribution task (the requirement that the ratio be simplified is equivalent either to the requirement that the total number of icons per cell be a minimum or that the number of boxes be a maximum); and by changing the allowable actions, one can force a different solution strategy for a missing values problem. See (Kaput & Gordon, 1987) for full details of these learning envoronments.

CONCRETELY BASED MULTIPLICATION AND DIVISION

As a further step backward towards what we feel is the conceptual bedrock that will support viable thinking patterns for the longer run, we devised a single icon calculation environment that utilizes similar primitive acts on a single set of icons that are active concrete versions of multiplication of intensive by extensive quantities, and partitive and quotative division. Again, the computer monitors the student's actions and, on request, provides feedback and opportunity to alter the chosen distributions. Our last Figure, below, is a screen after successful completion of a quotative division task, but before a progress request.

Note that, alternatively, by providing the the number of boxes instead of the total number per box, one has stated a partitive division problem. As is the case with the two-icon environment, the computer stores all the numerical information about successful problem solutions in a table which can be called up for inspection and hypothesis-building.



CONCLUDING REMARKS

The series of learning environments introduced here extends to include the more formal representations embodied in equations and actions on equations. Indeed, the overall approach is to link such abstract representations explicitly with those already established to form a smooth ramp upward from the concrete to the abstract, where movement on that ramp is accomplished through student-initiated actions on those representations, and where the students themselves inspect the consequences of their actions in whichever representation they deem appropriate.

This sequence of multiplicative structure learning environments is intended to be used over several years, beginning with the single icon calculation environment soon after addition and subtraction have been introduced in the early grades. The two-icon calculation environment comes next, followed by the linking environment, the multiple representation missing value problems environment, and finally, the equations environment.

Each of these discrete intensive quantity environments has a continuous analogue, where the discrete objects are replaced by line segments, and where the primitive grouping, matching and counting acts have continuous analogues. They also embody parallel linkages to the more abstract representations, which then serve to represent both the discrete and continuous worlds.

Despite the widely acknowledged value of concrete manipulatives, they are not widely used in schools for two reasons: (1) They impose a difficult classroom management problem, and (2) It is difficult to use them in ways that adequately expose the connections between actions on the manipulatives and the corresponding actions on their formal mathematical counterparts. Cybernetic manipulatives solve both of these problems, although at some cost. (It is not yet clear how much prior experience with physical objects embodying certain properties is needed before these properties can be effectively used in a computer representation of those objects. It seems likely that this will depend largely on the familiarity of the objects and the properties that are being drawn upon for mathematical purposes. In the object-based environments that we have worked with, essentially no particular properties of the objects were being called upon except their discreteness. In other cases, e.g., Dienes Blocks (Thompson, personal communication, 1986), very particular properties are being used to represent mathematical structures.)

The more powerful microcomputers now becoming available in the schools will make possible a whole new effort to return action to student mathematical learning. And since **we are now able to link actions on concrete representations of mathematical ideas systematically and explicitly to the more abstract representations that are at the heart of mathematics' power, we may be more able to engender that power in students.**

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STRATEGIES USED BY COLLEGE STUDENTS ENROLLED IN DEVELOPMENTAL MATHEMATICS TO SOLVE PROPORTIONAL REASONING PROBLEMS

Laura Coffin Kooh

ABSTRACT

This study investigated the effect of two instructional rational number units on the strategies used by college students enrolled in developmental mathematics to solve proportional reasoning problems. Strategies varied from problem to problem, but the unit value strategy was the most widely used correct strategy. The use of the additive strategy did not carry over from one context to another, implying that the context of the problem influences the selection of strategies used. Although the unit value strategy was the most commonly used correct strategy on the pretest, students were unable or unwilling to apply this strategy in all situations. It appeared to be a successful strategy for students when the problem involved whole numbers, however, students did not use it when the problem contained fractions or decimals.

INTRODUCTION

This paper will report the results of a study that investigated the effect of instruction within the conceptual field of multiplicative structures on strategies used by college students enrolled in developmental mathematics to solve proportional reasoning problems.

Vergnaud (1983) suggested that interrelated concepts should be taught as an organized unit, which he calls a conceptual field. One such conceptual field is that of multiplicative structures. This unit includes the connected topics of: multiplication, division, fractions, ratios, rational numbers, functions, dimensional analysis and vector space. The teaching of these interrelated topics over a period of time would allow students to see the relationships between them and would give the students time to comprehend numerical relationships over a broader scope rather than smaller isolated units with little or no relationships.

Research in the areas of proportional reasoning has indicated that the numerical structure of the proportional reasoning task affects the subject's ability to solve that task (Rupley, 1981; Karplus, Pulus, and Stage, 1983; and Abramowitz, 1974). This would seem to indicate that rational number skills play a crucial role in proportional reasoning ability, although Heller, Post, and Behr (1985) found that the students saw no relationship between rational number skills and proportional reasoning. This may be due to the fact that in the schools topics such as fractions and ratio and proportion are taught in separate units and the relationship is not made obvious to the students.

College students enrolled in developmental mathematics courses lack rational numbers skills, although many of the students have taken courses in both elementary school and high school that focused on rational numbers. In addition, more than 50 percent of all college students are not yet formal operational (Chiapetta, 1976) and are unsuccessful on proportional reasoning tasks (Thornton and Fuller, 1981).

The subjects in this study were students enrolled in several developmental mathematics courses at the University of Minnesota. Students were pretested with respect to proportional reasoning ability. Subjects were then randomly assigned to either a sequentially based treatment of rational numbers topics or to a multiplicative structures treatment of rational numbers topics. Each of the treatments lasted four weeks. After completion of the instructional units, the subjects were retested with respect to proportional reasoning ability.

An investigation of the strategies that students use prior to instruction and after instruction should shed some light on the strategies that may or may not have been initially developed or enhanced by the instructional treatments. One question from the Karplus Stick Figure pretest and posttest and four questions from the proportional reasoning pretest and posttest were selected to analyze the strategies used by students both prior to instruction and after instruction. The questions were selected because they were representative of the entire test. The questions chosen include problems with: 1) integer components and an integer solution (I-I); 2) integer components and a non-integer solution (I-N); 3) decimal components and an integer solution (D-I); 4) fraction components and an integer solution (F-I); and 5) two separate contexts (rate of walking (W) and scaling (S)).

RESULTS

The results indicate that on the Karplus Stick Figure pretest many strategies were used in attempting to solve the problem, most of them were incorrect. The most widely used incorrect strategy was the additive strategy. This is consistent with earlier findings (Karplus, 1983a). On the posttest, the percent of students who correctly solved the problem increased and the use of the additive strategy decreased. Although there are no clearcut strategies used across class groups and across treatments, it does appear that the students were able to set up the problem using measure space notation and solve the problem with calculators or in their heads. The other posttest strategy that was used in correctly solving the problem was the rule-of-three strategy.

An examination of the students' changes in strategies from the pretest to the posttest on the Karplus Stick Figure Test indicates that 11 percent of the students in the multiplicative structures treatment group used the same strategy on the posttest and the pretest. Of the students that used the same strategy on both instruments, 7 percent used the incorrect additive strategy. In the sequentially based treatment group, 27 percent of the students used the same strategy on both instruments. Nineteen percent used the incorrect additive strategy on both the pretest and the posttest.

The results suggest that, on this particular measure, the multiplicative structures treatment helped limit the use of the incorrect additive strategy more so than the sequentially based strategy.

A variety of strategies were used on the I-I question of the proportional reasoning pretest. However, the unit value strategy appeared to be used by most of the students. The use of the unit value strategy decreased markedly on the posttest. The use of the rule-of-three strategy increased from the pretest of the posttest. These results would be expected since the instruction focused on three strategies: scalar, function, and rule-of-three. In addition, many students did not show their work. Very few students used incorrect strategies in solving this problem.

Unlike the results on the Karplus Stick Figure Test, the use of the additive strategy was used very rarely on either the pretest or the posttest on the I-I question of the proportional reasoning test. Only 7 percent of the students in the multiplicative structures treatment group in each of the three class groups used the same strategy on both tests. In the sequentially based treatment group, 12 percent of the students used the same strategies on both the pretest and the posttest. In both treatment groups, all of the students that used the same strategy on the pretest and posttest, used correct strategies.

The results indicate that the most widely used correct strategy on the pretest was the unit value strategy. On the posttest, the most widely used strategy was the rule-of-three strategy. The use of measure space notation was used by many students to help solve the problem. The use of the unit value strategy declined again for this problem from the pretest to the posttest.

The findings suggest that on the I-N question of the proportional reasoning test again the unit value strategy was the most widely used correct strategy on the pretest. Over all, more than 50 percent of the students were unable to solve this question on the pretest. The use of the unit value strategy decreased from pretest to posttest. There was no consistent method or strategy used across subjects on the posttest.

The additive strategy was rarely, if ever, used on the D-I question of the proportional reasoning test. On this particular question, 7 percent of all of the students in the multiplicative structures treatment group used the same strategy on both the pretest and the posttest. In the sequentially based treatment group, 12 percent used the same strategy.

The results of the D-I question suggest that most students who solved this problem correctly on the pretest were unable or did not explain their procedures in obtaining the correct solution. Although this problem contained fractions, more students were able to correctly solve this problem than the I-N question on the pretest. The I-N question used only integers in the problem, but had a non-integer solution. The F-I question used fractions in the problem, with an answer that was an integer. This would lend support to the theory that proportional reasoning problems with non-integer solutions are more difficult for students than problems with integer solutions. On the posttest, the majority of students either did not or were not able to give an explanation as to their procedures used to solve the problem.

On the F-I question of the proportional reasoning test, the additive strategy was not used. This could be related to the fact that this particular problem involved the use of fractions. Fifteen percent of all of the students in the multiplicative structures treatment group and 15 percent of all of the students in the sequentially based treatment group obtained correct answers on both the pretest and the posttest without showing work.

CONCLUSIONS

During instruction, students in both treatment groups were taught to solve proportional reasoning problems by setting them up in measure space notation and then using a scalar procedure. If the problem did not appear to have an integer solution, students were instructed to try a function procedure. Most problems were solved this way. However, in each of the treatment groups, some students recalled learning the rule-of-three method. This procedure was then reviewed, but related to the scalar or function procedure. These were the only methods used to solve proportion problems in the treatments. Unit values were discussed extensively and were found, but were not used to solve proportion problems.

A wide variety of strategies were found on the posttest. Many students were able to solve problems correctly on the posttest without showing any work, even though they were instructed to do so. The use of the calculator may have been a factor. Students were not always consistent in the use of strategies from one problem to another. This may be due to the context of the problems and/or the numbers used. In other words, the numerical structure of the problem and the context of the problem are factors in the selection of strategies chosen to solve proportional reasoning problems, even following instruction designed to increase proportional reasoning ability.

On the Karplus Stick Figure pretest, the most commonly used strategy was the incorrect additive strategy. This was true for both treatment groups. On the posttest, a larger portion of the students were able to correctly solve the problem, but there was no one predominate strategy. The use of the additive strategy did decrease from pretest to posttest.

Although the unit value strategy was the most commonly used correct strategy on the proportional reasoning pretest, students were unable or unwilling to apply this strategy in all situations. It appeared to be a successful strategy for students when the problem involved whole numbers. Students did not use it when the problem contained fractions or decimals.

In examining the students' strategies, it is evident that prior to instruction the most commonly correct strategy used was the unit value strategy. This result is consistent with that of Bezuk (1986), and Heller, Post, and Behr (1985). After instruction, however, the use of the unit value strategy declined and the use of the rule-of-three strategy increased. The results of this study do not support Vergnaud's (1983) findings with French students that the scalar procedure is widely used. Although some students did solve problems on the posttest using the scalar procedure, the number was not that great, especially in light of the emphasis made on the method during instruction.

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PROBLEM SOLVING WITH SCHEMATIC PROCEDURES ILLUSTRATED BY WORD PROBLEMS BASED ON PROPORTIONAL AND INVERSELY PROPORTIONAL FUNCTIONS

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Our investigation shows that schematic procedures for solving problems in direct and inversely proportions do not necessarily lead to greater success among the students. Often, schematic procedures even represent an additional subject of instruction, which remains obscure, and, therefore, is a source of additional errors.
Apparently the students don't concentrate on solving the word problems but on the schematic procedures to solve them. The students' concentration on numbers and ratios and their personal ways of thinking collide very often with the schematic procedures introduced in the classroom.

Solving word problems which are based on proportional and inversely proportional functions is mainly taught in the seventh grade in the schools in the Federal Republic of Germany. The students, who are about thirteen years old, have already obtained some knowledge of fractions and rational numbers.
In order to solve the word problems, i.e. to calculate the fourth data on the basis of three given ones, schematic procedures have been introduced; these procedures show the rules in a special optical way.

The common procedures - the rule of three, the method of fraction operators, and the fractional equations - can be demonstrated by means of a (proportional) problem:

14 kg apples cost 56 DM (German marks).
How many kilograms will you get for 24 DM?

rule of three:

1. determination of the type of function: proportional

2. 56 DM - 14 kg
   1 DM - $\frac{14}{56}$ kg
   24 DM - $\frac{14 \cdot 24}{56}$ kg
   = 6 kg

fractions operator:

1. determination...

2.

qu(kg)	pr(DM)
14	56
x	14

$$\cdot\frac{24}{56} \qquad \cdot\frac{24}{56}$$

3. $x = 14 \cdot \frac{24}{56} = \frac{14 \cdot 24}{56} = 6$

fractional equation:

1. determination...

2.

qu(kg)	pr(DM)
x	24
14	56

3. $\frac{x}{14} = \frac{24}{56}$

$$56 \cdot x = 14 \cdot 24$$
$$x = \frac{14 \cdot 24}{56}$$
$$x = 6$$

There are, of course, variations of the procedures. In particular, you could also start using the method of fractional equations with the following equation:

$$\frac{x}{24} = \frac{14}{56}$$

For some time the working group "Lern-Lehrforschung in der Mathematikdidaktik" ("Teaching-learning-research within mathematics didactics") at the University of Osnabrück has been engaged in the question in which way the schematic procedures influence the students' problem solving behaviour.

The investigation was started at the "Hauptschul"-level, which represents that kind of school within the tripartite school system in the Federal Republic of Germany, where less gifted students go to.

A first test was developed and later on given to nineteen groups of students in the eighth and ninth grade - a total amount of nearly 300 students.

The test is composed of twelve typical word problems with very little text (see above) just as they do occur in our

schools very often.
In Viet et al. (1988) and Viet & Kurth (1986) the influence of the text variables on the problem solving procedures is discussed. These first investigations show that on the average only 40 % of all the problems were answered correctly, the proportional problems are solved more easily (50 %) than the inversely proportional ones (30 %).

We are now mainly interested in the way students handle the procedures for solving problems.
It is striking that the students often do not fail to understand the content of the problem, but rather consider the form of the problem as a demand for using the acquired schematic procedure and then fail to remember ist.
This becomes very clear when looking at the interviews which were done with some students in order to get more information on the problem solving process. Most of the students simply have a quick glance at the text and proceed to the schematic procedure at once; they start recalling fragments of the procedure, which then are combined in a wrong way, or they mix up certain parts of the rules for solving proportional and inversely proportional problems. Futhermore the students can not explain or give reasons
for the steps they use to solve the problems. A student's solution procedure of our previous example by means of the method of fraction operators may be given as an illustration:

$$\frac{24}{56} \qquad \begin{array}{lcl} 56\ \text{DM} & \text{———} & 14\ \text{kg} \\ & & \\ 24\ \text{DM} & \text{———} & x\ \text{kg} \end{array} \qquad \cancel{\frac{x}{14}} \qquad \frac{14}{x}$$

$$\frac{\overset{6}{\cancel{24}}}{\underset{\underset{1}{\cancel{4}}}{\cancel{56}}} \cdot \frac{\overset{1}{\cancel{14}}}{x} = \frac{6}{1}$$

$x = 6$ kg

Students do not remember the principle behind the procedure but only the stencil, i.e. the shape. It even happens that the students use procedures belonging to totally different problems, e.g. they calculate percentages which have no connection with the word problems.
Apparently the schematic procedures didn't lead to a better problem solving behaviour, but represent an additional subject of instruction, which remains obscure, even diverts from the proper problem, and finally leads the students to a procedure with which he is not successful.

Speaking in support of schematic procedures teachers stated that classifying and organizing favour a better comprehension of the rules, avoid making errors, and, consequently, are a help for less gifted students. This point of view may derive from the fact that performance tests mostly take place at the end of a teaching unit when the students still remember the aquired procedures quite well.
Within three months - from the end of the teaching unit to the day of the memory test - two teaching concepts developped by our working group showed a remarkable decline of correctly remembered solution procedures (for further information see Freking & Handke (1987)). Due to our investigation it is to be doubted that schematic procedures do help less gifted students.

Now our working group is investigating differences concerning the behaviour in problem solving between those students who have already acquired such schematic procedures and those who have not. For that purpose a new test with ten problems (five proportional ones, five inversely proportional ones) was developped. As our first tests show, students are extremely dependent on numbers and special rations. The new test uses - referring to investigations of Noelting (1980) and Karplus et al. (1983) - different combinations of ratios for the three given data a,b,c in the text (e.g. b:a integral and c:a nonintegral etc.). Although the investigation has not yet been finished, we can already state the following:

- Students who have not been instructed to solve proportional and inversely proportional word problems go into the problem more deeply and do not look for a method.
- Their strategies for solving the problem are to a high degree adjusted to the chosen ratio; they make use of integral ratios in particular, that means, students
use the "within strategie" or the "between strategy" so called by Noelting (1980). We also noticed this behaviour among students who have already acquired a fixed method, but it was less distint.
- However, there are also students who follow a uniform strategy, i.e. they either look for any relations between the two co-ordinated data, that is preferring a "within strategy" (according to the rule of three) or relate the two data of the same units first, that is preferring a "between strategy" (according to the method of operators).
- There are types of errors which play a less important role among instructed students: the numbers and ratios lead to incorrect starts in that the students only use two out of three data, and use additive instead of multiplying strategies.

The role of schematic procedures within these processes has to be clarified by the current investigation. However, we can already state the following facts:

- Schematic procedures alone cannot increase the students' competence, because an important part of the solution, i.e. the question which type of function is in hand, can only be managed within the context of the actual problem.
The interviews often showed that the students tried to apply schematic procedures to this question, too, by concluding the type of function from a tabular representation of the three data.
- Much as schematic procedures signify clearness and structure for an expert, they often only mean

additional, sometimes very formal information to the laymen (in this case : the students).

- Schematic procedures unnecessarily complicate solving problems with "simple" ratios and, therefore, are incompatible with the fact that many students prefer following a simpler solution procedure. Those students don't think according to any schematic procedure that requires extracting the operations from the text first and working out the numerical values afterwards, but the very way of extracting is already influenced and regulated by numerical values and possible difficulties.
- Schematic procedures may conflict with a student's personal way of thinking; this may happen, e.g., when a student is looking for relations between co-ordinated data on his own initiative (i.e. he is considering the rule of three), but is to be determined to make use of the fraction operators.

However, according to our investigation we cannot and should not reject schematic procedures as such. On the one hand a part of the students does, of course, benefit by them, on the other hand schematic procedures are a means to carry out important didactical intentions, like organizing the solution procedure and emphasizing the conceptual background.

The reason why we only partly succeed in doing so is that we still have to investigate the relationship between variables of the students (for example preferring a special strategy), of the problems (for example special numbers and ratios) and of the schematic procedures (for example the strategy, on wich the procedure is grounded) more closely.

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THE ASSESSMENT OF COGNITIVE STRUCTURES IN PROPORTIONAL REASONING

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It is the goal to assess the conceptual background for proportional problems. In terms of cognitive science we have to distinguish between declarative and procedural knowledge. Although the conceptual background is part of declarative knowledge it is usually assessed by procedural tasks. We present a more direct way to cognitive structures in proportional reasoning using NOVAK's concept maps. The students have to describe the relations between concepts verbally and write them down on a poster. The students' implicit theories can be described pictorially. The individual student is viewed as a theoretician rather than a problem solver.

An increasing number of studies have concentrated on issues related to rational numbers and proportions (Behr & al., 1984; Noelting, 1980; Siegler & Vago, 1978, Hasemann, 1987). Traditionally, difficulties in this domain are assessed by interviews. Students solve rational number tasks and the authors describe the most common errors. These errors are the basis for analyzing strategies involved. Karplus & al. (1974) and Noelting (1980) suggested that faulty qualitative reasoning was the basis for many incorrect solutions. Hart's studies show that students tend to use additive operations where multiplicative operations would have been appropriate. The same author (1985) reported in her study about

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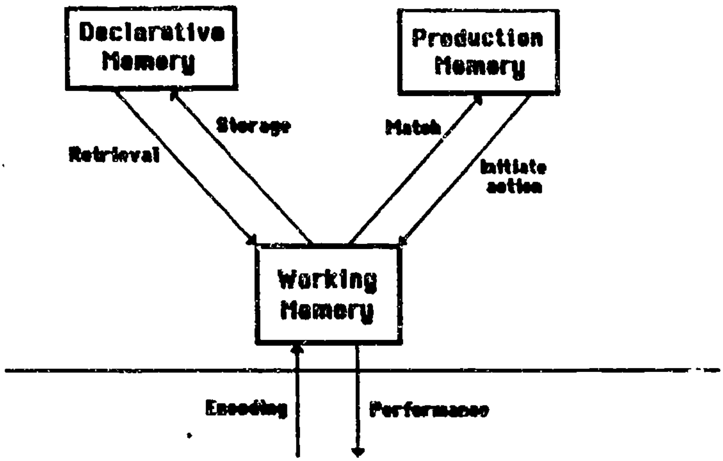
proportion tasks that students prefer whole numbers where the use of fractions would have been more adequate.

Hasemann (1987) emphasizes that already existing concepts and schemata are of central importance for mathematical learning. Often they exist before they become an issue in mathematics education. Therefore it can be productive to see the student not only in his or her role as a problem solver, but also as a theoretician. Often students have an implicit theory about mathematical concepts and they see relations between concepts not intended by teachers and textbooks. Students' errors may be seen in a different perspective taking this implicit theory into consideration.

A COGNITIVE VIEW ON PROPORTIONAL RELATIONS

In cognitive psychology a distinction is made between a procedural knowledge about rules and a declarative knowledge about facts. Anderson (1983) incorporates these two forms of knowledge into his psychological model of memory. He distinguishes between procedural rules formulated in condition-action pairs and factual knowledge in the form of strings, spatial images, and abstract propositions. The procedural rules make up a production system and are stored in the production memory whereas the factual knowledge can be retrieved from the declarative memory.

Anderson not only distinguishes between declarative and procedural memory but adds a third component, the working memory. The declarative memory contains rather general elements of knowledge and can be described as long term memory. In contrast, the working memory consists of rather volatile elements of knowledge that can be characterized as short term memory. Finally, the procedural memory includes productions, i.e. a set of rules which expresses the contingency between elements of knowledge in the form of condition-action pairs. In our case productions could be rules for handling fractions whereas the declarative knowledge would be involved while talking about the conceptual background of fractions.



(Anderson, 1983, p. 19)

In Anderson's model information originates in the environment and comes into the cognitive system via perception; it is encoded and stored in working memory. In our case the student perceives the fraction bar which is encoded in working memory. It does not have any meaning so far. The information from perception is transmitted to the declarative memory and the fraction bar becomes a signal for fraction tasks. Because the working memory only has a limited storage capacity, storage of perceptions is temporary and retrieval is fast. Perceptions are finally stored in the declarative memory for a long period of time. They are related to other objects and events, which is the basis for complex information retrieval from the declarative memory. More general aspects of the object or events are constructed so that information retrieval becomes more efficient.

It is not only the object that is stored, but this object as part of a whole class of objects. Fractions e. g. can have the attribute to be reduceable. This information is transmitted to the working memory and from there to the production memory. If the conditions for a reduceable fraction match with the numbers of a given fraction, the production memory initiates the action of reducing the fraction in the working memory. This is a cognitive activity which is finally performed by actually writing the reduced fraction on a sheet of paper. If the conditions do not match with a given fraction, more information about fractions in general and about the specific task is retrieved from the declarative memory and transmitted to the production memory via the working memory. This process of information retrieval and matching condition-action pairs is continued until the solution is reached.

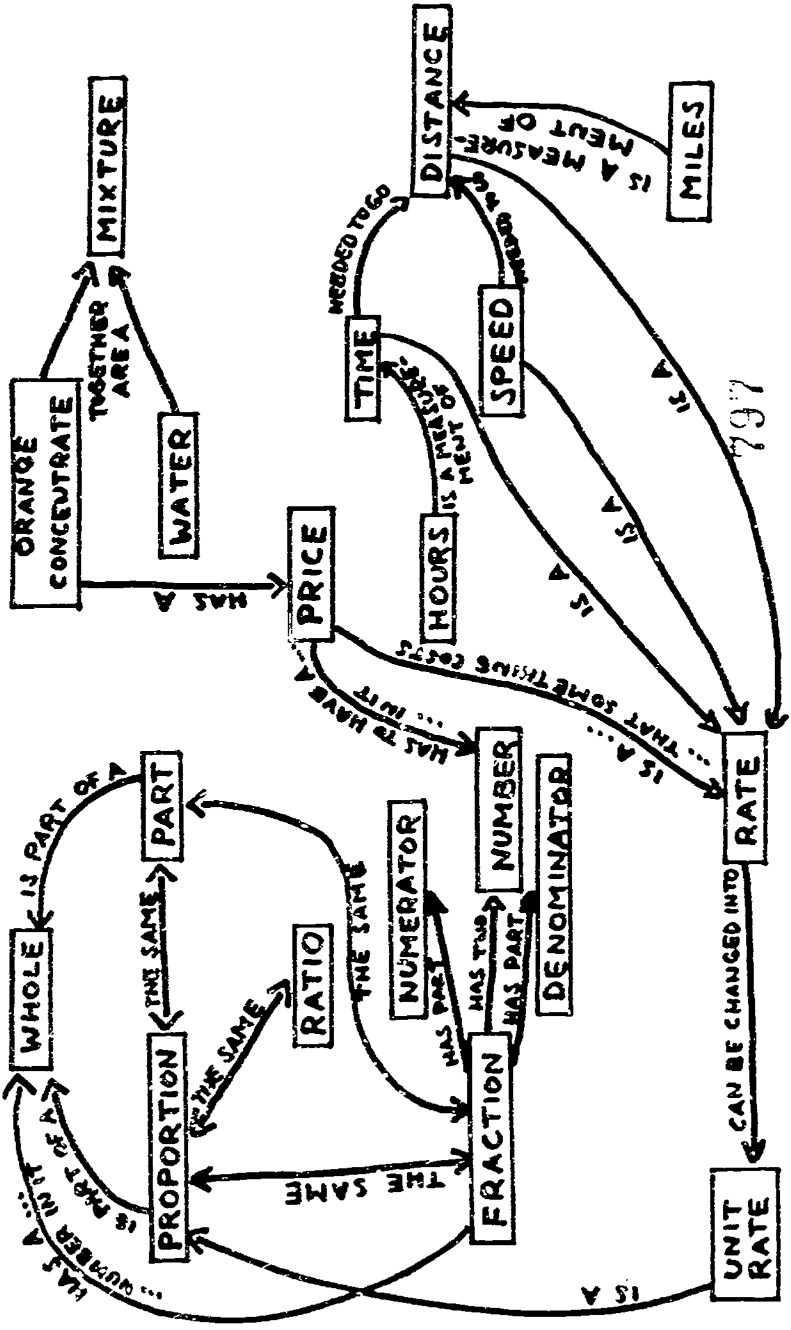
Problem solving in this model is conceptualized in terms of declarative and procedural components. As mentioned earlier research related to rational number concepts usually starts with the calculation procedure applied by the student. Errors in the procedure are interpreted as deficiencies in conceptual understanding. Using Anderson's terms, mathematics education usually starts with procedural knowledge and infers the structure of declarative knowledge on this basis.

The assessment of declarative knowledge about proportions

It appears to be worthwhile to try to assess declarative knowledge directly. In studies of artificial intelligence declarative knowledge is often described in the form of semantic nets. The concepts are nodes, the relations between the concepts are links. These relations are well defined: a concept can be a superconcept of another one (ISA link), it can be characterized by certain properties (HASPROP link), and it can have certain parts (HASPART link).

This form of describing declarative knowledge did not seem to be appropriate for the 7th graders we interviewed. The links in semantic nets are too well defined and do not allow space for ambiguities in cognitive structures. That is why we decided to make use of the experiences Novak & al. (1983) had in physics education. Novak developed an assessment procedure that he called "concept mapping". This involved concepts written on cardboard that had to be ordered on the table: similar concepts near each other, dissimilar concepts far away from each other. The students had to explain what they saw on the table and to describe relations between concepts orally. Finally these descriptions were written down in the form of arrows between concepts and sentences defining their relation to each other. The result was a map on the table that described the students' implicit theory about the domain on a conceptual basis.

In order to demonstrate the assessment method, a concept of one of our students is included. The concept map should be read in the following way: one has to start


ORANGE CONCENTRATE
TOGETHER ARE A
WATER
MIXTURE
HAS A
PRICE
HOURS
IS A MEASUREMENT OF
TIME
NEEDED TO GO
SPEED
DISTANCE
IS A MEASUREMENT OF
MILES
IS A
IS A
IS A
IS A ... THAT SOMETHING COSTS
HAS TO HAVE A ... UNIT
NUMBER
NUMERATOR
DENOMINATOR
HAS PART
HAS TWO
HAS PART
FRACTION
THE SAME
PROPORTION
THE SAME
RATIO
THE SAME
THE SAME
PART
IS PART OF A
WHOLE
IS PART OF A
HAS A ... NUMBER UNIT ...
RATE
CAN BE CHANGED INTO
UNIT RATE
IS A

where the arrow originates, read that concept, read the link, and finally read the concept where the arrow points to. In the case of double arrows it can be read in both directions.

The concept map on the preceding page is the product of an intensive interview (45 minutes) with Jon. He is a good student, he usually was one of the first students able to answer a difficult question. In spite of his remarkable qualities in solving numerical problems, his concept map reveals some misconceptions in the domain of proportions. He is able to describe what a fraction is, but he cannot relate the concept proportion to other concepts in a consistent manner. He seems to mix up proportion and portion. As long as he describes concepts within applications of proportionality concepts, he finds a more or less meaningful way to use them (distance, miles, time, hours, speed; water, orange concentrate, mixture). But rate, ratio, proportion, part and number do not really fit into the system and are used inconherently. Although for Jon proportion is the same as ratio and as part, he does not see a direct connection between ratio and part. Following this logic ratio had to be the same as part. The fact that he does not mention this connection reveals ambiguities in the use of these concepts.

Using this technique in the domain of proportions, it was possible to discover conceptual misunderstandings even in good students. Usually they did not connect all concepts that were related to each other. The results were islands of concepts which often were consistent in themselves, but proved to be relatively isolated from each other.

Concept maps reveal important aspects of declarative memory but certainly are not identical with this form of knowledge. They may be useful for diagnostic purposes as well as for monitoring the students' conceptual background. In contrast to Dörfler (1987) the underlying model assumes that it is possible to separate operative aspects and the conceptual background (or in Anderson's terms declarative and procedural knowledge). Dörfler argues that these are two sides of the same coin. Anderson's model views them as two components of a dynamic system interacting with each other. We used Anderson's model because we wanted to focus on the students' conceptual background and therefore had to separate it from operative aspects.

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CONJECTURAL SOFTWARE, VAN HIELE LEVELS AND PROPORTIONAL REASONING

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The development of new conjectural software, which allowed pupils to construct and validate ratio-tables, led to the construction of long-term courseware based on a proposed van Hiele model of proportional reasoning. The courseware was examined in terms of slow-learning adolescents' consequent thinking and insight into the proposed base-level van Hiele structure of proportion. Pupils response to the learning environment reflected their cognitive style. The proposed van Hiele base-level structure provided a framework in which pupils, teacher and researcher could communicate. These factors seemed to account for the pupils long-term acceptance of the learning environment and the demands it placed upon them.

Following the work of IOWO and the views expressed by an international panel of the 7th PME conference, Streefland proposed a 'theory' for the long-tern learning of ratio (Streefland, 1983, 1984 & 1985). In his 'theory', he outlined the possible key role of the ratio-table in the learning process.

That key role is an issue of the current research which entailed the development of courseware based on specially developed software which allowed pupils to type in the first line of a ratio-table and then check any subsequently entered lines against the initial line.

Since only one line needed to be displayed at a time, the software could be run on a calculator size device with a one-line display (Vincent, 1987a). To simplify keypad and display requirements, feedback and error-trapping techniques were developed which used the cursor to signal a correct item or a (syntactically) incorrect key-stroke, no other feedback being required.

Early Field Work

Early field work entailed pupils using an enhanced micro-based simulation called MATHSPAD (Vincent, 1987b) to check and accumulate numerical results derived from a variety of proportion based activities not connected directly with the computer (using 'ready reckoners', reading scales, reading graphs and so on).

Early exploratory work indicated that the simulation program could elicit sustained deep thinking from pupils of a wide ability and age range. It also became clear that the software could accommodate a variety of learning theories and that the environment it provided could be upgraded to support different levels of thinking (Dreyfus, 1984; Dreyfus & Thompson, 1985)

A van Hiele Model of Proportional Reasoning

This research took the view that slow learning adolescents who failed to gain insight into structures of proportion may have done so because their level of thinking did not match the instruction they were given.

Van Hiele (1986) was concerned that teacher and learner 'see' a topic in the same way; that instruction matches the level of thought of the pupil and that the pupil acts with insight.

Van Hiele models of teaching and learning require the stratification of thought into levels relating to the way that people 'see' a topic at each level. They contain prescriptions for moving pupils from one level of thought to the next, levels that are necessarily recursively related (Hoffer, 1983). They take as their starting point people's initial (naive) understandings and perceptions and, in so doing, focus on, and give legitimacy to, thought at the base-level. Van Baalen described this base-level as:

> the level at which people (including the pupils) think in their daily lives, with which they have their experiences, and with which they make their decisions. (van Baalen, 1980)

The current research proposes a van Hiele model of proportional reasoning in which base-level thinking corresponds loosely to the use of 'build up' strategies to solve missing value problems (Hart, 1984). Multiplicative and fractional comparisons, if used, mainly serve to determine the steps used to 'build up the answer'. The epistomological outlook is that proportion problems can be solved by the combination and 'fine tuning' of existing proportional pairs, the agents of which are addition, 'small' integer multiplication and division (seen as repeated addition and 'sharing' respectively). This outlook needs no justification by the pupil: it is 'seen to work' in intuitively understood contexts.

With respect to the proposed van Hiele model, pupils begin to think at the next level when they abandon attempts to build up toward the answer and, without using higher level conceptualisations of ratio (Karplus, Pulos & Stage, 1983; Noelting, 1980) meaningfully use multiplication and division to construct strategies, initially based on evaluating and applying multiplicative comparisons.

The Courseware

Courseware was designed which used the opportunity afforded by screen-based ratio-tables to take slow learning adolescents back to the base level (Vincent, 1987b). Van Hiele has called this 'telescoped reteaching'. It consisted of a series of narrowly focused proportion-based tasks designed to acquaint pupils with proportion structures across a wide variety of contexts viewed in terms of base-level thinking. These 'level specific' environments corresponded loosely to van Hiele's first two learning phases of 'information' and 'guided orientation' and mainly entailed the completion of (screen-based) ratio-tables. The ratio-table was used to record results accumulated from meaningful context-bound activities and not specifically to highlight operational features of proportion (Seeger, 1984).

However, as the screen accumulated only correct results (the software did not allow pupils to enter a new item until the current item was displayed correctly), pupils could use patterns discerned from earlier results as a basis for idiosyncratic algorithmisation as a first step toward higher-level thinking. This was further encouraged by providing only limited fall-back strategy or limiting the range of a table, scale ready reckoner.

Field trials of base-level courseware.

The courseware was used with two volunteer groups of slow-learning adolescents of about 10 pupils each in the 14 to 15, and 15 to 16 age range. The trials were respectively one term (1 hour a week for 13 weeks) and one year (1 hour a week for 40 weeks).

Pupils progression through the courseware was closely monitored. Pupils who experienced difficulties were given guidance. Taped interviews were conducted with pupils during and after the course. Tests were also administered after the course.

Pupils seemed to respond well to the learning environment offered by the computer simulation but for differing reasons. Despite following identical courses and performing identical tasks, pupils exhibited markedly different mathematical behaviours. They seemed to use the environment in ways that reflected their individual cognitive style. This may have contributed to pupils long-term acceptance of the learning environment.

Conclusion

Completing screen-based ratio-tables was shown to be an effective way for slow-learning adolescents to begin to explore structures of proportion. The proposed base van Hiele level seemed to provide a coherent structure in which these slow learning adolescents, teacher and researcher could communicate. A small number of pupils began to construct strategies that reflected higher level thinking in relation to the proposed van Hiele model.

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RATIO AND PROPORTION: A SYNTHESIS OF EIGHT CONFERENCE PAPERS

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The paper by Laura Coffin Koch investigated the question of whether two different treatments to teach strategies for solving proportion problems would differentially affect the types of strategies, particularly invalid strategies, that learner's use. Each of the two treatments considered the concepts associated with a multiplicative conceptual field (Vergnaud, 1983). One of the these treatments taught the concepts in a sequential manner, the other in a manner which apparently attempted to integrate the concepts as a conceptual field. An important finding is that the different instruction methods did differentially affect learners continued use of invalid solution strategies. It would thus seem that use of invalid solution strategies may be amenable to change under certain instruction treatments.

The pre-and posttest tasks reflect the structural variable of divisibility and the contextual variables of number type (integer, decimal, and fraction) and rate type (rate of walking and scaling). The results corroborate earlier findings of the strong affect that the variable of divisibility has on performance in solving proportion problems. The study also reports findings consistent with those reported by Heller, Post and Behr (1985) about how frequently given rates are changed to unit rates in solutions of proportion problems.

The results seem to suggest that problems which have a fractional answer (value for the missing value) are harder than problems that have an integer solution but involve fractions or decimals as problem components. Why might this be true? Unfortunately, the author's report does not give enough information about the problem structure to make any hypotheses. Nevertheless it would seem that there is lurking behind this result some important questions about variables that affect learner's performance on multiplication and division operations. This is suggested because any nonadditive st:ategy would require using either a multiplication or division computation. The finding that students are not always consistent in their use of strategies from one problem to another is consistent with earlier results. This finding highlights the important question of how a solver's choice of solution strategy depends on such matters as problem context, number type, and other variables such as the structure of the problem presentation and the structure of problem representations which are formed by the solver.

The paper by Wilfried Kurth investigated the strategies used by students at the "hauptschul level." These students had apparently been taught three schematic procedures for solving missing value direct and inverse proportion problems through their regular class instruction. These schematic procedures, using the terminology of Vergnaud (1983), were the rule of three, scalar operator, and function operator. The following observation is of interest not only to proportion problem solving research, but to problem solving research in general. The author suggests that these students did not seem to fail to understand the content of the problem, but nevertheless they considered the form of the problem as a demand for using one of the learned schematic procedures. Most of the students gave a cursory reading of the problem and then proceeded directly to the schematic procedure, often recalling only fragments of the procedure and mixing up aspects of one or more schematic procedure.

It would appear that we see in these results some difficulties which arise for students when instruction emphasizes the learning of solution strategies without conceptual understanding of what it means to reason proportionally. The issue of whether students are responding to the syntax or the semantics of a proportion problem likely underlies the students difficulties. This raises several questions: What is the semantic content of a proportion problem? What does it mean to reason proportionally; that is, what do students need to understand in order to solve proportion problems meaningfully? Another observation made by the author is that while schematic procedures may indicate clearness and structure for an expert, to the novice they may mean only additional information, for which additional instruction is necessary in order to use or understand the procedure.

The research reported in these two studies seems to have an emphasis on solution procedures. Moreover, it would appear that the instruction failed to emphasize the conceptual knowledge of what a proportion is and what solving a proportion problem involves. They fail to get at the basic issue of helping children to reason proportionally. I think more emphasis needs to be placed on the fact that what we do in solving a proportion problem is to determine a relationship between a pair of numbers and then: (a) transfer that relationship to an operation on a third quantity to find a missing value, or (b) determine whether or not the same relationship holds between a second, or successive pairs of numbers. It would appear that we have the cart before the horse. It is not that teaching solution strategies will lead to the ability to reason proportionally, but rather that conceptual knowledge about the meaning of a proportion will give the foundation for the development of valid strategies. Then, practice with the strategies, in order to automate them, and continual interaction with the conceptual knowledge about the fundamental issue of proportionality will lead to higher performance on proportion problems.

There is an issue of syntax versus semantics of proportion problems. It seems that the syntax of a proportion problem can be translated to a syntactical structure such as a measure space diagram without considering the semantics of the problem. What is the semantics of a proportion problem? Consider the problem

6 candies for 10 cents,

17 candies cost how many cents?

The conceptual knowledge for solving this problem is that there is a relationship between 6 candies and 10 cents and a way to express that relationship in terms of an operation or function -- multiple by 3 candies per 5 cents. The function rule, multiply by the extensive quantity 3 candies per 5 cents or 3/5 candies per 1 cent, is an "equalizer", "exchange rate", or "matcher", between 6 candies and 10 cents, and must also be the equalizer between 17 candies and "how many cents." The "equalizer" 3/5 candy per one cent is what allows for the equation

6 candies x (some quantity) equals 10 cents

to be formed.

The paper by Kaput, Clifton, Poholsky, and Sayer introduces an entirely new dimension on the matter of translating among several representational systems. Earlier work by Bruner (1966) and Dienes (1967) demonstrated the importance of having multiple types and levels of representation to facilitate the learning of mathematical concepts. Subsequent work by Behr, Lesh, Post, and Silver (1983) suggested that meaningful learning depended on flexibility in translating ideas between several modes or systems of representation. Since that time the concept of a representational system has been made more precise (Kaput, 1983; Lesh, Post, and Behr, in press; Goldin, 1985). The work reported here by Kaput et al. represents a giant step forward in providing a model of curricular material that makes the desirable use of several representational systems possible. Moreover, the work represents a major step forward in providing an exemplar of a medium in which translation among systems of representations is possible. The objective of the work is to provide cybernetically controlled learning environments which move learners from situation-bound representations to more abstract and flexible ones. The cybernetic control allows for the effect of some manipulation within one representational system to be observed by the learner in that system and one or more other systems simultaneously or sequentially. The specific materials reported on in this paper concern the matter of intensive quantity and related issues of ratio and proportion. The work represents a major stride toward providing a learning environment for students to learn these difficult concepts, yet it leaves unanswered or raises several questions. The paper repeatedly refers to the intensive value of a ratio or rate. Do learners perceive a ratio as a single entity with (intensive) value, or is a ratio a pair of distinct numbers? Do most novices to the concept of ratio see it only as a correspondence between two extensive quantities rather than as a third derived intensive quantity? Moreover, it appears to be an open question as to whether these materials move children in the direction of perceiving intensivity. These questions are likely related to the question raised by Behr, Wachsmuth, Post (1985) about whether or not children perceive a fraction as representing a single number, or value.

The paper by Hugh Vincent attempts to bring together the research on proportion problems and the levels of thought according to the Van Hiele theory. He defines base-level thinking in terms of the use of build up strategies which children are known to use to solve missing value problems. Transition to the next level occurs as the buildup strategy is abandoned and strategies began to depend on higher level conceptualizations of ratio. This is characterized by meaningful use of multiplication and division which is based on an initial evaluation and application of multiplicative relationships. A computer/calculator instructional package was developed to facilitate this transition which was based on the theory proposed by Streefland (1984, 1985). The instruction presented partial ratio tables and required learners to complete the table. A feedback mechanism in the instruction was designed to nudge students away from the buildup strategy to the more acceptable multiplicative strategy. The authors found that the ratio tables were an effective way for slow learning students to begin to explore structures of proportion and that the proposed base level from the van Hiele theory did provide a level at which teachers and students could communicate. This study provides another instance in which instruction on strategies for solving proportion problems facilitated learner's use of "higer level" strategies. Some questions which arise from these intervention studies are: What characterizes instruction that raises the level of strategies that a student applies to the solution of a proportion problem? To what extent does a student's choice of a "higher level" strategy have a commensurate improvement in the learner's understanding of proportion and proportional reasoning?

The semantic content in which a proportional relationship is embedded and the syntactical structure of this relationship are two classes of variables (problems context and problem structure) which are known to affect performance on proportion problems. The paper by Behr, Harel, Post, and Lesh outlines a theoretcal analysis of the problem structure variables for missing value problems. The intent of the analysis is to provide a theoretical foundation to guide systematic manipulation of this class of variables.

One structural variable included in the analysis, known from prior research to affect problem performance, is that of the divisibility relationsip among the problem components. Other variables included in the analysis, about which less is known from prior research, are the location of the missing value, and the unit of measure, which expresses the amount of a given quantity. The paper also alludes to, but does not pursue, the type of quantity to which the unit of measure refers.

The paper presents an information processing problem solving model for the class of missing value proportion problems. The model considers the fact that solvers take the problem structure of the problem presentation and transform that structure, due to the salience of certain structural variables, into mental problem representations. The problem representations are hypothesized to form three classes: for problem understanding, for exploring the relationship among problem components (intermediate representations). and for application of problem operators (procedural representations).

The analysis identifies a mathematical group of 8 transformations. These transformations are knowledge structures used by the solver to change the structure of the problem. Exactly 2 problem operators are identified, one is instantiated on the per statement which has two known quantities and the other on the per statement with the missing value. These problem operators are instantiated with either an addition subtraction, multiplication, or division equation. An instantiated operator is refered to as a solution strategy; these can be valid or invalid. Combinations of division and multiplication equations which instantiate the two operators lead to the identification of 14 valid multiplication solution strategies. A hierarchy for solver's preference for these strategies is hypothesized. Finally the set of 512 missing value problems (determined by the problem structure variables) are partitioned into 8 distinct levels by the transformations.

Each level consists of the 18 distinct problem structures; those structures of a given level are the 18 preimages which get mapped by the same problem structure transformation one-to-one onto the 18 procedural representations. An hypothesized difficulty hierarchy for the 18 procedural representations is used to hypothesize a difficulty hierarchy on the 18 problem structures within each level. The separate levels do not seem amenable, via this analysis, to an hypothesized difficulty hierarchy. These extensive analyses therefore lead to a partial difficulty hierarchy for the 512 missing value proportion problems.

In the domain of proportion problems, emphasis has been on the identification of solution strategies that solvers use to solve or attempt to solve proportion problems. No attention has been given in this research area to the problem representations which solvers form before they apply a solution strategy. Consequently no research has addressed the question of the relationship between the problem representation that a solver forms and the strategy which is applied. It appears that this would be a fruitful line of inquiry because of findings in other areas of problem solving research in which distinct differences are shown to exist between the problem representation and the procedures used by expert and novice solvers (Chi, Glaser, & Ress, 1981; Larkin, 1983).

Qualitative reasoning has been found to be an important component of successful problem solver's thinking. Successful problem solvers are known to reason qualitatively about the relationships among components of a problem before or instead of using quantitative procedures.

The paper by Harel, Behr, Post, and Lesh addresses the problem representation issue for a proportion problem task which emphasizes qualitative reasoning. This task, called the blocks task, requires the coordination of several weight and size relationships on a given pair of blocks to determine the weight relationshp between a criterion pair of blocks. The study identifies a hierarchy of 3 distinct problem representations used by grade-7 subjects and a hierarchy of 5 distinct solution strategies. A very close correlation was found between the levels of sophistication of the representation formed and the strategy used. The top representation was found "to call on" the top 3 levels of solution strategies, the middle level of representation "called on" the lower 3 levels of solution strategies (an overlap of the midle level strategy), and the lowest representation was found to "call on" only the lowest level of solution strategy.

Some questions which arise from these two papers are: What is the role of qualitative thinking in the development of proportional reasoning? Can children be taught to use qualitative reasoning? Will the development of qualitative proportional reasoning in children facilitate the development of quantitative reasoning about proportional situations? Can children be taught to form certain problem representations for proportion problems? If children are taught certain higher level problem representations will this improve performance?

The paper by Alan Bell and Barry Onslow concerns the concept of rate, that is, the concept of intensive quantity which they define as "the quotient of two extensive quantities." The paper reports children's responses on several carefully designed division problems. Two questions have to do with the phenomenon of "small-number divided by large-number." One of the problems was in the context of a whole of 27 units of measure being divided into 150 pieces (i.e. a whole is divided into parts) and the question is of the number of units of measure in each (or one) part, or how much of the whole is in each part. This partitive division problem has a partitioning behavioral model in a very natural sense, a whole is made into a number of equal sized parts, how big is each (or one) part? Moreover, the context of the problem involved the length of a ribbon, a quantity which is "obviously" partitionable, and partitionable as successively as necessary. The second problem on the other hand, involved two quantities: 44 pence was to be divided by a 250 gram pack of butter. This division, while also a partitive division (extensive quantity divided by extensive quantity), has much less of a natural behavioral partitioning of a whole, 44 pence, into parts. In this case, the whole of 44 pence was to be partitioned according to some exterior quantity, having no relationship to the 44 pence, such as a part-whole relationship, other than the relationship of correspondence. Moreover, the unit of pence, the smallest unit of British currency, is not partitionable in a behavioral sense, and is certainly not infinitely partitionable in this sense. Some important research questions arise from their finding that the first problem resulted in nearly twice as many correct responses for choice of operation as the second. To what extent are divisions which has a whole-divided-into-parts behavioral model easier than divisions without this model? The problem used in this study had the part-whole relationship of "part of the whole" relationship of "Part of the whole" in an inclusion sense; other types of part-whole relationships exist (Chaffin, Herrman, & Winston paper in preparation), a bedroom is part of a whole apartment in a different sense, a handle is part of a cup in still a different sense, are there partitive divisions which involve these different types of part whole relationships, and if so how do these differences affect children's performance? How does the partitionability of either, or both, of the dividend or divisor quantity affect children's performance on the second type of partitive division problem?

One finding of the study was that children performed much better on a rate problem when the problem question was put in the form of how much of quantity-1 in one unit of quantity-2?, rather than in the form how much of quantity-1 per unit of quantity-2? Another related finding is that children understood 20 francs per gram as meaning the same thing as 1 gram per 20 francs, for example. The first finding raises a linguistics question for investigation. What words best communicate the meaning of rate to children? Is there a developmental sequence in the words that communicate this meaning? From the second finding the question arises, Do children have a concept of intensive quantity? If so, at about what age is it acquired, or what set of experiences can bring it about? If not, what is children's understanding of rate? Behr, Harel, Post and Lesh (in preparation) suggest that children's perception of rate is simply that of correspondence. The rate 5 miles per 3 minutes is hypothesized to simply be the correspondence of 5 miles to 3 minutes. Since, a correspondence is likely to be considered symmetric by children, this would explain why 5 miles to 3 minutes is thought to be the same as 3 minutes to 5 miles. Some research questions in this context: How do we define intensivity so that it is learnable by children? Can we define appropriate experiences so children begin to learn the concept of intensivity? Do children who demonstrate knowledge of intensive quantity perform better than those who don't on the types of problems given by Bell and Onslow? Do children who demonstrate knowledge of intensive quantity perform better than those who don't on proportion problems?

In the paper by Reiss, Behr, Lesh, and Post the point is made that a technique to make direct evaluation of children's conceptual knowledge for proportional situations is needed. Moreover, this analysis technique should go beyond attempts that have been made to infer learner's conceptual knowledge from the strategies they are observed to use to solve proportion problems. The potential of the semantic net analysis used in cognitive science is discussed but rejected on the basis that the strictly defined links are to limiting to capture the richness and diversity of a student's knowledge structure. A recommendation to employ the concept map analysis developed by Joseph Novak (1984) is given and an example of one student's concept map for ratio and proportion is presented and discussed. It appears that this method of analysis holds promise for comparing the conceptual knowledge of successful to unsuccessful solver's of proportion problems, and for charting the change, in the conceptual knowledge within students for proportions over time and instruction.

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UNCOMMENTED
PAPERS

Arithmetic

CONFRONTATION D'APPROCHES CONSTRUCTIVISTE ET TRADITIONNELLE DE LA NUMERATION

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ABSTRACT

This presentation is a follow-up of a research study on numeration conducted in primary school during five years. This research has permitted to point out on one hand the main difficulties and misconceptions developed by children in current teaching of numeration, and on the other hand to develop a constructivist approach leading children to build a meaningful and efficient symbolism of number. This approach was experimented in a classroom from 1980 to 1983 with the same group of children from the time they were in first grade (6-7 years old) to the third grade (8-9 years old). To point out the real impact of this longitudinal study on the pupils, an evaluation of the understanding of numeration was conducted at the end of 1981, 1982, 1983 and after three years in 1986. The results are presented here. Confrontation of traditional and constructivist learning of numeration will be discussed.

INTRODUCTION

Plusieurs recherches récentes en didactique des mathématiques se situent dans un courant constructiviste de l'apprentissage. L'élaboration des connaissances y est conçue, en accord avec les perspectives piagétienne du développement (Piaget, 1975) et épistémologique du développement de la pensée scientifique (Bachelard, 1938), comme une suite de reconstructions successives passant par des périodes de déséquilibres, de déstabilisation. C'est en termes d'erreurs rectifiées, d'obstacles dépassés qu'ils caractérisent le développement de la pensée. Se situant dans ce courant constructiviste, plusieurs recherches en didactique des mathématiques ont permis de mettre en évidence les conditions dans lesquelles les enfants construisent des concepts fondamentaux en mathématiques. Ces recherches apportent un éclairage à l'analyse des erreurs produites par les élèves, aux conceptions inappropriées sous-jacentes ayant des conséquences sur la production d'autres connaissances. Un des problèmes qui se pose suite à ces études, et auquel les recherches en didactique fournissent peu de réponses, est le suivant: comment un apprentissage pourrait-il être organisé pour tenir compte de ce que l'on sait de la pensée mathématique des enfants, de leurs difficultés, conceptions, procédures...? Peu de recherches ont en effet effectivement élaboré et expérimenté une approche constructiviste des mathématiques. Les travaux réalisés dans cette perspective demeurent limités (Von Glasersfeld, 1983; Bergeron, Hercowitz, 1984; Steffe, 1977; Cobb, 1983).

Notre recherche est une contribution importante en ce sens. Nous avons en effet mené durant cinq ans un travail sur la numération et son apprentissage à l'école primaire, travail qui a permis: d'une part, de clarifier explicitement les conditions dans lesquelles l'enfant s'approprie le concept; d'autre part d'organiser un apprentissage reflétant ce que l'on sait de la pensée des enfants, apprentissage s'articulant autour d'une prise en compte des conceptions, procédures, représentations symboliques développées par les enfants. Cet apprentissage fait en sorte que l'enfant construise progressivement un système de représentation du nombre significatif et efficace.

APPROCHE TRADITIONNELLE DE LA NUMERATION; QUELQUES FAITS SIGNIFICATIFS REVELES PAR NOTRE PREMIERE ETUDE

Par numération, on entend traditionnellement un système cohérent de symboles régi par certaines règles permettant d'écrire et de lire les nombres. La numération est la partie de l'arithmétique qui enseigne à exprimer et à représenter les nombres. Ainsi, l'enseignement traditionnel de la numération se réduit souvent à la capacité de lire des nombres, de les écrire et à l'habileté de pointer dans un nombre donné les valeurs de position. Nous retrouvons alors un enseignement axé essentiellement sur l'écriture conventionnelle et sur l'acquisition des règles syntaxiques qui régissent cette écriture, et ceci très tôt (dès 6-7 ans) [9].

Afin de mieux comprendre la conception que nous avons de la numération et de son apprentissage, nous reviendrons sur les résultats les plus significatifs révélés par cette première étude [10].

- Quels sens les enfants accordent-ils à chacun des symboles intervenant dans l'écriture conventionnelle, voient-ils que l'écriture est un code relié à des collections "réorganisées pour faire apparaître des groupements"?
  Notre étude nous révèle que peu d'enfants accordent une signification véritable à l'écriture en termes de groupements. La plupart des enfants conçoivent l'écriture comme un alignement ou une séquence de chiffres (chiffres placés dans un certain ordre). Ainsi des mots comme centaines, dizaines, unités ne sont pas du tout pris en considération ou sont associés à un certain découpage de l'écriture.

De plus, l'écriture des nombres est associée pour beaucoup d'enfants au codage d'une collection d'éléments. Les enfants ont alors l'attitude à recourir au comptage un à un de la collection, même quand cette stratégie est inappropriée. Peu d'entre eux voient la pertinence de regrouper et voient que l'écriture est un code qui découle directement de ces groupements.

Quels sens les enfants accordent-ils au traitement de cette écriture dans les procédures de calcul? Voient-ils dans la retenue ou l'emprunt une action effective sur des groupements?

Peu d'enfants peuvent opérer sur les groupements lorsqu'ils ont à faire ou défaire ceux-ci. Ils ne peuvent illustrer ni expliquer avec un matériel les opérations effectuées sur l'écriture. Les conceptions erronées de la retenue et de l'emprunt que l'on retrouve alors illustrent que les règles utilisées dans les procédures de calcul ne correspondent à aucune action effective sur les groupements.

Nous pourrions poursuivre l'analyse de ces difficultés, la compréhension que les enfants retirent de la numération dans l'enseignement actuel est cependant suffisamment caractérisée. A la lecture des faits précédents, il est facile de noter que pour nous l'écriture conventionnelle n'est pas un but en soi mais est plus étudiée en regard de sa signification en termes de groupements et en termes de transformations effectuées sur ces groupements lorsqu'on a à opérer.

UNE APPROCHE CONSTRUCTIVISTE DE LA NUMERATION. BREVE CARACTERISATION DE NOTRE INTERVENTION.

Quand nous travaillons sur la numération, nous travaillons sur le processus de représentation du nombre, processus s'articulant sur des collections réorganisées pour faire apparaître des groupements. Un apprentissage de la numération doit provoquer ce processus de représentation et le mener à terme.

De 1980 à 1983, nous mettions en pratique une conception constructiviste de l'apprentissage de la numération auprès d'un même groupe d'enfants suivi pendant 3 ans (6-7 ans à 8-9 ans).

Toute notre stratégie a consisté à amener les enfants à développer leurs propres représentations du nombre et à les faire évoluer vers un symbolisme significatif et efficace. Pour accomplir ceci, l'enfant a été amené à vivre des situations qui le forçaient à opérer sur des collections sur lesquelles une relation entre les groupements était définie [11]. Les opérations sont essentielles dans notre stratégie, puisque en plus de donner une motivation aux transformations opérées sur les groupements, elles leur donnent une signification en termes d'action véritable (défaire les groupements, les faire, échanger...). Ces situations nécessitent que les enfants développent des moyens pour garder trace et communiquer de l'information sur les transformations opérées et sur les collections regroupées qui en résultent. Les représentations qu'ils développent sont alors significatives. Enfin, la sollicitation à devenir efficaces dans le traitement et la communication d'informations sur des collections force l'enfant à avoir recours à des représentations écrites qu'il raffine progressivement. Dans cet apprentissage constructiviste, les situations et interventions reposent sur une analyse constante des procédures, conceptions, représentations symboliques développées par les enfants et sur une analyse de l'évolution de celles-ci [12].

EVALUATION DE L'IMPACT DE CETTE INTERVENTION

Les effets de cette intervention sur la compréhension de la numération par les enfants ont été analysés à la fin de chacune des années: 1ère (6-7 ans) à 3e année (8-9 ans). Des items ont été élaborés à partir du cadre de référence développé dans la recherche précédente [8] et expérimentés sous forme d'entrevue auprès des enfants du groupe ayant suivi l'approche constructiviste et des enfants d'un groupe contrôle ayant suivi une approche traditionnelle de la numération (Bednarz, Janvier, à paraître).

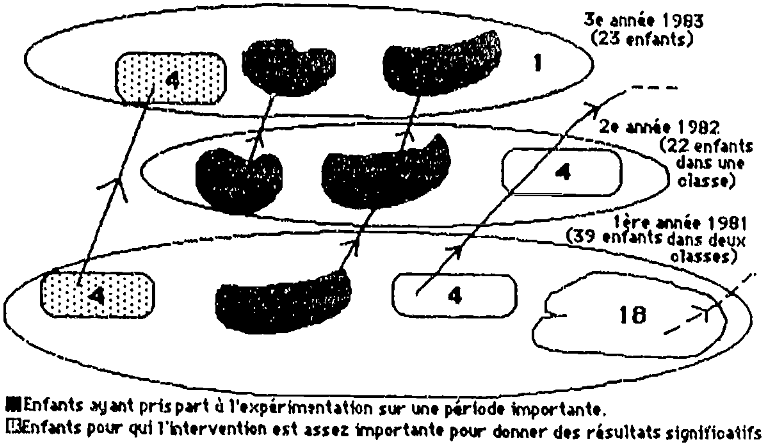


Figure 1 - Groupe expérimental

	Approche constructiviste	Groupe contrôle enseignement traditionnel
Fin 1ère année (6-7 ans)	39	25
Fin 2e année (7-8 ans)	27	23
Fin 3e année (8-9 ans)	23	26

Figure 2

Trois ans après notre expérimentation, nous cherchons maintenant à mettre en évidence ce qu'il reste de cette intervention auprès des enfants: les habiletés, procédures, façons de s'organiser, attitudes développées chez les enfants subsistent-elles?

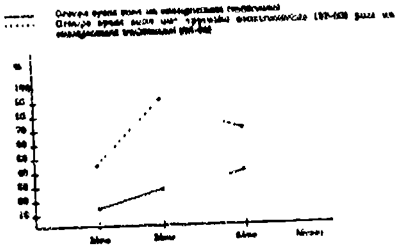
Quelles sont les influences chez l'enfant de la confrontation entre une approche constructiviste et un enseignement traditionnel des mathématiques suivi trois ans après notre intervention: maintien, régression ou disparition de certaines habiletés ou attitudes, conflits provoqués par l'interaction des deux approches?

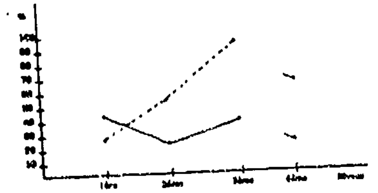
Une épreuve a été construite à partir du cadre de référence sur la numération développé dans la précédente recherche [8] et expérimentée sous forme d'entrevues. Des 23 élèves suivis au cours de l'intervention, il ne reste que 16 élèves (groupe expérimental). Afin de pouvoir répondre aux questions que nous nous posions, nous avons constitué un groupe contrôle formé d'élèves de la même école (même milieu, même niveau) que les élèves du groupe expérimental. Ces élèves ont suivi pendant tout le primaire un enseignement traditionnel des mathématiques.

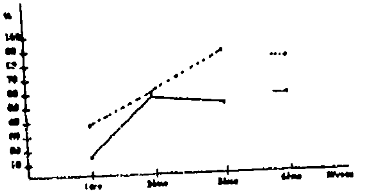
RESULTATS PRELIMINAIRES

Les graphiques présentés ci-dessous reprennent quelques-uns des résultats du groupe ayant suivi l'approche constructiviste à la fin de chacune des années de l'intervention, et trois ans après l'intervention. Des résultats plus détaillés et exemples seront fournis lors de la présentation.

Les items auxquels ces graphiques réfèrent de la 1ère à la 3e année, puis en 6e année, sont de plus en plus complexes en termes de cheminements, habiletés et représentations mises en jeu (cf. cadre de référence sur la numération).







CONCLUSION

En observant les différents graphiques, nous pouvons observer une évolution importante des enfants ayant suivi l'approche constructiviste entre la 1ère et la 3e année relativement à la signification qu'ils accordent à l'écriture et à l'habileté à opérer avec les groupements dans des tâches de plus en plus complexes. Trois ans après cette intervention, les habiletés, procédures développées par les enfants subsistent encore. Ainsi, 69% des enfants (fig. 4) accordent une certaine signification à ce qu'ils font en termes de groupements lorsqu'ils ont à opérer avec un matériel relativement complexe dans un contexte de division. Ces mêmes enfants (fig. 3) voient la pertinence d'utiliser les groupements pour coder une collection.

Nos résultats mettent cependant en évidence des conflits provoqués par la confrontation des deux approches. Dans des tâches complexes, deux conceptions différentes entrent alors en conflit chez l'enfant.

Ces résultats illustrent combien une approche constructiviste des mathématiques implique une intervention de longue durée dans la classe. Cette approche constitue en effet beaucoup plus qu'une variation pédagogique; elle est une remise en question complète de la conception traditionnelle que nous avons de la numération et de son apprentissage.

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KINDERGARTNERS' KNOWLEDGE OF NUMBERS: A LONGITUDINAL CASE STUDY

PART I: INTUITIVE AND PROCEDURAL UNDERSTANDING

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This paper reports the results of a longitudinal case study aimed at following a child's construction of the number concept during the kindergarten time-frame. The subject was met three times during the school year, in October, January, and May. Each assessment required four to five 20-minute interviews. The questionnaire used to this effect was developed within the theoretical framework provided by a model of understanding which identifies four levels in the construction of a mathematical concept: intuitive understanding, procedural understanding, abstraction and formalization. Results show that the child progresses simultaneously at many levels and that the questionnaire offers a perspective broad enough to study the evolution of his numerical profile. This study is reported in two companion papers dealing respectively with the first two levels of understanding (Part I), and with the last two levels (Part II).

The children's acquisition of number starts very early. Gelman & Gallistel (1978), as well as Fuson et al (1982), have shown that the first number words are learned as early as the age of three. But is is between the ages of five and six that one witnesses an outburst in their knowledge of the number-word sequence for, by then, most of them can recite it beyond 30 (Herscovics et al, 1986a). We have found that this major quantitative development is accompanied by major qualitative developments in their perception of number (A. Bergeron et al, 1986; Herscovics et al, 1986b; J.C. Bergeron et al, 1986). Our investigations were cross-sectional studies of different groups of kindergartners. And thus, they left open the question of how children evolve during this period. Only a longitudinal type of study would provide information on this subject.

While the mastery of the number-word sequence does not by itself constitute an understanding of natural numbers, it is nevertheless an essential prerequisite. Only when the child perceives that this sequence can be used to determine the quantity of objects in a discrete set or the rank of an object in an ordered set, is the notion of number involved. In this sense, the cardinal and ordinal aspects of number can be

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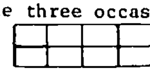
viewed as a measure of quantity or as a measure of the rank of an object. Of course, the child does not build overnight the notion of number as a measure of quantity or rank. In fact, four distinct levels can be identified in the construction of this conceptual scheme: intuitive understanding, procedural understanding, abstraction, and formalization.

We have been working for several years on the identification and refinement of criteria which might describe these four levels of understanding of number (Herscovics & Bergeron,J.C., 1983). These criteria have been used to develop a questionnaire made up of various tasks and questioning sequences aimed at probing the child's thinking processes. Running through these tasks and questions with children between five and six requires four to five interviews, each one lasting from 15 to 20 minutes, about the attention span of this age group. Four kindergartners were selected by their teacher on the basis of their willingness to work with us. In order to study their evolution, the questionnaire was administered individually at three different periods of the school year, October, January and May. Each set of interviews was video-taped. At present, we have completed the analysis of one of the four case studies. In this paper we report the criteria describing each level of understanding, the tasks and questions used to assess them, and the evolution of the subject.

INTUITIVE UNDERSTANDING

The level of cognition which we call "intuitive" relates to the informal knowledge acquired through life experiences, outside any formal instruction. For many mathematical concepts, one can find their embryonic presence in this informal knowledge. These germinal ideas can be considered as pre-concepts. For young children, these pre-concepts are quasi-physical and initially relate to concrete objects and their actions on these objects. Intuitive understanding results from a type of thinking heavily influenced by visual perception. For pre-concepts of an arithmetical nature, this translates into visual estimation and primitive actions which do not yet provide a numerical answer.

An intuitive understanding of number involves those notions which can be viewed as pre-concepts of its cardinal interpretation and its ordinal interpretation. Since in a cardinal perspective number can be viewed as a measure of quantity (the number of elements in a discrete set), the notion of quantity can be considered a pre-concept of number. Indeed, for discrete sets, one can deal with the notions of "more than, less than, or the same as" without any counting process, by simple visual estimation. For more accurate results, a one-to-one correspondence can be used. In an ordinal perspective, number can be viewed as a measure of the rank of an object. This presupposes that the set has been ordered and that the child perceives this order. Thus the notion of order can be considered as another pre-concept of number. For instance, in a row of objects, the child does not have to resort to any counting to determine if an object is placed "before" or "after" another one. Given two rows, he can use a one-to-one correspondence to determine if objects in each row have the same rank or not.

Our subject was a little boy, Philippe, aged 5:7 at the time of the first interview, in October. At the level of intuitive understanding no change was observed in the three sets of interviews (Oct., Jan., May). Comparing a set of 25 cubes with another set of 7, he could use the words "more" and "less" to identify the two sets. When asked to judge "just by looking", if one set of randomly disposed cubes (8) had more, less, or the same as a second set (8), he answered that they were the same. When further asked how he could make sure, on the three occasions he arranged each set into the same rectangular array. 

The classical Piagetian "conservation of number" task has been retained to further probe the child's intuitive understanding.In this task,subjects are asked to compare two rows of objects,before and after one row is elongated.Since no counting is involved,the comparison is purely in terms of quantity, not number. Our subject, Philippe, could use a one-to-one correspondence to lay out a row of seven cubes similar to a given one. However, at no time during the year did he succeed in this conservation task. Interestingly, he was focusing on the density of the rows rather than their length, for he always felt that the shorter row had more cubes. In January and May, he still had to resort to a one-to-one correspondence in order to determine if the two rows had the same quantity or not. At the time of this experimentation, no tasks had been set to investigate the ordinal aspect of number at the intuitive level. Ordinal tasks were included at the level of procedural understanding.

KNOWLEDGE OF THE NUMBER-WORD SEQUENCE

In order to investigate the subject's knowledge of the number-word sequence, the tasks we have used are those set by Fuson et al. (1982). The following table provides a description of the corresponding skills in each set of interviews:

	October	January	May
1.Can recite from 1 up to	13	14	19
2.Can recite from 1 and stop at a given number	yes(8)	yes(9)	yes(17)
3.Can recite starting at a given number	yes(4)	yes(4)	yes(9)
4.Can recite from a given number up to another one	no(4 to 7)	yes(3 to 7)	yes(11 to 18)
5.Can recite backwards from a given number	yes(6)	yes(5)	yes(6)
6.Can recite backwards from a given number down to another one	yes(5 to 3)	yes(5 to 2)	yes(6 to 3)

What the table does not convey is the remarkable change which occurred between the interviews. In October, Philippe was shy and retiring, and repeatedly complained that he did not like counting for he found it very difficult. At that time he was confusing two of the counting words "six" and "dix", since the difference in pronunciation is quite subtle, being limited to the initial consonants. By January, he found counting much easier, and since he was still confusing the two number-words we proceeded to distinguish them by producing a hissing sound for the first letter in "six": "sssix comme dans ssserpent" ('sssix like in sssnake'). He found this quite amusing and seemed relieved to discover that the two number-words were not the same. Curiously, the kindergarten teacher had never noticed this problem. Again in May, we found another great change, not only in his greater scope but also in the facility with which he handled the various tasks.

PROCEDURAL UNDERSTANDING

The informal mathematical knowledge we have described as intuitive can be considered as an initial level of understanding. It can be used to initiate an ensuing stage of mathematization, that of the acquisition of relevant mathematical procedures. Relating these mathematical procedures to the learner's intuitive knowledge justifies their need and helps prevent meaningless memorization. Conventional mathematical procedures are seldom discovered spontaneously by children. They are usually constructed by them following some socially transmitted information and convention, through schools, television, parents, peers,etc. The gradual mastery of mathematical procedures as well as their appropriate use constitutes a second level of cognition which we call procedural understanding.

We need to distinguish between the recitation of the number-word sequence and the act of enumeration. The number-word sequence is acquired by memorizing a set of conventional words in an appropriate order whereas enumeration is a procedure establishing a one-to-one correspondence between the number-words and a set of objects to be counted. Children may often know how to recite the sequence without necessarily coordinating it correctly in their enumeration procedure. They often count the same object twice or skip some others. At a young age, getting different results from different counts of the same set does not seem to bother them (Ginsburg, 1977). The learning of the number-word sequence and of the enumeration procedure will usually be mastered by the age of six. By that age, children have no problems in using the conventional number-words and perceiving their role in enumeration. In this paper the words "count" and "counting" will be used as synonyms for "enumeration".

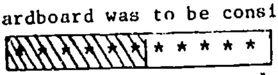
Simple counting tasks. In the first task the child was presented with a set of blocks, within the range of his known number words, and asked "how many" there were, in order to verify if these words were meaningful to him. To assess his ability to generate a required set, he was also asked to put a given number of cubes in a dish. In the second task, the subject was provided with a sample plate of seven cubes, and another four plates containing respectively 6, 7, 8 and 9 cubes were placed in front of him. He was then asked to identify the plate which had "one more", then "one less", then "the same". The third task consisted in finding the seventh cube in a given row of 15 cubes, and then in identifying the rank of an indicated cube. Finally, in order to compare the enumeration skills with his knowledge of the number-words, the child was asked to count up a set of cubes greater than the known sequence and told to count as far as he could. He was then asked to count and stop at a given number of cubes.

The following table describes the student's success in each set of interviews.

	October	January	May
1.Responds to "how many" by counting sets of	9 (T)	13 (P)	15 (P)
-generates sets of	7	9	10
2.Can identify set with "one more" than seven	does not understand	?	yes
Can identify set with "one less" than seven	does not understand	?	yes
Can identify set with "the same" as seven	yes	yes	yes
3.Can identify 7th cube in a row	yes	yes	yes
Can find the rank of indicated cube	yes (6)	yes (9)	yes (9)
4.Can enumerate cubes up to	13	14	19
5.Can enumerate and stop at	11	8	8

Our subject Philippe was able to handle most of these problems. The letters next to the first task indicate the specific way he counted the cubes. In October he counted them by touching (T) the cubes without moving them, while in the other two interviews he partitioned (P) the set into counted and uncounted cubes. On the second task, it was quite surprising to find that in October Philippe did not understand the meaning of "one more" and "one less". Although in January he succeeded in answering these questions, he seemed so hesitant as to make us wonder if he knew what he was doing. By May, however, he handled these tasks with great aplomb. It will be noted from the fourth task that for this child, the enumeration range corresponded exactly to his knowledge of the number-word sequence.

Counting partially hidden rows. The next set of tasks involved rows of chips with one end partially hidden by the interviewer. The reason for these was to provide a set of problems calling for more sophisticated counting strategies such as counting on or counting back from a given number. In the first task the child was shown a row of chips glued on a cardboard and told: THIS IS THE FIRST CHIP. I'M HIDING SIX OF THEM (covering the first six chips with another piece of cardboard). CAN YOU TELL ME HOW MANY CHIPS ARE GLUED ON THE WHOLE CARDBOARD? (gesturing to show that the whole cardboard was to be considered).



Within the above context, the student was also asked to identify the ninth chip. Since these problems could be handled by counting on from the hidden part, we verified if the subject had acquired this procedure. This was assessed by asking him to count on from the sixth chip in a completely uncovered row. The reason for delaying this assessment was to prevent us from suggesting any procedure to be used on this first task.

Another task was designed to provide a situation in which counting back would be a good strategy. The child was shown a row of 11 chips, three of them being hidden. The rank of a visible chip was given and the subject was asked to find the number of hidden chips. In a second question, he was again given the rank of a chip and was asked to find the rank of another one. Finally, his ability to count backwards was determined using a completely visible row of chips. The following table describes his performance during the year.

	October	January	May
1. With 6 chips hidden, can find total	no	no	yes
With 6 chips hidden, can find 9th	no	no	yes
2. Can count on from 6	-	yes	yes
Can count on from 6 and stop at 9	-	yes	yes
3. Given rank of a visible chip (9th), can find how many are hidden (3)	-	no	no
Given rank of a visible chip (9th), can find rank of another chip	-	yes (8 to 6)	yes (9 to 6)
4. Can count backwards from	5	5	6
Can count backwards and stop at given number	5 to 3	5 to 2	6 to 3

In October Philippe was unable to handle the first task. He initially counted only the visible chips. On a second attempt he counted them figurally (cf Steffe et al.,1983) that is, he pointed his finger over the hidden part and counted the imagined chips. Although he knew that six chips were hidden, his count did not correspond to that number. In the January interview, he did not fare any better although the assessment shows that he could count on from six (in task 2). By May he was still using figural counting but now, although the spacing between the imagined chips was too small, his finger jumped to the end of the hiding cardboard as he was pronouncing "six". Regarding the third task, perhaps it proved to be too difficult because the starting number was too high and/or the gap between the starting number and the hidden set was too large. For, as was found in task 4, the subject could only count backwards extensively when the numbers were below seven.

Double-counting. Well before children are formally introduced to subtraction, many of them can handle classes of problems in which they occur, by using a process of double-counting. For example, when asked how many numbers are between three and seven on a snakes and ladder gameboard, these children will count the number-words needed to go from three to seven. Double-counting procedures are fairly advanced since the things counted are no longer concrete, in the physical sense, but number-words one pronounces. Fuson et al. (1982) have identified four such procedures. The following table describes how our subject has handled them:

	October	January	May
1. Counts five number-words starting from	3	4	7
Counts the number-words between	5 and 7	3 and 5	6 and 9
2. Counts n number-words before a given number	no (3 nos before 5)	yes (2 nos before 5)	yes (4 nos before 8)
Counts (backwards) the number-words between	no (5 and 3)	no (6 and 3)	yes (6 and 2)

Philippe's counting skills are well below the average kindergartner's who can handle enumerations up to about 39 (Herscovics et al., 1986a). However, as the results in this table indicate, our subject can use very sophisticated counting procedures as long as one stays within his numerical range. The double-counting in the second task proves to be very difficult since the two counts move in opposite direction as when asking the child IF YOU START AT SIX AND COUNT BACKWARDS TO TWO, HOW MANY WORDS WILL YOU SAY ? Philippe was able to handle this question in May despite the confusing crossing of the sequences as in " 6 is one, 5 is two, 4 is three, 3 is four, 2 is five ".

An analysis of the remaining two levels of understanding, as well as a discussion of the results, and the references, appear in a companion paper, "Part II: Abstraction and Formalization", by Anne Bergeron, Nicolas Herscovics, and Jacques C. Bergeron.

KINDERGARTNERS' KNOWLEDGE OF NUMBERS: A LONGITUDINAL CASE STUDY

PART II: ABSTRACTION AND FORMALIZATION

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This paper reports the results of a longitudinal case study aimed at following a child's construction of the number concept during the kindergarten time-frame. The subject was met three times during the school year, in October, January, and May. Each assessment required four to five 20-minute interviews. The questionnaire used to this effect was developed within the theoretical framework provided by a model of understanding which identifies four levels in the construction of a mathematical concept: intuitive understanding, procedural understanding, abstraction and formalization. Results show that the child progresses simultaneously at many levels and that the questionnaire offers a perspective broad enough to study the evolution of his numerical profile. This study is reported in two companion papers dealing respectively with the first two levels of understanding (Part I), and with the last two levels (Part II).

In a companion paper, Part I: Intuitive and Procedural Understanding (N. Herscovics, J.C. Bergeron, A. Bergeron), we have described the theoretical framework and the methodological concerns which have led us to undertake a longitudinal case study. While that article deals with the first two levels of understanding, Part II presents an analysis of the remaining two levels of understanding, abstraction and formalization, as well as a discussion of the results, and the references.

ABSTRACTION

As mentioned in Part I, intuitive understanding results from the kind of knowledge acquired outside any formal instruction. On the other hand, the gradual mastery of mathematical procedures and the ability to apply them in appropriate situations characterize a second level of comprehension, that of procedural understanding. A third level involves mathematical abstraction. This must be distinguished from abstraction in the usual psychological sense which refers to a progressive detachment from concrete objects and also to a gradual interiorization of the procedures enabling one to anticipate the result. At the beginning, an emerging concept is often blurred and confused with the procedure leading to its construction (for instance, the notion of number is often confused with the counting procedure). It is only very gradually

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that the "outline" of a concept gains precision, that it separates from the procedure, and that it starts having an existence of its own in our mind. But even then, this existence is somewhat unstable and does not withstand various transformations. For example, a child may very well have detached himself from concrete objects, as evidenced by his success in double-counting when enumerating number-words, and yet may not have discovered that changing the configuration of a set does not change its cardinality. When the learner becomes aware of the invariance of the mathematical object (in this case number) under transformations as above, he achieves a third level of understanding, that of mathematical abstraction.

Piaget's work on the child's conservation of number (Piaget & Szeminska, 1941/1967) can be viewed in terms of mathematical abstraction. As mentioned in Part I, Piaget's classical conservation task deals with the invariance of quantity since no enumeration is required in the comparison of the two rows of objects. This was in fact recognized by Greco & Morf (1962) since they modified the conservation task by having children count the unstretched row and then asking them to predict the number of objects in the elongated row. Those who succeeded were said to conserve quotity, which is an indication of the invariance of the measure of quantity. Greco and Morf found that about 75% of the twenty 5-year-olds they had interviewed did not believe that the two rows had the same quantity even after having counted both the short and the elongated rows. Thus, for these children, the result of enumeration was not yet perceived as a measure of quantity. Perhaps the words "to conserve number" should be used to describe those who conserve both quantity and quotity.

The distinction between conservation of quantity and conservation of quotity can also be found in the much simpler context of a single set. In his case studies Ginsburg (1977) had found that some young children could enumerate the same set several times and obtain different results without experiencing any cognitive conflict. Our own results (Herscovics et al., 1986b) show that by the time they are in kindergarten, few children (3 out of 22) would accept the possibility of different counts for the same set. However, these same children may or may not conserve quantity even in the simplified context of a single set, in which there no longer is the conflict induced by the presence of two visually different configurations. The failure to conserve quantity for a single set was observed by Piaget (1973) and Gelman & Gallistel (1978) who noted that after counting a given set, many young children could not predict that a change in its configuration or a change in the order of enumeration would result in the same cardinality.

In our study, we used two tasks to investigate different aspects of the invariance of number using only one set. In the first one, we asked the subject how many cubes were in a given row (12) and then asked him how many he would find by counting them in the other direction. In the three sets of interviews (October, January and May), our subject, Philippe, could predict without hesitation that he would get the same number. This invariance of the result of enumeration with respect to the direction of the count can be viewed as conservation of quotity in the context of a single row.

The second task also dealt with the conservation of quotity, but with respect to changes in the configuration of a same set. The subject was asked how many cubes were in a randomly disposed set (9). After the blocks were counted the interviewer spread them out and asked: IF WE PUT THEM LIKE THIS, HOW MANY ARE THERE? Care was taken not to hide any cubes while spreading them out in order to prevent the child from thinking that some cubes might have been taken away or added surreptitiously. In the three sets of interviews, Philippe could not tell how many cubes were in the changed configuration and ended up counting them again. When queried about it, he was quite explicit in stating that he needed to count them. This was confirmed by another change in the configuration which induced a third enumeration of the set.

As mentioned in Part I, this child did not conserve quantity in the Piagetian test using two rows of cubes since he could not deduce that elongating one row would not change its cardinality. However, the effect of enumerating the rows proved to be interesting:

- In October, when counting the two rows he jumped from 6 to 9 on the elongated row as if attempting to reach a higher number in the count. The task was repeated with two rows of 6 cubes. Again, counting the elongated row, Philippe jumped from 5 to 9, but maintained that the shorter row had more "because they (the cubes) are closer together". Claiming that she had not seen how he counted, the interviewer asked him to do it again. On this second trial, Philippe found 6 for both rows but still claimed that the shorter row had more. Queried about what he would do to "have the same" (in French "pour en avoir pareil"), he simply aligned the cubes next to each other to get two identical rows.

-In January Philippe could not decide visually if the two rows had the same number of cubes and needed to count them. But he counted them in corresponding pairs pointing with his finger ("one-one, ..., seven-seven") and concluded that "they have the same" ("Il y en a pareil"). The ta-k was repeated with the elongated row now reduced to a length sl orter than the other row. Exactly the same responses were obtained. On a third trial, with the previously longer row shortened, Philippe did no longer count them stating "There are 7 and 7,...., the same,.. (because) I counted them several times".

-In May, when asked if the two rows had the same number of cubes, he stated that he could not know and that he had to count them. He started to count the cubes in pairs as he had done in January, then proceeded to count each row seperately, concluding that "they have the same (number)" and adding "It doesn't look like it". When the elongated row was reduced to one shorter than the other, he refused again to express an opinion based on visual perception for otherwise "I would be saying just anything", indicating that it would be purely a guess on his part. After counting the two rows he concluded that they had the same number.

A detailed description of Philippe's responses is important for it indicates some major changes at the level of abstraction. In October, his thinking based on visual perception dominates, and counting the two rows does not create any cognitive conflict. He still maintains that the shorter row has more. And it cannot be claimed that he does not understand the word "more" since this was tested in a previous task at the level of intuition. Furthermore, the child explains that he thinks the shorter row has more on the basis of its density ("they are closer together"). By January, two major changes have occurred. He no longer feels he can express an opinion based on visual perception. Moreover, the role of enumeration has evolved for it now becomes the criterion by which he will judge if the two rows have the same number of cubes or not. No other change could be observed in May, except for the subject's greater facility in verbalizing his reasoning.

While the changes observed between October and May are significant, one should not conclude that Philippe conserves number. We might be inclined to think that he does since after enumerating both rows he affirms that "they have the same (number)". But what does it mean? It means that when he enumerated the two rows he arrived at the same count. However, one cannot argue that for him enumeration is a measurement of quantity since he does not yet conserve quantity. He does not yet seem to realize that quantity can only be changed by adding or taking away some cubes. When the two rows are the same length he agrees that they have the same number but as soon as the spacing of one of the rows is changed, he refuses to state an opinion. Thus Philippe has not yet abstracted the notion of quantity ; hence the question of number as measure of quantity is simply irrelevant in his case.

FORMALIZATION

The fourth level of understanding, that of formalization, takes into account the particular importance of symbolization in mathematics. Several studies have shown that the symbolic representation of mathematics creates specific cognitive problems (Ginsburg, 1977; Carpenter & Moser, 1979). Since mathematical notation brings about an increase in cognitive problems, it would be tempting to identify symbolization as a fourth level of understanding. However, the work of Erlwanger (1973) has shown that children can learn to produce and manipulate symbols giving them purely idiosyncratic interpretations. This has led us to consider symbolization as relevant to a fourth level of understanding only if prior abstraction of the concept has occurred to some degree.

Thus, we consider the children's use of numerical notation as the formalization level of their understanding of number, only if they have achieved some degree of abstraction of this concept. In studying children's spontaneous written representation of small numbers (up to 9), researchers have identified four distinct stages. When asked to send a message describing the number of objects on a table, children will at first draw pictures of the objects in front of them and draw as many pictures as there are objects; at a more advanced stage, they will use tally marks; still later, they will write out the number sequence as far as the number of objects present; finally, when they no longer feel the need to rely on any written trace of their one-to-one correspondence, they will count up the objects in the set and write down the number indicating its cardinality (Sastre & Moreno, 1976; Allardice, 1977); Sinclair,A. et al., 1983; Bergeron,J.C. et al., 1986). Of course, there are many variations borrowing from each of the above representations.

Regarding the writing of numbers exceeding nine, their symbolization is complex since it necessitates the use of more than one digit. Ginsburg (1977) and M. Kamii (1980) have shown that children can write such larger numbers well before they understand the place value interpretation of the notation, that is the value associated with the numeral's position (e.g. the first digit in 12 represents ten). In writing and recognizing numbers greater than nine, the appropriate concatenation of two digits must be viewed globally (e.g. for "12" to mean twelve, it must not mean "one and two"). This is what is meant by positional notation. J.C. Bergeron et al. (1986) have identified three stages in the children's acquisition of positional notation. Initially, at the juxtaposition stage, they are aware that two digits are written side by side, but their relative position is not yet viewed as important. When they do become aware of the importance of their relative position, they may not as yet perceive it from a reader's viewpoint but instead focus on the order in which they write. This shows up when they are asked to write from right to left and write twelve as "21", first writing 1 and then 2. This stage, respecting the order of the writing, can be called the chronological stage. The third stage, the conventional one, is achieved when, regardless of the direction of the writing, the notation produced will convey the intended meaning.

The tasks we have designed for the assessment of this level of understanding involve either the generation of numerals or their recognition. At first, we asked our subject to write out the numbers he knew. In October, Philippe wrote all the numbers up to 14, which is interesting, since this exceeded his knowledge of the number-word sequence, which went up to 13. For small numbers under 10, he was inverting the numerals 3, 4, 7 and 9. For numbers greater than 9, the fact that he could write them up to 14 indicates that he was aware of positional notation. It was also evident that he was at the juxtaposition stage in his acquisition of this convention. This is shown by the fact that when he wrote double-digit numbers such as 10, he wrote from left to right but started with zero and got "01". Had he written this from right to left, starting with 1, he would have been at the chronological stage. In January he could enumerate sets of up to 14 objects, but he could write numbers up to 19 without knowing the number words past fourteen. He still inverted 3, 4, 6, 7 and 9, but he no longer interchanged the order of the double-digit numbers. By May, his enumeration skills went up to 19. The single digits 2, 3, 4, 7 and 9 were inverted while he still maintained the right order in writing the double-digit numbers. We have some evidence that he had reached beyond the chronological phase since in writing "16" and "17" he had forgotten to write the "1". He corrected himself spontaneously and wrote "1" on the left. Thus the order of the writing did not seem to have any effect on him. We might be tempted to consider that he had reached the conventional stage. However, when tested for the recognition of written numbers shown on flash cards, in the three sets of interviews he identified both $\boxed{12}$ and $\boxed{21}$ as "twelve". Although he could not conceive of numbers greater than 19, had he achieved the conventional phase,he would have known that $\boxed{21}$ was "twelve written backwards".

The second task on formalization consisted in laying out some cubes (9) in front of the child and asking him: I WOULD LIKE YOU TO WRITE A MESSAGE TO A FRIEND TO TELL HIM HOW MANY CUBES THERE ARE HERE. In October, Philippe wrote out the first nine numbers, a behavior which reflects his need to maintain a one-to-one correspondence between the set of cubes and the numerals he wrote. When asked to write how many chips were in two paper plates (3 and 7 respectively), he again wrote out the corresponding sequences of numerals. In January, his message consisted of a castle with six little man-like figures corresponding to the six cubes in front of him. Using these little figures as tally marks reflected his more fanciful mood. However, for the five chips in the dish question, he wrote " 1, Ƨ, Ɛ, 4, 5". In May, he went back to writing a message with the first nine digits to represent the nine cubes in front of him, and wrote the first seven digits to represent the seven chips in the dish. Regarding the recognition of numerals, his ability to generate sets of cubes corresponding to numbers written on flash cards was tested in the three sets of interviews. At all times, Philippe succeeded in producing the required sets (5, 8 and [illegible] respectively).

DISCUSSION

As mentioned in our companion paper, by the end of kindergarten, the average child can handle enumerations up to 39. Thus, our subject Philippe was well below this average. In fact, his counting scope did not increase by much since he could enumerate up to 13 in October and up to 19 in May. But the quality of his numerical thinking had greatly evolved. By May,

- he could solve all but one problem involving a hidden part;
- he could handle problems requiring double counting with counts going in opposite direction;
- he refused to express an opinion based on the visual comparison of two rows of cubes;

- he perceived enumeration as the criterion by which he could compare the two rows;
- he had progressed in his perception of positional notation from an initial juxtaposition stage to a level somewhere between the chronological and conventional stages.

This case study very clearly shows that within a fairly restricted range of numbers, the child's numerical thinking can become quite sophisticated. Furthermore, it also shows that between the ages of 5 and 6, we witness an important development in his construction of the number schema.

The model of understanding we have used as our theoretical framework suggests four levels of comprehension. However, it would be a mistake to perceive it as a linear model in the sense that a given level can only be achieved after all the steps of the preceding level have been covered. As our case study has shown, the child evolves simultaneously at many levels. In fact, Philippe expanded his procedural understanding of number at the same time as he was progressing at the levels of abstraction and formalization. The four levels of understanding and the various criteria used for each level can be viewed as a cognitive matrix. Each element in this matrix is reflected by a sequence of questions and a task in our questionnaire. The child's responses to the questionnaire provide us with an overview of his thinking which can be considered as his "numerical profile". Our case study involved three sets of four to five 20-minute interviews with the same subject and yielded an in-depth assessment at three different times, October, January, and May. Thus, we were able to follow the evolution of his numerical profile over a key developmental period in his construction of the number concept. Many specific aspects of the child's number knowledge have been investigated in prior work by many researchers. However, our questionnaire is a tool providing us with a more global perspective. And it is only from a global perspective that one can really follow the child's construction of this conceptual schema.

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KINDERGARTNER'S CONCEPTION OF NUMBERS

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We report upon a case study with 8 German preschool children (age 5;1-5;8). The investigation covers a broad variety of number activities and they are carefully listed (part 2). But we also discuss the purpose of this research (part 1) and the value of it for both prospective teachers and researchers (part 4). Therefore the evaluation (part 3) summarizes not along the investigations but along typical number concept skills.

1. Purpose of Our Research

As a preservice teacher training institute for mathematics education we combine our research responsibilities with our teaching responsibilities and conduct three types of investigations:

- o Informal observations of children together with our prospective teachers.
- o Systematic observations of children together with our prospective teachers (including protocols and videotapes).
- o Systematic research by the help of prospective teachers. (They have to write a final examination report, some of them work on a thesis, etc.)

Thus the prospective teachers get experiences in analysing learning situations and in designing research investigations. The protocols and tapes are also used for other teacher training courses. The repetition of similar investigations with different children (and often different prospective teachers) lifts step by step the quality of the research.

There is a basic difference according to "pure" research. We mainly do not concentrate on learning more and more about more and more specialized details. We rather try to learn as much as possible about the general process of learning in the broad field of mathematics education to develop suggestions for a better mathematics teaching. Our research is goal-directed and therefore comparable with the research of engineering sciences. We are not "pure" but "applied" mathematics education researchers. It is not necessary that our results always reach the same scientific level. The level is more determined by the purpose: Getting informations to a specific question in the field of teaching mathematics.

In this study we will report upon an investigation of the "research" type (Bremerich 1986). The investigation should collect data for teacher training courses about the knowledge of pre-school children to base the arithmetic teaching curriculum in grade 1 upon more realistic assumptions.

2. The Design

The investigation lasted 4 days in a kindergarten of a German city. The first day only was used to become acquainted with each other (the two interviewers played with the children, told stories, discussed with them and studied children books together). Starting on the second day three girls and five boys (age 5;1-5;8) were interviewed and video taped individually for about 20 minutes each according to the following outline.

A. Conservation of number

A.1. The interviewer (I.) puts 6 blocks in a row and places a plate with additional blocks in front of the child (subject, S.): "I'm sure you've a friend. What's her/his name? (F.) We now play, that these blocks (pointing to the row) belong to F. Take the same amount for you (from the plate). ... Do you both have the same now? ... Put your blocks in front of the blocks from F."

A.2. I. widens F.'s row, etc.

A.3. I. shortens F.'s row, etc.

B. Associations to a number

I.: "I'm sure you know already some numbers. What do you remember with 'four'?" (Paper and pencil and playing-cards with dots or with digits are available).

C. Counting

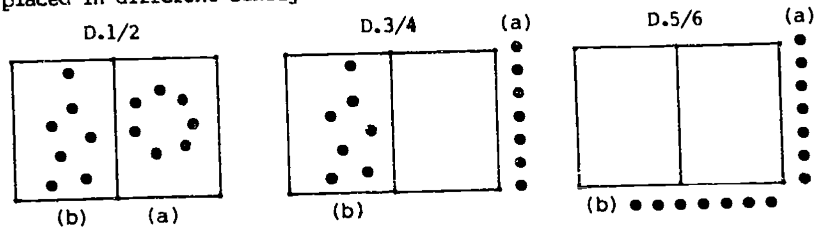
C.1. I.: "I'm sure you already can count." ... Stop at 42. If earlier: "Can you count on?" - If stopped at 42: "What do you think how far you can count?"

C.2. I.: "And which is the biggest number at all?"

C.3. I.: "Now start counting with 3." Help if necessary: "Which number comes after 3?" - Stop at 11. If successful: "And now start with 18." - Stop at 26. "Can you start with every number?"

D. Comparing Quantities

S. gets a box divided into two parts, each part filled with blocks placed in different configurations:



D.1. I.: "Are there both the same? ... Why?" (Or: "Which is more? ... Why?")

D.2. I. pointing to (a): "Please count, how many are there?"

D.3. I. replaces the blocks: "How many blocks are there now (in the row)? ... Are there as many as before (in the box)? ... Why?"

D.4. I.: "Count how many blocks there are (in box (b))."

D.5. I. replaces the blocks and continues like in D.3.

D.6. I. removes the empty box and points to the rows: "Are there both the same? ... Why?" (Or: "Which is more? ... Why?")

D.7. I. widens row (a), questions like in D.6.

D.8 I. shortens row (b), questions like in D.6.

E. Addition

E.1. S. gets a plate with 20 small green plastic frogs: "Do you know these animals? ... If you have 3 frogs and you get 2 more, how many frogs do you have then? ... How did you find out?"

E.2.a.: "And if you have 4 frogs and you get 5 more ..."

E.2.b. If S. does not use animals in E.2.a.: "Take 4 frogs. ... Take 5 more. ..."

E.3.a. I. removes the plate with frogs and replaces it by a plate with 15 larger red plastic bears. ... "Take 5 bears. ... Take another one. ... How many do you have now?"

E.3.b. I.: "Take another one. ... How many do you have now?"

E.4. I.: "You are pretty good, we'll try it now without animals (removes them). ... How much are two and one?"

E.5. I.: "And how much are three and two?"

E.6. I.: "And how much are four and three?"

F. Symbolization

F.1. I. "I'm sure you've already seen (written) numbers. ... Which number is this?" (I. shows a playing-card with the digit 3). ... "And this?" (playing-card with 7)

F.2. I.: "I've some more cards. (Order in the pile: 2, 4, 5, 1, 8, 6, 9). Could you put them in the right order?" ... After having all cards in a row ... "Please read the numbers and point to them."

F.3. Only if all numbers are in the correct order, I.: "We'll now play a game. I'll remove one card and you'll tell me which. Please close your eyes." ... I. removes the 4 and hides the blank by pushing the remaining cards together: "Which is missing?" ... The missing card then will be replaced. Another game with 6.

F.4. I. places with one grasp 3 frogs and 2 bears in front of S. and points on the row of playing-cards: "Where are they to place?"

F.5. I. presents a playing-card with the digit 0: "I forgot one card. ... What is on the card? ... Where shall we place it? ..."

F.6.a. I.: "I also have some other cards (with dots on it). ... Where is that card to place (3 dots)?"

F.6.b. I.: "And this (6 dots)?"

F.6.c. I. points to the 2 in the row: "And which (dot) card belongs to here?" S. gets the pile of dot-cards, order in the pile: 1, 5, 2, 4, 8, 0, 9, 7.

F.6.d. I.: "Try to place to each number the correct (dot) card."

F.7.a. All playing-cards are removed, I. shows a xylophone: "I'm sure you know that. ... I'll hit some sounds and you'll count how many there were." I. hits 6 times the same bar ... "How many were there?"

F.7.b. I. hits 6 different bars following to the gamut: "How many were there?"

F.8. I.: "Now it's your term. Hit as many bars as I show you on these cards." I. shows the digit-card 3, then 7.

F.9. I. shows a card with 4 dots, then with 6 dots.

3. Evaluation

The sequence of the questions was chosen to reduce interferences between the items. For an evaluation it is more useful to concentrate on the different number aspects. Here we can only report tendencies of our case study. Most of them can be generalized according to other investigations we did.+)

3.1. Associations with "Four" (part B)

Most children associate 5 or 3 or counting or counting on. Some write (more or less correct) numerals or point to digits they see.

+) But also with unlimited space we would not end up with statistical data. (What does it mean for example, "the mean was 86 and the standard deviation 37" when 12 children had to count, but not farer than 121?)

3.2. (Rote) counting (C.1, C.2)

All count till 10, most of them stop between 18 and 29. The biggest number sometimes is the last number of the last count, sometimes 100 or 1000. Some do not understand, but Henry (5;3): "There is no biggest number, because you always can take one more".

3.3. Counting on (C.3)

Most children can start with 3, but only a few with 18.

3.4. How many?, via counting (D, F.7)

In D about 80% manage the one-one principle and the stable-order principle (Gelman/Gallistel 1978), all succeed with the cardinal principle. In F.7 the gamut sequence is easier, most fail counting monotonic sounds.

3.5. Conservation of number (D.3, D.5)

About 50% conserve, about 50% count again.

3.6. Quantities to a given number (E.2.b, E.3)

About 35% take one after one while counting, about 40% grasp subsets of two or three and put them together or grasp two and then continue by ones.

3.7. Counting or subitizing? (F.6, see also 3.4)

All subitize till 3, 50% till 4.

3.8. Addition (E.1-E.6)

Most children have no real concept except "getting more".

3.9. Conservation of quantity (A.1-A.3), D.6-D.8)

25% use counting and 75% one-to-one-correspondence to produce their row in A.1, more than 50% do not conserve though some children count (correctly) several times (A.2, A.3). However in D.6-D.8 more than 50% conserve without any doubt arguing with the cardinal numbers being the same.

3.10. Comparing cardinality (D.1, see also 3.5 and 3.9)

About 30% count, about 30% compare "Gestalt".

3.11. Identifying numerals (F.1)

Most children recognize 3, only a few know 7.

3.12. Ordering numerals (F.2, F.3)

About 50% find the correct order and also the missing numbers.

3.13. Zero (F.5)

About 50% know "zero", but most of them put 0 behind 9.

3.14. Pattern recognition (F.4, F.6, F.8, F.9)

No child regards the animals as a quantity of 5 (F.4). Almost all children first count the dots and then place the dot card. Only a few start at a number and then look for an appropriate dot card. These strategies do not depend on the direction of the given task (F.6.a/b versus F.6.c/d). About 35% of the children finish the one-to-one-correspondence without mistakes. F.8 and F.9 seem to be of the same difficulty, about 50% hit the bars correctly.

4. Interpretation

The summary gives a small impression about the immense amount of informations we got. But what real knowledge did we get by that study? To order and to understand the details of our observations, the relations, the similarities and the contradictions, we need a more general view of the children's conception of numbers. We first discover that there are some "elementary" abilities like rote counting, unitizing, recognition of certain patterns, rote subitizing, simple magnitude discrimination, rhythmic motor activities, etc. They seem to develop more or less independent to each other. Other abilities are more complex like counting on, enumeration, counting objects, conservation of quantity, conservation of number, comparing quantities by counting, etc.

Studying the relations between all these abilities we can compare the development of the number concept with a growing network. The growth is characterised by the development of many independent or isolated small network pieces, which we called elementary abilities, and the growing together of these network pieces by developing relations and connections between them. The network gets expanded and structured by age step by step. Elementary abilities grow together to more complex networks, i. e. to complex abilities. Studying this network is like zooming a landscape, sometimes we concentrate on details and sometimes upon more global aspects.

We can "apply" this knowledge to improve both teaching and research. The teacher student learns observing, analysing and classifying the pupils' abilities. This enables him to adapt his future teaching more closely to the needs of the pupils. From this study the prospective teacher learnt, that school beginners know much more about numbers than the grade 1 school books assume according to the curriculum. He learnt that he probably should hurry through some parts of the books. His teaching of numbers might become less boring and he might get more time for other topics.

But we also learn to read and interpret other research reports. In our seminars we can discuss or compare different investigations and we can try to explain divergencies. For example Klahr/Wallace (1976) regard subitizing as an (in our terms) "elementary" ability, while Gelman/ Gallistel (1978) redefine subitizing as a "complex" ability. We feel both are right, but they speak about different parts at different developmental periods in our network.

Regarding the developing number concept as a network also explains a more general phenomenon which by Straues (1979) is called U-shaped behavioral growth: Some global and limited but very effective abilities ("elementary" ones) get into conflict with other growing up abilities and finally get replaced by more powerful "complex" abilities (e. g. subitizing, conservation of quantity, pattern recognition, simple magnitude discrimination, ...).

At the conference we will discuss these aspects in more detail.

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(For the lack of space in these proceedings a more detailed list of references will be distributed at the conference or mailed upon request)

CONCEPTUAL AND PROCEDURAL KNOWLEDGE AS RELATED TO INSTRUCTION ON SYMBOLIC REPRESENTATION

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Children's pre- and postinstructional concrete and symbolic representations of addition and subtraction word problems are examined according to growth of conceptual and procedural knowledge, and instruction on symbolic representation was designed to promote this growth. Results of instruction on the representations of 45 first-graders are presented and related to conceptual and procedural learning.

A recent focus of the literature on children's mathematical thinking is directed at distinguishing between children's conceptual and procedural knowledge (Hiebert, 1986). In terms of learning, Hiebert and Lefevre (1986) stated that, in essence, conceptual learning involves constructing relationships or connections between two pieces of information. They suggested that these connections can be constructed at either a primary or a reflexive level. At the primary level, the plane of abstraction is important to the construction of a connection, as the new information must be of equal or less abstractedness than the prior information to which the new is intended to connect. At the reflexive level, conceptual learning occurs when similar cores of different pieces of information are recognized and the connection is made between these similar cores.

A major difference between procedural and conceptual learning in mathematics, according to Hiebert and Lefevre, concerns the type or context of the information processed during procedural learning, specifically that of mathematical symbols and syntax and mathematical rules and strategies. They suggested that the appropriate aim of mathematics instruction is the promotion of both conceptual and procedural knowledge, but added that much of the content of current programs in school mathematics involves instruction to promote only procedural learning. They surmised that the linking of procedural to conceptual knowledge may have two potential benefits, a better understanding of procedures and a resultant ease in remembering appropriate procedures.

When young children come to school, they appear to have both conceptual and procedural knowledge in mathematics. The presence of both types of knowledge is apparent in their preinstructional performances on addition and subtraction word problems. With concrete items available, young children's strategies reflect a substantial amount of conceptual and procedural knowledge (Carpenter, 1986). Their conceptual knowledge is evident in their informal insights into the structure of various types of word problems as indicated by their predictable concrete representations. This knowledge is termed conceptual because children appear to recognize similar cores in the semantic structure of certain problem types; their informal strategies of solution appear linked to problem semantics and not to problem syntax. Their procedural knowledge is evident in their informal fluencies in translating the problem statement into mathematical symbols and performing accurate solution strategies on these symbols.

The detailed descriptions of children's thinking into conceptual and procedural categories are not intended as ends in themselves. These distinctions between conceptual and procedural knowledge are valuable for providing direction to prescriptive or instructional treatments and for better understanding students' failures and successes in the learning process (Hiebert & Lefevre, 1986).

This paper describes a recent instructional treatment that was planned to coincide with and capitalize on children's preinstructional conceptual insights into the structure of addition and subtraction word problems and their procedural knowledge about symbols and solution strategies. Instruction was designed to promote conceptual learning by establishing a connection between children's entering concrete representations based on problem structure and the number sentences that symbolically represented problem structure. Instruction was designed to promote procedural learning by making a translation between the entering concrete representation and the appropriate mathematical symbols and syntax.

METHOD

The sample consisted of 45 first-graders in two classrooms during the spring months. Instruction was provided to each classroom in fourteen sessions over a five-week period and was organized according to the effective mathematics teaching model of review, development, and practice.

The children were evaluated before and after instruction with word problem tasks in two settings, individual interviews and classroom group tests. Their preinstructional concrete representations during the individual interviews were used to categorize the sample into three entering levels, Basic, Direct

Modeling, and Rerepresentation. (See Bebout, 1986, for details.) The symbolic representations, number sentences, that children wrote before and after instruction were evaluated as to form and accuracy.

The eight word problems of instruction were Change and Combine types. (See Carpenter & Moser, 1983; Riley, Greeno, & Heller, 1983, for classification of word problems.) The order of instruction for each addition and subtraction problem pair along with the symbolic representation that coincides with the structure of each problem type are displayed in Table 1. These problem pairs and number sentences were introduced in the following instructional order: Change 1 and 2 ($A \pm B = ?$), Combine 1 and 2 ($A \pm B = ?$), Change 3 and 4 ($A \pm ? = C$), and Change 5 and 6 ($? \pm B = C$).

Table 1
Order of Instructional Presentation of Word Problem Types

Word Problem	Number Sentence	
	Form	Type
Change 1 & 2	$A \pm B = \square$	Result Unknown
Combine 1 & 2	$A \pm B = \square$	Result Unknown
Change 3 & 4	$A \pm \square = C$	Change Unknown
Change 5 & 6	$\square \pm B = C$	Start Unknown

Table 2
Summary of Instructional Steps for Each Word Problem Type

Instructional Step		Type of Representation		Type of Solution
(1) Verbal Problem	→	Concrete Model	→	Manipulative
(2) Verbal Problem	→	Concrete Model		
		Number Sentence	⇄	Manipulative Symbolic
(3) Verbal Problem	→	Number Sentence	⇄	Manipulative Symbolic
Verbal Problem	→	(4) Number Sentence		

A summary of instructional steps for each problem pair is displayed in Table 2. Briefly, the steps indicate that instruction for each pair proceeded in the following way: 1) concrete representation of the word problem; 2) concrete representation of the word problem replaced by symbolic representation; 3) symbolic representation of the word problem without the concrete representation; and 4) composition of a word problem to match a given symbolic representation. Direct instruction was provided on representation; indirect instruction was provided on solving the representation, thus allowing individual solutions of the number sentence with concrete, counting, or number fact strategies.

Each of these instructional steps focused on the development of conceptual and procedural knowledge. Step One was designed to emphasize and reinforce the conceptual and procedural knowledge that most children brought to instruction. Conceptual learning involved representing problem structure with concrete items. Procedural learning involved manipulating these items to determine a solution.

Step Two was designed to stress procedural learning by introducing a number sentence to match and thus replace the concrete representation, the previously established conceptual knowledge of Step One. Conceptual learning involved establishing a connection at the reflexive level between the similar elements in the concrete representation and the matching symbolic representation. Procedural learning focused on the formal mathematical symbols and syntax to replace the concrete representation.

Step Three was designed to promote conceptual learning at the reflexive level by emphasizing the similar cores between the word problem structure and the symbolic representation that matched this structure, without the intervening concrete representation. Conceptual knowledge at this point was planned to build on the procedural knowledge of number sentence elements and syntax that had been introduced in Step Two.

Step Four of instruction was designed to promote conceptual learning at both the primary and reflexive levels by teaching children to compose a word problem to match a number sentence. At the primary level, conceptual learning involved constructing a connection between an abstract symbolic representation and a realistic word problem. At the reflexive level, conceptual learning involved recognizing the core of a given number sentence and composing a word problem with a similar core. Procedural learning was promoted in reverse to most procedural learning in mathematics by providing first of all the mathematical symbols and translating them into text. Step Four was used daily as a closing to e instructional session.

In summary, instruction was designed to promote both conceptual and procedural knowledge. An interplay between the two was planned for each instructional step. Conceptual learning was designed to build on JJpreviously learned procedural knowledge, and procedural learning was plannea to obtain meaning through its connection with previously learned conceptual knowledge.

RESULTS AND DISCUSSION

Data on children's pre- and postinstructional representations for the Change 3 problem are presented in Table 3. (Data for other problem types will be presented in a detailed version of this paper.) The symbolic representations that children wrote before and after instruction are displayed according to their entering levels and types of concrete representations.

Table 3
The Relation Between Preinstructional Conceptual Knowledge and Symbolic Representation on Change 3 Type Word Problems

Preinstructional Concrete Representation			Preinstructional (Postinstructional) Symbolic Representation					
Strategy		Number Responding	*A+?=C	*C-A=?	A+C=?	A-C=?	NA	Other
*Add On								
	**B	0	- -	-	- -	-	-	-
	DM	21	4(19)	1	9 (2)	1	3	3
	RR	7	3 (7)	1	2 -	1	-	-
*Separate From								
	B	0	- -				-	
	DM	0	- -				-	
	RR	3	2 (3)				1	
Count All								
	B	5	- (5)		3	1	1	
	DM	0	- -		-	-	-	
	RR	0	- -		-	-	-	
Repeat Given								
	B	3	- (3)	-	2 -	1	-	
	DM	1	- -	-	- (1)	-	1	
	RR	1	- (1)	1	- -	-	-	
Guess								
	B	3	1 (2)		2 (1)			
	DM	0	- -		- -			
	RR	0	- -		- -			
No Attempt								
	B	1			1 (1)			
	DM	0			- -			
	RR	0			- -			

* Appropriate Representation

** Basic Level N=12
Direct Modeling Level N=22
Rerepresentation Level N=11

For all 12 children at the Basic level, their preinstructional concrete representations were inappropriate and displayed no conceptual insight into the structure of the Change 3 problem. Before instruction, one child wrote a number sentence that directly represented the problem structure and the remainder wrote inappropriate number sentences or made no attempt. The predominant number of inappropriate sentences were ones in which the two given numbers were added. After instruction, 10 of the 12 Basic level children wrote sentences that matched the structure of the problem.

For the 22 children at the Direct Modeling level, 21 of them displayed entering conceptual knowledge of the structure of the Change 3 problem through their use of the Adding On strategy. Before instruction, 5 of them wrote appropriate number sentences: 4 that represented the problem structure and one that represented the matching basic subtraction fact. Thirteen wrote inappropriate number sentences: 9 that added the given numbers and 4 that were of other forms. After instruction, 19 of the 22 children at this level wrote number sentences that represented the problem structure. Only 3 wrote inappropriate sentences that added the given numbers.

For the 11 children classified at the Rerepresentation level, 10 of them displayed entering conceptual knowledge for solving the Change 3 problem. Seven concretely represented the problem structure and 3 concretely represented the matching basic subtraction fact. Before instruction, 6 of these 10 wrote appropriate number sentences: 5 that represented the problem structure and 1 the matching basic fact. Three children wrote inappropriate sentences that added the given numbers and one made no attempt. After instruction, all 11 children wrote sentences that represented the problem structure.

In summary, 31 children from the total sample of 45 displayed conceptual knowledge about the Change 3 problem as evidenced by their appropriate preinstructional concrete representations. Eleven of these 31 were successful in writing appropriate number sentences before instruction, 9 that represented the problem structure and 2 that represented the matching basic subtraction fact. After instruction, 29 of these 31 wrote number sentences that represented the problem structure. The remainder of the total sample, 14 children, did not display conceptual knowledge with their concrete representations. Only 2 were able to write appropriate preinstructional number sentences. After instruction, 11 of these 14 wrote number sentences that represented the problem structure.

The postinstructional results indicate that children at all three entering levels of concrete representation were more successful in writing mathematically accurate number sentences for the Change 3 problem. Conceptual learning may have been enhanced by recognition of the structural core of the word problem and by connection of this core to procedural learning that focused on symbols and syntax of structurally-based number sentences.

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THE EFFECTS OF SEMANTIC AND NON-SEMANTIC FACTORS ON YOUNG CHILDREN'S SOLUTIONS OF ELEMENTARY ADDITION AND SUBTRACTION WORD PROBLEMS

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Over the past few years a substantial body of research has yielded evidence that the semantic structure of elementary arithmetic word problems influences children's strategies to solve them. In the present paper a study is reported in which three other, non-semantic task characteristics were involved, namely (1) the order of presentation of the two given numbers, (2) the order of presentation of the two given sets and (3) the size of the difference between the two given numbers. While data collected from collective as well as individual tests yield evidence in favour of the impact on children's strategy choice of the first and the second variable, they do not support the importance of the third factor.

INTRODUCTION

Recent research on simple addition and subtraction word problems (Carpenter & Moser, 1984; Riley, Greeno & Heller, 1983) has been mainly focused at the influence of one particular kind of task characteristics on children's problem solving, namely the semantic relations among the quantities described in the problem (Change, Combine and Compare) together with the identity of the unknown quantity. The results of research in our Center over the past few years are generally consistent with the findings of these investigations (De Corte & Verschaffel, 1987, in press). However, our work suggests at the same time that - in addition to the semantic structure - some other task characteristics may also have an important effect on children's problem-solving processes. In the present study three such characteristics were involved, namely :

(1) The order of presentation of the two given numbers (larger or smaller first), e.g. "Pete had 3 apples; Ann gave him 9 more apples; how many apples does Pete have now ?" versus "Pete had 9 apples; Ann gave him 3 more apples; how many apples does Pete have now ?"

(2) The size of the difference between the two given numbers (large or small), e.g. "Pete had 3 apples; Ann gave him 9 more apples; how many

apples does Pete have now ?" versus "Pete had 5 apples; Ann gave him 6 more apples; how many apples does Pete have now ?"

(3) The order of presentation of the two given sets in Change problems (regular versus inversed order), e.g. "Pete had 3 apples; Ann gave him 9 more apples; how many apples does Pete have now ?" versus "Ann gave Pete 9 more apples; he started with 3 apples; how many apples does Pete have now ?"

Some of these task variables have already been examined in previous research, but only with respect to their effects on problem difficulty. Surprisingly little research exists on their influence on children's solution strategies for word problems. Moreover, little or no effort has been made to relate the effects of these task characteristics to the problem's semantic structure.

DESIGN, TECHNIQUES AND DATA SOURCES

The present study consists of two parts : a paper-and-pencil test was administered collectively to eighty five children; afterwards a smaller group of twenty pupils was individually interviewed.

The collective test contained twenty elementary addition and subtraction word problems representing four different types from the well-known classification scheme of Riley et al. (1983) : Change 1 and Combine 1 (both involving addition) and Change 3 and Combine 2 (both involving subtraction). For each problem type, several variants were constructed by combining the three task characteristics mentioned above. However, one should note that the third characteristic - normal or inversed order of presentation of the two given sets - applies only for Change problems. Moreover, for Change 3 problems, this task characteristic coincides with the first one - the order of presentation of the two given numbers - as the start set necessarily contains the smallest given number. The following three number pairs were used for addition problems with a small and a large difference between the two given numbers respectively : (5,7), (7,8), (5,6) and (4,8), (5,9), (3,9). For the subtraction problems we used (13,9), (11,8), (12,9) and (11,3), (12,3), (13,4) respectively. The test was administered collectively in the beginning of the school year to four second-grade classes, with a total of eighty five children. They were not only asked to solve the problems, but also to write down for each problem a number sentence showing which arithmetic operation had been performed to find the solution.

In the middle of the next school year, a group of twenty first graders was individually interviewed using exactly the same set of word problems. Each problem was read aloud by the interviewer and the children were asked to solve it and to explain their solution strategy.

Both the collective and the individual test provided systematic data on the relative difficulty of the distinct types of word problems, on the strategies children used to solve them and on the nature of their errors. In the present paper we focus on the influence of the task characteristics on the solution strategies of the good problem solvers. In this respect we mention that the percentages of children who answered a problem correctly were much higher for the addition (Change 1 and Combine 1) than for the subtraction problems (Change 3 and Combine 2) during both the collective and the individual test. Consequently, the data on children's solution strategies for subtraction problems are based on a considerably smaller number of cases than those for addition.

Place restrictions preclude us to give an overview of the hypotheses and results concerning all three task characteristics involved in the study. Therefore, we leave out the third characteristic, only mentioning that no considerable differences in solution strategies were found depending on the size of the difference between the two given numbers.

We finally point to the fact that our data on children's strategy choice for the distinct problem types were not submitted to a statistical analysis. The main reason is that they are based on different numbers of children; moreover from problem to problem the children involved in the analysis were not the same.

HYPOTHESES AND RESULTS

With regard to addition problems, we distinguished between strategies in which the child begins with the first of the two given numbers (F-strategies) and strategies starting with the second one (S-strategies). For problems in which the first given number is the smaller one, S-strategies are more efficient. By disregarding the given order of the addends and starting with the larger one, the number of steps in the cognitively demanding "double count" is reduced to a minimum (Carpenter & Moser, 1984).

A first plausible hypothesis is that addition problems in which the smaller addend is given first will be solved more frequently by S-strategies than problems in which the larger number is given first (Hypothesis A1).

However, it is also expected that the ease with which children will alter the order of the two given numbers in their solution strategy will depend on the semantic structure of the problem. More specifically, it is predicted that Combine 1 problems starting with the smaller given number will elicit more S-strategies than the corresponding normal Change 1 problems (Hypothesis A2a). The argument underlying this hypothesis is that the dynamic nature of the Change 1 addition problems will invite children to model the described chronological sequence of events in their solution strategy. Because Combine problems have no implied action, altering the order of the two given sets and starting with the larger one seems less problematic. For a similar reason - namely the tendency to solve Change problems using a strategy that parallels the sequence of events described in the problem - it is hypothesized that inversed Change 1 problems will elicit more strategies starting with the second given number - i.e. the start set - than normal Change 1 problems, in which the start set is given first (Hypothesis A2b).

Table 1 gives the percentages of strategies starting with the first and the second given number on Combine 1, normal Change 1, and inversed Change 1 problems during the collective and the individual tests.

Table 1. Percentage of F- and S-strategies on addition problems during the collective and the individual tests

First given number	Structure	Collective test Strategies		Individual test Strategies	
		F	S	F	S
Smaller	Combine 1	86%	14	37	63
	Change 1 normal	91	9	52	48
	Change 1 inversed	77	23	39	61
	Total	85	15	43	57
Larger	Combine 1	99	1	83	17
	Change 1 normal	97	3	86	14
	Change 1 inversed	92	8	66	34
	Total	96	4	78	22

The results are in line with hypothesis A1. Children reported much more frequently that they had solved a problem with a S-strategy when it started with the smaller number than when the larger addend was given first both during the collective and the individual tests. This finding suggests that children's solution strategies are indeed strongly influenced by the location of the smaller and the larger given number.

It was further predicted that Combine 1 problems will elicit more S-strategies than normal Change 1 problems when the smaller number is given first (Hypothesis A2a), and also that inversed Change 1 problems will provoke more S-strategies than normal Change 1 problems both in the larger-given-first and the smaller-given-first condition (Hypothesis A2b). The results in Table 1 are in line with both hypotheses. With respect to problems starting with the smaller given number, children seemed to find it easier to use the more efficient S-strategies in the context of Combine 1 and inversed Change 1 problems, than when the problem had a normal Change 1 structure, especially on the individual test. The same trend occurs in the collective test results, although less strong. With respect to the problems in which the larger number is given first, inversed Change 1 problems obviously elicited the highest percentage of S-strategies in the individual test situation; the difference is again in the predicted direction on the collective test, but very small.

With respect to subtraction problems, we are especially interested in the influence of the distinct task characteristics on the choice of either a direct subtractive or an indirect additive solution strategy. In a direct subtractive (DS) strategy the answer is found by subtracting the smaller given number from the larger one; in an indirect additive (IA) strategy, the child determines what quantity the smaller given number must be added with to obtain the larger one (De Corte & Verschaffel, in press).

Based on arguments similar to those underlying our first prediction for addition problems, we hypothesize that subtraction problems starting with the larger given number will elicit more DS-strategies than those in which the smaller number is given first.

Second, we hypothesize that the influence of the order of presentation of the given numbers will interact with the semantic structure underlying the subtraction problem. More specifically, we expect that the implied joining action between the known start set and the unknown change set in Change 3 problems will elicit a large amount of IA-strategies, even when the order of presentation of the given numbers favours a DS-strategy. For Combine 2 problems, on the other hand, the choice of either a IA- or a DS-strategy will be influenced more obviously by the position of the given numbers.

The most remarkable finding for the subtraction problems was children's apparently very strong tendency to use IA-strategies, especially during the individual tests : on a total of 78 appropriate solution strategies, only four DS-strategies were observed. Consequently, the discussion is restricted to the data from the collective test (see Table 2).

Table 2. Percentages of DS- and IA-strategies on subtraction problems during the collective test

First given number	Structure	DS-strategies	IA-strategies
Smaller	Change 3	16%	84
	Combine 2	18	82
	Total	17	83
Larger	Change 3	22	78
	Combine 2	43	57
	Total	33	67

The results shown in Table 2 are in line with both predictions. First, we observed considerably more DS-strategies for problems starting with the larger given number than for problems in which the smaller number was given first. This finding supports the hypothesis that the order of presentation of the two given numbers has an influence on the kind of strategies children use to solve subtraction problems (see also De Corte & Verschaffel, in press).

Second, it was expected that the effects of the order of presentation would interact with the semantic structure of the problem. More specifically, we assumed that the influence of this task characteristic would be greater for Combine 2 problems than for Change 3 problems. The results show that Combine 2 problems starting with the smaller and the larger given number elicited indeed considerably different percentages of DS- and IA-strategies : while Combine 2 problems beginning with the smaller given number were solved much more frequently with IA- than with DS-strategies, the percentages of IA- and DS-strategies were much closer when the larger number was given first. For Change 3 problems starting with the smaller and the larger given number, the distribution of DS- and IA-strategies was almost alike : most children continued to apply IA-strategies even when the larger number was given first. These findings confirm the hypothesis that the influence of the order of presentation of the given numbers is not alike for all semantic problem types.

DISCUSSION

Over the past few years a substantial body of research has yielded evidence that the semantic structure of simple addition and subtraction word problems seriously influences children's solution processes. The results of this study are certainly not in conflict with this well-documented finding but rather complementary. Indeed, our data show that with respect to young problem solvers considerable differences in solution strategies can occur within a given semantic problem type, depending on other task characteristics, i.e. the position of the two given numbers and the order of presentation of the sets in the problem text. Moreover, the results reveal that the effects of these two additional task characteristics on children's solution strategies are not alike for all semantic problem types. These findings are not only helpful in explaining apparently conflicting results from different previous empirical studies involving the same types of word problems, but they also provide guidelines for improving and elaborating the available theoretical (computer) models of young children's skill in solving elementary arithmetic word problems.

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ELEMENTARY SCHOOLCHILDREN'S PROCESSING OF "CHANGE" ARITHMETIC WORD PROBLEMS

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ABSTRACT

The responses of 1195 children, aged from 5 to 12 years in 67 classes in 15 elementary schools to six arithmetic word problems of the "Change" variety were analyzed. While results agreed with recent research findings that Change 3, 5, and 6 questions are relatively more difficult than Change 1, 2, and 4 questions, the data were completely at variance with the information processing model, for Change problems, proposed by Riley, Greeno, and Heller.

BACKGROUND

By the now familiar classification system of verbal arithmetic word problems developed by Heller and Greeno (1978), arithmetic word problems which can be solved by the application of an addition or subtraction operation belong to one of three categories, **Change**, **Combine**, and Compare. Other writers have suggested other categories (e.g. an Equalize category - see Carpenter & Moser, 1984), but the Heller and Greeno system has greatly influenced recent research carried out by both North American workers (see, for example, Carpenter, 1983; Riley, Greeno, & Heller, 1983) and European workers (De Corte & Verschaffel, 1985).

In the present paper data are presented which cast doubt on the usefulness of influential addition /subtraction word problem-solving models put forward by Riley et al. (1983). While the authors have reported, elsewhere, data arising from a wider study involving **Change**, **Combine**, and **Compare** questions (Clements & Del Campo, 1986), we shall confine our attention, here, to an analysis of our **Change** data.

In **Change** questions there are three relevant sets (termed 'Start", "Change" and "Result") and depending on the context, the action involved in the question can cause an increase or decrease in the "Start" set. Thus, there are six different sub-categories within the major **Change** category. According to the Heller and Greeno (1978) classification the six **Change** type are as indicated in Table 1:

Table 1

The Six Change Types Distinguished by Heller and Greeno (1978)

Problem Type	Direction	Unknown
Change 1	Increase	Result set
Change 2	Decrease	Result set
Change 3	Increase	Change set
Change 4	Decrease	Change set
Change 5	Increase	Start set
Change 6	Decrease	Start set

Riley's Analysis of Processes for **Change** Problems

According to Riley's analysis the patterns of performance on Change problems are as shown in Table 2 (in which the **Change** questions are those used by the present writers - see Clements & Del Campo, 1936).

In Table 2 a "+" indicates that a correct answer is given, a "NA" indicates no answer, and numbers represent characteristic errors for the questions. Thus, reading vertically, a Level 1 respondent would answer Questions 1, 2, and 4 correctly, but would give the result set as the answer for the **Change** 3 question and would not attempt an answer for the **Change** 6 question. A Level 1 response to the **Change** 3 question is the result of the solution set not being available for inspection; instead, the focus is, erroneously, on the display of the result set (with 5 bananas). However, since for **Change** 4 questions, the change and result sets are physically separate, the Level 1 respondent can identify the change set (which is required).

It is reasoning such as this which is presented in the Riley et al. (1983) paper. The investigations carried out to test the theory have, typically, used small numbers of children aged from 4 to 8 years, usually in no more than two or three educational institutions. Because there has been a need to discover the children's patterns of thinking the investigations have been clinical in nature.

Table 2

Patterns of Performance on Change Problems (After Riley)

Examples of Questions	Levels of Performance		
	1	2	3
Result Unknown			
1. Barbara had 2 eggs Dan gave Barbara 1 more egg. How many eggs did Barbara have then?	+	+	+
2. Jack had 4 pens. Dianne took 3 of Jack's pens. How many pens did Jack have then?	+	+	+
Change Unknown			
3. Jeff had 3 bananas. Carmel gave Jeff some more bananas. Then Jeff had 5 bananas. How many bananas did Carmel give Jeff?	"5"	>	+
4. Anna had 6 books. Tom took some of Anna's books. Then Anna had only 2 books left. How many of Anna's books did Tom take?	+	+	+
Start Unknown			
5. Paul had some pencils. His father gave him 2 more pencils. Then he had 5 pencils. How many pencils did Paul have at the start?	"2"	"2"	+
6. Sally had some pictures. She lost 2 of her pictures. Then she had 3 pictures. How many pictures did she have at the start?	NA	NA	+

If, indeed, the proposed theories are close approximations to the truth, then data sets based on a large number of children in a range of educational institutions should be supportive. In the study which will now be described, which was part of a larger study investigating Change, Compare, Combine, and other arithmetic word problems (Clements & Del Campo, 1986) the Riley theories for **Change** problems were subjected to such a test.

METHOD

Sample

During the period April-June 1986, 1195 children in 67 classes in 15 elementary schools in the Eastern suburbs of Melbourne were asked to solve 24 arithmetic word problems. Children in each of Grades Prep, 1, 2, 3, 4, 5 and 6 were involved, the numbers of children at each level being 29, 65, 89, 258, 280, 256 and 218 respectively. The average age of children in Prep Grade was about 5½ years and the average age of children in the other grades increased steadily to about 11½ years in Grade 6.

The Research Instrument, and its Administration

The research instrument consisted of 24 questions, 6 of which were of the **Change** variety (these are given in Table 2). These 6 questions were randomly spread throughout the 24 questions. The other questions were of the Combine, Compare, Direct Comparison and Ratio types, and analysis of data pertaining to them has been presented elsewhere (Clements & Del Campo, 1986).

The 24 questions were administered to students in Grades Prep, 1 & 2 on a one-to-one basis, the average time of administration being about 45 minutes. Students in these grades were provided with relevant equipment (dolls, model cars, pencils, etc) and invited to use this in formulating their solutions. They were instructed to listen to each question carefully, and told that if they wanted to do so they could use the equipment to help them to get answers; then each question was read to them slowly, twice, and they were invited to give an answer; then, having given an answer, they were asked to repeat the story and the question, in their own words, and to show the meaning of the story by using the equipment.

Students in Grade 3 attempted the 24 questions as a class group. Each student had a copy of the 24 questions and was told to read each question silently as it was read to them by the person administering the research instrument. The questions were administered in a lock-step, one-by-one manner, with the students being allowed plenty of time to complete each question.

Students in Grades 4 through 6 were given written copies of the 24 questions and told they had to read and answer each question without any assistance being provided. The questions were not read to them.

RESULTS

Table 3 shows the percentage of incorrect or "no answer given" responses made by students in Grades Prep through 6 on the six Change questions.

By and large, the entries in Table 3 confirm the findings of recent research that Change 1 and Change 2 questions are relatively easy for young schoolchildren, that Change 4 questions are the next easiest, and that Change 3, Change 5 and Change 6 questions are relatively difficult. Table 4 shows the proportions of students in the present study who would be in the Riley levels and sublevels. According to Riley et al. (1983, p. 173), Level 1a children respond correctly to Change problems 1 and 2 only, Level 1b children respond correctly to Change problems 1, 2 and 4 (as for the Level 1 pattern in Table 2), Level 3a children give correct responses to all questions except either a Change 5 or a Change 5 question, and Level 3b children respond correctly to all six Change questions.

Now, since a 1a pattern requires that students get the Change 1 and Change 2 questions correct and all other Change questions incorrect, and the Change 3a pattern requires that all questions be answered correctly except either a Change 5 or a Change 6 question, and the Change 3b pattern requires that all six Change questions be answered correctly, it is not clear from the entries in Table 4 whether the proportions in the rows for sublevels 1a, 3a and 3b provide any support for Riley's hypothesized model. Other models could easily explain such results. In fact, qualitative data obtained from our interviews with the children in our sample in grades Prep, 1 and 2, and from our teaching of 22 classes each week over the period June-September 1986 (designed to elicit information concerning how children process

Table 3

Difficulty Indices (% Wrong) of Change Questions

	QUESTION	% Wrong in Grade ... Prep	1	2	3	4	5	6
(CH 1)	Barbara had 2 eggs. Dan gave Barbara 1 more egg. How many eggs did Barbara have then?	10	11	10	5	3	3	3
(CH 2)	Jack had 4 pens. Dianne took 3 of Jack's pens. How many pens did Jack have then?	10	14	11	5	3	1	1
(CH 3)	Jeff had 3 bananas. Carmel gave Jeff some more bananas. Then Jeff had 5 bananas. How many bananas did Carmel give Jeff?	52	37	27	20	14	7	8
(CH 4)	Anna had 6 books. Tom took some of Anna's books. Then Anna had only 2 books left. How many of Anna's books did Tom take?	24	22	19	15	6	6	6
(CH 5)	Paul had some pencils. His father gave him 2 more pencils. Then he had 5 pencils. How many pencils did Paul have at the start?	62	29	34	24	16	7	7
(CH 6)	Sally had some pictures. She lost 2 of her pictures. Then she had 3 pictures. How many pictures did she have at the start?	52	43	35	24	15	11	6

Table 4

Proportion of Response Pattern Consistent with Riley's Model for Change problems

LEVEL	P	1	2	3	4	5	6
1a	.03	.05	0	0	0	0	0
1b	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3a	.31	.22	.17	.16	.12	.07	.05
3b	.14	.29	.36	.52	.67	.80	.83
Residual	.52	.45	.47	.32	.21	.12	.12

verbal arithmetic problems - see Clements & Del Campo, 1986), indicate that the strategies hypothesized in Riley et al. (1983) were rarely, if ever, responsible for 1a, 3a and 3b response patterns. Furthermore, the entries in Table 4 for the sublevel 1b and Level 2 rows could not be more overwhelming in their rejection of the Riley model.

DISCUSSION

We do not know why our data should be so much at variance with those obtained by North American and European researchers. Perhaps, parents and teachers of young children in Victoria stress different language and thinking patterns in their everyday interactions with children. Perhaps, too, the Riley model has not been subjected to sufficiently rigorous testing by North American and European researchers.

While, in our study, the six **Change** questions were administered in different ways to children in Grades Prep through 2, 3, and 4 through 6, entries in Table 3 indicate that, generally speaking, student performance on the questions improved with increase in grade level and this suggests that the effect of different administration methods was not great.

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THE EVOLUTION AND EXPLORATION OF A PARADIGM FOR STUDYING MATHEMATICAL KNOWLEDGE

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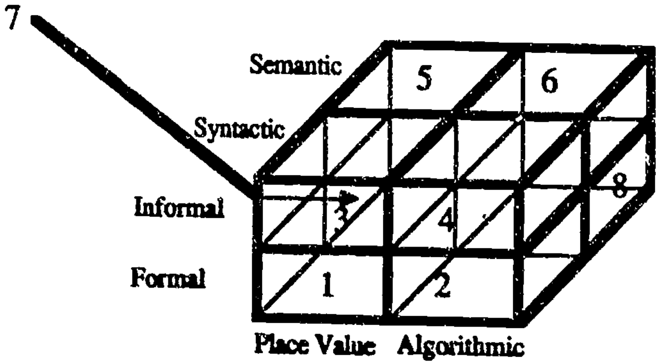
An analysis of the literature in the area of children's learning of subtraction led to an embryonic model of the types of knowledge that might be brought to bear on a given subtraction problem, a model with its roots in cognitive psychology, mathematics education and educational anthropology. Each of these fields contributes a dimension to such a model. To explore the usefulness of the resulting paradigm, a protocol procedure was developed to assess the extent to which these types of knowledge interact. A preliminary analysis of a case study using this procedure is reported. Some proposed extensions and improvements are also briefly presented.

The first part of this paper describes some of the foci of the model of long term memory when it is used to study children's knowledge of two-digit subtraction (see Figure 1). The model is depicted as a 2 (Place Value/Algorithmic) X2 (Semantic/Syntactic) X2 (Informal/Formal) matrix. The first dimension of this matrix, place value/algorithmic, was suggested by the Kent State Checklist of Mathematical Concepts (Underhill, Uprichard & Heddens, 1980) among others, to include both place value concepts and subtraction algorithm levels of complexity in diagnostic contexts. The second dimension, semantic/syntactic, has also been suggested in numerous sources and was articulated in this domain by Resnick (1982). In such work, the interrelationship between algorithmic performance on semantic (e.g., Dienes' blocks) and syntactic (e.g., symbol manipulation) subtraction tasks is explored. The final dimension, informal/formal, is used widely among cultural anthropologists (e.g., Carraher, Carraher and Schliemann, 1983; Lave, Murtaugh and de la Rocha, 1984) when conducting ethnographic research. People often have rich informal repertoires of arithmetical knowledge which they use in informal settings (e.g., market place or bowling alley). This knowledge is contrasted with formal knowledge which results from instruction in classrooms.

This application of the model was used to develop eight question types for exploring children's knowledge of two-digit subtraction. Relations among the eight cells will be presented. A more detailed discussion of the model will then follow.

Figure 1. A three-dimensional model for exploring understanding of whole number subtraction



Place Value/Algorithmic Dimension

Algorithmic knowledge is essentially a step-by-step procedure for completing a mathematical task (often depicted as a flowchart). In this case, the procedure for determining the solution to two-digit subtraction problems is the child's subtraction algorithm. The primary conceptual area included in the model is generally place value. Place value knowledge is a collection of concepts and sub-skills that are prerequisite to understanding the algorithm. In the domain of subtraction, the major place value concepts include representation of a number, tens and ones, and regrouping. These two types of knowledge were used to develop protocol questions for all eight cells in the matrix. The other two dimensions of the matrix define the context and task environment on which these concepts are explored.

Semantic/Syntactic Dimension

Syntactic knowledge (Resnick, 1982), as it is being used here, is the knowledge of the language of mathematics. This is usually demonstrated in a written format, but may also be verbal. The language of mathematics is a highly specialized problem solving language and is sometimes taught for the sole purpose of using the language. The semantics of mathematical knowledge refers to the underlying meaning of this language (syntax). Learners who are proficient mathematically are not only competent in the language of mathematics; they also understand from where the language is derived. Operationally, Resnick (1982) defines the syntax in terms of paper and pencil tasks involving symbol manipulations. Performance of such tasks using manipulatives is a way of observing and assessing semantic knowledge.

Informal/Formal Dimension

Informal and formal knowledge have been studied by ethnographers and other qualitative and quantitative researchers. Carraher et al. (1983) and Lave et al. (1984) describe people who have extensive informal knowledge in arithmetic, but who are unable to apply their knowledge in formal settings. Formal knowledge, on the other hand, is what teachers intend to teach.

The "formal" plane of the matrix has been extensively studied by diagnosticians, while the "informal" plane has been studied only minimally by mathematics educators. In addition, the rich set of interrelationships among the cells in the matrix have been studied hardly at all. The three dimensional framework facilitates the systematic, intensive study of children's understanding of not only subtraction algorithms, but many other areas of mathematical knowledge. By exploring the interrelationships among the eight cells, this model holds promise for facilitating the development of diagnostic procedures, delineation of prescriptions, and refinement of curricular decisions and teaching strategies.

METHOD

A structured clinical interview was conducted with a third grader in an elementary school in southwest Virginia. The subject was selected on the basis of her erroneous performance on a subtraction test designed to elicit error patterns (Van Lehn, 1982). The interview consisted of three questions from each of the eight cells in the 2 X 2 X 2 matrix (see Figure 1). Following are examples for each of the eight cells:

1. Formal/place value/syntactic: What number has 5 tens and sixteen ones?
2. Formal/algorithmic/syntactic: Find the difference 87 - 59.
3. Informal/place value/syntactic: How much money is 2 dimes and 4 pennies?
4. Informal/algorithmic/syntactic: A gum ball costs 10¢ and you give the clerk 25¢, how much change do you get back?
5. Informal/place value/semantic: Represent 79¢ with dimes and pennies.
6. Informal/algorithmic/semantic: Represent 50¢. Take away 13¢. In a store type role playing setting.
7. Formal/place value/semantic: Represent 86 using toothpicks.
8. Formal/algorithmic/semantic: Represent 50. Take away 13. Where the subject teaches the researcher how to do subtraction.

CASE STUDY RESULTS

Kate completed the Van Lehn (1982) test, and her responses indicated some inconsistent error patterns: 0-N=N (3 errors) and some problems requiring regrouping (5 errors). In total, she got 10 of the 20 problems wrong; 2 were fact errors.

Table 1. Results of the diagnostic interview for Kate.

'X' indicates an incorrect response and '√' indicates a correct response. In addition, the numbers indicate the cell and the progression of letters are in increasing complexity (i.e., 'a' is easier than 'c'). A double response indicates a two part question.

	Place Value		Algorithmic	
	Syntactic	Semantic	Syntactic	Semantic
Formal	3a √ b √√ c X	7a √ b √√ c X	4a √ b √ c X	8a X b X c X
Informal	1a √ b XX c X	5a √ b √X c X	2a √ b √ c X	6a √ b X c X

The place value knowledge that was selected for analysis in this study was: 1) ability to represent a number, 2) concept of 10 and 1, and 3) regrouping in the context of two-digit numbers. Algorithmic knowledge was defined in terms of the subtraction algorithm applied to two-digit subtraction. Three levels were selected: 1) subtraction without regrouping, 2) subtraction with regrouping, and 3) subtraction with zero in the units place of the subtrahend. Kate's performance is summarized in Table 1. Her performance on these tasks, in general, indicate that her best performance is on those tasks created to tap her formal/syntactic knowledge. Her performance on items designed to explore meaning (semantics) is not as strong. She has little meaning to tie to what she is doing nor does she have strong connections from her formal knowledge to her informal knowledge.

Kate's knowledge of representation is consistent across contexts. Whether she performs syntactic or semantic tasks, Kate is able to represent numbers easily. Indicated in these tasks is her knowledge of the convenience of grouping things in tens. When asked to represent some value concretely, she consistently counts out the tens then the ones. Also, she is able to represent a number abstractly given the number of tens and ones. However, her ability to extend this knowledge beyond representation is very limited.

When asked how many tens and ones there are in the number 52, Kate quickly responds, "5 tens and 2 ones." However, when she is told that the researcher has 79¢ in his pocket, all in dimes and pennies, she is unable to determine the number of dimes or pennies. She eventually guesses that there might be 40 dimes. When asked how many pennies would be in his pocket given that there are 40 dimes and a total of 79¢, she responds, "30 pennies." She obviously loses her ability to think in terms of tens and ones in this context. On the formal/semantic tasks, Kate has no problem determining which is greater between 1 ten and 1 one. However, she has problems knowing how much greater 10 is than 1 when using money.

The last place value question type assessed her ability to regroup tens and ones. Kate is unable to do this in any context or in any way the question is phrased. For the semantic plane, Kate is asked to represent a number using manipulatives. She performs the task as above (i.e., she would represent 32 toothpicks using 3 bundles and 2 singles). She is then asked to represent the same number in a different way (i.e., 2 bundles and 12 singles). Kate is unable to perform this task. In the syntactic plane, Kate also has difficulties. For example, when asked what number has 5 tens and 16 ones, she focusses on the typical syntactic structure. That is, she writes 5 in the tens place and 16 in the units place. She is then asked to read the number. She states "five-hundred sixteen," and does not see this as a problem. The last question in this plane deals with the value of 5 dimes versus 45 pennies. In the interviews, Kate becomes confused between the 2 facts - a dime is greater than a penny and 45 is greater than 5. She is unable to combine these two facts to compare value.

In the analysis of procedural knowledge, it becomes apparent that Kate is unable to make use of manipulatives. In fact, the manipulatives only bring confusion to the tasks. The first problem in the procedural plane is to subtract without regrouping. Kate manages to solve the manipulative problems at this level using a counting strategy. Basically, her procedure is to represent the subtrahend in tens and ones and remove the minuend in tens and ones. The result is stated by counting the remaining pieces. This procedure fails in the other problems because regrouping is necessary before you are able to count out the minuend. Kate is completely unable to do the last 2 problems with the toothpicks. In the last 2 problems where we role play a transaction in a store, Kate apparently focusses on the differences of the tens places when making change. That is, for one problem (43¢ for a 39¢ item) she gives a dime for change, the difference between the tens. On the second problem, 80¢ for a 57¢ item, she gives 3 dimes change, again the difference between the tens.

The other 2 cells in the procedural plane are solved in a similar manner. That is, cell 2 (Figure 1) are money word problems. She solves these problems by translating the problems into a column subtraction format, the task for cell 4. Therefore, the results of these problems are the same. Again, Kate is able to solve the problems that do not require regrouping. In addition, she was able to solve the problems at the second level, subtraction with borrowing. The interesting result in these problems is at the third level, subtract where there is a zero in the units place of the subtrahend. As described earlier, Kate occasionally exhibits the error pattern 0-N=N. In the interview data, Kate exhibits the error pattern 0-N=0. Apparently, she is unsure of what to do with the zero in these problems and makes repairs (Brown & Van Lehn, 1982) to her algorithm when she encounters such problems. In fact, it would appear that she initially makes such a repair on a given task and is able to retain this repair throughout the task. However, she needs to reformulate the repair across tasks on different days. This would be termed an unstable bug by Brown and Van Lehn.

CONCLUSIONS

This paper presented the first attempt in the specification of a conceptual model for studying mathematical understanding. A set of interview questions were developed and a case study was presented. Additional data are being examined and further attempts will be made to delineate the relationships among the cells in the model. In addition, we will report our attempts to extend and refine the model to make it more encompassing of other domains of mathematical knowledge. Although the model depicts a wide variety of knowledge structures, it is by no means intended to be comprehensive. We envision that the place value/algorithmic dimension of Figure 1 can be modified to distinguish between concepts, facts, and procedures. This modification may well add to the generalizability of the model as well as helping mathematics educators study relationships among specific types of knowledge.

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EXPLAINING DIFFERENTIAL PERFORMANCES ON ADDITION STORY PROBLEMS ACROSS GRADE LEVEL

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Abstract

Four variables (cues, transformation, test administration ratios, and success criteria) were manipulated to study grade level differences among 82 first grade and kindergarten children under carefully controlled experimental conditions. Even though there was a strong grade level main effect, none of these variables was found to be a main effect, nor were there any interactions. The most interesting result was the demonstration of further research support for the Brainerd/Kingma conclusion that short term memory and working memory do not share a set of scarce resouurces. It appears that the reduction of STM load does not induce higher success levels, contrary to expectations arising from the work of Pascual-Leone and Case.

Several researchers (Carpenter & Moser, 1984; Riley, Greeno, & Heller, 1983; Underhill, 1986a, 1986b, among others) have reported differential performances between kindergarten and first grade children on addition story problems. The sources of these differences have been explored and one variable is known NOT to have a significant impact: transformations (Joining) (Carpenter & Moser, 1984; Steffe & Johnson, 1970; Underhill, 1986a, 1986b). While others have found significant differences related to manipulatives, Underhill reported that when the conditions of problem presentation are optimized, the use of manipulatives, whether required, optional or not available are not significantly related to problem solving success (Underhill, 1986b). However, Steffe and Johnson (1970), Carpenter and Moser (1984) and Ibarra and Lindvall (1982) reported that manipulatives DID make a difference. Underhill (1986a) identified several sources of testing differences between the Steffe and Johnson, Ibarra and Lindvall, and Carpenter and Moser studies which had potential for leading to these differences: parsimonious versus chunked problem presentation; required versus optional use of manipulatives; individual versus group testing; exact answer versus ±1 criterion; and child naming of problem context (e.g., child supplying names and objects for join and part part whole contexts). In his 1986b study, Underhill reported that Rowe's (1986) concept of wait time DID influence young learners' successes and that learners performed better when problems were presented in a chunked format, one-sentence-at-a-time with pauses. He reported further evidence to support no significant difference on the use of manipulatives (none, optional and required), transformations (joining versus part part whole) and success criteria (exact versus ±1).

In the present study, five variables were manipulated: visual cues, problem transformations, success criteria, grade level and test administration. Visual cues were 3x5 cards with numerals to help learners remember the numbers in the problems; transformations were present in joining problems but not part part whole; answers were analyzed on the basis of exact numerical responses or within one (±1) of the correct response; subjects were kindergarteners or first graders, and children were tested individually or in groups of 3 or 4.

RATIONALE FOR VARIABLES

Visual cues were included because the work of Pascual-Leone (1970) and Case (1978) leads one to conclude that working memory and short term memory (STM) share the same resources. If the STM burden can be lightened, then learners will have more cognitive resources for processing or reasoning. The Pascual-Leone/Case perspective was refuted by Brainerd and Kingma (1985) when they reported a series of nine studies, each one involving over 900 protocols with kindergarten and second grade children. They concluded that STM and reasoning are independent when STM and reasoning capacities are manipulated in the contexts of transitivity, conservation, probability and class inclusion. By providing half the subjects in this study with visual problem number cues, these two positions could be examined within the context of addition story problems.

The inclusion of transformations permitted the investigator to explore whether any main effect or interaction would occur with visual cues. A number of studies have now rather clearly established that the differences on performances on part part whole and join problems are not statistically significant within age/grade levels.

Criterion scores of "exact" and "±1" were useful in Underhill's (1986a) study in explaining differences between first graders' and kindergarteners' performances. Since both could easily be collected, this, too, was used as an independent variable to attempt to explain observed similarities and differences.

Two grade levels were included because several previous studies have observed grade level differences, and these are the main differences this study sought to clarify and explain.

The last variable was test administration. It was conjectured that attention is probably better in a 1:1 setting, so a 1:3 or 1:4 setting might permit collection of data sought on the other variables while providing an opportunity to examine this conjecture.

METHOD

Eighty-two kindergarten and first grade children were tested in October and November in the following breakdown:

- 41 kindergarteners, 41 first graders
- 82 transformation, 82 non-transformation
- 41 cues, 41 no cues
- 52 individual, 30 group
- 82 exact, 82 ±1 criterion

Learners were randomly placed in administration and cue treatment groups. Each individual or group administration consisted of four transformational (join) and four non-transformational (part part whole) addition story problems with addends of 2, 3, 4 or 5. All subjects were in full-face view of the researcher, but partitions were placed between subjects in the group administration settings so that they could not see one another. Each child had a basket of chips which she was told she could use if she wished. Each child had a sheet with the numerals 2 through 13 written in approximate 2x2 squares. Children were instructed to point to (place a forefinger on) the numeral which was the answer to each problem. All questions incorporated wait time (pauses of 3 seconds after each sentence).

Data were coded as follows for analyses:

	Variable	Codings
A.	Grade level	0,1
B1,C1	Number of join correct, exact	0,1,2,3,4
B1,C2	Number of join correct, ±1 criterion -	0,1,2,3,4
B2,C1	Number of PPW correct, exact	0,1,2,3,4
B2,C2	Number of PPW correct,	0,1,2,3,4

±1 criterion

D.	Cues	0,1
E.	Administration	0,1

ANALYSES & RESULTS

A three-way ANOVA using "exact" scores was used to examine the effects of grade level, cues, and administration on addition problem solving success. The results are reported in Table 1.

Table 1. Three-way ANOVA using EXACT scores

Source	Sum of Squares	D.F.	Mean Square	F	P
Mean	1341.47	1	1341.47	649.47	0.00
Grade (G)	41.94	1	41.94	20.30	0.00
Cue (C)	0.49	1	0.49	0.24	0.63
Admin. (A)	0.84	1	0.84	0.41	0.53
GC	0.01	1	0.01	0.00	0.95
GA	1.11	1	1.11	0.54	0.47
CA	0.00	1	0.00	0.00	0.98
GCA	0.06	1	0.06	0.03	0.86
Error	152.85	74	2.07		

A three-way ANOVA using the ±1 criterion produced the same set of results.

Since Grade was the only significant main effect and there were no significant interactions, a series of cell comparisons were made using Scheffe's test with both exact and ±1 criterion to examine the combinations of these three variables (grade, cue, administration) on transformational and non-transformational problems and on the two criterion levels. None were significat at the .05 level.

DISCUSSION

On the variable ADMINISTRATION there were no significant main effect or interactions either within or across grade level. It appears that this variable holds little promise of explaining K, 1 differences.

On the variable TRANSFORMATION, there were no significant main effect or interactions within or across grade level. It appears that this variable also holds little promise of explaining K, 1 differences.

On the variable CUE, the results of this study agree with the conclusions reached by Brainerd and Kingma (1985) that STM and working memory use separate resources and that easing the STM load for information storage does NOT appear to enhance the information processing or reasoning capability of young learners. This finding runs counter to the assumption made by Pascual-Leone and, later, by Case that STM and information processing share a set of scarce resources which can be parceled.

On the variable SUCCESS CRITERION, there were no discernible difference which help explain K, 1 differences. Since this was a helpful distinction in Underhill's 1986a study, and since it can be readily employed within existing designs, it is probably useful to collect these data when comparing performances across age levels.

On the variable GRADE, one can only conclude from this study that differences do exist but that they cannot be explained by transformations, cues, success criteria or administrations in groups of 3 or 4 compared with 1:1.

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DIDACTICAL OBSTACLES IN THE DEVELOPMENT OF THE CONCEPTS OF MULTIPLICATION AND DIVISION

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Research shows that children at secondary school have many weaknesses in their understanding of multiplication and division as applied to (positive) rational numbers. In this paper some of the factors implicated in these weaknesses are discussed and alternative pedagogical strategies for tackling the problem are outlined. These include the possibility of developing "secondary intuitions" (as defined by Fischbein), the fostering of metacognitive awareness of the limitations of intuition and how formal methods may allow one to go beyond them, the exposure and discussion of cognitive conflict, and the use of appropriate situations within which to introduce the extension of multiplication and division from the domain of positive integers.

Research has shown that many children and adults do not develop adequate conceptualizations of multiplication and division in the domain of positive rational numbers. Among the evidence may be cited the following:

1. Results obtained by Mangan (Greer and Mangan, 1986; Mangan, 1986) showing that (i) even intelligent adults often fail to choose the correct operation for word problems involving multiplication and division when the combinations of numbers involved are of certain types, and (ii) there is remarkable consistency, for groups ranging from 12-year-olds to elementary teachers in training, in the differential effects for different number types.

2. Demonstrations among secondary school pupils of non-conservation of multiplication and division (Greer, 1987; Greer and Mohan, 1986). For example, Greer and Mohan found that a substantial proportion of 12-year-olds, shown a word problem with the numbers concealed by flaps, could correctly describe which operation should be used to solve the problem, but changed their choice of operation when the numbers were revealed (the numbers were chosen so that they were likely to provoke misconceptions).

A replication of that study currently being analysed showed that even when the initial choice of operation after the numbers were revealed did not change, most 12-year-olds and many of the 15-year-olds were susceptible to countersuggestion. Specifically, with a decimal less than 1 as operator, discussion about the size of the result relative to the operand led these children to choose the inverse of the operation which they had (correctly) nominated when the numbers were unknown. Initial impressions suggest that a higher proportion of 15-year-olds than 12-year-olds who exibited this behaviour were aware of the contradiction between their judgments.

In another ongoing study, similar manifestations of non-conservation have been found with rationals less than 1 written as fractions rather than decimals.

In this paper I consider some of the didactical obstacles which make it difficult for children to extend their conceptions of multiplication and division from the domain of positive integers to the domain of positive rationals.

OVERGENERALIZATION OF RULES

Extension of an operation defined in one domain to a more general domain is a characteristic development in mathematics. When such an extension takes place, some properties which hold in the more restricted domain will not hold in the more general domain, and this is a natural source of errors, particularly when rules applicable to the restricted domain but not to the more general domain are strongly entrenched. Examples of this, in the case of multiplication and division, are the misconceptions that multiplication makes bigger and division makes smaller, and that division is always of the larger by the smaller. No great theoretical sophistication is required to account for these; to quote Thorndike (1922): "If you are a youngster inexperienced in numerical abstractions and if you have had divide connected with 'make smaller' three thousand times and never once connected with 'make bigger' you are sure to be somehwat impelled to make the number smaller the three thousand and first time you are asked to divide".

LINGUISTIC ASPECTS

There are some specific linguistic aspects implicated in the errors children make. In English. "twelve divided into three" is notoriously ambiguous. In problems involving intensive quantities, there is often confusion between, say, "miles per hour" and "hours per mile" or between "miles per hour" and "miles". Here, however, I want to consider a more pervasive influence of language. To my mind, there is a fundamental difference between descriptions of situations in nature language and descriptions in mathematical language. Natural language descriptions are designed to refer to particular situations which obtain, whereas mathematical descriptions are designed to be uniformly applicable to all situations (within some structured class) which might obtain. (Is it too far-fetched an analogy to compare this to the difference between concrete and formal operational thinking?). Think of the situation where a photocopier can reproduce a page and either enlarge or reduce it. Note that last phrase - "either enlarge or reduce it". I can think of no single word which covers both possibilities. In mathematics, by contrast, we have the concept of multiplication by a constant positive rational, irrespective of its size. On the photocopier I use, there is a single control which allows you to set an enlargement factor at any value between 0.82 and 1.55 (by steps of 0.01). Similarly, it is not necessary in a computer program for working out cost given unit price and quantity to check if the quantity is greater than or less than 1 unit. It is clear, however, that many children (and adults) do not have this unified conception of a change in size. Rather, they think in terms of either making bigger, which is connected with multiplication (and addition) or making smaller, which is connected with division (and subtraction). Multiplying or dividing by a number less than 1 is a "different" case - children often say as much.

FAILURE TO DISSOCIATE CONCEPTS FROM ALGORITHMS

The problems children have are exacerbated by teaching which concentrates on extension of the algorithms for computation while paying no attention to extension of the conceptualization. The algorithms are often presented in a way which emphasizes their similarity to those for integer multiplication and division, either by converting the problems to an equivalent one involving only integers, or by having some rule for the final positioning of the decimal point. Results from studies where children have been presented with calculations, and asked to write stories which those calculations model, show that very often children are taught calculations with no idea of any situation in which they would want to do those calculations. Conceptualizations of the operations are often confounded with computational considerations. For example, a teacher said to one of my students recently that you cannot multiply by three-quarters - what you do is multiply by 3 and then divide by 4.

Even the notation used for division has been found to have an effect. In a choice-of-operation experiment a comparison was made between one group of subjects who had to choose the calculation from, say:

$8.2 \div 3 \qquad 8.2 + 3 \qquad 8.2 - 3 \qquad 8.2 \times 3 \qquad 3 \div 8.2$

and another group who had to choose from:

$$3\overline{)8.2} \qquad \begin{array}{r} 8.2 \\ +\ 3 \\ \hline \end{array} \qquad \begin{array}{r} 8.2 \\ -\ 3 \\ \hline \end{array} \qquad \begin{array}{r} 8.2 \\ \times\ 3 \\ \hline \end{array} \qquad 8.2\overline{)3}$$

It was found that for partitive division problems, relatively higher percentages of correct choices were made for the first of these, while for quotitive division problems, relatively higher percentages of correct choices were made for the second type.

THE PEDAGOGICAL DILEMMA

Fischbein, Deri, Nello and Marino (1985) identify a general dilemma facing mathematics teachers. For any mathematical concept, it seems inevitable that the early conceptualizations of that concept (either historically or ontogenetically) will be based on intuitions derived from practical experience. Such intuitive conceptualization is essential for initial understanding. However, when it becomes necessary to extend the concept to make it more general, more abstract, or more formal, the intuitive models continue to tacitly affect one's thinking. With respect specifically to multiplication and division, Fischbein et al contend that performance on choice-of-operation tasks is tacitly mediated by primitive models - repeated addition for multiplication, partition and quotition for division.

WHAT CAN BE DONE?

It may be possible to extend intuitions. Fischbein, in this regard, refers to "secondary intuitions". (There are cases, for example, of individuals who have been able to think spatially in four dimensions). Thus, it might be possible to develop the intuitive conceptualization of multiplication and division so that the choice of operation for the problem: "0.932 kilograms of cheese cost $2.50. How much is the cheese per kilogram?" would be as intuitively obvious as if the quantity had been 2 kilograms. However, evidence from an experiment with university students majoring in mathematics (Mangan, 1986) suggests that this rarely, if ever, happens spontaneously. When such students were shown word problems on a computer screen, and required to choose the appropriate operation, from 5 alternatives displayed, within 8 seconds, the usual patterns of results were found, namely that problems in which the operator was a decimal less than 1 led to more errors, or when answered correctly, required more time.

An alternative approach advocated by Fischbein is broadly metacognitive. The idea is "to attempt to provide learners with efficient mental strategies that would enable them to control the impact of these primitive models" (Fischbein, Deri, Nello and Marino, 1985, p.16). One general strategy along these lines would be to teach pupils to be aware of the limitations of intuition, to be able to recognize the "danger signals" and switch to more powerful formal methods which (to borrow a metaphor Bruner applied to language) are likely prosthetic devices. Thus, the university students referred to above would (presumably) have made very few mistakes given enough time because they could have resorted to algebraic formulae (say for speed/distance/time problems) or the "easy-number strategy".

Fischbein advocates also the exposure and discussion of conflict between judgments based on intuitions and results obtained through formal methods. This approach has been used in teaching experiments by Alan

Bell and others at the Shell Centre in Nottingham (e.g. Swan, 1983). The task used in the non-conservation tasks described at the beginning, where the numbers in the problem are initially hidden, has proved an effective way of exposing conflict and further work with this situation might extend it from a research methodology to a pedagogical method.

A different approach (which might help to produce the "secondary intuitions" referred to above) is to introduce the extension of multiplication and division from the integer domain to the domain of positive rationals within suitable contexts, in the spirit of Semadeni's (1984) "principle of concretization permanence".

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PRACTICAL WORK AND FORMALISATION, TOO GREAT A GAP

K.M. Hart, Nuffield Secondary Mathematics

'Children's Mathematical Frameworks' (CMF) was designed to monitor the transition from learning through using concrete materials to employing a symbolic, formal mathematical system. It followed the research projects CSMS and SESM at Chelsea College, which had shown that many secondary school pupils do not adopt the generalisable methods of solving examples, taught in the secondary school but instead cling to more naive methods. Volunteer teachers prepared teaching schemes which used practical work to lead-up to a formalisation. The lessons in which the formalisation was verbalised were recorded. This paper describes what took place in seven classes when a) the subtraction algorithm and b) equivalent fractions were introduced symbolically.

Children's Mathematical Frameworks (CMF) was a research project based at Chelsea College, University of London from 1983 to 1985. It was designed to monitor the transition between learning mathematics from practical or concrete work to using formalised (mainly symbolic) mathematics. The theories of Piaget which distinguish between concrete operations and formal level thinking have led to the acceptance by many British teachers of a method of teaching which is based on the provision of concrete experiences. The particular use of 'concrete' materials investigated in CMF was when the experiences were structured to lead to a generalisation which could be verbalised and made symbolic. Previous research at Chelsea (CSMS, SESM) had shown that many secondary school children ignore the generalisable methods of solving mathematical questions they are taught in the secondary school and instead revert to the more naive strategies they were taught in the primary school. Often these same children invent 'child methods' which are limited in application and depend on counting, addition and whole numbers. The CMF research was designed to provide information on the reaction of children aged 8-13 years, to the introduction of generalisable methods, rules or formulae.

The research was influenced by the theory of Piaget and more particularly by teachers' interpretation of this theory. However, the results of CSMS and SESM had led us to consider the 'framework of knowledge' suggested by Ausubel and to expect a novice learner who might initially be functioning in a manner which could be described as concrete thinking to progress to more powerful frameworks of knowledge after long term instruction.

The methodology of CMF involved observation of classes and 1-1 interviews with children before and after they were supposed to have made the transition between practical work and formalisation. A "formalisation" in this context means a rule, formula or general method which can be applied to a variety of mathematical examples (albeit in a limited topic area). Each teacher who took part in the research was interested in the teaching of mathematics and was in the process of obtaining further qualifications in mathematics education (a diploma or masters' degree). The scheme of work and methods of teaching to be employed were given considerable thought by the teacher and were not changed by the research team. They reflected the teachers' best efforts at teaching for understanding, moving from practical work to formalisation.

Six children in each class taking part in the research were interviewed before the teaching started, at the end of the practical work, just after the lesson in which the formalisation was stated and three months later. The interview questions were designed to find out whether the children had i) adopted the methods they were taught, ii) gained understanding because of the introductory 'concrete' phase and iii) appreciated a connection between the two phases and had a positive attitude to this style of teaching. The results showed overwhelmingly that the recipients of a series of practical experiences, leading up to a formalisation did not appreciate that the latter was a synthesis of the former. The most they could offer in explanation of a connection was that using a rule was 'quicker'. Two reasons for this discontinuity might be that the two types of experience were so fundamentally different that a third 'bridging' procedure was needed or that the teacher did not emphasise the connection between the practical and formal. The dat included transcripts of the teacher's statements in the 'formalisation' lesson(s) in which the practical experiences were brought together and a verbal or symbolic generalisation provided so that subsequently the child was expected to work with this format. A researcher was present at all such lessons and besides tape-recording the teacher she made notes of diagrams or other visual aids provided for the children. This paper gives a description of the teachers' approach to the formalisation and the reasons given to the children for its adoption. The CMF research monitored the teaching of area of a rectangle, volume of a cuboid, enlargement, algebraic equations, circumference of a circle, equivalent fractions and the subtraction algorithm (involving decomposition and place value). Reported here are the 'formalisation' lessons in the last two topics.

SUBTRACTION AND PLACE VALUE

Sample: Teacher A with 25 children aged 8-9 years.
Teacher B with two groups of eight children aged 9 years (thought to be 'ready').
Teacher C with nine 12 year olds considered to be in need of remedial work.

The period devoted to practical work varied according to the teacher's wishes. Teacher B spent two hours a week for nine weeks leading up to the symbolic form of the subtraction algorithm for three digits whereas Teacher C only spent forty minutes a day for eight days on the task (his pupils had, however, been taught the topic previously). The three teachers emphasised the exchange of tens for units etc. and in each case the meaning given to 'subtraction' was removal or 'take away'.

Teacher A used Unifix material with his class and his intention was that the pupils learn the algorithm for subtraction of two digit numbers with decomposition. In the 'formalisation' lesson the children were provided with unifix and a large sheet of paper on which the letters T (tens) and U (units) were written. The children were asked to set out, using bricks, the top line of a subtraction question written on the blackboard (42-19). The teacher manipulated figures on the blackboard explaining that a 'ten' was to be changed, whilst the children moved a column of ten bricks to the part of their paper labelled U. The teacher additionally drew a picture to represent the bricks. After three such examples the bricks were collected and the children and teacher worked symbolically.

The teacher's comments and questions guided the order of events and the children were encouraged to go through certain mental procedures described as 'questions we should ask ourselves'. These questions were couched in formal language and repeated in the same form by the children, e.g. Richard said "you ask yourself, can you take nine units from five units". After dealing with three examples in this way the children were given further examples to do on their own.

The transition method employed by the teacher was essentially that of using the materials to enact the algorithm. At no time did teacher A say why the class was engaged in this activity.

Teacher B taught two groups of children who were withdrawn from their classes and who were considered to be at the same level of attainment and equally ready to learn the subtraction algorithm for three digit numbers. He spent two lessons of nearly an hour, to formalise the algorithm, with each group of children. In each case his review of previous work was wide-ranging, led from the blackboard and mainly verbal. The children were asked (i) to think about the significance of the place in which a digit was written, (ii) to do addition examples together and (iii) when manipulating figures in the 'ten' column they were asked to say 'twenty' instead of 'two'. Again the children were encouraged to use special vocabulary such as 'the carrying figure'; this was at variance with the general linguistic style of the teacher, which was very casual. The groups had used a variety of materials in the previous nine weeks, including Dienes Multibase base ten. This material was mentioned by the teacher and the children were asked to remember how a 'ten' had been broken down into 'ones' etc. but no material was handled during the two formalisation lessons. The link between the two forms of experience was not specifically stated by teacher B nor was a reason given for learning the rule.

Teacher C's pupils had been taught the algorithm before but the teacher felt that they should be given Dienes' multibase material and build up to the algorithm for three digit numbers again. He often told the class that their understanding would be improved by using the bricks. At the start of the formalisation lesson teacher C asked the children to lay out Dienes' bricks to represent the two lines of figures in the subtraction 62-19. However, the bottom array of material was not used and on one occasion it provided exchange for the ten units needed at 'the top'. The children were then talked through the solution to the question using bricks, including the removal of a ten brick first (which does not match the algorithm). Next the teacher talked through the algorithmic method, writing on the board whilst the pupils manipulated the bricks in their display to match the verbal description and provided answers to factual questions concerning number bonds. Finally written examples were given to the children and they were told to put away the bricks and write in their books. Teacher C walked round the class offering advice and if a child was finding the subtraction example difficult he was encouraged to attempt a solution using bricks. Both child and teacher, however, when manipulating the materials, sometimes removed from the left rather than the right (which is fundamental in the algorithm). Teacher C often appealed to the superiority of demonstration using the material. He explained the algorithm was needed because the bricks were too cumbersome to carry around.

The three teachers adopted different methods of introducing the transition from materials to symbolism but none of them described what was happening as a summing-up of the previous experiences nor did they mention the generalisability of the algorithm.

EQUIVALENT FRACTIONS

Sample: Teacher D with 33 children aged 10-11 years.
Teacher E with more than 20 pupils aged 11-12 years (assigned to the top attainment set).
Teacher F with more than 20 pupils aged 11-12 years (assigned to the bottom attainment set).

The two secondary school classes were the top and bottom sets in the first year of a comprehensive school which took children from the primary school of teacher D. This fact was important in that the primary school teacher stated that although some members of the class did not grasp the rule during this teaching sequence, they would "get it again" in the secondary school. The secondary school teachers devoted five lessons to the topic; during this time teacher E also introduced addition of fractions. The attainment profiles of the groups interviewed were very similar and very few of these 18 children used a multiplicative method to generate equivalent fractions after the formalisation lessons although this was taught by all three teachers.

Teacher D was determined to provide more than one embodiment, so the children had cut-out and shaded regions of circles and rectangles, worked with a fraction wall and for homework found fractions of a set of drawn 'apples'. During the formalisation lesson they used cuesenaire rods or colour-factor rods to form a fraction wall and a set of counters. The teacher had provided, however, a definition of a fraction (which the children were supposed to know), which did not immediately match the material e.g.

[Teacher: T; Pupil: P]

T. Say I wrote up one sixth, that's writing it in numbers. On your sheet I wrote it in words, Now what does that actually mean?

P. One of six equal pieces that make a whole one.

T. Now if you look at our whole one this time, it's this group of apples...

During the two formalisation lessons the teacher built up 'fraction families' by asking the children to show the material form of a fraction and then to display another fraction which she knew was equivalent (all the examples were based on factors of 12). On the blackboard, the teacher wrote, in symbols, the fractions which appeared to be represented by the same amount of material. The transition appeared to be the formulation of a rule obtained from looking at the array of fractions "What sorts of things do you notice between those, those answers? Surely everybody noticed something, didn't they?". This general request produced non-multiplicative relationships (e.g. "Each answer goes up 1, 2, 3") besides the one being sought. A child suggested the top and bottom number could be multiplied by another number to provide another fraction in 'the family'. The teacher hinted that this way was 'quicker' and showed by multiplying, with the help of the class, that the fractions written on the blackboard could be obtained from each other by multiplication. The fact that multiplication did not in this case enlarge the amount was not mentioned and since the teaching of multiplication of fractions comes much later in British textbooks, multiplication by unity was not mentioned either. Other examples, not modelled with bricks, were solved by recourse to multiplication. The rule became much stronger as 'what you do to the top, you must do to the bottom' and in this form was used to correct David's view of the relationship between equivalents being subtraction.

In the second lesson rods were used to show equivalent amounts and again the fraction forms were written on the board. A child suggested one might obtain one fraction from another by division and the rest of the lesson was concerned with the mechanics of division. In summary teacher D obtained a set of equivalent amounts of material, wrote the symbolic names for these and then from a list required the children to state a relationship. From this was stated a rule. The gap between the two types of experience can be seen if one asks whether it is feasible to ask the child to set up the bricks or counters which will show an equivalent to 3/7 if the child does not already know one.

The secondary school teacher of the highest attaining set, let it be known that there were certain types of fraction or reply that he 'liked' and others that he considered crude or "harder to understand". Thus, he usually required fractions to be given in their lowest terms. The formalisation lesson recorded was concerned with the provision of a method of addition for two fractions with different denominators. The equivalence of fractions was assumed knowledge but if a child had difficulty in naming a fraction the teacher mentioned 'a cake'. The cake was not necessarily drawn on the board but when it was, the diagram was inaccurate and to show something already known rather than to demonstrate and 'prove'. The addition routine being taught was a more general method than the two previously in use and could have replaced them, but this fact was not mentioned by teacher E. When equivalent fractions were to be used the fractions were set out as follows and the process was concerned with division: $\frac{3}{4} \div \frac{5}{6} = \frac{}{12}$

Teacher F's class was recorded when he taught the formal method for finding equivalent fractions by multiplication. The first part of the lesson was devoted to children stating the number that was written in the denominator to label the drawing of a part of a 'cake'. Other ways of partitioning were shown on other 'cakes' and the amount to be eaten was mentioned "In all those four examples I've taken the same size cake every time". I've split it up differently, but in each case you're getting the same amount of cake. You're having to eat as much cake in that one - it's one large piece, as you are in this one, two smaller pieces ...". The link to the rule is stated by the teacher as "How many times as many parts did I split this one up into?" followed by ... "So we're having to multiply the top number by 2 as well". The teacher wrote $\frac{1 \times 4}{4 \times 4}$ which was in itself a new form of notation as previously $\frac{1}{4}$ was the name for a slice of cake.

CONCLUSIONS

The reason for the rule or algorithm being taught was not mentioned by any teacher except as being a quicker or convenient method. The bridge between the two types of experience received little attention, which is not necessarily the fault of the teacher since this stage is not mentioned in most school textbooks. The novice learner was assumed to possess quite a sophisticated knowledge of modelling and in particular what might be relevant when one moved from a model to symbols and vice versa.

DESCRIBING LEVEL OF UNDERSTANDING: RATIO

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ABSTRACT: Based on Taiwan students' performance on written test and the interviews, the meaning of some higher levels of understanding of ratio was studied. The difficulties of level 3 student in solving level 4 items were investigated in this paper.

INTRODUCTION: CURTURAL DIFFERENCES

As part of national project investigating levels of understanding in mathematics at junior secondary level in Taiwan, this study made of students' understanding of ratio. An adaptation of the CSMS methodology and ratio test (missing value problems; Hart,1981), together with an adaptation of Noelting's orange juice test (comparison problems; Noelting,1980) were used. Sampling was completed by two procedures: (i) stratified geographically Taiwan region into 11 strata, (ii) simple random sampling some number of schools and students from each statum according to the proportion of population. A final total of 2880 students,aged 13 to 15 years took part in this study. Samples of 13 and 14 years old were retested one year after to test the model of level of understanding which was built in the initial investigation. While the intention of the Taiwan study was not to investigate cultural differences between Taiwan and the U.K., nevertheless comparison of Taiwan results with those of the CSMS study showed both some interesting similarities and discrepancies which seemed to warrant further investigation.

To begin with, the Taiwan results showed a general consistency with the CSMS data in terms of the hierarchical grouping and sequencing of items in 4 levels of understanding (Lin,et.al,1986). This indicates that Taiwan children progress through essentially the same levels in terms of items successfully answered as do their British counterparts. However, the Taiwan findings differed from those of CSMS in the following aspects (see Fig.1):

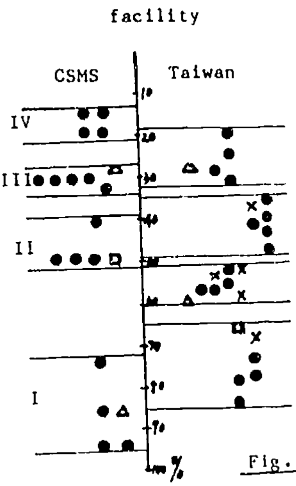

Fig. 1

● : item at the same level in both study

Δ,□,⌂ : item at a different level

x : item only included in Taiwan study

1. Differences in item facility. The item examplifying this most noticeably was the recipe question, given the amounts for eight people, to ask for the amounts for six people, (ref. Δ in Fig.1), where only 60% of the Taiwan sample solved this correctly, compared with 85% of the British sample. The CSMS analysis indicated that this item was "easy" for the British children because they were able to solve it by the use of informal "child -methods" (Hart,1981). The relatively difficulty of the item in the Taiwan sample therefore suggested that perhaps Taiwan children did not use the same kind of strategies. This suggestion receives support from the fact that the level 1 items were generally more difficult for the Taiwan sample than for the U.K..
2. Inclusion of all test items in the Taiwan levels of understanding defined, whereas some items were excluded from the CSMS levels. From the point of view of the mathematical nature of the items excluded by CSMS, there appeared to be no obvious reason why these should be omitted. This observation may also suggest that Taiwan students performance fits more closely to "mathematical expectation" than does that of the British students.
3. Narrower facility spread, distribution difference: There is marked difference in the facility bands defining the 4 levels of understanding, with the Taiwan levels spanning a narrower total range

than did the CSMS levels. This shows that the items corresponding level 1 were relatively more difficult for the Taiwan students and that a higher percentage (9-13% more) were failing to attain this level, but that each of the other levels 2 to 4 was relatively easier, with a substantially higher number of Taiwan students (10-20% more) performing at level 4 than reported for the British sample.

4. Also of interest was the lack of gap between levels 1 and 2 in the Taiwan data, showing that once Taiwan students have attained level 1, it is substantially easier for them to progress through the remaining levels than it is for the British children. The reasons as to why this may be so seemed to require further investigation.

QUESTIONS

It was therefore thought useful to examine three questions:

1. Why is it harder for Taiwan students to succeed on the level 1 items?
2. Why is it easier for Taiwan students to succeed on the level 4 items?
3. What are the difficulties of lower level student in solving higher level items?

The first question is being reported upon elsewhere (Lin & Booth, in preparation); this paper takes as its focus the second question. From developmental point of view, it is also necessary to consider level 3 students' performence to approach the second question.

APPROACH and METHODOLOGY

In discussing the different levels of understanding, the CSMS project placed considerable emphasis on the nature of the methods which students used in solving the different level items. This issue of method used is of particular importance in the case of the ratio test items, since examination of the test shows that most items can be successfully handled by knowledge and application of the formal rule $\frac{a}{b} = \frac{c}{d}$. In the case of CSMS research, Hart (1981) reported that "there was little evidence that the taught rule was remembered and used by the children"; and "no child quoted the unitary method" in the interviews. "In fact most children on interview changed the method they used continuously, adapting to what they saw as the demand of the question. Generally they avoided multiplying by a fraction and tended to build up and answer in small segments, adding them together at the end". The methods the Taiwan students used are quite different from the above description. It was therefore thought useful to examine the nature of the methods used by Taiwan students in handling the ratio test items. The examination was carried out by two steps.

1. Data Analysis

(i) analysing the frequency of different methods used to solve each item by level 3 and 4 students from retested data, which has better coding than the initial data.

(ii) analysing the association between Noelting's cognitive stages and the understanding levels in ratio with retested data.

2. Interviewing

(i) identified samples of level 3 and 4 by CSMS test.

(ii) interviewing with Noelting's orange juice test.

(iii) interviewing with a non-ratio missing value problem (Markovits, et.al.1986), the "Truck Problem": given the total weight of a truck with 3 tons of goods, to find the total weight of this truck with 6 tons of goods.

Taiwan data showed that there are 11%, 29%, 46%, and 42% of level 3 students used a wrong "addition strategy" to solve Mr. short problem and non-integer multiplier enlargement items in CSMS test, one L-shape and two K-shapes respectively. The reasons as to why those students fall back to "addition strategy" became our major concern while we investigated the difficulties of level 3 students in solving level 4 items. Six samples of this type of level 3 students were then interviewed.

RESULTS

A. Level 4 students' performance

- They have learned and known how to apply formal procedures, formula/multiplier/unitary method, to different number structures and contexts. 80%-90% of them solved 4/5-5/6 of items in each level by those multiplicative strategies.
- More than half of them have tendency of switching one method to another in terms of different situations. There are 40%/7%/0% of 15 years old group used formula/multiplier/unitary method consistently to solve more than 2/3 of all items. For 14 years old group, the corresponding percentages are 10/32/2. 15 years old group have received formal rule teaching, 14 years old group haven't.
- They have strong tendency of using one single method to solve items of different number structures in a context. Each student in the interviews applied only a single strategy to solve all Noelting's orange juice problem. The more advanced ability of choosing the "easier" ratio between scalar and functional ratio (Karplus et al, 1983) was not observed in this context.
- They have good ability of writing readable procedures. Only 4-10% of their correct answers in each item can't be identified by our coding system.
- The ability of differentiating ratio relation from non-ratio relation is relatively good. Most students on interview explained the truck problem correctly.
- Most of them are at late formal stage. 78% of 1599 retested samples are in Noelting's stage 3B.

B. Level 3 students' performance and learning difficulties

- Most of them can apply multiplier/unitary method to solve problems with rate as hard as 2:3 or 2:5 in different contexts. 50-70% (resp. 37-50%) of answers of 14 (resp. 15) years group were coded with those two methods in 4/6-4/5 (resp. 3/6-4/5) items of level 1, 2 and 3. 46% (resp. 27%) of 14 (resp. 15) years group are consistently used those two methods to solve more than 2/3 of items in level 1, 2 and 3.
- 14% of 15 years group used formula method consistently to solve more than 2/3 of items in level 1, 2 and 3. 12-25% of answers in

3/6-4/5 items in level 1, 2 and 3 were coded with formula method.

- 6% of level 3 students used "halving" and "building up using halving" strategies to solve the recipe question.
- The ability of writing readable procedures is relatively weak. 20-35% of their correct answers in level 1, 2 and 3 items can't be identified by our coding system.
- The ability of differentiating ratio relation from non-ratio relation is relatively weak. Only one out of six samples on interview explained the Truck Problem correctly.
- In terms of Noelting's stages, their performances are not stable.

Why many level 3 students used "addition strategy" as their fall-back method to solve enlargement problem with non-integer scales? Some possible reasons were observed on interview.

. do feel comfortable about addition strategy because it is simple and reasonable. The common difference relation is a strong support of their reasoning.

. do not understand the formal method of finding multiplier.

. do not view non-integer multiple as a multiple.

. can only recognize one kind of proportion, namely the proportion between the corresponding segment in two figures or the proportion between two parts in a figure. Be confused with proportion of curves.

They are likely working on the relation among numbers but ignoring the units which are accompanied with numbers in the context. e.g. in Mr. Short problem. This tendency is very prevalent among lower level students.

DISCUSSION

How does Taiwan students' learning of formal ratio procedures came about? Certainly the Taiwan education system is one which places considerable value on academic achievement, and students are relatively well-motivated to study hard. However this observation by itself does not seem to explain students' better performance in this regard. To say that it did would imply that the difficulties in this regard widely reported elsewhere in the research literature were motivational rather than cognitive in basis. Nevertheless, it would seen that some aspects of the teaching situation in Taiwan are contributing to children's better learning in this area. Studied elementary strategies used by Taiwan students, we had observed that they are always looking for number pattern and relations among given data (Lin & Booth), which are more formal and mathematical than by British students, where child-methods (Booth, 1981) is more prevalent. Therefore, the distance between elementary strategies and formal procedure is less for Taiwan students than for those in U.K.. In terms of Ausubel's "meaningful learning"-attaching new learning to what already learn, Taiwan students are on their ways of learning formal methods.

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DECIMAL MULTIPLICATION IN GRADE SEVEN

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The purpose of this paper is to report explanations of children's mental activity in multiplication as it applies to decimals. A small group teaching experiment was followed by tests and interviews. To assess student's thinking processes, individual interviews with six students are reported. Rational explanations beyond the stating of rules were forthcoming from only one of the six students.

Hiebert (1984) has distinguished two types of knowledge that children acquire about mathematics: form and understanding. Form includes symbols for numbers, operations and relations. Understandings are intuitions and ideas about how mathematics works that make sense to children. Some of these are learned in school and some arise from informal situations. Hiebert and Wearne (1983) concluded that the students they interviewed had a good grasp of decimal form but did not have a thorough understanding of decimal concepts. This is consistent with my previous research (Owens, 1986) in which students could very effectively compute in decimal multiplication by "counting" decimal places. However, when asked to estimate the product to place the decimal where the significant digits were given (but ending zeroes dropped), students persisted in counting to place the decimal.

The research reported in this paper is viewed as a teaching experiment with a small group of students under close observation. It is based in the view that students construct their own knowledge in an effort to order their experiences, whether informally or in school, individually or in groups, using manipulatives or textbooks. In this study students were presented with questions and problems in the presence of certain settings and stimuli intended to aid students to construct conceptual meanings for particular forms and symbols used to represent multiplication of decimals.

METHOD

The entire grade seven cohort of 41 students was given a pretest on the following topics containing decimal numbers: Writing a decimal between two given numbers with or without the aid of a number line, ordering, computing products, estimating to place the decimal in a product, and solving multiplication story problems. The investigator was then involved in instructing the cohort. Instruction was intended to help students construct meaning of and intuition about the size of products of rational numbers. As an experience base, students were asked to measure in metres or centimetres all sorts of things and parts of the classroom and building. We discussed partitioning such measurements into 10 equal parts. Students were asked to divide by 10 to get 1/10 of the measurements. Triads of patterns introduced multiplication of fractions in relation to division by a whole number: (a) 8 ÷ 4 = ____; (b) 1/4 of 8 = ____; (c) 1/4 x 8 = ____. The definition, "of means multiplication," was given.

At that point in the project, six students were selected for small group instruction and observation. Selection was made on the basis of performance on the pretest. Average students with a range of performance in mathematics were selected. With the group of six, more formal activities were undertaken in decimal notation, addition, subtraction, and multiplication. As feasible, activities included the use of base blocks, linear metric measure and diagrams. Students were asked to observe these phenomena and make conclusions.

Against this backdrop multiplication of decimals was introduced. From fraction multiplication 0.25 x 8 = 2. Using measurement, what is 0.1 of 1.5 m? It was reasoned that 0.1 of 1 m = 10 cm and 0.1 of 0.5 m = 5 cm. So 0.1 of 1.5 m is 15 cm, and 0.1 x 1.5 m = .15 m. Similarly 0.3 x 1.5 m = 0.45 m. This type of example eventually led to recognizing that tenths times tenths gives hundredths. Using a grid of 1000 squares, 1/2 of 1/4 (of 1000) is 125 squares. That is, 0.5 x 0.25 (of 1000) is 125 (of 1000). Hence, 0.5 x 0.25 = 0.125. In another setting, 0.5 x 0.25 m is (1.25 cm) 0.125 m. Examples like this led to the rule that "tenths times hundredths gives thousandths." Following instruction, the entire cohort wrote a posttest similar to the pretest. Following the posttest, the six students of the small group were interviewed to explore their thinking processes with regard to multiplication.

RESULTS AND CONCLUSIONS

Pretest and posttest subscale and total test scores of the six selected students are presented in Table 1. On the first subtest students scored lowest on the item "Multiply: .4 x .2. An example of second type, EPDP, is "Estimate the answer then place the decimal point in the given 'answer'. 7.342 x 0.5 = 3 6 7 1." The multiplication problems (Mul Prob) were story problems which included supplying the answer or equation, or selecting (multiple choice) the operation. The division story problems (Div Prob) asked for an equation or selection of an operation. The Order items asked that 3 decimal numbers be listed from greatest to smallest. "Between" requested the writing of a number between two given decimal nu.nbers.

Table 1

Summary of Written Test Data

		Items	Mean	Abe	Liz	Mick	Mary	Bill	Lena
Multiply	pre	8	4.0	4	0	7	4	4	8
	post	7	4.9	2	2	5	6	6	7
EPDP	pre	5	1.6	1	3	0	1	0	5
	post	5	.54	0	0	2	0	0	0
Mul Prob	pre	8	1.9	3	1	0	3	1	8
	post	10	4.7	4	6	6	7	7	11
Div Prob	pre	2	.18	0	0	1	0	0	0
	post	4	1.3	0	1	1	1	1	2
Order	pre	2	1.0	2	0	1	1	1	1
	post	2	1.2	2	2	0	0	1	2
Between	pre	3	1.0	0	1	1	1	0	0
	post	3	1.7	2	2	2	1	3	3
Total	pre	28	10.8	10	5	10	10	6	22
	post	33	14.3	10	13	16	15	18	25

The interviews were composed essentially of items selected from the posttest. Students were then asked about their use of "rules." Of particular interest are the issues of 1) estimation of products, 2) multiplication always makes bigger, division makes smaller, 3) "of" means multiply, and 4) shading a diagram. Numbers 2 and 3 were set in choosing the correct operation in a story problem. The diagram was a 10 by 10 grid with "Shade the diagram below to show 0.3 of 0.1." Table 2 contains a brief summary of the interview results.

Table 2

Interview Data Summary

	.4 x .2	Estimate	Choose Operation	Diagram
Adam	First .8 Later .08	Unable	Good guesses Unable to explain	First unable Later able.
Liz	.8	Unable	Multiplication makes bigger. Division makes smaller.	With diagram 0.3 of 0.1 is 0.03. But .3x.1 = .3
Mick	.8 corrected at end	Unable	Multiplication bigger. (Try it) Division smaller. Then adjust	With prompting
Mary	.8 What I think	"Forgot how", but done correctly	Uncertain or unsure.	With prompting. Not related to .3x.1
Bill	First .8 Then Noticed 2 places	Unable	Overgeneralized "of means multiply."	Unable. Compute: .3x.1 = .03
Lena	Counted places correctly	Satisfa-ctory	Guess and test. Estimate result and adjust.	Satisfactory

For <u>Adam</u> .4 x .2 = 0.8. Adam usually counted places, but seemed to have other rules for special cases. For example, on the posttest 13.0 x 0.6 = .78. When prompted, he estimated and was confident that the estimate was correct. (It was.) However, he was unable to estimate consistently. In choosing the correct operation, he appeared to make good guesses. He would respond correctly but was unsure and unable to explain. At first, Adam was unable to explain multiplication via regions but when we returned to it at the end of the interview, he did so quickly and correctly to conclude that 0.3 of 0.1 = 0.03. He was also able to fix 0.4 x 0.2 = 0.08 (formerly 0.8) without hesitation. Adam guesses that he uses rules, but he does not understand them.

For Liz .4 x .2 = .8. When prompted to estimate she computed 3.00 x 7.00 = 21.0000 but persisted to count decimal places and was unsuccessful in estimating. In a story problem about gasoline consumption, she estimated directly and correctly. However, she chose the operation based upon "when you multiply the answer is higher" and "when you divide the answer is less." With prompting she marked the grid to conclude 0.3 of 0.1 = .03. However, she calculated .3 x .1 = .3. She didn't know which of her answers to believe, and she was surprised ("Really?") when the interviewer suggested she should believe the diagram. Liz said that she does not use rules but she knows what to do by remembering "things like shortcuts," and she believes she understands.

Mick computed .4 x .2 = .8. When prompted to estimate, he wrote 3. x 7. = .21000 and was unsuccessful. In choosing the correct operation his first hypothesis is that multiplication makes bigger, division smaller. However when he multiplied 0.75 x $900.00, and got an answer which was smaller than $900, he decided that he should divide. He said "If you divide it by three-fourths, you find out how much each one-fourth costs, and add it to 900." While not precise, this statement shows considerable understanding. With prompting, "first get one-tenth", he was able to complete the grid for 0.3 of 0.1. After faltering he recovered to conclude .03. When asked, he realized "I made a mistake on the first question," and he corrected it. Mick uses rules and knows what they mean. When prompted he noted "hundredths times hundredths is ten-thousandths because 10,000 has four numbers after the decimal and 100 has 2, and because two times two is four."

When Mary was asked to explain how she knew .4 x .2 = 0.8, she replied, "Because, um, it's just what I think." She indicated, "I'm not sure how to do it using estimation - I forgot ..." In fact, she got both estimation items correct in the interview. Mary is uncertain in choosing the operation. In one division situation (divisor <1), she correctly said the answer would be larger. However if it were smaller she thinks it would be because it is cheaper if you buy more. For the question "If 16 friends got together and bought 4 kilograms of cookies, how much would each one get," Mary said divide because it's being divided into 16 groups. She wrote $4\overline{)16}$ because "you can't divide 16 into 4." With prompting "What is one-tenth of the page?... Now what is three-tenths of one-tenth?" She was able to shade the diagram. However, she found it necessary to compute 0.1 x 0.3, including a full row of zeroes to get 0.03. When asked about returning to .4 x .2 she corrected it (0.08) immediately. She also volunteered that she will check answers with estimation now. When given an example of a shortcut, "counting decimal places," she observed that she has been doing that since grade 5.

For .4 x .2 Bill first wrote .8. When he noticed there were 2 places in the factors, "I saw it was 2 over," he changed to .08. When asked to estimate, Bill did not remember how. He persisted in counting decimal places. In choosing the operation, Bill overgeneralized "of means multiply." For example, "Tom spent $900.00 for 0.75 kg of platinum. What would be the price of 1 kg ...?" Bill said this is a multiply because it has "of" in it. Strangely, he had not misused the rule this way in the posttest, but he did so very consistently here. This may have been because the giver of the rule was present at the interview. Bill could not shade a grid to show 0.3 of 0.1, but he could multiply 0.3 x 0.1 = 0.03. At first Bill said he doesn't use rules. Upon probing, "But you told me how to know where to place the decimal." He allowed, "Oh yes, that is a rule." However, he doesn't know why it works.

Lena is the one student who had reasons for her actions. She counted places to multiply. She estimated correctly when asked to do so. On the pretest she had done long multiplication to get the EPDP items correct. On the posttest, she appeared to have counted decimal places. She used a variety of strategies to choose the correct operation. In one case she chose the correct division equation by eliminating the other choices. In other cases she made a guess and tested it. For example, she multiplied (multiplier < 1) and got an answer less than $900, but she believed the answer should be greater than $900. She concluded that division might be correct but ended saying "I don't really know." She correctly shaded the diagram, solved the equation, and saw the relation between the two.

In summary, the rule for counting decimal places when multiplying is strongly entrenched. The item .4 x .2 was a particularly striking exception. Presumably, the Gestalt of aligning decimals in vertical form is particularly attractive in an item like this. Students who understood shading the grid in the interview, even with prompting, were able to then correct their earlier mistake. Both the written tests and the interviews confirm how extremely difficult estimation is for students. An important hypothesis to test is whether introducing and perfecting estimation skills prior to introducing the usual rule, would result in better understanding of estimation and indeed all decimal concepts. This study included no instruction on choice of operation or story problems per se. The results were consistent with previous studies (see Greer, 1987). Carefully designed instruction is needed to determine how the predisposition toward "multiplication always makes bigger and division always makes smaller" can be corrected.

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HOW SHOULD NON-POSITIVE INTEGERS BE INTRODUCED IN ELEMENTARY MATHEMATICS?

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This paper explores the possibility of introducing non-positive integers at an early level of mathematics instruction. It is proposed that negative and positive integers can be represented in a natural and concrete way as iterated actions oriented in opposite directions. Theoretical and formal justifications for the proposal are presented, and research is reviewed which suggests children do represent numbers in this fashion in some contexts. Further evidence from a recent study suggests that by about age 7, many children are able to solve problems involving negative quantities represented as action sequences in game-like activities. Similar activities, it is concluded, could be adapted for instructional purposes.

It is not unusual for children's introduction to the integers to take place in three steps: In preschool through first grade, children gain familiarity with the counting numbers and are encouraged to practice addition and subtraction of counting numbers using manipulable objects. Due to the absence of negative integers in these activities, children acquire the quite reasonable rule that larger numbers cannot be subtracted from smaller numbers.

Next, computation with written numerals is emphasized. These activities involve the concept of place value, and rules for carrying and borrowing. The latter rule partially contradicts the previous prohibition against subtracting a larger from a smaller number: subtraction of a larger from a smaller digit within a multidigit problem is now permitted, although subtraction of a larger multidigit number from a smaller one is not possible. The rules for borrowing may seem like arbitrary notational devices to children who acquire them without having a firm notion of place value (which may often be the case; cf. Kamii, 1985).

Negative integers are introduced last. Children learn that, in fact, it is always possible to subtract a larger from a smaller number. Indeed, it is possible to dispense with the idea of subtraction altogether, replacing it with a notion of adding negative numbers. The discontinuity of this aspect of children's formal mathematical knowledge with their original mathematical intuitions is now complete.

It is known that many children and adolescents feel alienated from mathematical activities, professing the heteronomous belief that math consists of following arbitrary conventions established by authorities (e.g., Erlwanger, 1973; Lee & Wheeler, 1986). Introducing the integers in a manner which requires students periodically to overthrow their previously well-founded intuitions could contribute to this mystification of mathematics. Legitimizing non-positive numbers from the earliest level of instruction might facilitate constructing a more coherent and autonomous view of mathematics.

The pedagogical strategy of initially emphasizing mastery of the counting numbers is based, at least in part, on the principle that children construct knowledge through the manipulation of objects. Since non-positive integers are not representable concretely as manipulable objects, their introduction is deferred until basic computational concepts and skills have been acquired. This principle is compatible with, but not the same as, the proposal by some scholars that children invent number knowledge through counting. For example, Gelman and Gallistel (1978) suggest that the zero concept may require formal tutelage in a way that knowledge of counting numbers does not, because zero is not represented concretely in the environment and is therefore not available in children's counting activities.

An alternative interpretation of the significance of manipulables derives from Piaget's (e.g., 1975/1985) proposal that logico-mathematical knowledge arises through a process of reflective abstraction. In this view, children construct knowledge of numerical invariance and numerical operations by reflecting on the general properties of their own actions, such as grouping objects into a collection, separating a collection into parts, and so forth. Accordingly, manipulables are significant primarily because they serve as a medium for performing actions, for distinguishing mathematically relevant from irrelevant actions, and for calling attention to the general forms of the relevant actions (i.e., adding, subtracting, multiplying, dividing). It is less significant for the growth of mathematical knowledge, although clearly it is convenient, that objects can be construed as empirical representations of positive integers (just as numerals can be used as notational representations by older students).

Unlike non-positive collections of material objects, non-positive actions performed on material objects not only exist but are familiar to children from infancy (e.g., Langer, 1980, 1986). Such actions include negations such as decrementing a collection, or dividing a collection into parts; and null actions such as leaving a set invariant by performing a quantitatively irrelevant maneuver, or restoring a collection's previous state by performing the inverse of a prior action. Both positive and negative integers are thus potentially representable in a concrete way; for example, as incremental and decremental actions performed on a collection, or as forward and reverse motions along a line.

(More formally, this is equivalent to representing numbers as unary operations or functions; thus, the integer 3 is identified with a function that increments 3 units [$\underline{fx} = \underline{x} + 3$]; and the integer -3 is identified with a function that decrements 3 units [$\underline{gx} = \underline{x} - 3$]. Cayley's theorem--any group is isomorphic to its transformation group--is the formal counterpart to, and provides a formal justification for, such an interpretation of the integers. The present interpretation can also be linked to category theoretic foundations, just as the interpretation which views objects as the primary representation of quantities is associated with set theoretic foundations.)

However, is there any evidence that children can in fact use iterated actions to represent quantities? The primary evidence for this possibility comes from studies of children's addition strategies (e.g., Fuson, 1982; Secada, Fuson & Hall, 1983). This research indicates that young children spontaneously perform addition by combining two count sequences, and progressively develop more efficient forms of this basic strategy. In the initial "counting-all" strategy, each addend is represented as a sequence of incremental actions (i.e., a function $\underline{fx} = \underline{x} + \underline{n}$), and addition is represented as the composition of these. In the subsequent "counting-on" strategy, one addend is represented as a constant and is taken as the argument of the other addend, which is represented as a function. This developmental sequence suggests that representation of numbers as iterated actions may precede, and even provide the foundation for, the representation of numbers as constants (at least in the context of computation).

A recent study (Davidson, in press) provides further circumstantial evidence of a link between numerical knowledge and understanding of functions in 5- to 7-year old children. Numerical tasks included number conservation and arithmetic problems. Function tasks from outside the numerical domain were used in order to disconfound children's ability to reason about the basic properties of functions (e.g., unique value and composability) from their general numerical competence; for instance, some tasks presented functions in the form of spatial transformations. The research design also permitted controlling other variables relevant to numerical competence, such as age and logical ability. The results confirmed a significant relation between numerical competence and ability to reason about the general properties of functions, even with these other relevant variables controlled.

My current research is concerned with whether children in the age range of 4 to 7 years can represent non-positive numbers in action contexts. In one task, children matched a set of toy bees to a set of toy flowers. They were then asked to "take away zero flowers" and to "add zero bees." The age trends for both problems were significant, with few younger children but about half of older children succeeding: performances on the two problems were highly correlated. On both, children who failed generally responded by removing objects from the display; those who succeeded usually performed a null operation by using an empty hand.

In another task, children were asked to count a set of objects starting with zero (i.e., 0, 1, 2, ...) and then count as usual starting with one; they were then asked to explain the discrepancy. All children counted correctly from one. Surprisingly, 85% were also able to count from zero, and without raising objections about using this method. Although all children could say which count yielded the correct number of objects, only 35% could explain the discrepancy by referring to the mistake of beginning a count with zero. Explaining this discrepancy was significantly associated with performance on both the adding zero and subtracting zero tasks.

Two tasks examined children's ability to combine positive and negative quantities represented as iterated actions. In the "Mailman" game, positive and negative quantities were represented as forward and backward movements along a cardboard street of houses which represented a number line from -4 to +4; movements were specified by colored arrows drawn from a deck of cards. Children combined positive and negative movements sequentially by moving a toy mail van. Their greatest difficulty was a tendency to skip over the zero house instead of treating it as a valid position on the number line, which resulted in incorrect answers. Nevertheless, about 40% of the subjects, at all age levels, used the line correctly and located addresses by combining forward and backward movements.

In the "Hippo" game, positive and negative quantities were represented as actions of incrementing and decrementing a set. Children played a zookeeper who puts food pellets into the hippo's bowl--or takes some out--according to instructions provided by a spinner. Colored marks on the spinner signified decrements and increments in order from -4 to +4. The first part of the task involved learning the decremental or incremental actions defined by these color codes; the purpose was to determine whether children would invent a number line ranging from negative to positive values in order to master the numerical meaning of the codes. In the second part of the task, the idea of the hippo owing food pellets was introduced, and children were given arithmetic problems such as adding 3 pellets when the hippo owes one pellet. Significant age trends were found for both the learning trials and the arithmetic problems; younger children showed understanding on under a third of the arithmetic problems, and older children on over half.

Because this research focuses on eliciting students' intuitions about non-positive integers, with no attempt to train them on the relevant concepts, these data are a conservative measure of children's potential competence. The pattern of results indicates that a practical ability to work with combinations of negative and positive quantities and zero--represented as action sequences oriented in opposite directions, and as performing a numerically irrelevant action, respectively--undergoes development between 4 to 7 years. The results do not imply that children spontaneously acquire explicit or formal conceptions.

For instance, few children could explain the difficulty in the counting task; also, when asked in the closing interview, only 4 students stated that there might be numbers smaller than zero (2 mentioned the hippo game and 2 mentioned the mailman game as giving them this idea). Nevertheless, the results do suggest a possible foundation for instructional activities that could lead to explicit conceptions about non-positive numbers.

CONCLUSIONS

The theory and research reviewed provide a framework for further exploring the possibility of introducing students to negative numbers, in a concrete way, earlier than is typical. The core of the present proposal is that there is a natural equivalence between integers conceived as constant quantities, and integers conceived as iterated actions. To take advantage of this, instruction could make use of game-like activities that require combinations of action sequences oriented in opposite directions, and that encourage students to reflect on the general properties of actions (e.g., that each action has an inverse). Explicit notions of negative numbers could be introduced by drawing attention to the link between, and by using the same word for, action sequences and constants; for instance, "if you have 0 and you take-away 4, then you have 4 take-aways left." (The idea of a negative constant could be further linked to the notion of debt.) Finally, conventional terminology for negative constants would be introduced.

The aim of this proposal is to provide a way for students to explore an interesting mathematical structure from the start, beginning with simple action-based intuitions that lead to the construction of increasingly formal understandings, without the mystification that may occur when early intuitions are in disharmony with later formal instruction.

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NON-CONCRETE APPROACHES TO INTEGER ARITHMETIC

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Building on previous work by the authors, the incidence and nature of the computational strategies used by upper primary school pupils, prior to formal instruction in integer arithmetic, was investigated through clinical interviews and a teaching experiment. The results indicate that about one third c' children in the age group 10 to 14 may spontaneously resort to using analogies with whole numbers when confronted with certain cases of computation with integers, acknowledging the excistence of negative numbers as "negative quantities" in the process. It also appears that a much smaller proportion of pupils resort to reasoning in terms of temperatures.

The research previously reported by Murray (Murray, 1984: 147-153) included the observation that, given appropriate stimuli, some upper primary school children readily and spontaneously resort to analog.cal methods of reasoning in order to perform calculations with integers e.g. $^-5 - {}^-2 = {}^-3$ "because when from minus 5 you subtract minus 2, you get minus 3", or, stating the analogy to whole numbers more explicitly, "because 5-2=3 and these are minuses...it is an ordinary subtraction sum". (The quotes are from Murray's interviews with upper primary school children who have not had any formal instruction in integer arithmetic). Some pupils in Murray's interviews also exhibited reasoning in terms of the "oppositeness" of negative numbers and whole numbers, e.g. "5 + $^-3$ was subtraction, then 5 - $^-3$ will be addition" and " $^-5$ means you have to get 5 more before you have 0". We decided to investigate this phenomenon further, with a view to the possibility that the exploitation of such non-concrete intuitions could be a useful adjunct to reasoning in terms of concrete and semi-concrete embodiments (e.g. temperature, the number line) during introductory teaching.

We now report on two further projects undertaken under our guidance, building on Murray's initial work. The first project, by Hugo, comprised a replication of Murray's interviews with upper primary school pupils, and more specifically with seventh graders. The second project comprised a teaching experiment in the upper primary grades, conducted by Malan.

Hugo (1987) conducted individual interviews with 97 7th grade pupils (age range 12 to 14), being the total 7th grade population of two fairly typical schools in a large country town (Kimberley). The purposes of this project was to check Murray's observation on the incidence of spontaneous non-concrete reasoning strategies in a different and more representative sample. The interviews were preceded by a short introduction to temperatures below zero, which included the setting of questions like "It is now 3 degrees C, where will the mercury on a thermometer be if it gets 10 degrees colder?" The interviews commenced two weeks after this introduction. One of the first questions asked during the interviews was "4 - 3 = ? ". Pupils who did not respond with $^-5$ were again reminded of temperatures below zero and asked a number of questions regarding temperature changes. No hints were given by the interviewer on how to answer the questions, nor whether the answers given by the pupils were correct. This procedure was also followed for the main part of the interview, which consisted of the questions in table 1 below being put to the pupil on a printed sheet, allowing the pupil to do them in any sequence. After answering a question, the pupil was asked to explain how he/she reasoned to obtain the answer. The facility levels obtained for some of these questions are given in table 1, along with Murray's results for 993 eighth graders, prior to formal instruction in integer arithmetic (1984:147), as well as the facility levels obtained by Malan's experimental classes after four half-hour periods of instruction in addition of integers. The important question with reference to these facilities is of course how pupils reasoned to obtain correct answers, since correct answers may result from quite meaningless manipulation of symbols, e.g.

(1) "$^-12 - {^-4} = {^-8}$ ..... $12 - 4 = 8$, put the minus before the answer too" (this strategy also produces correct answers for $^-a + {^-b}$, $^-a \times b$, and $a \times {^-b}$, but produces wrong answers for the other cases);
(2) "$8 + {^-5} = 8 + 0 - 5 = 8 - 5 = 3$", introducing an extra number in order to interpret the -sign as an operation sign

Our analysis of the strategies used by pupils have so far been completed for the cases $8 + {^-5}$ and $^-12 - {^-4}$ only. The results of this analysis is summarized in table 2. (The distribution of strategies for the addition of two negative numbers, e.g. $^-5 + {^-8}$, is very similar to that for the latter case above). The analysis shows that 35% of the 97 pupils managed to do $^-12 - {^-4}$ correctly in terms of a meaningful concept of negative numbers, while the corresponding figure for $8 + {^-5}$ is only 13,4%. These

TABLE 1: FACILITY LEVELS PRIOR TO FORMAL INSTRUCTION IN INTEGER ARITHMETIC

FACILITY LEVELS (%)						
Project	Hugo	Murray	Nalan			
Grade level	7	8	4	5	6	7
# of pupils	97	993	31	35	35	42
${}^{-}7 - {}^{-}7$	72					
${}^{-}12 - {}^{-}4$	56	57	69	73	66	61
$5 - 8$	54	48	50	60	60	90
${}^{-}5 + {}^{-}3$	53	61	42*	62*	63*	59*
$7 + {}^{-}7$	43					
${}^{-}6 \times 3$	41	45	39	49	66	51
$3 \times {}^{-}6$	40	54	44	57	74	56
$8 + {}^{-}5$	35	51	37*	38*	54*	73*
$3 + {}^{-}5$	34		42*	46*	60*	83*
${}^{-}5 - {}^{-}8$	20	34	19	35	17	27
${}^{-}8 - 3$	19	21	25	22	23	39
$5 - {}^{-}8$	3	17	3	3	0	2

(Facilities marked with asterisks indicate that formal instruction was given prior to testing) (In the table, the cases are indicated by examples, the actual numbers were sometimes different for the different groups of pupils. In most cases, the actual test contained more than one example of a particular case, the given facilities being medians. Differences in facility levels for different examples of the same case were negligible.)

TABLE 2: BREAKDOWN OF STRATEGIES USED BY HUGO'S 7TH GRADERS

STRATEGY	$8 + {}^{-}5$		${}^{-}12 - {}^{-}4$	
Total number of pupil giving correct answer (from a total of 97 pupils)		34		54
Correct answer obtained by simply putting -sign before answer, without any indications of a meaningful concept of negative numbers		0		13
Correct answer obtained by interpreting -sign as an operation sign (ignoring the + in $8 + {}^{-}5$ and subtracting: $8 - 5$ (14 pupils) or introducing a zero: $8 + {}^{-}5 = 8 + 0 - 5 = 3$ (4 pupils)	53% of 34	18	24% of 54	0
Reasoning in terms of the temperature scale (eg "go 8 degrees up from ${}^{-}5$")		8		0
Computing by analogy to whole numbers, interpreting negative numbers as temperatures below zero. (eg "$12 - 4 = 8$, but these are temperatures below freezing point")		0		20
Computing by analogy to whole numbers, making no reference to temperature but emphasizing that these are "minus numbers" (eg "minus 12 subtract minus 4 gives minus 8" or "$12 - 4 = 8$ but these are minus numbers")	38% of 34 13,4% of 97	0	65% of 54 36% of 97	11
Other methods exhibiting a meaningful concept of negative numbers (eg debt (1), inverses (3), points on number nine (1)).		5		4
Uninterpretable reasons		3		4

figures at the same time indicate that a much higher percentage of pupils spontaneously resort to analogical reasoning (which was never suggested by the interviewer), than the percentage of pupils resorting to reasoning in terms of temperature (which was strongly suggested by the interviewer through the introductory questions on temperature changes). Unfortunately no addition question of the form $^{-}5 + 3$ (negative number first) was included. We expect that more pupils would resort to temperature arguments in this case. We note though that the subtraction question $^{-}8 - 3$ yielded a rather low facility level of 19% (see table 1), although it can be interpreted very naturally as referring to a drop in temperature. Only 10 of the 97 pupils correctly answered this question by referring to temperature, while 9 pupils gave the wrong answer $^{-}5$ with reference to temperature.

Although our analysis of the data is as yet incomplete, we seem to find strong support for Murray's original hypothesis, though it is clear that high facility levels obtained prior to formal instruction in integer arithmetic are partially due to purely instrumental strategies. In particular, the data seems to indicate that while temperature is a useful context for introducing the concept of negative numbers, its employment as an embodiment to support calculations with integers is not readily accepted by pupils, and when used, often leads to mistakes. On the contrary, a large proportion of pupils seems to adopt analogical strategies spontaneously, and use them quite correctly.

In the second project, Malan taught integer arithmetic to full classes of 4th, 5th, 6th and 7th graders for one week

(5 half-hour periods) during April 1984, and again for the same period during October 1984. The teaching was strictly negotiative in nature, and at no stage were any computational strategies demonstrated by the teacher, except that pupils were periodically reminded that they may use a vertical number line (provided on a printed sheet) as an aid. During the first period pupils were introduced to negative numbers with reference to temperatures below zero, and a vertical number line as representing a thermometer. Further class activity consisted of the pupils doing exercizes involving the various cases of addition of integers during the April session, extended to the various cases of subtraction during the October session. The fifth period of the April session, as well as the first and fifth periods of the October session, were used to apply a written test on all cases of addition, subtraction and multiplication of two integers. Pupils were also asked to explain in writing how they obtained their answers, except the 4th graders who were questioned orally on their strategies. The results of the April post-test are given in table 1 above. The October pre-test reflected losses of up to 35% in facility levels for addition, yet increases of up to 30% (grade 4) in the levels for $^{-}a - {}^{-}b$, $a > b$. The October post-test reflected gains of between 15% and 40% for all cases of addition, as well as the abovementioned case of subtraction, which now yielded facility levels of above 84% for all the grades. A rather interesting feature of the data on the strategies employed by the pupils was a marked decrease in number line and temperature arguments from the April post-test to the October pre-test, followed

by an increase in embodimental reasoning from the October pre-test to the October post-test. This tendency was also very marked in a continuation of the experiment with the 4th and 5th graders in March 1985. It appears that when left to their own devices pupils tend to use analogical strategies rather than embodimental reasoning. This is further evidence for the conclusions reached in the earlier studies reported above.

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NECESSITY OF A TRIPLE APPROACH OF ERRONEOUS CONCEPTIONS OF STUDENTS, EXAMPLE OF THE TEACHING OF RELATIVE NUMBERS.(I) THEORICAL ANALYSIS

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A propos des représentations cognitives inexactes que la Didactique des Sciences a mis en évidence, on voudrait montrer la nécessité d'une triple approche Didactique, Mathématique et Psychologique pour étudier les problèmes d'enseignement. On mots successivement la nécessité d'une connaissance mathématique précise de l'écart entre le "savoir savant" et le "savoir enseigné", celle d'une étude des conséquences du "contrat didactique" et celles de deux sortes de stabilités des représentations cognitives, l'une "structurelle" et l'autre "fonctionnelle". L'argumentation s'appuie sur des travaux concernant essentiellement la sériation des décimaux, elle est illustrée, dans une seconde communication, par un travail sur l'enseignement des relatifs.

Many works in the Didactics of Sciences have shown student's erroneous conceptions that link to some situation or some notion. There are many examples of those in physics but we can find the same in other experimental sciences and many problems studied by Psychologists. Those conceptions are very stable : they can accept new experimental data without fundamental modifications, and so they are an obstacle to teaching.

We propose to study this important teaching problem by a triple approach, that would be at the same time a psychological, a didactical, and a mathematical approach. In a first paper, we present the theorical background, in a second paper, an application of this theorical approach to teaching additions and multiplications on directed numbers.

Mathematical approach

Erroneous conceptions are so stable because they are not always incorrect. A conception that fails all the time cannot persist. It is becaus there is a local consistency and a local efficiency in a limited area, that those incorrect conceptions have stability. For example, the multiplication is really a repetition of additions in whole numbers, but not in decimal numbers.

This local consistency explains why, sometimes, those incorrect conceptions are near historical conceptions that have marked some step of the construction of a notion. So that an epistemological survey is always useful before the teaching of a notion would be studied. Historical conceptions can be of help for the identification of students' erroneous conceptions.

A mathematical work is also necessary to understand the limits of the mathematical correctnes of students'conceptions. For what problems are those conceptions mathematicaly correct ? For what problems are they erroneous ? It is only when we know the mathematical limits of the student's conceptions, that we will be able to know when their conceptions will fail, to prevent them, and eventually to teach them to students .

Student conceptions do not make sens in the mathematical field alone but also in the teaching situation. Students use an erroneous conception becaus it allows some answer to the teaching problems. So this work cannot be done out of the teaching situation.

The mathematical and didactical approaches are also necessary for the definition of what is taught. It is generally impossible to teach perfectly correct notions, because perfect notions are too complex for students. The teacher must transform scientific knowledge into teaching knowledge, that is, a knoweldge that could be taught. It is a mathematical task to identify the difference between scientific knoweldge and teaching knoweledge, and a didactical task to define what knoweledge can be taught.

Didactical approach

In the classroom, the student is enclosed in a situation wich entails many obligations. He wants to resolve problems as efficiently as possible. He will be satisfied if he can use a conception that allows him a good percentage of correct answers. This is an effect of what Didacticians name "didactic contract".

There is an other effect of the same contract . The teacher must adapt his questions to the student's reactions : therefore he tends to avoid unusual problems or questions that decrease the class performance whithout clear reasons. But it is possible that the performance decrease because the problems proposed are problems that erroneous conceptions cannot solve. If the teacher avoids those problems, he prevents the students from seeing that their conceptions fail. Thus he gives better efficiency to erroneous conceptions.

We have found some traces of those effects of the "didactic contract" in school books, for ordering decimal numbers. Incorrect conceptions of the learner on ordering decimal numbers give the correct answer to the main part of exercises that we have found in school books

Another trace of the same adaptation of teaching to students' learning processes is the nature of the examples that are given to students. The teacher is generally happy when he finds a "good example" for new knowledge, that is when he finds some non mathematical presentation of the notion that permits students to give correct answers. But he ough to be disturbed when this happens! If a new knowledge is immediatly "understood" as a former one, the novelty of the knowledge is necessarily lost. It can be a learning step, but the teacher must pay attention to the fact that students must get out this first analogy. Teachers must not forget that students have not learned the new knowledge, but that they have recognized a former knowledge in the new one.

Psychological approach

Students'erroneous conceptions exist because they have a mathematical consistency, and because they take meaning in a didactic process. But if they had no psychological signification for students, it is obvious that they could not be able to exist. Erroneous conceptions exist because they participate in the learning process. So understanding students' erroneous conceptions implies an understanding of those learning processes.

A major problem concerns the stability of students'conceptions. Working memory has a very small capacity, and it cannot be enough for all informations that a new task contains for the learner. Learners who do not know the task do not know what is important and what is not. They do not know where the information is, and so all can be information and all is always too much. So the learner must use his former knowledge to organize and the reduce the amount of information. Problems arise when the learner cannot leave his first organization because it is too stable.

We have proposed elsewhere(Léonard 198?) to distinguish two forms of this stability : a "functional stability" that depends on the use of the organization, and its efficiency, and a "structural stability" that concerns its consistency and can be understood as Piaget's equilibrium of a scheme.

Former knowledge can have a strong stability through their use or because they have a very simple structure (like gestaltist good form) If the former knowledge seldom fails on the new set of problems, the functional stability of its conception will increase. For example, 92% of pairs of decimal numbers, with no more than three digits in the decimal part, are correctly ordered by the first erroneous conception of students (Sackur-Grisvard & Léonard 1985). So this conception is very stable.

The stability of an erroneous conception can come from another area. It is difficult to teach Newton'laws of forces because in the everyday world where friction is present there are many examples that support students'erroneous conceptions of forces and motion (Driver and al. 1985).

We have found a similar example with decimal numbers. An erroneous conception was very rare in French student's answers but frequent in other countries (Israel, Great Britain and the U.S.A.). That was because the cursus was different in those countries. Rational numbers are taught before decimal numbers in the U.S.A. but after them in France, and the erroneous conception for decimal number is correct for rational numbers. So in the U.S.A. erroneous conceptions would have a good stability for rational numbers (Resnick and al. 1987).

The teacher must give problems where erroneous conceptions fail, but that cannot be done at just any time. Correct conceptions are modifications of erroneous conceptions. They are built upon them, moreover erroneous conceptions must be sufficiently stable so that the construction could lean on them.

On those principle we have done a micro-computer program on the ordering of decimal numbers that identifies the erroneous conception of learner and gives him problems adapted to his level. In each set of problems one part of the numbers can be correctly ordered by erroneous concéptions of the learner, but one part cannot be. The subset of problems that cannot be correctly ordered bu the learner's erroneous conceptions is divided in two parts : in one the numbers can be correctly ordered by more elaborated, but erroneous,conceptions, and in another part they can only be ordered by the correct conception (Léonard and al. 1987).

To teach, it is important to know the first students' conception and, above all, the other intermediate conceptions before the correct one. We must know what conceptions are called by the examples that are given in the didactic situation, and we must estimate their stability. This work cannot be done from a mathematical, a didactical, or a psychological point of view alone. We hope that we have shown that this triple approach is a necessity to study such a difficult teaching problem.

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NECESSITY OF A TRIPLE APPROACH OF ERRONEOUS CONCEPTIONS OF STUDENTS, EXAMPLE OF THE TEACHING OF RELATIVE NUMBERS-(II) EXPERIMENTAL WORK

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Sur la base d'analyse psychologiques et mathématiques, nous avons analysé l'enseignement des relatifs dispensé dans les classes pour comprendre en quoi il pouvait être générateur de représentations inexactes et trop stables, de manière à proposer une progression minimisant ces inconvénients. Un nouvel enseignement propose aux élèves des activités (repérage dans le plan, exercices sur des couples d'entiers positifs, équations sur des grands nombres, exercices sur les trois significations du signe "moins" et opposé comme résultat de la multiplication par -1) dans lesquelles les relatifs sont des nombres nouveaux qui permettent de résoudre des équations, mais on ne propose pas de règle de calcul. Une analyse clinique et une comparaison entre groupe expérimental et groupe contrôle permettent d'estimer les résultats de cet enseignement.

We shall apply the analysis developed in the theorical part of our paper to the teaching of directed numbers and the learning of the rules of calculus of algebra. This teaching takes places during the first two years of Junior High School with students aged from 11 to 13 years old.

We have first been concerned with the evidence that stable, incorrect conceptions show up in the work of the students concerning directed numbers. These conceptions are of two main types :

When they start working with directed numbers, students are given simple concrete examples and build up representations that we can call "elevator type representations". These conceptions lead to simple procedures wich prove to be correct and efficient on additive

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We want to thanks M.Jean Louis Roux, teacher at the College Valeri in Nice who worked on this curriculum with us and welcomed us in his classroom as observers. Texas Instruments provided us calculators TI 30 Galaxy, used in this experiment.

(substractive) problems. After some work on addition, the students learn multiplication. The rule for multiplying directed numbers, i.e. $- \times - = +;\ - \times + = -$. is isomorphic to the gestalt of logic equivalence. Students "understand" quite easily this rule and produce correct answers on products of directed numbers.

These two conceptions are strong and efficient on numbers. Each one covers a certain area of problems for wich ut has been introduced. Nevertheless students lack a synthetic conception wich could cover the complete area of directed numbers. In the usual curriculum, directed numbers are merely natural numbers with a sign; there is a list of rules to deal with the sign, but the different rules have no conceptual basis in the mind of the children. Thus one can foresee difficulties for children when they have to combine additions and multiplications, and later when they start working with letters and not only numbers. The observations we made in the classroom confirm this analysis. A group of 175 students who followed the usual curriculum on directed numbers was tested three times during the first three years of Junior High School, on addition and substraction. The results of the tests are the following : (% of correct answers) 1rst year, before multiplication is introduced, 100%, 2nd year, after multiplication is introduced, 60%, 3d Year, 70%.

For the psychological point of view, we can conclude that if old and elementary conception are too stable, they impede students from proceeding in the process of learning. The new knowledge cannot be assimilated by their cognitive organisations and the students remain at an intermediate level, only capable of dealing with the class of problems that links to their cognitive organizations. As the teaching goes on, their performances decrease as they fail to solve mode advanced problems.

In the mathematical theory the status of directed numbers results from the necessity to solve equations wich prove to have no solutions with natural numbers. A mathematical analysis led us to give a proeminent role to the concept of opposite numbers. We can state that algebra can be reduced to a correct understanding of this concept. An obstacle to this understanding is the polysemy of the sign "-", wich is altogether the sign of substraction, the sign of negative numbers, and the sign of the opposite of a number. As we saw above, another obstacle is the existence of the two non-coordinated conceptions, one wich works on addition, the other one on multiplication.

A link between them can be made by considering the opposite of a number as the result of the product of the number by (-1)

On the basis of both the psychological and the mathematical analysis, and on the observations made about the results of the actual teaching, we built up a new curriculum on directed numbers. We tried to avoid the main inconveniences or at least tried to minimize them.

The new curriculum has been introduced into one class in 85/86, and into another one in 86/87, where it is still going on. Each group has twenty four students.

The main activities of this curriculum are the following :

-couples of coordinates in the plane rather than on an axis to avoid a too strong symetry between positive and negative numbers;

-concrete examples introduced directed numbers and their opposite as differences of natural numbers (3,7) = -4 = (10,14) ...

-equations on large numbers;

-introduction of letters as soon as possible and work on

$$-(a + b) = -a - b, \text{ where } -a = (-1).a$$

These activities lead the students to consider that directed numbers are really new numbers (not only natural numbers with a sign) While solving the proposed problems they build the set of directed numbers and the procedures to use them.We observed some unusual behaviours such as :

* (1) computing the sum (-31)+(-71)+(-136) they write :

$$\begin{array}{r} -31 \\ + \ -71 \\ + \ -136 \\ \hline -238 \end{array}$$

(2) comparing a large number of additions they sort them into three groups : those with only positive numbers, those with only negative numbers, those with both positive and negative numbers.

The additions in the two first groups are isomorphic to additions in $\mathbb{N}$, the third one is different.

Of course the students have not been given the rules for computation with directed numbers nor any procedure to deal with them.

We make systematical use of a pocket calculator during the lessons. The calculator we choose (TI 30 Galaxy) displays as well as the numbers, the sign of the calculations wich are being performed (+,-,x, %). Thus it displays two different signs "-", one attached to the number at the right hand side of the display area (-5), the other one at the left hand side of the display area. The student can distinguish between those two signs. The use of the +/-key completes the work on the polysemy of the sign "-".

Using calculator, students can perform some activities which turn out to be impossible without. The calculator "knows" how to calculate with negative numbers, human beings do not. They can only compute with positive numbers. To calculate 371 - 8146, they do 8146 - 371 and then change the sign. This capacity of calculators to works with negative numbers give them a reality which helps the students to construct the knowledge. On the other hand, and although that may seem contradictory, to handle large numbers (four or more digits) the students must create new prodedures. If the students solve 5 + x = 2 or -3 +x = 1, they can carry on very local procedures, like counting up or down which will not work on the equations 8736 + x = 3549, or -346 + x = 127. In fact we did observe that :

(3) the students use opposite numbers in algebric sums before they have been tought to do so;

(4) additive equations are solved without difficulties.

The clinical analysis wich is going on while the new curriculum is being taught allows us to think that our main goals have been achieve :

-the students have constructed a new set of numbers with specific rules for computing with them (observations 1,3 and 4);

- the polysemy of the sign"-" is mastered (observations 1,2,3)

Beside the clinical analysis a collective test is carried on comparing the experimental group and a control group.

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Cognitive development

EMPIRICAL INVESTIGATIONS OF THE CONSTRUCTION OF COGNITIVE SCHEMATA FROM ACTIONS

W. DÖRFLER, University of Klagenfurt, Austria

Abstract. The theoretical basis of the interviews reported about here is a Piagetian-like approach to the origin and genesis of cognitive schemata representing mathematical concepts. Such schemata are postulated to reflect the abstract and general structure of material, imagined or mental actions and of relations induced by these actions. The main cognitive tools for the mental construction of such schemata are seen to be: Actions, symbolic representations, prototypes of objects, reflection and abstraction, schematization, generalization. The interviews were devised such that the subjects were guided appropriately in their individual cognitive constructions. The mathematical topics treated are: Place value system, divisibility, word problems, geometric sequence, Riemann integral. In general the results support the view that the individual construction of cognitive schemata is possible and effective in the proposed way.

The research reported about here was carried out within a project at the University of Klagenfurt which was directed by the author. Members of the project team were H. Kautschitsch, G. Malle and W. Peschek. The interviews were partly made by three high-school teachers. The whole project was funded by the "Fonds zur Förderung der wissenschaftlichen Forschung (Wien)". Of course only a small part of the total work and of the results can be presented here, for a more complete overview see Dörfler (1987) and my contributions to former PME conferences.

1. Theoretical background

The basic starting point for the research was the constructivist position that individual knowledge is the result of a personal construction by the learner and is organized, stored and represented in structures modeled by cognitive schemata or frames. The general aim was then to get insight into the process of the construction of cognitive schemata corresponding to mathematical concepts: What can be means for and conditions of these constructions? Of course this would be to general a question for sensible investigations. Therefore the general problem was narrowed to the question: Which role can play material actions in the cognitive building-up by the learner of a mathematical concept or method? This research program is clearly related to Piagetian conceptions as interpreted for instance by Aebli (1980/81) or Papert (1980): Cognitive schemata (for mathematical concepts) reflect abstract and general structures (coordinations) of material (and mental) activity of the human being and they result from consciously reflecting on one's activity. The cognitive constructions are here considered not as being automatic or spontaneous but as intentional and controlled processes which are governed by the individual and the material which he/she is acting with. A main feature of these constructive processes turn out to be certain types of generalization and abstraction. Essentially the research question then was: What does it mean to generalize actions or to abstract from the action in cases of specific mathematical concepts? How can one initiate and guide such processes in a learner? The importance of these issues results also from the empirical research carried out by Krutetskij (1976). There is demonstrated quite clearly that mathematically successful pupils mostly recur to (imagined) actions and to far reaching generalizations and abstractions when solving for instance difficult word-problems.

On this basis we developed a theoretical approach and arrived at the following consequences which then served as hypotheses for the construction and organization of the interviews:

- An epistemological analysis of many mathematical concepts shows that the essential kernel-structure of these concepts consists of the formal and generalized schema of material actions and induced relations between the elements of the actions. The learning of these concepts has therefore to be organized as a cognitive process of abstraction, generalization and formalization by which the individual constructs a cognitive structure comprising the general schema or form of the actions and the induced relations.

- The individual construction of cognitive schemata representing mathematical concepts will be initiated, supported and guided by having the learner carry out adequate actions and reflect on them.
- Reflection on material (or imagined) actions needs guidance of the attention of the learner to let him recognize the relevant relations. Here the teacher (or the interviewer) has to come in, but also the kind of elements on which the actions are carried out play an important role.
- An indispensible tool for the cognitive construction of the formal schemata of actions and relations are representations of the elements of the actions by iconic (geometric) or algebraic symbols (media). These symbolic representatives play the role of prototypes of the material objects (elements of the actions). They exhibit in full clarity those characteristics (properties and relations) which are relevant conditions for the actions to be possible or are outcomes of the actions.
- The iconic or purely symbolic prototypes of elements of the actions yield abstraction and essential schematization of the actions, of their elements and of the induced relations. By that the prototypes and symbols serve as the basis for processes of generalization: The prototypes support the recognition of, the search for, and the construction of objects with which actions of the developed schema can be carried out resulting in formally identical relations and outcomes. The field of applicability of the type of action under consideration is thereby extended. In a way, the prototpyes represent the potential generality of the action-schema.
- If a concept (i.e. a representing cognitive structure) is built up in this way (i.e. via action-represen-

tation-reflection-abstraction and schematization-generalization) it will be possible to go back to its operative origin (when for instance solving a problem by use of the concept) and to think in terms of the constitutive actions (as carried out on respective prototypes).

2. The interviews

To get empirical support for the theoretical conclusions several interviews were developed. They were devised such that by the sequence of questions, hints and cues a cognitive process along the theoretically described lines could be induced in the mental activity of the interviewed subjects. According to the theory, guidance of the attention of the subject by the interviewer is needed (how to act, what to symbolize and so on). The thesis to be supported reads: If the actions are carried out and if they are appropriately symbolized and reflected then this results in an adequate cognitive structure and individual concept. Since we hold our theoretical approach applicable for all mathematical topics and all ages both of these variables were chosen from a broad range. The topics were: Place value system, rules of divisibility, word problems (following Krutetskij), geometric sequence, Riemann integral. Children of various ages (where possible) and adults were interviewed as well. The interviews on place value system and divisibility are amply documented in Peschek (1985).

The interview which was intended to lead to the construction of a cognitive structure representing the essential features of geometric sequences was organized in the following way. The subject was presented with a strip of paper, scissors, a ruler, a pocket calculator and asked to cut the strip such that 3/4 will remain. This action then was to be iterated several times and each time the relation between (the lengths of) the two successive strips was to be described. This could be done in a multiplicative ($x_{n+1}=(3/4)x_n$) or additive way

($x_{n+1}=x_n-(1/4)x_1$). Both possibilities occurred in the interviews. After some iterations and the corresponding notations the attention was led to the relationship between (the lengths of) the first and the last obtained strip. This relation was to be noted which naturally led to preferring the multiplicative over the additive notation. This first part of the interview could be called "action and representation". There have already occurred some generalizations when the relation of an arbitrary strip to its predecessor and to the starting strip was described by a formula. This description is the starting point for the next part "generalization" where the abstracted formal structure should gain referential meaning. This process is guided by questions and requests like: Describe verbally the essential features of what you have done before with the strips! What other fractional parts could you choose? Are the numbers bound to be less than 1? Can they be negative? On which other materials can you carry out similar actions? What can be varied among the constituents of the action? Give some examples! How else can you describe the actions and their outcome? Can you find a common notation for all the actions? At the end of this part the now generalized common structure is termed "geometric sequence". The third and forth part check to which extent a cognitve structure representing "geometric sequence" has been built up by the subject. For that 11 different numerical sequences were presented for which it was to be decided if and why they are geometric sequences. Then three word problems were posed which involved geometric sequences for their solution (interests, geometric increase, radioactivity). In the final part the subjects had to recall the meaning of "geometric sequence" and to give examples on their own. The last question was presented in written form: How do you judge the value of the "tools", especially of the strips? This interview was carried out with 10 subjects (age 13-22 years) and each took more than one hour. The subjects did not know the concept "geometric sequence".

Some results:

- The choice of the appropriate denotation and the adequate variables had often to be supported by the interviewer. But after overcoming these technical problems in all interviews the essential relations were abstracted from the actions on the strips.
- The generalization mostly is quite far-reaching; it is recognized what kinds of objects are suited as elements of such actions.
- The generalization to factors greater than one or to negative factors in some cases is impeded by too closly remaining at the generating actions. In other cases a factor greater than 1 is correctly interpreted as the action of adding to the strip and/or the generalization to negative factors is obtained at the level of the formal notation of the detected relations. Also for the referential generalizations the important role of the symbolic representations as a thinking tool gets quite clear.
- The numerical sequences in general presented no problems and the reasoning could often be carried out by the use of imagined actions, i.e. if needed the subjects can return to the starting point of their conceptual development. This is also true for the word problems though there were quite a few difficulties with recognizing how to apply the acquired knowledge.
- All subjects can summarize the essential structure and are quite positive about the cutting of the strips. Apparently the strips can serve if necessary as prototypes for the elements of pertaining actions.

Just to give a vague idea of the other interview series I quote the word problems used there:

1. Static, not action-oriented version: A man and a woman drinking water in equal portions every day can get by with a given supply 18 days. The man alone can do with the same supply 24 days. What is the ratio of the portions of the man and the woman? If the subject cannot solve it (what was always the case!) an action-oriented version was presented.

2. You lay squares with a red and a blue side in rows. If each row contains m blue and n red squares you need 6 rows. If each row contains just n red ones you need 9 rows. What is m:n? A solution by switching and shifting the blue squares was developed, described and then applied to version 1 which then (mostly with some troubles) could be solved. Here pictorial symbolizations were used which proved not to be very successful and it might be that the lacking of algebraic descriptions (like a(m+n)=(a+c)n) was a great obstacle for effective abstraction and generalization. Nevertheless two other versions (exchanging bank-notes of two different values, bottles of two different volumes) mostly could be solved along the abstracted pattern.

The interview series for the Riemann-integral follows a similar organization of examples where situations (from physics for instance) had to be interpreted by use of the integral. The relevant actions are here from the beginning (symbolic) actions with mathematical objects (numbers, functions, areas, volume). The abstract structure of these actions and the induced relations on the objects is just the standard definition of Riemann-sums.

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TESTING A MODEL OF COGNITIVE DEVELOPMENT WITH RESULTS FROM A LONGITUDINAL MATHEMATICS STUDY

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Results from a longitudinal mathematics study have been used to test a two-parameter model of cognitive development. The linearity of the data for students of widely differing ability from schools set in different socio-economic regions of two countries supports a model in which the development of mathematical concepts is largely independent of environment. Interviews with students indicate that those who show uneven rates of development have inconsistent approaches to different mathematics problems, in contrast with the consistent approaches shown by students whose rate of development is more uniform.

Few longitudinal studies concerning the development of mathematical concepts have been reported (see for example Carpenter,1980). Keats (1978) observed that no longitudinal data of Rasch ability measures was available to test theoretical relationships for studying the development of ability.

From a recent large scale study of the development of abstract mathematical reasoning (Ellerton, 1987), a sub-sample of 87 students from two secondary schools was followed longitudinally. The same set of mathematics questions was given to the students three times spread over a period of three years (average age at first test in Year 9 was 13-14 years). Twenty one students were from School A, set in a low socio-economic area of Wellington, New Zealand, and 66 students were from School B, set in a high socio-economic area of metropolitan Adelaide, South Australia.

A COGNITIVE DEVELOPMENT MODEL

Halford and Keats (1978) proposed a developmental curve which related ability (A_{ij}) of person i at time t_j to time as:

$$A_{ij} = \frac{M_i t_j}{t_j + k_i} \qquad \ldots . (1)$$

where M_i is the maximum ability for person i and k_i is the growth rate.

Keats (1982) pointed out that a two-parameter model for cognitive development can be written: $1/A_{ij} = c_i + d_i \,/\, t_j$ . . . . (2)
c_i, d_i are individual differences and rate of growth parameters respectively. Since equation (1) can be written as $1/A_{ij} = 1/M_i + k_i/M_i t_j$, it is clear that equations (1) and (2) are equivalent representations if $c_i=1/M_i$ and $d_i=k_i/M_i$. Time is taken from birth, so t_j corresponds to age.

Keats (1978, 1982) discussed the possibility of testing this model for cognitive development with ability measures found by applying the Rasch model to longitudinal studies. He pointed out, however, that no such results were available at that time. The longitudinal data described above provide the necessary data for testing the model.

TESTING THE MODEL

Rasch analysis was used to find the ability of the students at each age tested. The abilities of students over the study period fell into three main categories, showing: (a) an increase over the test period; (b) an increase at the second test, then no change or a small decrease and (c) a decrease or unchanged at second test, then an increase. Category (b) is a ceiling effect observed for some students with high scores in two of the three tests, and category (c) suggests environmental influences (such as illness) giving negative influences on development. Only the abilities of those students who fall into category (a) can be used to test the model described by equation (2) since a tacit assumption in its formulation was that the development of ability show a continuous increase with time.

As Rasch abilities can be <0 for an individual, later becoming >0 for the same individual on the same test, raw Rasch ability values cannot be used to test the model. A simple scaling procedure can convert raw Rasch ability values (which centre around zero) to A_{ij} values which would centre around 100, according to equation (3). Rasch ability values of 0.2 and -0.5, for example, give A_{ik} = 104 and 90 respectively.

$$100 + (20 \times \text{Rasch ability}) = A_{ij} \qquad \ldots . \quad (3)$$

By converting raw Rasch ability values to A_{ij} values with equation (3), the longitudinal data in this study can be used to test the model described in equation (2). It can be shown that the scaling of Rasch ability values is valid in the context of testing for this model, provided that the t_j values are large, as in the study here. Figure 1 shows typical results of plotting $1/A_{ij}$ against $1/t_j$ for students whose abilities showed an increase with time (category (a) above). A linear or close to linear relationship between $1/A_{ij}$ and $1/t_j$ was found for many students. Different slopes for plots for different students is consistent with the two parameter model for cognitive development which requires an individual rate of development parameter d_i as well as an individual differences parameter, c_i. The plots show that c_i and d_i are independent, with a wide range of c_i values for students with similar d_i values and vice versa.

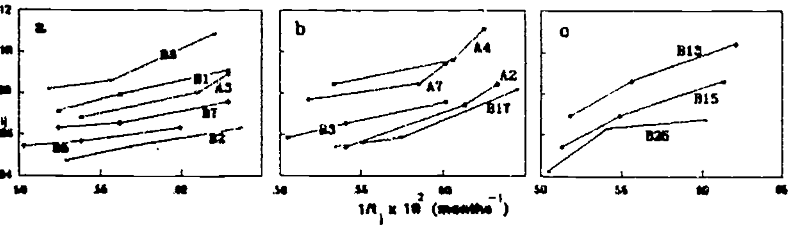

Figure 1: Reciprocal of ability vs reciprocal of age for students from Schools A and B.

The overall linearity of the 1/Aij vs 1/tj data for students of widely differing ability from schools set in different socio-economic regions in different countries is evidence to support a model of cognitive development in which the acquisition of mathematical concepts is largely independent of a particular environment. This is the first confirmation of this model with Rasch ability data.

INTERVIEWS WITH STUDENTS

Ability data alone, however, cannot be used to investigate the reasons for any departures from the linearity of the 1/Aij vs 1/tj plots such as those exemplified in Figure 1. Interviews with forty five students from School B focused on several of the test questions in an attempt to reveal any response patterns that may be associated with different attitudes and approaches to mathematics which, in turn, may exemplify particular types of 1/Aij vs 1/tj plots. Excerpts from interviews held with the students in Year 12 are discussed below; original question numbers are retained.

9. B is a number. What is B if B + 4 = 27 - 4?

18. Make up a short mathematical problem that you think would be quite difficult for a friend of yours to solve. Show how you would work out the answer.

37. A car firm has two types of car transporters available. Type A can take 9 cars, type B can take 8 cars. There are 77 cars to be carried to another city, and 9 drivers are available. The transporters must carry a full load.

☐ type A transporters and ☐ type B transporters will be needed to take the cars.

Student 2, School B - boy (B2 in Figure 1a)

Question 9: Mentally, twenty seven minus four is twenty three; subtract four again is 19. Then a quick mental check; nineteen and four is twenty three.

Question 18: If I had a problem which I'd seen before and I could remember and I didn't know the answer, I probably would give that problem to a friend and see if he could solve it and find an answer. And then go through his working and see whether I'd missed out on something.

Question 37: Now this is a nice one. I obviously see that I need simultaneous equations here, so I say let the x transporters, x Type A transporters. So in other words, nine times x, that's that amount plus eight times another amount y, has to equal seventy seven. . . And also, the number of transporters, assuming that one transporter has one driver, then x plus y equals nine because there are nine drivers . . .

I'll need four Type B transporters and, substituting that into the second equation because it would be easiest, they'll require five Type A transporters. And I'll just try a quick check; nine fives are forty five plus four eights are thirty two, that's seventy seven.

$$9x + 8y = 77$$
$$x + y = 9$$
$$x = 9 - y$$
$$9(9 - y) + 8y = 77$$
$$81 - 9y + 8y = 77$$
$$-y = -4$$

General Comments: . . . I enjoy maths, that is to say I occasionally have a tendency to get bored ...(when I had) twenty six questions all of exactly the same type and that sort of got boring, I would go through half of them, thirteen, and make sure I got them right, then try and program them onto my calculator. So I'd do the rest of them like that.

The high c_i value (238) for Student 2 is consistent with his command of spoken mathematical language, his thoroughness (he checked almost every answer) and the maturity with which he approached the question on making up a problem (he would effectively challenge a friend to solve a problem for which he didn't know the answer so that he could 'go through his working'). His fluency with mathematical language appeared to be intertwined with a rule oriented approach to the subject ('I obviously see that I need simultaneous equations here', Question 37).

Student 5, School B - girl (B5 in Figure 1a)

Question 9: Twenty seven take four is twenty three so that's twenty one and four. No it's not. It's twenty three add four. Hang on. Twenty seven take four is twenty three. So it's nineteen add four is twenty three, so it leaves nineteen.

Question 18: At the moment (I'd make up) some sort of limit problem because that's what I'm doing at the moment. I don't understand it!

Problem 37: . . . Nine nines is eighty one. No it's not that many. Nine eights are seventy two; no. I: What are you doing? S: Just trying to see, just trying different combinations, with the nine drivers. I've tried taking them all in Type A, and that doesn't work, so I tried taking A and taking the extra on the eights. I: So what might you try next? S: Try six, because eight was too far off. Nine sixes are fifty four, so x is twenty three. Doesn't go either. Fives; no, that leaves thirty two which can go into the eight, which goes four, so that works.

General Comments: (It's hard) when I see something that I don't know how to start, or that it's something that we don't usually do normally. . .Textbooks - they usually go through and have a million of one sort of problem and then a million of the next sort of problem, and when you just get one in front of you, it's pretty difficult to try and figure it out.

Student 5's approach to mathematics problems forms a strong contrast with that of Student 2's fluency with the subject; the c_i value for Student 5 is 185, but the slope of the plots for Students 2 and 5 in Figure 1a are very similar, implying that development is occuring at similar rates.

Although both Students 2 and 5 worked out that nineteen was the correct answer to Question 9, Student 5 only arrived at this answer by back tracking from an incorrect answer of twenty four, whereas Student 2 solved the problem in the minimum of time, using a very organized strategy. Student 5 appeared to be consistent,though,in her own approach to different problems.

Both Students 2 and 5 were cynical about all of the textbook exercises required of students, although the former used to get bored with the exercises and would program his calculator for half of them, while the latter found that she was unable to decide how to tackle a problem out of the context of a set of text book exercises. Both agreed, too, though they expressed this in different ways, that their made up problems would centre on something they were familiar with at present, and that this could mean that the solution would be difficult for themselves.

Student 8, School B - girl (B8 in Figure 1a)

Question 9: So I have to find B, and if B is B add four which equals twenty seven, take four. Well, twenty seven take four is twenty three (writes 23). In that case, that number plus four has to equal twenty seven so it must be - oh, no - (pause) That's right, then so from twenty three to find out what it is which you add four to you take four from twenty three so you get nineteen. So nineteen plus four equals twenty three. (Pause) Yes. Which equals twenty seven take four which is twenty three so I'll put nineteen.

Problem 18: I'm not sure. Umm, it has to be something I can work out myself . . . Something to do with finding the value of x or if you had so many things . . . which equalled such and such a price. something like how many children could be divided up between whatever.

Question 37: (Reads out question slowly. Writes down A=9cars, B=8cars, also 77 cars, 9 drivers.) I: Where would you start? S:You'd have to find out how many cars go into ... how many loads of A and B will fit into seventy seven. And divide it by the nine drivers. Well, 9 cars, (long pauses)... (she eventually gave up).

General Comments: I always tried, but I've never been a star at maths. I excell in other subjects.

Student 8 was of lower mathematical ability than Students 2 and 5; the plot for Student 8 in Figure 1 extrapolates to give $c_i = 125$. The rate of growth parameter, d_i, is, however, approximately the same as the that for Students 2 and 5. The contrast between these students, though, in the methods used and in the mathematical language spoken, is striking.

Student 8's approach to Question 9, for example, was full of stumbles and uncertainties, and took much longer to reach the tentative answer of nineteen. She kept trying to remember what she thought she was supposed to do to solve particular problems. She seemed to lose track of where she was or what the object of her calculations were. Perhaps many of her problems in mathematics stem from an inability to retain the necessary schema over both short and long terms. Whether this is associated mainly with mathematics is open to speculation, and to what extent the lack of short and long term memory for mathematics can be associated with her negative attitude to the field is impossible to gauge.

The difficulties she experienced in the test consistently involved those problems for which several facts had to be retained from one step to the next. Thus although the three students whose interview excerpts have been discussed above differ markedly in mathematical ability, they have all been consistent in their approach to the whole test. It can be hypothesized that this consistency is a major factor in determining a uniform rate of development; inconsistencies in approach would be expected to lead to rapid gains in some areas and slower ones in others, thus destroying any uniformity in the rate of development during different periods. Interviews with Students 13, 15 and 25 all revealed such inconsistencies.

Student 13, School B - girl (B13 in Figure 1c)

Question 18: Probably try an algebra one. I'd do one that I know if the answer was right or wrong. I might choose one that I might find easy and that I think my friend would find hard. I: How would you try to make it difficult? S: Use bigger numbers, I suppose. 9A + 36 = 77
Question 37: I think I'm completely off the track...
A can take nine cars so you say it's nine A and it would 9d + 77
be eight B. I know I'm doing it wrong because they've got nine drivers.
General: I'm not very good at maths. Never have been, never will be.

Student 15, School B - boy (B15 in Figure 1c)

Question 18: x squared plus four equals eight. . .First of all you'd take four over the other side which means that x squared equals four, and then you'd square root the four which gives you two so x equals two.
Question 37: That one would take a bit of thinking to do... What I've done is to put out the factors of nine and eight, and because you have to use nine drivers, you have to find a factor of nine and a factor of eight that will add up to seventy seven, which also has to be nine altogether, to make up nine drivers. So . . . Type A which has to have nine on each trip, carries forty five cars which nine fives are forty five so you've got five drivers and Type B carries thirty two which is four drivers, total of nine . . . (I used) more or less trial and error.
General Comments: . . . I like to make sure I know what I'm doing; I like to do the trial and error way as well as working out the mathematical way. It's a lot easier doing it the mathematical way, but you've sort of got to know exactly what to do.

Student 13 did not realize that she had correctly formulated one of the equations for Question 37. She had little confidence in her own ability. Although Student 15 was able to solve Question 9, he stumbled, at first, on another similar question of lower item difficulty. He missed the subtlety of the answer of ± 2 for his made-up problem. His desire 'to do the trial and error as well as working out the mathematical way' may have been his way of checking that he was on the right track before he used 'the mathematical way'. His approaches were not consistent across the test; he seemed to be considerably less secure with some concepts than with others.

Student 25, School B - boy (B25 in Figure 1c)

Question 18: It's an expansion of brackets, a cubic example...it was difficult when I first started; it's not at the moment. A few people I know wouldn't be able to do it.
Question 37: I do this by trial and error; it's probably wrong to do it this way, but I just experiment with numbers. Well, you've got nine and you've got eight cars available to be carried by each mode of transport, therefore you find out the numbers that will combine to give

you seventy seven. Chances are that it would be intermediate, between the two, because you've got nine. See with nine and eight, chances are it would be very close to carrying half each. Chances are the numbers would be either five or four, so could be nine times four and eight times five or nine times five and eight times four, which would give you nine fives are forty five, eight fours are thirty two, (writes these down). Yes, it's nine times five and eight times four gives you seventy seven. Therefore it's five Type A and four Type B.

Although Student 25 was confident in solving some problems, he made serious mistakes with others. He liked to experiment with numbers and use 'trial and error'.Student 15 used similar expressions.Students 13, 15 and 25 showed developmental plots of similar shape with a lower initial rate of development and a higher rate between the second and third tests consistent with a rapid increase in mathematical skills with insufficient time for the full acquisition of some of the underlying concepts. All three students solved Question 9 with ease, in contrast to their handling of other questions. The problems made up by these students were ones they felt confident to solve themselves, in contrast with Students 2 and 5. Ellerton (1986) has shown that students' made-up mathematics problems uniquely reflect their mathematical experiences and level of concept attainment.

The interview excerpts from students whose $1/A_{ij}$ vs $1/t_j$ plots are linear have provided evidence of consistencies in the students' approaches to different mathematics problems in the same test, in contrast to students whose plots are not linear. Students with 'trial-and-error' approaches and with poor symbolization skills show uneven rates of development. It is not clear at this stage whether an inconsistent approach to different mathematics problems is a consequence of an uneven rate of development in the acquisition of mathematical concepts or vice versa.

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TEACHING OF MATHEMATICS AND DEVELOPMENT OF OPERATING COGNITIVE STRUCTURES - AN EXPERIMENT WITH BRAZILIAN LOW-INCOME CHILDREN.

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In order to study the relationship between the construction of mathematics elementary notions and the development of its related cognitive structures, a research-in-action was developed following a psychopedagogical work with Brazilian children from low-income communities of Rio de Janeiro that presented learning disabilities and repeated 1^{st} grade several times. Data gathered has proved the treatment's efficiency and allowed for some statements to be made on the subject of psychology for mathematics education.

An investigation in the field of mathematics education is currently being developed at the Psychopedagogic Guidance and Counselling Center (NOAP) of the Pontifícia Universidade Católica do Rio de Janeiro (PUC/RJ/ - as a research-in-action - whereby a group of children from low-income communities of the city of Rio de Janeiro - slow learners-is being followed up and evaluated in a methodic and systematic manner. The whole job carried out was based on the assumption that every research on knowledge construction must consider the dynamics of mental structures involved. Therefore, in the specific case of this investigation, its main purpose was to investigate the relationship existing between the acquisition of basic elementary notions of mathematics and the development of cognitive structures which are essential so that the learning process would take place. The research sought to take into account the psychological, cultural and social variables involved in the act of teaching and learning mathematics.

The psychopedagogic performance plan selected to be developed among the children therefore always kept up a close link with children's modes of life and experiences, gathered while tending to them and therefore such plan was founded on Brazilian mathematics education which advocates, according to Lopes (1979), both the differentiation of knowledge of mathematics as a tool of life and science, in its strick sense, and the democratization of teaching. This was therefore a project whose aim was not only the implementation of the "corpus" of specific knowledge but also and chiefly a presentation of an explicit social and political commitment.

METHODOLOGY

The formative evaluation was chosen instead of the somative one as pattern of evaluation of the experiment because of the nature of this investigation, thus being able to differ and to discuss not only the most important aspects related to the mathematic knowledge construction itself, but mainly the psychological phenomena envolved while the whole process took place.

As a result of a previous psychodiagnosis of the children involved in the experiment, randomized groups were formed based on their cognitive, affective and social steps of development and submitted to the specific treatment. Groups were assisted on weekly basis during a school year by a psychopedagogic team made up of a coordinator and an observer, both permanent.

The work with the children was based on problem solving using group discussion, taking into account the children's own experiences of life and interests. Toys and other familiar objects pertaining to the same social level of these children were used to develop this work, but any other object, even junk, was also used once it enabled children to build up games. Systematic observation of group dynamics was selected as the main tool of this research.

Data gathered was submitted to regular discussion by the group coordinators as well as to a technical committee made up of psychologists and pedagogs. The former joined the latter and both evaluated the research's development being possible the re-elaboration of the system itself.

RESULTS

The experiment's first step has been accomplished and some results can be pointed out: 1) in a large approach, it is possible to consider that a significant change took place in children's general behavior as well as in their school performance, confirmed by their school's feedback; 2) in a strict point of view, this change was expressed by several skills, such as:

Corporal	Emotional & Social	Cognitive
Awareness of one's own body Confrontation between body space and the surrounding world	Construction of: . Identity . Autonomy . Group companionship	Construction of objects establishing: grouping, one to one pairing, classification patterns; use of words and of numbers for counting
Awareness of corporal limits according to one's own possibilities		Positioning of objects Symmetric and asymmetric relations
Awareness of the intentionality of movements; better corporal performance	Emotional control Initiative Industry	Posing problems

CONCLUSIONS

Data gathered allowed, at the first approach level, for some statements to be made on the subject of mathematics education psychology:

1) Results achieved strenghthen the thesis of authors such as Benjamin (1955), Piaget (1975) and Lopes (1979) among others, in the sense that cognitive structures refer chiefly to logico-mathematical activities while they are specific to the human being through some peculiar acts which he, and only he himself, can exercise over the objects, such as for example: to classify, put in order, modify and change. It was proven that these activities are actually implicit in children's action, in their playing and in their games and that they constitute the very essence for acquiring different school contents. It became very clear that logic and mathematics, at least in their genesis, are not constructed independently; they are linked to one another, and inasmuch as if an

individual is to operate, he must be able to establish simultaneously coordinated relations and to modify these same coordinations by association, inversion and reversion.

2) The experience carried out provided children with a space in which to use their own bodies as a basis to understand mathematics, since it advocates that an interrelationship between the body objects is essencial. It is by experimenting a multiplicity of possible body coordinations with the body itself, with the bodies of others and with different materials, within a given space and time, that mathematics is learned.

Results achieved showed that only by going through these steps, man will be able to advance qualitatively, by developing certain activities which will reflect complex mental actions derived from data gathered in the body-object relationship. School practices are some of those specific complex learning processes of men's abilities and therefore require psychomotor practices to be developed.

Acquisition of mathematical techniques and symbolic formulations, strictly speaking, preceded by body action which is related to an activity actually exercised in the world of objects, becomes then much more effective. It has become evident that it is by acting spontaneously that children, even having failed previously in school, can acquire their first knowledge of mathematics, such as for example, simple mathematical calculations, even though they may not be able to express then in a more systematic way. (Kamii, 1984, Vayer, 1973).

3) A third point covers the emotional link which is established between the individual and that which he learns. It was not considered, suggested by Piaget (1954) that the emotional behavior of mental functions is understood only into energetic terms as opposed to the cognitive behavior which constitutes the structural aspect. Likewise, it was not considered that affectivity cannot change cognitive structures, even though it may systematically interfere in the contents of that same structure (Piaget, 1954). On the contrary: a hypothesis was assumed in this investigation (with Barros, 1971 and Mamede Neves, 1977) that cognitive and affective aspects are both psychic structures which, if reenergized during the processes of evocation or (re)perception, will acquire a cer-

tain psychic intensity, by showing itself as an experience respectively ideational or perceptual, emotional or endo-perceptual. In the specific case of the experiment carried out, it was verified that many children bore a negative emotional link with learning of mathematics and that, through an adequate assistance, it was possible to restructure said link. These links became, without a doubt, structuring elements of the type of construction of mathematical knowledge, with all pertinent consequences, which goes to prove the assumption taken.

4) Finally, the experiment stressed the great importance of socio-cultural aspects in the act of learning. When child reaches the stage of formal learning it already bears in itself a whole structure of knowledge which reflects the culture of its family and its social milieu, which culture, however, is not always compatible with or used (and sometimes denied) by the culture of school. That is why oftentimes there appear certain insurmountable obstacles in the act of learning,which are, however, considered only as beig of a cognitive nature.

A great deal of the children assisted in the present experiment showed those problems. Only by respecting that knowledge, adjusting to language, to conduct and to socialization which they bore in themselves,it was possible to (re)construct the proposed knowledge.

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Combinatorics

THE COMBINATORIAL SOLVING CAPACITY IN CHILDREN AND ADOLESCENTS

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The following factors were considered: The type of combinatorial problem (arrangements, permutations and combinations); the number and nature of elements; the effect of age (grade 6, age 11-12 and grade 8, age 13-14); the effect of instruction. In order to check the effect of instruction, a pre-test, an immediate and a delayed post-test were administrated. The following findings have to be mentioned: a) The performances with digits are higher than those with concrete objects (colors, members in committees) b) The performances with arrangements and permutation problems are better than those with combinations. c) Both age and instruction have a net positive effect but in younger subjects (grade 6) the percentages of correct solutions for combinatorial problems remain under 40% even after instruction.

In the Piagetian theory the combinatorial capacity represents a fundamental component of formal reasoning. The propositional logic of adolescents is said to express basically the combinatorial resources of their thinking (Piaget and Inhelder, 1975; Inhelder and Piaget, 1958). On the other hand, combinatiorial analysis represents a powerful and important branch of mathematics related to probabilities, linear programing, the theory of games, topology, number theory, network analysis, etc. The literature concerned with the psychological aspects refer mainly to the development of formal operations (Neimark, 1975; Roberge, 1976; Kuhn, Ho and Adams, 1979; Pallrand, 1979; Wollman, 1982). Some other publications consider the didactical aspects (Kapur, 1970; Dessart, 1971; Ranucci, 1972; Hadar and Hadass, 1981; Quintero, 1985). A few deal with training strategies (Fischbein, Pampu and Minzat, 1970; Barratt, 1975; White, 1982).

The main aims of our research were to determine the optimal age for teaching cominatorial analysis systematically and the optimal strategies to accomplish this aim. An interesting cognitive -didactical problem was also to establish the effect of the nature of elements considered (abstract versus concrete elements).

THE DESIGN

The subjects were 84 elementary and high school pupils enrolled in two schools in Tel Aviv situated in a middle class urban area. Two age (grade) levels were investigated: age 11-12 (grade 6), 43 subjects; and age 13-14 (grade 8), 41 subjects.

The Lessons: Two junior high school and two high school classes participated in the teaching program. Each class received 6 lessons of 45 min. each distributed over four weeks. The topics were taught in the following order: Arrangements with and without repetitions; permutations; combinations. Arrangements and permutations were taught using tree diagrams. For solving combinations - which cannot be produced directly by the means of diagrams - one has taught the formula: $C^n_k = \frac{A^n_k}{P_k}$

The lessons had mainly an intuitive-experimental character but always lead to the solving formula.

Testing Procedures: Thirty-nine items distributed into thre groups of questions were used. The items referred to the following conbinatorial questions: $C^3{}_2$, $C^4{}_2$, $C^4{}_3$; P_3, P_4; $A^3{}_2$, $A^4{}_2$, $A^4{}_3$ (with repetitions); $A^3{}_2$, $A^4{}_2$, $A^4{}_3$ (without repetitions). Each of these problems was presented in three different embodiments (digits, colors, tasks in committees). These thirty-three items were distributed randomly in three questionnaires. In addition there were two items ($C^4{}_2$ and $C^5{}_2$) which appeared with the same embodiments in all three questionnaires. In order to neutralize as much as possible the effect of order, the various types of problems and embodiments appeared in different orders in the three questionnaires.

Procedure: The questionnaires were administrated collectively in the usual classroom conditions. Each subject received only one of the three questionnaires. There were three testing sessions: a) A pre-test session, a week before the beginning of the lessons; b) a post-test session, a week after the end of the lessons; and c) a delayed post-test session, 6 months after the instructional period.

Examples of Items: (1) Given 4 digits: 2, 4, 7, 9, how many numbers of 3 digits each may one obtain? Each digit may be used more than once, i.e.; 222. (The solution: $4^3 = 64$). (2) Given a three member committee (president, cashier and secretary) and 4 candidates. How many different committees may be formed? (The solution: $A^4{}_3$ without repetition = 24).

RESULTS

In Table 1 the data have been collapsed according to the mathematical type of problem and the sequence of the test administrations.

Table 1.
Percentages of Correct Answers*

	GRADE 6				GRADE 8			
	C^n_k	P k	RA^n_k	A^n_k	C^n_k	P k	RA^n_k	A^n_k
Pre-Test	35.62	16.11	16.47	17.35	49.57	21.52	27.08	35.83
Post-Test	30.48	58.17	43.92	48.52	39.19	76.38	59.40	75.46
Delayed Post-Test	39.68	46.27	43.86	44.60	60.09	62.91	52.87	49.07

* The numbers represent the percentages obtained by collapsing the data of the 13 items in the four main types of combinatorial problems. The symbol RA^n_k stands for arrangements with repetitions.

The type of combinatorial problem, order of difficulty.

Let us consider first the order of difficulty before the instructional sessions. The most difficult type at both age levels is that of the permutation problems, followed by arrangements with repetitions, arrangements without repetitions and finally (the highest proportion of correct solutions) the combination problems. This finding confirms that of Piaget and Inhelder (1975). At the post-test the picture is completely changed. At both age levels the combination problems provide the lowest frequencies of correct answers, and on the contrary, the permutations which seem to be the most difficult at an intuitive, pre-instructional level, become the easiest after the students have learned a systematic solution procedure (the tree diagrams and the formula $P_n = n!$). The explanation is obvious. The formulae for permutations and arrangements are simpler than that for combinations. In addition, for combinations there is no tree diagram which may be used directly.

On the other hand, it is important to observe that while at the sixth grade level the performances for combination problems are lower than the performances for the other types of problems, in grade 8 the combinational problems yield proportions of correct answers which are similar or even better than those obtained with other types of problems.

The nature of the elements: Let us recall that three types of elements have been used: digits, colors and individuals fulfilling tasks in committees. It is evident from Table 2 that the problems using digits yield the highest proportions of correct solutions. There is no systematic difference between colors and committees.

Table 2.
Percentages of Correct Answers*

	GRADE 6			GRADE 8		
	Digits	Colors	Commit-tees	Digits	Colors	Commit-tees
Pre-Test	28.54	19.65	17.67	46.78	25.47	31.52
Post-Test	45.19	46.88	40.21	69.83	58.3	55.84
Delayed Post-Test	49.44	41.04	39.61	67.08	53.27	55.44

* The numbers represent percentages obtained by collapsing the data so as to reveal the effect of the nature of elements.

We suggest the following explanations: Firstly, students are more used to operate mentally with digits and numbers than with flags and committees. On the other hand, a concrete embodiment represents a good productive model, if it leads easier to the solution by its own structure. This is not the case with flgas and committees in combinational problems. The arbitrary transformations have nothing to do with the nature of the embodiment.

The effect of age: The subjects belonged to the age groups 11-12 (incipient phase of the formal operational period) and 13-14 (the accomplishment phase of the same period). The results show (both Table 1 and Table 2) that the older subjects perform clearly better than the younger ones at each type of problem and at all the three administrations of the tests.

The effect of instruction: As expected, the students got better results at most combinational tasks after the instructional period. As it has been shown above, the picture with combination problems is more complicated - a drop of the proportions of correct answers at the first post-test and a rise again at the delayed post-test (see Table 1).

Generally speaking, one may conclude that at both age levels students evidently take profit from instruction. Except for combinations, 50 to 75 percent of them learn and remember the solution procedures.

<u>The solving strategies</u>: Each subject has also been asked to indicate explicitly the strategy and the formulae he has used for solving the problems. This has been done only for 4 problems out of the 13 given in order not to extend excessively the time for solving the problems.

Table 3.
Percentages of Correct Answers

		GRADE 6			GRADE 8		
		Pre-Test	Post Test	Del. Post-Test	Pre-Test	Post-Test	Del. Post-Test
C_K^n	Total Cor.	27.91	44.19	41.86	58.54	46.34	56.10
	Formula	---	11.63	6.98	---	7.32	2.44
	Sets *	27.91	32.55	34.88	58.54	39.0	53.66
P_K	Total Cor.	13.95	62.79	62.79	91.46	73.17	68.29
	Diagr/For.	---	13.95	4.65	---	31.71	11.20
	Formula	---	30.23	25.58	---	26.83	14.63
	Diagram	---	6.98	6.98	---	4.88	7.32
	Sets *	13.95	4.65	25.58	41.46	9.76	34.16
RA_K^n	Total Cor.	34.88	58.14	46.51	31.70	75.61	70.73
	Diagr/For.	---	16.28	4.65	---	14.63	12.20
	Formula	4	16.28	18.60	---	31.71	29.29
	Diagram	---	13.95	2.33	---	7.32	2.44
	Sets *	25.58	11.63	20.93	31.70	21.95	26.83
A_K^n	Total Cor.	27.91	75.61	75.61	51.22	80.49	58.54
	Diagr/For.	2.33	13.95	6.98	---	24.29	2.44
	Formula	---	32.55	30.23	---	34.16	9.76
	Diagram	---	9.30	4.65	---	4.88	9.76
	Sets *	11	16.28	30.23	51.22	17.07	36.59

* The term "sets" means here that the subjects did simply draw up groups of elements. "Diagram/Formula" means that the subjects have used both the diagram and the formula.

Three types of strategies are recorded in Table 3.: diagrams, formulae and the production of groups of elements. As expected, at the pre-test almost all the subjects simply wrote the various groups of elements. There is no spontaneous use of diagrams and formulae. At the first and second post-test, the image is changed. Many subjects use after instruction the taught diagrams and formulae. But there are remarkable differences between the various types of problems: The most frequent method used is the formula, especially for permutations and arrangements without repetition. Diagrams are also used, but less frequently. On the contrary, for solving combinations most of the subjects - at both age levels - continue to produce groups of elements without resorting to the taught formula.

Didactical implications

Elementary combinatorial analysis can certainly be taught systematically in grade six using tree diagrams and adequate formulae. We would not advise the teaching of combinations at this stage. In grade 8, students learn without special difficulties arrangements and permutations. The teaching of combinations (including the understanding of the formula) still raises certain difficulties, but it is possible that, with an increased number of lessons, one may succeed.

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Computer environments

PAPERS ARRANGED ACCORDING

TO AGE GROUPS

SCHEMAS DE CONNAISSANCE DES REPRESENTATIONS IMAGEES D'EXPRESSIONS RELATIONNELLES MATHEMATIQUES PRODUITES PAR DES ELEVES DE 9 A 12 ANS DANS UN ENVIRONNEMENT DE PROGRAMMATION OBJET

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Abstract

The main issue of the research is to get insight about the role and the development of knowledge schemas used by nine to twelve years old children in arithmetic word problem solving. The program we develop creates an object environment providing a visual representation of the kind of relational mathematical expressions frequently included in additive and multiplicative word problems. The program is written in Micro-Smalltalk and runs on a Macintosh. Children are invited to represent ten expressions with this environment and with two natural environments. The analysis of children behaviors is a first step in the identification of knowledge schemas used by children in the construction of visual representations of relational mathematical expressions.

Quels sont les schémas de connaissance appliqués dans la resolution de problèmes? Quels sont les processus de construction de ces schémas? Ces questions sont au centre des preoccupations d'un bon nombre de chercheurs en sciences cognitives, en intelligence artificielle et en didactique (Anderson, 1983; Chaiklin, 1985, Chi et al., 1981, Clement, 1980; Greeno, 1980; Hinsley, Hayes et Simon, 1977, Mayer, 1983)

Si l'illustration ou la representation imagee externe d'informations incluses dans les enonces de problemes constitue toujours un moyen privilegie par les pedagogues pour faciliter chez l'eleve l'interpretation de problemes arithmetiques concrets, de plus en plus de chercheurs s'interrogent sur les conditions d'efficacité de ce procede didactique. Les resultats des recherches conduites par Poage et Poage (1977), Campbell, 1981, Bednarz, Janvier et Poirier (1983, 1985) sur le rôle des representations imagees ou des illustrations dans la comprehension des problemes, des concepts ou des notions mathématiques soulèvent en effet de telles interrogations. Ils montrent que très souvent les interprétations de ces representations par les élèves diffèrent de celles des adultes qui les ont construites.

L'etude actuelle a pour objectif principal de préciser les schemas de connaissance à la base des representations imagées d'expressions relationnelles mathematiques produites par des élèves de 9 à 12 ans.

Methodes

10 élèves du second cycle du primaire participent a l'experimentation. 3 eleves de 4 ième annee, 4 eleves de 5 ieme annee et 3 élèves de 6 ieme annee. Ces eleves sont choisis en fonction de leurs performances a une epreuve de correction et de resolution de problèmes utilisee dans une etude anterieure sur les representations imagées (Lemoyne, 1985; Savard et Lemoyne, 1986).à celles des eleves de leur classe, un de ces élèves ayant participé a l'examen preliminaire de l'environnement objet.

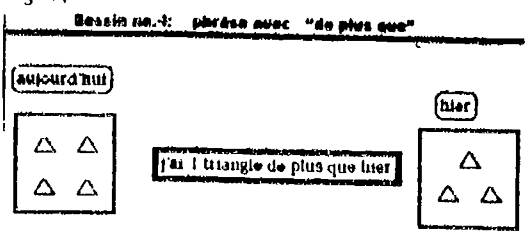
Chacun des élèves est invite a produire une representation imagee des expressions " autant que, moins que, fois plus que, de plus que, n objets par, plus que, n objets dans chaque, n objets pour, de moins que, chacun des... a n objets ", d'abord dans un environnement naturel non contraignant puis, dans un environnement naturel contraignant et enfin, dans un environnement de programmation objet contraignant. Dans les trois situations, l'espace qu'il peut utiliser pour le dessin est identique. Dans l'environnement naturel non contraignant, il peut choisir les objets qu'il désire et les dessiner sur le papier mis a sa disposition; dans l'environnement naturel contraignant, il ne peut utiliser que les dessins d'objets mis à sa disposition et n'ayant pas à dessiner lui-même, sa tâche en est une de sélection et d'arrangement des objets; dans l'environnement de programmation objet, l'éleve choisit parmi les objets a sa disposition ceux qu'il désire et place ces objets sur l'écran. Dans l'environnement de programmation objet et dans l'environnement naturel contraignant, les objets sont identiques. Nous décrivons maintenant de façon très succinte l'environnement objet.

L'environnement objet est constitue des classes suivantes d'objets: fruits (poire, ananas, fraise), humains (garçon, fille), animaux aquatiques (grenouille, poisson, tortue), formes géométriques (cercle, triangle, carré), liquides (eau, lait, jus de pomme, jus d'orange, jus de carotte, jus de tomate), fleurs (rose, primevère, marguerite), flèches (simple, double), textes (parler, decrire, étiqueter, penser). Les fruits sont placés sur des plateaux, les animaux aquatiques dans des aquariums, les fleurs dans des vases ou des pots, les liquides dans des verres; les formes géometriques sont encadrees par des carres. Diverses méthodes sont associées aux classes d'objets: a) il est possible de modifier la taille et le nombre de fruits, d'animaux aquatiques, de figures géometriques; les modifications sont accompagnées d'un changement proportionnel de leurs contenants respectifs; b) il est possible de modifier les quantites de liquides, la hauteur, la largeur et l'inclinaison des verres; c) il est possible de modifier l'orientation et l'expression des humains, d) il est possible de modifier la longueur et l'orientation des fleches, de convertir les fleches en traits, e) il est enfin possible de modifier le format des textes. L'eleve accede aux objets par un menu qui lui permet de choisir un objet, de le modifier selon les methodes associées à cet objet, de le déplacer, de le copier, de le supprimer. Les actions effectuees par chacun des éleves de même que les intervalles entre ces actions sont enregistrés dans un fichier "eleve", ce fichier constitue le protocole soumis a l'analyse. Afin de maitriser l'environnement objet, chacun des élèves est invité à appliquer à chacune des classes d'objets les actions offertes par le menu; au cours de cette activité préliminaire, l'experimentateur répond aussi à toutes questions relatives à l'utilisation de l'environnement. Pour illustrer cet environnement, nous reproduisons quelques extraits du comportement d'un sujet ainsi que le dessin produit pour l'expression "de plus que".

<u>Actions</u>

1. Fgeo: nouveau
2. Fgeo: modif. forme
   cercle--›triangle.
3. Fgeo: modif. déplacer
4. Fgeo. choisir - copier
5. Fgéo: modif: déplacer
6. Fgéo: modif: quantite
   4---› 3
7. Texte: nouveau
8. Texte. modif: bulle
   parler---› décrire
   "ajout de description
9. Texte: modif: déplacer
10. Texte. nouveau
11. Texte. modif. bulle
    parler---›étiqueter

<u>Dessin</u>



"ajout d'un texte"

12. Texte: modif: déplacer

. . . .

Resultats

Dans l'environnement naturel non contraignant, pour chacune des expressions relationnelles, la majorité des sujets n'ont recours qu'à des collections constituées d'un seul type d'objets (ex: deux collections de crayons); deux élèves de 6 ième année et 1 elève de 5 ième année utilisent au moins à une occasion des collections constituées d'objets n'appartenant pas au même ensemble (ex: une collection de crayons et une collection de pommes) ou des collections d'objets différents mais appartenant a un même ensemble (ex: une collection de pommes et une collection de bananes). Dans l'environnement naturel contraignant et dans l'environnement objet, la majorité des élèves se montrent au moins à une occasion capables d'utiliser des collections constituées d'objets n'appartenant pas à un même ensemble; ce comportement est observé dans plus d'un tiers des representations produites. Dans l'environnement naturel contraignant, on observe aussi chez la majorité des élèves le recours à des collections constituées d'objets différents appartenant à un même ensemble; ce recours est moins fréquent toutefois dans l'environnement objet et s'explique par l'emploi de procédés de reproduction d'objets, procédés économiques. Enfin, dans l'environnement naturel non contraignant un sujet de 5 ième année et un sujet de 6 ieme année utilisent au moins à une occasion des mesures de quantité de liquide pour representer une relation additive ou multiplicative tandis que dans les autres environnements, la majorité des sujets de 6 ième année et quelques sujets de 5 ieme et de 4 ième années utilisent de telles mesures au moins à une occasion.

Le recours à des procedés pour specifier les relations entre les objets des représentations est comparable dans l'environnement naturel non contraignant et dans l'environnement objet et plus fréquent que dans l'environnement naturel non contraignant. Dans l'environnement naturel non contraignant, pour les expressions additives, les relations sont précisées par les expressions faciales des personnages tandis que pour les expressions multiplicatives des textes descriptifs spécifient les relations. Dans l'environnement objet, des textes descriptifs expriment généralement les relations; les personnages dans cet environnement sont souvent objets des relations, faisant alors partie de collections. Les procedés sont surtout utilisés par les eleves de 5 ième et 6 ième années.

Tous les élèves utilisent dans la construction de plus de la moitie des représentations les méthodes de duplication mises à leur disposition; lorsque de telles méthodes sont utilisées, une fois la classe d'objets choisie, les actions sont réalisées très rapidement, sans hesitation. Un texte descriptif accompagne généralement les representations ainsi produites et ces représentations sont aussi dans l'ensemble adéquates. Les élèves de 4 ième année ont moins frequemment que ceux des autres niveaux recours à de telles méthodes et ceux qui y recourent ne complètent pas fréquemment leurs representations par un texte descriptif. Enfin, chez les élèves de 5 ieme et de 6 ième années, ces méthodes sont appliquées aux diverses classes d'objets; un souci de varier les objets des relations semble expliquer ce fait.

Si l'utilisation des objets de la classe textes est observée chez la majorité des élèves des divers niveaux scolaires, seuls les sujets de 5 ième et de 6 ième années se montrent capables de choisir les objets appropriés de cette classe; les sujets de 4 ième année se contentent généralement d'insérer leurs textes dans les fenêtres qui conventionnellement sont associées aux dialogues (parler) ou aux réflexions (penser). Chez les élèves des autres niveaux toutefois, le recours à des objets adequats de la classe textes se fait progressivement, ainsi, au cours des premieres représentations, le choix des objets est effectué apres un intervalle de plusieurs secondes (jusqu'à 125 secondes chez un sujet) et on observe souvent plusieurs modifications de ce choix.

L'examen des protocoles montre aussi que plusieurs des sujets qui recourent à des objets appartenant à diverses classes pour illustrer une expression ne parviennent pas à construire une représentation qui les satisfasse; ils ne gardent alors qu'un objet spécifique, généralement un objet de l'une ou l'autre des classes "fruits", "fleurs", "animaux aquatiques", "formes géométriques" et tres souvent procedent à une duplication de cet objet. Ce comportement est cependant plus fréquemment observé chez les élèves de 4 ième année que chez les élèves des autres niveaux.

Quelques sujets seulement ont recours à des flèches ou à des personnages pour spécifier le sens des relations entre les objets; dans l'environnement de programmation objet, ces objets sont considérés au même titre que les autres objets. Ainsi, deux sujets de 6 ième année utilisent des flèches a titre d'objets de la relation "n objets pour"; un sujet de 5 ième annee représente la relation "fois plus que" par 15 flèches et 5 prénoms.

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FROM INTRINSIC TO NON - INTRINSIC GEOMETRY

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Abstract. Turtle geometry, apart from being defined as intrinsic, has a special characteristic; it invites children to identify with the turtle and thus form a body syntonic thinking "schema" to drive it on the screen to make figures and shapes. This is a report of on - going research, whose aim is to throw further light on the nature of this "intrinsic shema", investigate the extent to, and the way in which children choose to use it, and whether it is possible for the children to use the schema to understand concepts belonging to Euclidean and Cartesian geometry. The research concerns 20 11 - 12 year old Greek children, with 40 to 50 hours of experience with Turtle geometry prior to the study.

The theoretical framework of the study is based on the role of Logo and turtle geometry within a specific view of mathematics education; i.e. learning mathematics is seen as an on - going re - organisation of personal experience, rather than an effort to describe some ontological reality. The child learns mathematics by building with elements which it can find in its own experience (Von Glaserfeld, 1984). Papert (1980) uses words like "doing" and "owning" mathematics to stress the dynamic and active involvement of the child. Hoyles and Noss ((b), in press) use the notion of "functional mathematical activity", i.e. the child using mathematical ideas and concepts as tools (rather than objects, Douady 1982) to solve problems in situations which are personally meaningful.

Logo is seen by more and more educators as a powerful tool for creating educational environments in accordance with the above perspective. Turtle geometry, a very important part of Logo, has a particular characteristic; when children "do" turtle geometry, they can identify with the turtle, and therefore use personal experience about their bodies to think about the shapes and figures they want to make. I am approaching this process, which Papert and Lawler call "intrinsic thinking", in two ways.

Firstly, in terms of the structuring of intuitive geometrical knowledge, i.e. the way children link very simple sets or "units" of such knowledge to the turtle's actions. They acquire these "units" from very early personal experience of movement in space. DiSessa might call these units "phenomenological primitives", although his study was in the context of physics (diSessa 1982). Lawler (1985) puts forward the notion of a microview to talk about domain specific fragments of personal experience.

He contends that the personal geometry microview is "ancestral" to the intrinsic geometry microview. Secondly, the approach stresses the importance of the interplay (Douady 1982) between the mental image, graphical output and formal symbolic code which is vivid when doing turtle geometry (Hoyles and Noss (b), in press). How does intrinsic thinking influence this interplay?

OVERALL OBJECTIVES.

The study is split into four parts (studies 1, 2, 3 and 4) and involves the observation of children engaged in Logo activities which require the use of the turtle to solve intrinsic, euclidean and cartesian geometrical problems. The aim is to gain insight into the ways the intrinsic schema can be used by the children for developing an understanding of concepts belonging to these three geometrical systems. In relation to the four studies, the research objectives can be split into;

a) illuminating the intrinsic schema, by creating environments where the children of the study can choose whether to use it to solve geometrical problems or not (studies 1, 2, 3 and 4),

b) investigating the extent to, and the way in which,the schema is used to construct geometrical figures whose properties allow the pupil to choose between interpreting them intrinsically or non - intrinsically (study 1), and

c) investigating whether it is possible for children to use the schema for making natural links between intrinsic geometry and other geometrical domains, namely euclidean and cartesian geometry (studies 2, 3 and 4).

OVERALL METHODOLOGY.

The research involves 20 11 - 12 year old children from the Psyhico College Athens, who, prior to the study, have had 40 - 50 hours of experience with turtle geometry in an informal setting, in pairs and threes to one machine. The research is split into four studies, for all of which the children were observed in detail, the data consisting of; audio taping of everything that was said, soft and hard copies of verbatim transcriptions translated in english, soft and hard copies of everything the children typed, hard copies and possibility for more graphics screen dumps, soft and hard copies of all the procedures the children wanted to save on disk, researcher's notes on anything of importance which might escape the rest of the data, and the childrens' required and non - required notes on paper. In study 1, 16 children working individually on structured tasks, were observed and interviewed. Studies 2, 3 and 4 employed pairs of children working in collaboration, my role being that of a participant observer. Two notions are central for the methodology of studies 2, 3 and 4. That of a "microworld", i.e. "Logo - based situations constructed so that the pupil will come up against embedded mathematical ideas in the context of meaningful activity" (Hoyles and Noss (b), in press), and that of a "conceptual field", i.e. "a set of situations, the mastering of which requires a variety of concepts, procedures and symbolic representations tightly connected with one another" (Vergnaud 1982). The studies involve children provided with microworlds constructed so that the use of the intrinsic schema is applicable for understanding non - intrinsic geometrical ideas, i.e. the "tools" can be used for both intrinsic and non -intrinsic geometrical representational systems. In this sense, microworlds with such characteristics may be described as "conceptual pathways" from the former representational system to the latter. The preliminary results presented in this paper, are examples of the children using the schema in specific geometrical situations.

USING EUCLIDEAN IDEAS FOR TURTLE GEOMETRIC TASKS (STUDY 1).

The study involved 16 children, individually attempting to construct certain geometrical figures in Logo, which were given to them on paper, one at a time. The figures were such that their properties allow the pupil to choose between interpreting them intrinsically or non - intrinsically. The procedure was as follows. They were given the figure on paper and time to think. They were then requested to talk about and write down their plan before constructing the figure on the computer. An interview followed, requesting verbal explanation of what they did, whether they saw different ways of construction, and if yes, which one they preferred and why. The childrens' activities are analysed with respect to:

a) Which properties they perceive explicitly and which they ignore.
b) How they interpret the properties, and
c) How the properties are used to construct the figure.

The example given concerns the childrens' verbal plan for the task of constructing a "window" figure which revealed a split in the way they organised the figure in their minds; i.e. how a "large square and cross" organisation of the figure (7 children out of 16) invoked the use of a non - intrinsic property (e.g. the centre of a square, which is away from the turtle's path) in contrast to a "four adjoining squares" organisation (9 out of 16). In the four square interpretation the children did not seem to use the centre apart from interpreting it as the screen's centre to talk about the turtle's initial position. Further analysis will show which properties they did use and how they interpreted them. In the square and cross interpretation, the children used their understanding of a non - intrinsic property to take the turtle to a specific point away from its path in a logical (vs perceptual Hillel 86) way. Is it possible to create environments rich in opportunities for children to integrate their intrinsic schema with the "logical" (Euclidean) or "analytical" (Cartesian) characteristics of other geometries? This issue is investigated in studies 2, 3 and 4.

PROGRESSING FROM AN INTRINSIC TO A EUCLIDEAN DEFINITION (STUDY 2).

The study involved a learning sequence, the design of which was influenced by the UDGS model for learning maths (Hoyles and Noss (a), in press). The aim of the sequence, which lasted 15 hours in total, was for a pair of children to see the need for, investigate and work out different ways of constructing a circle in tasks meaningful to them. In the end, they had written four distinct procedures for these constructions, progressing from an intrinsic "curvature" definition to that of a euclidean "centre and equidistant points" (figure 2, a to d), and used them for their own projects. Following the sequence, in a method similar to that of study 1, the children attempted to solve structured tasks, individually at first and then collaboratively. The tasks involved constructions of geometric shapes incorporating the circle (e.g. figure 4). The children could choose which procedure to use. They were then interviewed in a similar spirit as in study 1. The analysis concerns; i) the extent and the ways in which they see the need for and use Euclidean features of the circle while constructing it in the familiar intrinsic process in meaningful contexts, ii) the nature and the extent of the childrens' use of the Euclidean features employed in each construction in projects of their own choice, and iii) the influence of their experience on the way they interpret geometrical properties.

The following example is a case of an explicit use of the radius (and an implicit of the centre). The children had spent time trying to construct figure 3 using CIR4 (figure 2a), and as a result saw the need for and constructed CIR9 (figure 2b). They tried CIR9 a few times on the computer and then attempted to do figure 3 again. They typed the following without talking; CIR9 50 PU RT 90 FD 5 LT 90 PD CIR9...
Valentini: "How much shall we do it? (she typed 45)
Chronis: "Why 45?
V: "Because this is 45.
C: "So?
V: "Eh, the radius... 45 is the radius, since before it was 50, minus 5 is 45.
In CIR9, the turtle constructs the circle in the intrinsic way, but the input implies using a non - intrinsic feature of the circle. The interface between the two circles is again an intrinsic turtle move, but the quantification of FD requires an operation on the length of the radius. Thus, the children cause intrinsic actions with non - intrinsic quantifications.

CIR4 6 CIR9 50 CIR19 50 TC 50
(a) (b) (c) (d)

FIGURE 2.
FROM AN INTRINSIC TO A EUCLIDEAN DEFINITION

FIGURE 3.
A PROJECT

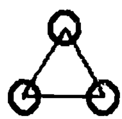

FIGURE 4.
A TASK.

USING TOOLS TO BRIDGE INTRINSIC AND EUCLIDEAN GEOMETRY (STUDY 3).

This study was designed to investigate how a POST/DIRECTION/DISTANCE microworld (Loethe 1985) could serve for developing an understanding of euclidean geometric concepts (e.g. an isosceles triangle). The ideas embedded in using the turtle's tools serve as a "bridge" from intrinsic to euclidean or cartesian geometry. The childrens' activities, which lasted for 15 hours in total, fall under 3 categories;

a) Making sense of the concepts involved in using the turtle's measuring instruments,

b) Constucting geometrical figures which require the use of these tools, and

c) Carrying out projects where the employment and choice of tool and how they would be used is left to the children.

The following extract is from a category a) activity. The task is to use the turtle to find the length of the sides and the size of the angles of a triangle formed by three points on the screen. The children had used the turtle's instruments to join up the points and measure the side lengths. Although they used the turtle's "protractor" to cause the action of the turtle turning to face a point, they had difficulty in realising how to use it to measure the internal angle. This is what Nikos said when he suddenly saw how to do it (figure 5).

N: "We turn it there, where it can do RT and then we do it...."
He then typed the following;

```
?PRINT DIRECTION :B
 136.705
?RT 136.705
?PRINT DIRECTION :C
 43.2961
```

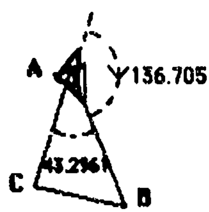

FIGURE 5.

After that he said;
N: "Since we couldn't find the angle when it (the turtle) looked towards AC, we put it to look towards beta (B) and like that we could find the angle and we found it."

The children distinguished the measurement from the turtle's action (it was their preferred strategy at the beginning), while trying to make sense of the situation. Their need to measure a "euclidean" angle (internal), led them to cause the turtle to change its heading to where it can make the measurement (i.e. to its right). This resulted in a need to integrate a static and dynamic interpetation of angle (Kieran 1986).

ATTEMPTING TO APPROACH CARTESIAN GEOMETRY (STUDY 4).

Lawler's research shows us how large the gap between intrinsic and cartesian geometry is by illustrating the failure of a 6 year old child to form a microview about the latter, based on her personal/turtle geometry microview. He also illustrates the child's reluctance to abandon this microview and use a different ancestral microview to make sense of cartesian concepts. Study 4 involved the developing of three separate learning paths from turtle geometry to cartesian geometry. Each of the three paths employed a different ancestor, thus building different bridges from intrinsic to cartesian geometry. Each pair of children (one for each path) started from a different set of activities. The structured tasks that the pairs attempted from then on were common and designed to probe the links each pair made to previous experience in order to understand the concepts which underlie driving the turtle in the coordinate plane.

The example given concerns a pair of children engaged in the first set of common activities. This pair's initial activities involved constructing a simple grid, using the POST/DIRECTION/DISTANCE microworld, and drawing several shapes on the grid using the turtle's instruments. The task now, was to take the turtle to a specific point on the screen, having the SETX, SETY and SETH commands at their disposal only. Prior to the following extract, they had taken the turtle to points on the first, second and third quadrant. For this point (-100 80), there was a difference on the screen; the axes were invisible. The children guessed the X coordinate wrongly, giving it a -120 value instead of -100. Their first reaction was to try and use the command BK 20, but BK would not function. They then spent time discussing which was the plus and the minus region on the axis and finally, typed in the following; SETH 0 SETH 90 SETX -20.

The first two commands illustrate the manner in which the children imposed a sequential "turning" property to the SETH command, and their resistance to see it as a placing of the heading in an absolute system. The third command seems even more interesting. It illustrates two kinds of confusion.

a) Perceiving coordinate values (names of places) as numerical.

b) Perceiving the sign as an operation rather than signifying a region on the axis.

The children also found difficulty in distinguishing the plus and minus regions, and in using the origin as a starting point for counting (they frequently counted from the edge of the screen to the centre). The following extract shows how Anna used her turtle identification schema to make sense of the situation;

A: "I think I got it. We told it (turtle) here 'lift me up and take me to minus 20' and -20 in relation to axis X is here (finger). Say that axis X is, lets say here, -20 is here."

CONCLUSION.

Preliminary results seem to indicate that children are ready to employ their intrinsic schema in turtle geometric situations, even when they have the option not to do so by using ideas from other geometries. Futhermore, examples have been given where the use of such ideas has occured as an extension of the schema. Further analysis will aim at illuminating the nature of the schema and the potential for children to use it for understanding the relations among the differential (intrinsic), euclidean and cartesian systems of geometry.

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RETOMBEES COGNITIVES D'UNE INITIATION A LA PROGRAMMATION EN LOGO

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De nombreuses recherches ont été entreprises pour tenter de mettre en évidence les progrès effectués par des élèves ayant eu l'occasion de pratiquer la programmation Logo. Les résultats de ces études sont assez contrastés et leur interprétation donne lieu à des controverses assez vives. C'est ainsi que si Pea et Kurland (1984) concluent à la quasi absence d'effets de transfert des acquisitions, Clements et Gullo (1984) ont eux trouvé des progrès significatifs dans différents domaines cognitifs. L'impression générale qui se dégage de la littérature est que les résultats sont plutôt en retrait par rapport aux attentes suscitées par ces nouvelles activités, même s'il convient de se garder de tirer des conclusions trop hâtives.

La spécificité de notre recherche est d'une part d'avoir combiné les aspects d'évaluation classiques avec une observation très attentive des activités des élèves, et d'autre part d'avoir porté sur un âge pour lequel relativement peu d'études avaient été entreprises, à savoir le début de l'école secondaire.

METHODE

Le cours d'initiation à la programmation en LOGO, proposé à une classe de 1ère générale (élèves de 12-13 ans) a remplacé durant 22 semaines des heures d'étude et d'information générale. Le cours était donné par la maîtresse de classe, enseignant par ailleurs les mathématiques. L'enseignement s'est déroulé en demi-classes par groupes de 2, exceptionnellement 3 élèves.

L'école possédant des micro-ordinateurs TI 99/4, le langage choisi fut le TI-LOGO. L'option pédagogique retenue reposait sur la notion de projets destinés à constituer à la fois une motivation et un fil conducteur permettant d'aborder différents aspects du langage.

Une cinquantaine d'élèves de section générale ont participé à cette recherche. Le plan d'expérience était conçu pour mettre en évidence d'éventuelles différences cognitives dans différents domaines susceptibles d'être affectées par l'activité de programmation.

Les épreuves retenues

Nous avons sélectionné six épreuves psychologiques sur la base d'une part de leur lien possible avec l'activité de programmation en LOGO, ce qui nous a incité à retenir des épreuves ayant une composante spatiale ou des épreuves de résolution de problème, et d'autre part en tenant compte de l'âge des sujets. Quatre épreuves ont été passées collectivement:

- Le **CATTELL** CFT3, test d'intelligence générale
- Les **BRIQUES**, test d'aptitude spatiale
- Les **ROULEAUX**, épreuve portant sur la représentation de la composition de deux mouvements.
- Le **GROUP EMBEDDED FIGURES TEST** (GEFT), épreuve portant sur la dépendance-indépendance à l'égard du champ.

Deux épreuves ont été passées individuellement lors d'un entretien clinique:

- La **SERIATION DE POIDS**, adaptée par Retschitzki (1978) sur la base de l'épreuve décrite par Piaget et Inhelder (1941).
- L'épreuve des **PERMUTATIONS** de jetons, portant sur les aspects combinatoires.

La maîtresse responsable du cours LOGO a mis au point une épreuve dont la plupart des items portaient sur des contenus proches de ceux touchés lors des cours de programmation. L'épreuve comprenait 6 items, dont les 4 premiers concernaient la géométrie, les 2 derniers d'autres domaines mathématiques.

Cette épreuve a été soumise avant et après le cours de programmation à l'ensemble des élèves des classes concernées par la recherche.

Enfin, pour évaluer les apports de l'enseignement de la programmation dans le domaine informatique, nous avons élaboré une épreuve comportant les quatre parties suivantes: représentation du système, compréhension de programmes, production de programmes, compréhension de concepts informatiques.

Observations

Les sessions Logo ont également été observées très finement; chaque groupe était soit directement observé par un collaborateur du projet, soit enregistré en vidéo. Ce dispositif était destiné à mettre en évidence les différences individuelles dans la participation au cours et le rôle plus ou moins dynamique de chaque élève dans l'évolution des projets et la résolution des problèmes en vue d'une mise en relation ultérieure avec les résultats enregistrés aux épreuves psychologiques en début et en fin d'année scolaire.

RÉSULTATS

Aspects qualitatifs

L'enseignante a choisi de laisser les enfants progresser à leur propre rythme. Cette option a eu pour conséquence de provoquer d'importantes différences entre les élèves au niveau des acquisitions dans le domaine de la programmation. Si certains élèves ont bien maîtrisé les primitives géométriques, la notion de procédure et ont pu de ce fait réaliser des projets interactifs (jeux élémentaires), voire recourir à des procédures récursives, d'autres élèves n'ont travaillé que dans le domaine de la musique à un niveau assez élémentaire.

Les observateurs ont été frappés par le manque de motivation pour l'activité proposée. Cet aspect de l'ambiance du cours s'exprimait de plusieurs manières : bavardages entre les élèves, attentes passives dès qu'un problème surgissait, plaintes par rapport au temps qui passe lentement, etc. Les élèves avaient donc de la difficulté à travailler pour eux-mêmes, à se motiver pour leur projet.

En dépit de ces impressions négatives, l'épreuve informatique passée dans le cadre du post-test a mis en évidence que les acquisitions des élèves en Logo étaient bien réelles pour la majorité d'entre eux.

Aspects quantitatifs

Nous avons appliqué une analyse de variance à deux facteurs à l'ensemble des épreuves pour lesquelles des données ont pu être obtenues tant au pré-test qu'au post-test. Les deux facteurs pris en compte par l'analyse étaient donc d'une part le statut du groupe (expérimental ou contrôle) et d'autre part le moment (pré-test ou post-test). Le Tableau 1 résume le résultat de cette analyse pour les épreuves psychologiques.

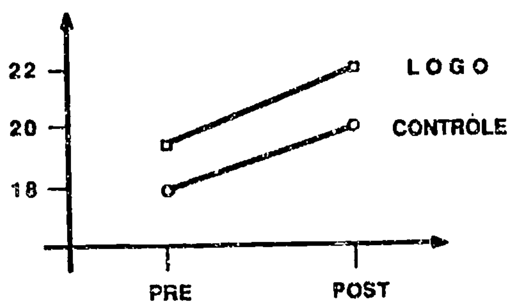
Il ressort de l'examen de ces résultats que globalement on n'observe que peu de progrès entre le pré-test et le post-test, sauf pour l'épreuve de Cattell. Les autres épreuves semblent donc mesurer des caractéristiques plus stables des sujets. On peut souligner qu'aucune

Tableau 1: Résultats de l'analyse de variance MANOVA à 2 facteurs

Epreuves	Sources de variation: Pré-Post	Entre groupes	Interaction
CATTELL CFT3	TS	NS	NS
G.E.F.T.	NS	TS	NS
ROULEAUX	NS	NS	NS
BRIQUES	NS	TS	NS
PERMUTATIONS	NS	TS	NS
SERIATION	NS	NS	NS

épreuve ne donne lieu à une interaction significative, ce qui indique que les différences observées ne sont pas attribuables à l'effet des variables introduites dans l'expérience, puisqu'elles sont présentes dès le début de l'année scolaire, comme le montre la Figure 1, qui illustre les résultats de l'épreuve des Briques.

Figure 1: Diagramme des interactions pour l'épreuve des BRIQUES



Analyse qualitative de l'épreuve de sériation

Si les effets globaux sont soit discrets, soit absents, on pouvait s'attendre tout de même à ce que la pratique de la programmation puisse manifester ses effets dans les épreuves présentant un lien plus étroit avec l'activité Logo. Cela nous semblait être le cas de l'épreuve de sériation de poids, proche des algorithmes de tri et se laissant bien décrire en termes de traitement de l'information (itération, optimisation, récursivité).

S'il était peu réaliste de s'attendre à des différences massives quant au niveau des solutions, en revanche on pouvait supposer que l'expérience de la programmation permettrait aux sujets de mieux percevoir la structuration de l'activité et leur fournirait une plus grande aisance pour ce qui est de l'explicitation de leur procédure.

Le Tableau 2 présente les niveaux attribués pour la conduite (plus ou moins systématique) de la stratégie.

Tableau 2: Sériation de poids / Conduite de la stratégie

		Groupe LOGO					Groupe CONTROLE			
		POST-TESTS					POST-TESTS			
		1*	2	3	4		1	2	3	4
	1	2	0	0	0	1	0	1	0	1
PRE-	2	2	1	3	1	2	1	1	0	0
TESTS	3	1	2	2	1	3	0	1	4	4
	4	0	0	0	7	4	1	0	4	6

* 1 Non systématique 2 Peu systém. 3 Début de système 4 Très systém.

Dans le groupe LOGO, 5 sujets progressent entre le pré-test et le post-test, alors que 12 sujets se situent au même niveau et que 5 sujets obtiennent un niveau moins évolué. Dans le groupe CONTROLE, ces effectifs sont respectivement de 6, 11 et 7 sujets. Une analyse similaire appliquée aux autres dimensions des conduites observées lors de cette épreuve donne des résultats comparables. On est donc amené à constater l'absence d'effet de la variable principale de notre étude sur ces dimensions plus qualitatives des conduites relatives à la sériation.

Epreuve de mathématiques

Le Tableau 3 résume les résultats de l'analyse de la variance appliquée à chacun des items pris séparément ainsi qu'à l'épreuve totale.

Tableau 3: Analyse de variance de l'épreuve de mathématiques

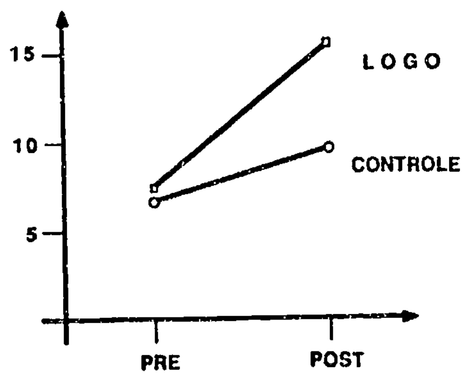
	Source de variation		
	PRE-POST	ENTRE GROUPES	INTERACTION
1 Estimation d'angles	TS	NS*	NS
2 Composition d'angles	S	S	NS
3 Calcul angles parallélogramme	TS	TS	NS*
4 Nombre côtés polygones	TS	S	S
5 Problèmes arithmétiques	NS	NS	NS
6 Séquence d'instructions logiques	NS	NS	NS*
Score total	TS	TS	TS

* Résultat presque significatif (P inférieur à .10)

On constate que des différences apparaissent surtout pour les items géométriques, aussi bien en ce qui concerne les progrès entre le début et la fin de l'année scolaire qu'entre les deux groupes.

Pour ce qui est de l'interaction, on peut noter que la différence est surtout marquée lorsqu'on considère l'épreuve totale. Dans ce cas, c'est le groupe expérimental qui a progressé plus que le groupe contrôle comme le montre la Figure 2.

Figure 2: Diagramme des interactions pour l'épreuve de mathématiques



CONCLUSION

L'ensemble des résultats obtenus lors de cette recherche confirme la tendance principale des études similaires. La pratique de quelque 30 heures de programmation en LOGO ne suffit pas à provoquer des progrès mesurables sur les différentes variables cognitives retenues dans notre étude. Le seul effet significatif a été obtenu pour une épreuve de mathématiques portant sur des contenus proches de l'activité Logo.

Cela ne suffit évidemment pas à dénier l'intérêt d'une telle activité ni à affirmer qu'aucun progrès n'a été accompli. Au contraire l'épreuve d'informatique a montré que les acquisitions des élèves en Logo étaient bien réelles. Ce que nos résultats suggèrent c'est qu'il s'agit de connaissances à caractère local, non transférables immédiatement à des domaines différents.

Par ailleurs on ne peut pas exclure que des progrès aient pu être accomplis sur des dimensions que nous n'avons pas considérées. Enfin des analyses plus fines tenant compte notamment des résultats des observations et permettant de différencier les sujets selon le degré de leur implication dans l'activité sont en cours et pourront infléchir quelque peu les conclusions provisoires de cette recherche.

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MENTAL REPRESENTATION OF RECURSIVE STRUCTURES

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In a West German research project we are investigating how recursive structures are represented by 7th and 8th graders. We use intensive interviews with different problems and analyse the student's way to the solution. Especially working on the "Tower of Hanoi" may reveal different recursive or non-recursive strategies.

Recursion is an important problem solving tool in mathematics and in many mathematical applications. But in mathematics as well as in computer sciences the concept is not at all easy to understand and to acquire (KURLAND & PEA 1983, HAUSSMANN 1986). In a West German research project we want to investigate how children learn recursive structures and how they develop a cognitive representation of recursion. In this investigation we are working with 7th and 8th graders aged 12 to 15.

It is evident that we have to use a method which is adequate for students of this specific age and this developmental stage. We decided to work with the students in a LOGO environment. LOGO is a computer language which permits an easy introduction to sophisticated programming concepts. So we suppose that working with LOGO may be a link between the more intuitive use of recursion in everyday life (HAUSSMANN & REISS 1986) and recursive structures used in mathematics. Moreover, learning to program is an example of the acquisition of a complex cognitive skill. The observation of this process seems to allow generalisations (ANDERSON 1984). So we presume that our investigation is of importance for the acquisition of recursive structures in mathematics instruction.

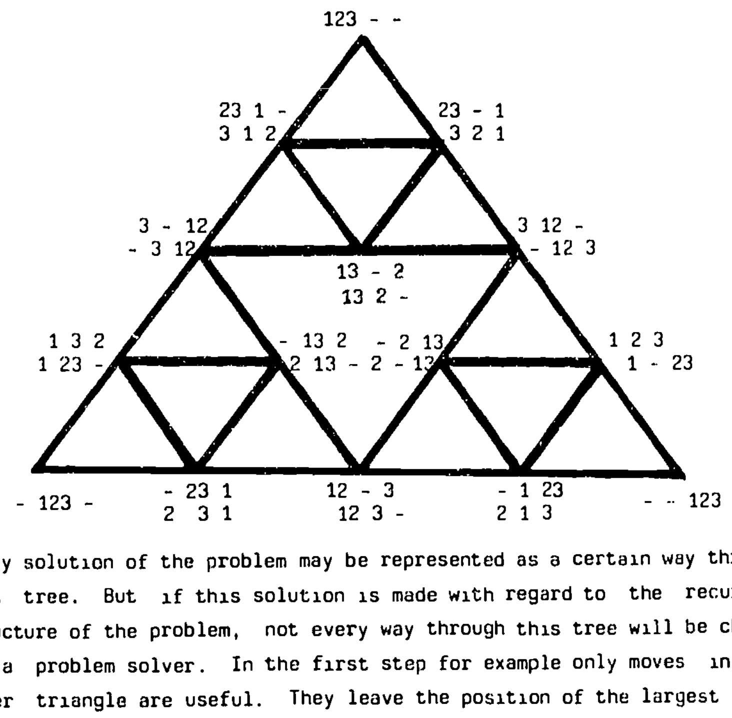
One of our most important research methods in this project is the use of intensive interviews. In such an interview only one student and the interviewer take part. The interviewer presents different problems to the student, and he or she is supposed to work on these problems. We are not primarily aiming at a correct answer. The main purpose is to get insights in the student's way towards the solution of a specific problem and the understanding of the student's explanations to his or her answers. We made the first interviews after about ten hours of programming instruction. At this time the students were familiar with some LOGO primitives and the most important syntax rules. They had almost no experience with the use of recursive calls in a computer program. The students did not only lack the experience of recursive structures. Moreover they did not have more than a vague knowledge of other control structures.

The problems presented during the interviews reflected four different tasks in the construction of a computer program. These tasks are planning, writing, understanding, and debugging of a computer program (see PEA & KURLAND 1983). All these problems are aimed at the identification of iterative and/or recursive structures in the representation of the problem's solution. The last problem was presented to the students with the same goal but was supposed to reveal another aspect of recursion. We asked the students to work with the "TOWER OF HANOI". This problem is well known in the problem solving literature. There is a tower made of N disks of decreasing diameter. This stack is set up on a rod A. It has to be moved to a rod B by using an intermediate rod C. There are two rules. The first one is that only one disk may be moved at a time, the second rule says that a big disk may never be put over a smaller one. If there is a tower made of three disks at least seven moves will be necessary. This number depends on the strategy. It is well known that the optimum strategy is recursive. During the interviews the students had to solve this problem with three and with four disks.

We think that a problem like this in advantageous in two respects. On the one hand we may have a look at the way students deal with recursive structures in a non-verbal way. If the students have difficulties with the programming language these difficulties will not interact with deficiencies in recursive thinking. On the other hand the students may be working on a recursive problem without having to be conscious of its recursive structure. They do not have to reflect their way of solution.

Let us take a look at the problem's solution while working with three disks. We have to start by moving the smallest disk. It may be moved to rod B or rod C, which is our own choice. Moving the second disk is different in a certain respect. We might once more take the smallest disk, and find a new position for it either on the free rod or on rod A. But this would not be a substantial alteration of the situation. Actually we cannot choose the next move freely. If the smallest disk is on rod B, we have to move the middle-sized disk to rod C. If the smallest disk is on rod C we move the middle-sized disk to rod B. Similar considerations are true as well for the following moves of the disks. We therefore distinguish between optional and compulsory movements of the disks.

KLIX (1971) describes a representation of all possible moves of the disks in a tree. He thereby combines the optional and the compulsory moves. In this way he gets a triangle which reflects in an interesting manner the recursive structure of the problem. The picture shows this tree with respect to the 3-disk-problem. A problem solver starts at the upper corner of the triangle. The nodes at the lower corners on the left and on the right represent the complete tower erected on rod B respectively on rod C. All other nodes in this tree represent the different possibilities for the intermediate stages of the disks.

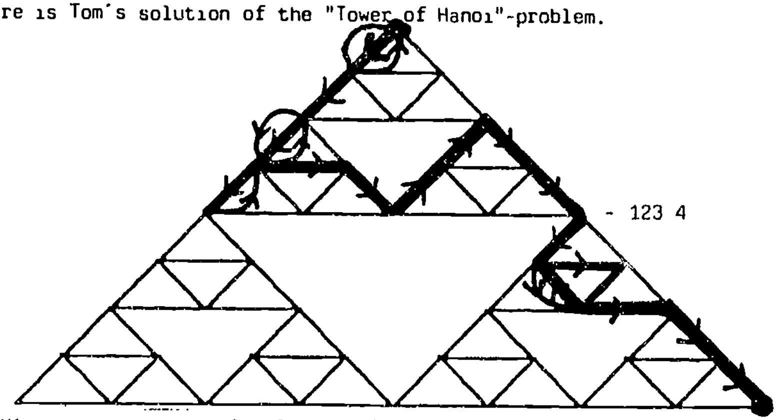


Every solution of the problem may be represented as a certain way through this tree. But if this solution is made with regard to the recursive structure of the problem, not every way through this tree will be chosen by a problem solver. In the first step for example only moves in the upper triangle are useful. They leave the position of the largest disk unchanged. It is clear that the 3-disk-tower has to be constructed on rod B in order to move the largest disk to C. A similar consideration is possible for the middle-sized disk. So we have a further restriction to a few moves in the upper triangle. The optimum strategy and so the optimum solution is represented by the movements along an edge. Thinking recursively while solving this problem may be regarded as the successive restriction of the problem space.

A similar tree may be made for the 4-disk-problem or for any problem with N disks to be moved. The diagram for four disks is made up out of three smaller triangles. All these three triangles represent 3-disk-problems which are part of the 4-disk-problem. The upper triangle includes all moves which lead to a 3-disk-tower on the intermediate rod. The lower triangles represent the movement of the 3-disk-tower to their final position. We analysed the solutions of all students in terms of this tree representation. Here are some of our findings for three students.

TOM (13 years old)

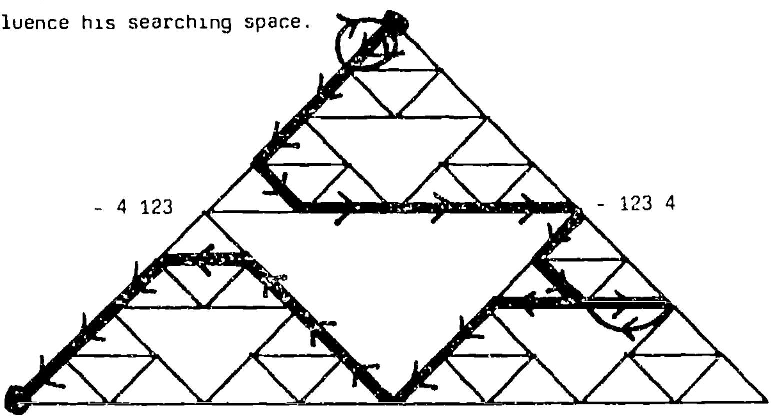
TOM is an 8th grader who is very much interested in computer programming. He had some knowledge of BASIC prior to the LOGO instruction. TOM has a computer of his own which enables him to work at home on the problems given during the course. Therefore he has more practice and a more profound knowledge of the LOGO syntax than the other students in the course. Here is Tom's solution of the "Tower of Hanoi"-problem.



TOM's strategy is a local one. At the beginning he aims at moving the largest disk from rod A to rod C. But he is making moves all over the upper triangle which means that TOM does not generalise his consideration made for the largest disk. He does not pay attention to the global objective of moving the whole tower. The hypothesis about TOM's strategy is hardened by looking at the lower triangle. After moving the largest disk he is erecting another local goal which is the move of the next disk to rod C. Once more we cannot find evidence for another strategy. He is performing moves all over the tree for the 2-disk-problem in a trial and error manner. Only after moving the third disk there are no more superfluous moves. The situation now seems to be evident for the student. It is obvious that TOM does not use a recursive strategy.

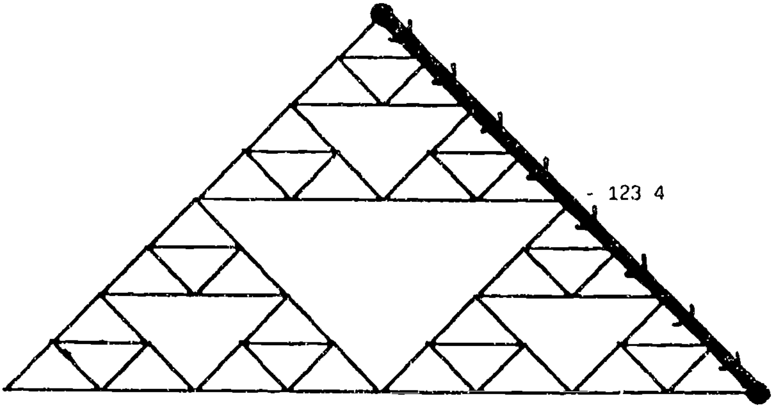
UDO (14 years old)

UDO is also an 8th grader. During the LOGO course he usually works with one or two class-mates. Prior to the interviews it was difficult to say anything about his abilities in problem solving with the computer because UDO is a very reserved and shy student. He hardly ever takes part in any discussions in the class-room. Let us look at his moves of the four disks. His solution of the problem seems to be without an overriding strategy. He moves the disks quite randomly. His local goals do not influence his searching space.



JAN (14 years old)

JAN is one of the students in the class of 7th graders. He does not have any opportunity to work with a computer except during the LOGO course. JAN is a good problem solver while working with the computer. Usually he develops further ideas while working in the course on the given problems. Therefore he creates quite elaborate LOGO procedures. JAN does his own work thoroughly but at the same time he helps other students in the class-room doing their work. His solution shows the optimum path through the tree. It is also interesting that JAN is very fast while working on the "Tower of Hanoi"-problem. There is only a break of a few seconds after moving the largest disk to rod C. We regard this break as part of restructuring the problem (see also COHORS-FRESENBORG 1985).



The goal of our investigation was to describe representations of the solutions of a recursive problem. We found out that students have different strategies which they apply while working on the problem. We could reveal a hierarchy of strategies following the description of KLIX (1971).

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A PROLOG DATABASE ON TAGUS ESTUARY PROBLEMS:

IMPLICATIONS ON THE MATHEMATICS AND LOGIC CONCEPTS

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PROLOG (PRogramming in LOGic) is a programming language in its own right, available in numerous implementations. It was chosen by the Japanese Fifth Generation Computer Systems researchers as the fundamental starting points in their development of a new generation of computer system. It is being applied internationally in research areas such as expert systems and databases.
The purpose of this study is to check: 1-to what extent PROLOG is a language accessible to secondary students; 2--the characteristics and potentialities concerning the learning of mathematics, of the environement supplied by the construction of a PROLOG database.
Students in the sample envolved (7th and 9th graders) did not show great difficulty in dealing with PROLOG and PROLOG was a source of motivation of question posing in maths learning and helped create a new pedagogical student-centred atmosphere.

INTRODUCTION

One of the fundamental and unseparable rights of Man is the right to a good education. Nowadays in our society where knowledge grows old at an unprecedented speed owing basically to the rapid evolution of new information technologies, school lacks adaptability. This results in a considerable decrease and, sometimes, in a total absence of the learner's motivation together with his failure.

Teaching is not only an art, it is also a whole science requiring sensitivity to react quickly and adequately to the learner's needs.
The use of computers can allow a wide range of changes on the teacher-student relationship. As a matter of fact, it appears as an element which works as the basis of an active learning, interacting with teachers and students, allowing storage, processing and systematic retrieval of information, thus ultimately contributing to a new pedagogical atmosphere.
Hartley (1981a, 1981b) emphasized the importance of the learner's role in this new environment supplied by the appearence of the computer: "Only by allowing the student himself to make choices, to justify them and see their facts, will he learn about the process of making educational decisions; only in this way will he become self-evaluative and learn how to learn".
The Artificial Intelligence (A.I.) is a special branch of informatics in permanent change. As far as A.I. is concerned, the Alvey Commitee, agreeing with the japanese initiative, has recommended a new emphasis on its study: "We want more powerful information processing systems with a more effective transfer of human intelligence and knowledge to the computer... The action must start in the schools" (Alvey, 1982). A. I. allows the focus on the analisys of human learning activity. Margaret Boden (1977) writes about this subject: :Artificial Intelligence is not the study of computers but of intelligence in thought and action...".
Hawkins (1981) adds: "An expert appears very much as an analytical tool, helping the users make well-informed decisions without forcing them to accept any particular interpretation or procedure". Among the languages available today, PROLOG (PROgramming in LOGic) seems to offer great opportunities in education. With a high declarative power, it can be used by the non-computer-specialist, ideally in his or her natural language, to solve problems concerning objects and their relationships.
The ease of interrogation of a logic database (qwery language) provides it with an interpretative and deductive characteristic, keeping a conversation between the user and the computer. In short, it is a formal language that can represent knowledge and it is suited both to the development of programs and to the storage of knowledge and its consultation.
Ennals (1981) argues: "PROLOG has a number of uses in work with

computers - as a database qwery language for specifying the information to be retrieved from databases; as a language for specifying problem prior to their solution by computer programs; as a formal language for the automatation of deductive reasoning; as a representation of information for natural language processing; and as a very high-level programming language in its own right".

More recently, Cabrol (1986) suggests that PROLOG is one of the higher--level languages with a great success.

Creating situations which lead to mathematization, in specified contexts familiar to learners and in close contact with aspects of real life, is one of the goals of the teaching of mathematics. The construction of a database, in a programming language, close to the natural one, and bearing a strong qwery power - PROLOG - will be able to open new perspectives which provide a contact with some meaningful applications of Mathematics, and are necessary to the construction of its theory.

So, the aims of this study are to check:

1) To what extent PROLOG is a language accessible to secondary students. More specifically:

a- which logical concepts in PROLOG programming 7th and 9th grade students show;

b- what effects PROLOG programming has on the development and consolidation of such concepts;

c- how 7th and 9th grade students differentiate as far as the precedent aspects are concerned.

2) The characteristics and potentialities, concerning the learning of mathematics, of the environment supplied by the construction of a PROLOG database. More specifically:

a- the difficulties inherent to the construction and understanding of a database;

b- the effects of such an environment on the acquisition and development of mathematical concepts;

c- the effects on the interpretation students make from the phenomenon represented in the database;

d- the implications in terms of exploration and creation of knowledge by students.

METHOD

The research study was conducted in a Lisbon secondary school. Eight students were involved: four 7th grade students and four 9th grade students, selected among thirty 7th-, 8th- and 9th-grade students who volunteered to submit to a pre-test for the purpose of assessing concepts concerning the predicate Logic.

The eight students selected belonged to both sexes - two girls and two boys in each grade -, and can be considered to have a medium level both in school achievement and as far as the results in the pre-test are concerned. Social status was formed to be similar in the eight students involved.

The hardware used in the study was the following: two Olivetti microcomputers M19 and M24, with 512kb and 256kb RAM, respectively. PROLOG, due to its demand of RAM, could only be used in the M19 unit. Software was also used such as word processing (Open Access), S.Calc3. Paper-and-pencil as well as calculators were used too.

In a first phase, which consisted in one-hour three sessions, students learned how to program roughly in ARITY/PROLOG. The phase comprehended a tutorial, containing some basic notions of ARITY/PROLOG such as variables, facts and lists. Such a tutorial was built up by an informatic technician who also participated in the study.

In a second phase, which consisted of one-hour twelve sessions, students dealt with some problems concerning the Tagus Estuary, having teamed up in subgroups of two elements: one subgroup of boys and one subgroup of girls for each level. All students were constantly required to organize the database or solving problems concerning their work.

Each student owned a note book where the information necessary to carry out his/her work was written down.

Thus, two databases on Tagus Estuary problems were built up by the 7th grade and the 9th grade students.

Assessment was made through observation as well as two tests on Predicate Calculus such as variable symbols, compound terms, function symbols, predicate symbols, logical connectives and mathematical concepts - variables, proportions, relations, equations, statistics, graphs.

DEVELOPMENT OF ACTIVITIES

After students have chosen some themes connected with Tagus Estuary problems, students posed some questions of two types: qualitative and quantitative questions. 7th graders showed a tendency to pose qualitative questions while 9th graders posed more quantitative ones, such as: "Which faunal species was of higher frequency in the sample considered?" The two subgroups of the same level worked in each session: while one group was working with the database, the other group used such means as bibliography, primary sources, and carried out adequate calculations and build up graphs in order to organize the information.

The activities involved concerned the units of the Tagus fauna, the relationship between environement and species, and polution in the Estuary. As these problems were posed, it was stated the need to calculate means, modes, measures of variability and solve equations, use variables, stablish relationships, built up graphs and interpret them, as well as the need to locate the spots under focus in maps and construct models also involving some geometry concepts.

CONCLUSIONS

About the organization of the PROLOG database, 7th grade students showed more difficulty in the use of this language than the 9th graders (they made more mistakes in notation - brackets, commas, full-stops were, sometimes absent, which caused a slower work pace).

However, they showed more ready - in terms of time - to seek information, while 9th graders organized information better having consulted fewer sources.

About the mathematical concepts, the notion of variable and its substitution did not bring any difficulties in whatever level considered. Issues concerning proportions and problem interpretation were satisfactorily solved by both levels. About the translation of problem into mathematical language 9th graders showed to be more at ease than 7th graders.

About the concepts of predicate logic, 7th graders showed more difficulty than 9th graders.

About the advantages of a computer representation questions can be asked which may involve exploring information from a number of different parts

of books, if we take advantage of the "qwery-the-user" facility the system can prompt us for missing information, and as a result can learn new knowledge during the interaction; the user can at any time add amend or delete information and create new facts or rules; questions can be asked which were not envisaged when the information was entered.
Information supplied by different sources can be compared, by analysing the information obtained and its source through the study of knowledge representation.
A new pedagogical, student-centred atmosphere is thus created. Many questions will require further information and computer motives the research and learning.
Students generaly understood better Tagus Estuary problems with the help of PROLOG as they were motivated by the issues aroused.

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CONSTRUCTING THE CONCEPT IMAGE OF A TANGENT

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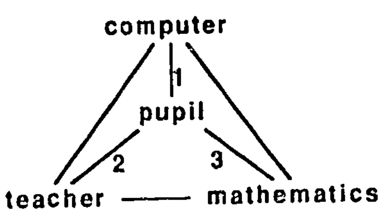
When a learner meets a new mathematical concept, it may be invested with implicit properties arising from the context, producing an idiosyncratic concept image which may cause cognitive conflict at a later stage (Tall & Vinner, 1981). The purpose of this empirical research is to test the hypothesis that interactive computer programs, encouraging teacher demonstration and pupil investigation of a wide variety of examples and non-examples, may be used to help students develop a richer concept image capable of responding more appropriately to new situations. Three experimental classes of sixteen year-olds were taught using computer packages capable of magnifying graphs to see if they "looked straight", and to draw a line through two close points on a graph. These formed the basis of class discussion and small group investigations to encourage the formation of a coherent relationship between the concepts of gradient and tangent. For comparison, five other classes were taught by more traditional methods. Two questionnaires administered during the course confirmed that the experimental students were able to respond more appropriately in new situations, for example in the case where a function is given by a different formula to the left and right. However, the notion of a "generic tangent" - an imagined line touching the graph at only one point (even where this is inappropriate) - persisted in both groups, though significantly less amongst the experimental students.

Building and testing Mathematical Concepts

The computer introduces a new factor into the classroom relationship between the pupils, the teacher and the mathematical concepts to be considered. It enables aspects of the mathematics to be externalised and manipulated on the computer VDU. In terms of Skemp's three modes of building and testing mathematical concepts (Skemp 1979), it offers a direct (mode 1) method of building and testing using the computer software, in addition to discussion with the teacher (mode 2) and internal consistency of the mathematics in the mind of the learner (mode 3):



This more immediate mode of building and testing can be highly advantageous in introducing new concepts that previously have seemed extremely abstract to pupils. However, there may be a danger that the computer introduces inappropriate factors that may cause difficulties of their own. For example, a "straight line" on a computer VDU is a coarse sequence of high-lighted pixels that, at best, may only look fairly straight.Such difficulties require careful handling by the teacher. However, the differences between the practical (and inaccurate) computer picture and the theoretical ideas can also provoke a great deal of discussion that can be most rewarding for the pupils. As Hart has observed:

> The brain was designed by evolution to deal with *natural complexity*, not neat "logical simplicities". (Hart, 1983, page 52)

Mathematicians analyse concepts in a formal manner, producing a hierarchical development that may be inappropriate for the developing learner. Instead of formal definitions, it may be better for the learner to meet fairly complicated situations which require the abstraction of essential points through handling appropriate examples and non-examples. Such complexity requires discussion and "negotiation of meaning" between teacher and pupils.

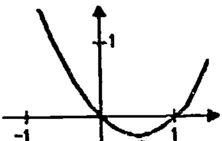
Vinner (1982) has observed that early experiences of the tangent in circle geometry introduces a belief that the tangent is a line that touches the graph at one point and does not cross it; this produces a concept image that causes cognitive conflict when more extreme cases are considered, such as the tangent at a point at inflection, where it does cross the curve, or the case of a tangent at a cusp, which is slighdy more contentious.

Classroom activities

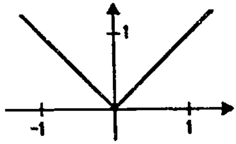
In three experimental classes, of 12, 14 and 16 pupils, the aim was to negotiate the meaning of the tangent concept through using the computer to draw a line through two very close points on the graph as part of a broader introduction to the idea of gradient of a graph in the calculus. This was to be demonstrated by the teacher leading a discussion centred on the computer, before encouraging the students to work with the computer in small groups. It was part of the brief for the experimental students to consider cases, such as $y=|\sin x|$, which have "corners" where they have neither gradient, tangent or derivative, though they visibly have different left and right gradients. One of the programs used purported to draw a "tangent", when it actually drew the straight line through $(x,f(x))$, $(x+h,f(x+h))$ for $h=0.0001$. This seemed to draw a "tangent" to $y=|\sin x|$ at the origin, providing a rich source of discussion. The researcher took an active part in the experimental group of 14 pupils, suggesting activities to be followed by the other two groups, whilst the five control classes followed a more traditional strategy assuming an intuitive knowledge of the meaning of a tangent. All teachers kept diaries of their activities.

The Test Investigations

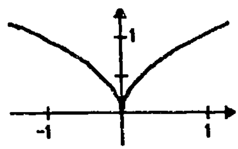
Two brief tests were administered to the students during the course of their work. The first followed immediately after they had studied the notion of the gradient of a graph at a point, the second after they had studied the notion of a tangent in greater detail. Both involved the same sequence of graphs:



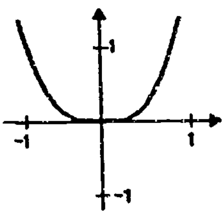
(1) $y=x^2-x$



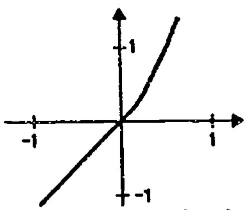
(2) $y=abs(x)$



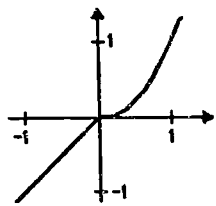
(3) $y=\sqrt{(abs(x))}$



(4) $y=abs(x^3)$



(5) $y=\begin{cases} x & (x\le 0) \\ x+x^2 & (x\ge 0) \end{cases}$



(6) $y=\begin{cases} x & (x\le 0) \\ x^2 & (x\ge 0) \end{cases}$

In the GRADIENT INVESTIGATION, for each graph the students were asked:

Can you calculate the gradient at x=0 ? YES/NO
If YES, what is the gradient, if NO, why not ?

In the TANGENT INVESTIGATION they were asked:

Does the graph have a tangent at x=0 ? YES/NO
If YES, please sketch the tangent, if NO, why not?

In each case the first question was to establish a base-line of performance, it being hoped that virtually all students would be able to answer the question correctly. The second question tested the concepts of gradient/tangent at a point with different left and right gradients, (where the experimental students would expect to have an advantage). The third tested the concept of gradient/tangent at a cusp (and here mathematicians may fail to agree over whether there is a tangent or not!) The fourth involved a function for which the students did not know the formula for the derivative, so they could not easily solve the problem in either case by differentiation. The fifth and sixth cause difficulties because there are different formulae on either side of the point under consideration. The fifth has the additional difficulty that it *does* have a tangent at the origin but, to the left, the tangent coincides with the graph and so causes conflict with those students who believe that a tangent touches the graph at one point only. The last two questions, in particular, would test a students' concept images in a broader context than they had previously encountered.

The tests were also administered to a group of first year university mathematics students, who are more highly qualified than the students in either control or experimental groups.

In the limited space at the disposal of this paper I shall report the total responses of the control and experimental students. In Tall (1986) there is a deeper consideration of matched pairs of students (matched on a pre-test not given here) with and without previous calculus experience which supports the same conclusions.

In each table the "correct" response will be given in **bold type** (though its "correctness" is sometimes a matter of opinion); other responses will be subdivided wherever appropriate. Where "statistical significance" is quoted, this will always be using a one-tailed χ^2-test, sub-dividing the responses of experimental and control groups into "correct" and "all other" responses, with the hypothesis that there will be more correct responses from the experimental group. The experimental students usually perform at least as well as the university group. Unless explicit mention is made, it may be assumed that the differences between the experimental students and the university students is not statistically significant.

Graph (1): $y=x^2-x$

		Gradient			Tangent		
		correct	incorrect	nr	correct	incorrect	nr
Experimental	(N=41)	**40**	1	0	**41**	0	0
Control	(N=65)	**59**	6	0	**64**	1	0
University	(N=47)	**47**	0	0	**47**	0	0

Although marginally more control students gave incorrect responses to the gradient question, this is not statistically significant.

Graph (2): $y=abs(x)$

		Gradient					Tangent					
		YES: 0	1	±1	other	**NO**	many	two	left	right	balance	**none**
Experim.	(N=41)	2	0	0	1	**38**	2	6	0	0	1	**32**
Control	(N=65)	23	14	4	3	**21**	8	9	0	2	17	**29**
University	(N=47)	7	1	2	1	**36**	2	0	0	0	8	**37**

More experimental students give NO responses than control ($\chi^2=34.73$, $p<0.000001$), and more say there is no tangent ($\chi^2=9.70$, $p<0.01$). The experimental students NO responses are significantly higher than those at university ($\chi^2=3.12$, $p<0.05$) whilst the numbers responding with no tangent are not significantly different ($\chi^2=0.03$).

The control students use their concept images to put forward a number of reasonable hypotheses, such as noting that the gradient has the two values ± 1, or averaging the two values to get zero, or calculating the derivative of abs(x) to get abs(1), or simply ignoring the *abs* symbol altogether to obtain the derivative 1. Several control students showed insight into the problem, asserting that there was no gradient with comments such as:

> "no, because the line is going in two directions at 90 degrees".

Note that five experimental students assert there are two tangents, almost certainly the legacy of discussion about "left" and "right" tangents.

Graph (3): y=√(abs(x))

	Gradient						Tangent						
	YES			NO									
	∞	0	other	∞	other	nr	many	two	vert	balance	other	none	nr
Experim. (N=41)	*8*	1	0	*5*	**27**	0	1	0	*3*	1	0	**36**	0
Control (N=65)	*10*	11	16	*6*	**20**	2	3	1	*23*	10	3	**24**	1
University(N=47)	*14*	5	4	*6*	**18**	0	2	1	*23*	2	0	**19**	0

This question is difficult to answer, for it even provokes debate amongst mathematicians. It does not magnify to look straight at the origin (with two superimposed half-lines), so a theoretical case can be made for *no* tangent and *no* gradient (noted above in bold type). Some would argue that there is a vertical (undirected?) tangent, with infinite gradient (noted in *italics*). A few students draw a "balance" tangent along the x-axis.

Significantly more experimental students respond NO to the gradient than control (χ^2=11.16, p<0.01), and more experimental students than university (χ^2=5.16, p<0.05). Grouping those who respond NO, or give the gradient as infinity (YES or NO), shows significantly more experimental students than control (χ^2=20.46, p<0.001), and more experimental than university (χ^2=4.52, p<0.05).

Significantly more experimental students than control say that there is no tangent (χ^2=17.71, p<0.0001), and more experimental than university (χ^2=7.43, p<0.01). Combining "no tangent" with "vertical tangent", there are significantly more responses in the combined category from experimental students than from control (χ^2=10.79, p<0.01).

Graph (4): y=abs(x^3)

	Gradient					Tangent			
	YES: 0	0 (?)	other	NO	nr	horizontal	other	none	nr
Experim. (N=41)	35	5	0	1	0	39	2	0	0
Control (N=65)	36	22	5	2	0	46	0	1	0
University (N=47)	35	9	1	2	0	46	0	1	0

Although most students state the gradient is zero, many perform an erroneous differentiation, such as giving the derivative of abs(x^3) as either abs($3x^2$) or $3x^2$ (a correct formula being 3x(abs(x))...) It may be that some giving the response 0 may have made the same error without writing it down. There is a significantly larger number of experimental students responding with gradient 0 without making an explicit error (χ^2=8.90, $p<0.01$). There is no significant difference between experimental students and university students and no significant difference in the drawing of the tangent at the origin between any of the groups.

Graph (5): $y=x$ ($x\leq 0$), $y=x+x^2$ ($x\geq 0$)

		Gradient				Tangent				
		YES: 1	other	NO	nr	standard	generic	other	none	nr
Experim.	(N=41)	39	1	0	1	31	8	0	2	0
Control	(N=65)	32	8	24	1	22	25	2	15	1
University	(N=47)	45	0	2	0	29	14	0	4	0

This is the most interesting example of all. The experimental students are very successful at calculating the gradient of the curve at the origin, even though all functions considered in the course were given as single formulae. The control students, however, find difficulties because they calculate the gradient by differentiation and are confused by the different formulae on either side of the origin. Comments include:

> "The line changes its characteristics - it is two graphs."
> "Because at x=0 is where two functions meet."

Significantly more experimental students give the gradient as 1 (χ^2=21.91, $p<0.0001$).

The tangent produces another difficulty because it coincides with the graph itself to the left of the origin. Coerced by their belief that a tangent touches the graph at one point only, many students draw the tangent a little off the curve, so that it seems to touch only once. This is termed a ***generic tangent*** in the table, a generic concept being defined as one abstracted as being common to a whole class of previous experiences. Even a minority of the experimental students draw the generic tangent including some saying the gradient is 1. However, the number drawing a standard tangent is significantly higher amongst experimental than control. (χ^2=15.91, $p<0.0001$).

Graph (6): $y=x$ ($x\leq 0$), $y=x^2$ ($x\geq 0$)

	Gradient			Tangent							
	YES	NO	nr	many	two	Left	right	balance	other	none	nr
Experim. (N=41)	36	5	0	0	3	0	0	1	0	37	0
Control (N=65)	39	24	2	3	4	2	4	7	4	38	3
University(N=47)	43	3	1	1	4	0	1	2	0	38	1

Significantly more experimental students respond correctly to the gradient question (χ^2=8.09, p<0.01). The tangent question has a wide variety of responses, with some seeing "many" or an "infinite number" of tangents touching the corner on the graph, others seeing two, or one (either left, right, or a line balancing at a rakish angle on the corner). Once again, significantly more experimental students explain that there is "no tangent" at the origin (χ^2=10.79, p<0.01).

Conclusions

The research emphasises the difficulties embodied in the tangent concept, but suggests that the experiences of the experimental group helped them to develop a more coherent concept image, with an enhanced ability to transfer this knowledge to a new context. For example, they were better able to interpret the tangent/gradient at a point where the formulae changed but left and right gradients were the same. However, potential conflicts remained, with a significant number of students retaining the notion of a "generic tangent" which "touches the graph at a single point", giving difficulties when the tangent coincides with part of the graph.

At the general level the research lends support to the theory that the computer may be used to focus on essential properties of a new concept by providing software that enables the user to manipulate examples and non-examples of the concept in a moderately complex context. This allows a curriculum development to be more appropriate cognitively by giving students general ideas of concepts at an early stage, to encourage discussion and the active construction of a shared meaning.

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LA STRUCTURE ITERATIVO-REPETITIVE COMME CHAMP CONCEPTUEL DANS L'ENSEIGNEMENT ELEMENTAIRE ET SECONDAIRE EN MATHEMATIQUES ET INFORMATIQUE

THE ITERATIVE-REPETITIVE STRUCTURE AS A CONCEPTUAL FIELD IN THE ELEMENTARY AND SECONDARY LEVELS IN MATHEMATICS AND COMPUTER SCIENCE.

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Many researches have been done in the field of cognitive difficulties of beginners, children or adults, when programming complex structures like the realization of a loop in various sorts of languages, imperative (Pascal) and applicative (LOGO). From a curriculum point of view, it is necessary now to develop a broader approach about learning and teaching this sort of structure. With this respect we have to take into account that Repetition, Iteration and Recursion belong to and define the same conceptual field integrating various aspects connected to other mathematics and computer science concepts, for instance multiplication at the elementary level, induction and aspects of the limit process at the secondary level.

In this presentation we are developing a first attempt in the description of the internal and external relations in the field of Iterative-Repetitive Structures (I.R.S.) and the informations we have about cognitive and didactical difficulties.

La notion de champ conceptuel a été proposée par Gérard VERGNAUD [8], [4], il y a quelques années pour fournir des moyens de construire des cadres interprétatifs, aussi bien pour l'enseignement que pour la recherche, dans le moyen et le long terme, des difficultés de nombreux élèves au cours de l'apprentissage de certains concepts. Les exemples qui ont été les plus développés sont l'addition et la multiplication à travers leurs extensions successives, extensions de domaines numériques de validité, extension de champs d'utilisation et de fonctionnement. Prendre en compte cette notion s'appuie au moins sur deux sortes de considérations :

- celle qui tient au concept ou au contenu étudié. Il s'agit d'identifier et de mettre en évidence la très grande variété de problèmes et de situations dans lesquels ils sont impliqués. C'est le cas de ceux que nous avons cité précédemment.

- celle qui tient aux élèves engagés dans des apprentissages à propos de ces notions, à l'analyse et à l'interprétation de leurs comportements dès lors qu'ils sont orientés par les caractéristiques épistémologiques du contenu.

Un des problèmes de la recherche en didactique des mathématiques, outre l'identification et l'utilisation des notions et concepts didactiques nécessaires, est celui du découpage du savoir qu'elle doit opérer. Celui que suggère l'enseignement existe comme objet d'étude. C'est un produit, le produit d'un travail de transformation proprement didactique qui modifie d'une certaine manière les objets de savoir [1]. De nombreuses recherches ont montré qu'on ne saurait le considérer comme un donné. Il faut donc opérer d'autres découpages ; la notion de champ conceptuel est un moyen de les effectuer.

<u>La structure de répétition-itération</u> : S.R.I.

Mathématiques et Informatique offrent à des notions communes des cadres conceptuels ainsi que des formalismes sensiblement différents. C'est le cas de ce que nous appellerons ici la structure de répétition itération (S.R.I.)

L'itération a un statut particulier en mathématiques. Stricto sensu, il n'est pas objet d'étude pas plus qu'elle n'est objet d'enseignement ni au niveau élémentaire ni au niveau secondaire. Elle est toutefois engagée explicitement dans des définitions (multiplication, nombres réels), dans des écritures et des manipulations de sommes et de produits, dans le calcul sur les limites. Elle a donc un rôle triple :

- Un rôle <u>producteur</u> : de nouveaux objets, de nouvelles représentations,
- Un rôle <u>prescripteur</u> : elle indique les opérations qu'il faut effectuer,
- Un rôle de <u>descripteur</u> : elle dit comment est constitué un objet ou un processus.

En informatique, la répétition-itération est un opérateur général qui représente une forme de coordination d'opérations élémentaires pour construire des objets de plus haut niveau. Elle intervient dans tous les langages de programmation, aussi bien ceux qui sont associés aux calculatrices programmables que ceux, de plus haut niveau, dans lesquels elle est codée récursivement. Nous inclurons ici, également, les outils de programmation et de calcul que sont les tableurs.

Le traitement algorithmique et informatique de l'itération va ajouter une quatrième caractéristique à nos trois propriétés précédentes, l'effectivité, autrement dit la dimension de réalisation du calcul. Elle va appeler et provoquer de nouvelles coordinations, donc une autre connaissance et un autre usage de l'objet qui nous intéresse ici.

Schémas et langages pour la S.R.I.

La structure répétivo-itérative est un opérateur essentiel pour l'analyse des objets complexes. Elle se rattache à un certain nombres de schèmes d'actions fondamentaux qui peuvent être associés à l'économie des désignations et des processus. L'économie des désignations, c'est de pouvoir représenter à l'aide d'une écriture condensée et limitée des objets et des processus pour lesquels cette écriture serait, a priori, de longueur égale à celle de la liste exhaustive des actions à accomplir pour les calculer ou pour les déterminer. L'économie des processus, c'est l'identification de leur manifestation dans un nombre important de situations pour qu'il soit nécessaire de le considérer en tant que tel comme un objet d'étude. C'est le cas de la multiplication, pour laquelle ce qui précède représente une description du schéma général de l'introduction à l'école élémentaire.

Certes, la multiplication ne représente pas le premier schème répétito-itératif qui puisse être rencontré par des jeunes enfants, mais sa construction dans l'enseignement réalise la première mise en forme mathématique d'une réalisation de la S.R.I. [2] .

C'est d'ailleurs en prenant appui sur cet exemple que nous pouvons parler métaphoriquement du "pouvoir multiplicatif" de l'itération. Il s'agit là d'une dimension conceptuelle qui nous paraît importante ; il y a changement d'ordre. Un ordre linéaire, ou séquentialisé est outil de base dans la construction d'un ordre de complexité supérieure (selon une épistémologie naïve de la complexité décrite par les mathématiques).

Les réalisations d'itération et leurs descriptions font intervenir un certain nombre de contraintes qui sont dépendantes du cadre dans lequel elles se situent : mathématique et/ou informatique et, dans ce dernier cas, de choix imposés par les concepteurs du langage dans lequel on est amenés à travailler. Autrement dit, les codages de répétition-itération vont devoir absorber un certain nombre de charges cognitives (qui ne renvoient pas seulement à les difficultés de nature seulement syntaxique).

Ainsi, la S.R.I. est prise dans un dispositif double :

- Le dispositif du problème que l'on souhaite traiter ou résoudre et dans lequel elle apparaît comme un élément de solution,
- Le dispositif de reconnaissance et de réalisation de la solution, qui comprend en particulier l'ensemble des opérations possibles leurs coordinations et les modes de description correspondants [3].

Les opérateurs les plus puissants ne sont pas forcément les plus souples:

Cette analyse des éléments de complexité propres au S.R.I. va donc prendre en compte une autre dimension de sa signification épistémologique. Il s'agit de l'identification des éléments de contrôle qu'il est nécessaire de construire pour utiliser et construire des répétitions et des itérations notamment dans le cadre informatiques ou dans un cadre qui permettra de problématiser l'effectivité. Ces éléments de contrôle sont de deux types :

- Ceux qui sont associés aux contraintes de formulation pour la réussite, c'est-à-dire pour l'écriture de procédures effectives.
- Ceux qui sont de type cognitif, c'est-à-dire qui sont de l'ordre des opérations du sujet pour la production des éléments du premier type.

Il n'y a pas que des purs effets de syntaxe à passer de l'écriture d'une répétition stricte de type suivant en LOGO et en PASCAL:

```
I.   POUR MOI
         REPETE 10 [ECRIS [I AM THE BEST]].
     FIN

II.  PROGRAM WELLDONE;
         var i:integer;
         begin
            for i:=1 to 10 do writeln ('I AM THE BEST');
         end.
```

puisqu'une modification de la consigne change très fortement les relations entre les éléments de ces programmes, le mode d'engagement des concepts et leur interprétation. Compléter en effet par une demande du type : *faire afficher l'ordre d'apparition de chaque phrase à l'écran* va interroger sur ce qu'on pourrait appeler le contrôle interne et le rôle de la variable i qui, en général, n'est pas perçue comme une variable. En effet les difficultés des élèves débutant se regroupent en trois classes:

1. *'i' n'est pas une variable*
2. *'i' n'est pas libre, non pas au sens classique de la logique, mais au sens où on ne peut pas l'utiliser librement.*
3. *On ne peut pas utiliser 'i' pour faire autre chose que le contrôle de la boucle, sinon le programme ne tournerait pas de la même manière.*

On pourra ne voir dans cet exemple que des indices de non--compréhension d'une réalisation particulière de la S.R.I. C'est une première interprétation.

La seconde interprétation nous irons la chercher du côté de ce qui nous occupe ici. A opérateur puissant (REPETE) occultation de fonctionnement et absence de souplesse dans l'adaptation à un problème "dérivé". La complexité, de fait, (et non pas seulement son étude) s'appuie et se nourrit sur le type de contraintes structurelles que montre cet exemple.

Il peut y avoir un débat sur la "puissance" des opérateurs que va offrir un langage pour coder la S.R.I. L'exemple précédent montre les limites même de ce débat. Par ailleurs, il montre aussi que la "puissance" est à mettre en relation avec une extension du champ de fonctionnement, ce qui est bien propre aux objets que catégorise la notion de champ conceptuel. Il s'agit d'abord de construire des éléments de contrôle qui ont été énoncés au paragraphe précédent, autrement dit d'intérioriser des schémas et/ou des règles d'utilisation.

Il y a eu un certain nombre d'études sur le sujet, études qui ont été effectuées dans des contextes déterminés. C'est le cas des travaux d'E. Soloway [7] dans le cas des boucles en Pascal. La problématique de l'intégration des plans est déterminante dans la mise à jour des mécanismes d'appropriation et de leurs étapes. Il y a construction de schèmes "autonomes", par exemple le schéma de compteur et le schéma de la variable d'accumulation. Schémas qui vont d'une part se coordonner, d'autre part correspondre à des conduites par lesquelles on va chercher à les appliquer en analysant le problème dans cette perspective. Il y a lieu, là, construction d'une compétence spécifique.

Dans le cas de l'écriture récursive des itérations avec le langage LOGO, nous avons pu mettre en évidence [5], dans un contexte d'enseignement bien défini, des opérateurs d'un autre type, internes à la forme récursive elle-même et permettant au sujet de décider sur la place relative de l'indicateur de l'action de base du programme et du moteur de l'itération (appel récursif).

La S.R.I. comme mise en ordre d'informations et d'opérations.

Du point de vue des études didactiques nous sommes directement intéressés aux conditions propres à permettre l'identification du jeu mutuel des éléments du problème et de la solution dans un contexte particulier de production et de mise en oeuvre de cette solution. On cherchera donc à faire varier des éléments du contexte autant que les problèmes eux-mêmes.

Par exemple, les relations entre les compétences réelles du dispositif d'exécution, les représentations que peuvent en avoir les élèves à un moment donné de l'apprentissage et les solutions qui vont être mises en oeuvre sont très importantes. Il y a en fait autant intériorisation des opérateurs que construction d'une conception des compétences du dispositif [3].

Lors d'un travail antérieur à propos de Pascal [6], une situation introductive de l'itération était composée d'un problème d'écriture de programme de calcul de multiplication d'entiers pour un dispositif composé d'un opérateur humain et de deux machines A et B. Sur la machine A seule l'addition pouvait fonctionner, sur la machine B, seule l'opération +1 était possible. Une fois écrit, le programme devait être communiqué à un récepteur qui recevant deux nombres devait exécuter le programme puis expliquer ce qu'il était censé calculer. Il s'agissait d'identifier des formes de structuration d'un type d'itération indéterminée.

Ce dispositif est insuffisant pour permettre l'identification des informations qu'il est nécessaire de coder dans un corps de boucle. D'un point de vue naïf, ce sont elles qui doivent être itérées. Elles sont donc de nature "coopérative" : que faut-il savoir si, s'arrêtant à une étape quelconque du calcul, on veut donner à quelqu'un les moyens de le poursuivre ?

CONCLUSION.

Il semble qu'il soit d'ores et déjà nécessaire de compter l'itération comme un des objets nouveaux de l'enseignement des mathématiques à l'école élémentaire et au niveau secondaire. Il ne s'agit pas d'une notion toute faite. Au contraire, tout donne à penser que par sa richesse notionnelle, la variété de ses domaines d'application, ses relations avec des concepts fondamentaux comme celui de variable, il faille l'engager dans un grand nombre de situations pendant une période de temps significative. A côté du mouvement, propre à l'évolution des curriculums qui va tendre à insérer dans l'enseignement des mathématiques des problèmes comportant plusieurs modes d'utilisation de l'itération, il paraît nécessaire de développer des études spécifiques à la place et au rôle de cette notion dans la construction des connaissances en mathématiques et en informatique. Nous avons essayé ici de soulever quelques éléments qui nous paraissent significatifs.

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Disabilities and the learning of mathematics

TEACHING REPRESENTATION AND SOLUTION FOR THREE TYPES OF ALGEBRA WORD PROBLEMS: A STUDY WITH LEARNING DISABLED ADOLESCENTS

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ABSTRACT

Recent theoretical work suggests problem solving in algebra consists of two phases: representation and solution. Adolescents, particularly learning disabled ones, have difficulty with algebraic word problems. The purpose of this study was to design and investigate the effectiveness of instruction in representation and solution for learning disabled adolescents.

Instruction for each phase was based on cognitive task analysis, and included declarative knowledge, modelling of procedural knowledge by thinking aloud, guided practice, and independent practice. Data were derived from problem-solving measures, think-aloud protocols, and interviews about metacognition. Students mastered representation and solution and maintained and transferred what they had learned. They acquired schemata for algebraic word problems.

OBJECTIVES

This study builds upon the cognitive theories of problem solving developed by instructional psychologists. For these researchers, problem solving in knowledge-rich domains consists of two phases: problem representation and problem solution (Mayer, 1985). Recent reviews show that while there are few differences in ability to solve, experts are better than novices at representation (Glaser, 1984). Despite this, most instruction has focused on solution. The focus of the study reported here is on representation as well as solution in learning disabled adolescents. Although many adolescents fail to understand and solve algebraic word problems, the learning disabled seldom suceed at this task (Lee & Hudson, 1981). Although these students have average ability, they evidence psychological processing problems and achieve poorly in particular curriculum areas.

The first purpose of this study was to design instruction in problem representation as well as problem solution of algebraic word problems, based on current cognitive theory. The second purpose was to investigate the effectiveness of this cognitive theory-based instruction for teaching learning disabled adolescents representation and solution of particular problem types, for the construction of problem schemata, and for transfer to related problems.

THEORETICAL FRAMEWORK AND INSTRUCTIONAL DESIGN

The theoretical underpinnings of the study enter in two ways: first the conceptualization of problem representation and solution; and second the delivery of the knowledge (both declarative and procedural) that students need to successfully represent and solve.

First, the conceptualization of problem representation and solution is described. This study addressed several convergent theories in instructional psychology in an integrated research design. Mayer's (1985) conceptualization of two phases of problem-solving instruction was adopted. Simon and Hayes' (1976) notion of problem isomorphs was employed in discerning three problem types for instruction. Reif and Heller's (1982) prescriptive task analysis of physics problems served as a model for analyzing these algebra problem types. Instructional procedures incorporated recent developments in declarative and procedural knowledge with principles of cognitive behavior modification. Together, these accounts suggest that students can construct problem representations if they are taught schemata for the most important algebra word problems.

There are two major systems for classifying algebra word problems: by the form of the underlying equations, and by the general form of the story line. The first is based on the mathematical relations present in the problem and refers to the structure of the solution equation. The second refers to contextual or surface details such as mention of money, age, or river current.

For many problems in the algebra curriculum, there is no necessary connection between the mathematical structure and the surface structure. Yet many of the investigations that provide the strongest evidence for instruction to develop problem schemata confound the mathematical structure and the story line (e.g., Hinsley, Hayes, & Simon, 1977). Mayer (1981) argued that instruction in algebra problem solving should be constructed around "templates" (specific propositional structures combined with a particular story line) because "... in general certain major categories of problems (based on story line) involve characteristic underlying equations." This is the case only in formula-based problems, however. Students who receive instruction for templates must then be taught to generalize from the template to other problems with the same mathematical-propositional structure and new story lines, or isomorphs. Isomorphs are problems in which the solution paths map directly on to one another in one-to-one fashion (Simon & Hayes, 1976). Three problem isomorphs were selected for instruction in this study: relational problems, proportion problems, and problems in two variables and two equations. Each is characterized by the mathematical relationships that have to be understood for constructing a thorough and accurate representation, and by the form of the equation that has to be solved. The three problem types are relational problems, proportion problems, and problems in two variables and two equations. An example of a relational problem is: "Sam has $18 more than Tom. Together they have $82. Find how much money each boy has." A proportion problem is: "Brian saved $50 in 18 weeks. At that rate how long will it take him to save $350?" An example of a problem in two variables and two equations is: "Andrew has 18 coins, some quarters and some dimes. The total value of the coins is $3.45. Find the number of each kind of coin." Five story lines or surface features were used with each problem type. These story lines concern money, age, distance, work, and number.

The second theoretical underpinning of the study is its instructional design. Recent intervention studies (e.g., Nuzum, 1983) have employed guided instruction based on cognitive behavior modification (Meichenbaum, 1977) to teach learning disabled students mathematical problems that require one operation or two operations. Problems like those taught by Nuzum (1983) are less complex, and involve representations requiring primarily a choice of operations. Principles of guided instruction were adapted for the complex cognitive task analyses of the three algebraic problem types of this study. Guided instruction was conducted for each of representation and solution, and involved the presentation of declarative knowledge, modelling of procedural knowledge by thinking aloud, guided practice in thinking aloud, and independent practice. Students mastered each task before proceeding to the next phase of the intervention. Students employed a self-questioning strategy to guide their construction of a representation for each problem, and their production of a solution. They completed the problems on a worksheet structured to parallel the self-questions. Detailed feedback was provided during the course of thinking aloud and at the close of each independent prac'ice. The self-questions and feedback were faded.

RESEARCH DESIGN AND METHOD

Two overlapping designs were employed to answer questions about the effectiveness of the instruction: a traditional experimental/two-group design and a single-subject design. In the two-group design, the instructed group's performance was compared with a control group on several dependent measures, including: instructed problem types, general problem-solving tests, understanding shown in think-aloud protocols, and a metacognitive interview. In the single-subject design, the course of intervention was recorded for individual students. Maintenance for the instructed problem types was measured six weeks after the end of instruction. Tests of near and far transfer were administered at the close of instruction. Near-transfer problems used the same mathematical structure with new story lines, while far-transfer problems used the same story lines and a more complex variation of the mathematical structure. The problems and instructional materials are available in Hutchinson (1986). The experimental group (N=12) received 12 weeks of the instruction described above, the three problem isomorphs being taught in succession.

Instructed problems were scored for thorough and accurate representation and solution, as well as correct numerical answers, using criteria based on prescriptive task analysis. Scoring procedures were developed for rating the degree of understanding of representation and solution in think-aloud protocols and for the metacognitive interviews (Ericsson & Simon, 1984). Interrater reliability coefficients for all scoring procedures exceeded .90.

Twenty learning-disabled adolescents in grades 8, 9, and 10 in two schools were randomly assigned to control (N=8) and experimental (N=12) groups. All students were familiarized with a structured worksheet designed to cue the steps in problem representation and problem solution.

RESULTS

Data were derived from problem-solving measures, think-aloud protocols, and metacognitive interviews. These were examined for convergent inferences respecting theoretical models in instructional psychology. As documented below, both the single-subject data and the group comparisons demonstrated the effectiveness of the designed instructional approach to representation and solution.

The single-subject data showed that individual students' problem solving ability increased dramatically for each problem type. For example, on the baseline measures of representation, the average percentage of problems represented correctly by all instructed students was less then 5%. During instruction in representation the average percentage of problems represented correctly was 84%. The students showed near transfer to problems with altered story lines in 83% of the cases and far transfer to problems with slightly more complex mathematical structure in 60% of the cases. Six weeks later, in 90% of the cases, students reached criterion for maintenance on problem types they had mastered during instruction.

All between-group comparisons favoured the instructed students over the control group. A significantly higher proportion of instructed students reached criterion in representation and in solution for each problem type. Fisher's Exact Test on posttest proportions was significant in all nine cases ($\underline{p}<.05$, $\underline{df}=1$). Analysis of covariance on adjusted posttest scores on an open-ended problem-solving test showed that scores of the instructed group were significantly higher, $\underline{F}(1, 17) = 37.79$, $\underline{p}<.05$. Similarly analysis of covariance demonstrated the superiority of the instructed group on the metacognitive interview, $\underline{F}(1, 17) = 32.38$, $p<.05$. On the think-aloud protocols, there was no overlap between the distributions of scores. The instructed group was clearly higher on each of representation and solution.

Think-aloud protocols showed shifts following instruction from mere reading comprehension of problem statements to expression and application of domain-specific algebraic knowledge, and shifts from use of arithmetic operations to algebraic procedures. Content analyses of protocols and interviews suggested that instructed students had constructed schemata for the three types of algebraic problems based on mathematical structure.

IMPORTANCE OF THE STUDY FOR THE STUDY OF THE PSYCHOLOGY OF MATHEMATICAL EDUCATION

The current instructional study was conducted with a sample of learning disabled adolescents and replication with a normal sample is necessary in order to draw general, widely applicable conclusions. This study affirmed the current models of problem solving composed of two phases: representation and solution (Mayer, 1985). It demonstrated that specific instruction in representation is feasible and effective (Reif & Heller, 1982). The notion of instructable problem isomorphs was refined and substantiated with qualitative data about the nature of schemata for algebraic problem types (Simon & Hayer, 1976). Recommendations were made for future research in instruction of problem representation and for the study of the construction of problem schemata, based on psychological theory applied to mathematics education.

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THE RELATIONSHIP BETWEEN MATHEMATICS LEARNING AND AUDITORY PERCEPTION

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This ongoing study examines the relationship between mathematics performance and auditory perceptual deficits. Twenty-one children identified as either mathematics impaired (N-13) or mathematics and reading impaired (N-8) were tested for auditory reception and perception deficits and mathematics performance. With the exception of some minimal central auditory processing deficits, all subjects tested within the normal range for peripheral acuity and central auditory functions. All subjects demonstrated abnormal cortical integration abilities as assessed by the GFW battery. The Arithmetic and Reading group demonstrated poorer performance throughout the battery with pronounced deficits on the auditory memory subtests as well as on arithmetic content.

The study examines the relationship between specific auditory perceptual characteristics and mathematics performance in children experiencing difficulties learning mathematics. The children included in this study can be defined as "mathematics deficient," rather than "learning disabled." None fit the definition of 'learning disability.' This distinction is essential, for evidence exists that a child with a mathematics learning disability will have visual perception deficits rather than auditory perception deficits. (Strang and Rourke, 1985)

Fuson and Hall (1983) have demonstrated that a wealth of information is known by the typical primary grade child. This knowledge is processed priarily in the tactile and visual modality but as the level of content abstraction increases, the child must shift to a auditory madality for learning with a decreased reliance on tactile and visual perceptual skills.

Thus it is possible for a child with an auditory perception deficit to possess a good understanding of the mathematics taught in the primary grades, e.g., counting, numeration, basic addition, etc., yet experience difficulty learning mathematics in the middle school. This occurs because the content to be learned in the early grades requires learning a limited number of graphemic symbols which can be acquired visually or tactually, without resorting to complex, auditory/visual linguistic transcoding. However, understanding mathematics in the upper grades requires effective perceptual integration of the auditory and visual sensory input systems. Hiebert (1984) terms this integration "site 2" and defines it as the stage where "form and understanding are linked when children connect a procedure or algorithm with the underlying concept or rationale that motivates the procedure."

If the transcoding is inefficient and requires additional processing time, then heavy demands upon the auditory memory system result in a loss of information from memory prior to completion of problem solving tasks. Perceptual learning is based upon experience, and is generally defined as the ability to "extract information from the environment." (Gibson) Children with auditory perceptual disorders can hear sounds, but may not be able to recognize that the sounds are relevant or needed for auditory association or other cognitive processing. (Gibson and Levin, 1975) As Sloan notes, defects in the auditory processing system become more apparent as the task at hand uses more complex sounds.

Cathcart (1974) identified listening ability as being the most significant non-mathematical variable for mathematics learning, but did not define what he meant by the term 'listening ability.' A positive relationship exists between the matching of modality i.e., Cross-Modality, and mathematics learning. (Sawada and Jarman, 1978) Freidea (1974) suggests an interaction between modality mathcing and information complexity. Cross-Modality refers to the ability to transform meaning between two different sensory systems.

Cross-modality matching occurs when a child relates a teacher's verbal instructions with the symbols which the teacher has written on a chalkboard. Mathematical algorithms are commonly presented as series of oral/visual instructions to be memorized. A child with an auditory/visual perceptual deficit may have trouble moving from the grapehmic system, i.e., blackboard demonstrations, or textbook explanations to accoustic system, i.e., the teacher's oral instructions.

Fletcher & Loveland (1986, p. 31) indicated the exact nature of the cognitive deficit in arithmetic-disabled children has not yet been established. One possible reason for the failure to identify these cognitive processes may be due to the failure to identify impairments of sensory reception, sensory transmission in the central nervous system. All of these systems must be functioning appropriately before normal cognitive operations can be performed efficiently.

METHOD

The assessment proceudre consisted of the following:

Auditory Three-phase auditory perceptual battery for evaluating the physiological status of the peripheral auditory mechanism, central auditory transmission network, and central auditory/visual perceptual-association systems.

Mathematics Administration of the Kay-math test

Auditory Perceptual Battery

I. Peripheral Acuity a comprehensive auditory test battery was administered to assess peripheral hearing acuity. This battery consisted of pure-tone threshold testing at 250-500-1000-2000-4000-8000 HZ. via air conduction for both ears, Speech-Reception thresholds bilaterally, and bilateral Speech-Discrimination percentage for undistorted words.

II. Central Auditory Assessment consisting of seven subtests was administered to evaluate physiological function of central auditory pathways of the brain-stem, primary auditory reception areas and auditory association areas of the cerebral cortex. Two of the subtests assess the function of the brain-stem, and the remaining five subtest evaluate function of auditory processing centers in the cortex.

III. Auditory-Visual-Perceptual Assessment was assessed through administration of the Goldman-Fristoe-Woodcock Battery of auditory skills (GFW) (finkenbinder, 1973). Three auditory memory tests measure short-term retention of acoustic information. Seven visual-linguistic tests measure auditory/visual integration functions and two test assess auditory figure-ground processing abilities.

Subjects

Group A. Mathematics only Ten subjects consisted of 3 females and 7 males ranging in age from 7 to 14 years.

Group B. Mathematics and Reading Eleven subjects consisted of 5 females and 6 males ranging in age from 10 to 13 years.

Each child in both groups had experienced difficulty learning mathematics but none qualified for special education placement. None were diagnosed as neurologically impaired or mentally retarded.

Results

Peripheral auditory results for both groups:

	Right Ear	Left Ear
Pure-tone averages	-8db to +10 dB	-5dB to +10 dB
Speech Thershold	0dB to 15 dB	-5db to +10 dB
Speech Discrimination	84% to 100%	92% to 100%

table 1

Central Auditory test results:

Fourteen subtest scores were obtained for each subject in both experimental groups. A total of seventy scores for each ear for the Group A and seventy-seven scores per ear for the Group B subjects.

Group A 14.3% of the right and left ear scores fell in the abnormal range. Abnormal scores occurred most often on the Dichotic sentences subtest (25%), on the competing conditions of the SSW (25%) and 15% of the errors on the Low-Pass-Filtered-Word test. The abnormal scores primarily occurred on the subtest which cause the greatest load on selective attention.

Group B 24.6% of the right ear scores and 23.4% of the left ear scores were abnormal. Combined abnormal scores totaled 24%. Abnormal scores occurred primarily on subtests that load heavily on selective attention skills (62%) subtests such as Competing Sentences, Dichotic sentences, and the competing conditions on the Staggered Spondaic Word test (SSW). In addition this group had difficulty with shifting directional attention between right and left ears. This pattern of directional processing deficit is seldom seen in other learning deficient populations according to Jack Willeford.

Auditory perception and Auditory/visual association results

All twenty-one subjects tested in the two groups performed well below the fiftieth percentile level on the Selective Attention subtest of the Goldman-Frestoe-Woodcock Auditory skills battery. The ability to screen out nonrelevant acoustic signals and attend to meaningful materiel is impaired in all of the subjects tested. Group A subjects tended to do slightly better than Group B subjects on this subtest. The GFW has three subtests that assess auditory memory function. They are; Recognition memory, Content memory, and Sequential memory. The poorest Group B performance occurred on the Content memory subtest where all scores fell below the 25th percentile level. In contrast the poorest score for the Group A on this subtest was the 39th percentile and only two of ten subjects socres fell below the fiftieth percentile level. The Group A demonstrated mixed performance on the two remaining auditory memory subtests (Recognition memory & Sequential memory) with almost an equal split above the 50 percentile level and below the 50 percentile level. It is apparent that both groups have inefficient auditory memory skills, however, the impairment is more pronounced in the Group B subjects. Group B are more likely to rank below the 50 percentile level on all three auditory membory tasks than are the Group A subjects. Only 18% of the Group B subjects had percentile socres on the Recognition memory and Content memory subtest above the 50 percentile level. This group did slightly better on the Sequential memory subtest where 27% of the scores were above the 50 percentile level. Performance by Group A was some what better for these three subtest on Recognition memory 50% of the scores were above 50 percentile level, 60% of the subjects ranked above the 50 percentile level on the Content memory subtest and 50% of the group ranked above this level on the Sequential memory task. Most of the twenty-one subjects had trouble with material on the Sound mimicry subtest where only 24% ranked above the 50 percentile level. In general Group A tended to do well on five of the six remaining GFW subtests that required auditory and visual transcoding abilities. Group A did show very poor scores on the Sound mimicry subtest where all but two subjects scored below the 37th percentile. This group has trouble imitating nonsense syllables received auditorally. In addition 60% of Group A did poorly on the Spelling of Sounds subtest.

Mathematics Testing

With three exceptions, all subjects tested below grade level on the Key-math Test.

Basic Areas-Addition, Subtraction, Multiplication, Division

	Basic Facts	Algorithms
Group A Mean %	100	85
Group B Mean %	94	49

table 2

While there was no consistent area of weakness among Group A students, the weakest area for Group B students was Word Problems and Mental computations.

Discussion

Results of the peripheral auditory assessment confirmed that all of the subjects had normal hearing acuity for nondistorted acoustic stimuli. Group A subjects performed in the normal range on 86% of the central auditory subtest scores, while Group B subjects scored normally on 76% of this battery. It is apparent from these findings that some of the subjects tested have central auditory transmission deficits which may adversely affect accuracy of processing at the perceptual and association centers in the brain. It is not possible with the current test battery to determine how much of the decreased performance is a result of the central auditory deficit or to determine how much might be due to accompanying auditory perceptual and/or association deficit.

The GFW assessment battery evaluated the auditory/visual perception skills on the subjects. All twenty-one subject demonstrated poor performance on tasks requiring selective screening of acoustic signals. Three areas of auditory memory were assessed by the GFW and generally the Group A subjects had less auditory memory difficulty than the Group B subjects. Both groups did show restricted auditory memory abilities, however these problems were more pronounced in the Group B subjects. Sound imitation of nonsense syllables and spelling of sounds heard as nonsense syllables were difficult skills for Group A subjects. In general the Group B subjects had more difficulty on subtest tasks which required transcoding between the acoustic symbol system and graphemic symbol system.

There was a marked difference between Group A and Group B subjects with respect to mathematics performance. While both groups performed equally well on basic facts group A subjects scores were markedly higher in the algorithms.

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Gender and mathematics

GENDER DIFFERENCES in ACHIEVEMENTS and in CAUSAL ATTRIBUTION of PERFORMANCE in HIGHSCHOOL MATHEMATICS

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Background

High school mathematics constitutes a critical filter for continuing education and for higher professional training in various areas of technology and the natural sciences. The existing occupational gap between males and females in these areas may therefore be traced to the issue of gender differences in mathematics education in high school which was found to be much wider than in other school topics (Movshovitz-Hadar 1984). The existing gap between the relative percentages of boys (26.4%) and of girls (12%) electing the study of math in highest level offered by school at grades 11, 12 in Israel (C.B.S., 1986) suggested this study.

Objectives

The study was aimed at answering two main questions:

1. Are there sex differences in the causal attribution of success and failure in the study of mathematics in grade 10 (where math learning is mandatory)?
2. How do mathematics achievements in grade 10 relate to patterns of causal attribution and to sex?

The Sample

The sample consisted of 384 students in 10th grade (178 male and 206 females) from three comprehensive highschools. The choice of 10th grade population was based on the assumption that causal attribution has direct implications on students' future choice of level of the study of mathematics, in grades 11, 12.

It should be noted that mathematics classes in 10th grade in Israel are ussually not grouped according to achievement level. At the end of 10th grade students chose the level the wish to take in 11th grade and are regrouped accordingly. Mathematics study is mandatory at the lowest level. Students who wish to study more than that continue in 12th grade.

The Instruments

The data pertaining to the level of achievement were obtained from school records.

Causal attribution was measured by means of the "Math Attribution Scale" (Fennema et al., 1979) which was translated to Hebrew revalidated and adopted for the purpose of this study. The design of this scale was based upon Weiner's Causal Attribution Theory (Wiener 1974). The scale presented four cases of failure and four cases of success in mathematics studies. Each case was followed by four causes, one of each of the following categories: ability, task, effort or environment. Examinee's task was to evaluate the personal relevance of each cause to each event. Thus, eight attribution patterns emerged, four of the type: (success, cause), and four of the type: (failure, cause).

Main Results

Most of the results obtained in this study support the results of an earlier study carried out in the U.S. (Wolleat et al. 1980).

1. Significant gender differences ($p < .05$) were found in seven of the eight causal attribution of success and failure in mathematics studies. Males attributed their <u>success</u> in mathematics to personal ability to a greater extent than females. On the other hand, females significantly more than males attributed their <u>failure</u> to (lack of) personal ability and their success to effort, task and environment. They also attributed their failure to task and environment significantly more than boys did. Table 1 shows the results.

Table 1:

Sex Differences in Causal Attribution Patterns (C.A.P)

(178 boys and 206 girls in 10th grade. * $p < .05$)

C.A.P.	Sex	Mean (S.D.)	t-value
Success-Ability	M	13.65 (3.52)	4.03 *
	F	12.27 (3.14)	
Success-Effort	M	14.15 (3.69)	-2.81 *
	F	15.17 (3.30)	
Success-Task	M	14.87 (2.64)	-3.24 *
	F	15.73 (2.53)	
Success-Environment	M	14.67 (2.99)	-2.02 *
	F	15.27 (2.81)	
Failure-Ability	M	10.38 (3.39)	-4.07 *
	F	11.82 (3.57)	
Failure-Effort	M	14.97 (2.68)	0.25
	F	14.89 (3.35)	
Failure-Task	M	13.11 (2.74)	-3.75 *
	F	14.13 (2.63)	
Failure-Environment	M	12.09 (2.99)	-3.31 *
	F	13.09 (2.87)	

2. Male and female students of equal achievement level differed significantly ($p < .05$) in their attribution patterns. Those differences increased in magnitude as the level of achievement went up. As shown in Table 2 below, gender differences in the upper achievement group were significant in seven out of the eight attribution patterns, (the direction of the differences were similar to the ones found with respect to question one). The gender differences in the medium achievement level group were significant in four attribution patterns. No significant differences were found in the lower achievement group ($0 \leq$ score ≤ 54, 38 boys and 52 girls).

Table 2
Sex Differences in Causal Attribution Patterns (C.A.P) Within Achievement Level (10th grade. *p < .05)

		High Achievement 75 ≤ score ≤ 100 (65 boys, 68 girls)		Medium Achivement 55 ≤ score ≤ 74 (75 boys, 86 girls)	
C.A.P.	Sex	Mean (S.D.)	t-value	Mean (S.D.)	t-value
Suc.-Ability	M	15.6 (2.9)	3.54 *	12.8 (3.6)	1.52
	F	13.8 (3.1)		12.1 (2.7)	
Suc.-Effort	M	13.4 (4.1)	-2.26 *	14.2 (3.3)	-2.79 *
	F	14.9 (3.3)		15.9 (3.0)	
Suc.-Task	M	15.2 (2.6)	-2.33 *	14.7 (2.5)	-2.72 *
	F	16.3 (2.7)		15.7 (2.2)	
Suc.-Env.	M	14.1 (3.0)	-2.21 *	15.0 (3.0)	-1.12
	F	15.1 (2.6)		15.6 (2.8)	
Fail.-Ability	M	8.9 (2.1)	-2.59 *	10.7 (3.1)	-2.21 *
	F	10.4 (3.7)		11.8 (3.1)	
Fail.-Effort	M	14.8 (2.9)	.12	14.9 (2.6)	.41
	F	14.7 (3.5)		15.0 (3.4)	
Fail.-Task	M	12.7 (2.7)	-1.99 *	13.3 (2.9)	-3.20 *
	F	13.6 (2.5)		14.6 (2.5)	
Fail.-Env.	M	11.7 (3.3)	-2.58 *	12.3 (2.8)	-1.85
	F	13.2 (3.4)		13.1 (2.6)	

Brief Discussion and Implications

According to the causal attribution theory and its implications for the field of learning, systematic attribution of failure in mathematics to lack of personal ability and attribution of success to the kind of task or to environment may lead to "learned helplessness" and consequently to reduced achievement or to avoidance of mathematics. The attribution patterns of 10th grade females, in particular for those of the higher achievement level, as described above, may have a deter imental effect on the actualization of their mathematical talent and to the avoidance of extensive studies of mathematics in the future. The existing low rate of enrollment of females in high level mathematic courses in grade 11 and 12, may therefore be related to their attribution patterns for success and failure in mathematics.

This study suggests that an active intervention, aimed at changing attribution patterns is needed for female students at the early stages of their high school edcation. Such an intervention, if proves successful, may improve the achievement level among female students and widen their mathematics education, which may in turn contribute to a greater vocational integration of women in the various fields of technology and natural sciences.

The discussion will be expanded in the conference.

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Geometry

SECONDARY SCHOOL STUDENTS' AND TEACHERS' UNDERSTANDING OF DEMONSTRATION IN GEOMETRY

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Abstract

The purpose of this study is (1) to describe, within a model of understanding, the secondary school students' and teachers' understanding of proof and demonstration in geometry and (2) to seek what kind of relationship could exist between the understanding of a demonstration and the van Hiele levels. Our sample consists of 220 students learning demonstration in French secondary schools and their 13 teachers. Each of the persons answered the "correction test", whose design refers to the model of understanding, and the Van Hiele Test. The results indicate that proof and demonstration are not synonymous for the teachers as well as for the students. Proof belongs to different modes of understanding but demonstration always pertains to the formal one, teachers emphasizing presentation and wording. Students seem not to favour such an emphasis. There is no obvious relationship between the understanding of a demonstration and the van Hiele levels.

RATIONALE (see note 1 at the end)

Even if it has not completely disappeared from secondary mathematics curricula, geometry is now a less important topic than it was before. This is especially true when one thinks of geometry with demonstration: nowadays, the geometry taught in classes is generally intuitive or descriptive. But in some countries, such as France, students are still taught formal demonstration. This usually brings out a lot of difficulties: whatever may be, the teachers' efforts and achievement in their teaching, it is well known that a large number of their students fail to understand demonstration (APMEP, 1979).

Some studies (Carpenter et al., 1980; Usiskin, 1982; Senk, 1985) indicate that fewer than 15 percent of highschool graduates in the

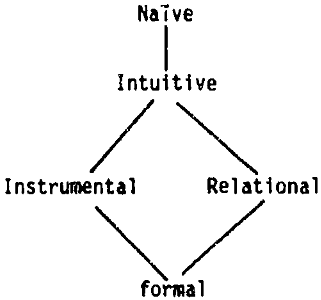
United States master proof writing. According to these studies, at the end of a full-year course in geometry, in which proof writing is studied, about 30 percent of the students reach a 75 percent mastery of proof while 25 percent of the same sample have virtually no competence in writing proofs. A canadian study (Williams, 1980) shows that only those students who were classified as high achievers by their teacher, less than 30 percent of the sample, exhibited any understanding of the meaning of proof in mathematics. However, having asked teachers their opinions about high school geometry course, Gearhart (1975) reports that half of the responding teachers think that 50 percent of the students are able to prove a medium difficulty theorem such as "the diagonals of a rectangle are congruent". Furthermore, 89 percent of these teachers indicated that proofs are not a too difficult topic.

According to Freudenthal (1973), "to progress in rigour, the first step is to doubt the rigour one believes in at this moment. Without this doubt, there is no letting other people to prescribe oneself new criteria of rigour." Therefore, the question arises: would not the problem be that teachers are unable to communicate their students the rigour criteria which are linked to their idea of demonstration? Teachers may be asking one thing and the students understanding something else, the same words carrying different meanings for the teachers and the students. This lead to the general objectives of this research:

1) To describe the students' and teachers' understanding of a demonstration;
2) To describe the students' and teachers' idea of proof;
3) To establish some kind of relationship between a student's understanding of a demonstration and his achievement in mathematics.

FRAMEWORK

Constructivist learning theories being the general framework of this study, references are made to the theory of levels of thinking in geometry initiated by P.M. van Hiele. A model of understanding of proof and demonstration was built altogether. This model although summarily sketched, is drawn from those by Skemp, Byers and Herscovics, Herscovics and Bergeron. This model includes five modes organised as on the figure below. These modes are defined in the following way:



- naïve: "you can tell from the figure". No other argument is necessary for the subject to be convinced;

- intuitive: the figure is not enough to convince the subject but any argument is convincing, whatever that argument is a mathematical one or not;

- instrumental: the subject lingers over each step of the demonstration without getting any global idea of it;

- relational: the subject is intercaled only in the important arguments of the demonstration;

- formal: the subject masters the instrumental and relational understanding.

This model being sketched out, a small experimentation was carried out in order to test its adequacy. That study revealed that, indeed, the model appeared adequate to describe any understanding showed by the students or the teachers. It revealed altogether that the word "proof" seemed to have a different meaning for the teachers than it has for the students and that some kind of relationship could exist between the kind of understanding achieved and the van Hiele levels. Therefore, the following questions arose and became the main objectives of this study:

- do "proof" and "demonstration" have different meanings for teachers and students and, if any, how could that difference be qualified?
- is there really a relationship between the students' mode of understanding of proof and demonstration and the van Hiele level he or she attained?

SAMPLES AND TOOLS

One of the objectives being to describe the students and teachers understanding of a demonstration in geometry, it was necessary to work with students to whom demonstration was actually taught and with teachers who were teaching demonstration. This is the reason why the study has been carried out in France with more than 200 students of 3e (fourth year of secondary school: the students were 15 years old) and with 13 teachers. Each student was initiated to demonstration the previous year and some of the teachers met were responsible for their initiation.

Two main tools were used for the experimentation, both with students and teachers. On one hand, the van Hiele Levels Test created ty Usiskin et al. (1982) was translated in French: this 25 questions test was designed to evaluate someone's level of thought in one or the other of the five levels described in the van Hiele theory. On the other hand, a "correction test" was built: it consist of a geometry problem taken from a text book of 4e in France and of 12 different solutions given by 12 fictive students. Each of these solutions was designed to belong to a precise mode of understanding: more precisely, there were 3 formal solutions, 2 relational, 2 instrumental, 2 intuitive, 1 naïve, 1 completely false solution and a solution which remained unclassified but that was considered interesting because of its content.

The students were asked to answer the van Hiele Test, to evaluate 6 of the 12 solutions (2 formal, 2 relational and 2 intuitive) and to justify their evaluation. For that evaluation, the students had to answer three questions: 1) which of the 6 solutions is the most convincing for you? 2) which solution looks like the one you would give if you were asked to answer that problem? 3) which solution looks like the one your teacher generally expects? The scores those students have had in mathematics the previous year were also recorded.

The teachers were also asked to answer the van Hiele test and they had, during interview, to grade all the 12 solutions of the correction test, their reactions being audio-taped. Before they started to grade the copies, teachers were asked to solve the problem and to write down the solution they would have given to their students. Finally, they were asked some general questions such as "how do you generally teach demonstration?" or "how important do you think demonstration is in mathematics education?".

RESULTS

In the following paragraphs, the results referring to the model of understanding and to the accuracy of its structure will first be presented. The interviews with the teachers will than be analysed and, finally, the students answers will be sum up.

The model of understanding itself and, particularly, its stromboïdal structure, have been corroborated by the teachers' reaction while grading the students' solutions, by the marks they gave and, above all, by the comments they made. Schematically, the means of the marks given the solutions grouped in each mode were the following: naïve: 0, intuitive: 0.5, instrumental: 2.5, relational: 2.5, formal: 4.5. But within each mode, the marks given by the teachers are more divergent than their comments, i.e. different marks derive from similar reactions and analyses of the same texts. The fictive students' solutions belonging to the naïve mode are systematically rejected by all the teachers whereas in texts of intuitive type, some teachers can see some justifications, i.e. for those teachers, the student has begun to analyse the figure and its properties. Following these reactions, we were led to reclassify two solutions. The first one, which was once located in the intuitive mode, was downgraded in the naïve mode because, in the teachers' view, the student had just read the figure. The second one was the one we had kept without classifying it: it is now located in the intuitive mode since it shows some analyzing in the figure. Finally, we also noticed that many teachers have spontaneously compared solutions which, according to our view, were actually belonging to the same mode of understanding.

From the interviews with the teachers, we clearly see that for these people, "proof" and "demonstration" are not synonymous. The teachers were convinced of the result of the problem as soon as they have finished to draw the figure: any other argument was unusefull even if some of them gave some more arguments to justify their conviction. However, their attitude became completely different when they were asked to write down the "type-solution": they then made a "real" demonstration, a "literary formal proof", that is a proof written down in "whole words" and complete phrases. Their requirements for rigour, presentation and wording, all these being very accurate in their opinion, were again felt in the scores they gave to each solution. As a general rule, half of the marks was reserved for presentation, the other half being given for arguments. Thus, the teachers tended to be more indulgent for the instrumental texts, congratulating the students' efforts in their attemp to reproduce the "type-solution" than they were for the relational ones where, although the problem was judged understood, "it was badly worded". The marks attribuated to the formal solutions give new evidences of their stringent requirements of presentation: the solutions where the formalism was not the literary one (the "two colums" or the "tree" formats) have been penalised, the student loosing half of a point because of his choice of presentation. Let us add that tne literary formal text was also occasionally criticized for lack of clearness of the connections between the different parts of the demonstration. However, the teachers recognized that a tree-format formal demonstration could be considered as a convincing solution even if they were not willing to accept it. Later in the interviews, the teachers explained that, according to them, demonstration is a very important activity in geometry and that it was a real educational task for the students, especially for the learning of rigourous deductive arguments. But there is a kind of contradiction here between the objectives and the means used to reach them because, the way demonstration is taught often emphasizes the presentation rather than the argument itself. It was thus easier for the teachers to speak about the "advantages" of their favourite presentation than about the way they used to teach how to construct deductive arguments. And the requirements for presentation were very different from one teacher to another but were always judged very important. So, one can ask oneself how do students feel with such an ambiguity.

The students' reaction to the test showed that two out of three were convinced by a formal solution rather than by any other type of solution. This choice is supported by the teachers' preference. But within the formal mode, the tree-format solution was prefered, 36% of the student sample having selected that type of solution as the most convincing. The same thing occured when the students were asked to identify what would be their teachers' prefered solution: they all knew that their teachers expected a formal solution but again, half of them choose the tree-format solution while, as it was seen, most of the teachers ask for a literary formal demonstration. The students' difficulties do not lie in a wrong perception of their teachers' requirements alone, since only about 37% of our sample would have given, on their own, a formal solution to the exercise. The other students choose a solution that would have been intuitive (30%) or relational (31%). The χ^2 test shows that the students'answers to all these question are very dependant one from the other: the type of solution the students judged convincing, the type of solution they would have given and the type of solution they thought their teachers expected are bounded together. However, the same χ^2 test indicates that those answers are independant of the van Hiele levels attained but shows that those levels attained and the students' marks in mathematics are strongly linked together.

CONCLUSIONS

Although it has been carried out with a rather small sample, this study made possible to examine a model of understanding of geometry proofs and demonstration and to check the accuracy of the stromboïdal structure of its different modes: naïve, intuitive, instrumental, relational and formal.

Furthermore, it has been seen that for the teachers who have participated to the study, proof and demonstration are not synonymous. Proof may vary but demonstration is always a formal presentation of a deductive argument. All the teachers agree with such a definition but their requirements of rigour, presentation and wording are very different from one to the other: for the same solution, the same comments can go with different judgements and different marks. All the teachers think that learning to construct deductive argument is very important for the students: according to their view, demonstration is the favoured activity for such a learning.

For the students, the difference between proof and demonstration does not seem to be as clear as it appears to be for the teachers: many students acknowledge to be unable to construct a formal demonstration, whatever the format should be. Furthermore, the students have rather badly identified the solution expected by their teacher, choosing correctly the type (formal) but not the format, preferring the tree-format to the litterary format. Finally, it seems that, although strongly linked to his or her marks in mathematics, the van Hiele level attained by a given student remains independant of the mode of understanding of a formal proof he or she manifests.

Note 1: In this text, "proof" and "demonstration" are defined or described as follow:

- a proof is an argument or a series of arguments which convince someone of something, whatever those arguments are mathematical or not;
- a demonstration is an answer to a geometry problem in which one is asked to give a formal proof of a statement.

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THE DEVELOPMENT OF GEOMETRIC THINKING AMONG BLACK HIGH SCHOOL PUPILS IN KWAZULU (REPUBLIC OF SOUTH AFRICA)

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Abstract

In this study various geometric thought categories (GTC's) typical of and necessary for high school geometry were distinguished. Although the Van Hiele model of geometric development provided a theoretical framework for these thought categories, a finer distinction was made between thought categories that normally appear on the same level Empirical data on these thought categories was collected via a questionnaire to provide an extensive data base on which may be founded further curriculum development in high school geometry Our study supports previous studies on the Van Hiele theory with respect to the hierarchical nature of the levels, and its explanation of pupils' problems in geometry However, the position of hierarchical classification with respect to deduction has to be clarified.

1. Objectives of the research

This talk is a discussion of research into pupils' problems in geometry and its relation to the Van Hiele theory The programme of enquiry was embarked upon with the following in mind

1 To find out if different geometric thought categories (GTC's) form Guttman scales and how they correspond with the Van Hiele model.

2 To ascertain levels at which pupils in different grades function

3 To identify the types of problems confronting pupils in high school geometry

2. Theoretical framework

2.1 The Van Hiele Levels

A number of sources are available on the different Van Hiele levels of understanding in geometry For brevity, only the general characteristic of each level is given For more information on the levels and the nature of the theory, consult Wirszup (1976), Mayberry (1981), Hoffer (1983), Burger & Shaughnessy (1985,86)

(1) Level 0 - visualization, (2) Level 1 - analysis of properties, (3) Level 2 informal deduction (ordering), (4) Level 3 - formal deduction, (5) Level 4 formal discernment of mathematics

It is perhaps important at this point to say something about the level at which hierarchical class inclusion of geometric figures is supposed to occur Although there seems to be a consensus amongst American researchers (e g Usiskin (1982), Senk (1983), Burger & Shaughnessy (1986)) that it occurs on Level 2 (Ordering), there is somewhat confusion in the Van Hiele literature itself For instance, in Van Hiele (1973, 92 93) Pierre Van Hiele argues that class inclusion can occur on Level 1 (Analysis of properties) since a child may then realize that a square is a rhombus because it has all its properties This same point is made by Dina van Hiele in Fuys et al (1984, 222) However, Pierre van Hiele in Fuys et al (1984, 245) contradicts himself and his wife when writing with reference to the First Level "But at this level . a square is not necessarily identified as being a rectangle "

2.2 Research on the Van Hiele Model

Research in the U S A on the Van Hiele Model has been done by the abovementioned researchers Their findings generally are: (i) that students can be assigned Van Hiele levels using interviews or written tests and (ii) that the hierarchical nature of the levels seem valid. However, Burger & Shaughnessy (1986) have questioned the discreteness of these levels.

3. Methodology

3.1 Sample

The sample consisted of 4015 high school pupils in grades 9 to 12, in May/June 1984. These were a random sample of pupils taking mathematics in high schools of the KwaZulu Department of Education, situated in the province of Natal The schools ranged from small, rural schools to big inner-city schools.

3.2 The instrument

The test consisted of 56 open-ended questions ranging from simple questions like indicating alternate angles when parallel lines were given, listing the properties of a given figure like a parallelogram, to questions requiring the interpretation of formal definitions and the construction of formal proofs. Most items dealt with content as commonly found in the high school syllabi like parallel lines, perpendicular lines, technical terms, isosceles triangles, congruent triangles, parallelograms, rectangles, logical inferences, significance of deduction, perspective on the difference between an axiom and a theorem. In this latter respect more general content was examined than in most other Van Hiele based research

3.3 Validity of the test

The Van Hiele Model of development in geometry was used as a guide in selecting geometric though categories and the items by which they were to be evaluated. A feature common to most items was that pupils had to give a motivation or reason for their responses In this way it was felt that more reliable interpretations to pupils' responses could be given

The preliminary test having been compiled, it was then given to mathematics educators, familiar with the Van Hiele theory, at the University of Stellenbosch so that they might check the validity and the adequacy of the items. After their comments, a final test was compiled which was then used in the investigation (a copy of the test and marking scheme is available on request)

3.4 Analyses used

3.4.1 Analysis of pupils' responses using GTC's

The present research was aimed at establishing a non prejudiced description of progress in geometric thinking This was facilitated by distinguishing a number of different GTC's each being reflected in a number of test items, and analysing the data in two ways

(a) analysing the differences between the performance of pupils in the different grades in the cluster of items belonging to a GTC by cross tabulating the different grades with the number of items from a category in which a pupil exhibited mastery

(b) performing Guttman analyses with respect to various categories (Guttman analysis is discussed at length in Mayberry (1981) and Njisane (1986)).

In the latter, Guttman analyses were done on various subsets of the various categories of geometric thinking This was to investigate the possibility that certain GTC's do not necessarily occur on the same level as theoretically formulated This was due to a hypothesis formulated by Malan (1986) that the development of an analytic perspective on properties (Level 1) and a logical perspective on the relationship between them (Level 2) may be independent of the development of a hierarchical classification of class concepts

3.4.2 The different geometric thinking categories GTC's

The different GTC's present in the test are identification of figure types, use of geometric terminology, interpretation of given definitions one step deductive arguments, verbal description of properties of figures, longer deduction and hierarchical classification of geometric concepts. First of all cross-tabulations of each GTC with the percentage of pupils obtaining a given number or more correct responses on a GTC was done.

3.4.2.1 Performance of pupils under different GTC's

3.4.2.1.1 Identification of figure types and drawing examples (Z)

The table of percentages of pupils obtaining a given number or more correct responses follows

Table 3.1 Number correct

Grade	0	1	2	3	4	5	6	7	8	9	10	11	12
9	100,0	99,9	98,8	93,9	81,2	61,5	38,8	16,7	5,2	1,1	0,2	0	0
10	100,0	100,0	99,6	97,4	91,7	78,5	58,8	36,2	18,8	6,8	0,9	0	0
11	99,9	99,6	99,6	99,6	98,5	94,5	54,3	64,5	31,5	19,2	6,7	0	0
12	100,0	99,0	99,0	98,7	97,1	95,5	90,0	79,4	60,9	36,4	13,7	0	0

On the whole, at least about 40% of the pupils in each grade were able to recognise figures and draw them, in 6 or more out of 12 cases.

3.4.2.1.2 Visual recognition of properties (F)

Table 3.2 Number correct

Grade	0	1	2	3
9	99,9	77,2	25,2	4,1
10	100,0	91,3	53,2	19,4
11	99,8	96,0	82,3	42,1
12	99,9	97,0	85,7	56,7

For the 2 or more correct responses out of 3 as criterium, pupils in grades 10 12 clearly perform better than those in grade 9

3.4.2.1.3 Use and understanding of descriptive terminology (A)

For this GTC only about 12% of Grade 9 pupils and 39% of Grade 10 pupils were able to give 8 or more positive responses out of 17

Table 3.3 Number correct

Grade	0	1	2	3	4	5	6	7	8	9	10	11	12	13
9	100,0	100,0	97,7	90,4	75,1	53,5	34,6	20,3	11,6	5,1	2,5	1.4	1	0
10	100,0	100,0	99,3	96,6	89,7	77,6	61,9	45,5	30,9	18,7	10,9	4,5	1,6	0,4
11	99,9	99,9	99,6	99,0	97,8	94,6	88,5	80,6	68,1	53,7	37,4	23,4	11,3	3,4
12	100,0	99,7	99,4	99,1	98.6	97.8	96,7	92,2	84,0	73,2	56.8	32,5	19.8	8.2

(Maximum possible score = 17)

3.4.2.1.4 Verbal description of properties of a figure (or its recognition from a verbal description) (E)

Table 3.4 Number correct

Grade	0	1	2	3	4	5	6	7	8
9	100.0	87.3	54.8	21.7	5.3	0.9	0,0	0.0	0
10	99.5	96.2	78,7	61,1	23.5	8.8	2,3	0.6	0.1
11	100,0	100,0	100,0	90,0	65.4	44.2	23.4	4.6	0.6
12	100,0	97.7	92,4	85.5	68.3	45.1	26.4	10,0	3.7

If one examines the 4 or more out of 8 positive responces. only 5.3% are in grade 9; 23.5% in grade 10; 65,4% in grade 11 and 68,3% in grade 12. It seems as if pupils not only find problems with mathematics as such, but also with the language in which mathematics is being learnt.

3.4.2.1.5 One step deduction (C)

The percentages of pupils giving 4 or more correct responses out of 7, range from 0,6% in Grade 9 to 49,6% in Grade 12.

Table 3.5 Number correct

Grade	0	1	2	3	4	5	6	7
9	100	28,4	15,1	2,5	0,6	0,3	0	0
10	100	50,9	32,4	14,0	6,8	4,0	1,5	0,1
11	100	81,5	61,9	44,3	29,8	18,2	7,8	0,8
12	100	87,1	74,4	63,3	49,6	38,3	19,8	3,4

3.4.2.1.6 Longer deduction (D)

These results show that formal deduction (proof) is one of the most difficult activities for children. Only 0.2% grade 9 pupils; 2.9% in grade 10; 22.2% in grade 11 and 42,6% in grade 12 gave 3 or more correct responses out of 6.

Table 3.6 Number correct

Grade	0	1	2	3	4	5	6
9	100,0	25,7	2,5	0,2	0,0	0,0	0,0
10	100,0	53,3	13,1	2,9	0,7	0,1	0,0
11	99,9	81,7	48,9	22,2	6,5	2,1	0,0
12	100,0	87,9	66,3	42,6	19,6	7,2	3,2

3.4.2.1.7 Hierarchical classification (B)

Table 3.7 Number correct

Grade	0	1	2	3	4	5	6	7	8
9	100.0	66,2	28.1	9.4	0,5	0	0	0	0
10	87.4	50,5	29.1	7.1	1,7	0,2	0	0	0
11	98.0	63,8	30.8	14.5	5,6	0,3	0	0	0
12	100.0	74,1	35.9	18.0	5,1	1.9	0	0	0

Hierarchical classification seems to be much more difficult than the other GTC's as can be seen from the fact that only 0,5%-5.1% were correct in 4 or more items out of 8. It is also significant that compared to the other GTC's, very little improvement occurred through the grades.

3.4.2.1.8 Reading and interpretation of given definitions (G)

Table 3.8 Number correct

Grade	0	1	2	3	4
9	100,0	45,6	10,9	4.3	0,7
10	100,0	63,8	22,4	8,1	3,2
11	100,0	76,4	37,8	16,4	4,3
12	100,0	84,4	57,8	41,7	13,2

On this GTC only 10,9% pupils in Grade 9 to 57,8% in Grade 12 gave 2 or more correct responses out of 4.

3.4.3 Guttman analysis

For the determination of the division points in each GTC, the criterium of 50% was used. In cases where the division points fell on non-integer values, integer values just above or below were chosen as division points. Guttman analysis for the total set of GTC's yielded coefficients of reproducibility (C_{rep}) of 0,8907 and of scalability (C_{sc}) of 0.5159. When G was left out due to its low correlations with the other categories, C_{rep} = 0,9100 while C_{sc} = 0,6141, indicative of the presence of a possible learning hierarchy. Various subsets of the remaining GTC's were further chosen for Guttman analyses.

3.4.3.1 Ordering of GTC's

The various Guttman analyses yielded the following ordering of geometric thinking categories beginning with the easiest up to the most difficult

Symbol	Description of GTC	Van Hiele level
Z	Recognition and representation of figure types	0
F	Visual recognition of properties	0
A	Use and understanding of terminology	1
E	Verbal description of properties of a figure (or its recognition from a verbal description)	1
C	One step deduction	2
D	Longer deduction	2
H	Hierarchical classification	?

From the above, the following general observations were made

- pupils at Van Hiele Level 0 seem reasonably capable of recognizing certain obvious visual properties of figures like equal or parallel sides, 90° degree angles or equal base angles in isosceles triangles

- simpler one-step deduction is possible at lower Van Hiele levels as evidenced by the fact that E and C continually interchanged depending on the division points chosen

- hierarchical classification was clearly the most difficult GTC for pupils.

4.1 Conclusions

The data generally provides support for the Van Hiele Model, except that hierarchical classification barely emerges, even among pupils who performed quite well in questions requiring formal deduction The data suggests that contrary to the Van Hiele theory, hierarchical classification is not a prerequisite for formal deductive thinking These conclusions are furthermore supported by an independent study by Malan (1986) using a different experimental procedure Also in Usiskin (1982), an examination of students' fall performances on items 13 and 14, at 26% and 13% respectively, compare unfavourably with the other three items of the same level at 48%, 43% and 30% In another study presently underway, Smith (1987) has found similar results using a slightly adapted version of the CDASSG-test in Usiskin (1982)

4.2 Some implications for mathematics teaching and further research

Although the data shows clear evidence of progress through the grades, only about 20% of grade 12 students show signs of mastering deduction involving more than one step The low facility levels for the category, verbal description of properties of figures (45% for grade 12), strongly suggests that insufficient development of Van Hiele Level 1 takes place This conclusion is also supported by qualitative analyses of the actual responses of pupils to questions, which indicate serious difficulties in mastering English geometric language (English being a second language for the pupils, their mother tongue being Zulu).

Obviously the relationship between hierarchical thinking and the basic Van Hiele theory should be clarified by further studies before firm conclusions can be drawn Furthermore the

Van Hiele theory needs refinement in regard to the levels at which deduction can occur. It is felt that simpler intuitive deductive reasoning is possible at levels lower than Level 2.

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CONSTRUCTION D'UN PROCESSUS D'APPRENTISSAGE DE LA NOTION DE SYMETRIE ORTHOGONALE POUR DES ELEVES DE 11 ANS

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Abstract
A teaching process about the notion of Reflection has been developed here from results of previous investigations which had shown particularly that erroneous conceptions persist after school teaching and that some didactical variables have an influence on pupils'answers. First, a theoretical analysis of the process will be presented, in order to explain our didactical choices according to the teaching aims, and the pupils'conceptions brought into play. Then, some results of the progress of the teaching sequence are given, especially the pupils'procedures and difficulties, and the changes with regard to the designed aims.

Cadre général de la recherche
L'objectif général de cette recherche est l'étude des conceptions des élèves sur la symétrie orthogonale et des conditions didactiques de leur évolution. Une étude préalable menée auprès d'élèves de 11 à 15 ans a mis en évidence d'une part, l'existence de conceptions erronées qui semblent résister à l'enseignement et d'autre part, l'influence de certaines variables sur les réponses des élèves dans une tâche de construction à main levée. Ces résultats ont été présentés à PME 9 (Grenier, 1985).

Cette étude a servi à la construction d'un processus d'apprentissage en classe de 6ème (11 ans) ayant pour objectif d'apprendre aux élèves à utiliser et expliciter les propriétés de la droite de symétrie d'une figure, et à construire la transformée d'une figure par une symétrie. Nous décrivons ce processus sous les aspects suivants :

I - Analyse théorique du processus : explicitation des variables de situation et du choix des valeurs données à ces variables, analyse des conséquences de ces choix sur les conceptions mises en jeu et le type d'apprentissage qu'elles doivent provoquer;

II - Analyse du déroulement du processus : caractérisation des procédures des élèves et mise en évidence des difficultés qu'ils ont rencontrées et du rôle de l'enseignant.

I - Analyse théorique du processus

1ère phase : repérage des conceptions initiales des élèves de la classe

L'objectif de cette phase est de connaître les conceptions initiales des élèves de la classe sur la notion de droite de symétrie. Nous avons proposé aux élèves une tâche de reconnaissance et de tracé à main levée qui met en jeu la perception globale de la droite de symétrie d'une figure (position et orientation dans la figure, forme et dimension). Les figures étaient déterminées de façon à faire intervenir les conceptions erronées repérées dans les expérimentations précédentes, en particulier :

- le symétrique est situé sur une même ligne horizontale dans la feuille que la figure objet, ce qui induit que les droites de symétrie ont une direction privilégiée, la verticale;
- le symétrique est une figure identique obtenue aussi bien par translation, symétrie glissée ou demi-tour que par une symétrie orthogonale; ce qui induit que toute droite partageant la figure en deux parties identiques est une droite de symétrie.

Cette phase de diagnostic nous permettait d'organiser la classe de façon à favoriser les échanges, en formant des groupes contenant des élèves qui avaient manifesté des conceptions différentes dans cette tâche.

2ème phase : déstabilisation des conceptions erronées

Nous avons de nouveau donné aux élèves la tâche de construction à main levée des droites de symétrie si elles existent. Cette tâche ne constitue un problème que si le pliage est interdit : dans ce but, un moyen de dissuasion a consisté à tracer les figures sur carton. Pour déstabiliser les conceptions erronées, il fallait jouer sur les variables pertinentes dans cette tâche, en leur donnant des valeurs reconnues comme étant sources d'erreurs chez les élèves. Ces variables sont liées à certaines propriétés des figures et à leur orientation dans la feuille. Nous avons ainsi retenu :

- des figures pouvant être décomposées en deux parties identiques mais ne présentant pas de symétrie orthogonale : il s'agissait de déstabiliser la conception erronée qu'une droite qui partage une figure en deux parties identiques est une droite de symétrie; ces figures sont de deux types, figure régulière autour d'une ligne centrale (figure 1) et figures présentant une symétrie centrale (figures 4, 5, 7, 11), ces dernières étant orientées dans la feuille de manière à provoquer des procédures de "rappel horizontal";
- des figures ayant plusieurs droites de symétrie, donc des droites horizontales ou obliques par rapport aux bords de la feuille : il s'agissait de déstabiliser la conception très répandue qu'une droite de symétrie est verticale (figures 2, 3, 6, 8, 9, 10).

Les figures étaient dessinées dans cette orientation dans la feuille.

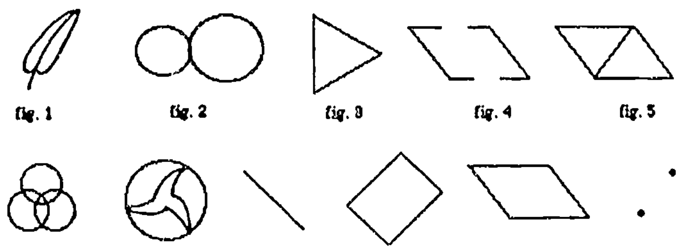


fig. 1 fig. 2 fig. 3 fig. 4 fig. 5

fig. 6 fig. 7 fig. 8 fig. 9 fig. 10 fig. 11

L'organisation du travail est choisie pour favoriser les interactions dans la classe. Les élèves travaillent d'abord par groupes de quatre et doivent se mettre d'accord pour chaque figure sur une réponse commune, le représentant d'un groupe devant pouvoir éventuellement défendre sa solution en l'argumentant, au moment de la mise en commun. L'enseignant a alors pour rôle de veiller au bon déroulement de l'activité, sans porter aucun jugement sur les productions des élèves. Puis une mise en commun est organisée sous le contrôle de l'enseignant : celui-ci doit alors gérer le débat et contrôler les apprentissages, il doit intervenir si toute la classe est d'accord sur un résultat faux ou pour proposer une solution à un problème non résolu. L'enseignant récapitule ensuite, dans une phase d'institutionnalisation, les propriétés qui permettent de réfuter les conceptions erronées, en particulier, pour cette tâche :

- une figure peut avoir plusieurs droites de symétrie et donc celles-ci peuvent être "obliques".

- une droite de symétrie passe par le milieu de deux points homologues et elle est orthogonale à la droite joignant ces deux points.

La tâche de tracé "à main levée" met en jeu les propriétés du milieu et d'orthogonalité de la droite de symétrie. Elle ne met en oeuvre, ni les relations d'incidence (points invariants) ni l'équidistance des points de la droite de toute paire de points symétriques.

3è phase : utilisation analytique des propriétés de la droite de symétrie

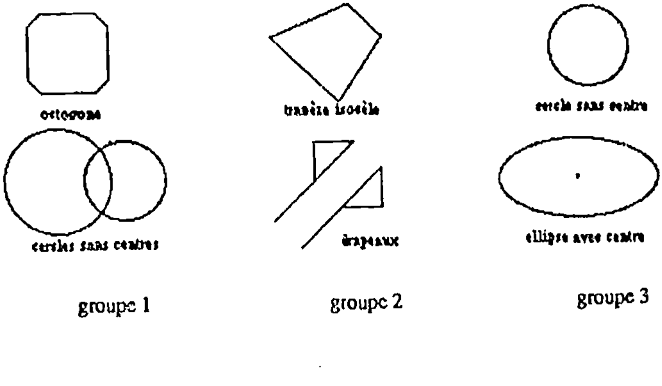
Nous avons élaboré une activité de construction avec instruments des droites de symétrie de figures, de telle façon qu'un jeu sur les variables des figures et sur les instruments permis privilégie ou même rend obligatoire l'utilisation des propriétés moins connues des élèves. A ce niveau, les élèves savaient utiliser quatre instruments : la règle graduée, la règle non graduée, l'équerre et le compas.

Analysons les actions possibles avec chaque instrument dans cette tâche. La règle graduée permet de construire la droite de symétrie dès que l'on connaît deux paires de points symétriques. C'est alors uniquement la propriété du milieu qui est mise en oeuvre. La règle non graduée ne met en oeuvre que les propriétés d'incidence, par la construction de l'intersection de deux segments symétriques. Elle permet de construire la droite de symétrie dès que l'on connaît deux paires de points symétriques, à condition que ceux-ci ne forment pas un rectangle. L'équerre permet de déterminer la direction d'une droite perpendiculaire à une autre droite et passant par un point donné. La propriété d'orthogonalité est alors utilisée conjointement avec une autre propriété, celle qui sert à déterminer le point par lequel est mené la droite perpendiculaire. Le compas permet de déterminer au moins deux points de la droite de symétrie dès qu'on a une paire de points symétriques. C'est la propriété d'équidistance des points de la droite de symétrie de toute paire de points symétriques qui est mise en oeuvre.

Dans chacun des cas, il faut exhiber au moins une paire de points symétriques, ce qui nécessite une anticipation de la position de la droite de symétrie. Ces points symétriques doivent être, soit reconnus, soit tracés (ils sont alors le résultat d'une construction qui relève des propriétés de la symétrie, ce qui rend la tâche plus difficile pour les élèves).

L'organisation de la classe est la même que dans la phase précédente. Les figures (dessinées sur carton rigide) et les instruments fournis sont les suivants :

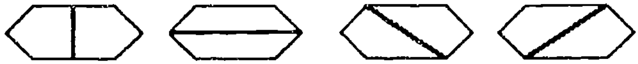
groupe 1 : règle graduée et équerre, octogone et cercles sécants sans centres
groupe 2 : règle non graduée et équerre, trapèze isocèle et drapeaux
groupe 3 : règle non graduée et compas, cercle sans centre et ellipse avec centre.



groupe 1 groupe 2 groupe 3

explicitation des propriétés mises en jeu dans les ensembles figures-instruments
La règle graduée suffit pour tracer les droites de symétrie de l'octogone (propriété du milieu), mais on peut utiliser aussi l'équerre (propriétés du milieu et orthogonalité). Par contre, la droite de symétrie des deux cercles ne peut être tracée qu'en utilisant à la fois la règle graduée et l'équerre.
Pour les figures du groupe 2, deux types de construction sont possibles : avec la règle non graduée et l'équerre (propriété d'incidence et d'orthogonalité) ou seulement la règle non graduée, en déterminant les points d'intersection de deux paires de segments symétriques (propriétés d'incidence).
Pour ces quatre figures, les paires de points symétriques sont déjà construits.
Le tracé de droites de symétrie du cercle s'obtient avec le compas et la règle non graduée en utilisant les propriétés d'équidistance de deux points quelconques du cercle. Il faut donc savoir au préalable que deux points quelconques du cercle sont symétriques. Pour construire une droite de symétrie de l'ellipse avec le compas, il faut déterminer un point, autre que le centre déjà donné, qui soit équidistant de deux points symétriques situés sur l'ellipse. Mais ces points sont eux-mêmes à construire comme points équidistants du centre de l'ellipse, ce qui rend la tâche difficile.

4ème phase : une phase de communication
Les trois premières phases de ce processus sont des situations d'action, c'est-à-dire que les connaissances investies dans les problèmes n'ont pas à être formulées. Les expressions utilisées par les élèves pour désigner la droite de symétrie, comme par exemple "droite du milieu" ou "droite qui partage la figure en deux" sont en général comprises par les autres élèves, parce qu'entre eux les implicites fonctionnent bien. Mais elles peuvent aussi provoquer des erreurs du fait de leur ambigüité. Par exemple, les droites partageant un rectangle en deux parties identiques sont de deux types : droite de symétrie ou diagonale. La capacité à formuler une connaissance fait partie de l'apprentissage de cette connaissance. Nous avons donc élaboré une situation de communication dans laquelle d'une part, la description des propriétés ou l'utilisation de l'expression "droite de symétrie" sont des outils de résolution du problème et, d'autre part, les implicites familiers aux élèves peuvent conduire à des réponses erronées.
Organisation de l'activité : Les élèves sont mis en binômes dans deux salles A et B, sans aucun contact direct, chaque binôme Ai étant en correspondance avec le binôme Bi. Le binôme Ai a une feuille sur laquelle est dessinée une figure avec un élément (droite de symétrie ou une diagonale) tracé dans une autre couleur. Le binôme Bi a une figure semblable de dimensions différentes, mais sans l'élément supplémentaire. Chacun des deux binômes sait que l'autre a une figure semblable à la sienne et que Ai a un élément en plus. La tâche de Ai est de décrire cet élément dans un message écrit (sans dessin) pour Bi, afin que celui-ci puisse le reconstruire. La réussite de l'activité pour le couple (Ai, Bi) est liée à la qualité du message codé par Ai, et à la justesse du décodage par Bi. Le choix des dimensions différentes pour les deux figures interdit une description par des mesures. Le choix des figures possédant aussi une symétrie centrale oblige de plus les élèves à expliciter la présence ou l'abscence des propriétés d'orthogonalité pour ne pas produire un message ambigu basé sur la propriété de partage de la figure en deux parties identiques. Dans le cas où l'élément est droite de symétrie, la formulation de ses propriétés ou l'utilisation de l'expression "droite de symétrie" sont des outils performants.
Les figures proposées sont les reproduites dans la page suivante.



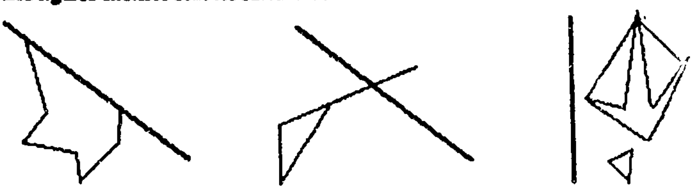
élément de type S (droite de symétrie) élément de type NS (diagonale)
les figures étaient dessinées dans différentes orientations dans la feuille

5ème phase : transformation de figures et transformation ponctuelle

L'objectif de cette phase est de mettre en oeuvre avec les élèves les procédures de construction du symétrique d'une figure par rapport à une droite donnée. Pour mener cette tâche à bien, il est nécessaire au préalable de décomposer la figure en sommets dont il suffit de déterminer les symétriques pour reconstruire entièrement la figure. Ceci nécessite que l'élève reconnaisse que le tracé de ces points permet de construire la figure complète, par exemple que l'image d'un segment est un segment dont les extrémités sont les symétriques des celles du segment objet. Il faut aussi traduire les connaissances des propriétés de la droite de symétrie d'une figure vers la construction du symétrique d'un point par rapport à une droite.

Nous avons donné aux élèves les quatre instruments de construction, puisque dans cette phase, il s'agissait d'exhiber les principales procédures de construction du symétrique.

Les variables de la figure qui jouent dans cette tâche sont essentiellement la position relative de la figure et de la droite et l'orientation des éléments de la figure par rapport à la droite de symétrie. Les figures choisies sont les suivantes :



L'organisation de la classe est la même que dans les phases 2 et 3, c'est-à-dire un travail de groupe suivi d'un débat dans la classe, et de l'institutionnalisation par l'enseignant des propriétés principales du symétrique d'une figure par rapport à une droite et de la médiatrice d'un segment.

Enfin, pour mettre en évidence la propriété de retournement, nous avons proposé aux élèves des activités (individuelles) utilisant le "papier calque" sous deux aspects :

- comme outil de différentiation des statuts des droites ayant la propriété de "partager en deux parties égales" (mise en jeu de la propriété de symétrie centrale);
- comme outil de construction du symétrique d'une figure par rapport à une droite. Nous avons fait jouer la variable "position relative de la figure objet et de la droite": si la figure ne rencontre pas la droite de symétrie, il est nécessaire pour réussir la tâche de relever la position de la droite par au moins deux points avant de déplacer la feuille de papier calque; il faut ensuite poser la droite sur elle-même après avoir retourné le calque (propriété d'invariance de la droite de symétrie dans la transformation).

Nous décrivons maintenant quelques aspects du déroulement du processus : les difficultés rencontrées par les élèves dans les différentes tâches, les conceptions et les apprentissages mis en jeu, enfin les objectifs atteints par rapport à ceux prévus.

II-Analyse du déroulement du processus

La tâche de construction à main levée a mis en évidence une conception de la droite de symétrie comme droite des milieux ou du "milieu de la figure", partageant la figure en deux parties égales. Cette conception conduit au tracé des droites suivantes :

fig. 1

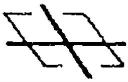


fig. 4

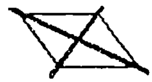


fig. 5

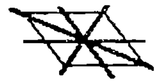
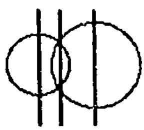


fig. 10

La prévision du résultat du pliage suffit pour provoquer la remise en cause des lignes de symétrie courbes. Mais, pour chacune des autres figures, la conjonction de l'orientation de la figure dans la feuille (horizontale) et du parallélisme de ses éléments rend difficile la reconnaissance du statut erroné des droites tracées : ces attributs provoquent des procédures de type "rappels horizontal ou vertical" et "parallélisme" qui sont difficiles à déstabiliser. La propriété d'orthogonalité n'est mise en oeuvre que lorsque les figures ne présentent aucun des attributs perturbant la conception de symétrie. Enfin, la conception très répandue de la verticalité de la droite de symétrie, associée à celle du partage de la figure en deux parties identiques a empêché, dans un premier temps, la reconnaissance de la droite de symétrie pour la figure :



L'élève doit dans ce cas mener une analyse différente de la figure. De même, lorsque le nombre d'éléments identiques est impair, le "double statut" des droites de symétrie (droite de symétrie de paires d'éléments et de l'élément restant) n'est pas reconnu.

Un des apprentissages constatés lors de cette phase est la reconnaissance des différentes orientations des droites de symétrie dans la feuille. La situation géographique des quatre élèves autour de la figure joue en faveur de cet apprentissage, parce qu'elle met en jeu la conservation des propriétés de la figure par rotation et relativise les directions privilégiées propres à chaque élève.

L'activité de groupe pour la construction avec instruments de la droite de symétrie n'a pas atteint les objectifs d'apprentissage prévus, parce que la propriété d'égale distance est la seule disponible en tant qu'outil pour les élèves dans cette tâche. La règle graduée est donc le seul instrument qui permet de résoudre le problème, quel que soit le type de figure proposée. L'utilisation des autres instruments, l'équerre et le compas, pose le problème de leur adéquation aux figures, l'équerre servant prioritairement pour l'élève à vérifier si la figure comporte des angles droits et le compas comme outil de tracé d'arcs de cercle plus que de report de longueurs égales. Sans la règle graduée, les élèves ont tenté de graduer les autres instruments ou bien ont tracé "au jugé".

Le rôle de l'enseignant s'est révélé primordial dans cette phase : les relations d'incidence et les propriétés d'équidistance n'ont été exhibées qu'au moment de la mise en commun et de l'institutionnalisation par l'enseignant de ces propriétés. Le problème de la finalité de cette tâche de construction reste posé : s'il n'y a pas d'autre but que le tracé, les élèves ne comprennent pas

l'intérêt de construire avec les instruments alors que les procédures "au jugé" ou "par tâtonnements" donnent de très bons résultats.
Par rapport aux apprentissages réalisés, l'analyse de la situation de communication témoigne, à cette étape du processus, de la reconnaissance par les élèves des propriétés de la droite de symétrie et de leur capacité à les formuler : les élèves ont en effet majoritairement utilisé les propriétés de la symétrie ou la définition "droite de symétrie" pour résoudre le problème de description de l'élément de type S. Mais cette situation de communication n'a pas atteint son objectif de remise en question de la conception erronée qu'une droite partageant une figure en deux parties identiques est une droite de symétrie. En effet, la stratégie majoritairement utilisée pour décrire le segment de type NS a consisté en un "repérage géographique", par rapport à l'orientation de la figure dans la feuille (horizontale, verticale, oblique, incliné), ou par rapport aux bords de la feuille (bas, haut, gauche, droite). Or, ces messages ont été très bien décodés, parce que les implicites correspondant fonctionnent bien chez les élèves et que les figures Ai et Bi avaient la même orientation dans la feuille. Pour que la tâche mette en jeu la différenciation entre les éléments de type S et NS, il faut faire jouer la variable "orientation dans la feuille" de la figure.
Dans cette phase aussi, le rôle de l'enseignant a été primordial. Il a permis de reposer le problème de l'ambiguïté de la propriété de "partage en deux parties égales" au moyen de l'élaboration d'une activité supplémentaire dans la classe, qui prenait en compte les résultats observés.

Pour terminer, nous donnons ici seulement quelques résultats de la phase de construction du symétrique d'une figure. Celle-ci a révélé la durabilité de certaines difficultés chez les élèves, en particulier :
- la prise en compte erronée de la propriété d'orthogonalité, qui se traduit par exemple par la règle d'action suivante : la transformée d'une figure est perpendiculaire à la figure objet;
- les difficultés des élèves à tracer le symétrique lorsque la figure coupe la droite de symétrie, difficultés qui font réapparaître les procédures de prolongement ou de parallélisme.
Dans la phase d'institutionnalisation, l'enseignant doit donc insister sur la propriété d'orthogonalité. Les propriétés de conservation par la transformation de la nature des éléments d'une figure et de leur dimension ne posent pas de problème pour les élèves.

Une nouvelle séquence prenant en compte ces résultats est en cours de réalisation.

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ESTUDIO DE LAS CARACTERISTICAS DE LOS NIVELES DE VAN HIELE

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ABSTRACT. The theory proposed by P.M. and D. van Hiele have given rise to a model of teaching and learning whose three main characteristics are discreteness and hierarchy of levels and usefulness of the theory for prediction.
We present the results of an empirical research which deals with the hierarchical structure and the predictive property of the model. The comparison of results obtained by administering three tests (on polygons, measurement and solids) to a group of preservice elementary teachers allows us to formulate the following conclusions:
a) Levels 1 to 4 form a hierarchy, but level 5 has some particularities that need an in deep investigation.
b) There is not relation between the individual results in the different tests, so the assessment of pupils´ level in a topic cannot predict their level in other topic.

INTRODUCCION

Una teoría de aprendizaje en matemáticas, que actualmente está siendo considerada con gran interés, es la desarrollada por P.M. van Hiele y D. van Hiele-Geldof. Esta teoría se basa en la definición de varios niveles de pensamiento, a través de los cuales progresan los estudiantes, estando caracterizado cada nivel por un tipo de conocimiento, un vocabulario y una forma de razonamiento particulares. Brevemente, las capacidades adquiridas por un estudiante en los diferentes niveles son:

Nivel 1 (reconocimiento): Reconoce los objetos y conceptos matemáticos por su aspecto físico y de forma global, sin distinguir explícitamente sus componentes.

Nivel 2 (análisis): Reconoce las componentes de un objeto o concepto matemático. Es capaz de establecer relaciones entre objetos y/o entre componentes, pero sólo de forma experimental. No puede establecer relaciones lógicas ni hacer descripciones formales.

Nivel 3 (clasificación): Realiza relaciones lógicas y es capaz de seguir razonamientos deductivos simples, pero no comprende la función de los elementos de un sistema matemático (axiomas, definiciones, demostraciones, etc.) y, por lo tanto, no sabe manejarlos.

Nivel 4 (deducción): Comprende y realiza razonamientos deductivos, pues ya entiende el valor de axiomas, hipótesis, definiciones, etc., pero todavía no ha adquirido un conocimiento global de los sistemas axiomáticos y no comprende la necesidad del razonamiento riguroso.

Nivel 5 (rigor): Comprende la necesidad del razonamiento riguroso, es capaz de realizar deducciones abstractas a partir de sistemas axiomáticos diferentes y de analizar y comparar esos sistemas.

El mayor interés de la teoría de van Hiele radica en la posibilidad de construir a partir de ella un modelo de enseñanza de la geometría, en el cual cada nivel lleva asociados un tipo de actividades, un lenguaje y una organización del aprendizaje que permiten alcanzar el nivel siguiente. Ejemplos de programas basados en los niveles de van Hiele los encontramos en la Unión Soviética, Holanda y U.S.A. (ver Wirszup (1976), Hoffer (1983), Freudenthal (1973), Mathematics Resource Project (1978) y Fuys, Geddes (1984)).

Hay tres características del modelo que deben ser estudiadas en profundidad: La discretitud de los niveles, su jerarquía y su capacidad de transferencia de unos campos de las matemáticas a otros. Durante los últimos años se han realizado importantes investigaciones en cada una de ellas (Usiskin(1982), Mayberry (1983), Fuys, Geddes (1984) y Burger, Shaughnessy (1986)), cuyos resultados más destacados están comentados en Senk (1985), pero no ha sido posible determinar de forma satisfactoria ninguna de dichas características del modelo.

EL ESTUDIO

A continuación exponemos los resultados de un estudio iniciado en este curso, con el fin de evaluar la validez de las hipótesis sobre la jerarquización de los niveles de van Hiele y la globalidad de los niveles respecto de diversos campos de la geometría.

Hasta ahora, la mayoría de las experiencias desarrolladas para determinar las propiedades de la teoría de van Hiele se han basado en cuestiones de geometría plana relacionadas con los polígonos. La primera parte del trabajo que hemos realizado ha consistido,en diseñar tres tests basados en los tres campos más importantes de la geometría: geometría plana (principalmente polígonos), medida de magnitudes (longitud, superficie y volumen) y geometría espacial (principalmente poliedros).

La comparación de los niveles alcanzados por cada estudiante en los diferentes tests nos permite observar la correlación existente entre ellos y, por tanto, conjeturar sobre la globalidad o localidad de los niveles de van Hiele. Por otra parte, el análisis independiente de cada test ha hecho posible medir el grado de jerarquización de los niveles.

Tomando como referencia la estructura del test de Usiskin (1982), cada test consta de 5 items para cada nivel, con 5 respuestas en cada item. Con el fin de hacer más fiables las comparaciones de los resultados, se ha mantenido la misma estructura gramatical en todos los tests. Respecto del contenido matemático de los items, es fácil enunciar modelos de cuestiones válidos tanto para geometría plana como espacial, pero en la medida de magnitudes se presenta una organización conceptual mucho más simple que en el caso de polígonos o poliedros, lo cual hace difícil mantener la semejanza entre los items.

Por otra parte, el diseño de tests basados en conceptos diferentes de los relacionados con polígonos puede ser útil para abrir nuevas líneas de investigación sobre el modelo de van Hiele para la enseñanza de la geometría.

LA MUESTRA

Hemos administrado los tests a 563 alumnos de los tres cursos de la Escuela de Formación de Profesorado de E.G.B. (enseñanza elemental española) de la Universidad de Valencia. La tabla 1 indica la cantidad de estudiantes que ha contestado los diferentes grupos de tests.

	Test P	Test M	Test S
Test P	409	276	232
Test M	---	392	241
Test S	---	---	318

Tests P, M y S: 193 estudiantes

Test P = polígonos
Test M = medida
Test S = sólidos

Tabla 1

La realización de los tests ha tenido lugar en tres sesiones diferentes, con un intervalo mínimo de tres días entre una y otra, si bien en la mayor parte de los casos este intervalo ha sido de 1 ó 2 semanas. Los alumnos han dispuesto de tiempo libre para contestar los tests (entre 30 y 45 minutos por término medio).

RESULTADOS Y CONCLUSIONES

Es evidente la importancia de los criterios, de paso de niveles y de asignación de nivel mínimo, elegidos en el análisis de las respuestas a estos tests. Habitualmente se toma como criterio para superar un nivel que el estudiante conteste bien 2/3 de las respuestas, lo cual supone en nuestro caso superar 3 ó 4 de los 5 items correspondientes a un nivel. La observación directa previa de nuestros alumnos y otros estudios realizados con estudiantes de niveles semejantes (Mayberry (1983) y Matos (1985)) nos hicieron suponer a priori que la mayoría de alumnos estarían en los niveles 2 y 3 (para nosotros, el primero de los niveles descritos por van Hiele es el nivel 1 y consideramos en el nivel 0 a quienes no alcanzan el nivel 1). Por este motivo, hemos utilizado dos criterios de superación de niveles (ver Usiskin (1982), pg. 23). Hay que procurar:

a) En los niveles bajos, que los alumnos no queden por debajo de su nivel real, por lo que usamos el criterio 3/5.
b) En los niveles altos, que los alumnos no superen un nivel por respuestas al azar, por lo que empleamos el criterio 4/5.

Hemos asignado los niveles de dos formas diferentes: mediante el criterio 33344 (es decir, 3 de 5 para los tres primeros niveles y 4 de 5 para los dos últimos niveles) y 33444. Además, a un alumno que supere los niveles 1 y 2 y falle el nivel 3 se le ha asignado el nivel 2 (independientemente de los resultados obtenidos en los niveles 4 y 5). La tabla 2 muestra un resumen de los niveles obtenidos.

Nivel	Test P 33344	Test P 33444	Test M 33344	Test M 33444	Test S 33344	Test S 33444
0	4.40	4.40	24.74	24.74	45.28	45.28
1	11.98	11.98	20.92	20.92	44 97	44.97
2	24.45	57.95	25.00	41.58	1.89	6.29
3	56.72	23.47	22.96	8.67	7.55	3.14
4	1.71	1.71	2.30	1.02	0.00	0.00
5	0.73	0.49	4.08	3.06	0.31	0.31

***Tabla 2**: Distribución, en %, de los alumnos por niveles*

El análisis de estos resultados ha sido realizado calculando varios coeficientes. Para la verificación de la jerarquía de los niveles se ha empleado el coeficiete R de reproductividad del Análisis Escalográfico de Guttman; este coeficiente, que evalúa la cantidad de estudiantes que han fallado un nivel pero han pasado otro superior, ha sido utilizado de forma eficaz en trabajos anteriores, como el de Mayberry (1983) o el C.S.M.S. Project del Chelsea College (Hart y otros (1981)). La tabla 3 contiene los valores obtenidos al medir las respuestas correspondientes a los niveles 1 a 5.

Test	Tipo de corrección 33344	Tipo de corrección 33444
P	0.940	0.939
M	0.868	0.861
S	0.853	0.903

***Tabla 3**: Coeficiente R para los niveles 1 a 5*

Test	Tipo de corrección 33344	Tipo de corrección 33444
P	0.976	0.990
M	0.959	0.966
S	0.858	0.935

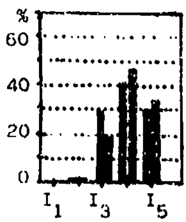
***Tabla 4**: Coeficiente R para los niveles 1 a 4*

Se suele considerar válida la jerarquía de los niveles si el coeficiente R no es inferior a 0.90 (Mayberry (1983)) o a 0.93 (Hart (1981)). Según esto, habría que rechazar la jerarquización de los niveles de van Hiele en medida de magnitudes y en sólidos. No obstante, un análisis más detallado de los resultados pone de relieve de forma clara la influencia del nivel 5 en los coeficientes anteriores. La tabla 4 muestra los coeficientes obtenidos al medir sólo las respuestas correspondientes a los niveles 1 a 4.

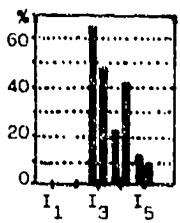
Estos resultados nos llevan a la conclusión de que los cuatro primeros niveles del modelo de van Hiele forman una jerarquía, pero que el quinto nivel presenta características especiales que deben ser estudiadas detalladamente, con el fin de reformular sus características o de considerar la posibilidad de eliminarlo del modelo, como sugiere el propio van Hiele (1986, pg.47).

Para evaluar la capacidad de predicción del modelo de van Hiele hemos utilizado dos coeficientes. Con el coeficiente C de consenso de Leik se mide el grado de dispersión de los niveles alcanzados por una persona en los diferentes tests, mientras que el coeficiente γ de Kruskal mide la correlación entre las respuestas del total de alumnos a dos de los tests.

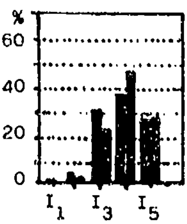
El coeficiente C varía entre 0 y 1, C=0 indica disparidad, C=0.5 indica aleatoriedad y C=1 indica concordancia entre las respuestas. Hemos agrupado los valores obtenidos en varios intervalos: I_1= [0, 0.15[, I_2=[0.15, 0.30[, I_3=[0.30, 0.70[, I_4=[0.70, 0.85[, I_5= [0.85, 1]. Las gráficas muestran los porcentajes de alumnos en cada uno de los intervalos; en cada caso, la columna izquierda corresponde al criterio 33344 y la columna derecha corresponde al 33444.



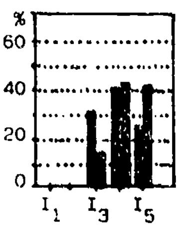
Tests 1 y 2



Tests 1 y 3



Tests 2 y 3



Tests 1, 2 y 3

El coeficiente γ toma valores entre -1 y +1 y su significado es el habitual en los coeficientes de correlación. La tabla 5 muestra los valores obtenidos para cada par de tests.

	Tipo de corrección	
Tests	*33344*	*33444*
P y M	*0.46*	*0.52*
P y S	*0.45*	*0.40*
M y S	*0.41*	*0.44*

Tabla 5: Coeficiente γ

Tanto el coeficiente C como el coeficiente γ indican que:

1) No hay divergencia entre los diferentes tests.
2) Hay cierto grado de concordancia entre los resultados de los tests, pero ésto es insuficiente para apoyar la hipótesis de la globalidad de los niveles de van Hiele.

Tanto nuestra experiencia como las otras a las que nos hemos referido antes presentan un porcentaje muy reducido de estudiantes en los niveles 4 y 5, por lo que creemos que no son válidas para obtener conclusiones respecto de estos niveles. Una línea que debería ser seguida en futuras investigaciones es la de realizar experiencias con grupos en los cuales la mayoría de las personas estén en los dos niveles superiores; es muy probable que en este caso se obtengan resultados positivos acerca de la capacidad de predicción del modelo de van Hiele, pues una persona situada en los tres primeros niveles tiene una visión local y fragmentada de las matemáticas que dificulta la transferencia de conocimientos de unas áreas a otras, mientras que aquellos estudiantes que han alcanzado el cuarto o quinto nivel son capaces de tener una visión global de las matemáticas que facilita dicha transferencia. Nuestra hipótesis al respecto es que los niveles 1, 2 y 3 son locales y no permiten realizar predicciones, mientras que los niveles 4 y 5 son globales y sí permiten realizar predicciones.

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High school

mathematics

THE ROLE OF COGNITIVE CONFLICT IN UNDERSTANDING MATHEMATICS

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ABSTRACT

Cognitive conflict is regarded as a factor that could promote students' understanding. Focusing on instrumental understanding and relational understanding, this paper aims to clarify the role of cognitive conflict in students' mathematical understanding, using the problem of solving a linear inequality. A model of students' understanding and cognitive conflict was built to analyze students' answers collected from a lower secondary school. Finally, implications for teaching were raised to provide teachers with some idea of how to utilize cognitive conflict questions.

I. INTRODUCTION

In mathematical education, a deep understanding is desirable rather than a superficial understanding. Typical of the latter is to know what to do without knowing why, and of the former, is to know what to do and why. In the teaching/learning process, teachers should make an effort to attain a deep level of mathematical understanding within students. The author points out the most effective way of achieving this is to provoke cognitive conflict within students and let them resolve the conflicts by themselves.

The purpose of this paper is to clarify the role of cognitive conflict in students' mathematical understanding and to provide teachers with some idea of how to use cognitive conflict questions to promote students understanding.

The present study focuses on the concepts of 'instrumental understanding' and 'relational understanding', initially used by R. R. Skemp. As it is crucial to distinguish between which conceptions are in conflict with each other, a model of the structure of understanding and cognitive conflict was developed. These models provided a basis for the following analysis of the students' understanding of solving linear inequality.

II. THE MODEL OF UNDERSTANDING AND COGNITIVE CONFLICT

(1) The Structure of Instrumental and Relational Understanding

According to R.R.Skemp, instrumental understanding is characterized by "rules without reasons" or "the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works". Namely, it is the understanding focused on 'how'. In this paper, we distinguish between two aspects of the 'how':'how in action' and 'how in expression'. The former is the action aspect of how called 'behaviorized how', that is, it is oriented to finding an answer. The latter is the verbal aspect of the how called 'verbalized how', that is, it is an expression of action procedure in words. Whereas instrumental understanding is "knowing what to do", relational understanding is characterized by "knowing what to do and why". This structure of understanding is shown in fig.1.

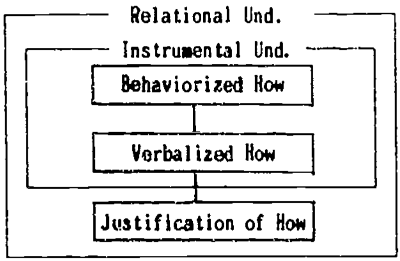

Fig.1

(2) Cognitive Conflict

We may say that not all processes of understanding are developed by cognitive conflict, but we consider that understanding may be promoted by resolving the cognitive conflict, especially from instrumental understanding to relational understanding. It is crucial to identify which conceptions are in conflict with each other. According to this model of understanding, we can identify at least three cognitive conflicts:

C_1: conflict in 'behaviorized how'

C_2: conflict in 'verbalized how'

C_3: conflict in 'justification of how'

In this study, the following problem concerning linear inequality was given to students to provoke cognitive conflict.

"Usually we would solve a linear inequality such as $X-2>5$ by adding 2 to both sides, that is, $X-2+2>5+2$, and the answer is $X>7$, or, if $a>b$, then $a+c>b+c$. On the other hand, if we were to add 2 to the left side and only 1 to the right, that is, $X-2+2>5+1$, we would attain the answer $X>6$, or, if $a>b$ and $e>f$, then $a+e>b+f$. What do you think about the difference ? "

Below is a detailed description of what took place in class when the author taught a Math class using cognitive conflict as the motivation for understanding.

At the very start of the fourth math class session, the students were asked to solve three inequalities without prior explanation as to how to do it. (The first investigation) One of the students solved the problem "X 2 > 5" this way:

X-2+2 > 5+2. The final answer was "X > 7". His explanation is written below.

'There is no change in the inequality 2 < 3 if we add 1 or 100 or 3.3 to both sides.

2+1 < 3+1 3 < 4 Therefore, I added 2 to both sides as: X-2+2 > 5+2.

X > 7 .'

When the teacher asked the student what he meant by "no change", the reply was "the direction of the inequality sign". Then the teacher made sure that this answer was supported by most of the students. Next, the teacher asked again, "If we add 2 to the larger side and 1 (which is smaller than 2) to the smaller side, we can be sure that the direction of the inequality sign never changes. This is because we add the larger number to the larger side and the smaller number to the smaller side. But if we do so,we get X > 6 which is different from the former answer. What do you think of the difference?"

Based on this type of interaction, the cognitive conflicts which may rise within individual students were as follows:

C_1: With regards to the answer, which is correct , X > 7 or X > 6 ?

C_2: With regards to the procedure to get the answer, which is proper - "to add the same quantity to both sides, or to add a larger quantity to the larger side and a smaller quantity to the smaller side?"

C_3: With regards to the validity of the procedure, which is valid even if both procedures result in no change in the direction of the inequality sign?

These conflicts are shown in figure 2.

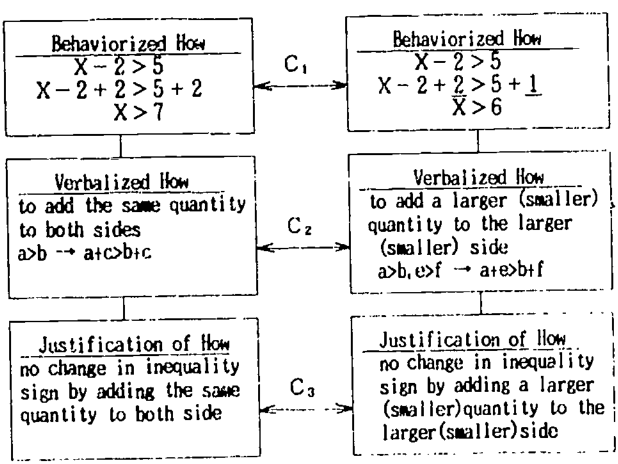

Fig.2

As a result of resolving these cognitive conflicts, we can expect the students' understanding to be more profound than before. C_1 will provoke the student to investigate the correctness or the incorrectness of the answer. Therefore, by resolving C_1 the student may move from instrumental understanding to relational understanding (R_1).

R_1: 'behaviorized how' and why; namely, "why" is focused on the correctness (or incorrectness) of the answer

Similarly, conflicts C_2 and C_3 will lead to relational understanding R_2 and R_3 respectively.

R_2: 'verbalized how' and why; namely, "why" is focused on the correctness (or incorrectness) of the procedure

R_3: "why" is focused on the validity of the reason of how

III. EXPERIMENTAL STUDY

Focusing on the students' understanding of linear inequality, we carried out an experimental study to clarify the role of cognitive conflict in the learning process. The subjects used were eighth graders (N=40) at a national secondary school in Tokyo. The author started with a three-hour introductory session with the students which consisted of the following: a. citing a situation and translating it into a mathematical expression (linear inequality); b. clarifying all possible implications of the said linear inequality; c. knowing the relationship between the solution set of the linear inequality and the number line; d. determining the smallest possible value by treating the linear inequality as a linear equation.

Then, without prior explanation as to how to solve the problem, the students were asked to solve three linear inequality problems: $X-2 > 5$, $1/2X+2 > 4$, and $2-3X > 8$. The number of students who obtained the correct answer for problems were 40, 38 and 23 respectively. These results are actually very satisfactory considering that the students had not yet been taught how to solve linear inequalities. In the third problem, the most common mistake was the incorrect use of the inequality sign. Using the inequality '$X-2 >5$', the teacher then tried to provoke student conflict by asking if that inequality could be solved by adding 2 to the left side and 1 to the right side of the inequality, thus resulting in '$X > 6$'. The teacher asked the students to write what they thought about this suggestion and collected their papers. The students were then asked to discuss this matter with their classmates, after which they wrote their final opinions.

Based on the students' opinions, the author constructed 10 categories, 5 levels for understanding and 5 for misunderstanding, to evaluate students' opinions. (see fig. 3 and its note)

	Misunderstanding			Understanding
Inst. Und.	$\overline{I}$			I
Rel. Und.	$\overline{R_0}$			R_0
	$\overline{R_1}$			R_1
	$\overline{R_2}$			R_2
	$\overline{R_3}$			R_3

Fig.3

NOTE : Besides categories of R_1,R_2 and R_3 mentioned above, R_0 and $\overline{I}$, $\overline{R_0}\sim\overline{R_3}$ are established.

R_0: student's description was too vague to classify as either category.

$\overline{I}$, $\overline{R_0}\sim\overline{R_3}$: misunderstanding ,that is, inclusion of some incorrect terms or explanations in student's written answer.

IV. RESULTS

This paper focused on students who were identified in the first investigation as having an instrumental understanding level. It may be concluded that through provoking and causing students to resolve cognitive conflicts, students' understanding can be promoted from instrumental understanding to relational understanding. However, a closer examination of the results reveals that the students' state of understanding greatly varies, implying that cognitive conflict can be provoked in various ways (depending on which conceptions are in conflict). The author tried to categorize the state of understanding in terms of which conceptions are in conflict with each other.

Here are some typical students' answers. The first type, represented by students #37, #11, and #24, is shown in fig. 4; the second type, represented by students #5, #7, and #40, is shown in fig. 5.

(1) In the first basic type, conflicts C_1,C_2 and C_3 seem to lead to an understanding level of R_1,R_2 and R_3 respectively, thus producing a parallel line image as shown in fig. 4.

Student #37 arrived at R_1 by resolving cognitive conflict C_1

I : treated it as an equation

$\overline{R_1}$: "actually I substituted 8 for X, then the inequality seemed OK "

R_1: "according to Mr. O he substituted 7 for X and found that the answer was strange so I thought that that couldn't be right (ie. X couldn't be greater than 6)."

Student #11 arrived at R_2 by resolving cognitive conflict C_2

I : "if I subtract 2 from X, the result will still be larger than 5; therefore it means that X is larger than 7."

R_2: "it is necessary to add the same quantity to both sides to avoid making an unnecessary part($6 < X \leq 7$)."

R_2: "we can find the correct answer for X by adding equal quantities to both sides."

Student #24 arrived at R_3 by resolving cognitive conflict C_3.

I : "I regarded '>' as '=' and treated it as an equality."

R_3: "it is okay not to change the direction of the equality sign but the meaning of the mathematical sentence itself is strange."

R_3: "when we simplify the inequality we have to maintain value of X"

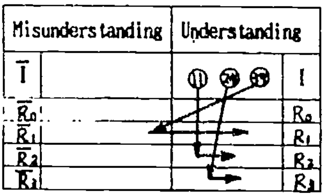

Fig.4

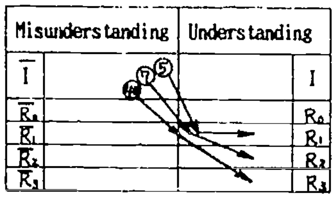

Fig.5

(2) The second type shows that the state of understanding changes diversely. Even the students all started from $\bar{I}$ but moved in different directions thus producing a scattered-image as shown in fig. 5. These three students were considered to be within category $\bar{I}$ because they got the correct answer for problem 1, solving it in the same way as an equation. However, they got the wrong answer for problem 3 when they applied the same logic.

Student #5

$\bar{I}$: treated it as an equation

R_1: "substituting 6.5 for $X-2 > 5$, I got $4.5 < 5$ with a change in inequality sign.

R_1: "I understand that the value of X is important. I understand it through the number line.

Student #7 moved from I to R_1 then to R_2.

$\bar{I}$: treated it as an equation

R_1: "'$X > 6$' is different from the original value of the inequality. I substituted 6.2 , which is larger than 6, and got wrong expressions, (ie $6.2-2 > 5$, $4.2>5$).

R_2: "What we add to both sides of the inequality must be the same because if we add or subtract different quantities we get the wrong value of X".

Student #40 moved from $\overline{I}$ to R_1 then to R_3. This could mean that conflict C_3 was the motivation tool which led him eventually to understand R_3 .

$\overline{I}$: treated it as an equation

R_1: "We can substitute any number which is larger than 6 for X. (6.1 was substituted for X)."

R_3: When we change the sentence we should maintain the direction of the inequality sign and the range of the value of X.

At level R_1 students #5, #7 and #40 are all in the same category of understanding. However, their final states of understanding all differ that is, R_1, R_2 and R_3. This suggests that students focused on different aspects of the cognitive conflict. These differences were, in turn, reflected in their approaches taken to resolve the problem, producing varied answers.

V . CONCLUSION

1) Through provoking and causing students to resolve cognitive conflicts, students' understanding may change from instrumental to relational understanding ($R_1 \sim R_3$). Therefore cognitive conflicts can be regarded as motives which promote students' understanding from instrumental to relational understanding.
2) Cognitive conflicts $C_1 \sim C_3$ often lead to the understanding of $R_1 \sim R_3$ respectively.Therefore, it is crucial to recognize which cognitive conflict(ie, C_1,C_2 or C_3) the student has as this is thought to largely determine his ultimate level of understanding.

IMPLICATIONS FOR TEACHING

1) In the teaching/learning process, the questions which may provoke conflict in students are useful especially when the teacher needs to know the state of understanding of the students. In fact, even if the answers to the problem 'X 2 >5' are all correct, behind this efficiency lies varied states of understanding which can be revealed through the use of cognitive conflict questions.
2) The cognitive conflict questions should be contrived so that they contain 'behaviorized how', 'verbalized how' and 'justification of how' because this kind of question will activate students' thought at various levels.

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Conflictual conceptions of transformations.

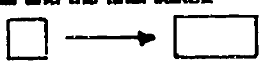
Catherine Girardon and Claude Janvier.

The notion of transformation is characterized as a process through which entities are changed from an initial state to a final one. Transformations are considered in the paper as internal representations (or conceptions) linked with the use of arrows. Temporal conceptions are distinguished from procedural conceptions. It is assumed that problem solving abilities require a flexible use of both. It has been shown that a special type of mathematics teaching can prevent such a flexibility to materialize.

A definition

A transformation is a process through which an entity is changed into another one. Such a definition encompasses several cases. An object can become another object: a) a square being stretched into a rectangle, b) taking the square root of 4 and obtaining 2, c) squaring 4 to get 16 ,... One object can be changed into other objects: d) a loaf of bread being cut into two parts, e) a molecule split into its components . Several objects can also be reorganized into many objects. One notes that the objects considered vary on a spectrum of abstractness, in the examples we gave: bread, molecule and 4 are located at different levels of abstraction. What characterizes any transformation is that an **initial state** can be distingued from a **final state**.

A transformation is often symbolically represented by an arrow linking the initial and the final states.

a) 

b) $4 \longrightarrow \sqrt{4} = 2$

c) $4 \longrightarrow 4^2 = 16$

e) $2\,H_2O \longrightarrow 2\,H_2 + O_2$

A classification

In a classification of the types of arrows that one finds in mathematics and science textbooks made by a few interesting facts were singled out. Arrows are often used in order to express **a duration**, **a lapse of time**. Rather than using two different frames as in a comic strip, some authors link two drawings by an arrow. One of the technique of analysis used in the paper just mentioned consists in dynamizing static diagram comprising a final and an initial state in order to assess the role of the arrow and its indistensable presence in the diagram. This dynamization involves creating a scenario of an animated movie sequence. This method shows that sometimes intermediate states can be inserted in between the initial and final states, sometimes the change is sudden and no in-between states can be introduced.

Transformation and conception

This suggests that the actual use of arrows (external representation) may evoke mental images or elements that are linked with some sensori-motor experience (internal representation) which can be assigned to two irreducible categories. In this paper, we examine only the transformations as conceptions, as they are mentaly processed by students' mind. Transformations and conceptions of transformation are consequently interchangeable in this paper. We shall call **temporal** a transformation that involves intermediate states and **procedural** the others for which no intermediate state can be introduced. We accept the fact that there still remain doubtful cases and also that it is not easy to detect students' nmental constructs. We believe it is the price to pay for exploring the domain.

A recurrent difficulty

Janvier and Mary (in preparation) and Bednarz N., Janvier B. and Poirier L(1983) have shown that describing a transformation was difficult to achieve. With young children, identifying how a number is changed when someone adds or takes away objects often amounts toproviding the final state. For example, let us consider the following easy problem:" You have 5 candies in your bedroom desk drawer, your mummy when you're at school added some candies in the drawer. When you come back home you find 9 candies in your drawer. How many did she add? Most children aged 5 or 6 would answer 9 or 14 to this question describing the transformation by the final state. The idea of squaring a number brings about a dilemma of the same kind. Indeed, x^2 represents a variable, an individual entity that changes when x changes. At the same time, it shows as well how x can be transformed into x^2 (by squaring it). This fact explains partially why the transformation as such is identified to the final state object which sometimes (as it is the case here) symbolically depicts or evokes it.

Experimental design and questions researched.

The subjects of the experiment were 20 boys and girls of year 4 of secondary level. They were selected on the basis of their general ability in mathematics demonstrated by their belonging to a) six (6) from a group taking a strong concentration in math and science, b) eight (8) from a regular teaching in mathematics and c) six(6) taken from a group of weak students. They all had followed a biology course the previous year including some instruction on the phenomenon of "mitosis". The results obtained in the biology course were checked. With two exceptions from the group B, they fit the classification already done with the ordering done using their mathematics ability.

As it will be noted in the description of the task, the experiment was presented as a project aimed at improving the diagrams used in textbooks. We hoped in doing so to avoid that the task be identified to a kind of school test. The twenty students were divided equally into an experimental group (A) and a control group (B).

In the experimental design, both group A and group B were administered the mitosis diagram task. Group A had worked previously on a series of exercises that can be considered as a perturbation treatment. Group A students worked individually on those exercises during four fifteen to twenty minute sessions over a period of one month. All interviews were videotaped so that the hand gestures could be observable.**The general objective of the research** (that will be made precise later) **was to observe and describe the effects of this treatment which we shall now describe on the mitosis task.**

The procedural transformation exercises.

The students had to complete seven "arithmetic network" similar to the one presented in figure 1. Calculators were allowed. The loops included in the network allow the student to automatically check their results. Each arrow plays the role of a transformation. In one network the nature of the transformation was asked, in another a part of the network was absent.

Description of the mitosis diagram task.

The task was developed by Janvier and Mary. It is build "around" a diagram that appears in a biology textbook and that shows... The designer (either the book author or the illustrating artist) has systematically described all the transformation of a chromosome by pointing to the results of the transformation appearing in the diagram at the end of the arrow. Doubling or splitting was indicated by exhibiting the results namely, two chromosomes.After minor adjustments to remove ambiguities carried by the wording, the biology textbook diagram became the one on which the interviews were based that was used in the interviews (see figure 2).

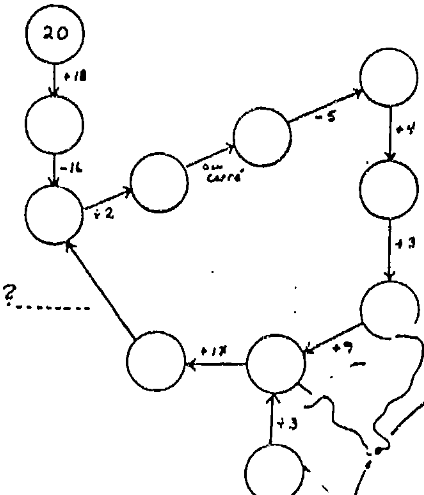
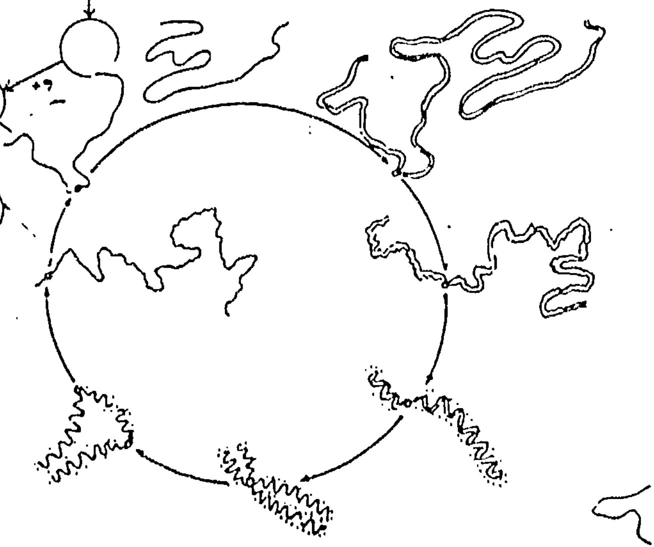


figure 1.



It is subdivided into seven parts.

figure 2.

A) Introduction.

A central point at the outset is to state clearly that the diagram shows what happens to a chromosome during mitosis and is not a description of mitosis. A short conversation follows on how the cells split.

B) Story with the words (on figure 2)

After having explained the words of the diagram, the student is asked to tell the story of the chromosome or what happen to it during the chromosome

C) Story without the words.

The transparency on which appear the writen inscriptions is then taken away, and the student is asked to tell in his own words what does the chromosome during the mitosis.

D)The phases.

The following definitions are given to the students. They are read from a sheet that will stay in front of them for the rest of the interview.

METAPHASE: period during which the chromosome splits into two parts that remain connected to the centromere.

ANAPHASE: period over which the centromere divides and renders each chromosome totally separate.

The student is asked to use a sticker on which appears a pointing index to show on the diagram where are those phases.

E) Doted arrows.

The student is presented with a new diagram but this time the plain arrows (⟶)are changed for dotted arrows(- - - -). The interviewer then asked which diagram he (she) prefers the new one or the one presented at first.

F) Descriptions on top of the arrows.

Another diagram is shown to the student. On this one, the words describing the transformation are put on the top of each arrow rather than next to the final state. For example, we would find:

$$\xrightarrow[\text{THE CHROMOSOME}]{\text{DOUBLING OF}}$$

G) The ideal diagram.

The student is asked to take any elements whatsoever of the previous diagram in order to create the diagram he (she) thinks to be the best and to say why?

Interpretation of the students' responses.

The main idea behind the task is to offer the student several situations in which, he (she) can reveal how evolves his (her) conception of transformation. This is made possible thanks to the particular structure of the task which consists in a continuous attempt to divert the subject from identifying the transformation with the final state to using the arrow as such to do it. As one can see, every new question is an attempt to do so through sollicitations or an inducements which become more and more "irresistible", compelling.

We distinguish in part B), C), and D) gestures with their finger (or indication suggested from their speech) which would amount to pointing the final state from gestures such as moving the finger along the arrows which would implicitly reveal the "presence" of intermediary states. As for parts E), F) and G), the answers of the students were analysed on this basis.

Results and discussions

The effects of the "treatment" was astonishingly high. We will report simply on a few parts of the tasks, the rest being available at the conference.

In part B) and C), stories were really different in nature. For the groug A, as expected, the arrows seem to be considered as ways to go from a state to another. The stories are told "with the fingers on the states. For group B, we have noted more dynamic stories which suggest an unfolding action.

Such a pattern of response is also confirmed with the answers to part D):

The phases., The table presented below is very significant.

Classification of the 40 answers given to part D) of the Mitosis Task.

	group A	group B
A phase identified with one state	19	12
A phase identified "along a region" (one case special)	0	8

The arithmetic network has certainly brought about a fixation on a procedural conception that was not dislodged or shaken by the first attempts of the interview.As the interview progresses not much change in the students' patterns of response can be identified as the next table for part E) shows.

Classification of the 20 answers given to part E) of the Mitosis Task

	group A	group B
Arrows of the diagram	9	4
Arrows with the description on top	1	6

The other results are similar and the comments provided by the students all converges in the same direction. The availability of a temporal conception seem to have been greatly reduced the arithmetic network exercises.

<u>Conclusions</u>

We think that the consequences of such a study are major. A recent survey we have made has shown that transformations in mathematics can be likewise labelled as procedural or temporal. Is it not possible that some form of instruction turns out to be counter-productive. In geometry, for instance, as we will show at the conference some transformations are procedural, others are temporal. But, there exists at the moment a pedagogical creed which assumes that manipulations guarantee the elaboration of the right mental support to deal with algebraically defined transformation. This is contrary to our findings since we believe that temporal conceptions cannot be generalized to procedural ones.

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LA NOTION DE FONCTION A TRAVERS LES REPRESENTATIONS GRAPHIQUES DU MOUVEMENT. UNE EXPERIMENTATION SUGGEREE PAR L'HISTOIRE.

(Conference will be in english)

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U.Q.A.M. et UNIVERSITE DE GRENOBLE I

ABSTRACT

The notion of function has been initially considered as a "dependent variable". An historical and epistemological study showed that it is through the integration of quantitative measures into qualitative representations of the motion that the concept of dependent variable emerged. With this analysis as a starting point, we did an experiment with children aged 12-15 to see how they graphically represent different movements using a qualitative or a quantitative approach, and so, how they construct their concept of function as a dependent variable.

La notion de fonction a grandement évolué au cours des siècles derniers. On retrouve un vaste échantillion de définitions et, avec le temps, le domaine sémantique de fonction s'est modifié. On remarque, en étudiant quelques définitions, qu'au départ la fonction se résumait en définitive à la variable dépendante puis, peu à peu, entre autre avec l'introduction de la théorie ensembliste, elle est devenue une règle de correspondance. La fonction est donc passée de $y=f(x)$, ie la variable = la fonction, à $f(x)=$ ..., ie la fonction = la règle de correspondance. Voyons quelques définitions qui nous aideront à mieux saisir cette nuance très importante.

"Une fonction f d'un ensemble A dans un ensemble B est une règle de correspondance qui associe à des éléments de A un et un seul élément de B." (Manuels scolaires)

Cauchy (1821): "Lorsque des **quantités variables** sont tellement liées entre elles que, la valeur de l'une d'elles étant donnée on puisse en conclure les valeurs de toutes les autres, on conçoit d'ordinaire ces diverses quantités exprimées au moyen de l'une d'entre elles, qui prend alors le nom de variable indépendante et les **autres quantités** exprimées au moyen de la variable indépendante sont ce qu'on appelle des **fonctions de cette variable**." (Phili, C.)

Euler (1755): Des **quantités dépendent des autres** de manière que si les autres changent, ces quantités changent aussi." (Youschkevitch, 1976, p.61)

C'est à Leibniz, dans son "Methodus tangentium inversa, seu de fonctionibus" (1673) qu'on doit le premier emploi du mot fonction pour désigner les **grandeurs** dont les **variations** sont liées par une loi.

Ces définitions nous montrent bien que pour Leibniz, Euler et Cauchy, la fonction est la variable dépendante, tandis que dans la première définition on a un exemple où la fonction est une règle de correspondance.

Cette distinction nous a amenés à retenir pour notre étude la notion de fonction comme variable dépendante.

HISTOIRE

Les limites imposées nous empêchent de retracer ici l'histoire de la notion de fonction. Il est tout de même important de retenir que jusqu'à la fin du Moyen Age, à cause de l'obstacle de l'incommensurabilité, il était impossible d'avoir des relations numériques entre deux variables étant donné le caractère non continu des nombres, ou pour être plus précis, étant donné le caractère non numérique ou incommensurable de certains rapports de grandeurs. Ainsi, si on voulait exprimer la relation qui existait entre deux choses continues, on ne pouvait utiliser les nombres, et les proportions entre grandeurs devenaient alors la solution. Cette utilisation presque systématique des proportions dissimulait le lien fonctionnel qui pouvait exister entre les éléments considérés, car on comparait toujours les choses 4 à 4. On ne pouvait ainsi établir de relations entre les variations de deux éléments; les proportions permettaient seulement de relier deux rapports. Cet obstacle de l'incommensurabilité a crée une incompatibilité entre numérique et continu.

Cette conception est encore présente au XIVe siècle comme l'illustre cette première phrase du traité "Tractatus de configurationibus qualitatum et motuum" de Nicolas Oresme (1323-1382).

"Omnis res mensurabilis exceptis numeris ymaginatur ad modum quantitatis continue." (Every measurable thing except numbers is imaginable in the manner of continuous quantity.) (Clagett, 1968, p.167)

Dans ce traité, Oresme nous fournit une méthode permettant de représenter les qualités changeantes au sein d'un sujet. Les intensités des qualités (des vitesses) sont représentées par des segments (non des nombres!) érigés perpendiculairement à un autre segment représentant, lui, le sujet (le temps). On obtient ainsi un graphique illustrant les intensités d'une qualité ou d'une vitesse à différents points du sujet ou à différents temps.

Cette méthode graphique offre l'avantage de relier deux éléments sans passer par les proportions. Elle se rapproche donc de considérations plus fonctionnelles, cependant, elle demeure totalement qualitative et théorique, on pourrait presque dire imaginaire. Oresme, en effet, n'a jamais mesuré de quelque façon que ce soit les intensités des qualités ou des vitesses. Ses configurations étaient ainsi purement théoriques et ses graphiques n'illustraient que la façon dont il imaginait les configurations des qualités.

C'est avec Galilée et Descartes qu'on assiste à l'introduction du quantitatif, des mesures numériques dans les représentaitons d'Oresme. Et c'est de cette introduction du quantitatif dans les représentations qualitatives du mouvement qu'est née la fonction. C'est ainsi qu'on a pu établir des relations précises entre deux variables. Nous avons donc retenu pour notre expérimentation ces deux voies d'étude du mouvement.

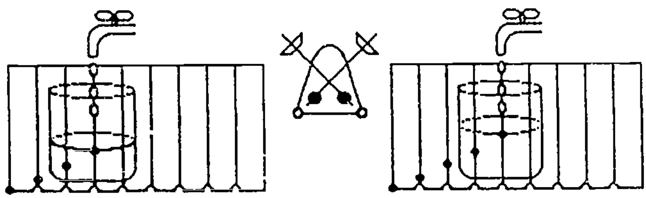
EXPERIMENTATION

Nous venons de voir rapidement que les représentations qualitatives mettaient en évidence l'aspect continu du mouvement puis, l'introduction du quantitatif a permis, en discrétisant parfois le phénomène, d'établir des liens de fonctionnalité entre les variables du mouvement.

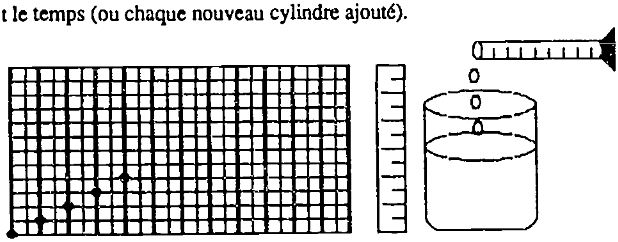
Notre expérimentation est construite à partir de ces deux types d'étude du mouvement. Nos sujets, âgés de 12 à 15 ans (1e et 2e années du secondaire au Québec) ont été divisés en deux groupes de dix élèves. Un groupe appelé qualitatif et l'autre quantitatif. L'expérience se divise en deux parties.

Dans la première partie, on demande aux enfants, groupés par deux, de représenter graphiquement comment l'eau monte dans différentes bouteilles en fonction du temps. Le matériel fourni aux équipes diffère selon qu'elles sont qualitatives ou quantitatives. (5 équipes qualitatives et 5 équipes quantitatives)

Pour les équipes qualitatives, il s'agit d'un plexiglass, placé devant la bouteille qui se remplit d'eau, dans lequel on a fait des coches à intervalles réguliers. Un métronome est placé sur la table. A tous les deux battements de métronome, les enfants doivent faire un point sur le plexiglass à la hauteur où l'eau est rendue, puis déplacer le plexiglass d'une coche.



La procédure des quantitatifs est différente. Dans leur cas, on remplit la bouteille avec un cylindre gradué. Après chaque ajout d'eau, ils doivent mesurer avec une règle la hauteur de l'eau et l'inscrire sur une feuille quadrillée où on a tracé des lignes verticales plus foncées représentant le temps (ou chaque nouveau cylindre ajouté).



Dans cette première partie d'expérience, on demande d'abord aux élèves de tracer expérimentalement, avec leur matériel respectif, le graphique de la hauteur de l'eau dans un bécher en fonction du temps (variation linéaire). Puis, on leur demande de prédire l'allure du graphique pour un plus petit bécher qu'on leur présente (variation linéaire). Ils vérifient ensuite expérimentalement leur prédiciton et commentent. Enfin, il y a de nouveau prédiction et vérification pour une bouteille conique ou Erlenmeyer (variation non linéaire).

Afin de voir comment les élèves utilisent leurs conceptions dans d'autres types de problèmes nous avons réservé la deuxième partie de l'expérimentation à l'observation. Dans cette deuxième partie, on demande aux équipes de tracer les graphiques de simulations vues à l'ordinateur sans faire de vérification. Il n'y a plus à ce moment de différences de matériel entre qualitatifs et quantitatifs.

Les deux premières similations illustrent des tiges qui montent; une première à vitesse constante, une deuxième à vitesse constante lente puis rapide. On peut comparer ces tiges à l'eau qui monte dans les bouteilles. Les équipes doivent tracer le graphique de l'eau, ou de la tige, qui monte en fonction du temps. Les graphiques sont tracés sur du papier millimétrique. Les enfants peuvent revoir le phénomène aussi souvent qu'ils le désirent.

Les deux dernières simulations représentent les mêmes types de variations, ie une linéaire puis une linéaire par morceaux. Ce qui est vu à l'écran sont de petits carrés qui apparaissent dans un rectangle. On explique aux enfants que cela peut illustrer des gens qui emménagent sur un terrain vague. Ils doivent faire le graphique de comment la population, ou le nombre de carrés, varie en fonction du temps. On fait ici une vérification après la présentation du premier phénomène.

Partie I Apprentissage			**Partie II** Observation			
Bécher I	Bécher II	Erlenmeyer	Tige I	Tige II	Pop I	Pop II
Exemple	Prédiction	Prédiction	Prédiction	Prédiction	Prédiction	Prédiction
	Vérification	Vérification			Vérification	
Différences de matériels			*Même matériel pour tous*			

Tableau I: Déroulement de l'expérimentation

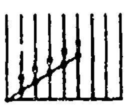
ANALYSE

Nous présentons ici rapidement les principales conclusions et stratégies de réponses utilisées. Nous nous attarderons plus longuement sur cette analyse lors de la présentation.

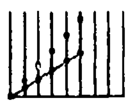
N.B. Tous les graphiques suivants illustrent la prédiction. Pour le Bécher II la ligne pleine ponctuée de points indique l'exemple.

Partie I Apprentissage

Bécher II



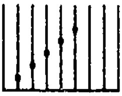
Plusieurs groupes ont répondu de façon parallèle à l'exemple. Cette stratégie est due à un raisonnement additif.



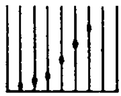
"Bonne réponse"

Il semble que plusieurs groupes aient répondu en ajoutant toujours une même quantité par rapport à la courbe de l'exemple. Ils obtiennent donc un graphique de points parallèle à la première courbe. Cette stratégie est imputable à un raisonnement additif plutôt que proportionnel. Des recherches sur les rapports et proportions (Karplus, Kieren, Nelson ...) ont fait état de ce même problème. Il est intéressant de noter aussi qu'aucun groupe n'a pensé à faire moins de points étant donné que le petit bécher se remplit plus vite que le gros, ou encore, qu'il est moins haut.

Erlenmeyer



Réponse linéaire: prise en compte de la grosseur de la bouteille et non de la forme.



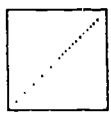
Réponse linéaire par morceaux: difficulté à dessiner des points selon une courbe parabolique.

Outre le "bon graphique", deux types de réponses sont apparues. La première, linéaire, provient d'une stratégie où seule la grosseur de la bouteille est prise en compte, indépendamment de sa forme. L'autre, linéaire par morceaux, est due à la difficulté de s'éloigner du linéaire et surement aussi au fait que les enfants ne sont pas toujours conscients qu'une plus grande vitesse signifie une plus grande distance pour un même intervalle de temps.

Partie II Observation

Tiges I et II

D-V-T
Qualitatifs: plus proche signifie plus lent.



D-V-T
Quantitatifs: plus proche signifie plus vite

A l'épreuve des tiges, nous avons obtenu, en plus des "bons graphiques", plusieurs réponses où les enfants représentaient non pas la hauteur en fonction du temps, mais simplement la vitesse(1). Parmi celles-ci, nous voulons souligner les réponses du type D-V-T, c'est-à-dire les problèmes de relations Distance-Vitesse-Temps. Ce qui est particulièrement intéressant dans ces réponses, c'est que pour le même problème D-V-T, les qualitatifs et les quantitatifs répondent de façons opposées. En effet, pour une perception d'un mouvement lent puis rapide, les qualitatifs font un graphique où les points plus proches signifient que la vitesse est plus lente. Pour les quantitatifs, les points plus proches signifient une vitesse plus grande. Comment interpréter ces représentations?

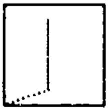
Nous croyons que pour les qualitatifs le temps est un élément important étant donné le métronome à la Partie I. Ainsi, pour eux chaque nouveau point indique qu'un même intervalle de temps s'est écoulé et la distance entre les points représente la distance parcourue pendant cet intervalle.

Pour les quantitatifs l'élément frappant de la Partie I était la distance constante. Plusieurs ont remarqué avec les béchers que ça "montait égal". Il est donc possible que pour eux, chaque nouveau point indique qu'une même distance a été parcourue et la distance entre les

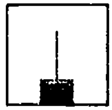
---

Note 1 Notons que pour la tige I, 3 groupes ont perçu une vitesse lente puis rapide alors qu'il s'agissait d'une vitesse constante.

points indique le temps pris pour parcourir cette distance. Nous ne prétendons pas que les enfants soient conscients de ce théorème en acte, mais il est tout de même possible qu'ils l'utilisent.



Un changement d'angle indique un changement de vitesse.



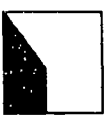
Les lignes indiquent une vitesse lente, les points une plus rapide.

Un autre graphique intéressant obtenu à l'épreuve des tiges est celui où un changement d'angle dans la courbe indique un changement de vitesse. Dans un cas comme celui-ci le temps et la distance n'interviennent plus du tout en aucune manière. On ne considère que la vitesse. On retrouve encore ce phénomène dans un autre graphique où la vitesse lente est illustrée par des traits et la plus rapide par des points. Dans ce dernier cas, il s'agit presque d'une photo, d'une reproduction de ce qui est vu à l'écran. En effet, la tige à l'ordinateur était formée de petites lignes semblables à celles dessinées par les élèves.

Toutes ces stratégies se retrouvent à l'épreuve des populations, soit telles quelles, soit sous d'autres formes.

Populations I et II

On observe pour les populations, en plus des types de réponses décrits pour les tiges I et II, un nouveau graphique pour la population I. Il s'agit de la réponse dite de Fechner.



Fechner
Considération de l'augmentation par rapport au tout et non pour elle-même.

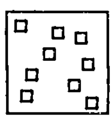


Photo
Reproduction du phénomène plutôt que sa représentation.

Cette réponse reflète une conception du mouvement présenté où chaque nouvelle augmentation de population est considérée par rapport à ce qui est déjà rempli plutôt que pour elle-même. Ainsi, au départ, les premiers carrés qui apparaissent font augmenter de beaucoup la population, puis, plus il y a de carrés, moins chaque nouveau carré a d'importance par rapport à ce qui est déjà là. C'est de cette stratégie que provient ce graphique dit de Fechner. On ne retrouve plus cette réponse à la population II, elle a été remplacée par D-V-T, probablement à cause de la vérification qui présentait une droite. Les enfants ont essayé de produire un graphique se rapprochant plus de ce qui était "bon".

Trois équipes ont répondu aux deux populations par une photo ou une reproduction du phénomène. Dans ce cas, ni le temps, ni la distance, ni la vitesse ne sont représentés. On n'a fait que reproduire ce qui était vu à l'écran.

CONCLUSION

La conception globale de la vitesse qu'ont plusieurs enfants fait obstacle à une représentation de la hauteur en fonction du temps. Les élèves conçoivent souvent la vitesse comme une qualité, une entité, et non comme un rapport distance/temps. Ils ne peuvent donc considérer le lien fonctionnel entre distance et temps.

De plus, certains graphiques des enfants (cf. réponse de Fechner) nous incitent à nous questionner sur leurs interprétations du mouvement et du changement qualitatif ou quantitatif. Nous avons vu qu'ils considéraient parfois un accroissement par rapport à l'état précédent de l'objet qui varie, plutôt que pour sa valeur pure.

Ces conceptions du temps et de la vitesse nous semblent très intéressantes d'un point de vue psychologique. Plusieurs études ont été réalisées sur ce sujet, dans des cadres différents, entre autre par Piaget et Crépault. Cet obstacle des relations distance-vitesse-temps s'est fait moins sentir chez les qualitatifs que chez les quantitatifs. Bien que nous n'ayons étudié qu'un petit nombre d'élèves, il semble que l'importance accordée au temps dans l'approche qualitative puisse être responsable de leur meilleure performance.

Malgré cela, nous croyons que la conception quantitative du mouvement reste indispensable à une bonne conception de la fonction afin de pouvoir traduire précisément, de façon numérique, les liens fonctionnels. En fait, c'est par une interaction entre les approches qualitative et quantitative, qu'à notre avis, les notions de fonction et de variable peuvent être construites; l'approche qualitative aidant à bien saisir l'aspect de variabilité, de continuité du phénomène et le quantitatif permettant de préciser la loi de dépendance.

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TWO CONCEPTIONS OF MATHEMATICAL NOTIONS: OPERATIONAL AND STRUCTURAL

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An analysis of different mathematical definitions and representations brings us to the conclusion that abstract notions, such as function or set, can be conceived either structurally (as static constructs) or operationally (as processes rather than objects). On the grounds of historical examples and in the light of the cognitive schema theory we claim that an operational conception is for most people the first step in acquisition of a new mathematical idea. This supposition is confirmed by the results of our two experimental studies, which show significant predominance of the operational conceptions over the structural in secondary school students.

When a math teacher talks to her students about numbers, functions or sets - what do all these words mean to her? And do they mean the same to her students?

Some people, especially professional mathematicians, refer to abstract concepts as if they were real objects, existing outside the human mind. Indeed, most mathematicians have a kind of static image of sets and functions and talk about their properties in much the same way as a scientist talks about the structures of atoms and crystals. We shall say that these mathematicians have developed structural conceptions of the mathematical notions.

There are however accepted mathematical definitions which reveal another kind of conception. Function can be defined not only as a set of ordered pairs, but also as a "method for getting from one system to another" (Skemp, 1971). Symmetry can be conceived as a static property of geometrical form, but also as a kind of transformation. The latter type of description speaks about processes and actions rather then about objects. We shall say therefore, that it reflects an operational conception of a notion.

These claims about the nature of mathematical perception can have important educational implications. Our current research deals with this issue both from theoretical and practical points of view.

THEORETICAL ANALYSIS OF THE ROLE OF OPERATIONAL AND STRUCTURAL CONCEPTIONS IN DEVELOPMENT OF MATHEMATICAL NOTIONS

Of the two kinds of mathematical definitions, the structural descriptions seem to be more abstract. Indeed, in order to speak about mathematical objects, we must be able to deal with products of some processes without bothering about the processes themselves. In the case of functions and sets (in their modern sense) we are even compelled to ignore the very question of their constructivity. According to this we claim that the structural approach should be regarded as the most advanced stage of concept development.

Careful analysis of several historical examples confirmed us in this opinion. It brought us to the conclusion that most mathematical notions had been conceived operationally long before their structural definitions and representations were formulated. For instance, let us consider the notion of number. The meaning of this concept has been broadened several times in the course of the last three thousand years. For long periods mathematicians did perform some special manipulations with already known numbers, before they were able to accept the products of these manipulations as a new kind of mathematical objects. Indeed, a ratio of two integers was first regarded as a short description of measuring process, rather then as a number. Analogically, the term "negative number" was initially considered nothing more than an abbreviation for certain "meaningless" numerical operations. It came to designate a full-fledged mathematical object only after mathematicians got accustomed to this strange but useful kind of computation. (Cardan's prescriptions for solving cubic equations involved subtracting positive rationals from smaller ones and even finding roots of what is today called negative numbers; despite the widespread use of this algorithm, mathematicians refused to accept its by-products and for some centuries refered to them as "absurd" or "imaginary"; see Cajori, 1985).

In the light of this and many other examples we conclude that most (if not all) of contemporary structural definitions evolved gradually from operationally conceived notions. On the grounds of the cognitive schema theory we conjecture, that learning processes must follow a similar pattern. Procedures, which can be actually performed by the learner himself, are no doubt much more tangible then abstract mathematical constructs. It seems plausible, therefore, that formation of an operational conception is for most people the inevitable first step in the acquisition of a new notion. In many cases it may be also the last.

Apparently, there is nothing wrong with the purely operational approach to mathematics. The "operational" knowledge, however, although seemingly sufficient for problem solving, can not be easily processed by the learner. This kind of knowledge can only be stored in unstructured, sequential cognitive schemas, which are inadequate for the rather modest dimensions of human working memory. Consequently, the purely operational ideas must be processed in a piecemeal, cumbersome manner, which may lead to a great cognitive strain and to a disturbing feeling of only local - thus insufficient - understanding. Also, in the sequential cognitive schemas there is hardly a place for assimilation of a new knowledge, or for what is usually called meaningful learning.

It is the static object-like representation which squeezes the operational information into a compact whole and turns the cognitive schema into more convenient, hierarchical structure. The structural representations constitute the upper levels of such a hierarchy (the top of a pyramide), while the operational information is stored at its bottom. The deeper and narrower the hierarchy, the greater the capacity of the schema. Thus within the structural approach there is much more room for meaningful learning. Also, problem solving processes become more effective, when they can be "navigated" by the help of compact, if not detailed, overall representations. To sum up, transition from an operational to a structural conception is a qualitative change, which renders all the cognitive processes much more effective and thus enhances a person's feeling of understanding mathematics.

OPERATIONAL VS. STRUCTURAL CONCEPTIONS IN SECONDARY SCHOOL

In the two experimental studies, that will be now described in detail, we tried to find out which kind of conception prevails in secondary school students. On the grounds of our theoretical assumptions we expected to find that many pupils conceive mathematics operationally rather then structurally.

First study: the notion of function

Modern mathematical textbooks define function as a correspondence between two sets - as an aggregate of ordered pairs. This structural definition, however, is relatively new. Until the second half of the nineteenth century function has been conceived mainly as an "analytical expression" (Euler, 1748), representing an algorithm for computing one changing magnitude by the help of another. Our conjecture was that despite the intentions of modern curricula, similar operational conception may still be found in today's schools.

To test our supposition we presented 60 secondary school pupils (S_1: age 16, N=31; S_2: age 18, N=29) with the questions given in Box 1. At the time of the experiment all our subjects were already well acquainted with the notion of function and with its formal structural definition.

The results of our exploration are presented in Box 2. The response to the first question clearly shows that the majority of the pupils conceive function as a process rather than as a static construct. This conclusion finds its further confirmation in the answers given to the second question: literally every subject responded affirmatively either to 2a or to 2b. Thus all the pupils felt that there must be certain algorithmic process behind every function. [We were somewhat puzzled by the answer "no" to 2a, accompanied by the answer "yes" to 2b. This strange inconsistency may be due to a contradiction between the teacher's claims and the student's own convictions.] The intresting thing is, that according to the numbers these conclusions apply to the older students even more than to the younger. The operational conception may strenghten with time as a result of a long experience with only one kind of function (the numerical functions given by equations).

Box 1: Questionnaire on function

1. Which one of the following sentences is, in your opinion, the better description of the notion of function?
   a. Function is a computational process which produces some value of one variable (y) from any given value of another variable (x).
   b. Function is a kind of a (possibly infinite) table in which to every value of one variable (x) corresponds certain value of another variable (y).
2. True or false?
   a. Every function expresses a certain regularity (the values of x and y can not be matched in a completly arbitrary manner).
   b. Every function can be expressed by a certain computational formula (e.g. y=2x+1 or y=3sin(π+x)).

Box 2: Percentages of different answers

Question 1

	S1	S2
a	58	79
b	42	21

S1: N=31, S2: N=29

Question 2

a	b	S1	S2
yes	yes	35	79
yes	no	45	--
no	yes	20	21
no	no	--	--

Second study: algebraic notation

Two different approaches to mathematics - operational and structural - can be found not only in concepts' definitions but also in various kinds of mathematical representations. The modern algebraic notation, taught at school, supports the structural approach. Indeed, to translate a "real life" problem into an equation the student has to combine several changing magnitudes into one static whole. Let us remind ourselves, that the algebraic symbolism is not much older than the modern definition of function. Medieval Arabic and Italian scholars described all kinds of computational processes only verbally. The transition from "rhetorical" to "syncopated" to symbolic algebra, which was completed in seventeenth century, was an important stage in development of a structural approach to computational mathematics.

Everyday classroom experience (as well as some recent research, e.g. Mayer, 1983) shows, that many pupils can handle the algebraic symbols only with great difficulty. As has been recently shown by Soloway et al.(1982), these students may still be able to cope with word problems by writing short computer programs. These facts can be regarded as the first evidence for the predominance of operational conceptions.

In our experiment, 96 seccondary school pupils (S_1: ages 14-15, N=44; S_2: ages 16-17, N=52) were asked to translate four simple word problems into equations. In another multi-choice questionnaire they were required to find verbal prescriptions (algorithms) for calculating the solutions of the same problems (for the second questionnaire we altered the numerical data of each problem). Each age-group was divided into halves and the two questionnaires were administered to the sub-groups in reverse order. Two sample problems together with their operational and structural solutions are given in Box 3.

The table in Box 4 shows how many right answers were given to the different questions by the two groups. Both groups succeeded in the

BOX 3: Sample problems

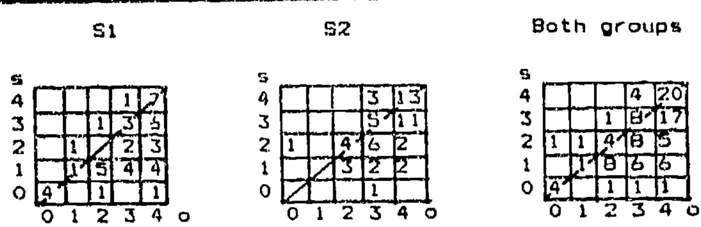
Problem	Operational solutions	Structural solutions
1. In a class the boys outnumber the girls by four.	To find the number of girls we have to: a. add 4 to the number of boys b. subtract 4 from the number of boys c. none of the above	x = number of girls y = number of boys a. x + 4 = y b. x = y + 4 c. y > x + 4
2. The number x is 3.5 times as big as y.	To find y we have to: a. multiply x by 3.5 b. divide x by 3.5 c. none of the above	a. 3.5x > y b. 3.5x = y c. x = 3.5y

BOX 4: Scores in groups

Problem	S1, N=44		S2, N=52		Total, %	
	O	S	O	S	O	S
1	37	28	51	42	92	73
2	32	22	43	33	78	57
3	37	21	42	36	82	59
4	26	20	38	36	67	58
average, %	75	52	84	70	80	62

O - number of right answers on operational task,
S - number of right answers on structural task

BOX 5: Individual scores



o - individual score on operational task
s - individual score on structural task

operational tasks (verbal prescriptions) much better then in the structural (equations). According to our expectations, in each group and in every problem the gap between the two scores was definitely significant, even if in the second group this gap was usually much smaller than in the first (which is also consistent with our theoretical claims).

The diagrams in Box 5 present the individual scores obtained by our subjects in the two questinnaires (p and s are, respectively, the numbers of the right "operational" and "structural" answers given by a subject; every square of a diagram corresponds to a certain pair (p,s); a number in a square shows how many subjects belong to this square according to their scores). Since almost all the data concentrate in the lower right-side half of the diagrams, the advantage of the operational approach becomes even more obvious.

Conclusions

Our results may seem quite surprising, considering the fact that the operational representations were never explicitly taught to our subjects. If so, these findings provide all the more convincing evidence for our claims: they show that the operational conceptions develop at an early stage of learning even if they are not deliberately fostered at school.

"OPERATIONAL" APPROACH TO TEACHING MATHEMATICS

The results of our experiment invoke some important questions about the traditional ways of teaching mathematics. The symbols and the definitions taught at schools are clearly structural, not operational (not surprisingly so, as the structural approach obviously predominates in the most developed branches of contemporary mathematics). Pupils are required to absorb the structural definitions of new notions before they become fully aware of the processes and algorithms underlying these notions.

In the light of our former claims, this method of teaching is not necessarily the most effective. If an operational conception is indeed the necessary first step in an acquisition of a new mathematical idea, we can probably precipitate the learning by fostering the student's understanding of processes and algorithms before translating them into structural definitions. This can be done by incorporating computer programming into mathematics courses. Indeed, many experimental studies (e.g. Johnson and Harding, 1979) have already shown, that in mathematics "a computing experience can be considered highly beneficial" for the learner.

Our own teaching experiments (which can not be presented here in detail) confirmed the advantage of the "operational" approach. In one of these experiments we have developed a teaching unit on mathematical induction. According to our program, at the first stage of learning the pupils get acquainted with many kinds of recursive processes. The student's task is not only to understand and to execute recursive calculations, but also to formulate recursive algorithms explicitly in a simple formal language. The principle of mathematical induction is then formulated in "operational" terms (we speak about equivalence of algorithms instead of dealing with equality of infinite sets). Our material on induction has been taught to three groups (55 students) in the Centre for Pre-academic Studies at the Hebrew University. At final exams on induction the participants of the experiment did much better than the control group (who learned the subject from a traditional textbook), while solving either standard or non-routine problems.

Almost every chapter of school mathematics can be taught in the "operational" manner. This special approach is certainly promising enough to deserve further investigation.

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ATTRACTIVE FIXED POINTS AND HUMANITIES STUDENTS

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Abstract. The research reported in the paper aims at elaborating didactical situations favouring the overcoming of epistemological obstacles relative to functions in 16-17 y.o. humanities students. It concerns a series of situations in the context of iterations of functions and fixed points. It is conjectured that the visual presentation of the idea of an attractive fixed point deepened the "geometrical" obstacle in students and weighed heavily on their disposition to discoursive thinking. The topic, although rather difficult for humanities students had the advantage to reveal all their obstacles. It also offered them an opportunity to get involved in a genuine mathematical activity.

This is a part of a wider research aiming at exploring the possibility of elaborating didactical situations favouring the overcoming of epistemological obstacles relative to limits in 15-17 years old students. It concerns two groups of 16 and 17 years old humanities students (ZII and ZIII, resp.) and a series of situations in the context of iterations of functions and fixed points. This context was meant to help the students to overcome e.o. relative to functions which are at the source of e.o. relative to limits. The students of these groups had previously participated in a series of sessions on infinite sums and decimal expansions of real numbers (partly described in Sierpinska, 1986). It was observed at the end of these sessions that, in the students' conceptions of limit the notion of function is either absent or hidden in the idea of time necessary to make a construction (e.g. to compute the terms of an infinite sequence). Some students preferred to think in terms of sets rather than function, and thus speak about e.g. the number 0.999... as "an infinity bounded by 0 and 1" rather than an infinite sequence approaching 1. Therefore the important problem seemed to make the notion of function appear in the students' conceptions of limit.

1.- Description of sessions (fragments)

Session 11. A computer-aided lecture by A.S.. A question is formulated:

given a mapping f: A $\longrightarrow$ B, what happens with an $x \in A$ if we iterate the mapping f infinitely many times: $x \rightarrow x' \rightarrow x'' \rightarrow \ldots$. Examples of iterations of several mappings are given. The fixed point is introduced as an argument that does not change under the mapping. The names of "attractive" and "repulsive" fixed points are suggested without definitions being explicitely given. Students are asked to describe the behaviour of sequences with their own words. Examples of plane mappings were not included in the lecture for the group ZII (more details in the Appendix).

Session 12. A problem to solve in subgroups. Both in ZII and ZIII there were two subgroups of three persons: Monika, Darek, Konrad and Aga, Gutek, Lukas in ZII, and Anita, Ewa, Agnès and Thomas, Przemek, Jacek in ZIII. The problem: find a real function such that x=2 gives rise to the sequence $2 \rightarrow 3 \rightarrow 2 \rightarrow 3 \rightarrow \ldots$, and have an attractive fixed point in the interval (2,3). How many such functions are there?

In both groups, the name "fixed point" stood for the intersection point of the graph of the function and the line y=x. Moreover, in my discourse, the "attractiveness" and "repulsiveness" were properties of the fixed point. But the students would say either "it attracts" (just as one would say "it rains") or "she is attractive", the "she" standing for "the function" which is feminine in Polish ("point" is masculine). In the group ZIII the first expression was prevalent, in ZII - the second. In the first case, the "attractiveness" is perceived as dynamic phenomenon, in the second - it is the property of the function (cf. Granas, 1983).

ZII: the first idea in both subgroups is to "draw any function" and experiment on it by imitating the movement on the screen. If it "closes over itself" or "turns around" then it is O.K.. Consecutive trials in both subgroups are shown in Figures 6,7&8 .

ZIII: the plane transformations must have impressed very much the students. The first idea in both subgroups was to find a homothety or an axial symmetry or some combination of the two to transform 2 into 3 and vice versa. The "2" and "3" appeared in the students' discourse to denote points rather than numbers. The problem that Agnès wanted finally to discuss was to find a ratio of a homothety that would send the point distant by two from the centre onto a point distant by 3 from the centre. It turns out to be difficult to find such a ratio but she feels satisfied with two: $H^{3/2}(2)=3$, $H^{2/3}(3)=2$. There are two ratios but the mapping is the same; it is a homothety, so everything is O.K.. Ewa is not happy with this solution, and she looks for a mapping that would send the point (2,2) onto (2,3), (2,3) onto (3,3), (3,3) onto (3,2), (3,2) onto (2,2) in order to obtain a square like those shown on the computer. The problem was never read to the end in the group of girls.

Session 14. Which linear functions have an attractive fixed point? A problem to be solved in groups.

ZII found a correct solution in a quarter of an hour.

ZIII; Anita, Agnès, Ewa: they start by considering a homothety, find a sequence convergent to zero: $(-1/2)^n$, and then look for a function (not necessarily linear) that would give such a sequence. They come to $y=-1/2\ x$. I suggest to find conditions on the coefficients a and b of the general formula of the linear function $y=ax+b$ that would guarantee that the function have an attractive fixed point. The girls conjecture that a be negative and $b=0$. They verify the conjectures and find out that, indeed, a cannot be positive (in one of the further sessions, Ewa made it explicit that she considers the "spiral" representation as a necessary condition for a fixed point to be attractive), and b may be different from zero - this changes only the position of the fixed point.

Thomas, Przemek and Jacek: the boys experiment with $f(x)=2x+1$,   . Thomas says it does not make sense in the last case, and finally Przemek tries a constant function. The boys say that re it gets immediately into the point. They laugh because they find the example trivial. So Przemek proposes to incline the line a little, then perhaps it will not hit the point so quickly. They experiment with one position of x_o and conclude that it is all right now. They come to the conclusion that the attractiveness of the fixed point depends upon the inclination of the line. Now Thomas starts asking questions that may lead him to the understanding of the linear function, of the role of coefficients in the formula, and of its graph. He asks about the inclination: relatively to what should this inclination be considered.

Session 15. ZIII only. A regular lesson by A.S.: recapitulation of the previous sessions, exchange of criticisms, development and justification of conjectures concerning attractiveness in linear functions.

Session 18. Problem: communicate in written form, without using drawings, the idea of the attractive and repulsive fixed points to someone who was absent from the last sessions. The problem was given to the group of girls in ZIII. In ZII two groups were working: Darek and Monika, and Aga, Gutek. There were two kinds of attitudes towards the task: one

(ZIII, Darek, Gutek) tending to formulate a general definition of an attractive fixed point; and a second (Monika, Aga) aiming at giving a practical method of checking whether a particular fixed point is attractive. Gutek's first attempt of formulation: "The fixed point is contractive if, by consecutive transformations, any point of OX, except the value of the fixed point, gets closer and closer ... until infinity ... And we should get the fixed point". Darek and Monika's final text: "If on the plane there exists a rectangular coordinate system and an arbitrary function has been traced in it / To find the fixed points, the function given by the formula $y=x$ has also to be traced. The common points of the function $y=x$ and the given one are the fixed points / All fixed points are divided into two groups : the attractive fixed points and the repulsive fixed points. To check what group does the determined fixed point belong to, it is necessary / 1^o to choose an arbitrary point G on OX and execute the transformations: / x_o / $x_1=f(x_o)$ / $x_2=f(x_1)=ff(x_o)$ / $x_3=f(x_2)=fff(x_o)$ etc. / That is to say, / draw a road from the chosen point G parallel to the axis OY until the point on the graph of the function. From this point draw a road parallel to OX until the graph of the function $y=x$ / Go on further analoguously / Remark that the road of this sequence may run away or come closer and closer to the fixed point. If it is running away then the fixed point is called a repulsive fixed point, and if it is coming closer and closer to it then it is an attractive fixed point". Final text in ZIII: "0^o-45^o - attractive point / 45^o-135^o - repulsive point / 135^o-180^o - attractive point/ forget it / Attractive point - the point of intersection of the graph of the function with the auxiliary line $y=x$ to which tend the values of the function trying to reach it. In our opinion, they will never reach it because they run to infinity. Repulsive point - the point of intersection (...) from which the values run away to infinity / Would you wish to make a graph, then, in the formula of the function, if the sign of the coefficient of the function is negative then the values reached by this function are alternatively negative and positive / The distance of the points on the graph of the function from the attractive point tends to zero".

Session 19. Confrontation

ZII. Confrontation with Irena. Irena and the group ZII are sitting opposite one another separated by a curtain. After Irena has read and discussed the communications with the group, she is given several functions to examine. The first is: $f:R \to R$, $x \mapsto -2x+1$. She transmits the function to the group and solves it by herself. On the other side of the curtain, Darek comes to the conclusion that the solution depends upon where you go from the starting point: "You say it is repulsive because you have started by going down. If, at the beginning you go up then the point is attractive. Go up now, Irena!".But Irena KNOWS BETTER now: "No, listen, you are wrong! Your reasoning is wrong because we have a graph of a concrete function ... $-2x+1$. If you take a point x_o ... it is some real number ... and you put it in place of x in the formula and you compute! And if you take $x_o=1$ then you don't get +1 but -1! And you are bound to go down!". Darek: "AAAAaaa!"

Session 20. The magnifying glass criterion

ZIII. A regular lesson by A.S.. I suggest the following criterion which, I say, might describe the idea of the attractive fixed point: Take a magnifying glass and keep its centre over the fixed point. If the fixed point is attractive then whatever the radius of the magnifying glass and whatever the initial point of the sequence, you should see through it almost all terms of that sequence. An example is given to stress that "almost all" does not mean "infinitely many". But Thomas, who, all this time, with his colleagues, was trying hard to rebuild (or, rather, to build by himself) the criterion for a linear function to have an attractive fixed point, says: "What's the use of such a criterion!? With it I can only prove what I know already! What I need is a criterion that would tell me what are the functions that have an attractive fixed point and what not!"

2.- Some remarks

R1. At the example Bb. in Session 11 the formula of the function was neither shown on the screen nor told to the students. It was done intentionally to stress the graphic method of "finding values of the function". But, as I understood only much later, this last expression might have had no meaning for the students in this context. When talking to each other, the students would rather say "put ... in place of x in the formula and compute". Therefore they might have not associated the operation on the graph with finding the value of the function. It is possible that the graph is for them not a set of points $(x,f(x))$ but a line obtained by taking different x's and calculating the y's.

R2. One aspect of the notion of function seems to be missing from the students' image of it: the function as a dependence between variables. It is a pity, because this historically important aspect is capable of making the link in the students' minds between all the other aspects: algorithmic, algebraic, geometrical ...

R3. Didn't the early introduction of geometrical representation deepen the obstacles in the students? The visual presentation made them grasp the idea of the attractive fixed point in an immediate, intuitive, global way which seemed to block their discoursive thinking. After Irena's brilliant performance at Session 19, one wonders whether it would not be better to start by asking the students to find or propose their own geometrical representations of the sequence of iterations .

3.- Some general conclusions

C1. The topic may seem too difficult for humanities students. But it certainly has the advantage to surface all their misunderstandings and obstacles relative to the notion of function, thus giving the opportunity for the teacher to help his students to overcome them. Simple exercises like sketching graphs out of a formula don't seem to have taught them much and, worse, kept hidden all their difficulties.

C2. It is perhaps not the main goal in teaching mathematics to humanities students to make them become acquainted with mathematical results: definitions, theorems, theories, algorithms. It is highly probable that they will not need these in their adult lives. But they will most certainly need some intellectual training like conjecturing, verifying hypotheses, finding rational arguments, formulating ideas. The girls in ZIII have involved themselves in such activities, and perhaps it does not matter very much that they arrived to false conclusions. The teacher may feel that it is necessary to correct the error but this may result merely in calming down his own conscience while the student would remain in his error: will he or she remember the conclusions that are not his or her own? Thomas, Ewa and Agnès did not.

C3. Thomas' last remark taught me one important lesson: "Never come up with general and formal definitions when important problems still wait to be solved. Precising definitions and axiomatising are activities of putting into order a wealth of results, facts already discovered; they do not make sense if there is little or nothing to order". The history of calculus from Newton to Cauchy should have taught me that. But it did not. I also, learn by my own experience.

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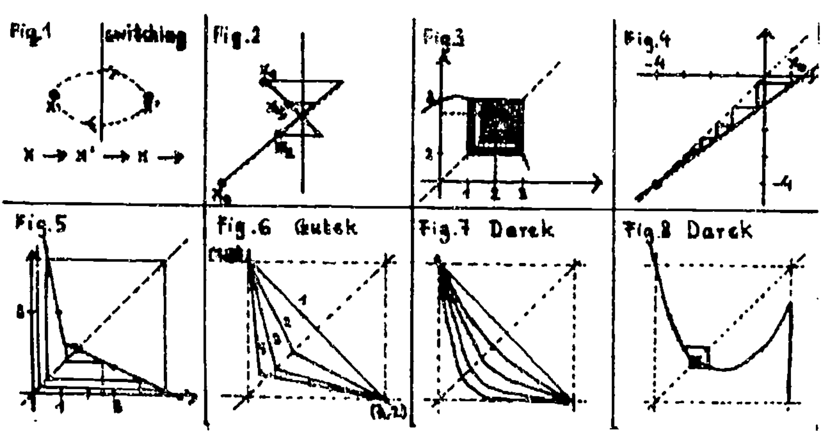
Appendix: Session 11. Description of examples.

A. Plane mappings (ZIII only). a. Axial symmetry. It is observed, among others, that points on the axis are not moved by the mapping (Fig.1). b.Homotheties with ratios greater and smaller that 1. Students are asked to comment on the behaviour of the points. The names of "attractive and "repulsive" fixed points are proposed. c. Compositions of axial symmetries and homotheties with different ratios (Fig.2).

B. Real functions. a. $f(x)=2x+1$. The formula is shown on the screen. Some iterations of f are performed mentally starting with different x_o. Also for $x_o=-1$. It is observed that -1 is not changing under the mapping and it is called the fixed point of f. b. $f:(-1,4) \rightarrow \mathbf{R}$, $f(x) = (1-x)x/3+3$. Only the graph is shown. It is said that given a graph of a real function, the iteration has an interesting graphic representation. The construction of it is shown slowly on the screen and explained with $x_o=2$ (Fig.3). The role of the line is explained, and it is said that the intersection of this line with the graph of the function shows us that the function has a fixed point and tells us what is its approximate value. Next, numerical values of 100 terms of the sequence with $x_o=2$ are shown on the screen. It is observed that the sequence splitts into two subsequences, one tending to 1, the other to 3. The fixed point is not attractive in this case, it is said. c. $f(x)=.75x-1$, $x \in \mathbf{R}$ (Fig.4). A graph is drawn on the screen. Numerical sequences for different x_o's are shown. d. $f(x)=-x+1$, $x \in \mathbf{R}$. The graphic representation of the iteration is observed and it is proved (on the blackboard) that wherever we start, we get a sequence of a repetitive kind. e. $f(x)= 2x+1$, $x \in \mathbf{R}$. It is observed graphically and numerically that sequences run away from the fixed point. f. $f(x)=-5x+8$ for $x<9/7$, and $f(x)= -1/3\ x+2$ for $x>9/7$. Some students say: "O, but here, there are two functions!". Discussion follows. It is observed that the function has a periodic sequence $1 \rightarrow 3 \rightarrow 1$ ..., and that if x_o is outside the interval $[1,3]$ then the sequence runs away from the fixed point, and if it is inside, then the sequence comes closer and closer to the fixed point (Fig.5).



CONTINUOUS FUNCTIONS - IMAGES AND REASONING IN COLLEGE STUDENTS

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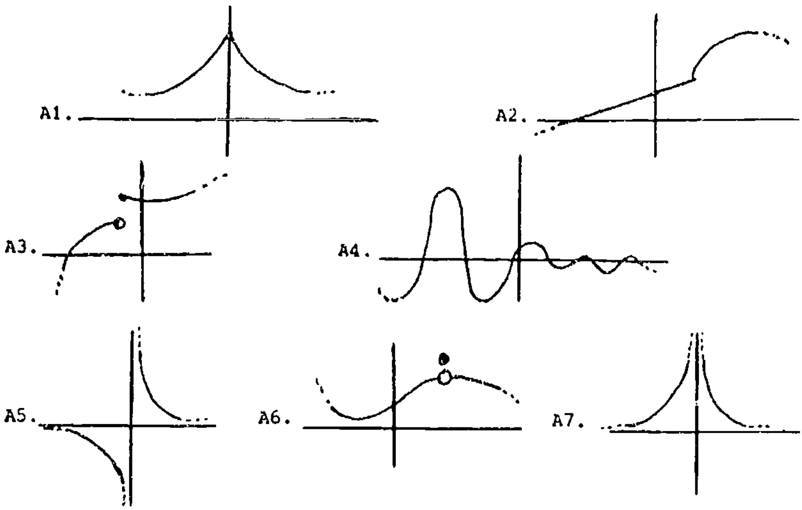
This study investigates the concept of continuous functions in Science students at college level. Students were asked to identify continuous and discontinuous functions and to justify their answers. It turns out that although they succeed in the identification task in the common cases they fail very often to justify their answers. 40% use the argument "the function is defined for every number" in order to establish the function continuity. Their use of the limit concept is quite fuzzy and they often rely on nonrelevant arguments. Altogether, the level of their mathematical reasoning is quite inadequate.

In 1976, Skemp suggested the distinction between knowing how and knowing why. To know how means knowing how to carry out an algorithm. To know why means knowing why the algorithm works. But knowing why can be related to additional situations in Mathematics learning. We shall distinguish also between *knowing that* this is the case and *knowing why* this is the case. What we have in mind is the situation of identifying examples and nonexamples of certain mathematical concepts. A student may identify successfully examples or nonexamples of a given mathematical concept. After doing it, we can ask him why. Why is this an example or a nonexample? We can ask him to justify his answer. This is not necessarily asking how he made up his mind. The act of identification does not have to follow any process of mathematical reasoning in the student's mind. It might be based entirely on mental pictures or other non-verbal inner representations. Nevertheless, being asked why, the respondent is forced to think mathematically. In order to justify his answer he has to refer to some mathematical knowledge that he (hopefully) has. He has to give some convincing arguments to support his claim which was possibly made on a nonverbal ground. The purpose of this study was, first, to investigate the images of continuous functions in college students after this concept was taught to them and, second, to characterize their mathematical reasoning in this context; namely, to characterize the arguments they use in order to justify their answers. This approach has already been used in some studies in geometry (Vinner and Hershkowitz, 1983 and Hershkowitz and Vinner, 1984). The concept of continuous functions was studied quite briefly in Tall and Vinner, 1981. It was examined in Mathematics majors arriving at university in England. In this study we examined 406 Science students after the concept had been taught to them. We used a questionnaire which was supposed to uncover the main images of continuous functions in these students and also their mathematical reasoning associated with this concept. For some of these Science students the calculus course, where we administered the questionnaire, is the last Mathematics course in their entire life. Others will take one or two additional more advanced courses. But whether a student completes his mathematical education with the calculus course or not, it is important to know his level of mathematical reasoning. It may help to evaluate the outcomes of the entire enterprise of Mathematics education of students who do not major in Mathematics. After all they are the decisive majority of the Mathematics students.

METHOD

Sample: our sample consisted of 406 Science students at the Hebrew University, Jerusalem. All of them studied the concept of continuous functions in calculus courses. Their mathematical background was relatively strong. They took a short calculus course in high school. Most of them were immediately after their military service (3 years for men and 2 years for women) so that they had forgotten a lot. The concept of continuous functions was taught to them in several ways. All the teachers used the visual approach, speaking about the possibility of drawing the graph without lifting the pen from the paper. Some of the teachers used also the ε, δ definition, the limit definition ($f(x)$ is continuous at x_o if $\lim_{x \to x_o} f(x) = f(x_o)$ and the intermediate value definition ($f(x)$ is continuous in $[a, b]$ if for every x_1, x_2 such that $a \leq x_1 < x_2 \leq b$ and for any intermediate value c between $f(x_1)$ and $f(x_2)$ there exists ζ, $x_1 < \zeta < x_2$, such that $f(\zeta) = c$.) The differences in the formal definitions did not have any impact on the students' responses, therefore we do not distinguish between subgroups in our sample.

The Questionnaire: our questionnaire had 2 parts. In the first part we presented to the students the following seven graphs:



We asked the students to determine whether the corresponding functions are continuous or discontinuous and to explain their answers. In the second part of the questionnaire we presented to the students 5 functions by means of their defining formulae (no graphs). They were: B1. $y = 1/x^3$. B2. $y = |x|/x$. B3. $y = \sqrt{|x|}$

$$\text{B4. } y = \begin{cases} \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \qquad \text{B5. } y = \begin{cases} x^2 \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Procedure: The questionnaire was administered in the students' regular classes a few weeks after the concept of continuous functions was taught to them. It took 20 to 25 minutes to complete. The explanations were classified into 10 different categories. In this paper we will relate statistically only to the major categories. The small categories did not get more than 3% and we will describe them very briefly.

RESULTS

First, we would like to bring the distribution of the correct and the incorrect answers in our sample. This is shown in Tables 1 and 2.

Table 1

Distributions of Answers to Questions A1 - A7.
(The numbers indicate percents.) N = 406

	A1	A2	A3	A4	A5	A6	A7
Correct	89.9	86.5	89.7	95.6	86.5	80.3	78.6
Incorrect	8.1	10.8	7.9	2.7	10.6	17.0	17.7
No Answer	2.0	2.7	2.5	1.7	2.7	2.7	3.7

Note that in Questions A5 and A7 the functions are not defined for $x = 0$. This point is usually considered as a discontinuity point. There is, however, another approach which associate neither continuity nor discontinuity to points at which the function is not defined. No student in our sample took this approach.

Table 2

Distributions of Answers to Questions B1 - B5.
(The numbers indicate percents.) N = 406

	B1	B2	B3	B4	B5
Correct	79.1	74.1	83.0	28.3	50.7
Incorrect	17.0	19.7	10.1	53.2	24.4
No Answer	3.9	6.2	6.9	18.5	24.9

The reader should not be impressed with the high percentages of success, especially in Questions A1 - A7 since (as already indicated in Tall and Vinner, 1981) many students got correct answers for wrong reasons. In order to get the right impression one must consider the arguments students used to justify their answers. We will relate to the 5 following categories:

I <u>Continuity is considered as being defined and discontinuity is considered as being undefined.</u>

It is hard to tell for sure the origin of this confusion without interviewing some students. Our guess, however, is that students conclude from cases where being undefined (like $y = 1/x$ in 0) is considered as discontinuity. The "logical conclusion" from this is "derived" as follows: If $f(x)$ is not defined at a certain point then it is discontinuous. Hence, if $f(x)$ is defined at every point then it is continuous. (This mistake, the false contraposition, is quite common in college students as reported in O'Brien, 1973). Students in this category claimed for instance that:
"The function is continuous because it is defined for every x".

"The function is discontinuous because it is not defined for every x".

II Continuity or discontinuity are related to the graph.

Students in this category referred to the graphs in their explanations:

"The function is continuous because its graph can be drawn in one stroke"

"The graph has no jumps"

"It is in one piece"

"The function is discontinuous because its graph has two parts which do not meet"

"There is a gap in the graph".

III There is a certain reference to the concept of limit.

This category was used much more in Questions A1 - A7 than in Questions B1 - B5, where it ought to be used. Our impression was that it was not used in a meaningful way. For instance:

"The function is continuous because it tends to a limit for every x"

"The function is continuous because $\lim_{x \to x_o} f(x) = f(x_o)$"

"The function is discontinuous because $\lim_{x \to x_o} f(x) \neq f(x_o)$"

The last two statements were made without any specification of x_o in the particular questions.

IV No explanation.

Quite many students gave no explanations in some of their answers. We believe that this is a weakness that should be noted since we should expect students at this level to be able to justify what they do. This is part of mathematical thinking that we look for so much, very often with very little success (see for instance, Burton, 1984). This category includes also tautological explanations like "this function is continuous" or "this function is discontinuous".

V Other

Here we included several small categories that did not get more than 3% of the answers. Here we have references to the concept of one to one correspondence, confusions between continuity and differentiability, wrong applications of mathematical theorems (like " $x^2 \sin(1/x)$ is continuous because it is a product of two continuous functions"), the claim that a function like in A2 or B5 is discontinuous "because it consists of two functions", the claim that a function is continuous "because it has no inflection points" and other irrelevant explanations. Tables 3 and 4 show the distribution of explanations to the correct answers into the above categories. The numbers indicate percentages out of the correct answers. (The percentages out of the entire sample can be obtained from Tables 1 and 2 by multiplication.)

Table 3

Distribution of explanations to the correct answers to Questions A1 - A7. (The numbers indicate percentages out of the correct answers.)

	A1	A2	A3	A4	A5	A6	A7
I	46	44	20	32	47	22	50
II	22	22	33	21	20	35	21
III	15	15	17	13	15	13	4
IV	10	14	12	25	14	25	19
V	7	15	18	9	4	5	6

Table 4

Distribution of explanations to the correct answers to Questions B1 - B5. (The numbers indicate percentages out of the correct answers.)

	B1	B2	B3	B4	B5
I	82	69	60	25	46
II	2	8	4	7	3
III	4	4	4	22	8
IV	8	13	25	30	31
V	4	6	7	16	12

Since the success percentages in Questions B4 and B5 were relatively low it is interesting to see also the distribution of the explanations to the wrong answers to these two questions. This is shown in Table 5.

Table 5

Distribution of explanations to the incorrect answers to Questions B4 and B5. (The numbers indicate percentages out of the incorrect answers.)

	I	II	III	IV	V
B4	48	4	6	36	6
B5	27	10	8	41	14

The reader can see that something strange happened with Questions B4 and B5. The percentages of the incorrect answers to B4 is about the same as the percentage of the correct answers to B5 (about one half). We could expect that the fact that both functions are well defined everywhere will cause about half the sample to say "continuous" in B4 (wrong answer) and continuous in B5 (right answer). However this is not true. Only about one quarter of the sample has an incorrect answer to B4 which is justified by Category I and only about one quarter of the sample has a correct answer to B5, justified by Category I. We do not know yet whether these two quarters consist of the same people. Anyhow, to claim about B4 or B5 that the functions there are not defined for every x shows inability to understand the formulation in B4 and B5. Luckily enough, this is true only about 6.5%-7% out of the entire sample as can be deduced from Tables 2, 4 and 5. Note that the increase of Category III (the only relevant category) in the correct answers to B4 is rather accidental since in the answers to B5, which is similar to B4, it went down again.

It is worthwhile to mention that nobody tried to draw the graphs in B4 and B5. Very few (about 2%) drew graphs in B1, B2 and B3, which very often were incorrect.

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Mathematics
instruction

LES DEFINITIONS COMME OUTILS DANS LA RESOLUTION DE PROBLEME

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We give account in this communication of the analysis of pupils' behaviour in a problem solving activity with respect to the problem of the definition of some mathematical concepts. We have observed pupils 13 years old in two contexts for the same problem : in the first they had no information available, in the second a text provided them with the definitions and properties of the concepts concerned. Our analysis gives evidence of the way in which pupils consider (or not) the problem of definition and the basis of the decisions they take.

INTRODUCTION

L'objet des recherches, dans le cadre desquelles se placent les résultats que nous rapportons ici, est l'étude des processus de preuve et du traitement d'un contre-exemple dans la résolution d'un problème de mathématique. Nous avons été conduit à développer particulièrement la question de la définition car elle est soulevée par un des traitements possibles d'un contre-exemple : la réfutation d'une solution peut en effet avoir pour conséquence de poser le problème de la nature des objets en jeu dans la résolution d'un problème et donc de leur définition. C'est par ailleurs dans cet aspect de la dialectique des preuves et des réfutations que Lakatos (1976) voit un moteur essentiel de l'évolution et de la construction des connaissances mathématiques.

Nous prenons ici la notion de définition au sens naïf, par opposition à une acception formelle selon laquelle la définition est une abbréviation permettant une économie de mots. Cette acception que nous retenons pour la notion de définition, que Vinner (1976) appelle sa valeur lexicale, est vraisemblablement la plus répandue chez les étudiants : une définition explique un mot avec d'autres mots, elle est un discours visant à éclairer et à fixer le sens d'un mot (Vinner ibid.). Ce point de vue et par ailleurs cohérent avec celui en vigueur dans la pratique des mathématiciens : la définition permet que deux interlocuteurs se comprennent, c'est à dire parlent de la même chose (Borel 1948, p.2070).

Nous avons examiné comment est posé et résolu le problème de la définition par des élèves de 13 ans (8° année de l'enseignement obligatoire en France) à propos du problème suivant : *donner un moyen qui permette dès que l'on connaît le nombre des sommets d'un polygone, de calculer le nombre de ses diagonales.* L'observation a été conduite dans deux modalités. Dans la première le élèves ne disposaient d'aucun document, dans la seconde un texte leur donnait la définition de «polygone», «polygone croisé», «diagonale» et de la concavité. Dans les deux cas le dispositif expérimental a consisté à demander à deux élèves de proposer une solution commune à ce problème. De telle situations d'interaction sociale dans un contexte de communication invoquée nous donne l'accès, à l'occasion des débats entre les élèves, à la genèse des questions qu'ils se posent et aux fondements des décisions qu'ils prennent (Mugny 1985). Dans une première phase de l'observation, dont la durée n'est pas fixée a priori (les élèves décident eux-mêmes que le problème est résolu), l'observateur n'intervient en aucune façon. Dans une seconde phase il soumet aux élèves des contre-exemples (Balacheff 1985). Les élèves sont enregistrés. Les analyses et les résultats que nous présentons ici sont tirés des protocoles obtenus lors de ces observations dont la durée moyenne a été de 1h30.

PREMIERE MODALITE : LES ELEVES NE DISPOSENT D'AUCUNE SOURCE DOCUMENTAIRE.

Dans cette modalité nous avons observé 14 binômes : 6 ne sont à aucun moment entrés dans une problématique de la définition, 2 binômes ont seulement évoqué le problème de la définition sans véritablement s'engager dans son traitement (nous les considérons avec les précédents), les 6 autres binômes ont en revanche posé d'emblée le problème de savoir ce qu'est un polygone, une diagonale.

Les binômes qui n'entrent à aucun moment dans une problématique de la définition ont en commun le fait de soutenir une solution correcte au problème proposé. Ces binômes partagent une conception classique de polygone et de diagonale, lorsqu'il s'agit de polygones convexes. Lorsqu'un polygone concave se présente, certains d'entre eux écartent la diagonale extérieure, amendant leur solution d'une condition de convexité (Geo-Oli, Lau-Lio, Nad-Eli). Dans certains binômes un échange a eu lieu à propos de diagonales «extérieures», mais cet échange n'a jamais conduit au problème de l'expression d'une définition. Tout se passe comme si les élèves disposaient de conceptions assez **robustes et partagées** pour que leur explicitation ne soit pas nécessaire.

Pour les six binômes qui ont posé, dès les premiers instants de la résolution du problème, la question de savoir ce qu'est un polygone, ou ce qu'est une diagonale,le problème de la définition a ensuite joué un rôle essentiel à la fois dans la résolution du problème et dans leur démarche de validation. De plus, ces six binômes, au terme de leur activité, soutiennent la conjecture que le nombre des diagonales d'un polygone est la moitié du nombre de ses sommets, pour les polygones ayant un nombre pair de sommets, amendée éventuellement d'une solution spécifique pour les polygones ayant un nombre impair de sommets.

Pour ces binômes le problème de la définition est entretenu par des conflits que les élèves ne parviennent pas à dépasser : (i) **conflit entre les conceptions** mobilisées de "polygone" et de "diagonale" ; pour certains binômes le retour à l'éthymologie du mot "polygone" conduit à une conception classique en conflit avec une conception de "diagonale" issue de l'exemple prototypique du parallélogramme (Ant-Dam, Pie-Mat, Bla-Isa) ; (ii) **conflit de conception entre les élèves** (Lyd-Mar, Pie-Mat) ; (iii) **conflit suscité par une réfutation** : pour sauver leur conjecture les élèves se placent sur le terrain de la définition, cherchant à ne retenir que les seuls objets pour lesquels elle est valide (Eve-Chr, Naï-Val).

Le caractère dominant de cette problématique est parfois renforcé par une conception globale du rôle de la définition dans une activité mathématique, ou par la lecture du contrat expérimental par les élèves : (i) **la définition fixe des bases communes** pour la résolution d'un problème ; ainsi par exemple : *faut qu'on leur explique c'que c'est pour nous un polygone [...] parce que dans c'cas là pour eux un polygone pour eux ça peut être ça... heu... j'sais pas quoi* (Dam 275-277) ; (ii) la situation est lue comme **un jeu de la définition** ; par exemple : *la règle du jeu dit que c'est nous le professeur. C'est eux [l'observateur et une autre personne présente durant l'observation] on leur dit ... un polygone c'est c'qu'on a donné et lui [l'observateur] doit trouver si notre définition est valable* (Dam 509).

Dans les binômes où elles sont explicitées, les définitions sont une forme d'institutionalisation. Elles fixent le statut des conceptions relativement à la résolution du problème. Mais la recherche du contenu de la définition peut avoir deux types fondements : (i) **des fondements endogènes** : les élèves cherchent à expliciter une conception de polygone et de diagonale en référence essentiellement à des exigences de fondements de la conjecture ; (ii) **les fondements exogènes** : les élèves cherchent à reconstituer des références reconnues culturellement ou par rapport au savoir scolaire. En quelque sorte, la définition qu'il « faudrait» connaître.

Les marques principales de fondements exogènes sont le recours à l'**éthymologie** (Ant-Dam 21, Eve-Chr ?4, Pie-Mat 2) et le recours à l'**exemple prototypique** (Ant-Dam 65 et 70, Bla-Isa 67, Eve-Chr 706, Naï-Val 10, Pie-Mat 15). Dans ce dernier cas il s'agit d'asseoir la conception de diagonale. L'exemple prototypique le plus souvent utilisé est celui du parallélogramme. Notons que le recours à l'éthymologie devrait conduire à une conception classique de polygone, en fait, dans la situation que nous avons observée il n'en est rien. Le prototype du parallélogramme conduit, pour la diagonale, au respect de la contrainte : *faut qu'elles passent toutes par le milieu. Par le même milieu* (Ant 70) qui ramène à la conception de polygone quasi-régulier et éventuellement régulier : *on croyait que ça f'sait une espèce de roue, avec simplement des sommets* (Eve 847). Quant aux fondements endogènes, ils confirment la prégnance du modèle prototypique du parallélogramme : *en général ça coupe en un point au milieu* (Val 65), *il faut qu'ça passe toujours au même endroit [les diagonales]* (Mat 15).

De même que leurs conceptions, les définitions qu'élaborent les élèves ne restent pas figées aprés leur première formulation. Elles peuvent évoluer dans le cours de la résolution du problème, notamment sous la pression de conflits de conceptions entre partenaires, ou sous celle de réfutations rencontrées dans l'une ou l'autre des deux phases.

SECONDE MODALITE :
LES ELEVES DISPOSENT D'UN *TEXTE* DE REFERENCE

Tous les binômes observés, dans cette seconde modalité, ont abordé la résolution du problème par une lecture préalable du *Texte* qui leur était fourni. Mais ce comportement peut très bien être interprété comme un effet du contrat expérimental. L'usage qui en est fait ensuite, dans le cours de la résolution, a deux types d'origine, exogène ou endogène, de façon analogue avec ce que nous avons relevé pour l'entrée dans une problématique de la définition dans la première modalité.

Les origines endogènes correspondent à un recours au *Texte* à la suite de questions soulevées par la résolution du problème, notamment pour ce qui concerne la définition de diagonale, ou par des conflits d'interprétation entre les élèves, mais cela est beaucoup plus rare.

La principale origine d'un retour au texte sur un problème de définition est le traitement du cas du triangle. Dans la première modalité le fait que le triangle soit un polygone a été plusieurs fois remis en question parce qu'il n'avait pas de diagonales (il était alors soit rejeté, soit traité comme une exception). Dans la seconde modalité, l'absence de diagonales est expliquée par la définition fournie par le *Texte* : *quand y a un triangle...y a pas de diagonales parce que les côtés sont tous consécutifs* (Rem 500). Une telle explication est proposée par la moitié des binômes (Rém-Chr, Loï-Mar, Ann-Lau, Chr-Oli, Mab-Sih) Ceci écarte bien sûr le traitement du cas du triangle par son rejet comme polygone, il s'en suivra sa mise à l'écart comme exception ou par l'introduction d'une condition.

Les origines exogènes participent du contrat expérimental, elles attribuent au *Texte* une fin relativement à la résolution du problème.

Certains élèves ont cherché les marques d'une intention des auteurs directement liée à la résolution du problème : le *Texte* doit bien servir à quelque chose. Ainsi la mise en page, l'usage des italiques, de caractères gras ou d'autres procédés, la typographie, peuvent être pris par les élèves comme autant d'indices. Les exemples donnés dans le *Texte* prennent une valeur prototypique tendant à limiter le niveau des procédures de validation des solutions envisagées.
Une figure (un polygone à 7 côtés concave) donnée «en exemple» par le *Texte* pour «montrer» les différents éléments d'un polygone : sommet, côté, diagonale, est utilisée par la majorité des binômes comme un polygone quelconque «type». Par ailleurs, le recours à la cette figure a pu inciter certains élèves à ne pas distinguer le cas des polygones convexes de celui des concaves, au moins parce que les diagonales extérieures sont explicitement admises : *le segment AF est une diagonale, AD aussi. Ah ouais, donc ça peut aussi être extérieur* (Rémi 15).

La distinction convexe/concave est cependant faite plusieurs fois comme une conséquence des distinctions apportées par le *Texte* au niveau des définitions. Ainsi Stéphanie et Julie se posent d'emblée la question après la lecture du *Texte* : "*il est convexe ou non convexe le polygone ?* " (Jul 13-15), pour s'en tenir d'abord aux convexes : "*d'abord vaut mieux déjà essayer d'trouver avec convexe* " (Syl 342) et lorsque leur solution sera réfutée :

- *il est pour tous les convexes, mais pas celui-là, celui-là, regarde, c'est la figure 3.* (Jul 646)
- *ah ben voilà !* (Sté 647)
- *c'est un polygone croisé !* (Jul 648)
- *Ah ouais, il est pas pour les polygones croisés ... ben voilà* (Sté 649).

Relativement à la définition, la fonction essentielle du *Texte* est d'avoir fourni un point de départ à la résolution du problème. La résolution a ensuite eu lieu sous le contrôle de la définition pour la plupart des binômes (Rém-Chr, Loïc et Marc, Chr-Oli, Bén-San, Céc-Emm, Isa-Jul, Ann-Lau, Béa-Syl). Elle conduit essentiellement les élèves à analyser ce qui se passe en un sommet d'un polygone. Le cas d'Isabelle et Juliette en est un bon exemple :

- *alors on va faire les diagonales [...] donc faut procéder par sommet, ou*

prendre un sommet... (Jul 11-15)
Puis elles reviennent à la définition :

- *regarde, on appelle diagonale un segment joignant deux sommets non consécutifs* (Jul 55)

Elles recherchent alors les diagonales sous le contrôle de cette définition :

- *bon d'après la règle* (Jul 113)
- *on appelle diagonale un segment joignant deux sommets non consécutifs* (Isa 114)
- *alors, définition d'une diagonale...donc AE, AD, AC, sont des diagonales* (Jul 115)
- *donc il y a 3 diagonales issues d'un sommet* (Jul 117)
- *à chaque fois y en aura pareil* (Isa 118)

Cette orientation de la résolution explique que la moitié des solutions observées soient du type "le nombre de diagonales en un sommet multiplié par le nombre des sommets". Plusieurs binômes s'écartent de cette voie parce qu'ils sont confrontés au problème de ne compter qu'une fois chaque diagonale. Ce problème apparaît le plus souvent sous la forme du problème des « diagonales confondues » sur lequel, comme le dit Olivier : "*ils [les auteurs] disent pas* ". En l'absence d'éléments du *Texte* permettant de décider, certains binômes accepteront de compter deux fois les diagonales (Mab-Sih, Ann-Lau), d'autres ne trancheront pas (Chr-Oli, Céc-Emm).

CONCLUSION

Le problème de la définition se pose de façon très différente dans les deux modalités de notre expérience.

Dans la première modalité une problématique de la définition apparaît nettement chez plusieurs binômes. Ce phénomène est moins lié à la nature des fondements rationels de la résolution du problème, qu'à la robustesse des conceptions des élèves. C'est ainsi que la moitié d'entre eux, s'appuyant sur des conceptions réputées classiques de polygone et diagonale, n'abordera pas la question de la définition. En revanche tous ceux qui s'appuient sur la conception « fragile » type de diagonale-diamètre associée ou non à polygone-régulier, soulèveront le problème de la définition sous l'impulsion soit d'un conflit de conception entre les élèves, soit de réfutations « sévères » (par exemple de la conjecture "le nombre des diagonales est la moitié du nombre des sommets" par des polygones impairs). La problématique de la définition, et sa résolution éventuelle, se développent alors dans un système de contraintes que sont : les conceptions des élèves, la conjecture en question, l'existence d'un savoir ou de pratiques mathématiques de référence.

Dans la seconde modalité, les définitions qui ont été données jouent un rôle déterminant puisque nous voyons disparaître toute trace des conceptions erronées de polygone et diagonale si fréquentes dans la première modalité. Des conflits de conception surgissent encore, par exemple à propos de ce qu'est un polygone quelconque, sur la nature du croisement en dehors de leurs extrémités de deux côtés d'un polygone croisé, etc. , mais ils ne conduisent plus à un problème de définition en tant que tel : la définition est donnée, elle n'est pas remise en question. En revanche se posent des problèmes d'interprétation du *Texte* qui peuvent opposer les deux élèves d'un binôme, et même faire apparaître un tiers : l'auteur du *Texte* . En effet dans un première phase, pour résoudre un conflit de conception ou un question d'interprétation issue de la résolution du problème, les élèves paraissent s'en tenir à ce que nous pourrions appeler « la lettre du texte ». Les élèves cherchent à comprendre le texte, ou encore à ajuster leurs conceptions à un sens du texte qui exiterait en soi et qu'ils s'efforcent de reconstruire. Si la pression se fait importante, par exemple parce que les élèves ne débouchent pas sur une accord sur le sens du texte, alors sont prises en compte les intentions de l'auteur qui seront invoquées pour ou contre une interprétation donnée.

En conclusion, nous pensons que le fait de fournir une information de référence, s'il modifie sensiblement les « performances » éventuelles au regard de la justesse des solutions fournies, en revanche il ne modifie pas sensiblement (dans le cadre de notre expérience) les processus de preuve engagés dans la résolution du problème ; nous avons même remarqué ici le renforcement des preuves empiriques principalement à cause de la présence du *Texte* et du caractère prototypique des exemples qu'il fournit.

Une recherche à venir sur ce thème devrait prendre en compte deux types de phénomènes dont nous avons ici observé des indices trop faibles pour une conclusion ayant une portée générale :

- l'évolution des élèves d'une lecture du texte comme porteur d'une signification « objective », vers le texte comme reflet des conceptions de son auteur. Ou encore une évolution d'une problématique que peut traduire la question « qu'est-ce que cela veut dire ? », vers une problématique qu'exprimerait la question « qu'est-ce que l'auteur veut dire ? ».
- la prise en compte du texte comme porteur d'indications sur les intentions de l'auteur, ou peut être de celui qui a fourni le texte, relativement au problème soumis ou à la situation dans laquelle prend place sa résolution.

L'hypothèse, que nous paraît soutenir notre expérience, est que la construction du sens d'un texte (la lecture) dépend de la finalité de cette élaboration. Il n'y aurait donc pas un sens a priori d'un texte mathématique engagé dans une relation didactique, mais un sens spécifique à la fois des conceptions du lecteur et des caractéristiques de la situation de lecture (notamment au sens du contrat didactique).

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LECTURE DE TEXTES MATHEMATIQUES PAR DES ELEVES (14-15 ANS) : UNE EXPERIMENTATION

PUPILS READING MATHEMATICS : AN EXPERIMENTATION

Colette Laborde

Reading or listening are fundamental activities in learning mathematics The object of the present paper is to study how pupils construct a meaning of mathematical notions from written texts and to investigate the influence of the linguistic features of these texts on the pupils' understanding. Pupils of the grade 9 were placed in an experimental situation where they were confronted with the task of reading four teaching texts introducing operations on squares roots. They were instructed to write a new text presenting the same content meant for other pupils who have to be introduced to this content. Pupils were working in pairs and observed.The analysis of the observation and of the of the written texts produced by the pupils gives insight on the pupils' understanding of these texts.

L'étude présente a été menée par un groupe de didacticiens (M. Dupraz, M. Guillerault, C. Laborde) et linguistes (R. Bouchart, J. Lachcar) grenoblois.

Les recherches sur l'apprentissage des mathématiques ont essentiellement porté sur les processus de construction de connaissances ou d'utilisation de connaissances déjà disponibles chez les élèves. Mais elles n'ont pour ainsi dire pas abordé la question de la prise d'information par les élèves comme si elle allait de soi. Or depuis une vingtaine d'années, le caractère naturel, évident de la lecture ou de l'écoute est remis en question, ces activités ne sont plus considérées comme un simple transport d'informations du texte ou du discours à l'ensemble des connaissances déjà disponibles du sujet lecteur ou auditeur. Elles sont maintenant analysées comme des activités complexes au cours desquelles le lecteur ou l'auditeur s'engage avec ses connaissances.

La prise d'information est pourtant fondamentale dans l'enseignement qui sollicite fortement des activités d'écoute ou de lecture de la part de l'élève :

écoute du discours de l'enseignant en classe, lecture de textes de problèmes, de passages du manuel, d'énoncés écrits au tableau par l'enseignant.

Cette étude concerne la lecture de textes mathématiques par des élèves de fin de scolarité obligatoire (14-15 ans). A ce niveau d'enseignement, la prise d'informations à partir d'un texte écrit est considérée comme devant faire partie des compétences des élèves. Selon un constat devenu banal cette compétence n'est partagée que par un nombre restreint d'élèves. Nous avons donc cherché à connaître les interprétations construites par les élèves lors de la lecture de textes mathématiques, et à dégager l'incidence des caractéristiques linguistiques des manuels de mathématiques sur ces interprétations. Notre approche est expérimentale mais elle se situe au sein d'un cadre théorique et elle est fondée sur les hypothèses que nous précisons ci-dessous.

I. Cadre théorique

L'analyse de l'activité de compréhension d'un texte a pu être centrée sur l'appréhension par le lecteur du contenu mathématique sous-jacent et négliger les aspects linguistiques. Inversement d'autres recherches n'ont pris en compte que les aspects de surface du texte cherchant à définir des critères de lisibilité indépendants du contenu et du lecteur. Il nous paraît au contraire important de prendre en compte à la fois la complexité cognitive des contenus en jeu dans le texte et sa complexité rédactionnelle. Cela signifie que notre analyse des conduites de lecture prend en compte et de façon interdépendante les éléments suivants:

-**le contenu mathématique** sous-jacent

-**le sujet lecteur**, ici l'élève, en tant que sujet cognitif avec ses conceptions tant des objets mathématiques que de la langue dans laquelle est rédigé le texte à lire

-**le modèle langagier** en vigueur dans l'enseignement mathématique et les **caractéristiques rédactionnelles** du texte à lire

-la situation de lecture : le statut du texte pour le lecteur, et surtout **la**

finalité de la lecture : à quoi sert la lecture, quelle activité ultérieure lui est-elle subordonnée?

Dans ce schéma, le sens voulu par l'auteur du texte n'est pas automatiquement appréhendé par le lecteur mais est reconstruit par ce dernier en fonction de ses conceptions des objets mathématiques sous-jacents, de ses connaissances linguistiques, de ses représentations de la situation de lecture et de sa finalité et des caractéristiques linguistiques du texte.

II. Méthodes d'analyse des conduites de lecture

Comment savoir ce que le lecteur a compris, ou n'a pas compris ? Une observation directe d'un sujet en train de lire ne fournit guère de données dans la mesure où la lecture est une activité individuelle destinée à soi-même. Un autre point que nous tenons à souligner concerne le caractère illusoire d'une analyse de ces interprétations indépendamment d'une prise en compte de la finalité de la lecture.

Pour obtenir des observables liés à l'activité de lecture nous avons donc choisi de la finaliser au sens où l'activité de lecture a conditionné une activité ultérieure de formulation écrite de l'élève lecteur. Nous avons donc en fait évalué la lecture en évaluant la production écrite fournie par l'élève qui nous a permis de dégager des informations sur l'interprétation et la compréhension du texte par ce dernier.

III. Dispositif expérimental

On a donné quatre extraits de manuels en vigueur actuellement en France (pour la classe de 3ème, élèves de 14-15ans) portant sur les opérations sur les racines carrées à deux élèves qui devaient dans une première phase les lire attentivement pour dans une deuxième phase écrire ensemble un texte commun pour d'autres élèves du même âge ne connaissant pas les opérations sur les racines carrées et destiné à leur permettre de les apprendre. Les élèves travaillant à deux ont été observés et enregistrés. Le travail durait environ deux heures. L'expérimentation s'est déroulée avec douze paires d'élèves qui avaient reçu un enseignement sur les racines carrées six mois auparavant.

IV. Choix des extraits de manuels à lire

Il répond à notre objectif de repérer l'effet des variables de présentation d'un même contenu mathématique sur la compréhension des élèves lecteurs.En effet les quatre textes retenus ont été choisis parcequ'ils diffèrent essentiellement sur les six aspects suivants :

-utilisation de **diagrammes, schémas, tableaux** : un seul manuel en utilise

-choix du **code** dans lequel sont formulés les énoncés récapitulatifs des propriétés sur les racines carrées : langue naturelle, écriture symbolique, usage conjoint et plus ou moins imbriqué des deux codes. Ainsi un manuel formule-t-il systématiquement une même propriété, une première fois en langue naturelle, une seconde fois en écriture symbolique :

"La racine carrée d'un produit de réels positifs est égale au produit des racines carrées de ces réels : $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, $a \in R+$, $b \in R+$ "

La même propriété est ainsi formulée par un autre des 4 textes :

"Quels que soient les réels positifs a et b, $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$"

Le premier manuel énonce volontairement deux formulations redondantes mais chacune homogène du point de vue du code utilisé; le second manuel a recours aux deux codes au sein d'un énoncé unique hétérogène.

- **existence de démonstrations** des propriétés des racines carrées : un seul manuel ne fait aucune démonstration

- **le type d'objets** sur lesquels porte la démonstration : nombres fixés ou nombres indéterminés désignés par des lettres ou les deux types à la fois.

- importance en nombre et place des **exemples**

- existence **d'exercices** soit d'application, soit introductifs à une propriété.

V. Analyse de la tâche à laquelle sont confrontés les élèves

Dans cette tâche la lecture est finalisée par l'activité de production qui la suivra; puisque dans cette dernière les élèves ont à s'exprimer pour des pairs, on peut penser qu'ils vont retenir les informations qu'ils jugent importantes et qu'ils vont les transmettre sous la forme qu'ils pensent la plus accessible pour eux mêmes. On pourra aussi avoir accès à leurs interprétations des textes à lire, dans la mesure où ils vont développer à l'intention des élèves lecteurs de leur production des explications susceptibles d'aider à la compréhension du contenu.

Le travail à deux permet l'extériorisation des démarches de pensée de chacun des partenaires grâce aux échanges verbaux; la confrontation des points de vue des deux partenaires est aussi un élément qui contribue à la dynamique de l'activité.

Le choix de donner à lire quatre textes rend la tâche particulièrement complexe. Il a été conçu pour nous permettre de repérer l'effet des variables de présentation d'un même contenu en particulier au niveau des choix faits par les élèves pour la présentation qu'ils adoptent dans leur texte.La lecture de plusieurs textes exige une plus grande activité de construction de la part des élèves et évite une recopie possible s'ils n'avaient eu qu'un texte à lire.

VI. Analyse de quelques données de l'expérimentation

. 1.Représentation globale des textes

La compréhension d'un texte ne consiste pas seulement en l'extraction d'informations mais en la construction d'une représentation générale du texte et de son articulation globale entre ses différents éléments

- *Identification d'un contenu commun aux quatre textes*

Il semble que la majorité des élèves aient reconnu que tous les textes portaient sur les mêmes propriétés des racines carrées : seules trois paires d'élèves sur les 12 observées n'ont pas toujours reconnu le même contenu dans ces textes

Deux paires d'élèves ont répété deux fois, dans le texte qu'ils ont écrit, les règles de calcul du produit et du quotient, prises à chaque fois à partir d'un texte différent car ils ne les avaient pas interprétées comme la même règle.

Une autre paire d'élèves a tiré deux informations contradictoires du même texte et les a consignées dans son texte écrit :

"$\sqrt{5+4} = \sqrt{5}+\sqrt{4}$ accompagnée de la mention "on emploie la même méthode pour les 4 fonctions" et plus bas "$\sqrt{2}+\sqrt{8} \neq \sqrt{2+8}$ ".

On peut penser que la coexistence des deux propositions

$\sqrt{5+4} = \sqrt{5}+\sqrt{4}$ et $\sqrt{2}+\sqrt{8} \neq \sqrt{2+8}$ ne revêt pas pour ces élèves le caractère contradictoire qu'elle présente pour nous. Chacune de ces propositions est interprétée par les élèves comme valide dans son contexte d'énonciation, les élèves n'ayant pas reconnu l'actualisation dans chacun de ces contextes d'une règle générale sur la somme de racines carrées.

- structuration du texte et contrat

Il est apparu que la structuration du texte a une incidence sur le statut accordé par l'élève lecteur à ce qu'il lit. Il semblerait qu'un contrat se noue entre auteur et lecteur en particulier au niveau des titres de paragraphe ou de la place dans la page et de la typographie. Par exemple, les énoncés en plus petits caractères ou en marge du texte même peuvent être considérés comme épisodiques et n'ayant pas d'incidence sur le contenu du texte central.

Le titre d'un paragraphe annonce le contenu du paragraphe et conditionne déjà l'interprétation que le lecteur va faire de ce contenu. Une rupture non explicitée du contrat annoncé par le titre de paragraphe d'un manuel a conduit ainsi à plusieurs interprétations erronées.

Un texte annonce en titre de paragraphe :"Transformer des écritures contenant des radicaux" et commence immédiatement par un avertissement vague sur les sommes de radicaux puis continue par une remarque de notation sur la suppression possible du signe de multiplication dans une écriture algébrique et termine enfin par les règles de calcul sur produit et quotient de radicaux. Certains élèves ont alors interprété la remarque de notation comme une transformation d'écriture et ont alors attribué aux règles de calcul sur les radicaux le même statut de notation. Il suffit de déplacer les signes √ dans les écritures de produits et de quotients.

- l'homogénéité du texte

L'homogénéité d'un texte porte à plusieurs niveaux; elle peut être relative au découpage du contenu en parties d'organisation interne semblables. Le texte peut être découpé en paragraphes, chaque paragraphe étant relatif à une opération sur les racines carrées, l'organisation de chacun d'entre eux restant du même type comme par exemple, démonstration, énoncé de propriété, exemples, exercices.L'homogénéité peut descendre à un niveau plus profond, par exemple sur le déroulement des démonstrations et le type d'objets utilisés.

En particulier une démonstration faite par un des manuels portait d'abord sur un exemple numérique puis sur le cas général à l'aide de nombres désignés par des lettres. Elle a en général été très mal comprise, les élèves n'ayant pas reconnu qu'il s'agissait de la même démonstration conduisant au même résultat.

Deux des textes proposés présentent une plus forte homogénéité : ils ont davantage été utilisés par les élèves pour produire leur propre texte et les élèves ont à l'issue de leur travail clairement exprimé leur préférence pour ces deux textes.

. **2.Usage des deux codes langue naturelle et écriture symbolique**

Les énoncés des propriétés des racines carrées présents dans les textes des élèves font plus fortement appel à l'écriture symbolique que ceux des manuels. Certains élèves ne retiennent que les règles de calcul sans les conditions de validité dont l'expression nécessite soit un énoncé en langue naturelle, soit un énoncé hétérogène dans les deux codes. Par exemple, ils ont seulement écrit : "$\sqrt{a}\,\sqrt{b} = \sqrt{ab}$"

Le faible usage de la langue naturelle est dû aussi dans les productions des élèves à la relative absence de méta-discours explicatif ou introductif malgré la consigne d'écrire un texte qui permette à d'autres élèves de comprendre les propriétés des racines carrées. De l'ensemble du contenu présenté dans les textes les élèves ont retenu les règles de calcul sous forme symbolique et jugé important de les transmettre.

. **3.Rôle des exercices**

Le type de présentation par exercices n'a pas été retenu par les élèves. On perçoit très nettement au contraire le choix par les élèves d'exposition institutionnalisant clairement la propriété à retenir sur la somme des racines carrées. Seul un manuel indique la propriété $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$, les autres ne donnant que des exercices ou des exemples numériques. C'est ce texte qui a inspiré massivement les productions des élèves.

A YEAR IN THE LIFE OF A SECOND GRADE CLASS: COGNITIVE PERSPECTIVE

Paul Cobb, Purdue University

The paper first outlines the organization of a constructivist second grade research and development project that focuses on a) individual children's cognitive development in the context of classroom instruction, b) small group interaction patterns, and c) whole class interaction patterns. The cognitive goals of the experimental curriculum are then discussed. Finally, attention is given to the ways that models of children's conceptual development are applied to instruction. These include a) guiding the development of instructional activities, b) assessing the pedagogical value of particular activities, and c) accounting for the children's mathematical learning.

I and my colleagues Erna Yackel, Terry Wood, and Grayson Wheatley are in the second year of a three year research and development project. The constructivist theory of knowledge in general and the theory of children's counting types (Steffe, von Glasersfeld, Richards, & Cobb, 1983) and its extension to thinking strategies and child-generated algorithms (Steffe, Cobb, & von Glasersfeld, in press) constitute the theoretical framework of the project. As constructivism appears to mean a variety of different things to different people (Kilpatrick, 1986), it is as well to stress that the variant to which we subscribe rejects the notion that cognitive reorganizations occur when students somehow apprehend or intuit the structures said to be found in problems. Consequently, the curriculum activities we are field-testing are not designed to present mathematical relationships in an implicit or `transparent' form (cf. Resnick, 1983, for an alternative view). For us, the process of substantive mathematical learning and of constructing a mathematical reality are one and the same (Piaget, 1980; von Glasersfeld, 1984). They are but different aspects of the reflective abstractions students that make as they reorganize their sensory-motor and conceptual activity. From the researcher's point of view, the result is new, more powerful conceptual structure. For the child, it is a mathematical object that is experienced <u>as though</u> it were there all along. In short, mathematical structures are given to rather than extracted from problematic situations.

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Current Activities

We are currently developing, implementing, and refining prototypical instructional activities in a second grade classroom for the entire school year. Although the implemented curriculum consists of both whole class and small group problem solving activities, lessons do not fit the typical review-development-seatwork format (Good, Grouws, & Ebmeier, 1983). Instead, the children attempt to solve problems and then the teacher leads a discussion of their solutions. No attempt is made to evaluate students' contributions or to "steer" them to a desired official solution. The general instruction format of activity-discussion is used in all curriculum areas including arithmetical computation.

All lessons are video-taped and eight children have been selected for intensive study as they interact in small groups. The project staff's individual responsibilities are:

Cobb: document and account for the target children's construction of arithmetical knowledge.

Yackel: document and account for a) the target children's beliefs about, attitudes towards, and motivations for doing mathematics and b) the changing nature of the small group interaction patterns they establish.

Wood: document and account for a) the nature of the total class interaction patterns, focusing particularly on how a "problem solving atmosphere" is created, and b) the dynamic relationship between the teacher's knowledge and her practice.

Teacher: a) develop and teach lessons, and b) keep diary of classroom life from the practitioner's perspective.

Cognitive Goals

To put it simply but vaguely, the cognitive goal of the curriculum is to encourage the construction of powerful conceptual operation. When we negotiated with the cooperating school corporation, we agreed to address all their stated second grade mathematics objectiv s. As an example, one of these is to add and subtract two-digit numbers with regrouping. Interviews conducted during the first year of the project indicated that only children who have constructed a system of conceptual operations that we call the part-whole system can construct their own efficient algorithms of this type (Cobb, 1987). These conceptual operations are also crucial to activities that are treated as separate topics in traditional intended curricula (e.g., elementary multiplication and division, money, estimation). Consequently, the construction of this system of operations is a central goal of the curriculum. We are attempting to achieve this and other cognitive goals by implementing curriculum activities that give rise to genuine mathematical problems for the children. Just as scientific progress is precipitated by the perception of anomalous phenomena, we expect the children to make progress by reflecting in either problematic situations or situations in which two solution activities are "seen" to give rise to the same conceptual result. In both cases, the crucial element is that of surprise--surprise at encountering an unanticipated difficulty or at two apparently different ways of operating leading to the same result. Both situations give the children opportunities to reflect and reorganize the conceptual base of their mathematical actions. The instructional activities can be solved in a variety of different ways by children at different conceptual levels and thus attempt to take account of individual differences. Their design and implementation is guided by models of early number learning that specify the sensory-motor and conceptual activities that young children might be able to reflect upon and by the on-going initial analysis of the video-recordings. Thus, the cognitive aspect of the project is, in part, a process of formative assessment that attempts to judge the pedagogical value of specific instructional activities. Criteria include a) the reflectiveness of the children's mathematical activity, b) opportunities for dialogue about mathematics that arise within groups and during whole class discussion from the construction of alternative and, at times, conflicting solution methods, c) the acceptance of a difficulty as personally challenging problem (i.e., persistence), and d) personal satisfaction achieved by resolving a difficulty or accounting for an unanticipated convergence of results. In short, the value of instructional activities is assessed in terms of the quality of individual children's problem solving activity and the quality of dialogue. (Clearly, these qualities depend on far more than just the instructional activities and, as Voigt (1985) noted, attention must also be given to the social context within which the children give meaning to the activities.)

The Role of Cognitive Models

As Romberg and Carpenter (1985) observed, "we currently know a great deal more about how children learn mathematics that we know about how to apply this knowledge to instruction" (p. 859). Although we are far from achieving a satisfactory resolution of this issue, some initial reflections on the role that cognitive models play in our current work can be offered.

We had anticipated that the teacher would construct an understanding of the models of children's counting types, thinking strategies, and self-generated algorithms as she interacted with other members of the project staff and use this knowledge to guide her interactions in the classroom. During the induction process, she viewed video-recordings of children solving problems using a variety of self-generated methods and discussed the various conceptual levels at which they were operating. Further, the teacher frequently asks about the levels at which the eight target children are operating and asks clarifying questions during weekly project meetings. Knowledge of these levels is, however, completely irrelevant to her as she interacts with children. There is no indication that she uses the constructs of the theory in an attempt to build models of individual children's mathematical thinking. This, we believe, tells us more about the proposal that teachers should develop such models in the classroom than it does about the project teacher. A conviction that children can construct mathematical knowledge for themselves combined with a desire to find out how they are attempting to solve problems seem sufficient for her to make appropriate interventions.

Nonethelass, the models are proving invaluable in a) guiding the development of instructional activities, b) assessing the pedagogical value of particular instructional activities, and c) accounting for the children's mathematical learning. In contrast to most recommendations, we are not attempting to explicitly teach children methods or strategies associated with more sophisticated conceptual levels (e.g., Case, 1983). Although children's problem solving activities do express their current understandings, the relationship between concepts and solution methods is many-to-many. Consequently, children who have been successfully taught to use a particular method to solve a certain range of tasks have not necessarily constructed the intended concepts.

As an example, consider again the case of constructing efficient algorithms for adding and subtracting two-digit numbers. Typically, textbook instruction first attempts to teach place value concepts and then introduces the standard algorithms. Unfortunately, this approach assumes that the construction of a place value numeration system is a matter of empirical abstraction and figural representation rather than reflective abstraction from and reorganization of problem solving activity (Kamii, 1986; Steffe, 1983). As a consequence, most children construct ten as an abstract singleton, if that is not a contradiction in terms (Cobb, 1987; Ross, 1986). In other words, 54, say, is composed of five singletons of one type called "tens" and 4 unrelated singletons of another type called "ones." The so-called "tens" are distinguished from the "ones" in terms of figural imagery--they are not composite structures composed of units of one.

Mapping instruction in which children are taught standard algorithms by manipulating blocks in parallel with writing symbols has also been of limited success (Nesher, 1987). Once again, the emphasis in on figural representation and children are somehow expected to "take in" place value knowledge from physical objects that the adult can "see" as expressions of his or her own understanding. This and other evidence leads Nesher (1987) to conclude that "no one has succeeded in demonstrating that understanding improves algorithmic performance" (p. 6). The question that immediately comes to mind is whether researchers who believe they have taught children to understand have in fact done so. From the perspective of Piagetian constructivism, there is no reason to assume that they have.

The one exception to the finding that algorithmic performance does not necessarily depend on understanding is that of counting. Researchers operating within a variety of different paradigms have all argued that the construction of counting methods is related to conceptual development. Further, none has suggested that this is merely a matter of empirical abstraction and figural representation. It appears that conceptual development and the construction of increasingly sophisticated counting methods go hand in hand and the opportunities for children to solve problems by using their current counting methods are essential to these developments. From this, we derive the general contention that the relationship between conceptual understanding and observable problem solving behavior is analogous to that between theory and experimentation in science--it is dialectical. Problem solving activity is an expression of and is constrained by current concepts. On the other hand, the activity, as an instantiation of understanding, makes possible unanticipated surprises and constitutes material upon which to reflect. The activity is therefore essential for and constrains subsequent conceptual developments. The value of the models of counting types, thinking strategies, and self-generated algorithms resides in the unique emphasis they place on sensory-motor and conceptual activity (Cobb, in press). The models trace the eventual objectification of particular arithmetical concepts from their sources in activity. Colloquially, these initial beginnings might be called concepts-in-action. In Searle's (1980) terms, the child's intentions are in and cannot be separated from his or her activity and the child acts in order to create meaning. This in no way implies that the child's activity is meaningless or that the child is merely performing a rote procedure. Instead, the meaning is an integral part of and can only be analyzed in conjunction with the activity. Meanings of this sort are rarely captured by cognitive models, particularly those produced by researchers who strive for the stamp of scientific respectability by limiting themselves to the formalisms of currently available computer languages.

Meanings-in-action should not be confused with performing an activity that one has been taught to use. Activities of the latter type can be mathematically meaningless (Hiebert & Wearne, 1985). It is a matter of trying to remember what an authority told the students they are supposed to do rather than acting with meaning. It might well be that the failures to demonstrate that understanding improves algorithmic performance tell us more about instruction that ignores students' reflections on their meanings-in-action than it does about the process of constructing algorithms.

Our approach to helping children invent algorithms for adding and subtracting two-digit numbers has been to try and build on the children's meanings-in-action. The models we are using specify both the activities from which children might abstract and the activities that might be objects of reflection for children at particular conceptual levels. We have simply developed instructional materials that the children typically attempt to solve by engaging in these types of activities. At the same time, we have tried to ensure that the construction of undesirable concepts such as ten as an abstract singleton lead to contradictions. The activities we are currently interested in are a) counting by ones, which can be segmented into modules and, eventually, units of ten, b) making finger patterns such as two open hands as embodiments of the result of counting ten one's or one ten, and c) recognizing and visualizing spatial patterns as embodiments of counting ten ones or one ten. Materials such as hundreds boards and multilinks arranged in bars of ten are always available if a child chooses to use them. However, the children reflect on and abstract from their activity of counting with these materials to solve problems, not from the materials themselves.

The next phase of the cognitive aspect of the project is to construct detailed models of the target children's learning in the classroom.

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A YEAR IN THE LIFE OF A SECOND GRADE CLASS: A SMALL GROUP PERSPECTIVE

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Preliminary indications are that use of a problem-centered mathematics curriculum in a second grade class results in development of moral and intellectual autonomy. Children have developed relational rather than instrumental beliefs about mathematics, are task- rather than ego-involved, view problems as challenges, persist in problem solving and are aware of their cognitive capabilities. Further, they are able to establish productive cooperative working relationships with each other, taking others' viewpoints into consideration and are successful in resolving conflicts.

Recently mathematics educators have posited that students' mathematical performance and learning are influenced by their attitudes toward and beliefs about mathematics (Cobb, 1985; McLeod, 1986; Silver, 1985; Wheatley, 1984). Closely related factors include motivation for engaging in activity, the nature of the learning environment and the social context in which learning takes place. The purpose of this paper is to discuss observations about the attitudes toward and beliefs about mathematics and the nature of small group interactions from a second grade classroom using a problem-centered mathematics curriculum. The study discussed here is part of the Purdue Mathematics Problem-Centered Curriculum Project for second grade, a comprehensive project which includes investigation of individual children's construction of mathematical knowledge, investigation of the total class interaction patterns including establishment of a non-evaluative, risk-free, problem solving environment, and investigation of non-cognitive growth that results from use of a problem-centered curriculum, in addition to development of the curriculum activities. The constructivist theory of knowledge and recent studies in achievement motivation, cooperative learning and students' mathematical beliefs and attitudes form the theoretical basis for this study.

Classroom learning occurs within the context of on-going social interactions. Consequently, children do much more than learn mathematics as they attempt to make sense of the instruction they receive. They develop general beliefs, attitudes, and affective feelings about mathematics and themselves as students and construct expectations for their own and the teacher's role. Traditional

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curriculum designers' failure to consider these developments has resulted in the unanticipated and usually undesirable side-effects of instruction that are documented in the literature (e.g., instrumental beliefs, mathematics anxiety, negative attitudes towards mathematics, learned helplessness). Simply stated, the non-cognitive goals of this project are intellectual and moral autonomy. Autonomy means governing oneself. Thus moral autonomy means "the ability to make moral judgments and decisions for oneself, independently of the reward system, by taking into account the points of view of the other people concerned" (Kamii, 1985, p. 40). Intellectual autonomy refers to making judgments for oneself in the intellectual realm. These constructs subsume such notions as relational beliefs (Skemp, 1976), task- rather than ego-involvement (Nicholls, 1983), viewing problems as challenges, gaining personal satisfaction by solving a problem for oneself, persisting on challenging problems, taking others' viewpoints into consideration, and being aware of one's cognitive capabilities.

The implications for the curriculum of these non-cognitive goals concern the process of implementing the instructional activities as well as the activities per se. Specifically, the instructional strategy must establish conditions which encourage construction of knowledge and result in creation of a non-evaluative, risk-free setting in which children's ideas and solution attempts are valued over correct answers and numbers of problems solved. Small group problem solving has the potential for establishing these conditions. In a recent review of the literature on small group (cooperative learning) instructional methods, Slavin (1986) has identified two general instructional techniques that express differing perspectives, the developmental perspective and the motivational perspective. The developmental perspective, based on Piagetian and Vygotskian theories, holds that "task-focused interaction among students enhances learning by creating cognitive conflicts which they must resolve and by exposing students to higher quality thinking that is within their proximal zone of development" (p. 1). Our approach to small group instruction falls within this category. The motivationalist approach, which is based on reward structures which emphasize extrinsic rewards, is incompatible with our non-cognitive goals. Slavin reported that the motivational approach has been more successful in field experiments than the developmental approach. However, the curricula that were used in those studies follow the traditional pattern of presenting knowledge as isolated facts and sets of rules and procedures that emanate from an authority and are to be acquired by repetitive practice. The curriculum activities in this project are not of this traditional nature. Instead they are designed to present students with problematic situations. As students work in groups to solve the problems, cognitive conflicts arise both because of the problematic situations themselves and also when partners use different solution methods or arrive at different answers. Resolution of conflicts between partners has the potential of resulting in significant learning as partners reflect on their own and other solution methods.

ORGANIZATION OF THE PROJECT

This project involves development and field-testing of a complete second grade mathematics curriculum. The curriculum is field-tested in a classroom with 20 students. Project staff observe and video-tape every mathematics lesson. Video-tapes are analyzed from three different perspectives, a) cognitive growth and conceptual development of the students, b) the nature of teacher-student interaction patterns and the implementation of the curriculum by the teacher, and c) non-cognitive growth which includes the nature of small group interactions and students' attitudes and beliefs. The classroom teacher participates in weekly meetings in which activities are designed and sequenced and the previous weeks' activities are discussed.

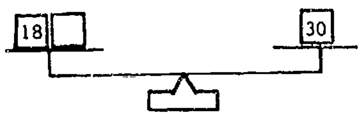
The primary instructional strategy used in the project is small group problem solving followed by class discussion. Students work in teacher-assigned groups of two or three. Typically students work with the same partner for several months at a time. Four pairs, identified for in-depth study, kept their same partners for the majority of the school year.

STUDENTS' ATTITUDES AND BELIEFS

Preliminary results indicate that students attitudes towards mathematics have become much more positive, as assessed by a questionnaire. Students' attitudes towards mathematics reflect their beliefs. Consequently, observations concerning children's attitudes lead to consideration of their beliefs.

The impact of the problem-centered mathematics instruction on students' beliefs about what mathematics is, what it means to "do" mathematics, and how mathematics is learned are being investigated.

Observations to date indicate a number of changes in the children's beliefs. First, the children are less dependent on the teacher. At the beginning of the year children expected the teacher to tell them what to do in order to complete tasks. Such behaviors reflect the childrens' beliefs, based on their first grade experiences, of how mathematics is to be taught and learned. This expectation is illustrated by Amy when she and her partner encountered the problem of deciding what number goes in the empty box in the example below.



Amy: "I'm going to ask Mrs. M if we're supposed to add or subtract." Now it is common to hear students say, "We don't need help. We can figure it out ourselves" or "It won't help to ask Mrs. M. She won't tell us. We're going to have to figure it out for ourselves."

Persistence has increased greatly. Recently almost all of the children spent two class periods of one hour each working on one particularly challenging problem. There are numerous other examples in which one member of a pair tenaciously insists that they not go on to another problem until s/he has a solution method that makes sense to her/him. Their growing persistence and self-reliance indicates progress toward the goals of intellectual and moral autonomy. These appear to be a direct consequence of the instructional setting that has been created in the classroom.

Competition between groups and between children within groups which was common at the beginning of the year has almost disappeared. At the beginning of the year many students put up large folders on their desks as screens to prevent others from seeing their work or finding out which problem they were on. Students were concerned with how many problems or activity sheets they had completed in comparison with others. In one extreme case, one member of a pair spent all of his time finding out how many problems other groups had completed and telling his partner to "Hurry up." He spent no time on the mathematics tasks themselves. In fact, he did not even read them. Now the norm is that students are relatively unconcerned with how many problems they have completed in comparison to others. It is not uncommon for children to report in the group discussion, "We didn't get to that problem. We got only __ problems done."

Competition due to ego-involvement is illustrated by the following example. At the beginning of the year one child was difficult to work with for any partner because she insisted on dominating the group work. Frequently she would not even let her partner see or touch the paper that described activities and on which solutions were to be written. Her dominance resulted from extreme ego-involvement that required that she expend maximum effort to insure that a large number of problems were completed with a high probability of success. She would not take time to consider her partner's views or to engage in reflective activity. Further evidence of this student's extreme ego-involvement at the beginning of the year is shown by a video-taping episode. When the camera was directed on her group as they were engaged in a problem solving task she suggested that they trade in the problem for one they had previously solved "because we did that one the best." This student now freely admits in the total class discussion, "I did it wrong at first" or "My partner and I disagreed but after she explained it to me I found out that I was wrong." This decrease in competition indicates that the children are becoming more task-involved. Further evidence of task-involvement is the fact that pairs often continue working on the activities after the class discussion has begun. In some cases they quietly continue their discussions and collaboration, pausing now and then to participate in the class discussion.

Virtually all competition within groups has disappeared. For the most part pairs view themselves as a single unit. When one member of a pair is called on to provide an explanation the partner frequently assists either by directly joining in the discussion or by prompting the partner. Students often go to the front of the room to explain their solutions. When one member of a pair is called on the partner usually follows along to the front of the class. Unity is usually demonstrated in such solution explanations even when there was considerable disagreement between partners while they were completing the task.

Further evidence of changes in student beliefs indicate that, in comparison with the beginning of the school year, now mathematics has less to do with getting done or generating correct answers than it does with thinking things through for oneself. Initially children made excuses when they came up with wrong answers, such as, "I couldn't see the picture from where I was sitting" or "I said it wrong." Having the answer correct was very important. This is in contrast to a recent episode in which one student told another pair the answer to a problem.

The pair was very disappointed because they had been deprived of the opportunity to figure it out for themselves. In a traditional classroom students are happy to get answers from one another because their goal is to get finished. In this case the disappointment of the pair was so great that they reported it to the teacher.

SMALL GROUP INTERACTIONS

Initially, the children typically divided up assigned activities and then worked separately on particular problems. For example, some children agreed that they would solve alternate problems. Most of the conflicts observed resulted from one child giving an answer to a problem they previously agreed would be solved by the other child. Now this division of problems is rare. Children either assume complementary roles, for example one child physically operates with manipulatives while the other child observes and checks the solution, the children solve the same problem using their own methods and compare answers and solution methods, or they work cooperatively to solve the problem.

Conflicts arise primarily when one child dominates the activity either by not allowing the other child to see and handle the activity sheets, when one child does not wait for a partner who is using an alternative solution method, or when partners disagree on a solution method or an answer. In the four pairs being studied in depth, domination of the activity sheet occurs when one child perceives that the partner does not want to work or when one child thinks that he/she is "smarter" than the other. The experience of this project is that in most cases the child that does not have access to the activity sheet protests to the partner and they mutually resolve the conflict. Only in the case of one pair are protests frequently made to the teacher. The teacher's typical response to a protest is, "What are you going to do about it?" This response is consistent with the goal of development of moral autonomy. Conflicts that arise when one child does not wait for the partner who may be using a different solution method are usually resolved in a similar fashion.

The third type of conflict is commonly observed in the project classroom. Such opportunities for peer challenge typically elicit extensive dialogue, sometimes with very determined exchanges of ideas. Our experience is that almost invariably the partners achieve a mutual understanding of the problem and agree on a solution.

CONCLUSION

Preliminary evidence is that students who participated in the second grade problem-centered mathematics curriculum project believe that mathematics makes sense, it consists of ideas and relationships that can be figured out, mathematics is learned by solving problems, and that they are capable of constructing mathematical knowledge for themselves. As early as second grade, children are able to form productive cooperative working relationships which facilitate their learning. The non-cognitive goals of development of moral and intellectual autonomy are attainable with appropriate curriculum activities and instructional strategies.

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MEANING CONSTRUCTION IN MATHEMATICAL PROBLEM SOLVING

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ABSTRACT

This paper reports early results of a program of research that aims to improve children's mathematics learning by developing attitudes and strategies that support processes of interpretation and meaning construction in mathematics. We are examining processes of socially shared problem solving, in which an adult and other children provide scaffolding for individuals' early problem solving efforts. Different ways of scaffolding problem solving efforts and building self-monitoring strategies are explored in early studies. These studies also show that the intimate relationship between conceptual knowledge and problem-solving in mathematics sets special constraints for instruction and learning.

Considerable research now shows that many children learn mathematics as symbol manipulation rules. They do not adequately link formal rules to mathematical concepts--often informally acquired--that give symbols meaning, constrain permissible manipulations, and link mathematical formalisms to real-world situations (Resnick, in press a). Widespread indications of this problem include buggy arithmetic algorithms, algebra malrules, and a general inability to use mathematical knowledge for problem solving. However, hints exist that strong mathematics students are less likely than other students to detach mathematical symbols from their referents. These students seem to use implicit mathematical principles and knowledge of situations involving quantities to construct explanations and justifications for mathematics rules--even when such explanations and justifications are not required by teachers.

This conjecture is supported by research in other fields of learning. For example, it has been shown that good readers are more aware of their own level of comprehension than poor ones; good readers also do more elaboration and questioning to arrive at sensible interpretations of what they read (e.g., Brown, Bransford, Ferrara, & Campione, 1983). Good writers (e.g. Flower & Hayes, 1980), good reasoners in political science and economics (e.g., Voss, Greene, Post, & Penner, 1983), and good science problem solvers (e.g., Chi, Glaser, & Rees, 1982) all tend to treat learning as a process of interpretation, justification and meaning construction. In a few instances intervention programs have improved both the tendency and the ability of students to engage in meaning construction. The best developed line of such research is in the field of reading. Palincsar and Brown (1984), broadly following a Vygotskian analysis of the development of thinking, proposed that extended practice in communally constructing meanings for texts should eventually internalize the meaning construction processes within each individual. Their instructional experiments, in which small groups of children worked cooperatively to interpret a text, showed broad and long-lasting effects on reading comprehension.

We report here on a program of research that is aiming to improve children's mathematics learning by developing attitudes and strategies that support processes of interpretation and meaning construction in mathematics. Our choice of collaborative problem-solving as a means for meeting this goal reflects an analysis of the nature of cognition that we share with a small, but growing number of psychologists, anthropologists, linguists and sociologists who have been analyzing socially distributed cognition in various applied and school settings (see Resnick, in press c, for a review and interpretation of some of this research).

Socially shared problem-solving sets up several conditions that may be important in the development of mathematical competence. The social setting provides occasions for *modeling* effective thinking strategies. Thinkers with more skill (often the instructor, but sometimes more advanced fellow students) can demonstrate desirable ways of attacking problems and constructing arguments. It also permits *critiquing and shaping* of thinking because processes of thought as well as results become visible. The social setting is also *motivating*; through encouragement to try new, more active approaches, and social support even for partially successful efforts, students come to think of themselves as capable of engaging in interpretation and meaning construction. Finally, collaborative problem solving can provide a kind of *scaffolding* for an individual learner's initially limited performance. Instead of practicing small bits of thinking in isolation, so that the significance of each bit is not visible, a group solves a problem together. In this way, extreme novices can participate in actually solving the problem and can, if things go well, eventually take over all or most of the work themselves.

INITIAL STUDIES: SPECIFIC KNOWLEDGE AND GENERAL STRATEGIES

Our initial efforts were aimed at examining the extent to which the method of reciprocal teaching, developed by Palincsar and Brown to teach reading comprehension skills, could be applied to mathematics learning. Palincsar and Brown use a highly organized small-group teaching situation, in which children took turns playing the role of teacher, a role in which they pose questions about texts, summarize them, offer clarifications and make predictions. These four activities are thought to induce the kinds of self-monitoring of comprehension that are characteristic of good readers. The adult's role in these sessions, in addition to keeping the general process flowing smoothly, is to model problem-solving processes (including encountering and overcoming difficulties); to provide careful reinforcement for successively better approximations to good meaning construction behaviors on the part of the children; and, above all, to provide scaffolding for the children's problem-solving efforts.

Knowledge-dependence of Mathematical Problem-solving

We began with a series of four sessions with a group of five fifth grade children. In these sessions, word problems involving some aspect of rational numbers were to be solved collaboratively, with children taking turns serving as leader of the discussion. Sessions were tape recorded and full transcriptions prepared. Study of the protocols revealed two fundamental problems that would have to be met in adapting the principles of reciprocal teaching to mathematics. Both are rooted in the fact that mathematics problem-solving is more strictly knowledge-dependent than is reading.

First, in our problem-solving sessions, children frequently foundered on sheer lack of knowledge of relevant mathematical content--despite our having chosen rational number problems in order to match our sessions' content to what children were studying in their regular mathematics class. This contrasts sharply with conditions in reciprocal teaching groups in reading, where children are rarely outright *wrong* in their summaries and questions; their responses may not enchance comprehension very much, but they do not drive it off course, either. An example of the dramatic ways in which insecure basic mathematical knowledge blocked successful problem solving is a situation in which the children had drawn a "pizza" and divided it into six parts, each called "a sixth"; they then shaded three parts, after which they asserted that each shaded part was "a third." In situations like this, the adult must choose between interrupting attention to problem-solving processes to teach basic mathematics concepts and attempting to continue problem-solving with fundamental errors of interpretation. Neither choice seems likely to foster the development of appropriate meaning construction abilities.

Second, part of what makes reciprocal teaching work smoothly in reading is that the same limited set of activities (summarizing, questioning, predicting, clarifying) is carried out again and again. It is not as easy to find repeatable activities of this kind for mathematics, because specific knowledge plays such an important role in solving each problem. We used some very general repeated questions--introduced and repeated by the adult leading the sessions--such as "What is the question we are working on?" "Would a diagram help?" "Does that [answer] make sense?" or "What other problem is like this one?" However, as is also often the case for more mathematically sophisticated Polya-like heuristics, these appeared too general to adequately constrain the children's efforts. For example, they did not know *what* diagram to draw (or drew it incorrectly), or could not decide whether an answer was sensible because they had misunderstood basic concepts.

Using Strategies Versus Talking About Them

In a second effort, we attempted to respond to each of these problems in a systematic way. The children were fourth graders; they worked in a group of five for 13 sessions, each led by the same adult. To control for children's lack of specific relevant mathematical knowledge, we chose problems that invoked concepts from the previous year of mathematics instruction rather than the current year. This control for unmastered mathematical content was successful. We encountered very few occasions in which fundamental mathematical errors or lack of knowledge impeded problem solving.

On the basis of cognitive theories of problem solving, we identified four key processes that should be repeated in each new problem-solving attempt. These functions are (1) planning--i.e., analyzing the problem to determine what kinds of procedures are appropriate; (2) organizing the steps for a chosen procedure; (3) carrying out the steps of the procedure; and (4) monitoring each of the above processes to detect errors of sense and of procedure. For each problem to be solved, the four functions were assigned to four different children. The Planner was to take responsibility for leading a discussion of the problem, in order to decide what particular strategies and procedures should be applied. Once a procedure was chosen by the group, the Director's task was to explicitly state the steps in the procedure. These steps were to be carried out by the Doer at a publicly visible board. The Critic was to intervene whenever an unreasonable plan or an error in procedure was detected.

The tactic of dividing mental problem-solving processes into overt social roles was not initially a success. The research community has shared meaning for terms such as planning, directing and critiquing/monitoring. But, with the exception of the Doer role, these meanings were not conveyed to children by the labels, and we were not successful in verbally explaining them to the children. As a result, the roles became instruments for controlling turn-taking and certain other social aspects of the sessions, but they did not successfully give substantive direction to problem-solving. Children discussed the roles a great deal, but they did not become adept at performing them. This points to a fundamental problem with certain metacognitive training efforts that focus attention on knowledge *about* problem solving rather than on guided and constrained practice in *doing* problem-solving. Such efforts are more likely to produce abilities to talk about processes and functions than to actually perform them.

In session 8, we attempted a modification of one of the roles, the Critic, in order to deal with this problem. The critic's function was distributed to two children, who were each given "cue cards" that they were to use to communicate their criticisms. The cue cards read:

1. "Why should we do that?" [request for justification for a procedure]

2. "Are you sure we should be adding (subtracting, multiplying, dividing)?" [request for justification of a particular calculation]

3. "What are we trying to do right now?" [request for clarification of a goal]

4. "What do the numbers mean?" [insistence that attention focus on meanings rather than calculation and symbol manipulation]

The cue cards served to scaffold the critic function by providing language for a limited set of possible critiques. At first the children used the cue cards more or less randomly and in a rather intrusive fashion. However, during the course of the succeeding seven sessions, children's use of the cue cards became more and more refined, so that they used them on appropriate occasions and in ways that enhanced rather than disrupted the group's work.

CURRENT AND PLANNED STUDIES

In studies currently underway and planned, we are examining more restricted forms of shared problem-solving, in order to gain greater experimental and analytical control. We will study groups engaged in collaborative solution of various classes of mathematics problems. We will also study groups whose task is to construct story situations that could generate particular arithmetic expressions or equations (cf. Resnick, Cauzinille & Mathieu, in press; Putnam, Lesgold, Resnick, and Sterret, this volume). Finally, we will study groups whose task is to instruct new (to them) mathematical procedures and algorithms.

Planning and Means-Ends Analysis

A study currently underway examines pairs of children solving problems that are particularly suited to classical "means-ends" problem-solving strategies (cf. Newell & Simon, 1972). Participants in the study were 12 pairs of children, 3 pairs each in grades 4, 5, 6 and 7. Each pair of children met three times for 40 minutes and solved two to six problems.

To scaffold the means-end problem-solving strategy, children were given a Planning Board to work with. The board provides spaces for recording what is known (either given in the problem statement or generated by the children) and what knowledge is needed (goals and subgoals of the problem). Using the board, children can work both "bottom-up" (generating "what we know" entries) or "top-down" (generating "what we need to know" entries). A space at the bottom is provided for calculation. Each child writes with a different color pen, so that we can track who is responsible for each entry. Full verbal transcripts of each session are also prepared.

At each grade level, one pair of children was assigned to each of three conditions. The conditions were:

1. Planning Board With Maximum Instruction. The children solved problems using the planning board. During the first session, the adult demonstrated use of the planning board, and then participated in the first two sessions as a provider of hints and prompts to further scaffold the problem solving process and the use of the board.

2. Planning Board With Minimum Instruction. The children solved problems using the planning board. The adult demonstrated the board and provided hints and prompts during the first session only.

3. Control. The children solved problems without the planning board during all three sessions.

Preliminary inspection of the data suggest that older children and children with more training come to use the board more efficiently. They also generated more goals and inferences on the board. However, in three sessions, there was no effect on accuracy of solutions.

Protocols of the sessions are now being coded in a form that allows us to plot the logical structure of the joint problem-solving effort--i.e., what goals are generated and in what order, what inferences are made from data that is given in the problem statement, how what is known is mapped to goals. Our coding will also permit us to examine the nature of the social sharing of the problem-solving effort. For example, we will be able to determine whether the two children work together on a particular goal or whether they work in parallel; and whether role specializations arise, such as one child working "bottom up" and the other "top down."

GENERAL DISCUSSION

The use of a social setting for practicing problem solving is shared by a number of other investigators, including some in the field of mathematics (see Resnick, in press b). Lampert (1986) conducts full-class discussion in which children invent and justify solutions to mathematical problems. Lampert's discussions are like those of reciprocal teaching in that they are carefully orchestrated by the teacher, and include considerable modeling of interpretive problem-solving by the teacher. Schoenfeld's (1985) work with college students shares many features of the Lampert class lessons, but with considerably more focus on overt discussion of general strategies for problem solving than Lampert uses. Lesh (1982), by contrast, shares reciprocal teaching's small-group format for collaborative problem solving, but has no teacher present. This means that Lesh's problem solving groups benefit from the debate and mutual critiquing that children give each other, but do not have the opportunity to observe expert models engage in the process and are not taught any specific techniques for problem analysis or solution. Scaffolding will also be limited to what children are able to provide spontaneously for one another. The kinds of analyses that we are developing for our data could also be applied to problem-solving groups functioning in these alternative modes. Eventually, comparative studies should help us understand more how these alternative approaches to collaborative problem-solving actually function in supporting and developing mathematical competence.

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L'INGENIERIE DIDACTIQUE,UN INSTRUMENT PRIVILEGIE POUR UNE PRISE EN COMPTE DE LA COMPLEXITE DE LA CLASSE

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Abstract: Didactical engineering is one of the components of a methodology which takes the classroom as an object of study. For this it is necessary to define the possible means of action of the teachers and to determine the contraints to which they are subject. The problem is to delimit a small but significant part of a very complex system. We propose a methodology in three stages. First, there is an a-priori analysis. As an exemple M.Artigue presents a study of the role of the differential equations in scientific knowledge, its teaching until now, and the gap between the two. In the second stage didactical hypotheses are made, and a teaching process based on these is conceived. This process is then realized in the classroom. The final stage consists of an analysis and critique of the results of stage two. C.Comiti will present an example illustrating some didactical problems which are encountered when teachers attempt to construct and interpret a didactical sequence.

I - Introduction

L'un des objectifs de la recherche en didactique est de produire des connaissances sur le système didactique à tous les niveaux de l'enseignement, qu'il soit obligatoire ou postobligatoire. La classe est un lieu de vie où se nouent des relations complexes entre le maître et les élèves dont l'enjeu est le savoir que l'un a la charge de transmettre et que les autres doivent s'approprier. Des facteurs d'ordres différents influent sur ces relations parfois de façon contradictoire. On y distingue des contraintes explicites -programme, horaire, nombre d'élèves- des facteurs dont l'enseignant doit tenir compte -l'attente des parents, des inspecteurs, des élèves eux-mêmes, les coutumes didactiques de l'environnement professionnel- mais aussi des marges de manoeuvre quant au découpage des savoirs, à leur forme de présentation (transposition didactique), quant à la répartition des responsabilités entre l'enseignant et les élèves et aussi aux règles qui vont régir les interactions maître-élèves ou élèves-élèves au sein de la classe à propos des contenus de savoirs sélectionnés (contrat didactique), quant à la prise en compte des diversités cognitives et plus largement de l'hétérogénéité des élèves (point de vue socio-cognitif).

Situer les marges de manoeuvre d'un maître dans sa classe compte tenu de l'ensemble des contraintes auxquelles il est soumis, déterminer l'exploitation qui peut en être faite pour obtenir un résultat désiré au niveau de l'apprentissage des élèves, voilà des questions que le chercheur en didactique va pouvoir aborder par une méthodologie d'ingénierie didactique. Pour cela, il doit préciser ses questions, les transformer en hypothèses se situant dans un cadre théorique éprouvé de manière à pouvoir construire une expérience les prenant en compte et confronter les résultats de l'expérience aux prévisions.

2 - Problématique

2-1 Position épistémologique

Avant de présenter notre cadre théorique, nous faisons l'hypothèse préalable, qu'au-delà des diversités des élèves et des situations, *il existe des régularités dans les processus d'apprentissage scolaire* (y compris universitaire, mais les régularités ne sont pas nécessairement les mêmes) qu'il est possible de construire des enseignements les prenant en compte et que ces enseignements améliorent l'efficacité des acquisitions pour beaucoup d'élèves. Par ailleurs, pour des raisons épistémologiques issues de l'histoire passée, récente ou de la vie scientifique actuelle, nous adoptons le point de vue selon lequel *l'activité mathématique a essentiellement pour but de résoudre des problèmes et de poser de nouvelles questions.* Aussi dirons-nous qu'*un élève a acquis des connaissances en mathématiques s'il est capable d'en provoquer le fonctionnement dans des problèmes*(point de vue *outil*), que l'énoncé y réfère explicitement ou non, pourvu qu'il s'agisse d'outils adaptés de résolution. Nous retenons aussi que des périodes fondamentales de la vie mathématique ont été consacrées à réorganiser les connaissances, à construire des théories et à assurer les fondements (point de vue *objet*).

2-2 Cadre théorique

Nous nous situons dans le cadre de la théorie constructiviste de Piaget. Nos références théoriques sont les suivantes: *transposition didactique* (Y.Chevallard 1985), *contrat didactique, caractéristiques* - action, formulation, validation et aussi institutionnalisation- *des situations*, les concepts de *variable didactique, saut informationnel* (G.Brousseau 1986), *organisation de l'enseignement*, à partir de problèmes de mathématiques répondant à certaines exigences, par *dialectique outil-objet* dont la phase "recherche" est activée par des *jeux de cadres* (R.Douady 1985), effet producteur des *conflits cognitifs* , en situation de communication (A.N.Perret-Clermont 1979, A.Robert et I.Tenaud 1987, R.Douady et M.J.Perrin 1987).

2-3 Exemples de problèmes didactiques

Les problèmes didactiques qu'un chercheur est amené à étudier ne relèvent pas tous d'une méthodologie d'ingéniérie didactique, tant s'en faut. Nous allons donner des exemples d'études didactiques qui relèvent de cette méthodologie et plus précisément d'une *macro-ingéniérie*, c'est à dire d'une ingéniérie portant sur beaucoup de séances car la durée est un paramètre significatif pour ce qu'on veut observer.

Une catégorie concerne l'étude des processus d'apprentissage d'un concept donné. Citons par exemple la numération en début d'école primaire (A.Bessot, C.Comiti 1985 p.305-345), les aires de surfaces planes en fin d'école primaire et début du collège (R.Douady et M.J.Perrin 1987), les équations différentielles en début d'université (M.Artigue PME11,1987). D'autres études sont transverses aux contenus: apprentissage de méthodes portant sur un domaine comme la géométrie (en terminale C section mathématique ou en 4ème année d'université préparation à un concours de recrutement de l'enseignement secondaire) ou d'un mode de travail en situation scolaire ou universitaire comme le travail en groupe (qu'il va bien falloir étudier à travers l'apprentissage de quelquechose: c'est ce qu'étudie A.Robert en collaboration avec I.Tenaud à l'occasion d'un enseignement de la géométrie). Dans cette dernière recherche, A.Robert et I.Tenaud mettent en évidence qu'un changement du contrat de classe entre l'enseignant et les élèves change de façon qualitative les comportements et par suite les productions. Précisons par un de leurs exemples. Une des hypothèses était: le travail en petits groupes sur des problèmes de géométrie choisis pour être résolubles de multiples manières permet de faire fructifier les différentes conceptions des élèves en présence et favorise le développement de *preuves pour convaincre l'autre* . Premier contrat: au cours du travail, l'enseignante circule dans les rangs et demande aux élèves où ils en sont. Les élèves à l'approche du professeur s'arrêtent de travailler. Elle décide de changer de contrat et l'annonce. Deuxième contrat: l'enseignante ne vient que si le groupe l'appelle en cas de conflit. Elle constate alors que chacun reste sur ses positions en comptant sur le professeur pour apporter la vérité. Troisième contrat: l'enseignante demande qu'on ne l'appelle qu'après avoir résolu et prouvé quelque chose. C'est seulement à ce moment que les preuves et démonstrations sont apparues dans leur fonctionnalité. Néanmoins nous estimons que ces contrats locaux prennent leur sens au sein d'un contrat plus global attaché à la classe, ce qui rend, entre autres éléments, incontournable le développement de recherches expérimentales sur le terrain.

Le problème auquel s'est attaqué M.Artigue et qu'elle va vous présenter dans sa communication, est celui de l'adaptation d'un enseignement obsolète, en début d'université de la théorie des équations différentielles, à l'évolution scientifique et technique. Généralement, le point de vue adopté est celui d'un apprentissage dans le cadre algébrique, lequel permet de résoudre certaines équations différentielles en exhibant des formules. Or le cadre qualitatif, qui offre des méthodes qualitatives d'étude, enrichit considérablement le champ des équations différentielles susceptibles d'être traitées. L'ingéniérie construite par M.Artigue a parmi ses objectifs de cerner la contribution que chaque cadre - algébrique, informatique qualitatif - peut apporter à l'étude des équations différentielles non résolubles et de préciser la gestion didactique qui peut être faite des changements de cadres pour qu'ils soient un instrument producteur d'information entre les mains des étudiants. Qu'est ce que les étudiants doivent savoir ou savoir faire dans chaque cadre, pour que la mobilité puisse devenir une habitude et qu'ils n'aient plus qu'à se soucier de l'efficacité qu'elle peut produire? Quels contrôles peuvent-ils avoir sur leurs conjectures et convictions? Pour d'autres problèmes, une micro-ingéniérie (portant sur une ou un petit nombre de séances) plus légère de mise en oeuvre et plus aisément contrôlable peut être la méthodologie adaptée .

3 - méthodologie

La méthodologie se décompose en trois temps. D'abord, une *analyse a priori* qui doit permettre de formuler des hypothèses cognitives et didactiques. Ensuite, la *conception* d'un enseignement qui les mettent en oeuvre, puis la *réalisation* et l'*observation* des séquences construites. Enfin l'*analyse* et la *critique* des productions par rapport à la problématique de départ.

3-1 Analyse a-priori

Elle est un point clé de la méthodologie pour élaborer des séquences d'apprentissage pertinentes vis à vis à la fois des élèves et du problème didactique posé. Cette analyse comporte plusieurs composantes:

a) une *étude épistémologique*: caractériser la place du concept dans sa genèse historique, sa place actuelle dans la diversité des problèmes où il intervient comme outil adapté, repérer les autres concepts avec lesquels il entre en interaction dans les problèmes retenus et qui contribuent à lui donner de la signification, les cadres utilisés et la façon dont ils sont exploités.

b) la *signification mathématique*: point de vue objet.

c) une étude du *point de vue généralement adopté dans l'enseignement*, voire de son évolution à travers les changements de programme.

d) un relevé des *conduites des élèves* compte tenu de l'enseignement habituel: repérage des erreurs, lien éventuel avec l'enseignement reçu au moins au niveau d'une hypothèse, repérage des procédures, des conceptions, des performances.

3-2 <u>Ingénierie didactique proprement dite</u>

a) *élaboration de séquences*: les études précédentes (dont il s'agit d'évaluer pour chaque recherche la pertinence compte tenu du coût) permettent de cerner des variables potentielles sur lesquelles l'enseignant peut agir (variables didactiques). Elles fournissent les moyens de poser des hypothèses à partir desquelles se font des choix didactiques (contraintes et valeurs de variables didactiques, cadres de travail...) qui débouchent sur la conception de séquences d'apprentissage les mettant en oeuvre et satisfaisant les contraintes de la classe, notamment en ce qui concerne le temps et l'hétérogénéité des élèves. Elles permettent de faire des prévisions sur les comportements des élèves.

b) *réalisation* des séquences et *observation* des élèves, de ce que dit ou fait l'enseignant et quand. Est ce bien la séance prévue? sinon en quoi diffère-t-elle? pourquoi? Quel contrat règle les échanges entre les différents acteurs de la classe? Peut-on repérer des règles stables (coutumes) et des règles variables et alors en fonction de quoi?

On se rend compte que les *représentations* des enseignants sur les mathématiques, sur ce qu'est l'activité mathématique, sur l'accès au savoir parfois modulé selon le profil qu'ils conçoivent de leurs élèves, comme d'ailleurs les représentations des élèves eux-mêmes sur ces mêmes facteurs, interfèrent fortement avec les conditions de déroulement prévues pour la situation. D'où de nouvelles hypothèses à tester dans de nouvelles recherches selon une méthodologie à déterminer. Je fais là référence à des recherches récentes d'A. Robert sur les représentations.

c) *évaluation des élèves* : confrontation des comportements initiaux avec ceux en cours d'apprentissage et à la fin, par épreuves écrites et/ou par entretiens individuels ou à deux selon qu'on envisage la possibilité de conflits cognitifs et qu'on veut les observer et en analyser les effets.

3-3 <u>Analyse des produits de l'expérience</u>

Le matériel expérimental soumis à l'analyse est constitué de la description des séquences telles qu'elles ont été prévues en fonction de l'analyse a-priori, des chroniques de classe rédigées par des observateurs, des transcripts d'enregistrements audio ou video, des productions des élèves en cours d'apprentissage et en épreuves d'évaluation. Il se peut que pour un corpus donné une méthodologie particulière d'exploitation soit à définir: par exemple, grille de dépouillement de copies, découpage du texte des transcripts en référence aux concepts didactiques du cadre théorique...l'analyse doit prendre en compte à la fois l'analyse a-priori et les conditions réelles de réalisation de l'expérience.

3-4 Interprétation des résultats et retour sur l'ingéniérie didactique.

Il se peut qu'apparaissent des phénomènes qui obligent à modifier la première élaboration et peut-être de façon fondamentale. Citons à ce propos l'importance des situations d'institutionnalisation, celle du contrat didactique. Il se peut que les observations confirment les hypothèses. On peut alors espérer tenir de bonnes conditions de reproductibilité .

4 - Evaluation et limites de cette méthodologie

Elle est très lourde et ne peut être que le travail d'une équipe. Elle est donc à économiser en l'associant à d'autres méthodes telles que entretiens, questionnaires, études statistiques, mais aussi simulation de classes par ordinateur. Par ailleurs, elle ne permet de tester que des hypothèses composées avec des facteurs dont certains ne peuvent être contrôlés lors de la réalisation. Toutefois, si les résultats analysés de l'observation grossissent durablement les effets attendus, on peut estimer que les hypothèses sont validées. Par ailleurs, on peut trouver des éléments de conviction dans la convergence de résultats expérimentaux variés, dans la cohérence entre des résultats nouveaux et des résultats plus anciens, mais surtout dans la reproductibilité quand c'est possible.

5 - Application

Les outils forgés pour les besoins de l'ingéniérie didactique ou issus des résultats des différentes recherches y recourant, sont des instruments précieux pour expliquer des comportements d'élèves dans une situation mise en place par un enseignant dans sa classe en dehors de tout contexte expérimental. C.Comiti dans son intervention, va vous présenter une analyse des comportements d'un enseignant et des élèves d'une classe de grade 10 (classe de seconde en France) dans une situation d'étude de la pertinence mathématique d'une représentation graphique de phénomènes économiques. On reconnaîtra au déroulement de la séquence les différentes phases de la dialectique outil-objet s'appuyant sur une situation-problème répondant aux conditions qu'elle demande. On reconnaîtra dans les problèmes qui ont apparu au cours du déroulement ceux posés par la transposition didactique et le temps didactique , par le contrat didactique, dévolution du problème, représentations des élèves et de l'enseignant sur ce qu'est l'activité mathématique...

6 - Conclusion

Si lourd et ambitieux que soit l'instrument, c'est le seul qui prenne en charge l'ensemble des composantes du système éducatif. Il est indispensable pour étudier ce qui est propre à la classe, à savoir *le fait collectif*, et ce, à deux niveaux: entre pairs et entre maître et élèves. Nous repérons un point sensible de la responsabilité du maître et pourtant crucial par ses conséquences: c'est l'articulation entre les activités des élèves et le cours du maître, i.e, ce que le maître institutionnalise et le moment qu'il choisit pour cela. Un autre point sensible, mais de la responsabilité des élèves est l'articulation entre savoir de la classe dans son ensemble et appropriation individuelle. Un problème peu étudié jusqu'ici est la construction du savoir de chacun dans le *temps réel* de la classe.

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ANALYSIS OF A CLASS-SESSION FOCUSING ON THE MATHEMATICAL RELEVANCE OF A CHART ILLUSTRATING ECONOMIC DATA

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Starting with the analysis of a class-session (15 years-old) focusing on the mathematical relevance of a chart concerning the distribution of local taxes (see the last page), a demonstration will be made of how didactic research tools developed within the framework of "didactic engineering" can provide valuable support in

. explaining learner behaviour on a normal classroom situation
. facilitating teacher awareness of the different learner appropriation levels of a given mathematical concept and at the same time of the different cognitive paths to be taken in consideration in the collective elaboration of a new knowledge.

ANALYSE D'UNE SITUATION DE CLASSE PORTANT SUR L'ETUDE DE LA PERTINENCE MATHEMATIQUE D'UNE REPRESENTATION GRAPHIQUE D'UN PHENOMENE ECONOMIQUE

A partir de l'analyse d'une situation de classe, portant sur l'étude, par des élèves de 15 ans, de la pertinence mathématique d'une représentation graphique présentant la répartition de l'impôt local (voir dernière page), on montrera comment les outils forgés pour les besoins de la recherche en didactique sous le nom d'ingéniérie didactique peuvent être des instruments précieux pour

. expliquer des comportements d'élèves dans une situation de classe mise en place hors de tout contexte de recherche
. permettre à l'enseignant une meilleure prise de conscience des différents niveaux d'appropriation par les élèves d'un concept mathématique donné et par la même de la diversité des chemins cognitifs des élèves à prendre en compte lors de la construction collective de la connaissance.

LA DEMANDE DES ENSEIGNANTS

Le travail qui est présenté ici est le résultat d'une collaboration entre enseignants de lycée et chercheur en didactique, collaboration qui s'est instaurée à la suite d'une demande de deux enseignants de mathématiques de classe de Seconde (Grade 10). Ces enseignants avaient essayé, les années précédentes, de mettre en place, en liaison avec un professeur d'histoire-géographie et un professeur d'économie, une innovation dont les objectifs étaient les suivants :

* décloisonner l'enseignement des mathématiques par une prise en compte de l'environnement des élèves,

* diversifier les cheminements possibles de l'apprentissage afin de mieux prendre en compte l'hétérogénéité des classes,

* favoriser l'appropriation par les élèves des concepts de croissance, de proportionnalité,

* les rendre capables de comprendre, construire et utiliser comme base d'argumentation des tableaux ou graphiques.

La demande des enseignants de mathématiques répondait à un besoin de prise de recul par rapport aux pratiques mises en place en liaison avec cette innovation ; elle mettait l'accent sur une volonté de recherche d'outils méthodologiques permettant d'objectiver leurs pratiques, de mettre en évidence les modes d'appréhension par les élèves des situations proposées, de mesurer les apprentissages effectués au cours de ces situations.

Au delà de cette demande précise, se profilait un questionnement

- sur le rôle ainsi que la place des problèmes ouverts et des travaux en petits groupes dans une construction et une structuration du savoir par les élèves qui prennent en compte la diversité des chemins cognitifs de chacun,
- sur la meilleure façon pour le maître de gérer les rapports entre constructions individuelles et construction collective des connaissances.

LE PROBLEME POSE AU CHERCHEUR

Il s'agissait, pour le chercheur, de mettre à la disposition des enseignants les concepts et outils, forgés pour les besoins de la recherche en didactique sous le nom d'ingéniérie didactique, de façon à permettre aux enseignants d'élaborer eux-mêmes des réponses à leurs propres questions en les amenant à

* une explicitation des choix théoriques sous-jacents

* une analyse des situations-problèmes retenues par eux, en ce qui concerne les contenus mathématiques en jeu, les conduites et procédures attendues chez les élèves, leur propre rôle...

* une mise en place d'outils d'enregistrement et d'observation des conduites d'action, de formulation et de validation de élèves au sein de chaque groupe de travail

* une confrontation du déroulement effectif des séances avec les prévisions effectuées avant la réalisation de la séquence

LES CHOIX THEORIQUES SOUS-JACENTS

Un travail préalable a permis une clarification des hypothèses retenues en ce qui concerne la construction de la connaissance mathématiques chez les élèves :

1) accord sur les hypothèses constructivistes et sur la théorie des rééquilibrations : la connaissance se construit dans l'action qui se traduit essentiellement en mathématiques par la résolution de problèmes où les interactions entre sujet et situations-problèmes jouent un rôle fondamental ;
2) existence de diversités de cheminements cognitifs des élèves dans l'acquisition d'un concept mathématique donné et rôle fondamental, dans l'appropriation des connaissances par l'élève, des interactions et des conflits socio-cognitifs entre pairs.

PRESENTATION DE LA SITUATION DE CLASSE RETENUE

La classe concernée étant une classe de 36 élèves, l'emploi du temps prévoyait une séance hebdomadaire dédoublée. C'est cette séance de 1h ½ en demi-classe qui a été utilisée pour l'expérimentation. Au sein de chaque demi-classe, les élèves ont été répartis par groupes de trois ou quatre.

La situation-problème retenue consiste en l'étude d'une page, issue du Journal Municipal de Grenoble, présentant, sous le titre "Où va l'impôt" une représentation graphique de la répartition de l'impôt local (voir dernière page).

La consigne donnée était la suivante :

"vous allez lire une page du journal de la ville de Grenoble, Grenoble Actualité. Il s'agit, non pas de discuter comme le ferait le professeur d'économie, pourquoi si peu à la culture... ? Mais, les chiffres étant ce qu'ils sont, il s'agit de réfléchir sur le dessin, c'est-à-dire sur l'illustration graphique qui a été faite de ces chiffres. Si vous n'êtes pas d'accord avec ceux qui ont fait ce journal, vous cherchez tous les défauts ! A vous de proposer, si vous le souhaitez, une autre illustration. Je vous laisse vous débrouiller : vous êtes des critiques de journaux".

ANALYSE DE LA SITUATION-PROBLEME

Cette situation a été choisie car il s'agit d'une séquence d'enseignement où l'enseignant n'apporte pas de connaissances directement, mais prend en charge la construction et la structuration d'un certain savoir par les élèves eux-mêmes.

Du point de vue cognitif, il s'agit de mobiliser les connaissances antérieures des élèves en ce qui concerne les notions de pourcentage, de croissance et de proportionnalité pour engager une procédure de résolution de problème dans un cadre différent de celui dans lequel ces notions fonctionnaient jusque là : celui des représentations graphiques de phénomènes économiques. Sont de plus en jeu dans cette situation-problème des notions de géométrie (carré, triangle, quadrilatère), la mesure des angles, le calcul d'aires. Enfin la situation se prête à la mise en place de démonstration par contre exemple.

Du point de vue des rapports des élèves à la situation-problème, le problème proposé est un problème ouvert par la diversité des questions et des stratégies possibles et par l'incertitude qui en résulte pour l'élève, un problème suffisamment riche pour que tous les concepts énumérés ci-dessus puissent y être impliqués mais pas trop pour que les élèves puissent en gérer la complexité. La résolution de ce problème doit amener les élèves à produire des actions dans leur recherche de solution, à créer des signifiants, un langage, pour assurer l'échange avec leurs camarades, avec le maître, enfin à prouver leurs affirmations pour convaincre leurs pairs et le maître de leur justesse (phases d'action, de formulation et de validation).

Du point de vue de contrat didactique, le déroulement de la séance dépend essentiellement de l'activité que déploient les groupes d'élèves et des connaissances antérieures qu'ils mobilisent. Le travail est proposé avant tout cours et sans rappel antérieur sur les notions mathématiques en jeu. Chaque élève travaille à son propre rythme. Le maître est dans une position d'assistant, il passe de groupe en groupe, pose des questions, demande des explications, débloque par des interrogations certaines situations...

Ceci crée des modifications importantes en ce qui concerne le contrat didactique :

- pour les élèves : il n'est plus question pour eux d'appliquer des résultats du cours ou de répondre à des questions posées par le maître, il leur faut ici trouver eux-mêmes les questions auxquelles ils doivent répondre pour décider de la correction mathématique du graphique.

- pour le maître : il ne connait pas à l'avance les directions de travail que prendront les élèves, ce qui crée chez lui une incertitude en ce qui concerne le déroulement de la séance et les activités produites par les élèves ; ceci modifie profondément le contrat habituel où c'est lui en général qui guide les élèves en imposant un rythme commun à tous.

Ce choix de situation nécessite par la suite une séance collective d'institutionnalisation dont les objets dépendront du déroulement effectif de la 1ère séance et des divers cheminements mis en place au sein de chaque groupe de travail.

LES INTERROGATIONS DES ENSEIGNANTS

Elles peuvent être regroupées en trois grandes catégories, qui concernent

. la phase d'entrée dans le problème :

Combien de temps faudra-t-il aux élèves pour faire le tour du problème et entrer dans l'action ? Les élèves s'arrêteront-ils à des remarques d'ordre général (sur le centre du carré, la somme de pourcentages...) ou s'intéresseront-ils à la proportionnalité de la figure aux pourcentages ? Porteront-ils alors leur attention aux angles ou aux surfaces ?

. le travail de groupe lui-même :

Aidera-t-il les élèves les plus faibles à comprendre et à résoudre le problème posé ? Permettra-t-il au maître de découvrir des difficultés insoupçonnées chez un élève donné ? Suscitera-t-il l'émergence de questionnements nouveaux chez certains élèves ?

. la mobilisation des notions mathématiques en jeu :

Comment les élèves maîtriseront-ils les notions géométriques en jeu, les problèmes de mesure d'angles, de calcul d'aires ? Utiliseront-ils le mot "proportionnel" ? N'y aura-t-il pas confusion entre proportionnalité et croissance ? Y aura-t-il recherche de contre exemple ?

LES PRINCIPAUX APPORTS DE L'EXPERIMENTATION

L'analyse du déroulement du travail de chaque groupe a été basée sur l'étude des protocoles obtenus par dépouillement des feuilles d'observation et des cassettes enregistrées (un magnétophone par groupe d'élèves avait été utilisé lors du déroulement de la séance). Le déroulement réel a ainsi pu être confronté aux prévisions effectuées antérieurement par les enseignants ainsi qu'à leurs interrogations de départ. Nous en donnerons des exemples précis lors de l'exposé oral. Nous nous contenterons ici de lister les principaux apports repérés par les enseignants eux-mêmes, à la suite des analyses effectuées :

- remise en question de leurs représentations individuelles des élèves, du degré d'attention de ces derniers, du sens qu'ils accordent aux paroles du maître, de leur intérêt pour une activité mathématique...

- prise de conscience :

  . du fait que les élèves déploient une activité effective et soutenue tout au long de la séance, n'hésitent pas à demander aux autres membres du groupe les explications qui leur sont nécessaires et qu'ils n'oseraient jamais demander au maître en situation de classe habituelle, ce qui favorise leur appropriation individuelle des notions en jeu,

  . du rôle de cette activité des élèves dans le travail de transposition, la tâche proposée par l'enseignant étant en fait transformée par les connaissances effectives qu'y investissent les élèves,

  . de l'illusion qui est celle de la plupart des enseignants en ce qui concerne la transparence de certaines formules ou encore l'évidence de certaines propriétés...,

  . du décalage entre le projet didactique de départ et ce qui est effectivement réalisé.

EN GUISE DE CONCLUSION

Six mois après la fin de l'expérimentation, les enseignants précisent comme suit les principaux changements qu'ils estiment observer dans leur propre pratique :

- écoute nouvelle par l'enseignant de ses propres discours
- écoute différente des interventions et questions des élèves, dans le but de déceler ce qui fait, chez chaque élève, obstacle à la construction du savoir en jeu,
- prise en considération des erreurs des élèves non plus comme signe d'incompréhension ou de travail insuffisant mais comme élément à utiliser dans la construction même de la connaissance
- mise en mémoire des difficultés de chaque élève donnant lieu à un essai de suivi individuel parallèle à la conduite collective de la classe.

Comment évoluera dans l'avenir la pratique professionnelle de ces enseignants ? Parviendront-ils à gérer d'une manière plus satisfaisante à leurs yeux la construction collective du savoir tout en prenant davantage en compte la construction individuelle de ce savoir par chaque élève ?

Ne peut-on espérer qu'après s'être trouvés, pour répondre à leurs propres questions, utilisateurs actif de la didactique, ils seront motivés pour une formation plus approfondie en didactique et pourraient alors devenir un relai privilégié entre la recherche en didactique et leurs collègues enseignants ?

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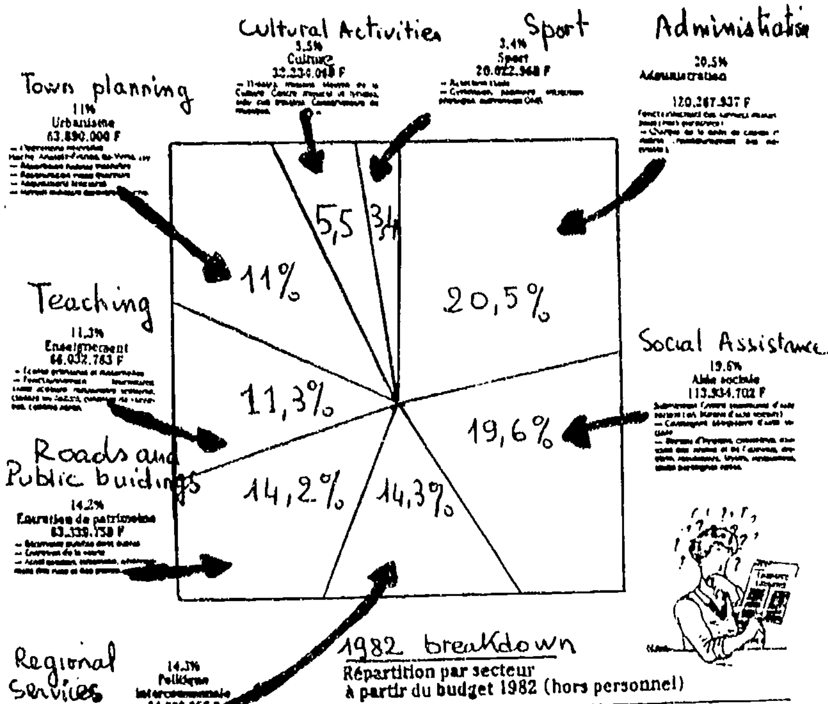


1982 breakdown

Répartition par secteur à partir du budget 1982 (hors personnel)

Budget de la Ville (1982)

Teacher Instructions : "This is a page from a town council newsletter. You are not expected to hold discussions on the municipal budget like the économics teacher might ask you to do. What I want you to do is to look carefully at this chart, i-e, the graphical presentation of these figures. Assuming you don't agree with the people who made up this news sheet, try and find all the mistakes you can and then sugget another way of doing it. Right, off you go. Just imagine that you are newspaper critics."

INGENIERIE DIDACTIQUE A PROPOS D'EQUATIONS DIFFERENTIELLES

M.ARTIGUE UNIVERSITE PARIS 7 - IREM PARIS 7

DIDACTICAL ENGINERY ABOUT DIFFERENTIAL EQUATIONS

The report will be given in English

Abstract · In France, teaching about differential equations for undergraduate students has not been influenced by mathematical and technological developments. As a result, this teaching is, at the present time, obsolete. The research reported here is concerned in its renewal, specially in the viability of a qualitative approach at this level. After an epistemological study, the research was led devising and experimenting a didactical enginery in this conceptual field, using microcomputers. We present this methodology in a fairly detailed way and analyse the first experimental results. They show partial viability but, at the same time, strong difficulties related to qualitative proofs. The conclusions given then concern both this specific didactical enginery and the whole methodology.

I - INTRODUCTION

I-1 L'évolution du champ scientifique

La théorie des équations différentielles s'est développée depuis le 17ème siècle dans plusieurs cadres, au sens défini par R. Douady [1] : le cadre "algébrique" de la résolution exacte, le cadre "numérique" de la résolution approchée, le cadre "qualitatif" enfin qui vise l'étude globale, géométrique du flot des équations.

Il est bien connu que le cadre algébrique a dominé pendant plusieurs siècles ce champ scientifique et que le cadre qualitatif ne s'est développé qu'à partir du début de ce siècle avec les travaux de H. Poincaré. Mais il est clair que la théorie des systèmes dynamiques (du point de vue mathématique), le développement des moyens informatiques (d'un point de vue technologique) ont remodelé profondément, ces vingt dernières années, le paysage de ce champ scientifique au profit des cadres qualitatifs et numériques.

I-2 La stabilité de l'enseignement

L'enseignement supérieur élémentaire (1ères années d'université), en France, n'a pas suivi cette évolution. Il reste centré sur le cadre algébrique : intégration des cas classiques intégrables, développement en série de solutions à l'occasion de l'étude des séries entières ou des séries de Fourier. Le problème didactique auquel nous nous sommes attaquée dans cette recherche est donc celui de l'adaptation d'un enseignement obsolète à l'évolution scientifique et technologique.

II - LES DEBUTS DE LA RECHERCHE

II-1 L'analyse épistémologique

Notre problématique relevait en premier lieu du processus de transposition didactique (Y. Chevallard [2]) : identification, dans le savoir constitué et les pratiques mathématiciennes, des objets et outils au coeur du champ conceptuel considéré et mise en forme de ces objets et outils pour l'enseignement, en fonction du public visé. Cette phase épistémologique fut menée au début des années 1980 avec le concours d'experts mathématiciens (M. Artigue, V. Gautheron [3]).

II-2 L'expérimentation empirique

Cette première phase fut suivie d'expérimentations empiriques ayant pour objectif d'explorer la viabilité d'un tel enseignement. Les difficultés rencontrées dans cette seconde phase nous convainquirent d'une telle viabilité n'était pas évidente, du fait de l'interaction de différentes contraintes : contraintes d'ordre cognitif mais aussi contraintes liées à la gestion du temps, aux coutûmes didactiques et aux représentations des enseignants. Mais il nous apparût aussi qu'une utilisation adéquate de l'outil informatique pouvait modifier suffisamment certaines de ces contraintes pour autoriser un autre point d'équilibre que celui de l'enseignement traditionnel.

C'est cette hypothèse que nous avons cherché à tester dans la troisième phase de la recherche que nous présentons ici. Ceci nécessitait un cadre expérimental qui n'évacue par la complexité du système didactique ; d'où le choix effectué d'une méthodologie d'ingénierie didactique, se situant dans le cadre théorique présenté ici même par R. Douady [4].

III - L'INGENIERIE DIDACTIQUE

III-1 Les choix didactiques

Vue notre problématique : adaptation d'un objet d'enseignement obsolète au développement scientifique et technologique, deux points nous ont semblé fondamentaux :

a) concevoir dès le premier contact avec la notion un enseignement dans les différents cadres répertoriés et organiser entre ces cadres un réseau relationnel.

b) se situer dans une problématique générale de production, prévision et contrôle de tracés de solutions sur micro-ordinateurs.

* Pour prendre en compte le point a) nous avons élaboré un enchaînement de situations didactiques favorisant les jeux de cadres, notamment l'interaction algébrique/qualitatif.

* Pour prendre en compte le point b), nous avons eu recours à une utilisation interactive et une utilisation differée de l'outil informatique, l'utilisation différée, c'est à dire le travail sur des tracés fournis, étant privilégiée, dans les phases de justification et de contrôle, compte-tenu

des contraintes de temps et de matériel.
Il faut souligner que l'outil informatique a été aussi utilisé pour découper et réduire la complexité des tâches. En effet, une étude qualitative d'équation différentielle "à la main" suppose le tracé d'isoclines, le régionement du plan suivant le signe de y'....Les expérimentations de la phase 2 avaient montré que les capacités limitées des étudiants débutants dans le domaine graphique constituaient un obstacle à la viabilité des approches qualitatives à ce niveau. L'utilisation de l'outil informatique nous a permis de dissocier les différentes étapes de l'approche, donc de modifier sensiblement ce type de contrainte.

III-2 L'interaction qualitatif/algébrique dans l'ingénierie

Les études préliminaires laissaient prévoir que le point sensible de l'ingénierie didactique concernerait l'approche qualitative et ses relations avec l'approche algébrique. C'est donc cette partie que nous avons choisi de présenter ici. Elle est organisée en quatre phases :

- *phase 1* : introduction à la problématique de l'approche qualitative ; mise en place d'outils qualitatifs élémentaires (isoclines, régionement associé au sens de variation des solutions, invariances par symétries, translations) ; interprétation dans le cadre algébrique.

Situation support : tracé de champs de tangentes pour des équations très simples ; exploitation pour le tracé de solutions ; association entre des tracés de champs fournis et des équations données.

- *phase 2* : exploitation des notions et relations introduites ; institutionnalisation.

Situation support : association de tracés de familles de solutions fournis et des équations données.

- *phase 3* : comparaison des moyens d'action respectifs des approches qualitatives et algébriques.

Situation support . prévision de l'allure des solutions de l'équation intégrable $y'=(x-2)(y^*y-1)$.

- *phase 4* : notions et théorèmes fondamentaux de l'approche qualitative élémentaire (barrières, zones pièges, entonnoirs et anti-entonnoirs).

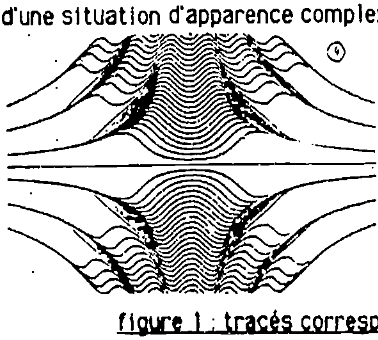
Situations support : justifications puis prévisions de tracés.

Du point de vue de la gestion, les phases de recherche sont prévues sous forme de travail en petits groupes, et ce pour prendre en compte au niveau expérimental les hypothèses théoriques concernant le fait collectif.

III-3 . Un exemple d'analyse a priori de situation didactique

Nous voudrions, à la suite de cette présentation d'ensemble, illustrer par l'analyse a priori d'une situation la méthodologie didactique de l'ingénierie. En effet, un élément clef de cette méthodologie est l'analyse a priori des situations didactiques, précisant et hiérarchisant le champ des possibles. C'est par rapport à cette analyse que sont ensuite analysées et interprétées les observations, dans un processus qui vise, à travers la validation de l'analyse a priori, la validation indirecte des hypothèses qui la fondent.

Nous avons choisi la phase 2 du processus.
La situation problème support de cette situation didactique est la suivante : on fournit aux groupes d'étudiants n tracés de familles de solutions d'équations différentielles et une liste de p équations Il s'agit pour eux d'associer équations et tracés en justifiant par écrit les choix réalisés. Après une phase de recherche par groupes, les productions sont regroupées et analysées et utilisées pour l'institutionnalisation prévue.
L'analyse a priori de la situation prévoit :
- la détermination des variables didactiques de la situation (nombre de tracés, d'équations, complexités respectives, caractéristiques....), les choix effectués et leur motivation ;
- la détermination des procédures de résolution possibles, une estimation de leurs coûts respectifs, les procédures attendues, correctes ou erronées ;
- des prévisions sur les dynamiques de classe possibles, sur le rôle des interactions élèvés/élèves et enseignant/élèves dans leur évolution.
Pour la situation étudiée ici, résumons en brièvement les données.
Le nombre de tracés est de 8, celui d'équations de 7, deux tracés correspondant à la même équation, avec deux cadrages différents (cf. figure 1 ci-dessous). Ce nombre est un compromis entre des contraintes diverses : rendre la situation d'association réellement problématique, fournir un éventail suffisamment large de tracés et équations pour que les critères d'association visés aient chacun une occasion d'emploi de coût faible, permettre d'autre part aux groupes d'aboutir en un temps raisonnable.
Les caractéristiques de l'ensemble des tracés sont choisies pour rendre difficile des associations par simple analogie de formes (deux cadrages pour une des équations, trois équations présentant des expressions trigonométriques, deux présentant des singularités pour x=1 et x=-1 par exemple) et favoriser des argumentations mathématiques variées.
Enfin la complexité, autorisée par l'utilisation de l'outil informatique, joue aux niveaux précédemment décrits mais aussi au niveau psychologique : intérêt esthétique, satisfaction retirée de la maîtrise d'une situation d'apparence complexe.



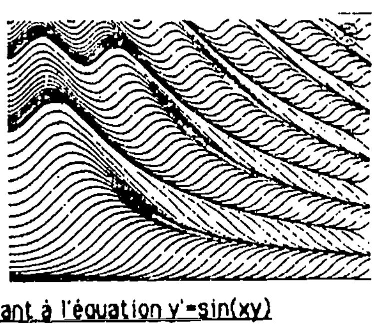


figure 1 : tracés correspondant à l'équation y'=sin(xy)

Les principaux critères d'association retenus dans l'analyse sont les suivants :

cadre qualitatif	*cadre algébrique*
invariance par translation	y' ne dépend pas de x ou de y
invariance par symétrie	y' fonction impaire de x ou de y...
tangentes verticales	y' n'est pas définie pour .....
isoclines horizontales	y'=0 pour ....
tracé ondulant	présence de sinus ou de cosinus
solutions particulières	la fonction .... est solution
sens de variation des solutions	signe de y'

On fait l'hypothèse de l'apparition spontanée de ces différents critères, vu les tracés et équations retenus, chacun étant d'un coût raisonnable pour au moins deux équations.
Du point de vue de la gestion, on fait l'hypothèse qu'une gestion quasi-isolée de l'enseignant est possible pendant la phase de recherche.

IV - LES PREMIERS RESULTATS EXPERIMENTAUX

Cette ingénierie didactique a été expérimentée, en Janvier 1987, dans le cadre d'un enseignement de première année à l'Université de Lille I, avec 90 étudiants répartis en trois groupes pour les travaux dirigés. Elle a occupé quatre semaines d'enseignement c'est à dire 32 heures. Elle avait été préparée par un travail sur les représentations graphiques de fonctions. Pour ce qui concerne l'approche qualitative décrite en III-2, les conditions expérimentales ont été globalement respectées, la phase 4 étant toutefois limitée pour des raisons de temps.

IV-1 : les observations

L'observation des séquences didactiques confirme dans ses grandes lignes l'analyse a priori menée pour les phases 1 et 2, en particulier la viabilité des séquences dans les cadres de gestion décrits. Les critères d'association repértoriés apparaissent, avec une dominante de formulation géométrique. On note également l'utilisation de critères plus locaux comme la valeur ou le signe de y' en des points particuliers, souvent liés à des stratégies consistant à opérer une première classification grossière, puis à trier à l'intérieur des classes ainsi créées. Les solutions particulières sont peu utilisées.
En revanche, des difficultés sous-estimées apparaissent dès la phase 3. Le jeu collectif ne suffit pas à garantir la viabilité quasi-isolée de la phase de prévision et l'enseignant doit d'autre part prendre en charge toutes les justifications concernant le comportement asymptotique des solutions. Au cours de la phase 4, la situation s'améliore concernant les prévisions mais demeure critique au niveau des justifications. Ceci est confirmé par les résultats des étudiants à l'examen passé à la fin de cet

enseignement.

IV-2 : Les résultats de l'examen

Cet examen comportait notamment une résolution algébrique d'équation différentielle linéaire et l'étude qualitative de l'équation $y'=(1/1+x^2)^2-y^2$. On demandait : 1° de régioner le plan suivant le sens de variation des solutions et de tracer l'isocline horizontal ; 2° de tracer sans justification les solutions C0,C1,C2, passant par les points (0,0), (-2,1) et (0,2) (cf.figure 2) ; 3° de montrer que si une solution f définie sur]a,+∞[vérifie : lim f'(x)=-l en +∞ avec l>0, alors lim f(x)=-∞ en +∞ ; 4° de justifier le tracé de C1 sur]-2,+∞[; 5° de déterminer s'il existait une transformation géométrique simple conservant la famille de solutions. Les résultats (cf. tableau 1) sont éloquents.

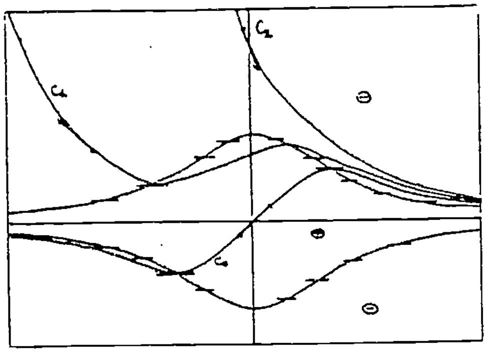


Figure 2 : prévision associée à l'équation de l'examen

	Groupe 1(29)	Groupe 2(30)	Groupe 3(30)	Total (89)
Résolution équation	18	23	21	62
Régionement	26	25	24	75
Isocline horizontale	28	27	27	82
Au moins deux tracés	22	25	18	65
Prévision correcte	13	17	9	39
Limite de f(x)	3	6	0	9
Justifications partielles	1	4	4	9
Comportement en +∞	0	6	0	6
Symétrie famille	14	16	17	47

Tableau 1 : Réussites à l'examen final

Par prévision correcte nous désignons un tracé équivalent à celui de la figure 1. Il faut noter que 27 étudiants seulement ne respectent pas le sens de variation des solutions ou les font se croiser. Par justification partielle nous entendons les justifications concernant les positions respectives de C1 et de l'isocline horizontale. Sont comptés ici les étudiants ayant à peu près réussi deux justifications sur les 4 attendues.

V- CONCLUSION

Au niveau de cette ingénierie particulière :

L'expérimentation menée nous a permis de mieux cerner les problèmes initialement posés, notamment les possibilités offertes par le jeu collectif et leurs limites. Les résultats obtenus tendent à prouver que, sous des contraintes comparables, le point critique concerne non pas toute la phase qualitative, mais la composante justification de cette phase. Elle nous aide également, à travers l'analyse du comportement des étudiants, des enseignants et de leurs relations, à comprendre la nature des difficultés rencontrées. Dans les phases 1,2 et même dans la prévision des tracés, les tracés de fonctions sont gérés suivant les règles usuelles : le tracé est toujours le plus simple possible répondant à un ensemble de contraintes. La problématique de la justification est opposée: il s'agit de prévoir toutes les possibilités compatibles avec les contraintes initiales et donc de mettre en doute les prévisions faites suivant les règles usuelles. Comment instaurer une confiance suffisante dans les tracés pour rendre l'approche qualitative opérationelle et en même temps ne pas sacrifier la rigueur ?

Au niveau général de la méthodologie

Dans la problématique qui était la nôtre ici et au point où nous en étions de la recherche, l'ingénierie didactique nous est apparue comme une méthodologie incontournable pour prendre en compte la complexité du système didactique, notamment au niveau du jeu collectif et de la gestion du temps. Mais il faut souligner que c'est une méthodologie, difficile à mettre en place et à gérer. Elle veut, par exemple, prendre en compte la complexité de la classe de façon contrôlable. Elle contraint de ce fait son fonctionnement dans un cadre qui, par le biais de l'analyse a priori, permette ce contrôle. Mais ce faisant, ne court-elle pas le risque de tuer la vie complexe qu'elle prétend étudier ? Nous pensons que ce sera le cas si l'ingénierie ne ménage pas un espace de liberté suffisant à l'enseignant. Comment alors gérer cette liberté ? Il s'agit enfin d'une méthodologie lourde qui doit pouvoir, pour être rentable, s'appuyer sur et être relayée par d'autres approches notamment épistémologiques et cognitives.

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MATH A AND ITS ACHIEVEMENT TESTING

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Summary

In 1985 a new curriculum in the Netherlands for upper secondary education was introduced aiming at non-mathematical majors.
In this article we analyze this curriculum, discuss its consequences for achievement testing and present results on an explorative study to 'other ways' of achievement testing.

1. Introduction

In August 1985 a new curriculum was introduced in the Netherlands at upper secondary level by means of the Hewet-project: Mathematics A. This curriculum was aiming at students that were preparing for a study at university in psychological, social and economical sciences. A curriculum that was considered by many as a revolution because it broke away from traditions in math-instruction. (See these proceedings Verhage; De Lange; [1]).

> "The most important subjects from an applied point of view are: calculus, linear algebra and geometry, probability and statistics, computers. These topics should be treated extensively. They must be integrated into one coherent course."

This was pronounced by Engel in 1967 and shared by Pollak. [2]
It is remarkable how closely the Hewet report approached what after years of research came out to be that kind and level of mathematics as used as a tool in the 'soft' disciplines.

The Hewet-report recognizes that at first sight the broad program may look confusing, but it argues that the unity of this curriculum is routed in its applications.
In an even stronger way Klamkin [3] claims that one of the reasons why students have difficulty in applications is that most of mathematics is learned 'vertically', that is that its various subjects are taught separately, neglecting the cross-connections. Usually in applications one needs more than just algebra alone or geometry alone.
Consequently, courses should be designed 'horizontally' cutting across several different mathematical branches.
Or, as Hilton stated:

> "We must break down artificial barriers between mathematical topics throughout the student's mathematical education." [4]

According to the Hewet report the unity in the mathematics A program should be aspired at via its applications.

2. Analysis of Math A

Analyzing the experimental material is a very complex task, but one can delineate a rough schema that represents the main aspects of the Math A curriculum as operationalized in the experimental material.

One thing is clear in all materials: the large role played by the context.
The role of the context is two-fold: the start of any sub curriculum takes place in some real world situation. This real world is not restricted to the physical and social world.
Also the 'inner' reality of mathematics or the real world of the students imagination

provides sources for developing mathematical concepts.
The second role for the context is in the applications: they uncover reality as source and domain of application.

The real world situation or problem is explored intuitively in the first place, with the view on mathematizing it. This means organizing and structuring the problem, trying to identify the mathematical aspects of the problem, to discover regularities.
This initial exploration with a strong intuitive component should lead to the development, discovery or (re)invention of mathematical concepts.

As our classroom observations made clear depending on such factors as interaction between students, between students and teachers, the social environment of the student, the ability to formalize and to abstract, the students will sooner or later extract the mathematical concepts from the real situation. This phase we like to refer as *conceptual mathematization*.
At the same time reflexion on the process of mathematization is essential.
The next phase recognizable in the material is the description of the desired and resulting mathematical concepts, followed by a more strict and formal definition.

We can put it another way:
In the first stage of mathematization we develop our tools, and after formalization we use them in the second stage. And by applying the concepts to new problems one of the main results is reinforcement of the concepts and developing mathematization skills.
Finally, at the other hand problems solved will influence the student's view on the real world.

3. Mathematization in Math A

Mathematizing is an organizing and structuring *activity* according to which acquired knowledge and skills are used to discover unknown regularities, relations and structures.
We may distinguish two components in mathematization, according to Treffers and Goffree [5]: the horizontal and vertical components.
First we can identify that part of mathematization that is aiming at transferring the problem to a mathematically stated problem. Via schematizing and visualizing we try to discover regularities and relations, for which it is necessary to identify the specific mathematics in a general context.
Activities with a strong horizontal component are:
- identifying the specific mathematics in a general context;
- schematizing;
- formulating and visualizing a problem in different ways;
- discovering relations;
- discovering regularities;
- recognizing isomorphic aspects in different problems;
- transferring a real world problem to a mathematical problem;
- transferring a real world problem to a known mathematical model.

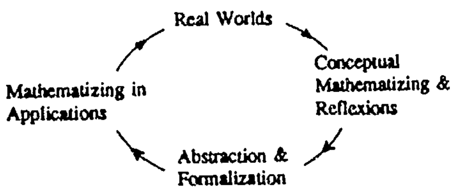
As soon as the problem has been transferred to a more or less mathematical problem this problem can be attacked and treated with mathematical tools: the mathematical processing and refurbishing of the real world problem, transformed into mathematics.
Some activities that have a strong vertical component as:
- representing a relation in a formula;
- proving regularities;
- refining and adjusting models;

- using different models;
- combining and integrating models;
- formulating a new mathematical concept;
- generalizing.

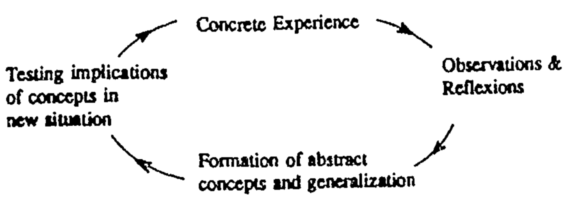
Generalizing may be seen as the top level of vertical mathematization. We mean, with Hilton, that when we are reasoning within the mathematical model we may feel compelled to construct a new mathematical model which embeds our original model in a more abstract conceptual way. [6]
Mathematization always goes together with reflexion. This reflexion has to take place in all phases of mathematization. The student has to reflect on his personal process of mathematization, discuss his activities with other students, has to evaluate the product of his mathematization, and to interpret the result.
Horizontal and vertical mathematizing comes about by students *actions* and their *reflexions* on their actions.

4. The learning cycle

The learning cycle for Mathematics A may be described in the following way:



In this way the learning cycle shows a remarkable similarity with the Experiental Learning Model of Lewin. [7]



Two aspects of this learning model are particularly noteworthy:
First its emphasis on concrete experience to validate and test abstract concepts: in mathematics A this is the phase of applied mathematization in the problem solving process.
Second, the feedback principle in the process. Lewin used the concept of feedback to describe a social learning and problem solving process that generates valid information to assess deviations from desired goals.
It is clear that we can find similar abilities in mathematics A. The weakest link in the cycle seems to be the active experimentation. Students do work with real world problems. But it seems worthwhile to consider measures to improve this link. One way to achieve this seems to have students make more productions - not only mental contributions. As we will point out later, when describing the results and products of alternative tasks, this production seems to have a very beneficial aspect on the learning process. This point is also stressed by Treffers. [8]

He stresses the fact that by producing simple, moderate, complex problems the student reflects on the path he himself has taken in his learning process, and the same time anticipates its continuation.

We conclude this part of our framework with Kolb's definition of learning:

> Learning is the process whereby knowledge is created through the transformation of experience.

For a description of a framework for instruction theory we refer to the contribution of Gravemeijer. (These proceedings [9]).

5. Achievement Testing

During the teacher training courses it became clear that testing Math A involved many problems.

Typical questions from teachers:

- How to prepare students properly for the exam?
- How to know what to test?
  (What goals should I try to operationalize?)
- Should the role of the test differ from the traditional one?

This last question arose when teachers found out that they needed the test as an integral part of the learning process. Rather than 'sampling' marks to give a judgement at year's end, the teachers should see to it that the students learned through the tests as did the teacher. Achievement testing should be a learning aid, as formulated by Gronlund. [10]

On the other hand the Hewet team posed themselves the question:

- Are our goals for Math A manifest in the experimental textbooks?
- Knowing the limitations of the restricted-time written tests like those at the exam how do we prevent this exam from dictating the program (and the intermediate tests)?

A good answer on this last question is essential for the survival of Math A if the original intentions of the curriculum are to be met.

Our research shows that from the tests designed by teachers (restricted-time written tests) roughly 80% of the exercises looks more or less alike the exercises in the student-booklets. That means that only 20% of the exercises try to test higher (process-oriented) goals of Math A.

From our analysis of tests of twelve schools it is clear that testing tends to emphasize the 'lower' behaviour levels, such as computation and comprehension. This is not a specific Math A problem. As Wilson states:

> "Mathematics teachers often state their goals of instruction to include all cognitive levels. They want their students to be able to solve problems creatively. But too much of their testing consists only of recall of definition, facts and symbolism." [11]

But in Math A the problem is even more serious because of its quite specific goals.

Mathematization, Reflexion, Inventivity and Creativity are essential activities in Math A which are hard to be tested in the restricted-time written test.

The teachers in the experimental schools were hindered in several ways. In the first place the goals of Math A had not been clearly stated, which made teachers rely heavily on the textbook materials.

In the second place no good exercises outside the booklets were available and only few teachers created 'new' exercises as we noticed.

In the third place the teachers were under heavy time pressure as there were hardly any provisions for teachers' participation in the experiment: preparations for teaching Math A took much time, compared with the old program.

In the fourth place the tradition of testing mathematical skills in Dutch schools did not inspire teachers to get involved in tests other than those similar to the textbooks.

During the experiment the goals of Math A showed up more clearly, as did the problem of assessment testing. The demand of the Cockcroft report that assessment (at 16+) is to reflect as many aspects as possible, including those which need other means of assessment than restricted-time written tests, was taken seriously by the Hewet team as well as by some teachers. [12]
The team that carried out the experiments, aided by teachers, tried to counter the problems as follows.

In the first place an attempt was made to clarify the goals of Math A.
Secondly, and of more direct help to the teachers, exercises were sampled and distributed in order to confront teachers with new exercises. The effect was positive as well as negative: the pressure on the teachers was softened, but at the same time teachers fell back on the new exercises rather than creating their own ones.
Thirdly a discussion with the teachers was initiated about validity and limitations of written timed test, eventually resulting in the development of alternative tasks.

6. Alternative Tasks

The alternative tasks were developed with the following principles in mind:
1. Tests should improve learning.
2. Tests should allow the candidates to show what they know (positive testing).
3. Tests should operationalize the goals of the Math A curriculum.
4. Test-quality is not in the first place measured by the accessibility to objective scoring.
5. Tests should fit into the usual school practice.

Alternative tasks that were developed were:
- take home task;
- two-stage task;
- oral task;
- essay task.

Our research led to the following conclusions:
1. Girls perform less than boys in restricted-time written tests.
2. Girls perform more or less the same as boys on oral tasks or take-home tasks.
3. From the above one is tempted to advise more oral and take-home tasks in order to offer girls fairer chances.
4. Oral tests results have a somewhat higher correlation with restricted-time written test results than take-home tasks.
5. Students perform best with take-home tasks. The constructive and productive aspect seems to offer students a fair chance to show their abilities (creativity, reflexion, etc.). Positive testing is at its maximum in this way.

As we have pointed out we started our explorative study to alternative tasks because of the fact that restricted-time written tests as carried out by the teachers did not meet the intentions and goals of Math A. Not only did the teachers stay very close to the exercises in the book, but when they did not they ran into trouble because of time-restrictions encountered with those timed tests.

Mathematics A is strongly process-oriented; the mathematization process needs time to develop, time to reflect, time to generate creative and constructive thoughts.
Those 'higher' goals are not easily operationalized with timed-tests.

On the other hand it is our opinion that the tests or tasks should play a more constructive and productive role in the learning process. Especially formative tests are well suited to improve the learning process.
Furthermore we should try to offer the students ample opportunities to show their abilities. In timed tests we usually notice negative testing.

In our efforts to find other ways of assessment testing we should not be hindered by the strict rules of objective or even mechanic scoring. Too often the influence of those rules has a very negative effect on the way of testing.

Finally teachers should be given the opportunity to carry out those task without disrupting schoolpractice too much; such tasks should be developed by a central institution with help of teachers.

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THE HEWET PROJECT

The psychological aspects of a large scale innovation project.

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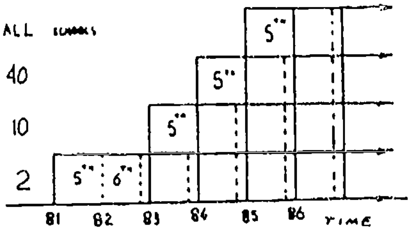
In the Netherlands recently a new math curriculum based on the philosophy of realistic math education has been introduced at upper secondary level. Real life situations are not only used for doing applications, but also for developping new mathematical concepts.
One of the main problems of the implementation of the new syllabus was how to change the attitude of the teachers, because the teachers had to change their image of mathematics.

Recently the mathematics curriculum of upper secondary level (age group 16-18, fifth and sixth grade) in the Netherlands has been changed in a radical way. At this level math is optional and the students can choose between two different courses: math A and math B. There was a growing need for a math course for non-science students at pre-university level. For this reason the new math A course has been developed.
The math A curriculum is based on the philosophy of the so called Realistic Math Education. Some characteristics of math A are:

— Modelling and applications;
— Introduction of new math concepts by real-life situations;
— Horizontal structure, unified approach;
— Integrated use of computers;
— Collaboration and interactivity;
— No unique right answers.

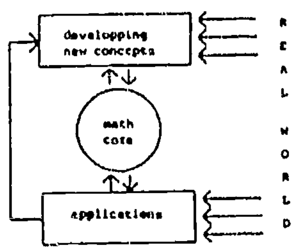
The new syllabus has been developed and tried out in an experimental setting during five years: the Hewet project. The innovation started in 1981, when two schools started experiments with the new materials. These materials were developed by a small team of math educators of the research group OW & OC of Utrecht University. During the next five years step by step the implementation of the new course took place. During the experiments all lessons were observed by the team members. Discussions with students and teachers led to adjustment of the materials. After two years the experiment spread to another ten schools; these were joined by another 40 the next year. In 1985 all schools started with the new math curriculum and in 1987 the first final examinations will take place all over the country.

Basically we have the following schema of the project:



Realistic math education.

In the Netherlands math education is developping in the direction of *realistic math education*. Real life situations are used both for developping new concepts and for doing applications.
The relation between the world around us and math education according to the philosophy of realistic math education can be schematized as follows:



There is an important difference between *realistic* math and *applied* math. In the latter case math is used to do some (real life) applications in the final stage of the learning process, but it is not an aim to use real world situations while developping new concepts. The sophisticated use of real life situations (contexts) is one of the characteristics of a realistic math curriculum. An other aspect can be found in the way in which the students are involved in their learning process. They make large contributions to the course by their own productions and constructions.

The content of the math A syllabus.

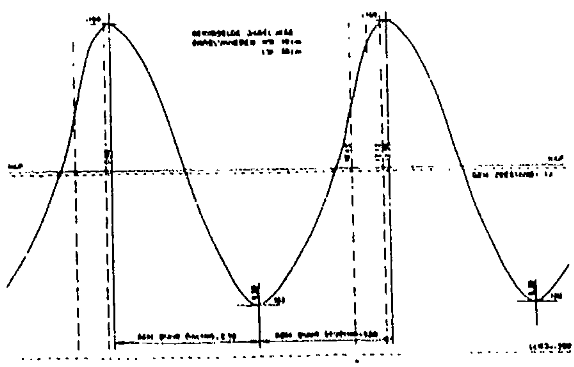
Very few among the students who take the math A course will become professional mathematicians. Many of them, however, will specialize in economics, social sciences, medicine and will use mathematics as a tool. This means that at any time the usefulness of mathematics should be preponderant.
The three main streams of the syllabus are:

— applied calculus
— matrix algebra and linear programming
— probability and statistics

The use of the computer has been integrated in this subjects.

Example 1
Very important within math A is the activity of mathematising and modelling. This is a complex and difficult matter and offers lots of discussions.
The following example of this process has been taken from the subject *periodic functions*, part of the applied calculus.

The yearly average tide-graph of a coastal town in the Netherlands (Vlissingen) is given by this graph:



Gemiddelde getijkromme te Vlissingen

The question: Find a simple (goniometric) model to describe the tidal movement.
Initially three rather different models were found by the students (17 years of age):

$$f(x)=2\sin\frac{1}{2}x$$

$$g(x)=190\sin\frac{1}{2}x+8$$

$$h(x)=190\sin\frac{\pi}{6.2}x$$

Of course a lively discussion was the result:

f(x), that was clear was a very rough model: the amplitude was 'more or less equal to two meters' and the period was 4π or 12.56 which is not 'far away' from 12 hours 25 minutes.

g(x), as the girl explained, was better in respect to the amplitude: the amplitude of 190 cm, together with a vertical translation of 8 cm gave exactly the proper high and low tides, which was very relevant to her.

h(x), was more precise about the period. This boy considered the period more relevant 'because you have to know when it is high tide.' The period proposed by this model was 12 hours and 24 minutes, which really is very close.

After a long discussion it was agreed that:

$$k(x)=190\sin\frac{\pi}{6.2}x+8$$

was a nice model although some students still wanted to make the period more precise.

The goniometric functions are embedded in the more general periodic functions and used to model real life situations. From the experiments we got the impression that this makes the subject much more motivating to the students.

Example 2
As mentioned before, new math concepts are introduced by real-life situations. This can be illustrated by the following example, taken from the booklet 'Matrices'. The mathematical concept of the multiplication of two matrices is introduced by means of the context of a jeansshop.

5

MULTIPLICATION (2)

In the previous chapter we multiplied in a usefull way by means of the jeansshop a 4 x 5 matrix by a 5 x 1 matrix:
his can be summerized as follows:

W L CF Bo Ba $\xleftarrow{\text{'cancelled out'}}$ W L CF Bo Ba

$$\begin{array}{c} \\ 28" \\ 30" \\ 32" \\ 34" \end{array}\begin{array}{c} \begin{array}{ccccc} W & L & CF & Bo & Ba \end{array} \\ \begin{pmatrix} 1 & 3 & 0 & 1 & 2 \\ 5 & 8 & 6 & 1 & 2 \\ 2 & 3 & 5 & 6 & 0 \\ 0 & 1 & 1 & 0 & 3 \end{pmatrix} \end{array} \times \begin{array}{c} W \\ L \\ CF \\ Bo \\ Ba \end{array}\begin{array}{c} \text{profit} \\ \begin{pmatrix} 30 \\ 35 \\ 40 \\ 25 \\ 40 \end{pmatrix} \end{array} = \begin{array}{c} \\ 28" \\ 30" \\ 32" \\ 34" \end{array}\begin{array}{c} \text{profit} \\ \begin{pmatrix} \ldots \\ \ldots \\ \ldots \\ \ldots \end{pmatrix} \end{array}$$

$4 \times \boxed{5 \qquad 5} \times 1 \qquad 4 \times 1$

This way of multiplying can be extended to e.g. a 4 x 5 matrix by a 5 x 3 matrix in the following way:

$$\begin{array}{c} \\ 28" \\ 30" \\ 32" \\ 34" \end{array}\begin{array}{c} \begin{array}{ccccc} W & L & CF & Bo & Ba \end{array} \\ \begin{pmatrix} 1 & 3 & 0 & 1 & 2 \\ 5 & 8 & 6 & 1 & 2 \\ 2 & 3 & 5 & 6 & 0 \\ 0 & 1 & 1 & 0 & 3 \end{pmatrix} \end{array} \times \begin{array}{c} \\ W \\ L \\ CF \\ Bo \\ Ba \end{array}\begin{array}{c} \begin{array}{ccc} \text{profit} & \text{cost price} & \text{selling price} \end{array} \\ \begin{pmatrix} 30 & 45 & 75 \\ 35 & 40 & 75 \\ 40 & 40 & 80 \\ 25 & 40 & 65 \\ 40 & 45 & 85 \end{pmatrix} \end{array} = \begin{array}{c} \\ 28" \\ 30" \\ 32" \\ 34" \end{array}\begin{array}{c} \begin{array}{ccc} \text{profit} & \text{cost price} & \text{selling price} \end{array} \\ \begin{pmatrix} .. & .. & .. \\ .. & .. & .. \\ .. & .. & .. \\ .. & .. & .. \end{pmatrix} \end{array}$$

'cancelled out'

$4 \times \boxed{5 \qquad 5} \times 3 \qquad 4 \times 3$

➤ 77. Compute the total matrix: profit, costprice and selling price per size.

➤ 78. On what condition for the orders can you multiply matrices in this way?

The result of the multiplication is a matrix with the profit, costprice and selling price per size. Within the context of the jeansshop this way of multiplying matrices is very natural and meaningful.
A context meant to introduce a new concept, has to be chosen very carefully. Finally

the mathematical concepts and the formal methods have to be clear without the contexts, but a well chosen context makes it possible to the students to reconstruct the concept again themselves if necessary.

The innovation strategy.

Text-books is a variable that we - as math educators - can influence easily. This is supported by the fact that the commercially developed text books are very close to the materials developed during the experiments.

But a new curriculum cannot be implemented by new text books only. A very important but less easy to influence variable is the attitude of the math teacher. For many teachers the new math A course did not fit in with their image of mathematics, at least in the beginning. Math A appears to be intuitive, realistic and subjective instead of axiomatic, formal and objective.

So how to overcome the resistance the teachers will probably have? The teachers have to be convinced that although math A is different, it is still mathematics. It appeared to be very important to keep the distance between the Hewet-team and the teachers as small as possible, so it was a matter of great concern how to keep in touch with them. At the 2-school level this was no problem: all lessons were observed. For the 10 schools the distance was close as well: the teachers of this schools followed an inservice teacher training course organised by the team members. The teachers of the remaining schools (40 + 430) followed a course organised by the regular teacher trainers. Generally spoken, during this courses the attitude of the teachers changed from waiting to positive.

A lot of articles about the progress of the project and the classroom experiences were published. It appeared that written information only was not enough to inform the teachers well, it was necessary to have personal contact with as many teachers as possible. For this reason a number of activities like hearings, conferences and workshops were organised by the team. Because we have a small country, it was possible indeed to have personal contact with teachers of most of the schools.

Student reactions.

Another variable of importance is the student. How did they react on the new curriculum? In the beginning they had partly the same problems as the teachers, for example with the characteristic 'no unique right answer'. But very soon they were used to these changes.

During an inservice teacher training in the second year of the experiments teachers interviewed a panel of students of the first two schools. This discussion was videotaped.

The following was said about the use of contexts and mathematization:

> Teacher: "Math A contains many more of those story-sums, compared with how it used to be. With story-sums you often get a long text to read, and than you have to do something with it. We often hear the complaint that, although it's fun, you have to read lots of those stories before you get enough practice. I've noticed that none of you have mentioned that - have you had enough of that to be able to say 'I understand that section' ?"

> Marius: "Well, it's not true that there are pages and pages of text, there's just a short introductory

text. It's not true that reading takes so much time that we don't get around to the theory."

Judith: "With normal math you first got the theory and then the sums. Here you usually begin directly with a sum and a story, but the first sum is usually pretty easy. Then they build it up without you noticing so that the difficult things emerge naturally."

Annemarie: "The way to do it, if you know that - the method - then you can do it if you ask me."

Jan: "First of all, it's great that you don't have to do one sum after another, it's kind of relaxing if you get a story once in a while, then you're not working so intensively all the time. It's less tiring. When you get those stories all the time you get practice in searching out the essential bits."

Wim (the teacher of the students): "If you look at this morning's sum, with the proceeds and prices, do you find it difficult because it's a story or do you say that's not so bad?"

Jan: "It's not so bad at all. Maybe it's kind of confusing now and then, but I think it's a good idea to learn to separate the main issues from the side issues in those stories.
(...)

Laura: "It's because of those stories that it's clear to me what the idea is, what it's about. In my case the lack of anything being told was why I didn't understand it. I didn't know what it was about and had to learn it by heart. Here there's a story around it and it just happens as a matter of course."

Marius: "It's just a lot more appealing. For instance, in tenth grade we had logarithms. You learned it, but I couldn't follow it very well in class because it just didn't interest me, while now it does. Logarithms become much more clear through the examples."

Later on it appeared that the questions of this teachers as well as the reactions of the students were paradigmatic for nearly all teachers and all students.

Conclusion.

Altogether the Hewet project appeared to be rather successful. Due to math A the math courses are more popular now: in the new situation about 85 to 90% of the students chooses math, in the old situation this was 72%. Especially the percentage of girls choosing math has risen considerably.
But two aspects of the implementation of the new curriculum has been underestimated:
- the problem of changing the attitude of the teachers;
- the problem of developping achievement tests: how can the higher goals of math A be tested?
During the five years of the project the team members learned a lot about these two problems.
At the moment two other large scale innovation projects are started. The are both based on the same philosophy of realistic math education. The experiences of the Hewet project will be used is these new projects.

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THE IMPLEMENTATION OF REALISTIC MATHEMATICS CURRICULA.

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abstract. About 50% of the Dutch primary schools is implementing realistic mathematics curricula. This means a far-reaching change, because the teachers were used to mechanistic mathematics instruction. On the basis of the characteristics of realistic mathematics instruction is shown that the change to realistic mathematics instruction implies a change in the role and beliefs of the teachers. A framework of an in-service program is presented. This program takes into account, both the development of the levels of use of the innovation and the necessary change in teacher beliefs.

introduction

The launching of the Russian Sputnik invoked, as we all know, the New Math-movement in the US, which spread all over the world. This New Math-wave however never reached the Netherlands. This was mainly the result of the dikes built against this wave by the Dutch Wiskobas-group. This Wiskobas-group also saw the necessity of the development of an alternative for the proposed renewal of the math curriculum.
The research and development started in the early seventies and was carried out at the IOWO (Institute for the Developement of Mathematics Education). This resulted in the so called Wiskobas program for the primary school, which, a decade later, highly influenced the modern textbooks in mathematics education. Now there are several programs of a 'realistic' signature commercially available. These new programs are already being implemented in about 50% of the primary schools.
But these schools were used to completely different programs. Since the start of the IOWO more than 15 years passed and the schools sticked to their programs during this period. However, the way the traditional textbooks were used changed over time: it went towards a so called mechanistic approach. So there is a sharp discrepancy between the traditional math education the teachers are used to and the new math curriculum now implemented.
This indicates the need of an in-service training. Before discussing the in-service training, we will first take a closer look at the innovation to get a beter idea of the consequences.

characteristics of realistic mathematics education

The Wiskobas program is based on Freudenthal's view on mathematics and mathematics education. Freudenthal (1970) distinguishes the idea of mathematics as a human activity from the image of mathematics as a complete ready-made system. For mathematicians mathematics is an activity and the main part of it is organizing.

> "This can be a matter from reality which has to be organized according to mathematical patterns if problems from reality have to be solved. It can also be a mathematical matter, new or of results, of your own or of others, which have to be organized according to new ideas, to be better understood, in a broader context, or by an axiomatic approach." (Freudenthal;1970;414)

Treffers (1987) shows that it is useful to distinguish horizontal and vertical mathematisation in order to account for the difference between transforming a problem field into a mathematical problem on one hand, and processing within the mathematical system on the other hand. With the help of this distinction it is possible to describe four trends in mathematics education, according to the extent that the horizontal and vertical aspects of mathematising are present.

Table 1. Trends in Mathematics Education

trend	aspects of mathematizing	
	horiz.	vert.
realistic	+	+
structuraliatic	-	+
empiricist	+	-
mechanistic	-	-

The mechanistic approach appears to be the opposite of the realistic approach, it is characterised by the weakness of both the horizontal and the vertical component.

In realistic mathematics instruction both the horizontal and the vertical component of mathematising are used to shape the process of progressive mathematisation. How these components are combined is described by the five tenets of the process of progressive mathematising (Treffers; 1987, Treffers & Goffree; 1985):

1. *phenomenological exploration:* The real phenomena from which the mathematical concepts and structures arise are explored to acquire a rich collection of intuitive notions. In this way the essential aspects of concepts and structures are constituted. This, then, is laying the basis for concept formation.

2. *bridging by vertical instruments:* A variety of 'vertical' instruments

such as models, schemas, diagrams and symbols are offered, explored and developed. This is done to bridge level-difference between the context bound operatingon the first Van Hiele level and the reflective, formal systematic one on the third Van Hiele level.

3. *selfreliance: pupils' own constructions and productions:* The children make an active contribution to the course by their own productions and constructions. The individual solutions to contextproblems, notation patterns, short cuts etc. determine the progression in the learning process.

4. *interactivity:* The pupils informal methods are used as a lever to attain the formal ones. Such a teaching method requires a specific didactical shaping of interactive instruction, in which individual work is combined with consulting fellow students, etc.

5. *intertwining:* The contextbound introduction of mathematical concepts and structures implies an intertwining of related learning strands, because of the interrelatedness of different mathematical domains in reality.

From the characteristics described above emerges a constructivistic conception of mathematics instruction. Treffers (1987) shows that the Wiskobas program fits into a global framework for instruction theory, built by the level-theory of Van Hiele (1973) and the didactical phenomenology of Freudenthal (1983). This global theoretical framework corroborates the impression of a constructivistic view. We stress this constructivism because it has important consequences for the role of the teacher in the learning process.

The main point is the active contribution of the pupil to his own learning process. The teacher has to support the developement of mathematical concepts and structures in the above described interactive instruction. This means that the teacher has to be able to anticipate on the solutions and constructions of the pupils. He has to be able to evaluate them; to give the kind of hints etc., that will help the pupils to elaborate their own findings.

It is impossible to embody the required teaching strategies in teacher guides. So the role of the teacher is much more complicated and much more important, than in mechanistic mathematics instruction.

the beliefs of math teachers

The research of Thompson (1984) under math teachers shows a remarkable agreement between the teachers' views on mathematics education and their instructional practices. And these views appear to fit into the classification of four tenets in mathematics instruction, which Treffers found in textbooks.

Following this line of thought we may expect to find a lot of teachers with a mechanistic view on mathematics instruction, who are working with realistic textbooks. According to the findings of Thompson we may presume that these teachers will use the realistic textbooks in a mechanistic wey. So we meet the question of how to bring these teachers to ways of use that will be in harmony with the ideas underlying the textbooks.

It will be quite clear that this change will not be restricted to external charecteristics. The ideas of the teachers have to change as well. Fullan (1983) indicates that there are at least three dimensions at stake in implementing any new program. They involve the possible alternation or use of new:

- *materials:* the use of new or revised instructional materials or technologies;
- *teaching approaches:* new strategies, activities, practices, etc. engaged in by the teacher.
- *beliefs:* pedagogical assumptions and theories underlying new policies or programs.

Fullan stresses the importance of the change in beliefs:

> "... a teacher could use new materials, alter some teaching practices or behaviors without coming to grips with the conceptions or beliefs underlying the change." (Fullan; 1983;454)

The importance of coming to grips with the beliefs underlying the change is supported by Leithwood's analysis of curriculum dimensions. Leithwood (1981) indicates that the apropriateness of teaching strategies depends on a number of situational factors. This means that the teacher has to make his own decisions on the spot.

These decisions will be based on the beliefs of the teacher. Fullan & Pomfret (1977) distinguish, among other things, the variables 'knowledge and understanding of the renewal' and 'value internalization'. The first variable describes a rather technical knowledge of the renewal, while the second is connected with accepting the goals of the change. In our opinion the concept 'beliefs' goes deeper. It includes convictions; views on teaching and learning, views on the essence of the subject of the course, etc. In the end it will be these beliefs that will determine the way the teacher arranges his instruction.

It is clear that it will not be easy to change the beliefs of the teachers. It is not a simple matter of telling them what it's all about. So we have to follow Fullan in his conclusion that

> "... educational change along three dimensions - in materials, teaching approaches, beliefs - involves changes in what *people do and think* and as such, it represents a complex adult learning experience." (Fullan;1983,455)

The next question is how to support this learning process of the teacher in such a way that there is an optimal chance for the desired change. In doing so we will have to consider the fundamental difference between pre-service and in-service teachertraining. In the case of in-service training we are dealing with experienced teachers. They want to be aknowledged in their expertise and their responsability for their own teaching. This implies some specific constraints for the form and content of the in-service program.

stages in the process of implementing a new curriculum

If we want to establish the learning process indicated above, we will have to take into account how the process of implementing a new curriculum develops over time. Fullan (1984) takes up the position that - in contrast with the usual idea of adoption as a solitary decision - adoption has to be seen as a prolonged process of choosing. The longer the new program is in use, the more the teacher knows and the better the teacher can see what he is choosing for (or does not want to choose). Now the process of implementing a new curriculum can be seen as a process of learning and choosing. This corresponds with the findings in research that show that there are several levels of use of a new curriculum, which Hall & Loucks (1977) label as: *non-use; orientation; preparation; mechanical use; routine; refinement; integration;* and *renewal.*

Research, like the research of Hall & Loucks (1981) and from Van den Berg & Vandenberghe (1981), shows that these levels of use can be seen as different stages in the implementation proces.

In the first stage when the curriculum is rather unknown to the teacher, the teacher will follow the teacher guide rather mechanical, without understanding the meaning of activities completely. After some time the familiarity with the curriculum grows, which gives the teacher the possibility to change small parts of the curriculum to adjust them to the actual situation (routine). A growing understanding of the concepts underlying the practices of the curriculum will enable the teacher to come to a more flexible use of the curriculum. In the end this will lead to adaptation of the curriculum on basis of the gained insight and experience (renewal).

This sequence is, of course, an ideal one. Not all teachers start at the same level, and there are differences in the development towards the higher levels of use. But, nevertheless, the research evidence still reveals a learning process of the teacher. It also shows that this learning process needs some support.

guiding the learning process of the teacher

The in-service training has to level out with the stage of implementation of the new curriculum. The focus will be on the use of the new materials in the first phase. In the following phase a change of teaching strategies is at stake. The teachers will have to be informed about the teaching strategies and their theoretical background. These teaching strategies are at last legitimated by the views on learning and instruction.

In the case of a change from mechanistic mathematics education to realistic mathematics education the main piont is the opposition between a task analytical approach and the idea of progressive mathematising, based on the active contribution of the children. This implies a choice with regard to the way children should learn mathematics; a choice between learning mathematics by copying adults, or learning mathematics on ones own legs (by 'taking responsibility'; Whitney;1985).

The position taken by the realistic trend is quite clear. It is also obvious that it will not be sufficient to tell this to the teachers, they have to be convinced. We think that the best way to do so, is to give the teachers the opportunity to find out for themselves.

We developed an in-service program that is so designed, that it will provoke this kind of learning. We try to make the teachers aware of the possibilities and desirability of interactive instruction. In doing so, several ingredients are being used. We will discus the following ingredients:

- working on problems;
- being informed;
- becoming conscious.

Working on problems the teachers find out that they all use different strategies to solve the same arithmetic problems. It also appears that most of these strategies differ from the strategies that have been learned. We hope that the teachers will get interested in solution procedures, and especially in reference to the solution procedures of children.

They are *being informed* of the strategies of children found in research. And they are stimulated to investigate if the children of their classes use the same strategies.

They are also informed about the bad results of the traditional mathematics education, that come forward in national and international assessments.

And they are informed on learning theories that support the realistic approach, based upon the own constructions of the students.

The last point concerns the question for remedial strategies, which is elaborated in such a way that the teachers *become concious* of their own theories, or beliefs.

The aim of these activities is to give the teachers more insight in the different views on mathematics education, in connection with the possible teaching practices. We hope to give the teachers the opportunity to make their own choices about the implementation of several aspects of the innovation. We found that the learning process which we try to set going, shows a great similarity with the learning process of the pupil in realistic mathematics education.

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OBSTACLES CONFRONTING THE IMPLEMENTING OF CONSTRUCTIVST INSTRUCTION AND THREE MODELS FOR ADDRESSING THESE OBSTACLES

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The paper has two main objectives. The first objective is to outline some of the difficulties involved in implementing constructivist instruction in secondary and college classrooms. The second objective is to bring into the realm of mathematics education three models which can be useful in minimizing these obstacles. Each model offers a different perspective through which to understand the dynamics involved in implementing a new method of instruction. Each model focuses on a key issue: components of "personal investment" in learning, teaching as a developmental process, and significant events during the course of a semester.

Much attention has been devoted to understanding how students construct their understanding of mathematics and to describing constructivist instruction. However, less attention has been devoted to examining some of the difficulties commonly encountered when teachers attempt to implement this unorthodox mode of instruction in their classrooms. This paper will first focus on various obstacles facing the secondary or college level teacher. The paper will then focus on three models, developed in other disciplines, which can enhance the teacher's implementation of constructivist instruction.

The challenge such a teacher faces is analogous in many ways to that of a new baseball coach introducing a rather different system. A major goal of the coach is to teach new skills and strategies. However, the coach will invariably encounter players who resist the new system and continue to run or to bat or to field in old ways. The coach will also encounter some players who, despite liking the new system, find it difficult to break old habits. The underlying theme of this paper is that a thorough understanding of constructivist principles alone does not ensure a successful classroom. In other words, the teacher must necessarily focus on both cognitive and affective issues.

OBSTACLES TO CONSTRUCTIVIST INSTRUCTION

Before examining various obstacles facing the teacher, let us examine some common characteristics of constructivist instruction which differ from traditional instruction: the teacher does less explaining and "showing" and more questioning, more attention is given to the processes involved in solving problems, students do most of the talking and writing, and the teacher challenges the usage by students of rotely-learned terms, rules or algorithms (Confrey, 1984). Such a departure from traditional teaching practices is often met with varying amounts of resistance (active and/or passive) on the part of the students. For example, some students may insist that that they cannot explain how they got their answers and that they should get credit as long as their answers are correct. Four kinds of obstacles, considered together, help to explain why the implementation of constructivist instruction can be difficult: students' attitudes toward learning, in general; students' attitudes toward learning mathematics; systemic constraints; and students' cognitive limitations.

Obstacles arising from students' attitudes toward learning include (1) resistance because this represents a new way of learning (students are often reluctant to give up old ways for a new and untested way); (2) the perception that such a teaching approach represents more work on the part of the student ("it's like we have to teach ourselves"); and (3) the fact that the teacher, at first, is generally not as polished, consistent, and authoritative as when teaching in a more traditional mode.

Obstacles arising from students' attitudes toward learning mathematics include: (1) attitudes about the nature of mathematics ("this is a math class, not a problem solving class"); (2) anxieties about word problems, especially for slower students who may see even the brighter students struggling, at least at first, with problems requiring higher order thinking skills; and (3) concerns about evaluation--the different format of tests, how the grade will be determined, etc.

Systemic constraints include: (1) heavy content pressure (i.e., pressure to cover the whole book); (2) more remedial students, anxious students, and struggling students than at the lower grades; (3) very often little support, or even resistance, from colleagues, administrators, and parents; and (4) inadequate curriculum materials; in fact, conventional or required texts often run counter to constructivist principles.

Cognitive limitations have been well-documented: the robustness of students' algebraic misconceptions (Rosnick & Clement, 1980), poorly developed diagram drawing skills (Simon,1986), and poorly developed metacognitive skills (Schoenfeld, 1985), to name but a few.

The point is that, partly because it represents a change and partly because it is more work for the student, the teacher implementing such a different method of teaching is likely, at least at first, to encounter varying degrees of resistance. Realizing that such resistance is not uncommon can enable the teacher to anticipate and act rather than react. Being aware of the different kinds of obstacles can enable the teacher to direct his/her energies more effectively.

A MOTIVATIONAL MODEL

For constructivist instruction to be successful, the students must function at significantly higher levels of cognitive activity than are normally found in classrooms. Therefore, a first task for the teacher is to convince the students that this way of learning mathematics is more effective, more powerful than the ways to which they are accustomed. Although the value of constructivist instruction may be clearly meaningful to the teacher, this is not necessarily so for the students.

Maehr (1984) asserts that meaning is the critical determinant of motivation. Each student comes to the classroom with a package of meanings derived from past experiences. Whether or not the students will invest themselves in a particular activity depends on what the activity means to them. According to Maehr, the extent to which students will invest themselves depends on several interrelated factors: (1) judgments about self, especially beliefs about one's competence or ability to master the material, (2) their perceived goals, (3) the subjective cost of success, and (4) judgments about the options or alternatives available for reaching these goals. In Figure 1, these factors have been characterized with respect to the issues and questions they raise.

Issue	Question
1. competence	Can I do it?
2. goals	What will I get out of this activity?
3. cost	Will it be worth the effort?
4. autonomy	Will it be my learning?

Figure 1. Components of personal investment.

The desire for competence has long been established as an essential motivating element in the learning process (White, 1959). This was brought home to me during my first year of teaching. So many of my pre-algebra and algebra I students who plaintively asked, "Why do we have to learn this?" stopped complaining when they were able to master the material.

The term goal refers to the motivational focus of the activity. Maehr discusses four categories of goals: task goals (e.g., becoming competent), ego goals (e.g., performing better than others), social solidarity goals (e.g., pleasing one's parents) and extrinsic goals (e.g., grades). He asserts that "to create the kind of spontaneous learning pattern of continuing motivation, a task-goal orientation must be fostered. Only as one is oriented toward doing a task, apart from the evaluation placed on it by others, will one continue doing it when there are no others evaluating it" (Maehr, 1984, p. 130).

The cost of attaining the goals is also important. If the cost is very low, students tend to become bored and dissatisfied; if the cost is very high, students tend to become frustrated and discouraged. Furthermore, those who are less sure of their competence need special encouragement if they are to confront the challenges presented. This is especially true at the beginning of the year or semester.

Finally, with respect to autonomy, as the individual sees herself as the initiator or cause of her own behavior, she will more likely find her own reasons for engaging more deeply in the activity. This is one of the reasons for more attention to "real" problems in mathematics classes nowadays.

A high degree of personal investment produces a student characterized by continuing motivation, as opposed to the more common stop-start cycle of motivation exhibited by many students. It is important that the teacher focus on nurturing and developing continuing motivation, for this kind of motivation is not commonly seen in traditional classrooms and is crucial in the constructivisit classroom with its higher focus on problem solving and other higher order thinking skills.

Maehr's motivational model has much to offer the constructivist teacher. First, it educates the teacher out of the commonsense notion of motivation as a monolithic construct. Attending to motivation means more than simply getting the students to see the new mode of learning as meaningful. It also means addressing issues of competence, goals, cost, and autonomy. Second, the model explains the need for attending to motivational issues: a high degree of personal investment is essential for meaningful learning to occur, especially in the constructivist classroom where there is greater focus on higher order thinking skills.

A DEVELOPMENTAL MODEL

A common mistake made by teachers implementing constructivist instruction is to try to do it all at once, especially if the teacher had a very positive experience in a workshop or in a summer program. The teacher may forget that the students have not been similarly changed. Rather, the students see the teacher behaving in unfamiliar ways. However, just as the construction of one's understanding of mathematics must develop, so too construction of new ways of learning (and teaching) need time to develop.

For many students, there is an enormous difference between their present beliefs and attitudes toward learning and what we would term mature beliefs and attitudes. The figure below, adapted from Argyris' (1957) characterization of the difference between the immature and mature worker, illustrates the differences.

<u>Immature</u>	<u>Mature</u>
Passive	Active
Dependent	Independent
Behaves in few ways	Capable of behaving in many ways
Erratic shallow interests	Deeper and stronger interests
Short time perspective	Long time perspective
Lack of awareness of self	Awareness and control over self

Figure 2. Differences between immature and mature students

After eight to twelve years of schooling, many students have become conditioned to "learning" in passive and dependent ways. They are not flexible learners but rather have become accustomed to doing math in few ways, mostly focusing on learning the "correct" procedures in the "correct" ways. For a variety of reasons (e.g., little relevance to their lives and the greater focus on extrinsic than intrinsic goals), their interests in mathematics have become shallow and focused on the short-term (e.g., the next test and the final grade). In order for a constructivist teacher to be successful, (many of) the students must make the large jump from immature to mature learner. In many respects, the task of such a teacher is similar to the task of a physical therapist working with a stroke victim. At first, the therapist must be very directive and supportive, not only providing words of encouragement but also providing explicit instruction on how to relearn to use the paralyzed portion of the body.

The situational leadership model (Hersey & Blanchard, 1982), developed for use in all types of organizations, incorporates these ideas into a developmental framework which enhances the teacher's ability to pass through the transition from traditional to constructivist instruction more effectively. Although, it is beyond the scope of this paper to give a thorough treatment of this model, the basic features can be outlined and the relevance for constructivist instruction discussed.

The model, as adapted to the classroom, consists of four teaching styles on a continuum from most to least directive and four developmental levels of the student, also represented on a continuum. The terms for each teaching style in the figure below are intended to give a feeling for what each level "looks like." As one goes from S1 to S4, the amount of student autonomy and choice increases. Another way of saying this is that the amount of one-way communication (i.e., the teacher spelling out the students' role and telling the students what to do) decreases and the amount of two-way communication increases. The choice of teaching style is determined by the students' developmental level which, in turn, is determined by examining two factors: (1) ability--do the students have the necessary knowledge and skills to perform at the desired level, and (2) motivation--do the students have the necessary confidence and willingness to perform at the desired level?

Teacher:	S1 directing showing telling	S2 consulting guiding coordinating	S3 participating coaching	S4 facilitating collaborating
Students.				
Ability	lower	lower	higher	higher
Motivation	lower	higher	lower	higher

Figure 3 Teaching styles and developmental levels of the students

A basic principle of this model is that the teaching style must be consistent with the developmental level of the students. If the teacher is too directive (as is often the case in classrooms), the students can become bored, and motivation is likely to be low. On the other hand, if the teacher moves too quickly in the other direction, there may be much frustration and dissatisfaction on the part of the students. For example, the teacher might introduce significantly more difficult problems before the students' problem solving skills have sufficiently developed.

Like Maehr's motivational model, the Hersey-Blanchard model has much to offer the teacher. First, the model serves to expand the teacher's focus from cognitive issues to affective issues which can affect the amount of time and energy which the student brings to bear on learning mathematics. Second, the model focuses on the distance between the present developmental levels of the students (knowledge, skills, confidence, and willingness) and the desired levels. Such a perspective can enable both the teacher and the students to realize that the transition from immature learner to mature learner does not happen immediately. Finally, the model offers both the structure and the vocabulary to assist the teacher in the transition process. The teacher can minimize student resistance by providing support for the students, who are learning to take more active responsibility for their learning.

THE CRITICAL INCIDENT MODEL

The third model comes from the field of organizational development in which the dynamic nature of groups (of all kinds) has been widely studied. A major contribution from this field is the realization of the group as an entity in its own right, as opposed to simply being a collection of individuals (i.e., the whole being greater than the sum of its parts). A negative way ating this is that the effectiveness of the leader or teacher who is . sensitive to the dynamics operating in a group is often seriously diminished.

The critical incident model (Cohen & Smith, 1976), derives from observations that certain critical incidents (significant events) can substantially affect the development of a group (class). The critical incident concept evolved with the observation that certain critical situations emerge and repeat themselves time and again in different groups and at different developmental stages. The manner in which the leader (teacher) responds generally has a large influence on the direction and the development of the group (class).

With respect to mathematics classes, there are several critical incidents to be aware of: at the beginning of the semester--when the teacher sets the tone for the semester; just before first test--when the teacher can stress how to study and perhaps offer a make-up test for the first test or two in this new way of instruction; and just after the first test--especially for students who did poorly. A skillful teacher will make good use of these situations to maximize the students' continuing motivation and to deepen the students' understanding this new way of learning--what is expected of the students and how the students can develop new skills.

The main value of the critical incident model lies in its interaction with the previous models. With respect to motivational issues, the focus on critical incidents can enable the teacher to respond (with the class as a whole or with individuals) to motivational issues at appropriate or "ripe" times. With respect to developmental issues, the focus on critical incidents can keep the teacher sensitive to the developing skills and motivation of the students.

SUMMARY

Implementing constructivist instruction in our mathematics courses is not an easy task. In this paper, a number of obstacles confronting the teacher have been discussed. Three models from other disciplines have been explicated and their relevance to mathematics classroom discussed. Each model focuses on a key issue: components of "personal investment" in learning, teaching as a developmental process, and significant events during the course of a semester.

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INSTRUCTIONAL STRATEGIES AND ACHIEVEMENT IN GRADE 8

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This study sought to identify teaching behaviours which are related to significant gains in mathematics achievement over the grade 8 school year. Of 23 classes studied, two were found in which the mean student achievement gain was significantly greater than in other classes with comparable pretest achievement. Analysis indicated that the significant gain in mathematics achievement in these two classes could not be attributed to differences in factors such as students' home background, class size, number of lessons per week or teacher workload. From the aspect of instructional strategies, however, analyses of teacher questionnaires and classroom observations revealed differences between the two effective teachers and their less effective colleagues. Approaches which emphasized an organized presentation of new material followed by extensive practice in its application to new situations seemed to have contributed most to greater achievement gains.

This study builds upon previous empirical research on teacher effectiveness (Berliner,1976; Brophy, 1979; Treiber, 1981) that has shown consistent relationships between certain teacher behaviours on the one hand and student involvement and achievement on the other. Its primary objective was to identify those patterns of behaviour in the teaching of mathematics which are associated with significant gains in student achievement.

Data

The study examined mathematics instruction in 23 classes, using data collected in the Ontario IEA Classroom Environment Study. The data consisted of direct observations of teacher behaviour in the classroom, teacher responses to questionnaires and the results of a student achievement test.

The observation component of this IEA study focussed on the direction of classroom interaction (e.g., teacher to group, individual student to teacher); the context of the interaction (e.g., small group, large group, private); the nature of the interaction (e.g., asks high-order questions, gives directives, manages only) and the event (e.g., instruction, question, feedback). These observational data were collected in each class on eight different occasions. On each occasion, records were made of the interactions that occurred during each of five different five-minute intervals, one at the beginning of the class period, one at the end, and three spaced more or less evenly over the rest of the period.

The teacher questionnaire consisted of 106 questions on such matters as classroom organization, teacher strategies, testing procedures, use of curriculum materials, and the proportion of time spent on various activities.

The achievement test consisted of a 40-item multiple-choice mathematics test administered as a pretest at the beginning of the school year and as a posttest seven months later.

Method and Results

1. Identification of Effective Teachers

Since the objective of the study was to find teaching patterns associated with significant student achievement gains, it was first necessary to identify classes showing such gains during the school year.

Complicating this identification were the differences among the 23 classes with respect to initial achievement level. To take this factor into account, the 23 classes were separated into homogeneous groups on the basis of their mean score on the pretest, after ascertaining through analyses of variance carried out for each group separately that there were no differences among the classes within each group. There were three homogeneous groups: (1) low entry (5 classes), (2) average entry (7 classes), and (3) above-average entry (6 classes). (The remaining five classes were eliminated from further study, since they could not be added to any of the groups, nor could they form a separate grouping with similar pretest achievement scores.)

Within each of the three groups, additional analyses of variance were carried out on the posttest results to identify those classes for which the achievement gain was larger than for the other classes in the same group. In each of the first two groups there was one class with a significantly higher posttest mean. As shown in Table 1, the students in Class 331 exhibited a gain of 15.6 at the end of the school-year, well above the average gain of 7.2 for the other four classes of the low entry group. Similarly, class 101 had the highest gain in the average-entry group. In the third group there were no differences in posttest achievement among the classes.

Table 1

Mean Pre and Posttest Mathematics Achievement Scores by Classroom within Each of Three Groups Identified on the Basis of Equivalent Student Entry Performance

Classroom Group		Class Number	Mean Pretest Mathematics Score	Mean Posttest Mathematics Score	Mean Achievement Gain
Group 1:	Low Entry Performance	201	14.1	21 8	7 7
		222	14.0	17.3	3.3
		301	13.5	19.5	6 0
		312	14.1	18.2	4 1
		331	13.6	29.2	15.6
	Mean of Means		13.8	21 1	7.2
Group 2:	Average Entry Performance	101	16.7	26.6	9.9
		111	16.8	21.3	4.5
		112	16.4	22.2	5 7
		151	16.5	20.6	4.0
		202	15.1	23.0	8 0
		302	15.0	18.5	3.5
		421	16.9	22.5	5.6
	Mean of Means		16.2	22.1	5.9
Group 3:	Above Average Entry Performance	121	19.7	22.5	2.8
		131	18.5	25.5	7 0
		141	17 3	22.9	5.6
		161	18.0	23.7	5.7
		311	19 8	24.8	4 9
		411	19 8	24 2	4.4
	Mean of Means		18.8	23.9	5.1

2. Background Variables

To identify any differences in achievement gains from pretest to posttest which might be attributed to factors other than teaching effectiveness, background variables relating to student characteristics and classroom conditions were examined.

2.1. Student Characteristics

A comprehensive set of 40 student characteristics was employed, including the socio-economic status of the parents, the father's and mother's educational level, the language spoken at home, the student's attitude toward mathematics, the student's perception of abilities in mathematics, and the teacher's assessment of the number of students needing remedial help. There were no differences among the classes with respect to these characteristics.

2.2. Classroom and Teacher Characteristics

A comprehensive set of 20 classroom and teacher characteristics was also studied, including class size, time allocated for instruction, proportion of girls and boys in the class, amount of homework assigned, teaching experience, teaching workload, and number of subjects other than mathematics taught by the teacher. There were no differences among the classes with regard to these characteristics.

3. Instructional Strategies

Both the classroom observational data and the teacher questionnaire data were then examined for possible identification of effective teaching strategies. In particular, the study compared the behaviour of each of the two teachers whose classes displayed exceptional achievement gains to that of the other teachers in their respective groups.

3.1. Time Allocated to Various Instructional Practices

On a four point scale (no time, a little, a fair amount, and a great deal) the teachers reported the amount of time they allocated to the following six instructional and managerial practices:

1. Questions directed to the whole class
2. Monitoring or giving help to individual students
3. Teaching the whole class
4. Teaching small groups
5. Monitoring while students work in small groups
6. Disciplining students

Teacher 331 spent much more time on activities 1, 2, 3, and 5 than did the other teachers in Group 1. However, this was not the case for Teacher 101, who did not differ from the other teachers in Group 2 on any of these activities.

3.2. Planned Lesson Emphases and Lesson Intentions

The teachers were asked to state the frequency with which they made use of four specified emphases: (a) introduction of new material, (b) extension of new material, (c) review to refresh memory, and (d) review to correct misunderstanding. The first two emphases were used much more frequently by the effective teachers (about 80 to 100 percent of the time) than by the others (about 50 percent). Emphases (c) and (d) were seldom used by the successful teachers, but were used about 50 percent of the time by the others.

Four lesson intention were investigated: (a) comprehension of a concept, (b) teaching rules, (c) applying previous learning to new situations, and teaching social and interpersonal skills. The two effective teachers reported a higher frequency of use of (a) and (c), and a lower use of (b), than did the other teachers. None of the teachers reported using (d).

The lesson emphases and intentions used by the effective teachers appear to reflect a teaching model that emphasizes the presentation of new material followed by a great deal of application of that new material to ensure that the students have grasped its content.

4. Classroom Interaction

Data pertaining to classroom interaction had been collected through the Five Minute Interaction (FMI) observational instrument, on eight different days or observation occasions. To cast this large and unwieldy volume of data into a form more amenable to display and analysis, the original interaction categories were grouped into four major interaction types: (1) Classroom Management (4 categories); (2) Instruction (6 categories); (3) Teacher/Student Exchanges (12 categories); (4) Seatwork (2 categories).

These four interaction types were then graphed as a function of time (strictly speaking, as a function of the ordinal number of the observation record within each observation period). The resulting graphs proved very effective in presenting a large amount of information in a small space (Hanna, Postl, Truab, and Wolfe, 1986).

To assess the bearing of observed classroom interaction on achievement, the two classes with high gains (classes 331 and 101) were compared to the two corresponding classes with the lowest gains, (classes 222 and 302).

For both of the achieving teachers, the patterns of observation showed considerable inconsistency from one observational occasion to another. There were in fact, more similarities among the four teachers than there were among the various observational occasions for any individual teacher. In sum, there was nothing in this display of the observational data that could have predicted the differences in mean achievement gains.

Summary

The significantly greater gains in mathematics achievement in the two classes with the largest gains could not be attributed to factors other than teacher behaviour. Analysis of teacher questionnaires revealed differences between the effective teachers and their less effective counterparts. The teaching strategies that seemed to have contributed most to greater achievement gains were (a) an extremely organized approach to teaching, wherein material is taught until the teacher feels it is mastered, thus reducing the need for frequent review, and (b) an approach in which every presentation of material is followed by extensive practice in applying the material to new situations.

Analysis of the classroom observations, on the other hand, did not yield any insight into the influence of classroom interaction on achievement gains.

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THE USE OF EXPLICITLY INTRODUCED VOCABULARY IN HELPING STUDENTS TO LEARN, AND TEACHERS TO TEACH IN MATHEMATICS

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Abstract

We claim that explicit use of terms for technical aspects of mathematical thinking and teaching can help, and we describe the activities which led us to this conclusion. More importantly, we describe our way of working and justify our claim by reference to it, by expecting our readers to recognise what we are saying in their own situation, and thereby informing, clarifying, and perhaps altering their own activity.

THE PROBLEM

When teachers are invited to talk about their teaching, they tend to focus on lesson content and to shy away from discussing techniques, tensions and difficulties. As one tutor said at one of our meetings, *"People want to talk about teaching, but they don't have the vocabulary for it. So it comes over that they don't want to."* In the related domain of feelings, Lewis and Michelson (1983) note that although some studies have focussed on how children and adults talk about their feelings, very little is known about how labels are actually acquired for feelings.

The problem therefore, is how to improve the effectiveness of teachers' discussions of their teaching, and correspondingly, of students' discussion of their learning. Our belief is that these two domains are connected, and that a sensible place to begin is to work on helping teachers to talk effectively with their students about the learning of mathematics.

Scardamalia and Bereiter (1983) suggest that (many) students are interested in analysing their cognitive processes, and propose some techniques for generating reflection in children. Our approach with adults, is to develop and introduce meaningful vocabulary for the learning and doing of mathematics, as well as strategies for negotiating or rejecting that vocabulary. However, words in themselves do not carry meaning. Meaning arises in an individual as a result of a desire to make sense, which acts upon some recent or vivid experiences, and which is integrated by a crystallizing word or phrase. For example, most students have found themselves stuck on a mathematics problem at some time. They have plenty of experience of getting unstuck, as well, through teacher or peer questions or suggestions. The single word STUCK, used as an acknowledgment of a state, can, we claim, integrate a number of vivid experiences of being stuck, and aid recall of strategies for getting unstuck that have worked in the past.

HYPOTHESES

1. People can develop their mathematical thinking, learning, and teaching by reflecting on their experience;

Comment: Despite the adage, experience alone is insufficient to guarantee learning. Play with Dienes blocks does not automatically lead to a deep sense of base ten notation. As Brophy (1986) observes, "Mere engagement in activities will not facilitate learning, of course ...". Hart (1985) studied transitions between practical activity and formalisation, found enormous gaps, and questioned the myth that 'practical work is a good thing' in and of itself. Schon (1983) and Kilpatrick (1986) highlight the role of reflection in learning from experience.

2. Even where change in behaviour, attitude and perspective takes place without conscious overview, it follows a definite pattern which is elaborated in Mason (1984) and Mason (1986), but can be briefly described as follows:

   * Attention is focussed on some aspect of teaching, or learning, or mathematics, as a result of reading, discussion, or observation (*Attending*);
   * That aspect seems to make sense in terms of past experience (*Resonating*);
   * Subsequently, that aspect comes to attention in or soon after an event, either spontaneously, or as a result of intentionally looking out for it (*Noticing*);
   * An alternative behaviour-response is triggered as a result of noticing an opportunity (*Choosing*).

Comment: Our radical constructivist perspective (Mason and Davis (1987), von Glasersfeld (1984)) leads us to concentrate on studying how we and other people construe and reconstruct ideas for ourselves, so that ideas become 'our own' and integrated into our behaviour, attitudes and perpsective. Being involved in in-service and mathematics teaching, we act upon our our beliefs by constructing events, both face-to-face and at a distance, which are designed to foster and support sense-making.

DEDUCTION

Assuming Hypothesis 2, it makes sense to invoke the same pattern to investigate Hypothesis 1. Baldly described, our method is

* Offering activities (Wertsch 1981) or tasks in which a potent distinction is often noticed, or to which attention can be drawn (*Engaging*);
* Drawing attention to such a distinction (*Attending*);
* Asking people to interrogate their experience and see if they recognise the distinction (*Resonating with own experiences*);
* Inviting people to set themselves to notice the distinction and to report such noticings vividly to colleagues (*Noticing*);
* Suggesting or identifying specific teaching techniques or mathematical processes which may be effective or appropriate in such situations (*Choosing*);
* Inviting people to share vivid reports of the use of such techniques or processes as part of negotiating and enriching the meaning for vocabulary (*Deepening through Resonating with others*).

For example, as a result of our teaching tasks, we were led to a particular case of Hypotheses 1 and 2:

THESIS

Teachers and adult students *can* improve the effectiveness of their reflection by being encouraged to adopt a vocabulary for significant thinking processes, such as the use of

Specialising to refer to a wide range of acts in which abstract and general statements are particularised to concrete, specific, confidence inspiring examples (all relative to the individual of course), both in mathematics and in mathematics education, and summarised in the expression, *Seeing the Particular in the General.*

Generalising to refer to the act of focussing on similarities in several situations, leading to a sense commonality and generality, and summarised in the expression, *Seeing the General in the Particular.*

KNOW and WANT to refer to the act of crystallising and recording what you know that is relevant to the problem, and what you want. These tend to change as you begin to make progress.

An elaboration of the technical uses of these and other words, and the way such technical uses are introduced in a manner corresponding to Hypothesis 2, can be found in Mason et al (1982) and Mason (1984). Michener (1978) offers additional vocabulary, but does not elaborate particularly on a method of introducing the vocabulary to students.

COLLECTION OF EVIDENCE

The context of our activities consists of students studying Open University courses and of running in-service workshops for O.U. tutors, and for advisory teachers, teachers, and pupils. One O.U. course is in Mathematics (3500 students per year), and two others are in Mathematics Education (300 students each per year). These are studied at home by correspondence, supported by television and radio broadcasts, tutorials, and in the case of mathematics, a week long summer school in the middle of the course. Since 1982 we have engaged in the following:

Activity 1 Informal discussions with O.U. students about learning mathematics.

Activity 2 Meetings with O.U. tutors to stimulate reflection on and questioning of their tutoring, especially with regard to conducting mathematical investigations with students, and to suggest vocabulary as indicated above, for use by them with students. The meetings were conducted along the lines indicated in Hypothesis 2.

Activity 3 Structuring of investigative workshops for O.U. undergraduates at face-to-face summerschools (3500 students each for a week at one of 3 sites over a 10 week period) in a style corresponding to Hypothesis 2.

Activity 4 Conducting INSET workshops for teachers, and workshops for pupils in the same style.

Activity 5 Course texts written for adult O.U. students, in the style of Hypothesis 2. Our main study coincided with the redraft of the original text (Mason (1980), and strongly influenced the new version (Mason 1984).

Activity 6 Course texts were written for mathematics teachers (300 per year) in the style of Hypotheis 2 (O.U. 1980).

Information and evidence was collected in a disciplined (Mason 1984) but informal manner as follows:

1. Audio recordings were made of meetings, from which pertinent anecdotes and quotations were selected, to use as stimuli in subsequent meetings. For example, one student said "*I study a text for a week, do the assignment, score well, but three weeks later I can't remember anything about it.*" We have found that when quoted to students at summerschool, there is an immediate moan of recognition, and a sense that they appreciate that we understand something of what it is like to be an O.U. student. This is a good example of *resonance*, and its use to communicate effectively with people.
2. Audio recordings were made and used analogously to activity 1.
3. Anecdotal but unsystematic observation suggests that over the four years since we began, tutor and student attitude to investigations has changed markedly from *negative, barely tolerant* to *accepting or even enthusiastically positive*, as reported in Davis and Mason (1984). Tutors are found using the suggested vocabulary, not just in investigation sessions, but in lectures and tutorials; course authors have picked up the vocabulary and used it in subsequent redrafts of mathematics texts for students.
4. We use live workshops for the honing and precising of activities which are then used in texts as described in the **DEDUCTION**. A major force in our methodology is the search for resonance - first in past experience, and then subsequently in noticing that aspect in new situations. (See O.U. 1982 for examples.)
5. We look for spontaneous utterances from students at summerschools during the particular or subsequent courses. For example, Davis reported a student in a tutorial group for an advanced O.U. mathematics course as follows:

One student was invited to go to the board and co-ordinate suggestionsfrom the others. ... Then he said out loud, and wrote, "I KNOW ..., I WANT ...". I was surprised to observe such explicit use of one of our frameworks which I had not yet myself used with this group, and asked him where he had come across this approach. "Oh, I found it some years ago in a book (the O.U. text (1980) referred to earlier), and I've used it ever since because it's so useful. It helps me to get started on questions, to organise my thinking, and to get me out of being stuck."

The spontaneity and richness of experience suggested by the remark is most gratifying. Our criteria for success do not depend on achieving any particular density of spontaneous uses of the suggested vocabulary, because we do not want to get trapped into trying to prompt students to 'give us what we want to hear'. We have recorded enough such utterances to convince ourselves that students, and teachers *can* become more reflective and more effective. We make no claims of necessity.

6. Remarks under 5 apply to activity 6 as well. In November 1986 we conducted a survey of ex-students on the mathematics education courses. Of the 896 students approached, replies were received from 625 (69%). Of these

   40% said they now use *Do, Talk, Record* (one of the course frameworks), most of the time;

   92% said they now use it some or most of the time;

   56% said they were using one or more of the frameworks *explicitly* with their pupils.

   On the surface, such responses seem encouraging, but whether the respondents all mean the same thing by "use most of the time", is impossible to tell, and rather unlikely. We choose to interpret the results simply as indication that there may be some potential in our approach. Our methodology validates its findings in other ways.

VALIDATION

Our Radical Constructivist approach leaves validity to the practitioner. We seek resonance in those we work with face-to-face. We then write our texts in a corresponding manner to try to promote resonance and effective reflection. This report is not presented in as self-consistent a style as we would wish, due to limitations of space and in order to meet the usual research requirements. Our preference, and our methodology, require us to present exercises through which readers might attend to some pertinent noticing, recognise it in their experience, and so be moved to notice something in the future which otherwise might have gone unnoticed. Validation of our thesis resides for each person in the extent to which they recognise something of what we describe, in their own situation, and find their own experience enriched or clarified as a result of trying to construe our report.

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Measurement
concepts

YOUNG CHILDREN'S UNDERSTANDING OF NUMBER AND UNIT IN A CONTINUOUS DOMAIN

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Kindergarten and first-grade children's developing conceptualization of number and unit in the domain of length was assessed using a computerized estimation game format. Assessment occurred after 2, 4, and 6 hours of Logo keyboard experience. Preliminary results indicate significant effects due to grade, unit of measure and magnitude.

Much research has focused on the development of counting skills (Fuson, Richards & Briars, 1982; Gelman & Gallistel, 1978) and the importance of counting in mathematical development (Carpenter & Moser, 1984; Fuson, 1982, 1984). However, number in its most general form is a measure of quantity. What distinguishes number from sequencing, as in rote counting, is a unit of measure. In assigning a number to a collection of discrete, discontinuous items via counting, the units are the individual items in the collection. In that case, quantity is determined only by the number of unit entities, regardless of their size. However in the most general case of measure, quantity is a function of both the number of units and the size of the units. Further, different units may be utilized when determining quantity. Accordingly, in this domain, the effect of unit transformation is much more complex. Young children appear to understand measurement as a relationship between increased quantity and increased unit numerosity. However the inverse relationship between unit size and the number of units is difficult for young children (Carpenter 1975; 1976; Gal'perin & Georgiev, 1969; Hiebert, 1981).

In their information-processing model, Klahr and Wallace (1976) hypothesize that individuals generate an internal representation of a component of the total space or quantity to be measured. This representation serves as an internal unit; numerical values may be associated with it. Further research is needed to determine how children determine unit size, coordinate number words with unit

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iteration, and maintain consistent unit sizes during measurement and estimation tasks.

Logo provides an arena in which young children may define, create and manipulate units, maintain or predict unit size, and create length rather than end point representations through either iterative or numeric distance commands. Further, Logo permits a child control of transformations of unit size and number without the distracting processing and dexterity demands associated with measuring instruments and physical quantity. This study utilized Logo as a controlled setting in which young children's developing understanding of distance and the inverse relationship between unit size and number was examined.

METHODOLOGY

Subjects. The subjects for this study were 26 kindergarten and 23 first-grade children from a diverse racial and ethnic population of middle/upper-middle class socio-economic status.

Experimental Design. The dependent variables were children's understanding of distance in terms of a unit of measure and children's conceptualization of the inverse relationship between the number of units and unit size. These variables were measured by an Estimation of Distance task that varied both unit size and partitioning constraints. The independent variables were grade, Logo experience and the spatial factors inherent to distance representation (the length to be measured [long or short] and the orientation of the distance path in the plane [horizontal, vertical or oblique]).

Procedure. Each of the first-grade children received Logo instruction in small groups of 12 children during one-hour sessions once a week. Each of the kindergarten children received Logo instruction in small groups of six children during 15- to 20-minute sessions approximately two to three times per week. Following each instructional session, the Logo instructors recorded the number of minutes of individualized Logo exploration keyboard time made available for each child.

Logo instruction for the first-grade children utilized the Turtle Graphics component of Logo; the kindergarten children utilized a version of Instant Logo which required the RETURN key to be struck with each single keystroke. The kindergarten children spent approximately two months completing pre-Logo movement activities followed by exploration with a Logo robot prior to introduction to the triangular cursor on the monitor screen.

Estimation of Distance Task. The Estimation of Distance Task utilized a hypothetical setting that required the child to move the triangular cursor from an initial location to a targeted position on the monitor screen in either one move (go all the way to the target) or in two moves (first go half way to the target, then go the rest of the way). Because this assessment was to measure only distance estimation, not direction estimation, the initial heading of the cursor was always positioned toward the target. A line segment representing the distance traversed by a single forward unit was visible during each item presentation.

Each of the three Estimation of Distance assessments were individually administered in two 15-20 minute sessions. The two sessions, each presenting 24 items, were conducted within a one week period after 2, 4 and 6 hours of individualized Logo keyboard time. The first 12 items in each session consisted of 12 targets whose distances were to be estimated using a unit length of 10 turtle steps (U_1). The next 12 items in the session utilized a forward unit length which was either half ($U_{1/2}$) or double (U_2) the original unit. Children in each grade were randomly assigned to either the U_1 $U_{1/2}$ or the U_1 U_2 condition. During the second testing session of each assessment, the last 12 items were estimated using the other unit length. In both testing sessions, the items alternated between the Single Move and the Two Moves condition.

For half of the items, the distance to the target was short (6, 8, or 10 U_1 units); for half of the items the distance was long (12, 16 or 20 U_1 units). Children in each grade were randomly assigned to one of 12 distance presentation patterns.

The path between the initial cursor position and the target location was either a horizontal, vertical or oblique directed line segment. Children in each distance pattern were randomly assigned to one of six possible patterns for orienting the target path on the plane of the monitor screen. Because pairs of items were balanced for the three angles of orientation, this assignment controlled for the representation of the Move condition across alternating items. In addition, each path orientation occured for both long and short distances. Each student's path/distance pattern for the first 12 items was repeated for items 13-24.

ANALYSIS AND RESULTS

The following repeated measures analyses reflect data collected after 2 and 4 hours of Logo keyboard time. Two dependent measures were utilized. The Numerical Estimate of Distance measure is the number of units entered as the initial estimate of distance in response to each item in the U condition. The Estimation Accuracy measure is the ratio of the number of units initially estimated to the correct number of units. This measure was calculated for each item under all three unit conditions (U_1, $U_{1/2}$, U_2). Both of these measures were analyzed separately across the Move conditions. In each analysis the covariance structure was checked to determine if assumptions for repeated measures was satisfied; if necessary, adjusted p values were computed using the Greenhouse-Geisser correction.

The initial analysis sought only to determine if the children did in fact estimate different lengths with differing numbers. This analysis utilized the Numerical Estimate of Distance measure with a Grade(2) x Time(2) x Session(2) x Length(6) ANOVA with repeated measures on the last three factors. Under the Single Move condition, significant effects due to grade ($F(1, 47) = 9.75$, $p = .003$) and length ($F(5,235) = 26.06$, $p < .0001$) were noted. The first-grade children were more accurate estimators overall; as the distances increased, the children estimated the length with larger numbers. No significant grade effect was noted in the data collected under the Two Moves condition. The significant effect due to length noted under the Single Move condition was also present under the Two Moves condition ($F(5,235) = 20.79$, $p < .0001$). Under the Single Move condition, the children generally tended to underestimate the total distance to the target; when estimating half the distance to the target, the children tended to overestimate the distance.

Subsequently, a Grade(2) x Time(2) x Session(2) x Orientation in the Plane(3) x Distance(2) ANOVA with repeated measures on the last four factors was completed using the Estimation Accuracy data collected under the U_1 condition. Because the ratios of the children's initial estimation of length to the actual length were not normally distributed, this analysis was completed on the logarithms of the ratios in order to yield homogeneity of variance. Because of the use of logarithmic data, geometric means were computed.

Under the Single Move condition, significant main effects due to time ($\underline{F}(1,47) = 5.3$, $\underline{p} = .0259$) and grade ($\underline{F}(1,47) = 18.37$, $\underline{p} < .0001$) were noted. Significant interactions were also indicated between time and distance ($\underline{F}(1,47) = 6.07$, $\underline{p} = .0174$) and between orientation and distance ($\underline{F}(2,94) = 3.77$, $\underline{p} = .0346$). The first-grade children were more accurate than the kindergarteners with accuracy for all children improving over time. After two hours of Logo keyboard time, the children tended to underestimate the distance to the target by 32%; after four hours of Logo keyboard time, the children were underestimating the distance by 24%. Examination of the time x distance interaction means revealed that after two hours of Logo keyboard time, the children were underestimating the longer distances by 38% and the shorter distances by 27%. However, after four hours of Logo keyboard time, both longer and shorter distances were being underestimated by 24%. The interaction means also indicated that while longer distances were more difficult to estimate than were shorter distances when positioned as either horizontal or vertical lines, this pattern reversed for oblique lines. A significant time x distance x grade interaction was also indicated ($\underline{F}(1,47) = 5.12$, $\underline{p} = .0283$). Under the Two Moves condition, the grade effect persisted ($\underline{F}(1,47) = 8.5$, $\underline{p} = .0054$) along with a significant effect due to distance ($\underline{F}(1,47) = 14.06$, $\underline{p} = .0005$). As the distance to the target increased, so did the children's underestimation of half of that length. The shorter lengths were underestimated by 7%; the longer distances were underestimated by 20%.

The analyses thus far have considered children's ability to associate a numeric value with either a total or a partitioned distance for a fixed unit (U_1) of length. In addition, the children also estimated the lengths of these same distances when the unit of measure was either doubled (U_2) or halved ($U_{1/2}$). The Estimation Accuracy measures under these conditions were evaluated with a Grade(2) x Time(2) x Unit Size(2) x Orientation in the Plane(3) x Distance(2) ANOVA with repeated measures on the last four factors. Under the Single Move condition, significant main effects due to grade ($\underline{F}(1,47) = 11.46$, $\underline{p} = .0014$) and unit ($\underline{F}(1,47) = 145.99$, $\underline{p} < .0001$) were indicated as was a significant interaction of unit with grade ($\underline{F}(1,47) = 11.82$, $\underline{p} = .0012$). Overall the children tended to overestimate the distance to the target by approximately 2% when using the doubled unit (U_2); however, with the halved unit, the children underestimated the length to the target by approximately 50% (62% for kindergarten children; 35% for first graders). When the doubled unit was present, the kindergarten children were very accurate, averaging only a 3% underestimation. Conversely, the first-grade children tended to overestimate with the doubled unit by about 9%. A time x unit x grade interaction was also significant ($\underline{F}(1,47) = 4.88$, $\underline{p} = .0321$). Under the Two Moves condition, the significant main effect due to unit size persisted ($\underline{F}(1,47) = 111.76$, $\underline{p} < .0001$) as did the unit x grade interaction ($\underline{F}(1,47) = 6.53$, $\underline{p} = .0139$); however, the overall level of accuracy diminished. Under the halved unit condition, the children tended to underestimate the position half way to the target by about 42% (50% for kindergarteners; 31% for first-graders) while they overestimated the half-way position with the doubled unit by approximately 10% (9% for kindergarten children; 12% for first-grade children). In addition, a significant main effect due to distance was indicated under the Two Moves condition ($\underline{F}(1,47 = 38.65$, $\underline{p} = .0001$) as was a distance x grade interaction ($\underline{F}(1,47) = 8,42$, $\underline{p} = .0056$). The partitioning of longer distances was underestimated by about 30% (40% for kindergarten children; 18% for first-grade children) while the partitioning of shorter distances was underestimated by only 8% (10% in the kindergarten; 5% in the first grade).

DISCUSSION

The analysis described above is preliminary as data from the third assessment period is currently being collected; however, some trends seem to be present. The children did understand that a distance could be traversed by iterating a given unit of measure; as the unit of measure decreased in size, the estimation task became more difficult. The children did understand that a compensatory relationship existed between the unit size and the number of defining units. Further analysis is planned to examine whether the children's estimations reveal application of the inverse relationship between unit size and unit numerosity and whether the children realize that equal distances remain equal when they are measured with a different number of units (Carpenter & Lewis, 1976).

Although longer distances were more difficult to estimate, the children gained proficiency over time. Further analysis is planned to investigate whether having visible versus invisible/self-determined targets (Single Move versus Two Move condition) influences the length effect. Even the kindergarten children understood that a distance could be partitioned into two components and that the numeric value for length assigned to any one component would be less than the numeric length value assigned to the original distance. Howver, many of the children had difficulty determining what that smaller numeric value should be.

Although significant main effects due to Logo keyboard time are indicated in these results, further analysis is pending. A control group of first-grade children who have had no Logo experience will be administered the Estimation of Distance Task at the time that the first-graders with Logo are completing their third assessment. Subsequent comparison of these two sets of first-grade data will serve to control the confounding effect of development and maturation.

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MECHANISMS OF TRANSITION IN THE CALCULATION OF VOLUME DURING THE CONCRETE SYMBOLIC MODE

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Forty Eight primary school children were tested individually on a series of tasks related to the calculation of the volume of physical or pictured rectangular solids, some constructed from individual cubes, others undivided. A developmental sequence was found which was analysed using the SOLO Taxonomy. This showed clearly how the young child starts by focusing on the external aspects of the figure and gradually moves to an appreciation of its internal structure. The results could not be explained simply by the child's number skills, and the developmental sequence was justified by a Guttmann reproducibility coefficient of r = 0.90, obtained over several tasks. The results were related to performance on transformation tasks used by Piaget et al. (1960) and Lunzer (1960) in order to clarify questions raised by Lunzer concerning Piaget's theory of the primacy of topological notions of volume.

INTRODUCTION

The aim of this study was to validate and to extend the theory of the SOLO Taxonomy (Biggs and Collis, 1982) through its application to a new set of mathematical data, and to analyse the transition mechanisms by which children move between successive levels of development. A pilot study involving detailed examination of some of the closed format mathematics items used by Collis, Romberg and Jurdak (1986) revealed an interesting sequence of development on a series of questions related to the volume of pictured three dimensional rectangular solids constructed from individual cubes. A sequence of changes in the way in which children organised individual cubes for counting appeared to reflect successive steps in their ability to conceptualise and integrate the three dimensions. This developmental sequence proved amenable to analysis using the SOLO Taxonomy, and allowed investigation of the way in which children come to use the three dimensions in understanding volume. This latter is a subject of long-standing controversy between Piaget, Inhelder and Szeminska (1960) and Lunzer (1960). Piaget et al. contend that the child's first understandings of volume are topological. Lunzer (1960), who replicated the general stages of development discovered by

Piaget et al., disputed the theory of the primacy of topological notions. The present research examines the mechanisms involved and provides some further insights into the child's understanding of the concept of volume.

METHOD

Forty eight primary school children in grades 2 to 6, and from middle class areas of Hobart, were tested individually on a series of tasks related to the calculation of volume. The same testing procedure was used with each child. The items included:

(A) Pictured constructions: Three sets of questions in the format developed by Collis, Romberg and Jurdak (1986) required the child to calculate the number of cubes used to build pictured constructions made from individual cubes.

(B) Transformation tasks: Four transformation tasks were used. Two involving physical constructions were those used by Piaget et al. (1960) and Lunzer (1960). In the Piaget et al. task children had to build on a 2x2 base, a tower made with the same number of cubes as a 3x3x4 construction also made from individual cubes. In the task developed by Lunzer, the original construction (equivalent to 4x4x3 cubes) had no visible divisions into individual cubes, but one row or column of cubes was placed along each dimension. Children had to use this information to calculate the number of cubes needed to construct the whole figure, and to build a new tower of equal volume, but on a 2x3 base. Equivalent pictorial representations of these two transformation tasks were also included.

In addition several other tasks were included to throw more light on the behaviours under examination. These were: -

(i) Conservation of interior volume: A 3x3x4 "house" was transformed into a 1x36x1 "bungalow", and appropriate conservation questions asked.

(ii) Invisible cubes: Children who did not include invisible cubes when counting a pictured construction were asked (a) whether, if they were to build the item with real cubes, they would need to use any cubes not shown in the picture; (b) whether any cubes would be needed under the back corner cube, and if so, how many.

(iii) Faces and cubes: All children were asked about how individual pictured faces combined to form cubes, and to give an opinion about an inappropriate combination of faces.

(iv) Impossible figure: The final pictured construction was an "impossible" figure in that two elevated cubes were depicted with no supporting cubes. Children's reactions were gauged as they were asked to count the cubes, and then make the construction.

(v) Memory questions: Children were asked to memorise, and then to build from individual cubes, immediately upon removal of the original, four constructions, each presented individually. Two were easy and two more difficult. Two were presented physically, two pictorially.

(vi) Number skills: To investigate the relationship of numerical skills to ability to calculate volume, children were asked to solve, at the start of the interview, equivalent numerical items which contained no reference to the counting of cubes in three dimensional constructions.

RESULTS AND DISCUSSION

The results will be discussed in three parts:- (A) The developmental sequence observed on the pictured constructions; (B) The transformation tasks; and (C) Numerical items.

(A) Pictured Constructions. The following sequence of development in counting the cubes used in pictured constructions was observed. The developmental sequence is described in terms of the SOLO Taxonomy and, based on two criteria, all steps are within the concrete symbolic mode: (i) All children conserved interior volume; (ii) The tasks involve quantitative abilities.

Steps within the Unistructural Level.

(1) Visible cubes only are counted, individually, and in a relatively disorganised manner.

(2) Visible cubes are counted, individually, but organised into (the visible aspects of) rows, columns or layers.

(3) Visible cubes are counted first, individually; some invisible cubes are included afterwards - incorrectly and in a disorganised manner.

Steps within the Multistructural Level.

(4) Visible and invisible cubes are organised together in a non-optimal way, and are counted individually.

(5) Visible and invisible cubes are still counted individually, but they are well organised into a consistent arrangement of rows, columns or layers, which reflects the internal structure of the building.

(6) Visible and invisible cubes are chunked together within a group (that is a row, column or layer), and successive chunks are added.

(7) Doubling is used to combine two similar layers.

(8) Multiplication is used to find the total number of cubes in several similar groups.

Some, but not all of the children who multiplied had reached the relational level. The criterion of relational level ability used here, is the generalisation of these multiplication skills to constructions where the individual units (division into cubes) are not shown.

Identical sequences of development were found for constructions made of physical cubes, except in these cases "visible" cubes corresponded to all the cubes which could be seen on the outside of the construction. This would seem equivalent to the topological appreciation of the object described by Piaget.

Coping with Invisible Cubes. Some additional tasks were included to investigate why children at the earliest stages fail to include invisible cubes. First, in 77% of cases where invisible cubes were not included the child later stated that cubes which could not be seen in the picture would be required to construct the building. In addition, in 79% of these cases, children could give the correct number of cubes needed under the back corner cube. When asked why these cubes were not included in their calculations they replied "I forgot" or "I didn't realise I had to" etc.; but when asked to recount the cubes, many still omitted the invisible ones, or at most moved to step 3 in the unistructural level. Thus, in general, children who do not include invisible cubes when counting the total construction, can focus correctly on them when asked about a particular part of the construction. This is typical unistructural behaviour. What they cannot do, is remember the parts sequentially to provide a "construction" of the whole. This is a multistructural ability.

The ability to operate correctly on part of the construction is also apparent in the results from the "impossible" figure. All children showed by questions or exclamations while counting or building the item, that they recognised the need for supporting cubes. In addition, on questions about the faces of a pictured cube, all children showed appreciation of how faces combined to form cubes. No child agreed with a statement proposing an alternative structuring of faces into different cubes.

Finally, on the memory items 94% of children, including all children responding at the unistructural level, solved the two physical memory items correctly. Eighty percent of children operating purely within the unistructural level (Steps 1, 2 and 3) failed both pictured memory items. No other children failed the easy pictured memory item, although there was a progressive ability to solve the difficult pictured memory item with developmental level. It would seem that all the children in the sample could form an image of a physical construction with which to guide building from memory. Often children were not accurate on their first attempt, and altered their construction until it "looked right". This indicates probable use of the Ikonic mode of operation. In contrast on the pictured memory items, children responding unistructurally could not use the pictured information to form a suitable representation in memory of either the easy or difficult items; while children in the early stages of the multistructural level continue to have problems with the difficult pictured memory item.

(B) Transformation Tasks. A regular sequence of development was found across the four transformation tasks. Their order of difficulty for 44 of the 48 children tested was:

(a) physical transformation task - individual cubes;

(b) pictured transformation task - individual cubes;

(c) physical and pictured transformation tasks - undivided solids; solved simultaneously.

This sequence occured because the first two transformation tasks could be solved by multistructural level strategies; while the tasks, involving undivided solids could only be solved at the relational level, as defined for this project.

(i) Physical transformation task - individual cubes: Thirty one children in the sample (65%) solved this task correctly. Incorrect strategies included building the new tower higher than the original without counting cubes, and the counting of "curtain walls". All children at the unistructural level in Part A used such incorrect strategies. Correct strategies reflected the previously established developmental sequence. First the cubes in the original construction were counted individually, then by successive addition, and finally by multiplication.

(ii) Pictured transformation task - individual cubes: Twenty two children in the sample (46%) solved this task correctly. No children at the unistructural level (steps 1, 2 and 3) were able to develop an appropriate strategy to find the height of a building, built on a 2x2 base with the same number of cubes as a pictured 3x6x2 construction. Once children gave multistructural level responses in part A, they attempted to calculate the height by counting successive groups of 4 cubes, usually by dividing the original building into groups of 4 cubes and counting the number of groups. Children in the early sta es of the multistructural level (Steps 4, 5 and 6) lost track when applying this strategy, and only at the final stages of the multistructural level could this strategy be followed through successfully. It was succeeded by the use of multiplication.

(iii) Physical transformation task - undivided solid: Thirteen children in the sample (27%) solved this task correctly. No children at the unistructural level in Part A had any appropriate idea of how to calculate the number of cubes equivalent to the solid. Many children responding multistructurally in Part A counted "curtain walls", where the four side walls only were counted, with corners included twice. A progression of strategies was used from counting imaginary cubes individually, to sequential addition and then multiplication for the four walls. The next step was for the child to realise the inadequacies of the "curtain wall" approach, and to modify the calculation to include the middle, or to count corners once only; inevitably a cumbersome and inaccurate procedure. Finally, children who imposed an appropriate 3 dimensional structure, used a correct multiplication strategy.

(iv) Pictured transformation task - undivided solid: Twelve children in the sample (25%) solved this task correctly. Children responding unistructurally in Part A had no idea how to start solving the problem, while children at the multistructural level attempted to calculate aspects of the outside of the figure only - either adding edges or surface areas. A correct solution was only achieved by multiplication of the three dimensions.

Developmental Sequence. The results of both Parts A and B present a clear developmental sequence which can be tested empirically, using a Guttman reproducibility coefficient. This sequence can be represented as follows:

Concrete symbolic mode

(a) No transformation task solved correctly: this group included all children classified as operating at the unistructural level in Part A.

(b) Physical transformation task with individual cubes solved successfully: these children perform at or above the start of the multistructural level (Step 4) in Part A.

(c) Pictured transformation task with individual cubes solved success - fully: these children perform at or above multistructural level Step 6 in Part A.

(d) Two undivided solid transformation tasks solved successfully: relational level: these children also use multiplication in Part A

(Step 8) and on the other transformation tasks.

Out of 48 children, 5 were exceptions to the above sequence: r=0.90. This sequence, together with the detailed developmental steps given in Part A, appear to add weight to Piaget's claim that the child's first understandings of volume are topological. That is the child starts with the external aspects of the figure and gradually comes to an appreciation of its internal structure. This occurs earlier for constructions where division into individual cubes makes the internal structure easier to grasp, and on these the "curtain wall" approach is abandoned at the end of the unistructural level. With undivided solids, calculation of "curtain walls" continues throughout the multistructural level, and children's first attempts at sequential addition or multiplication are applied to this inappropriate structuring. Resolution of the problem, for undivided solids marks entry to the relational level.

(C) Numerical Skills: Several additional items were included to eliminate the possibility that the developmental sequence merely reflects the child's progressive mastery of numerical skills. When children were asked to count arrays of "X"s there was no difference in counting accuracy across grades or developmental levels. The developmental sequence cannot thus be explained by simple counting ability. Nor can it be seen as depending only on the child's proficiency with addition and multiplication. While grade 2 and 3 children had lower scores on these tasks than older children, within grades 4, 5 and 6 there was no relation between developmental level and addition skills. Furthermore, half of these children did not use multiplication to solve any cube problem, and yet obtained correct answers on most multiplication items. This suggests two separate sets of skills are necessary for solving the cube problems: (i) mastery of appropriate numerical operations, (ii) correct understanding of the internal structure of the solid. Relational level skills would appear to include appropriate integration of both domains.

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CONCEPTIONS OF AREA UNITS BY 8-9 YEAR-OLD CHILDREN

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Many investigations have shown the difficulties experienced by children in understanding the area concept. The present study has focused on a fundamental sub-concept, that of the unit of area. Twenty-five third graders who yet had never been subjected to any formal teaching of this notion, were questioned individually. They had to cover up different figures using a variety of unit shapes. It was found that the majority of them are naturally inclined to select only one type of units, and that this type is intimately tied up to the shape of the figure being measured. This study thus raises the problem of using exclusively the square unit, since this choice does not impose itself as forcefully as one usually believes.

THEORETICAL FRAMEWORK

According to the three successive evaluations of NAEP (National Assessment of Educational Progress), elementary school children are far from having acquired the most primitive notions related to the concept of area. For instance, the last report (Lindquist et al., 1983) confirms the preceding results and shows that only 24% of 9-year olds have been able to determine the area of a rectangle, although subdivided into square units, and only 8% of the same children have been able to find the area of the rectangle using the given dimensions. Hiebert (1981) stresses the fact that these poor results point to a primitive understanding of the unity concept, a key notion which underlies all measure operations.

Many investigators have shown interest for this concept and have tried to identify the related problems of understanding and learning. For example, from their work on conservation and on the measure of surfaces, Piaget et al (1948) have shown that recourse to common units of measure, in order to compare two different surfaces, is not a natural procedure for children. Moreover, when faced with the problem of enumeration, children perceive as equivalent units which are certainly not (squares, triangles, rectangles). More recently, following their identification of misconceptions of the concept of area, Hirstein et al (1978) have shown the importance of understanding the concept of a unit of area and its spatial characteristics. Rogalski (1982) has studied the covering of a plane figure with similar figures and she claims that the operations of enumeration develop in a context of constant interaction with the properties of the figures involved.

Recently, Maher and Beattys (1986) have studied the problem of choosing an appropriate measuring unit. They found that 11-14-year olds prefer a square unit to cover up a rectangular shape, but not to cover up a circular one. But in that study the type of shapes to be covered was quite limited (rectangular or circular) and so was the choice of unit shapes, the only rectangular unit being the square.

The aim of the present investigation is to look more closely at the problems related to the choice of an appropriate unit in the measurement of plane figures. More specifically, we think it is important to find out to what extent children, who have no school knowledge of the concept of area, are inclined to choose the square as a covering unit. This study should also provide an answer to two more specific interrogations:

1) When covering a given figure, are children naturally inclined to resort to only one or to several types of shapes?
2) Does the shape of the figure to be covered influence the choice of a unit? Are these units similar in shape to the figures?

EXPERIMENTAL DESIGN

Twenty-five third-grade level children aged 8:9 years old on the average and living in a middle-sized city, were individually interviewed. Each interview lasted on the average 20 minutes. The notion of area had not been studied in class yet.

Each child was presented with four geometrical figures of about the same size: a circle (40 cm. diameter), an equilateral triangle (40 cm. side), a square (40cm. side), and a rectangle (48 x 32 cm). These dimensions were chosen so that, on the one hand, the different areas be approximately the same, and that on the other hand, they did not take too long to cover up with unitary shapes, the reason being not to discourage the children facing the task. For each figure to be covered, children were supplied with six elementary shapes: the square (8 cm. side), the equilateral triangle (8 cm. side), the circle (8 cm. diameter), the rectangle (made up of two squares), the rhombus (made up of two triangles), the isosceles trapezium (made up of three triangles). The dimensions of these unitary shapes were chosen so that children could easily manipulate them and also that as far as possible, a whole number of these different shapes could be juxtaposed to cover up the figure to be measured.

The experimentation has been conducted by a single investigator using a semi-standardized form of interview. At first, each child was asked how many (unitary) shapes was needed to cover up as best he/she could each one of the cardboards. Care was taken to insure that the question had been well understood. Then, it was made clear to the child that he/she was free to start with any one of the cardboards, and that to achieve this, he/she could take the pieces (i.e. the units) he/she wanted. Thus, there were no constraints imposed regarding the choice of the figure to cover up, or the units to take. Once the covering of a figure was done, the unitary shapes covering it were taken away so that the child's choice would not influence his/her next covering.

After having completed their covering tasks, children had to give the reasons for each one of their choices. Finally, they had to do a last covering, that of a curvilinear shape The analysis was based on the videotapes of the inverviews.

RESULTS

Considering the first question: **Do children choose a unique type of unit shape** (not necessarily of the same shape as the figure to be covered)?, we found that the majority of children adopted this strategy (cf. Table).

shape to be covered / units chosen	▭	□	△	○	
only ▭	64%	28%	-	8%	4%
only □	12%	40%	-	4%	4%
only △	-	-	36%	8%	4%
only ⏢	-	-	20%	-	8%
only ◇	-	-	8%	-	16%
only ○	-	-	-	44%	32%
Total (question 1)	76%	68%	64%	64%	68%
▭ and □ (both)	12%	28%	-	-	N/A
△ and ⏢ or ◇	-	-	24%	N/A	N/A
Total (question 2)	88%	96%	88%	N/A	N/A
other types of units	12%	4%	12%	36%	32%

Type of unit chosen in relation to the shape to be covered.

Thus, in 68% of the total number of coverings a single type of unit has been chosen. Considering the different types of figures, the highest incidence occured for the rectangle (76%). The lower incidence, in the case of the square (68%), could be attributed to the fact that quite a number of children (28% - cf. question 2), for efficiency reasons, were inclined to use a rectangular unit along with the square unit (the rectangle being twice the size of the square). It will also be noted that the results obtained for non-rectangular figures (triangle and circle) are only slightly inferior to the rectangular ones (rectangle and square).

In analyzing the children's strategies, we find that more than half of them (56%) have systematically proceeded in this way (choice of only one type of unit) for the four types of figures to be covered. And we find only 16% of them who, for all cases, always choose at least two different types of unit. It thus seems quite natural for children in this age group, to resort to only one type of unit shape.

As an answer to the second question: Is the choice of a unit dependent on the shape to be covered?, we see from the table that children are inclined to choose a type of unit that somewhat matches that of the figure to be covered.

Thus, as far as the rectangle is concerned, we find that 64% of the children have chosen the rectangular unit as the sole shape to cover up the figure, and that 88% of them have selected exclusively rectangular units (rectangles or squares). Similarly in the case of the square, although only 40% of the children have chosen the square units as covering pieces (the influence of the rectangular unit has been mentioned above), we find that in all 96% of them have restricted their choice to rectangular units only.

Considering the triangle, we found that 36% of the subjects resorted to triangular units to cover up this type of figure which is very significant if we consider that this kind of covering proved to be very difficult and long to realize. It must also be noted that no child restricted his choice to either the rectangle or the square as unit pieces to cover up the triangle. On the other hand, 88% of them have chosen units of a triangular type (triangles, trapeziums or rhombuses) to cover up this kind of figure.

Finally, it is evident that the circle was the type of figure having been the most problematic to children. Nevertheless, when compared to the other shapes, the circular unit has been clearly prefered (44%). For instance, the square unit has been chosen by only 4% of the subjects! This result is most remarkable if we consider the fact that the circular unit has never been utilized alone in covering non circular figures. Moreover, in combination with other shapes of units, it has only been used in a few cases to cover up the non circular figures (8% for the triangle, 4% for the rectangle and the square).

Thus, we can conclude that whatever the type of figure to be covered, there exists a direct link between it and the type of unit chosen.

THE CURVILINEAR FIGURE

We found it interesting to know what strategy the children would adopt when faced with a curvilinear figure. Globally (cf. table), the results obtained agree with our previous findings. Thus, in relation with question 1, we find that 68% of the subjects have chosen only one type of unit, which corresponds to the averages found for the other types of figures. In relation to question 2, our global results are nearly equivalent to those found for the circle: 32% of the children have chosen exclusively the circular unit because it is better suited to cover up a round thing. Let us also notice that very few subjects (8% total) have chosen either the rectangular or the square units. Nevertheless, we note that 28% of them have shown a preference for a unit of the triangular type. This could be explained by the fact that these shapes having acute angles made it easier to follow the edge of the figure to be covered.

IN CONCLUSION

From this study we see that children who have not yet formally learned the concept of area are naturally inclined to resort to only one type of shape when choosing appropriate covering units. This has been true whatever type of figure had to be covered. On the other hand, the choice of units is strongly tied to the shape of the figure to be covered. For example, the units of a certain shape (e.g. rectangular units) will mostly be used to cover up a figure of the same type (e.g. the rectangle or the square). Nevertheless, we cannot claim that the square naturally appears as a unit of area. This is especially true for non rectangular figures, but it also holds for rectangular ones, in particular for the rectangle.

From these observations it appears that the traditional approach, whereby the measure of a surface area is presented solely in terms of the square units, has to be questioned. It would probably be preferable to construct a different introductory teaching unit that would start with the utilization of a variety of shapes, and that would finally bring the child to a rational choice of the square unit.

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Philosophy,
epistemology,
models of
understanding

THE CONSTRUCTIVIST

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This is the first one of a series of dialogues between Socrates and a researcher in mathematics education who holds constructivist assumptions named Leggos. It is not necessary to understand the constructivist position before reading the dialogue. Leggos, the constructivist, begins with naive constructivist assumptions and with the assistance of Socrates, he moves towards more radical ones throughout the course of the dialogue.

Socrates: Good day, Leggos. I have heard rumors that you have invented a new way to teach mathematics to the youth of our country. I have always held a keen interest in the teaching and learning of mathematics, so I should like an opportunity to learn more about these methods.
Leggos: I would enjoy discussing these matters with you, Socrates, for I consider you to be the patron of these methods.
Socrates: Tell me, what are you engaged in?
Leggos: We have developed ways to teach the students to understand their mathematics.
Socrates: What mathematics is this?
Leggos: The arithmetic, algebra, geometry, and trigonometry taught in our schools, of course Socrates.
Socrates: Ah, school mathematics, then, and how is it that you know that they are not understanding this mathematics?
Leggos: The evidence is abundant. They can not solve problems which deviate only minimally from the traditional presentations. Their knowledge is fragmented and unstable. They propose solutions without assessing their adequacy and whose credibility is negligible. When problems are embedded in real-world contexts the students' performance on them is significantly diminished.
Socrates: Oh, the situation is indeed dismal. How has it come about?
Leggos: Through the teaching-learning process. The students imitate the examples in the text or spoken by the teacher. They memorize formulae and carry out techniques in a mechanistic way.
Socrates: And is the teacher engaging them in a conversation such as we are engaged in today.
Leggos: No, Socrates, the students are like scribes in the classroom, diligently copying and recording the lessons, answering brief product-oriented questions and returning home to carry out the calculations which will produce the required answers quickly and mechanically.
Socrates: So, what you seek to do is to improve the students' performance on these problems?

The Naive Constructivist Claim

Leggos: Oh no, Socrates, not on those exercises but on real problems, what we call problem solving problems. Problems which entail multiple steps and which are set in a word problem context.
Socrates: Can you give me a simple example?
Leggos: Well, for instance, we ask the student to write an equation for the problem, *there are six plebians for every orator*.
Socrates: And what happens?
Leggos: They write 6P=O

Secrates: And do you claim to have a way to overcome such persistent errors in translating from the word problem context to the algebraic expressions?
Legges: Yes, Socrates
Secrates: How do you intend to do this?
Legges: We involve them actively in learning. Our classrooms are student-centered. Our teachers do not teach; they provide a rich environment filled with hands-on activities and the children take part in active ways manipulating the objects and discovering the important concepts for themselves
Secrates: **So you claim to be able to overcome the errors made by students by using manipulatives and involving the students actively in the learning process. By doing so, you claim that the students will learn how to solve problem-solving problems with real world contexts so as to learn school mathematics. The teacher's role is to create such an environment**
Legges: Yes Socrates
Secrates: It is from this point that we shall begin then. It sounds quite simple and so, let's examine these ideas a bit more closely. I have heard that your methods have evolved from your conduct and analysis of clinical interviews with students. How are these interviews conducted?
Legges: Tasks are developed to allow us to examine students' conceptions of particular concepts. The interviewer poses the problems and the students speak aloud as they solve the problem. The interviewer explores the students' ideas flexibily pursuing the students' line of thought whereever it may stray. The fundamental requirement is for the interviewer to attempt to reveal the rationality in the students' methods.

Errors, Misconceptions, and Children's Mathematics

Secrates: What have you learned from these?
Legges: We have found all the things I described before: poor understanding, imitation, blind repitition of formulae, and weak transfer to different contexts.
Secrates: And so, the students perform quite poorly?
Legges: Well, in fact, we have found some positive results as well.
Secrates: And what are those?
Legges: We have found that the students' errors seem to be systematic, not random. They also exhibit some original attempts to solve the problems in potentially innovative ways, they show great diversity in their methods, and they frequently express delight as a result of their participation in the interviews
Secrates: Now this is interesting indeed. Do the students see the systematic character of their errors?
Legges: Well, they perform them repeatedly, so they must be aware of them
Secrates: Ah, Legges, have you been aware of your tendency to rub your hands together nervously?
Legges: No I was not
Secrates: Suppose a child were walking about in a unfamilar room, blindfolded, and reporting the shape of the things s/he encountered to an experienced blindperson who had thus been in many similar rooms before. Perhaps that blindperson had even been the same room, though s/he had no way of knowing if it were the same one, except through the child's description. Isn't it entirely possible and indeed likely that the blindperson would be sensitive to patterns in the child's reports which the child did not notice?
Legges: It is

Socrates: And would the child's experience in the room mirror the blindperson s were they both to walk about the room?
Legges: Not if it were the first time s/he had encountered it S/He would be awkward and uncomfortable I suspect The child's reports would be very confused
Socrates: Could the blindperson know what the child s experience was even after many reports from the child?
Legges: Only as it appeared like his/hers. or in ways which it explicitly differed Let's go back to considering the interview Can you give me an example of a systematic error pattern?
Leggos: Well. in fractions. the students will claim the 5/4. 2 and pi are not fractions
Socrates: Why?
Leggos: It seems that to the student. a fraction is a part of a whole You see they have certain misconceptions
Socrates: And what is it that they cannot do with such a misconception?
Legges: What do you mean Socrates?
Socrates: Well, surely Leggos, you would not be concerned about such a thing unless it impeded the child's progress, would you? Does it cause the child difficulty in solving problems?
Leggos: I hadn't thought about that Socrates I simply judged it as mathematically errant.
Socrates: I see. Leggos I only asked because I thought you had indicated that your goal was to teach the child to solve problems I see now that you are also concerned that the child's knowledge of mathematics is correct
Legges: Surely Socrates. that is what schooling is about
Socrates: Leggos. let us return to this question of what correct mathematics is in a few moments I am captivated by your example of this child's misconception How would a child know that 5/4 is a fraction?
Legges: For a child a fraction is one number over another
Socrates: And what kind of problems is it used to solve?
Leggos: Problems dividing quantities into equal shares
Socrates: And does the answer come out sometimes to be 5/4?
Leggos: Surely Socrates for if 5 apples were shared equally by 4 children each child would get 5/4
Socrates: It is so. Leggos How would a small girl solve such a problem if asked to take 5 apples and to divide them among 4 children?
Leggos: She would probably give each child an apple and then struggle to divide the last one into 4 pieces
Socrates: So each person would have 1 and 1/4 apples
Legges: But Socrates that is the same as 5/4!
Socrates: To you perhaps. how will it appear to the child?
Leggos: As 1 apple and 1/4 But Socrates 5/4 means 5 divided into 4 equal parts and 1 1/4 is the same as five 1/4 portions
Socrates: I agree Leggos. for I know the relationships among these multiple meanings but the child has solved her problem in an entirely adequate way has she not?
Leggos: She has
Socrates: And so does this child possess a misconception when she denies that 5/4 is a fraction
Legges: Yes Socrates it is wrong
Socrates: Indeed Leggos. it is wrong from the perspective of the more expert knower Is it also wrong from the perspective of the child? Does it not meet the child s purposes?
Leggos: Yes I see that the child cannot know unless she/he is told she/he is wrong

Socrates: And if she is told she is wrong can she make sense of why she is?
Leggos: Perhaps not but we cannot leave her with this wrong idea
Socrates: Is there another way? Another problem which might leave her wanting to assert that 5/4 is a fraction?
Leggos: Hmm perhaps a problem with a ratio of 5 yellow apples to 4 red ones. In that, she will not want to write it as 1 1/4
Socrates: Good. Leggos, now you are on your way to exploring interesting curricular ideas But let us stay with this idea of misconceptions a bit longer Perhaps, we ought to declare that the label "misconceptions" is unfortunate for it neither conveys who is aware of the limitations (the observer) nor does it emphasize that despite its limitations, the concept does function to assist the student in doing something
Leggos: I think I agree Socrates, and I can see from our example that it conveys a judgment that the child has not managed to learn the concept which s/he could have been expected to learn. What I wanted required a different problem perhaps But, Socrates, if "misconception" is misleading, then a phrase needs to be developed which conveys what the primitive or naive conceptions of a student allow him/her to do and how they are restrictive
Socrates: Restrictive, primitive, naive from whose point of view?
Leggos: The teacher, researcher or mathematician So perhaps we should name it from the perspective of the child... say *child's mathematics*
Socrates: But would not my own mathematics, unsophosicated as it is appear to be a childish mathematics to a research mathematician
Leggos: Mine might as well, Socrates, there are many subset's of the population who might make claim to their own mathematics, and perhaps there are as many mathematics as there are people ultimatiely But for my situation, where I am interested in educating the child, perhaps such a phrase, though limiting serves some purpose
Socrates: Leggos, at the beginning, you claimed that your methods eliminate errors and misconceptions If you were to cast these as limited but sensible conceptions, as *a child's mathematics*, would you still seek to eliminate them?
Leggos: No, I see that I would not, Socrates I would seek to build from these limited conceptions towards more sophisticated conceptions
Socrates: Is there a way to assist the child in seeing the limitations in her beliefs?
Leggos: I am not sure that one can ever *see* the limitations in one's own perspective before one has achieved a more sophisticated perspective But, I can imagine that the child could recognize that she was bumping up against a problem which she could not solve with her perspective
Socrates: And how might you do that with 5/4, 2 or pi?
Leggos: Each one might cause a different difficulty Socrates Not understanding that pi is a ratio of circumference to diameter could hide the recognition that for all circles that ratio is constant, not recognizing that in the number 2 as a hidden ratio to the unit 1, can cause students difficulty in solving problems with quantities such as miles per hour, feet per second, pounds per inch, the "per" quantities And not seeing 5/4 as a fraction might cause the student to lack an understanding of simple ideas of probability I see that the name, fractions, hides a whole host of different conceptions, each one complementary but different
Socrates: So in each case there might be a different problem circumstance that could be devised for extending and modifying the concept of a fraction There is something more that intrigues me about these interviews You said that the interviewer comes to the interview with a certain set of tasks, and then depending on how the student responds, the interview evolves
Leggos: Yes

Modelling, Self- Reflection and Interdependent Communication

Socrates: And what is the interviewer finding out?
Leggos: What the students concept is
Socrates: Indeed, and is that indelibly in place in the students' head, awaiting discovery by the interviewer?
Leggos: No, it takes a great deal of skill to interview You must enter with the expectation of seeking these stable belief systems in students
Socrates: What you have said is that through these tasks, the interviewer can reveal these conceptions held by students which appear limited to the interviewer Can the interviewer also reveal the child's conceptions which do not result in obvious limitations, or seem to differ from those held by the interviewer?
Leggos: What do you mean Socrates?
Socrates: Well, it seems that one can examine where the student's activities and responses differ from the one's expected by the interviewer When the responses do not differ, can the interviewer presume that the student's concept is the same as his/her own?
Leggos: I do that, but I must admit that when I listen closely, I am often surprised by the diversity of the ideas I often presume agreement only to learn that I am mistaken
Socrates: How do you find out that you are mistaken?
Leggos: By the unexpected responses given by the child
Socrates: So in the absence of evidence to the contrary, you assume agreement?
Leggos: I suppose
Socrates: And so, Leggos, can you in fact claim that you are revealing the child's conception?
Leggos: But how am I to describe the results of our investigations if I am not uncovering the child's conceptions For documenting these provides the evidence for our claims
Socrates: What is wrong with what you have said yourself? As you have said through the use of particular problems, designed to highlight certain aspects of a concept, you are attempting to characterize *children's mathematics* In doing so, you build *a model* of what a student believes by creating an outline of the boundaries of the belief system, from your own more sophisticated perspective What is the child doing in the interview?
Leggos: Being interviewed Telling us what s/he knows
Socrates: But Leggos, did you not say that the students in the interviews often express pleasure at their participation? Why is this?
Leggos: Perhaps because they seldom get the opportunity to be listened to in mathematics
Socrates: That may be so, but perhaps there is more to it than that What is it that they say more precisely?
Leggos: They say that they learned more in the interview than they do in class It is always an embarrassment Socrates for we claim to be examining their conceptions, not changing them
Socrates: I see Suppose you took that as a strength of the interview how could you understand their role? Recall, you have just finished revising the role of the interviewer into a model builder modelling *children's mathematics* So I pose the question, if the child's active participation in the interview process were highlighted how might you characterize it?
Leggos: Well, I doubt they are trying to examine the interviewer's conceptions, if you are implying that Socrates
Socrates: Could they do this if they wished?
Leggos: Not at a very deep level for we decided that to build a model one relies

heavily on the discrepancies with expected responses and the interviewer seldom makes errors or deviates from the students' expectations

Socrates: Well figuring out the interviewer's mathematics would be quite difficult then for the student. although you allow this might occur at some level But is there something else that the student might be inclined to do?

Leggos: They do seek to figure out what it is that the interviewer wants

Socrates: So indeed, they too are in the process of model building but their task is far more difficult, is it not?

Leggos: Well, with careful analysis of the conduct of interviewers, it is often the case that the student seems to respond to subtle cues on the part of the interviewer

Socrates: I see, and so is this what students gain from the interview?

Leggos: No, I do not believe it captures it all, for the student's also seem to be impressed by their own attempts to solve problems They seem to act more automously in the interview, maintain a sense of responsibility over their activities

Socrates: What is this you are saying?

Leggos: Well, the students often are surprised by their ability to explain their methods, and by describing them to an expert listener. they seem to become more aware of them It provides quite a contrast to the docility we witness in class

Socrates: Don't they feel exposed?

Leggos: At times, but they also seem to gain some insight into the concepts through their process of examination

Socrates: Is that because they solve the problems successfully?

Leggos: Perhaps, in part, because their level of success is higher in an interview than with paper and pencil But the process of verbalizing their methods to another person and the need to be complete and careful in their explanations also influence their level of insight.

Socrates: I see, so the interview contributes to their ability to see their own thinking

Leggos: Yes, Socrates, it does seem to be a process of *reflection* encouraged by a situation in which they wish to *communicate*

Socrates: Ah, Leggos, this business of the social interaction in the interview seems far more complex that we had expected. It seems to be a process of two people building models of the other and of themselves Is the process the same for the two people?

Leggos: No. for the student, the model building of the interviewer is difficult, and they are encouraged to focus back on themselves, for the interviewer. the model building of the student is primary, and the model of him/herself is elaborated as the differences in expected responses are explored.

Teaching and Learning Revised

Socrates: And now, Leggos, how do you use the results of this research?

Leggos: From it, we can reveal students' conceptions and we design materials to build from them With these materials, the students can develop more powerful conceptions

Socrates: Are you saying that you treat the interviews as providing model of *children's mathematics* and using these you create materials Does not our discussion of the conduct of the interview provide any influence on the design of the teaching and learning?

Leggos: Yes, Socrates as I said we expect students to construct the ideas actively, for we know active participation is necessary

Socrates: Active participation Leggos, is just a phrase What happens in that interview that makes it active participation? How can these things be encouraged in the teaching- learning process?

Leggos: Socrates, these questions are quite difficult, but I will try to answer them succintly In the interview, the student attempts to solve a problem with a conceptual basis, s/he engages actively devising strategies to solve that problem, trying different methods, working from his/her prior knowledge, and in doing so s/he becomes aware of his/her own commitments, beliefs, and strategies
Socrates: An adequate description of the individual's process of solving the problem Notice that your portrayal omits the part of interviewer Who in the classroom can act the role of the interviewer?
Leggos: The teacher
Socrates: Are there differences?
Leggos: Yes the teacher is responsible to a number of students
Socrates: Is there anyone else?
Leggos: No, Socrates, for the other students lack the necessary knowledge to act the part of the interviewer
Socrates: I agree, but I wonder if they can be taught to play some role to assist in the process of active participation as you describe it Perhaps we can define this process a bit more.

A Constructivist Cycle

Leggos: Well, Socrates, at first the student interprets the problem
Socrates: Shall we call that finding the **problematic**, using that term to describe what problem the student wants to solve?
Leggos: Okay, and we must recall that there are many different problematics for a given problem. Then the student plans and executes a course of action to resolve the problemmatic using his/her child's mathematics
Socrates: And that shall be the **action**?
Leggos: And the student becomes aware of his/her own beliefs and strategies as s/he talks out loud?
Socrates: A process of **reflection** is it not?
Leggos: Yes, I suppose it is a process like that, a problematic action and reflective cycle repeating again and again in the company and encouragement of an interviewer.
Socrates: So, Leggos, originally you claimed to overcome the errors made by students by using manipulatives and involving the students actively in the learning process You said the teacher's role was to create such an environment Have you changed your position?
Leggos: I believe my original statements needs some improvement indeed Socrates I now see that it is not a case of overcoming the errors of the student but it is a case of modelling a child's mathematics In doing this discrepancies between the expected range of responses and the responses of the child assist us in devising our models, it does not, however give us access to the child's conceptions in any direct sense By engaging in a cycle of construction, including defining the problematic, acting using prior knowledge and reflecting a student constructs his/her ideas And if the teacher is indeed to act in the role of the interviewer, a role which seems essential in the process of construction s/he will have far more responsibility in teaching than to simply create the appropriate environment Finally our discussion suggests that the role of the other students in the process might be important to attend to

What is Mathematics: Alternatives to Idealism and Realism

Socrates: You are a very able learner Leggos Notice that so far it is not obvious that the ideas we have examined are unique to mathematics Are there implications

of your interviews that are specific to mathematics?
Leggos: Socrates, though our example of the children's mathematics was mathematical, I do not see that these methods are limited
Secrates: Leggos, there are a few ideas in your original statement which we have not examined as of yet Let us visit those very briefly You said that you use manipulatives and that these manipulatives would assist students in solving real-world problems How do you see that as coming to pass?
Leggos: **The manipulatives we use are designed to embed the concepts of mathematics in them. When working with these, these ideas will emerge. By learning meaningfully in this way, the child will be able to recognize the mathematical ideas in the real- world. By doing so, our goal is to assist the student in developing an understanding of mathematics.**
Secrates: Let us explore your ideas about mathematics Can you give me an example?
Leggos: Well, suppose we stay with fractions since our explorations have been with that
Secrates: What is a correct understanding of a fraction, Leggos?
Leggos: A fraction is a number in the form a/b where a and b are integers.
Secrates: Oh, then, 2 and pi are not fractions
Leggos: A fraction is a number which <u>can be put</u> in the form of a/b where a and b are integers
Secrates: I see, then a fraction is a rational number and pi is not a fraction, but it is a ratio I was mistaken before Do the students understand the difference between fractions and ratios? These definitions can be very subtle We must explore the place of such formalism at a later time, for I see that this neat and tidy stucture underlies f your concern for "correct" mathematics. For now answer me this, why are fractions important?
Leggos: Oh, because they have real-world applications Students encounter them all the time
Secrates: Please Leggos, tell me where in the world a student will stumble over a fraction?
Leggos: Suppose a young boy measures himself and finds that his height falls between 45 and 46
Secrates: And where is the fraction?
Leggos: It is the point between 45 and 46 where he marks his height
Secrates: The point is the fraction?
Leggos: No, well, the fraction is the length from 45 to that point
Secrates: And how is the child to know this?
Leggos: He must estimate what proportion of the distance is covered
Secrates: So Leggos, where is the fraction?
Leggos: It is the result of comparing the distance between 45 and the point to the distance between 45 and 46
Secrates: Is the fraction the distance or is it a comparison?
Leggos: It is a comparison <u>made by the child</u>
Secrates: So the fraction is an action by the child in order to describe a distance which is less than one unit long Ah, it is a part of a whole Can it be determined that this is really his height?
Leggos: Someone else might measure as well
Secrates: Would you then know his real height?
Leggos: No, you would have other measurement
Secrates: Leggos then is the real fraction knowable
Leggos: Perhaps not, but error is always a part of measurement
Secrates: Is it error you need to speak of or is it variability? Can you know if it is

error ever?
Leggos: I see your point for measurement Socrates perhaps a more formal example would allow us to demonstrate the perfection and eternal qualities of mathematics.
Socrates: Okay, let's try one
Legges: 1/2 + 1/3 = 5/6/ That is a statement about fractions which is really true Socrates We can prove it
Socrates: Suppose I got 1 out of 2 problems correct on a quiz (or1/2) and 1 out of 3 on the other, how many correct will I have of my total?
Leggos: 2 out of 5
Socrates: Can that be written 2/5?
Legges: Yes
Socrates: So, is 1/2 + 1/3 = 2/5?
Legges: Yes, but that is not what we mean by 1/2 + 1/3 = 5/6
Socrates: Precisely my dear Leggos The meaning is not **in** the mathematics though it is convenient to speak that way Just as when we say an object is symmetrical we mean that with our bilateral way of viewing and a mental act of folding, one side of the object will land on top of the other To say an object is symmetric, is to embed in the object, the mental act of folding Suppose we were to differ on whether an object were described as symmetric, how would we decide?
Leggos: We would each try to demonstrate its symmetry to each other
Socrates: And could we know **really** who was right?
Leggos: No, although we could appeal to an expert
Socrates: Let us return to the question of the room again with the blindperson and the child If no one with vision were allowed ever in the room, could you come to know the real interior of the room?
Legges: If enough people were to report on it, and if we were to test out the level of consensus in our reports by experiments conducted within and outside of the room we could achieve a measure of consistency and confidence in our description Though, no doubt even within that apparent consensus, disagreements would likely arise They always do
Socrates: And would any of those descriptions be definitively the way the room really was
Leggos: Not necessarily, but if we could communicate to enough people about our experiences would it matter?
Socrates: Indeed it would not, if you are prepared to accept that no way of knowing how the room **really** was is possible
Legges: But, Socrates, you would simply remove the blindfold for those who could
Socrates: And how would you know another more figurative blindfold were not in place?
Legges: Because we would all agree
Socrates: Though my experience denies that level of agreement would happen it would not solve the problem, for even consensus is achieved through another act of human cognition No escape from this predicament is possible We cannot know **reality** for that in a matter of ontology Our epistemological boundaries require that all knowledge requires acts of **construction**.
Leggos: And is that the assumption behind constructivism?
Socrates: It is, and Leggos is the blind person like any other person but s/he lacks sight? Are not his/her other capabilities more acute, developed and perhaps different in quality than ours from the absence of sight, as we speak of it?
Leggos: Yes I believe that is so
Socrates: So canyou indeed declare that removing the blindfold is a initiation into 'true' vision or is it only another a switching of systems of perception?
Legges: I am embarrassed at my neglect of the blindperson's strengths Socrates And Socrates are you also implying that a researcher, a mathematician or teacher is

in some way akin to the blind person. are you suggesting that becoming expert can also lead to a blindness? For whereas blinders can restrict one's field of vision. there is also freedom from distraction Perhaps it is specialization we ought to speak of at the same time as we speak of expertise

Secrates: I had no intended that Leggos. but perhaps it is so

Leggos: It may be true that knowledge of our experience of the world is bounded by our conceptions. Socrates. but surely. it is not the case for the queen of the exact sciences. mathematics?

Secrates: How you will attempt to secure this special status? And is it possible in the arena of human knowledge to ever escape the boundaries of human knowing?

Leggos: Not in any way. though I must confess to believing there is a lot about human knowing that we do not know yet How feelings and intuition work. how we communicate and so on But I see that mathematics seems to require a human activity.though I cannot say that when I see a fraction I go through the complete process of construction

Secrates: That is so. Leggos. for automaticity and stability are certainly necessary for using a construct efficiently But we will explore this further at another time

Leggos: I do hope so. Socrates. for I fear you are leading me down the path of the skeptics And no son or daughter of the ruling class will attend my school should I be cast in the company of such critics But for now I must admit I find your arguments compelling.

Secrates: Ah so Leggos. perhaps mathematics is also cultural and political. if you fear such reprisals.

Leggos: Socrates. all knowledge is essentially political. as you well know

Secrates: Yes. but to establish that the content of mathematics is political. its concepts. and structures as is its means of access and its dispersal will take another conversation Leggos

Leggos: I will look forward to such a conversation

Secrates: Also. keep in mind that a constructivist way of looking is only as correct. as it is helpful to you in solving the problems you have described about students' learning in mathematics. It would be contradictory for me to assert any stronger position than that. while establishing the boundaries of knowledge

Leggos: I see

Secrates: And. Leggos. one last line of questionning will complete our discussion today You have claimed the manipulatives you use are designed to embed the concepts of mathematics in them When working with these these ideas will emerge By learning meaningfully in this way. the child will be able to recognize the mathematical ideas in the real- world Do you wish to revise that portrayal?

Leggos: Yes Socrates. I must. for as you have led me to see. the mathematics is not in the manipulatives but the actions and operations enacted on the manipulatives by the child.

Secrates: And how shall the child come to act on the manipulatives?

Leggos: Through his/her interpretation of a problematic which she finds in her/his interest to solve

Secrates: And how shall s/he gain facility in these actions. to recognize opportunities for their use?

Leggos: S/He must reflect on those actions and see how they were useful in solving the problematic

Secrates: And is this activity done all alone?

Leggos: No. Socrates. in this the lesson is the clearest of all. for I shall have to educate my teachers in the ways I have learned today. though I must confess I have a need to reflect on all of this myself For Socrates. I see that I did not pay close attention to what it was I had learned through my own methods of interview It is clear there is more to this than what is easily noticed from the interview and I must stregthen my awareness of my own commitments and my own understanding of the problem before we can continue to work

Socrates: Leggos thank you for an enjoyable exchange and as you are reflecting on our conversation you might wish to include another item in your deliberations

Legges: And what is that?

Socrates: Do not forget to consider my role in this exchange for what is contained in the container has the same shape as the container Perhaps next time we can talk more about teaching. Good day

AS A CHILD LEARNS, SO MUST ONE TEACH (ELEMENTARY SCHOOL MATHEMATICS DIAGNOSIS AND INSTRUCTION)*

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Diagnosis and instruction in elementary school mathematics have tended to focus more on mathematical content and on error patterns than on the conceptual misunderstandings within the learner. The present review cites examples of weaknesses inherent in such approaches and contrasts these with the strengths and practical applicability of a growing body of research that has been built on insights from genetic epistemology. Exemplary research projects offering promise for the advancement of knowledge regarding meaningful and helpful diagnosis and follow-up instruction are described and discussed.

A natural outgrowth of the kind of behaviourist thinking that has dominated North American in education is the tendency to focus almost exclusively on the mathematics content being learned by children and to alter mathematical "stimuli" according to observed error patterns while providing considerable "drill-and-practice" to "extinguish" inappropriate responses. A more realistic approach attempts to understand how the child processes the information presented, adapting to feedback from real-life experience. When knowledge and understanding of mathematics are seen in terms of ongoing thinking processes rather than as products, traditional tests prove less valuable than diagnostic techniques like those that have grown out of "genetic epistemology." For example, the Chelsea Diagnostic Mathematics Tests (Hart et al., 1985) were constructed using information from numerous Piagetian and neo-Piagetian interviews of large numbers with 10- to 15-year-olds. However, tests like these have only been developed relatively recently. What about their predecessors?

*Excerpts from a review of more than 300 diagnosis/instruction research documents prepared for the Student Evaluation Branch, Alberta Education

BEHAVIOURIST APPROACHES

Prior to the popularization of "genetic epistemology," North American educational test design was (and, in many ways, continues to be) dominated by behaviourist learning theories. Th? one impossible weakness in behaviourist theories is that the "learning" which these theories explain bears no necessary logical relation to the events in the environment. "It is always possible that the reward has come for reasons that have nothing to do with the response. The animal never understands the nature of the problem, never knows why the 'operant' operates and therefore, all his learning is potentially superstitious" (Mayer 1961, p. 75).

Many diagnosis and remediation studies do not appear to have adequately considered underlying cognitions. One such study was carried out with 36 Grade 3 pupils. The Key Math Diagnostic Arithmetic Test was administered with items in ascending or descending order of difficulty. There were 15 addition and 14 subtraction items. Four feedback item-order treatments were given: two gave item-by-item feedback of correctness of response with either ascending or descending item difficulty, two did not give feedback. In any event, when feedback was given it was with respect to answer correctness only. The researchers found it "puzzling" that pupils who attributed success or failure to external causes like "task difficulty" or "luck" performed better in subtraction when receiving feedback. "Somehow, contrary to the hypothesis, it appears that such individuals found both positive and negative feedback as facilitating. We have no reasonable explanation for this result. . . ." (Englehardt, Van Wengenen & Thomas 1982, p. 56-60). Such "puzzles" underscore the importance of taking into account underlying cognitions and processes, especially in relation to realistic, concrete constructions of meanings. More reliable evidence is needed than that drawn from naive student introspection, and the kind and quality of feedback is an important consideration.

An often-cited guidebook for diagnostic teaching of arithmetic (Reisman 1972) suffers from a preoccupation with correct mechanical procedures and correct answers, to the neglect of ways to encourage children to construct meaningful ways of thinking about arithmetic and its concepts and processes. It focuses almost exclusively on the mechanics of computational algorithms and on the "patching up" of erroneous procedures. While it is unlikely that anyone would contend that computation is unimportant in arithmetic there is a growing body of evidence to support the claim that the best way to improve student performance in this area is not to focus on the algorithmic mechanics and the vaguely conceived symbols involved. An indicator of the power of hands-on activities with concrete materials has come from an analysis of 64 elementary school research studies which reported that pupils who had used manipulative materials in arithmetic scored at approximately the 85th percentile on achievement tests whereas those who had not had such experiences scored at approximately the 50th percentile (Suydam 1986, p. 32 citing Parham 1983).

Research on the effects of "low-stress algorithms" appears to have focused primarily on computational mechanics and memorized shortcuts. Examples of student work with low-stress algorithms give the impression that they are heavily dependent on pictorial organizational "props" (that appear to have little to do with the underlying meanings for the computations being carried out) and that they are geared toward special-education, severely remedial, or disadvantaged children. It is claimed that such children can reach "a normal level of mastery in critical numerical competencies," while normal children could reap time savings by using such procedures (Hutchings 1980, pp. 244, 245). But "numerical competency" only means being able to compute accurately with simplified algorithmic procedures. What about promoting understanding and realistic problem solving? Imitative teaching of operations and solution methods for one-step "word problems" is not enough (Cooper & English 1984, p.111).

COGNITIVE APPROACHES

Learning theories are being developed that take account ". . . of the systematic development of an organized body of knowledge, which not only integrates what has been learnt, but is a major factor in new learning . . ." (Skemp 1962, p. 133). Any teaching/learning design which can ensure that each individual will be able to build the prerequisite concepts, before or while tackling new learning tasks, promises to facilitate effective, enjoyable, useful, and transferable learning. Such an approach encourages the development of relational thinking ("knowing both what to do and why") rather than instrumental thinking ("using rules without reasons"), which is likely to occur when one's mathematical diet is limited to a steady stream of narrowly focused "explanations" and "exercises" (Skemp 1978, pp. 9, 14).

Thornton (1978) reported the results of using experimental materials in the teaching of basic facts of addition and multiplication to students in Grades 2 and 4. The outcome performances used to compare the effectiveness of the experimental and control groups were timed posttests and retention tests of student achievement in recalling number facts, but the experimental treatment was essentially a cognitive one in which students were encouraged to organize their thinking and create their own thinking strategies or adopt strategies suggested in the instructional materials. Superior performances by the experimental groups were taken as indicating that an emphasis on the cognitive can demonstrably improve students' achievement, even in routine computational skills.

One way to provide supplementary diagnosis and instruction for a classroom teacher's mathematics program is to set up a mathematics lab that includes provisions for clinical assessments. The Diagnostic-Prescriptive Program in Mathematics ". . . was designed to improve mathematics achievement by directing the intervention toward diagnosed deficiencies in number concepts, computational skills, relationships among measures and problem solving" (Knight 1979, p. 1). Mathematics laboratories were set up in seven participating schools with Stanford Diagnostic Tests and/or Metropolitan Mathematics Survey Tests as diagnostic pretests and progress posttests. Mathematics activities were jointly prescribed and planned by the mathematics resource teacher and the regular teacher. A variety of physical materials such as Unifix cubes and base-ten blocks were available for hands-on activities. The pupil participants were 420 2nd through 8th graders who had been identified as needing remedial help. Significantly higher posttest scores were achieved by the participant students than were statistically predicted from the pretest scores. Over the seven month treatment period the smallest grade-equivalent gain was 11 months and the largest, 14 months. (Knight 1979, pp. 1, 3, 7-11).

A mathematics lab can also be made an integral part of the instructional program for all students. The formal diagnostic assessments can be replaced with daily and weekly individual anecdotal appraisals in the context of ongoing learning activities. This was the approach taken in the Highwood Bilingual Elementary School Mathematics Lab (Harrison & Harrison 1983). The Lab was started in 1976 to help students develop a better understanding of mathematical concepts through purposeful use of a wide range of manipulative materials. A Project Teacher worked with the classroom teachers over the first six years of the Project to develop student-centred, concrete, mathematics-learning-activity approaches for six- to eleven-year-olds. By observing how real objects are manipulated and by encouraging discussion among students, the classroom teachers were able to gain valuable insights into their pupils' understanding of the concepts being developed, enabling them to further that understanding through supportive activities. A frame of reference based on contemporary cognitive learning theories was used in the selection of manipulative materials and in the design of the teaching approaches used in the Lab. (Harrison 1982) Just before the Highwood Math Lab project began, the Stanford Achievement Test median arithmetic score for the eight-year-old students was 5 to 10 percentiles below the median reading score, approximately the 50th percentile. After the first three years of development of the Lab, the eight-year-olds' median arithmetic score was then 15 to 20 percentiles above that of the reading scores, which had remained near the 50th percentile. (Harrison & Harrison 1983, 1986, Highwood 1979)

Several studies (e.g., Booth & Hart 1983, p. 80, and Romberg & Collis 1985, p. 376, 377) have reported on the importance that children ascribe to their own intuitively devised "child methods" in arithmetic and how these can interfere with the development and use of paper-and-pencil algorithms that should prove more efficient in the long run. It has been suggested that the current emphasis on paper-and-pencil procedures as early as Grade 3 may be inappropriate in terms of pupil cognitive development, partly because "...children's decisions to use taught algorithms to solve these problems appear to depend more on the semantic structure of the problems than on either instruction or cognitive capacity...." (Romberg & Collis 1985 p.376). Educators would do well to invent ways of building on these child-methods instead of trying to subvert them with algorithmic procedures that the child finds meaningless, at least initially. With so many electronic machines around that can handle algorithms consistently, accurately, iteratively, and endlessly, why promote anything less than the intelligent learning of mathematics?

CONCLUSION

A major weakness in research in the area of diagnosis and instruction in mathematics has been the preoccupation with mathematical content and computational algorithms to the exclusion of considerations of the processes by which pupils build abstractions and generalizations from their own actions with physical objects. It is necessary to understand how children learn mathematics and how to determine the levels of abstraction and sophistication that the children concerned bring to the particular mathematics topics at hand. Approaches that have taken into account insights from cognitive assessments of child responses to mathematical tasks have proven particularly successful in providing effective diagnosis and instruction.

One source of interview task assessments of children's mathematical cognitions in the major mathematical topics from Grades 1 to 3 is the Assessing Cognitive Levels in Classrooms (ACLIC), Final Report (Marchand, Bye, Harrison & Schroeder 1985). For older children, cognitive assessments can be made by means of paper-and-pencil adaptations of individual interviews or by drawing from existing sources of cognitive assessment items (e.g., Cornish & Wines 1977 1978; Hart et al. 1985; Marchand, Bye, Harrison & Schroeder 1985).

Instructional activities suitable for providing follow-up to the cognitive assessments need to be mathematically sound and relevant to curricular expectations and must make provision for the child to respond naturally at or near the level of cognitive sophistication so far reached in the particular context. Provision must be made for the student to build concepts and strategies from direct experience, by communicating with others, and from within through imagination and intuition. A particularly rich source of instructional activities in mathematics for 5- to 12-year-olds has been developed by Richard Skemp in accordance with these principles just stated. (Skemp 1982-84, Primary Mathematics Project, in press) Exemplary process-oriented materials for mathematics teaching, learning and evaluation have been developed for secondary school students at the Shell Centre for Mathematical Education, University of Nottingham. Each of these sources of cognitively-oriented learning materials describe field-tested, inservice strategies that can be used to lead teachers, evaluators, and researchers to a greater appreciation of cognitive assessment procedures and activity-based instructional methods.

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EPISTEMOLOGICAL DETERMINANTS OF MATHEMATICAL CONSTRUCTION, IMPLICATIONS IN ITS TEACHING

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Abstract

In this work an analysis of the epistemological constituents of mathematical knowledge is attempted. Its involves a discussion over the relationship between the roles played by the epistemic object and subject, as well as, within the first but with its own status, the social role in the cognitive result.

Before beginning completely the chosen set of problems, an attempt is made to place it in a historico-philosophical context that allows a better understanding of its nature. This is given especially by the existing contraposition between Empiricism and Rationalism, and in particular the crisis in the latter brought on by the results of Gödel in the thirties. The author affirms that these results show the need for an epistemological and methodological renewal in the understanding of the nature of Mathematics. It reveals a weakening in the rationalist paradigm which emphasizes the deductive-formalizing aspects, and the **a priori** sense, dominant in the history of mathematics.

An epistemologico-psychological analysis is made starting from the concept of Piaget of "reflexive abstraction". The essence of this concept is described, along with some premises and limits of the horizon where it is inscribed, which is to say in that of the so-called genetic epistemology.

So, by way of conclusions, we study the consequences of this interpretation in Mathematics Education and its psychology, its implications in the orientations, but above all in the methods of teaching. Also the importance is affirmed of the history of Mathematics to give structure to the priorities and orders in Mathematics Education.

1. In the 1950's and 60's took place the now famous reform which introduced the so-called "modern mathematics". This reform sought to emphasize the axiomatic, deductive and abstract character of mathematics. A structuralist and formalist vision of mathematics was favored over empirical, heuristic, intuitive or constructive approaches. These reforms corresponded completely to paradigms about the nature of mathematics which had predominated for centuries (in different manners) and had been in vogue in the period known as the "Crisis of Mathematical Foundations" at the end of the 19th and beginning of the 20th centuries.

In reflecting about mathematics, we will call those paradigms which affirm the a priori character of mathematics and emphasizes the role of the mind as a producer of absolute and eternal truths **rationalist paradigms**. The rationalist paradigms have combined, althought not always, two additional visions about mathematics: that of Plato, which makes mathematical entities citizens of an abstract and independent world, accesible only by means of reason, and, on the other hand, the vision which emphasizes the axiomatic and formal as determinant characteristics of the nature of mathematics. An integrated combination of these ideas, in some proportion, constitutes what can generally be called the Rationalist Paradigm in Philosophy of Mathematics (I emphasize rationalist because it constitutes the decisive theoretical element in this conceptual conglomeration).

The reforms previously mentioned were made in large part on the basis of this intellectual paradigm. The interesting thing is, however, that they were made 20 or 30 years after this paradigm was severely criticized by Gödel in his famous article "On Formally Undecidable Sentences of **Principia Mathematica** and related systems" in 1931. The conclusions of Gödel are extraordinary. On one hand they imply that any formalism sufficiently strong for expressing the basic parts of the elemental theory of numbers is incomplete. As such, he concludes that mathematics cannot admit any absolute formalization and that formalizable parts do not guarantee consistency. On the other hand, he cast doubt on the Hilbertian pretentions for proof of consistency and mathematical fundamentation (pretentions which monopolized the interest of many mathematicians and philosophers after the "inexcusable" weakness of the Logicist project of Frege-Russell). Gödel's conclusions were directed toward breaking the scheme of an absolute and closed system for any discourse and the pretention of Rtionalism of explaining any reality on the basis of reason. They were a reminder that no rational system can explain the totality of the real, as well as a call for seeking a bridge between Rationalism and Empiricism, giving a just role in mathematics to intuition and experience. They constituted a hard blow to the rationalist paradigm about mathematics to Formalism, to Platonism, as well as the infallibility of this.

The reading of the implications of the results of Gödel, however, was not undertaken in an ample and decisive manner. The best proof of this is probably the reality of the reforms in the teaching and learning of mathematics, which might be explained by the Platonist position of Gödel himself, or by the weak traditions of applied mathematics in North America, or simply by the enormous life of a paradigm which, in order to die, needs to be replaced first by another. What is clear is that no alternative paradigm to Rationalism exists. The conventionalist and linguistic proposal of Logical Empiricism does not appear a totally satisfactory option.

2. The crisis of Rationalism rising from the results of Gödel should lead to the denying of the principal premises of the majority of thinkers in this tradition. Now it can't be said, for example, that the theorems of mathematics are true in the real world and that these infallible truths are accesible to human thought; or that intersubjectivity is a non-practical and non-material metaphysical fact. These have been, during a long period of time, theoretical suppositions beyond discussion 1. The results of Gödel have created the necessity for a theoretical rethinking. It is not strange, then, that Lakatos

insisted since the 1960's on the "rebirth of Empiricism in recent Philosophy of Mathematics" 2. The paradigm of the injection of truth from the summit to the base entered in crisis with Gödel's results. The intents (which Lakatos called "Euclidian") to incorporate the nature of mathematics in its scheme failed. Lakatos affirmed the existence of a "... genuine revolutionary turning in the philosophy of mathematics"3. To what point a reactivation of Empiricism has extended is difficult to know; it is not easy to change a paradigm when this has been a fundamental pillar of western epistemology for many centuries. But it is obvious that Empiricism has received greater attention.

Alongside Cartesian rationalism, the empiricist tradition developed together with the experimental sciences. This gave birth to a different rationality, which gave preeminence to the sensory in the knowledge of the world. Empiricism has stolen much terrain from Rationalism in that which has to do with knowledge in general. However, mathematics, until Gödel, had been the last redoubt of Rationalism. Before Gödel, at times Empiricism was flirted with, but the predominant mathematical philosophy was rationalist. The ideas of Mill about mathematics were always strongly attacked.

The two traditions could both be characterized as rationalist, in the sense that both work on the basis of deduction and the axiomatic, but in their bodies of theory they differ about the injection of truth (in one case from top to base, and in the other the reverse). This is the approximation of Lakatos 4. Both traditions have been mutual antagonists in the latest centuries of epistemology. It is only natural that with the weakening of Rationalism, Empiricism would be fortified. However, the situation hasn't been so simple. The classic vision of Mill isn't so easy to accept; the character of mathematics a product of the generalization of physical experience? Go back to almost mechanical inductivism in reflecting on mathematics? This difficulty had led the great majority of empiricists to deny real material content to mathematics, and like Ayer, Pap, Carnap, and up to a point the late Russell, concede it a conventional and linguistic character: "There are three feet in a yard". Parting from the considerations we have mentioned, the options which appear are: mathematics has to do with infallible truths **a priori**, without material content, or these refer to some ideal or material world; or mathematics does not have to do with infallible truths and so the options which appear are inductivism like Mill or conventionalism. This appears to be the crux of controversy in modern mathematics. If the scheme of **a priori** truths is soffocated by the "Gödel factor", then everything indicates a tendency to submersion in some type of Empiricism.

A revision of reflection on the nature of mathematics has been needed for several decades. In this terrain the analysis of its constituent epistemologies is important. We will use ideas of Piaget to develope an epistemological proposal.

3. In the theoretical analysis of mathematics, it is necessary to treat the central theme: What type of mental activity does mathematics engender? How is mathematical thought generated? Are we talking about the inductivist generalization of Mill? If not, are we not affirming typical rationalism? For Piaget, in relation to mathematical thought, there has been an evolution in human beings in steps from birth to maturity. Each of these steps manifests itself in **mental** structures 5. These structures are **operational** in character 6. The passing from one to another takes place through an abstraction, different than that postulated by Aristotle, which is called **"reflexive"** 7. We are talking here about an operatory as well as combinatory generalization 5. Each new structure adds elements to the former in a new synthesis 7. This leap forward occurs because of necessity for filling voids in the avoidance of contradiction 10. This lack of contradiction, or coherence is, according to Piaget, a particular case or operatory "reversibility" 12.

As such, **the abstraction is produced parting from the actions of the subject,** not of **the object** 12. The reflexive abstraction is identified with the power of the subject to coordinate actions. Moreover, we are speaking of an abstraction parting from the combination-coordination of actions 13. This power of the subject is not exactly hereditary, but it is biological 14. It is part of the "cognoscitive functions" in general. According to Piaget his cognosctive functions are the differentiated organs of the "autoregulation" of the subject in relation to the object; it is something which affects; then, behavior. This vital autoregulation associated with any living thing has to do with the organization of the organic structure of the subject in assimilation of conditions produced by the outside environment. As such, the "reflexive abstraction" refers to the most general organizing function in living things: the "reconstruction convergent with superation" 15.

According to Piaget, the reflexive abstraction, the evolution of structures and preeminence of the subject, within the basic biological framework, are the key to mathematical reflection. Mathematical entities are produced by the subject in steps parting from the reflexive abstraction which refers to the form of organizing content. Moreover, the subject, in this process, separates form and content. Form is what corresponds rightly to mathematics. **The formal character of mathematics is no accident, but rather according to Piaget is intrinsic to the epistemological nature of mathematics.**

In all this, exists a basic supposition that, parting from an analysis of the psychogenetical processes, understanding can be obtained about the general conditions of the epistemology. This is a point of departure. The reflexive abstraction permits us to explain how mathematical structures advance from one level to another. Piaget says that in this process the subject is active factor and the object is only an environment. Of course, all this is only an interpretation. In order to do Piaget justice we should avoid the interpretative and seek the objective. The probable is, in our opinion: (1) the subject needs action in order to generate change in mental structures, and (2) the concrete material object is necessary in all the first steps in psychogenetic evolution. At this point, knowing with certainty how much corresponds to the subject and how much to the object is difficult. Piaget's interpretation is: the emphasis should be put on the subject, along with a recurrence of biological hypothesis nor proven.

At this point, then, a methodological and philosophical discussion becomes extraordinarily important. Piaget, through his genetic epistemology, seeks a range of statistical results about the development of mental structures in children, but these are inadequate for arriving at more general epistemological conclusions. In any case, the processes of knowledge can only be explained in any depth parting from a physical analysis (in the general sense). Behavioral data cannot be conclusive because they are indirect. They cannot explain the material phenomena which engender them.

The power of abstraction of the human mind over the real and objective cannot be apprehended with a single method. Different types of abstraction exist, not only related to different objects, but also according to their form of approximation. The reflexive abstraction of Piaget, which emphasizes the operative does not represent a complete explanation of how to determine mathematical knowledge epistemologically. It is even possible that the most definitive answer is related to deeper knowledge than the science we possess today. In any case, it is necessary to supply some methodological indicatives to this discussion. **In the first place, an epistemic subject is a dynamic factor of central importance in mathematical knowledge, but always dependent upon the object.** At this point it is also necessary to point out that the social is not only a part of the object, but also intervenes in the configuration of structural/mental conditions which participate in the relation object-subject. We can, then, speak with security of three functional factors in the process of knowledge: the subject, the society and the material object. These are simply three categories that refer to intricate overlapping aspects of the "non-null intersection" of a unified totality. The three affect the epistemological behavior in different ways. The behavior of the object, in its most general sense, affects the other two. Knowledge is a dialectic fusion which encloses three different functional dynamics in a proportion difficult to establish by merely speculative means. I even believe that, up to a point, the biological interpretation of cognoscitive functions of Piaget cannot be ignored a priori. On this subject, the point which should be understood is that there exists at this time an impossibility for sufficient scientific evidence and a methodological inconvenience for accepting it. Seen from the classic framework of reference: the subject, whose ultimate determinants are biological and even more so physical, is active; but not in a vacuum, but in a conjunction-fusion with the object, whose independent movement incides over the subject. The social is a bridge which penetrates the interior of the subject, supplying part of its theoretical supports. The epistemological relationship subject-object established within the general social framework defines a different reality to the object and subject. Thus, the object is not the object itself, but rather the **subjectivized object**; the characteristics of the object are learned and understood by the subject within the conditions which establish their own limits. This **subjectivized object** is the departure basis of consciousness. **The subject is, at the same time, objectivized.** In this epistemological relationship, while it is true that both poles do not remain as they were, this does not mean that their independent conditions disappear. These conditions are points of support for the relationship. This is a methodological interpretation. From this point we need scientific and precise empirical support.

In the classic empiricism of Mill, the subject is reduced to a blank piece of paper on which the object prints its message. In Kantian apriorism the object almost disappears (is subjectivized) and the subject is the only active factor which produces mathematical knowledge. In Piagetian apriorism, some of the problems of Kant are inherited and are attempted to be resolved through a biological interpretation. **The analysis of the mental factor as operatory, as advanced by Piaget, is more adequate than the mere inductivist generalization.** The epistemological problem with the vision of Piaget is, however, the reduction in the active role of the object, and the almost total absence of the social as a special factor; the limitation of the types of mental abstractions. But through this analysis we have identified a possible point of methodological reference: the constituent epistemological factors of mathematical knowledge.

4. It is not necessary to insist upon the importance and implications of an epistemological and philosophical revision of the teaching of mathematics or its implications upon the psychology, programs, and methods of teaching. If we adopt the point of departure identified here, some consequences are inevitable:

a) Mathematics should be taught in direct relation to material and social reality. This implies, apart from a direct approximation to objects, a strong linkage with other natural and social sciences.

b) The content of mathematics should be taught with an understanding that a dialectic relationship exists between the abstract and the empirical. It is necessary to understand the **specific abstract dimension** of mathematics integrated in a special subject-object epistemological relationship which determines its character.

c) Mathematics should not be taught as absolute, eternal and invariable truths.

d) Axiomatics and deduction should be introduced in the teaching of mathematics as important and useful instruments, but not the heart of mathematics. (This implies, for example, determining precisely when certain formalism are specifically useful or necessary and when they are not).

e) A new teaching of mathematics should emphasize heuristic, intuitive, concrete and constructive theoretical methods.

One of the decisive resources in teaching mathematics which assumes this type of vision is the use of the history of mathematics. Not only as a source of anecdotes which give "color" to mathematical content, but rather a key factor in the structure of the teaching of these contents, the programming and method teaching. This does not mean, for example, that the order and logic to be decided should simply be historical and the deductive logics: an adequate convergence between sociogenetic and psychogenetic orders.

It is probable that at this time no defined program of how to achieve a new strategy of teaching mathematics exists, along with its respective philosophy, methods, programs and texts. I believe that we live in a transitional period in which different concrete ideas and plans are being tested. It is difficult to know how long this will last. But I believe that in this period, discussions about methodology and philosophy are particularly important. (On many occassions, there is a tendency to dismiss philosophy as mere empty and useless sets). In this sense, perhaps, all this work constitutes a defense of the Philosophy of Mathematics.

NOTES

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9. Cf. Beth, Piaget. **Op. Cit.** p. 212.
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11. Piaget, Jean. **Introducción a la epistemología genética.** p. 286.
12. Cf. **Ibid.** p. 38 cf **Biología y Conocimiento.** p. 282.
13. Piaget, Jean. **Introducción a la epistemología genética.** pp. 296 cf 300, 301.
14. Cf. Piaget. **Biología y conocimiento.** pp. 287, 288, 289 cf 296 ss.
15. Cf. **Ibid.** p. 303.

STUDENTS' UNDERSTANDING OF MATHEMATICS: A REVIEW AND SYNTHESIS OF SOME RECENT RESEARCH.

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Models of understanding and conceptual frameworks for building such models that have been developed over the past decade are described in three broad categories: (1) The instrumental vs. relational distinction and evolving epistemological and constructivist models, (2) Piagetian, neo-Piagetian, and related psychometric models, and (3) human information processing and cognitive science models. Some examples of studies of students' understanding of mathematics are reviewed and discussed in terms of the practical decisions researchers have made in choosing models and making operational definitions, the feasibility of reinterpreting the studies' findings in different theoretical terms, the generalizability of their findings, and the potential benefits of the studies for curriculum evaluation, teachers' professional development, and the improvement of instruction.

Mathematics educators have been concerned about students' understanding for many years (Brownell's work in the 1930's and 40's on "meaningingful arithmetic" is relevant to this issue, for example), but the past decade or so has seen substantial activity and progress in this area. The numerous models and conceptualizations of understanding which have been developed can be put into three broad categories.

EPISTEMOLOGICAL AND CONSTRUCTIVIST MODELS

In a seminal article, Skemp (1976) distinguished between instrumental understanding, "rules without reasons," and relational understanding, "knowing both what to do and why." Skemp commented that formerly he would not have considered instrumental understanding to be understanding at all, but he gave reasons for believing that "understanding" is a case of *faux amis*, where one term is used with two entirely different meanings. Skemp also discussed mismatches between the goals of students and the goals of their teachers with respect to type of understanding they sought, and argued that the term "mathematics" itself could be a case of *faux amis*. Playing Devil's Advocate, Skemp listed some supposed advantages of instrumental understanding, and gave several arguments in favour of relational understanding. The theoretical formulation Skemp used in the discussion was based on the distinction between fixed plans for solving particular problems and schemas from which plans can be produced; throughout the article he was concerned with the nature of the knowledge students have, their goals in developing it, and their abilities to use it in solving problems.

Responding to Skemp, Byers and Herscovics (1977) proposed two additional types of understanding: intuitive understanding, the ability to solve a problem without prior analysis of the problem, and formal understanding, the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning. In their model the four types (or "modes") of understanding were represented as the vertices of a tetrahedron. Byers and Herscovics argued that formal understanding is not merely a special case of relational understanding, by giving the example of an algebra problem solved "with understanding" and correctly but with literally incorrect symbolic expressions. They also noted that effective learning cannot be based only on a single type of understanding, that at any given time a student's understanding of a particular piece of mathematics will be of a mixture of types, that the types cannot be arranged in any strict temporal sequence, and that the system is dynamic because the four types interact at times reinforcing or hindering one another.

Tall (1978) criticized the tetrahedral model because it can only represent the ratio between types of understanding and suggested repesenting understanding as a point in 4-space. Acknowledging that regardless of representation understanding is a function of time, he urged consideration of the dynamics of understanding and suggested that the kinds of understanding should be seen as facets of a single development. He noted that schemas grow and become more versatile, they decay, and they are reformulated consciously and unconsciously as the individual attempts to make a coherent pattern out of the world. Understanding which results from such a search for coherence Tall identified as relational understanding. Instrumental understanding is characterized by compartmentalization of ideas, not wishing to make an overall pattern, preferring a limited closed system. Thus, type of understanding was related to attitudes and goals. Tall argued that intuitive understanding occurs when a developing schema is not yet sufficient for the task at hand, but there are facets of the problem that seem to link with it. Formal understanding, he contended, has both individual and corporate aspects. The individual reflects on his schema and rationalizes his thinking to fit it together coherently, a process reminiscent of "formal operations" in Piaget's use of the term. The individual also puts mathematics into a public context using conventional notations appropriately. Turning to the case of non-understanding, Tall gave examples illustrating the proposition that, if understanding is assimilation into an appropriate schema, then non-understanding may mean either assimilation into an inappropriate schema or failure because no schema is available.

While accepting a model containing four types of understanding, Tall expressed the concern that this might lead to the development of more and more refined categories, some of which might not be universally accepted or uniformly interpreted. He also wondered whether the presence of "intuitive" and "formal" suggested the need for a kind of understanding to correspond wih Piaget's "concrete." Finally, Tall expressed the hope that focussing on the distinctions between types of understanding would not obscure the connections between them, and he gave the separation of "cognitive" from "affective" as an example of an unhelpful distinction.

Another article by Skemp (1979), offered as a synthesis of the articles mentioned above and others, was based on his model of intelligence in which human behaviour is seen as goal-directed, and two director systems are posited. Delta-one receives information from the environment and acts upon it, comparing the present state of the environment with the goal state, constructing plans from available schemas to take the operand to the goal-state and keep it there. Delta-two, whose operands are not in the external environment but in delta-one, has the function of optimizing the functioning of delta-one. The goal of instrumental learning is to give right answers. The overt operands are symbols, mathematical and verbal, spoken and written; the hidden operand is the teacher or examiner, and the goal is to gain approval and avoid disapproval. These operands are in the environment, so the activities are delta-one activities. When pupils learn a set of rules (degenerate schemas) that are appropriate to a limited class of tasks, the rules are rules for manipulating symbols, and the connections are connections between symbols, not between concepts. The goal of relational learning is the construction of relational schemas. The operands may be newly encountered concepts, and the goal to connect them with appropriate schemas; the goal may be to deduce specific methods for solving particular problems or rules for classes of tasks; or the goal may be to improve existing schemas by reflecting on them to make them more cohesive, better organized, etc. These operands are concepts and schemas within delta-one, so in relational learning delta-two activities are dominant.

Skemp proposed to resolve the proliferation of models problem with a new model containing three kinds of understanding: instrumental, relational, logical (equivalent to formal) and two modes of mental activity: intuitive and reflective. He argued that the two modes of mental functioning do not correspond to different kinds of understanding, but that they occur in all combinations with them. Skemp offered examples to show that the six cells of this model are meaningful. In the cells instrumental/intuitive, relational/reflective, and logical/reflective the examples are straightforward, but the example for instrumental/reflective seems contrived. Skemp himself expressed dissatisfaction with his attempt at an example for the logical/intuitive cell. The relational/intuitive combination Skemp matched with insights which develop slowly, as opposed to leaps of intuition.

Olive (1982) designed a group-administered test for 9th and 10th grade students to test for relational or instrumental understanding of various mathematics topics, in both the reflective mode and intuitive modes. Some items were found successful for these purposes, and some strong correlations between item responses and interview assessments were observed.

Bergeron and Herscovics (1981) reported using a model that described understanding of mathematics (as intuitive, instrumental, relational or formal) in a project with elementary school teachers. They found that the teachers could identify different modes of understanding with various topics in elementary school mathematics, and that doing so changed their perception of mathematics and their view of their own mathematical competence. Furthermore, it led to the development of a constructivist approach to learning, a deemphasis of the importance of written answers, a focus on thinking processes, and an awareness that only through an appropriate form of questioning can children's reasoning be uncovered. Noting certain weaknesses of the model and difficulties using it, these researchers set themselves the task of constructing a new model better suited to the analysis of concepts, a hybrid model that would apply both to states of understanding as well as the construction of understanding. In a companion paper (Herscovics & Bergeron, 1981), they concluded that some of the criteria for classifying understanding were quite useful in describing concept formation, and they stated that they "now perceive[d] these criteria as levels of understanding which in fact constitute the backbone of a constructivist model of understanding." (p. 69) The new model (Herscovics & Bergeron, 1982) included the intuitive and procedural levels and the stages of abstraction and formalization.

PIAGETIAN AND NEO-PIAGETIAN MODELS

Many mathematics educators consider the main value in Piagetian research to be the extent to which it characterizes students' understanding or potential for understanding of important mathematical ideas. For example, one might say that a child giving a pre-operational response in the conservation of number task lacks understanding of certain number concepts, even if he does have abilities with counting, or one might argue that a child giving a concrete operational response to a task has a poorer level of understanding than a child who responds at the formal operational level. The power to reveal students' understanding of mathematics results from the non-routine nature of the assessment tasks and the ability of researchers to pursue a line of questioning in a task-based interview. While a great many mathematics topics have been addressed in the Piagetian literature, many others (such as multi-digit numeration or algebra) have not, but a number of researchers inspired by the Piagetian approach have worked in these areas with similar goals and tools.

The CSMS project (Hart, 1981) established heirarchies of understanding in a variety of topic contexts exhibited by English schoolchildren aged 11 to 16 years. The project's methods involved conducting tape-recorded interviews with groups of about 30 students and developing written items that were eventually used with large samples. The interviews served not only as a means of assessing the suitability of the written items, but also to inform the interpretation of the results of the written tests. In measurement and ratio and proportion, the initial interviews and written items were based directly on Piagetian tasks. All the interviews and tests were designed to reveal the methods and errors of students when they were confronted with mathematical problems, rather than to assess abilities to use methods taught in school. One of the project's main findings was that the methods students used were often not "teacher-taught;" but nor were they idiosyncratic, for similar "child-methods" were observed repeatedly. Another important conclusion was that mathematics is very difficult for most students, since the levels of understanding observed were often much lower than teachers and syllabi expect.

The CSMS hierarchies were constructed within topics; the number of levels per topic varied from three to seven. A student's level in a heirarchy was determined as the highest level of item-group in which the student obtained two-thirds correct. Criteria used to group items into levels included similarity of difficulty level, acceptable homogeneity and scalability, and mathematical coherence. In the discussion, each heirarchy of levels was described both in terms of content and processes used to deal with content.

The Piagetian stage-theory model of cognitive development has often been criticized for the fact that the same subject may appear to be at different levels of development in different topic contexts. This phenomenon, which Piaget recognized and called *decalage*, suggests that the results of this type of study need to be reported with reference to the particular contexts and tasks used. While students' *responses* can be rated, students themselves should not be labelled "pre-operational," "concrete operational," etc.

A useful concept when applying Piagetian constructs and methods to mathematics curriculum contexts is the concept of cognitive demand. By this is meant the nature of the thinking that is required in order for a learning activity to "make sense" to the student, for the student to assimilate it into his available schemas. In the CJHMP project (Bye, Harrison, & Brindley, 1980) and the ACLIC project (Marchand, Bye, Harrison, & Schroeder, 1985), for example, a great deal of effort was expended developing demand criteria and analyzing the cognitive demands of different aspects of mathematics curricula such as objectives, textbooks, tests, and teachers' classroom presentations. By using the same cognitive level criteria both in assessing the responses of students and rating the cognitive demands of aspects of the curriculum, it is possible to investigate whether there is a match or mismatch between distributions of demands and of responses. Details of procedures for carrying out this sort of analysis have been given by Bye, Harrison, and Brindley (1980), and by Marchand, Bye, Harrison, and Schroeder (1985).

The SOLO taxonomy (Biggs & Collis, 1982) may be considered to be both an application and an extension of the Piagetian model of cognitive development. It is based on two constructs: hypothetical cognitive structure (HCS) and structure of observed learning outcomes (SOLO). The former is closely related to the Piagetian stages (sensorimotor, intuitive/preoperational, concrete operational, formal operational); the latter are concerned with describing the structure of a given response as a phenomenon in its own right, without the response necessarily representing a particular stage of intellectual development (Collis, 1982). In some versions of the theory (Collis, 1983) the structures of learned responses are regarded as occurring *within* each stage (or "mode"), and they become increasingly complex as the "cycle of learning" develops. Uni-structural responses represent the use of only one relevant aspect of the mode; multi-structural, several disjoint aspects, usually in sequence; relational, several aspects related into an integrated whole; and extended abstract moves the process into a new mode of functioning, i. e. extended abstract functioning in one mode is identified as equivalent to unistructural functioning in the next higher mode (Collis, 1982).

In a series of reports, Collis (1982) explained the theoretical role for the SOLO taxonomy in assessing reasoning in mathematical problem solving, Jurdak (1982) gave details of the construction of superitems (clusters of test items with a common stem where each individual item corresponds to a different SOLO level) for this purpose, and Romberg (1982) gave interpretations of clusters of superitems. The undertaking described in these technical articles seems quite complex, especially as compared with the more familiar Piagetian terminology, constructs, and assessment procedures. But the SOLO taxonomy has appeal because it avoids the problem of *decalage* and focuses on the structure of responses which can be observed, rather than on structures of the intellect which cannot.

HUMAN INFORMATION PROCESSING AND COGNITIVE SCIENCE MODELS

A relatively new theoretical approach to mathematics education research questions, especially those having to do with problem solving, is the "human information processing" or "cognitive science" approach. Robert Davis (1984) has recently produced a major book which defines key terms, presents methods and findings of relevant research carried out in this framework, and shows how the concerns of mathematics education researchers are addressed by this theory. Understanding of mathematics was the focus of a report by Davis, Young, and McLoughlin (1982) which used human information processing in considering the question: "What would be lost if understanding was eliminated as a goal of instruction?" In the studies included in the report, episodes with a wide range of students were analyzed and categorized as indicating either a presence of lack of understanding in some particular form. Identified behaviors were related to basic conceptualizations of human information processing such as sequential processes, procedures, knowledge representation systems, frames, retrieval and matching, pointers, transfer of control, sub- and super- procedures, metaphor and isomorphism, critics, planning space, planning language, and meta-language.

Cases of good understanding were associated with characteristics such as knowing necessary techniques, having appropriate descriptors to specify what each technique can accomplish, having a collection of recognizable sub-goal candidates, having mechanisms for recognizing appropriate sub-goals and retrieving appropriate labels, having mechanisms for retrieving a tool from its tag, and having mechanisms for assigning correct inputs for each tool or sub-proceedure. It was observed that the best problem solvers were exceptionally skillful in setting goals and sub-goals, using a powerful meta-language to describe and analyze what they were doing, and making quick revisions to their strategy when they got a glimpse of a new possibility, or when they saw a dead end looming ahead.

On the other hand, lack of understanding was characterized by lack of critics, failure to relate mathematical processes (e.g., borrowing and carrying) to pre-matheatical schemas (e.g., making change and fair trades), failure to acknowledge that concrete embodiments (even familiar ones) had anything to do with related problems, and failure to appreciate the nature of the task. The report concluded that "the overall patterns of what it means 'to understand' are strikingly similar at both ends [of the continuum from Grade 3 arithmetic to calculus], and everywhere in between." (p. 35).

DISCUSSION

The conceptualizations of understanding described above differ somewhat in the terms they use, but less so in the ideas that these terms seek to express. A useful exercise for researchers investigating students' actual understanding (as distinct from those who have proposed to conceptualize it) is to reinterpret their work using each of several of these ways. This has value in focussing the observations, and new insights often result.

A second worthwhile activity is to compare their findings with those of others, an activity that would serve (if repeated often enough) to test Davis, Young, and McLoughlin's (1982) conclusion that what it means to understand is similar across mathematical contexts and populations of students.

There is already evidence that studies of understanding have had positive impacts on teaching practice and curriculum development, and every reason to believe that if practitioners continue to be invloved further research will also result in additional benefits.

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PREFERRED LEARNING STRATEGIES AND EDUCATIONAL/MATHEMATICAL PHILOSOPHIES: AN HOLISTIC STUDY

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This paper reports an exploration of the relationships between philosophies of education and mathematics and the preferred learning strategies of students currently undertaking courses in mathematical education. This exploration represents the extension of an earlier discussion about the implications of Gordon Pask's Holist/Serialist dichotomy for the learning and teaching of mathematics. As well as providing a theoretical rationale for the hypothesis being tested, the report presents early results from an empirical study currently being conducted in London. Directions for further research and implications for teacher education are also indicated.

INTRODUCTION

The motivation for the research reported here stemmed from two primary sources: the author's earlier theoretical discussion of the implications of Pask's Holist/Serialist dichotomy for the learning and teaching of mathematics [1], and results of her doctoral research programme on pupils' views of their teachers [2], which clearly demonstrate that the pupils involved in the study valued teacher behaviour which could be interpreted as reflecting a particular view of how pupils learn (serialistically) and/or a specific philosophy of the nature of mathematics and the way in which it should be taught (as a body of knowledge, known to the teacher and to be conveyed to the pupils). These results prompted a review of the literature relating mathematical to educational philosophies and practices, and reflection upon the interrelationship of these areas of study.

Before presenting the hypothesis which was derived from these reflections, and describing the current empirical study aimed at testing this hypothesis, the underlying strands will be considered separately.

IMPLICATIONS OF PASK'S SERIALIST/HOLIST DICHOTOMY

Pask and his colleagues have established a strong case for the existence of two distinct learning strategies - serialist and holist. Learners with a tendency to adopt exclusively serialistic strategies are labelled 'operational' learners, whilst those who in a free situation consistently employ a holistic approach are called 'comprehension' learners [3].

There is a clear link between the level of uncertainty at which a learner is prepared to work and their preferred learning strategy. Operational learners are characterised by a preference to proceed from certainty to certainty, learning, remembering and recapitulating a body of knowledge in small, well-defined and sequentially ordered 'parcels' [4]. They are confident that the necessary expertise will be gained steadily. Comprehension learners, on the other hand, prefer to start in an exploratory way, guessing ahead and working first towards an understanding of an overall framework, before filling in the details. In order to ensure complete mastery of complex topic areas interventions must be made which encourage the learner to explore the material in a variety of ways.

A consideration of reported gender differences in mathematics led the author to a theory about the implications of Pask's theories to mathematics learning and teaching [5]; a brief outline of the argument is given below:

The evidence suggested that some pupils (both boys and girls, but more girls than boys) may be adhering to serialistic strategies which have led to their success in mathematics at the primary level, but which, when used exclusively, have negative implications for these pupils' later mathematical development. It is argued that children who are predisposed to a serialistic approach are less likely to develop into versatile learners within the mathematics classroom than those who are inclined to adopt holistic strategies. This is held to be directly attributable to the fact that input from primary teachers tends to be serialistic in nature, based on their own rule-based experience of mathematics, and lack of confidence in their own mathematical ability.

The effects of these teacher interventions will be different for comprehension and operational learners; the operational learners will become increasingly committed to the view that a step-by-step approach leads to success in mathematics; even those with enough versatility to become more flexible in other curriculum areas, where they are actively encouraged to adopt other strategies, will fail to do the same in mathematics, because the value of alternative approaches will not have been demonstrated; the effect on the comprehension learners, on the other hand, will be mediatory; by showing the effectiveness of techniques associated with serialistic skills, the teacher provides the impetus for them to supplement their self-developed strategies to produce the versatility of approach which underlies complete understanding of mathematical topic areas.

PHILOSOPHICAL PERSPECTIVES

Within the field of mathematics education, a great deal of recent research has focussed on the relationship between teachers' views about the nature of mathematics and their educational philosophies and practices [6]. There is not room here to discuss this work in detail, and it is, in any case, readily accessible. There seems to be agreement about the importance of these interrelationships [7]:

> "There is strong reason to believe that in mathematics, teachers' conceptions...about the subject matter and its teaching play an important role in affecting their effectiveness as the primary mediators between the subject and the learners."

Some researchers have gone further, [8], identifying a particular mathematical philosophy as being the most compatible with current beliefs about what constitutes good mathematical education practice [9]:

> "It is concluded that each of the views (of mathematics) provides insights as to the nature of mathematics, but that Falliblism is perhaps the only viewpoint compatible with humane mathematical education."

INTERRELATIONSHIP BETWEEN PHILOSOPHICAL AND PSYCHOLOGICAL PREFERENCES

Although some writers have considered philosophical and psychological constructs as separate factors influencing teaching strategies [10], there has been little attempt to focus on links between them. One exception to this [11] concludes that there is a relationship between Platonism and Logicism with right hemisphere processing and between Formalism, Nominalism, Constructivism and Intuitionism with left hemisphere processing. No account is taken of the 'Falliblist' stance taken by, for example, Lakatos.

This paper differs from that above in several ways. Firstly it is believed that the distinction between certainty and 'fallibility' in mathematics is of fundamental importance; secondly, it is believed that many people do not have clear-cut views about the nature of mathematics, so that imposing a rigid framework in terms of established schools of thought may distort the true picture; thirdly, it is believed that categorising learning strategies and styles under the headings right and left hemisphere may encourage a simplistic view, and lead to unwarranted conclusions about the relationship between achievement and 'inherent ability'.

The hypothesis being put forward here is that relationships between philosophical views and cognitive strategies will not be clear-cut, but that there will be a correlation between a view of mathematics as a fixed body of knowledge and a tendency to adopt serialistic learning strategies. A possible rationale for this situation would be that an individual's own learning experience within the mathematics classroom had fostered both tendencies.

THE EMPIRICAL STUDY

The research, which is on-going, focusses on initial and in-service students of primary and secondary mathematics education. The first stage of the study, now completed, consisted of establishing the students' preferred learning strategies, by means of the Clobbit learning task

[12], which requires students to learn, under restricted conditions, about the taxonomy of a fictitious Martian species, and subsequently to 'teach back' what they have learnt to the researcher; and eliciting their beliefs about mathematics and mathematics learning using two questionnaires. The first of these is based on bi-polar constructs:

e.g.

Mathematics learning relies almost entirely on experts' thinking and knowledge.	A	B	Gives opportunity for thinking things out for oneself in learning.

The second asks for a response to statements like:
Creative work in mathematics is possible only at research level.
or Mathematics is accurate and precise - free of ambiguity and vagueness.

PRELIMINARY RESULTS

Overall results of the research to date will be circulated at the Conference. To give some idea of the information elicited, the results obtained from a subset of the sample - 3rd year students on a 4 year Primary B.Ed. course - are summarized below. These particular students have not elected to study mathematics as their specialist subject.

Analysis of the strategies employed in the study phase, and the protocols of 'teachback', led to five students being categorised as operational learners, and five as comprehension learners.

The questionnaire analysis revealed a high level of agreement between all subjects on a variety of areas:-

Subjects agree overwhelmingly that mathematics learning is fascinating, exciting and stimulating, challenging and satisfying.; it was unanimously agreed that mathematics learning gave scope for imagination, and provided scope for finding things out for oneself and for being creative at all levels. There was consensus, too, on the view that mathematical problems in the classroom invite a variety of appropriate methods of solution, and different 'correct' answers.

It was only with regard to questions about the nature of mathematics itself that there was disagreement between the two groups.

Three of the five operational learners gave responses which were consistent with a view of mathematics as an irrefutable body of knowledge. They selected most, if not all, of the following statements:

1. Truths have existence independent of peoples' discovery of them.
2. Mathematics is essentially hierarchical and cumulative.
3. It has developed through consolidation and extension of earlier work - previous knowledge is not rejected as untrue in the process.
4. Mathematical truths are not susceptible to revolutionary change in the way that scientific truths are.
5. Mathematics is consistent - free of conflicting ideas, results and conclusions.
6. Mathematical truths have an absolute quality about them.
7. Mathematics is a 'tidy' subject: there are no 'loose ends', no ambiguities, no uncertainties.
8. I do not think of mathematics as a changing field of knowledge.
9. Mathematics is accurate and precise - free of ambiguity and vagueness.

Three of the comprehension learners were also consistent in selecting statements indicating a more dynamic model of mathematical knowledge:

a. Mathematics knowledge is hypothetical and potentially subject to modification or falsification.
b. I think of mathematics as a changing field of knowledge.
c. Mathematics is not consistent.
d. Mathematics is not accurate and precise.
e. Mathematical truths are susceptible to revolutionary change.
f. Mathematics is not a 'tidy' subject; there are ambiguities and uncertainties.
g. Mathematical truths are not absolute.

The other four subjects were inconsistent in their responses, selecting the following statements from the lists above:

Operational Learner 4	1,2,3,5,7,a,b,e,g
Operational Learner 5	1,2,3,b,c,d,e,f,g
Comprehension Learner 4	2,3,.,6,8,c,d,e,f
Comprehension Learner 5	1,2,3,4,8,c,d,f,g

DISCUSSION

The evidence so far collected from this and other student groups does suggest a strong link between philosophies of mathematics and preferred learning strategies. It is surprising, given this, that professed views about the nature of mathematics learning are so similar. Interviews with these students will probe reasons for this. Many of the students have indicated that their views have changed dramatically since they arrived at college; the nature of these changes will be explored, and related to earlier learning experiences, and their current teaching practices.

In the longer term, a longitudinal study will be made to monitor such changes in students of a particular cohort from their arrival until the end of the course.

It is important to encourage students to explore the interrelatedness of their beliefs and preferences, and those of their future pupils. In this, as in so many areas, increased versatility can only begin with an awareness of current beliefs and levels of understanding. It is the responsibility of teacher educators to foster such awareness, and to emphasise the necessity of adapting teaching approaches to take account of the views and learning styles of others.

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A SOCIAL CONSTRUCTIVIST THEORY OF INSTRUCTION AND THE DEVELOPMENT OF MATHEMATICAL COGNITION

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This paper presents a theoretical framework for viewing mathematics instruction and development of mathematical cognition from a social constructivist point of view. The paper describes the philosophical underpinnings, taken from Wittgenstein, and the psychological foundation, drawn from Vygotsky's work. In addition, it describes implications for research in mathematics education.

INTRODUCTION

The purpose of this paper is to present a theoretical framework for viewing the teaching and learning of mathematics (and its relationship to the development of mathematical cognition) as a process of social construction. The paper will present its theoretical framework which is informed by the conceptions of mind and knowledge of Wittgenstein's philosophy and Vygotsky's psychology. The paper will also discuss the implications of a social constructivist view for research in mathematics education and teacher education in mathematics.

The social constructivist views knowledge not as a reflection of some reality but as an artifact of communal interchange. Cognition is something social in its very essence. Knowledge is based on social interaction. Research in mathematics education grounded in this perspective focuses on the social construction of mathematical meaning that arises from teacher-student or student-student interaction in classroom settings.

WITTGENSTEIN

The philosophical framework for a social constructivist view of learning, instruction, and cognitive development is informed by the writings of Ludwig Wittgenstein. He assumed that mind and knowledge are constructed through social interaction. The properties of mental states derive from the fact that they are really properties of groups of people which have been imputed to individuals (Bloor, 1985).

One of Wittgenstein's main concerns was the use of language and its role in meaning and knowledge. Meaning is located in the function words have as signals used by people in the course of shared activity.

Language games are systems of communication; languages which are complete in themselves. Wittgenstein's social theory of mind derives from his social theory of meaning. Thinking is an activity of operating with signs. The mental experiences which accompany the use of signs are a result of our patterns of usage of those signs in a particular language. "When I think in language, there aren't meanings going through my mind in addition to the verbal expressions; the language itself is the vehicle of thought (Wittgenstein, 1958, p. 107).

Instruction plays a central role in Wittgenstein's conception of mind and mathematics for we must be taught to use these signs in a manner consistent with social practice. Our knowledge depends on our patterns of training. Wittgenstein saw teaching as a constituent part of the "forms of life" of which he speaks. It was thought of as one of the activities that go to make up a form of life and of an activity that shapes how that form of life evolves. "An education quite different from ours might also be the foundation for quite different concepts...." (Wittgenstein, 1958, p. 128).

Wittgenstein spoke of mathematics as one of many language games. Mathematics is an anthropological phenomenon -- a system of signs and procedures for manipulating these signs which are established through convention. As conventions these procedures are not accepted because they correspond to some ideal; they are correct because they are accepted. And they are accepted because they have proven functional. Wittgenstein's point is not that anything goes, but that there is no mathematical reality which guarantees the results we get. Mathematical objectivity is a function of human practice. "What I am saying comes to this; that mathematics is <u>normative</u>. But norm does not mean the same thing as ideal." (Wittgenstein, 1956, p. 190).

Mathematics creates concepts. Mathematics is the process of inferring one statement from another, and the criterion of correctness is found in the collective behavior of humans, in the results of their calculations. Mathematical statements do not state facts of any sort, but provide us with a linguistic framework in which we can classify and organize the empirical observations we make. If mathematical propositions do not state facts, do not tell us the properties of numbers, what do they do? Wittgenstein claimed that they make a linguistic point. "Rather than unfolding the <u>properties</u> of the number 100, for example, what I unfold may be said to be the role which 100 plays in our calculating system" (Wittgenstein, 1956, p. 26). Wittgenstein emphasized that it is essential to mathematics that there be agreement in the results of calculation among those who use the system. "The mathematical must is only another expression of the fact that mathematics forms concepts (Wittgenstein, 1956, p. 190).

VYGOTSKY

If Wittgenstein provides the philosophical framework for the social constructivist view, then Vygotsky provides the psychological underpinnings. Both men believed that thinking is a social activity and that knowledge is a collective achievement. Vygotsky believed that all higher mental processes, such as logical memory, selective attention, and comprehension of sign systems, occur at a social, interpsychological level before they are internalized by the individual. The vehicle for the the development of higher psychological functions is the mastery of sign systems such as language and mathematics (Wertsch, 1985). This mastery alters the nature of cognitive functioning. Another important aspect is the process by which the meaning of signs becomes less dependent upon the context in which they are used (Wertsch, 1985), such as the use of numbers abstracted from representation of concrete objects. Thus the activities associated with cultural learning play a leading role in the development of the individual.

Vygotsky believed that instruction plays a major role in leading the child to new developmental levels. "What the child can do in cooperation today he can do alone tomorrow. Therefore the only good kind of instruction is that which marches ahead of development and leads it" (Vygotsky, 1985, p. 188). The zone of proximal development (ZPD) is defined as the region of sensitivity to instruction in which the transition from interpsychological to intrapsychological functioning can be made. It is the distance between the child's level of cognitive functioning during independent problem solving and the level of potential development during problem solving with the guidance of an adult (Vygotsky, 1978). Adults and more capable peers provide instruction in the ZPD by such means as directing attention to salient features of a task or assuming responsibility for parts of the task beyond the child's capabilities. As the child masters previously instructed skills, the adults or peers provide successively less assistance with the task. In this way other people provide instruction in skills slightly in advance of the child's current abilities.

Central to the issue of instruction in the ZPD are situation definition and intersubjectivity. Situation definition is the way objects and events in a situation are represented, and intersubjectivity exists when participants share some aspect of the situation definition (Wertsch, 1985). When the participants in instruction are adult and child, the communication is asymmetric, with the adult managing the control of interaction. In an ideal situation, control of the situation moves from teacher, to joint teacher-student, to student as the student becomes more competent with mathematical processes.

Saxe (1982) defines cognitive development as the transformation and elaboration of systems of knowing that are progressively more comprehensive and powerful as relationships develop among concepts. Vygotsky, in stating that higher mental processes of the individual originate in social processes, claimed that we must consider two forms of cognitive development; changes in ontogenesis and changes in sociocultural history (Wertsch, 1985). To understand the development of mathematical cognition in the child, we must understand the cultural and historical development of mathematics as social practice.

If one holds that mathematical meaning is found in the shared understandings of human beings, then we have to account for how this understanding, initially external to the child, becomes part of the child's own cognitive processes. One way in which internalization might occur is that children first acquire part of their culture's set of number terms in playful activities. As children use these terms to solve numerical problems, their use of the terms is regulated by adults. The adults' conventionally defined system becomes a means of number representation for the child through further constructive process; the attempts to understand the organization of its own enumerative activities. With progress in this understanding, the child would be increasingly capable of using historically and culturally determined number terms to solve numerical problems (Saxe, 1982).

A SOCIAL CONSTRUCTIVIST MODEL OF RESEARCH IN MATHEMATICS EDUCATION

The issue in research in mathematics education which takes a social constructivist perspective is the specification of the processes which make possible the transition of mathematical cognition from the interindividual to the intraindividual plane. The teacher's definition and the student's definition of the task might be totally different. In addition, the definition of the task might change for any of the participants across the course of instruction (Bauersfeld, 1979). Intersubjectivity, or a shared task/situation definition, is developed through communication and negotiation of meaning. The communication takes place on two levels; communication about classroom processes and routines, and communication about mathematical content. Bauersfeld (1979) points out that mathematicians have invested much effort in producing universal statements, and most school mathematicians would claim any mathematical statement as universal and objective. However, mathematical meaning is developed in the context of social interaction, and "it inescapably becomes dependent upon interpretive, indexical, and reflexive constitution of meaning" (Bauersfeld, 1979).

Articles by Bauersfeld (1979), Bishop (1985) and Campbell (1986) suggest ways of using discourse analysis as a means of examining the development of mathematical cognition. Bishop (1985) suggests a new orientation for viewing social interaction in mathematics classrooms. He suggests that mathematics teaching be viewed as controlling the organization and dynamics of the classroom for the purposes of sharing and developing mathematical meaning -- knowledge which connects with the individual's current knowledge about mathematics, knowledge about other subjects, and knowledge about the real world. His analysis focuses on three aspects of the classroom: 1)mathematical activities, focusing on the learner's involvement with mathematics; 2) communication, emphasizing the process and product of shared meanings; and 3) negotiation, focusing on the nonsymmetry of the teacher/student relationship in the development of shared meaning.

As a prerequisite to communication, participants have to share common understandings, which they take as a basis for reference when speaking to each other. What a participant says not only carries the intended message, but over and above that, the utterances contain information about understanding of the topic, interpretation of the situation, and expectations of what others might know (Bauersfeld, 1979). "The student's reconstruction of mathematical meaning is a construction via social negotiation about what is meant and about which performance of meaning gets the teacher's (or the peers') sanction." Negotiation is goal-directed interaction, in which the participants each seek to attain their respective goals. Negotiation includes the working out of both the rules of procedure in the classroom and the construction of a way of knowing, which the teacher is trying to develop in the students through his/her greater mathematical knowledge. This construct catches the imbalance implicit in the teaching/learning situation (Bishop, 1985).

Bauersfeld (1979) reanalyzed a portion of a dissertation by Shirk to demonstrate the process of negotiation of mathematical meaning and to demonstrate major shifts in student-teacher interpretation of the situation definition during a lesson on the use of slide arrows in motion geometry. Bauersfeld presents 116 lines of transcript, which he divides into four parts. In each of the sections, the teacher's definition of the task differs from the students' definition. In addition, the task definition of both the teacher and the students change across the four segments of instruction. Included in the analysis are the changing meaning of "slide arrow" for the students across the course of instruction.

Campbell (1986) suggests specific points of the discourse to examine to gain insight into student learning of mathematical concepts. He analyzes a lesson on set sentences and number sentences in a fifth grade classroom in the Phillipines. He uses the metaphor "going for the answer" to examine the manner in which the teacher and students collaberated to produce correct answers to the teacher's questions. He segmented the lesson into a series of "question on the floor" and associated "answer-established" pairs. From this he was able to illustrate, with examples, from the transcript, how the teacher used corrections, prompts, and hints to help the children build definitions to "set sentence" and "number sentence."

These three articles demonstrate the usefulness of discourse analysis in examining teacher-student interaction in mathematics classes. This, in turn, will help us gain insight into the way mathematical meaning is socially constructed in mathematics classes.

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Pre-service teacher training

THE MATHEMATICAL LEARNING HISTORY OF PRE-SERVICE TEACHERS

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Summary.Pre-service teachers from two consecutive academic years were investigated. Consistently in both samples it was found that future primary/junior teachers evaluate their own former achievement in mathematics as lower, have a less favourable causal attribution pattern for their achievement and accordingly are less confident in teaching mathematics than preservice teachers who intend to teach math at the junior/intermediate or intermediate/senior level. Differences in attributing failure in teaching were not found to be equally consistent. Moreover, gender differences for teachers of the latter divisions were investigated.

Motivation theory , based on attributions, provides a basis for understanding how former and future achievement are interlinked and how the achievement motive of a student develops. A summary of relevant research results demonstrating how motivation directs the learning process in general is provided by Alderman et al. (1985). A detailed description of the impact of a specific math-related achievement motive can be found in Schildkamp-Kuendiger (1982).

Yet little is known about:

a) teachers' mathematical learning history; that is, their evaluation of their own math achievement and its causal attribution; and

b) the relationship of this history to their confidence in teaching mathematics, and about the causal attribution they call in when their teaching is not successful.

The impact of what has been called the mathematical learning history on teaching-related performance seems to be of particular interest at the beginning of a teacher's career, that is, at the pre-service level. Pre-service teachers' perceptions about their own former achievement are well established as they are based on extensive experience. Applying motivation theory, it seems reasonable to assume that this body of experience forms a motivational set that has a high impact on their perceptions related to the successful teaching of mathematics.

It is generally the case in teaching that some students probably will not reach the objectives set by even the most successful teacher. The students' failure to reach these objectives provides the teacher with relative failure experiences. In a situation where one learns how to teach, the interpretation of these failure experiences is crucial for the development of the teaching-related self-esteem ..

a) The impact of the variable " divisions chosen by pre-service teacher"

Eccles (1986) points out that occupational choices are based on a positive attraction to a profession.

The students in this study, whether male or female , who decided to become junior/intermediate or intermediate/senior mathematics teachers (j/i/s teachers) made a positive choice for mathematics. This is not, however, the case for primary/junior (p/j) teachers, as mathematics is only one of the many subjects they have to teach. These considerations are in line with those of Aiken (1976), who reports that future secondary mathematics teachers have more positive attitudes towards mathematics than primary teachers.

Looking at these pre-service teachers as former learners of mathematics, it is assumed that the j/i/s teachers have a more positive learning history than the group of p/j teachers. More positive learning history means here: former mathematical achievement is perceived as relatively high, few reasons are called in to explain this achievement and the achievement is mostly attributed to ability and effort. Less positive learning history means : former achievement is perceived as lower , more reasons are called in for explanation; amongst which are external reasons and lack of ability.

No reseach results could be found that relate teachers' learning history to their perceived success in teaching. The line of reasoning outlined above leads to the following hypothesis:

Hypothesis

The two groups of pre-service teachers described earlier differ in respect to their mathematical learning history, in the direction outlined above. Accordingly these two groups of teachers differ as to their perceived success in teaching mathematics and their attribution of failure in teaching in the direction that p/j teachers are less confident and call in more reasons to explain failure in teaching.

b) The impact of the variable " gender of pre-service student"

The research of Eccles (1986) clearly indicates that because of sex - role stereotyping, women generally are not attracted to professions in which mathematics plays an important part.

The women in this study who explicitly chose to teach math obviously constitute a highly selective group that did not follow the general trend. Therefore one might assume that female j/i/s teachers have an even more positive learning history than their male peers. On the other hand, research studies focussing on attributions have revealed that sex-role perceptions mediate the attribution of academic achievement in general and that of mathematics in particular;, the direction of this mediation being unfavourable for women (see e.g. Hansen and O'Leary 1985;Schildkamp-kuendiger 1982). As no related research results could be found , no directed hypotheses will be formulated. Rather the following research question will be investigated:

Research Question

Does this selection process evoke that female j/i/s students differ from their male colleagues as to the variables considered here ?

Since in the sample considered in this research there were very few male students enrolled in the p/j division a comparable question for this division cannot be investigated.

Sample and Procedure

Subjects of this study are all students enrolled in the pre-service programme at the University of Windsor. and who

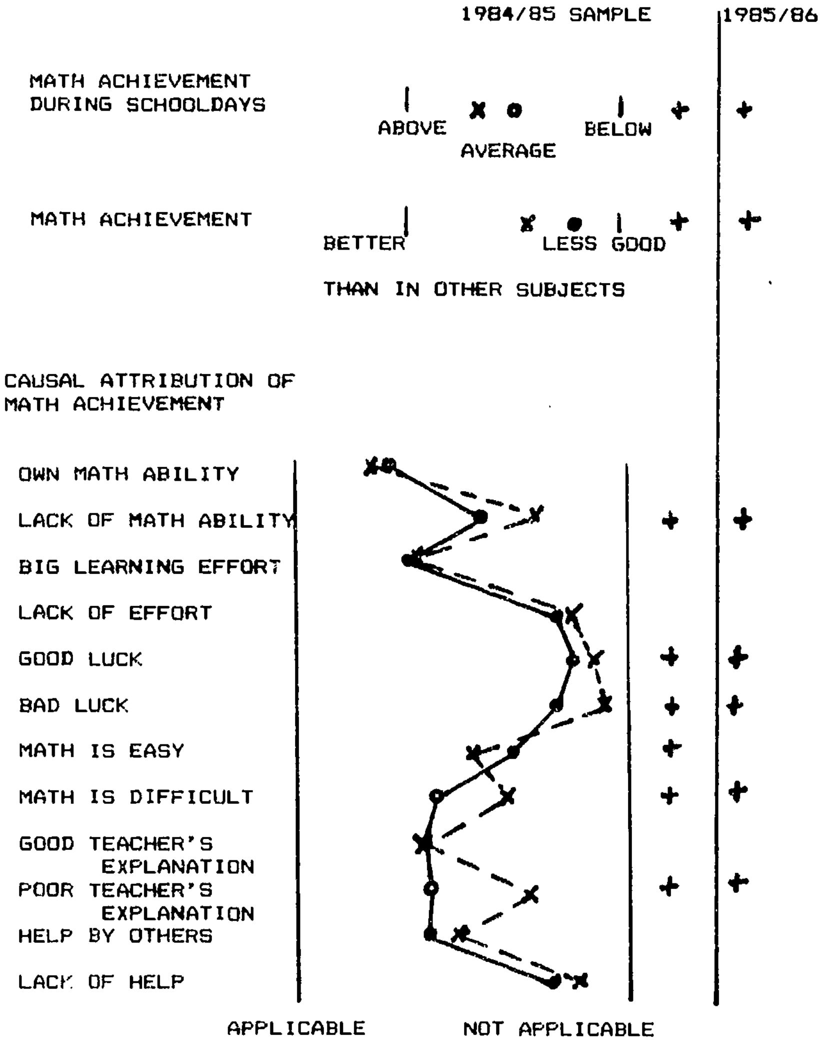
a) either had chosen to become primary/junior (K-6) teachers; referred to as p/j teachers (math education is a compulsory class for these students) or
b) had chosen to qualify to teach mathematics at the junior/intermediate (4-8) or intermediate/senior (7-13) level, referred to as j/i/s teachers (math education class is an optional choice of these students).

The pre-service programme is a one year programme. Data were gathered twice; at the end of the academic years 1984/85 and 1985/86 when students had gained the most experience possible in teaching mathematics.

Sample sizes:

1984/85	p/j teachers:	111	(96 female, 15 male)
	j/i/s teachers:	61	(36 female, 25 male)
1985/86	p/j teachers:	98	(84 female, 14 male)
	j/i/s teachers:	58	(35 female, 23 male)

GRAPH 1



o P/J PRE-SERVICE TEACHERS
x J/I/S PRE-SERVICE TEACHERS
+ $p < 0.05$ two-sided

Relevant information was gathered via a questionnaire, in which subjects were asked to make judgements on 3 point scales. The questionnaire contains sub-questionnaires developed in other studies that were slightly modified to fit the purpose of this research; these are:

a) Attribution of former mathematical achievement ; a questionnaire developed by Kuendiger , referred to in Schildkamp-Kuendiger 1980; this questionnaire was chosen as it has the advantage of having been developed from the reasons students actually call in for the kind of achievement considered in this research study. For further discussion of this issue see Hansen and O'Leary (1985,p.74).

b) Perceived reasons for failure in teaching mathematics; a questionnaire from the Second International Mathematics Study (part of the Teacher General Classroom Processes Questionnaire).
To investigate the above formulated hypothesis and the research question Chi2 tests on the item level were done with a significant level of 5% two-sided.
In the graphs showing the results, arithmetic means are used to characterise the distributions.
Data were analysed seperately for each academic year to inspect whether or not results were consistent over time.

Results and Conclusions

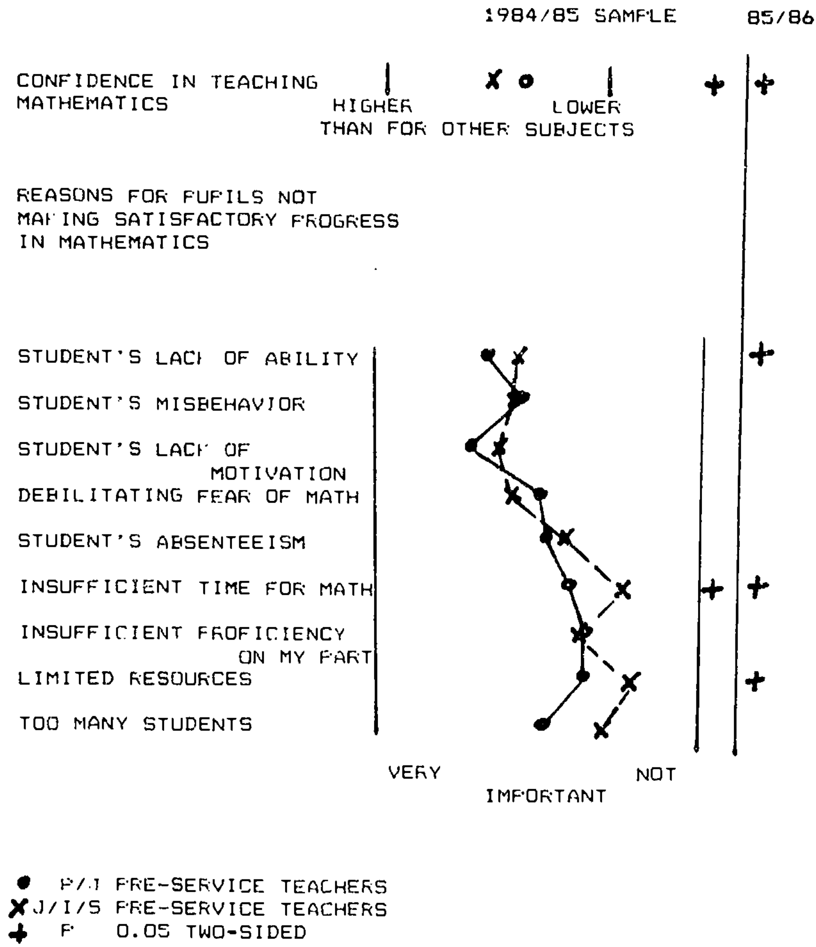
a) The impact of the variable "divisions chosen by pre-service teacher"

Graph 1 and 2 display the results for the 1984/85 sample. Significant results of the 1984/85 sample are added; all differences of this latter sample go in the same directions as those of the 1984/85 sample'

In agreement with the above formulated hypothesis p/j pre-service teachers as a group have a less favourable learning history than j/i/s pre-service teachers in both samples ; accordingly the former group feels less confident in teaching mathematics. As to the reasons that are called in for students not making satisfactory progress in mathematics, differences between the two groups are more distinct for the 1985/86 sample. Yet, if they accur they go in the expected direction, that is p/j teachers call in more reasons.

Future research is planned to identify more precisely the subgroup of p/j teachers who enters the teaching profession with a negative motivational set, and to investigate in what way lack of confidence in teaching mathematics moderates actual teaching. Findings will provide the basis for an intervention programme intending to establish a positive motivational set related to the mathematics, teachers will have to teach, in contrast to mathematics in general.

GRAPH 2


1984/85 SAMPLE
85/86
CONFIDENCE IN TEACHING MATHEMATICS
HIGHER
LOWER
THAN FOR OTHER SUBJECTS
REASONS FOR PUPILS NOT MAKING SATISFACTORY PROGRESS IN MATHEMATICS
STUDENT'S LACK OF ABILITY
STUDENT'S MISBEHAVIOR
STUDENT'S LACK OF MOTIVATION
DEBILITATING FEAR OF MATH
STUDENT'S ABSENTEEISM
INSUFFICIENT TIME FOR MATH
INSUFFICIENT PROFICIENCY ON MY PART
LIMITED RESOURCES
TOO MANY STUDENTS
VERY
NOT
IMPORTANT
P/J PRE-SERVICE TEACHERS
J/I/S PRE-SERVICE TEACHERS
P 0.05 TWO-SIDED

b) The impact of the variable "gender of pre-service teacher"

Although the whole group of j/i/s teachers remembers its math achievement during schooldays as above average, this is even more true for the female teachers ($p < 0.05$) in both samples. Moreover, female teachers of the 1984/85 sample evaluate their math ability and good teachers' explanations as more relevant a reason for their achievement than do male teachers ($p < 0.05$) ; whereas lack of effort is perceived as less a reason by female teachers ($p < 0.05$).
In the 1985/86 sample female teachers attributed their achievement significantly more often to good teacher's explanation and to help by others.
Overall,these results only partly indicate that female j/i/s pre-service teachers have a somewhat more positive learning history than their male peers. Further research is needed to get a clearer picture of attributional differences between these two groups.
Finally, in both samples there is another rather unexpected significant difference between male and female j/i/s teachers :when it comes to explaining why their pupils did not make satisfactory progress female teachers perceive insufficient proficiency on their part to be more relevant a reason ($p<0.05$), although both groups are equally confident to teach mathematics.
It is intended to further investigate the implications of this finding.

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INFINITY CONCEPTS AMONG PRESERVICE ELEMENTARY SCHOOL TEACHERS

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ABSTRACT

Because concepts of infinity held by preadolescent students are resistant to the effect of schooling and are contextually sensitive, there are two objectives for this research. First, infinity concepts held by preservice teachers, who may eventually teach infinity concepts, are described along three dimensions: arithmetic-geometric, convergent-divergent, and cardinal-limit. Second, the stability of concepts of infinity along the dimensions is systematically explored. To meet these objectives, an interview-based research model was developed to randomly assign clusters of tasks crossing the three dimensions to 48 subjects, optimizing comparisons along each of the three dimensions. Meny teachers provided responses which were finite in nature. In each task, over 50% of the responses reflected incomplete concepts of infinity. Teachers' iofinity concepts were not stable, with inconsistency of responses exceeding 45% in each dimension.

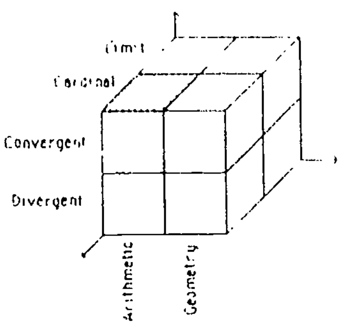
The mathematical concept of infinity contradicts personal experience, which is necessarily finite. Research, most of which has been reported within the last twenty years, has revealed various aspects of infinity held by elementary and junior high school students. Preschool and young elementary school children show intuitions of infinity when the questions are incorporated into a competitive game setting or are phrased as: "Is there a biggest number?", "Can you name a bigger number?", "Can you count forever?", "Can you successively halve a segment?", "Can you draw dots forever?" (Falk, et. al., 1986; Gelman, 1980; Langford, 1974; Piaget & Inhelder, 1949). Evans (1984) showed that children in the primary grades had knowledge of infinity, including recognition that there is no largest number. Fischbein, Tirosh and Hess (1979) found that older children, starting about age 11, have intuitions of infinity that are extremely sensitive to the conceptual and figural context of the problem posed. Langford (1974) found children able to conceive of indefinite iteration of addition, subtraction, and multiplication about age 9, but not until age 13 could students conceive of indefinite iteration of division.

It is interesting that the development of infinity concepts arises independent of formal schooling (Gelman, 1980; Evans, 1984), yet appears resistant to the effect of schooling (Fischbein, Tirosh, & Hess, 1979). Intuitive notions of infinity seem to cluster in developmental levels (Falk et. al., 1986, Gelman, 1980; Langford, 1974; Piaget & Inhelder, 1949) but do not expand to increasingly abstract and formal concepts through the process of schooling. One reason may lie with the concepts of infinity as developed through instruction. The relationship between instruction and students' concepts are unclear when increasingly sophisticated conceptual schema are considered. If, however, a teacher's schema are incomplete or inconsistent, concerns exist about the teacher's own students developing increasingly abstract mathematical concepts. The primary objective of this research is to examine the infinity concepts held by preservice elementary school teachers.

Previous infinity research has been neither comparative nor broad in range. Researchers have tended to focus on either an artihmetic context (e.g., Evans, 1984, Langford, 1974) or a geometric context (e.g., Fischbein, Tirosch, & Hess, 1979). They have not compared and contrasted the commonality of such contexts. Little attention has been given to differences in infinity used in both divergent and convergent contexts: the unboundedness of the whole numbers versus the bounded intervals of fractional numbers, or lines versus line segments. Infinity used in a cardinal sense has not been contrasted with infinity used in a limit context. A secondary objective of this research is to compare and contrast behaviors within and across these domains: arithmetic versus geometric, bounded versus unbounded, and cardinal versus limiting processes.

METHOD

Three dimensions, arithmetic-geometric, convergent-divergent, and cardinal-limit, were crossed to generate eight (2x2x2) tasks: ACC, GCC, ACL, GCL, ADC, GDC, ADL, and GDL. The shaded cell in the figure identifies Arithmetic-Convergent-Cardinal task named by the ordered triple ACC. For each task, three subtasks were systematically developed in order to examine the consistency of response within a particular task. These were a baseline subtask and two systematic variations. The first variation was additive in nature whereas the second



variation was multiplicative. The three constructive subtasks of the ACC task as printed in the protocol are as follows:

> Let's construct Set S.
> Write the number 8, the first element of the set.
> Multiply the element by 1/2. Include the product in the set.
> Multiply that product by 1/2. Include it in the set.
> Consider Set S, with all such numbers.
> Describe Set S.
>
> Let's construct Set T.
> Write the number 64, the first element of the set.
> Multiply the element by 1/2. Include the product in the set.
> Multiply that product by 1/2. Include it in the set.
> Consider Set T, with all such numbers.
> Describe Set T.
>
> Let's construct Set U.
> Write the number 8, the first element of the set.
> Multiply the element by 1/8. Include the product in the set.
> Multiply that product by 1/8. Include it in the set.
> Consider Set U, with all such numbers.
> Describe Set U.

A set of questions for each subtask was prescribed so that responses to questions concerning the baseline subtask could be contrasted by the subject to the other two subtasks. For each subtask of the ACC task, the subject was asked "How many elements are in the set?" Subsequently, the subject was asked to think about the elements in Set S and Set T (also Set S and Set U) and to identify the set which has more elements or if one set has as many elements as the other. Each response was probed by the interviewer.

With the ACL task, the three subtask settings were identical but the questions varied. "What is the smallest element in the set?" was a common question for each subtask. Similar to the ACC task, the subject was subsequently asked to consider Set S and Set T (also Set S and Set U) and to identify the set with the smallest element or if the sets have the same smallest element.

For testing purposes, eight task-clusters were developed. Each task-cluster was formed by clustering a task with the three tasks differing from it along a single dimension. For example, a task-cluster was formed by clustering the ACC task with the GCC, ADC, and the ACL tasks and the GCC task was clustered with the ACC, GDC, and GCL tasks. When the eight task-clusters are considered collectively, each task appears in four different clusters and each singular comparison appears in two clusters. This design optimizes singular contrasts along the three dimensions.

From approximately 120 advanced undergraduates enrolled in a methods course for the teaching of elementary school mathematics, a pool of volunteers was solicited. Forty-eight subjects were randomly selected from the pool. Six students were randomly assigned to each task-cluster. Subjects were individually interviewed by one of the two investigators using the protocols appropriate for the tasks within a particular cluster. The order of presentation of the tasks within a cluster was random. For each subtask within a task-cluster, a printed description of the setting was concurrently made available to the subject while the interviewer read aloud the description. The questions, presented only in oral form, were prescribed by the protocol.

During the 30-45 minute interview sessions, audio-recordings and interviewer's notes were made. Complete transcripts were prepared from the notes and recordings.

RESULTS AND CONCLUSIONS

Analysis of the data proceeded in two stages, corresponding to the two major objectives of the investigation. In the first stage, an overview of infinity concepts of preservice elementary school teachers was sought. To this end, responses to each task were categorized; the categories and results are summarized in Table 1. Six categories were used to describe responses: finite, four classes of infinite, and uncategorized. The Finite categorization includes responses in which specific numbers were employed or in which an unspecified number is indicated (e.g., "very small" or "a lot").

The four infinite categorizations include responses in which the terms "infinite" or "infinity" were explicitly used. Also included are responses which indirectly evoke the concept. In the cardinal context, these responses included "goes on and on" or "unending"; whereas in the context of limits, the responses include "you can't say [which is largest]" and "there isn't [a largest element]". The four infinite categorizations are differentiated from each other based on responses to the comparison subtasks. In Infinite, neither agree, the subject indicated that both comparison subtasks disagreed either in number (cardinality tasks) or in limit (limit tasks) with the baseline subtask. In the Infinite, both agree category, the subject stated that both of the comparisons agree with the baseline. In the Infinite, inconsistent category, the two comparisons receive unlike responses. In the Infinite, cannot tell category, when asked to compare the subtasks the subject stated that one "can't tell". Finally, Uncategorized task responses include responses which did not fit into any of the other categories.

Several observations based on the data in Table 1 are important. In only one task (GCL) are more than 50% of the responses categorized as finite. Moreover, when summing the percentages of the four categories judged infinite, the remaining tasks have rates of infinite responses of over 50%. It appears that while the majority of the students have some concept of infinity in many task settings, finite responses continue to occur with some frequency. Further, three of the four convergent tasks had rates of finite responses of over 20%, while no divergent task had a rate of finite categorizations 15%.

Table 1.
Percentages of Responses by Task

Categorization	Task							
	GCD	GDL	GCC	GCL	ADC	ADL	ACC	ACL
n	22	21	22	25	23	23	23	23
Finite	14	0	36	56	9	0	13	22
Infinite								
Neither agrees	23	5	5	0	43	0	22	0
Inconsistent	23	14	18	8	17	9	35	9
Both agree	23	43	36	16	17	57	26	39
Cannot compare	9	19	0	16	8	35	4	26
Not categorized	9	19	5	4	4	0	0	4

Table 2.
Percentages of Agreement of Responses to Tasks,
Within Dimensions

Dimension	n	Level of agreement		
		Consistent	Inconsistent	Not Classified
Cardinal-Limits	41	34	61	5
Convergent-Divergent	42	50	45	5
Arithmetic-Geometric	45	44	49	7

Thus, the convergent setting seems less conducive to use of the infinity concept than the divergent setting.

Responses in each of the first three categories (Finite, Infinite, neither agrees, and infinite, inconsistent) of Table 1 are not adequate for any of the tasks settings. Summing the percentages across these categories, five of the eight tasks (GDC, GCC, GCL, ADC, and ACC) have inappropriate response rates above 50%. Thus, many subjects appear still to have an incomplete concept of infinity. All of the Cardinal tasks have rates of inappropriate responses above 50%, while only one of the four limit tasks is over 50% inappropriate. It thus appears that comparing limits of infinite sets may be easier than comparing the cardinality of such sets.

In the second stage of the data analysis, behaviors were compared along the Arithmetic-Geometric, Convergent-Divergent, and Cardinal-Limit dimensions. To accomplish this, a subject's categorizations on each pair of tasks singularly differing along a given dimension were compared. For example, to explore differences along the Arithmetic-Geometric dimension, responses to the following pairs were compared: GDC - ADC, GDL - ADL, GCC - ACC, and GCL - ACL. A subject receiving the same categorization for both tasks, was judged Consistent. If the categorizations differed, the subject was judged Inconsistent. If either or both of the tasks were not categorized, the subject was Not classified. Results are summarized in Table 2.

In each of the three dimensions at least 45% of the responses were inconsistent. This suggests that the conceptions of infinity are context-dependent; the subjects do not have a generalizable notion of infinite. The percentage of inconsistent responses was greatest in the Cardinal-Limit dimension, while percentages of inconsistent responses were nearly balanced in the other two dimensions. This suggests that subjects are more likely to be inconsistent between cardinal and limit settings.

Prospective elementary teachers have difficulty with the concept of infinity. Some do not recognize infinite situations as being infinte. Many have incomplete conceptions of infinity in various settings. Their conceptions of infinity seemed frequently to be inconsistent when an arithmetic context is contrasted with a geometric context, when a convergent context is contrasted with a divergent content, and when a context involving cardinality is contrasted with a context involving limits. The deficiencies in the conceptions of infinity of these prospective teachers suggest that they will not have the knowledge necessary to develop and deliver effective instruction on the infinity concept. If such deficiencies exist among practicing elementary teachers, a partial explanation of the inefficacy of schooling in developing children's conceptions of infinity may have been identified. These results suggest that teacher education programs should devote more attention to the concept of infinity. In particular, care should be taken to discuss the concept of infinity in a variety of contexts. Highlighting the similarities between contexts may prove useful in extending the generalizability and richness of their infinity concepts. In this way teachers may develop in their future students broader conceptions of infinity.

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INTERVENTIONS TO CORRECT PRESERVICE TEACHERS' MISCONCEPTIONS ABOUT THE OPERATION OF DIVISION

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In this study, 32 out of 59 preservice elementary teachers experienced difficulties in solving division word problems with the divisor greater than the dividend. Two interactive computer instructional programs, a tutorial program and a drill and practice program, were developed to increase the preservice teachers' awareness of the source of their difficulties and to help them improve their performance on these problems. Half of the 32 preservice teachers worked with each program. After working with the programs, the performance of 25 preservice teachers in solving division word problems improved. Only the tutorial program, however, was effective in increasing the preservice teachers'awareness of their tendency to reverse the roles of the divisor and the dividend.

Arithmetic operations are central to the mathematics curriculum of all countries. A relational understanding of arithmetic operations - based on conceptual and operational connections - is essential to avoid misconceptions. Misconceptions, once learned, may be difficult to overcome. An understanding, based on constructs and relationships, of arithmetic operations is likely to facilitate students' transition to new material such as extending their conceptions from the domain of whole numbers to the domain of rational numbers.

Studies by Greer and Mangan (1986), Tirosh, Graeber & Glover (1986) have indicated' that a substantial portion of preservice elementary teachers have difficulties in solving division word problems with divisor greater than the dividend. The main purpose of this study is to better understand this particular misconception among preservice elementary teachers. Prototypic instructional computer programs were written to implement strategies to help preservice elementary teachers (a) to become aware of their tendency to reverse the role of the divisor and the dividend in solving division word problems with the divisor greater than the dividend, and (b) to improve their performance in solving division word problems.

PERSPECTIVE

A considerable body of research now exists on children's and adolescents' erroneous beliefs about the operation of division. One most widespread misconception is that the divisor must be less than the dividend (Hart, 1981; Bell, 1982; Bell, Fischbein & Swan, 1984). Due to this misconception, children and adolescents are faced with difficulties in solving division problems with divisor greater than the dividend. Tirosh, Graeber,and Glover (1986) found that preservice elementary teachers held this misconception. Many of them reversed the role of the divisor and the dividend in problems with a whole number divisor greater than a whole number dividend.

Several of strategies have been used to help children and adolescents improve their performance in solving word problems. These strategies include use of diagrams, estimation of answers, and substitutions, of simpler numbers (Bell, Swan & Taylor, 1981). Previous interviews with preservice teachers indicated that the majority of them accepted diagrams and estimations as appropriate strategies. The authors decided to implement the diagrams and the estimation strategies in an interactive computer instructional program designed to increase the preservice teachers awareness of their tendency to reverse the role of the divisor and the dividend to improve their performance in solving division problems.

A second computer instructional program, developed for this study, implemented a drill and practice mode with immediate feedback. This program provides the students with immediate feedback as to whether their responses are right or wrong and with opportunities to correct wrong answers.

METHOD

Subjects

The subjects were selected from 59 college students enrolled in sections of a mathematics content course or a methods course for early elementary education majors at the University of Georgia during the Spring Quarter 1986.

Instruments

1. Writing Expressions for Word Problems. This test included 21 word problems(13 division, 4 multiplication, 1 addition and 2 subtraction). Six of the division word problems had a divisor greater than the dividend. The remaining 7 division problems had a dividend greater than the divisor.

The 13 division items were interspersed with the other 8 problems to reduce the likelihood that correct answers would result from guessing. The students were instructed to write an expression that would lead to the solution of each problem.

2. Beliefs about Division. Students were presented with statements about multiplication and division. One of these statements was: "In division problems, the divisor can be larger than the dividend". The preservice teachers were asked to determine whether the statements were true or false and to defend their answers.

3. Tutoring Computer Program. This program includes eight division word problem, four problems with a divisor greater than the dividend and four problems that have divisors smaller than the dividend. Three distinct sets of cues were available for each of these problems: (a) diagrams that illustrate the word problems, (b) estimation of the quotients, and (c) completion of statement about the size of the the divisor and the dividend. The preservice teachers received these cues in the above order. The sequence of cues was terminated when the student gave a correct response and indicated with certainty that the answer was correct. If the student finished the set of cues without giving a correct response, a correct response and rationale was provided. After completing the work on each of the four problems with the divisor greater than the dividend, the student was shown a statement about the relative size of the dividend, divisor, and the quotient.

4. The Drill and Practice Computer Program. This computer program includes the same problems used in the tutorial program. The student has three opportunities to answer each problem. After each trial the student gets an immediate response as to the correctness of the answer. If the student gave incorrect answers in each of the three trials, the correct response was provided.

Procedure

At the beginning of an academic quarter, the preservice teachers completed the first two instruments. Preservice teachers who reversed the role of the divisor and the dividend in their answers to at least two of the division word problems with the divisor greater than the dividend were assigned to use one of the computer programs in the instructional stage. A short interview was conducted by one researcher immediately after the preservice teachers worked with the computer.

Three weeks after working with the computers, the preservice teachers that participated in the instructional stage were given two instruments similar, but not identical, to the instruments given to them before instruction.

RESULTS

Thirty-two of the 59 preservice teachers (54%) reversed the role of the divisor and the dividend in their answers to at least two out of the six division problems with the divisor greater than the dividend. Half of these 32 students used the tutorial program; the other half used the drill and practice program.

In the tutorial group, only one of the 16 preservice teachers correctly completed the four problems with the divisor greater than the dividend without using any of the assisting cues. Four of the 16 used the diagram in answering at least one of the problems, seven required the diagrams and the estimation cues, and four all three cues (diagrams, estimation, and completion questions). Eleven of the preservice teachers reported, immediately after working with the tutorial program, that the technique of estimating was the most helpful to them. In the drill and practice group, only one of the 16 students correctly completed all four of the problems with the divisor greater than the dividend on the first attempt, eight wrote appropriate expressions using not more than two attempts, and three needed no more than three attempts. Four students failed to give a correct answer to at least one of the four problems after the three attempts allowed by the program. Thus most of the students in both the tutorial and the drill and practice groups were able to respond correctly to division problems when assistance was available.

In an interview with each of the preservice teachers immediately after their work on the computer 10 of the 16 students in the tutorial group recognized they had a tendency to reverse the roles of the divisor and the dividend. Only four of the 16 students in the drill and practice group, however, became aware of this tendency. Most of the students in the drill group argued that they were assigned to "work with the computer" because they made "careless mistakes" on the pretest. Moreover, three weeks after instruction all the students from the tutorial group gave a correct response to the statement about the relative size of the divisor and the dividend whereas two students from the drill group still claimed incorrectly that the dividend must always be greater than the divisor.

Table 1 and 2 show that before instruction the preservice teachers performance on these division word problems was rather low. It was found that only 42 percent of the preservice teachers in the tutorial group and 41 percent in the drill group wrote appropriate expressions to the division word problems. After instruction, however, 70 percent of the preservice teachers wrote appropriate expressions. The number of reversed answers decreased from 44 to 23 in the tutorial group and from 41 to 18 in the drill group.

The difference in performance between the tutorial and the drill groups after instruction is rather small. Both of the computer programs helped the students write appropriate expressions for division word problems.

DISCUSSION AND IMPLICATIONS

The results show that both the tutorial and the drill and practice programs proved to be effective in improving students' performance in writing expressions for division word problems. The frequency of reversed expressions to these problems decreased by half. The potential of computer instruction programs such as these with children and adolescents and the lasting effects of such interventions needs to be investigated.

Preservice teachers that used the tutorial program were exposed to appropriate strategies of solving division word problems such as drawing diagrams and estimation of the answers. The majority of them identified the strategy of estimating answers, which provided the students with a means of checking their answers, as the one that was most helpful to them in monitoring their work. The estimation strategy may be useful for helping children and adolescents as well as preservice elementary teachers overcome other misconceptions about the arithmetic operations and for improving their performance in solving word problems.

Table 1 : Distribution of responses by students in the tutorial program to division problems with divisor greater than the dividend

problem number	operation	before instruct. % cor-rect	% rever-sal	% other	after instruction % cor-rect	% rever-sal	% other
1	4:16	44	56	-	81	19	-
2	3.25:5	75	19	6	75	25	-
3	3:15	44	50	6	81	19	-
4	5:20	25	44	31	69	12	19
5	.2:5	19	62	19	38	50	12
6	9:12	44	37	19	63	25	12
average		42	45	13	68	25	7

Table 2 : Distribution of responses by students in the drill program to division problems with divisor greater than the dividend

problem number	operation	before instruct. % cor-rect	% rever-sal	% other	after instruction % cor-rect	% rever-sal	% other
1	4:16	56	44	-	69	31	-
2	3.25:5	63	31	6	88	6	6
3	3:15	50	38	12	75	13	12
4	5:20	31	25	44	69	19	12
5	.2:5	25	56	19	44	44	12
6	9:12	19	56	25	75	6	19
average		41	41	18	70	20	10

teachers with strategies for solving problems. Immediate feedback, and the opportunity to try different solutions to the problems, helped the preservice teachers improve their performance. Further research on the effects of feedback in computer instructional programs is warranted. We need to understand more about the processes students may generate in such situations in order to correctly perform division problems.

The tutorial program was effective in helping preservice teachers become aware of their tendency to reverse the role of divisor and dividend. After working with the computer programs, the majority of the students assigned to the tutorial program were aware of the impact that their tendency to reverse the role of the dividend and the divisor had on their performance. Only 25 percent of the students assigned to the drill and practice program, however, were able to describe their misconceptions. The influence of awareness of misconceptions on students' understanding and performance on problems division needs to be investigated.

Today's preservice teachers are tomorrow's teachers. It is crucial that they become aware of and and develop strategies to overcome their misconceptions about the operation of division and about other arithmetic operations.

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Tertiary level

ALTERATION OF DIDACTIC CONTRACT IN CODIDACTIC SITUATION

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Exposé en anglais de Daniel Alibert

Formalism and lack of functionality of proofs are frequently observed in the first year student's mathématical productions at University: mathematics aren't acknowledged as a scientific subject playing a rôle in the understanding of reality.
Scientific process itself seems not to be an interesting learning subject to deal with.
Facing this problem, we set, within a definite theoretical framework, an experimental teaching method for mathematics whose characters (uncertainty, scientific debate among students, emphasis laid on epistemology...) presuppose an important alteration of the didactic contract.
This alteration produces changes in the student's relation with mathematical knowledge and learning.We analyze these alterations through observations of lectures and student's answers to questionnaires.

§1 Problématique et cadre théorique général.

Les productions mathématiques d'un grand nombre d'étudiants entrant à l'Université pour y suivre des études scientifiques longues consistent en textes ou en discours visant à reproduire la forme du discours de l'enseignant sans que le contrôle de la signification soit ressenti comme primordial: souvent le syntaxique prend ainsi le pas sur le sémantique. On observe par exemple ([7]), un contrôle insuffisant sur le reste d'un développement limité ("il tend vers zéro ") ce qui vide de sens la partie principale; cette difficulté est en particulier un obstacle important à la maitrise de la mise en équation par les différentielles ([7]).

Une autre observation est le désintérêt témoigné par beaucoup d'étudiants vis-à-vis de la preuve en général: vue comme un exercice de style propre à l'activité de l'enseignant, qu'il faut savoir reproduire devant lui sans en ressentir la nécessité profonde, les problèmes qu'elle vise à résoudre n'ont en général pas fait l'objet d'une véritable dévolution par l'enseignant, ni d'une appropriation par l'étudiant.

Chez ces étudiants entreprenant des études scientifiques universitaires, de telles constatations montrent que les mathématiques ne sont pas reconnues comme une discipline scientifique permettant, comme d'autres disciplines (sciences physiques par exemple), la résolution de certains problèmes, et en liaison avec elles, une certaine appréhension du réel: en particulier on peut constater un vide épistémologique quasi-total .

En réponse aux problèmes posés par ces observations, nous avons développé une expérimentation d'enseignement des mathématiques en première année d'Université (Deug A1) s'insérant dans le cadre théorique formé par les hypothèses cognitives et didactiques générales suivantes:

-les étudiants construisent leurs propres connaissances dans un jeu de déséquilibre et de rééquilibre dans lequel interviennent tous les constituants du milieu auquel ils sont confrontés: savoir mathématique, problèmes, maître, autres étudiants.([6],[3]).

-Ces connaissances seront d'autant plus stables et susceptibles d'un réinvestissement performant qu'elles auront été établies, et utilisées, dans plusieurs cadres de conceptualisation entre lesquels une communication est possible ([4]).

-En ce qui concerne l'apparition de la necessité de la preuve pour les étudiants, nous nous appuyons sur les travaux de N.Balacheff ([2]) pour étudier en quoi les situations mises en place sont favorables à cette apparition.

Nous considérons d'autre part, pour les étudiants en scolarité post-obligatoire au moins, qu'un véritable apprentissage mathématique dans sa dimension scientifique générale passe par la constitution d'une épistémologie des notions abordées.

§2 Description de l'expérimentation:

Ce système d'enseignement, qui a été expérimenté depuis trois ans à l'Université de Grenoble dans une section d'une centaine d'étudiants a pour principales caractéristiques les suivantes:

1) de grandes plages d'incertitude sont ménagées à l'intérieur de l'enseignement, incertitude institutionnalisée par le recours aux énoncés explicitement conjecturaux, dont la validation, et souvent la production même, est dévolue à la collectivité des étudiants:

En effet nous pensons que la fonctionnalité de la preuve ne peut apparaître réellement que dans une situation où l'incertitude constitue une variable fondamentale.

2)Les arguments de preuve apportés par chaque étudiant ne le sont pas dans une démarche s'adressant à l'enseignant, mais dans une situation de Débat Scientifique entre pairs, les étudiants eux-mêmes:

Nous avons été amenés à distinguer très nettement entre les argumentations de preuve "pour convaincre" un interlocuteur dont on sait qu'il ne possède pas la connaissance visée, comme connaissance institutionnelle, et les argumentations de preuve "pour adhérer" à la conviction d'une personne possèdant préalablement cette connaissance (ici l'enseignant). Cette distinction est la base de la description théorique élaborée a posteriori, sous le nom de "situation codidactique" : il s'agit d'une situation dans laquelle un étudiant souhaite convaincre la collectivité des autres étudiants de la validité d'une connaissance qu'il possède, sachant que ses camarades ne possèdent pas cette connaissance. Ceux-ci, de leur côté, savent qu'il s'agit d'un savoir non institutionnel à examiner. La situation codidactique est du type décrit par N.Balacheff comme potentiellement capable d'amener à l'émergence de contradictions et à la production de preuves.

3)L'introduction de certains outils mathématiques nouveaux (intégrale par exemple) est organisée de telle sorte qu'ils apparaissent comme necessaires à la résolution de problèmes complexes, souvent de Sciences Physiques. Cette necessité est rendue manifeste par l'impossibilité où se trouvent les étudiants de résoudre des situations-problèmes, par ailleurs très concrètes, sur lesquels ils peuvent avoir un certain contrôle ([5]). La complexité des problèmes à résoudre servira par la suite de justification à l'important effort théorique qui sera consenti pour établir les principales propriétés de l'outil construit.

A travers cette construction d'un outil nouveau, nous pensons que l'étudiant pourra se constituer une épistémologie véritable du concept visé:

En particulier nous avons choisi de repousser longtemps l'introduction d'algorithmes performants (calculs de primitives ici) pour ménager la possibilité de la réflexion en profondeur sur le concept lui-même.

4) Nous pensons qu'à ce niveau d'études, la prise de conscience par les étudiants qu'il leur incombe de construire, dans un processus réflexif, leurs propres connaissances est un facteur positif de réussite:

Les principales propriétés des concepts introduits seront établies au cours de débats autour de conjectures . Un certain nombre de considérations sur les connaissances enseignées ainsi que sur les processus d'apprentissage mis en oeuvre sont également développées dans les séances d'enseignement, visant à faire ressortir des connaissances à ce niveau, lorsqu'elles ont été produites par le système codidactique.

La mise en place de cette expérimentation passe necessairement par un changement radical du contrat didactique en vigueur, par l'instauration d'une nouvelle coutume constitutive du "système codidactique". En particulier il est necessaire de donner un statut institutionnel aux énoncés dont on n'a pas encore décidé du caractère de véracité, sous le nom de conjectures. Ce terme sera par la suite investi d'une signification supplémentaire de nature épistémologique: conjecture exploratoire pertinente, dont les conséquences seraient importantes...

§3 Méthodologie d'étude:

Les séances de Débat Scientifique en cours de mathématiques sont préparées par le groupe de recherche par une analyse a priori des situations-problèmes ou du domaine des conjectures proposées, en particulier de leur potentiel d'apparition de démarches jugées favorables, des techniques à utiliser pour échapper à tel ou tel blocage prévisible. On essaie de prévoir les principaux choix de l'enseignant, en particulier les moments où il ne doit pas intervenir.

Ces séances ont été enregistrées et observées par les membres du groupe, puis analysées a posteriori en comparant prévisions et réalisations pour mieux adapter les séances suivantes. Cette analyse a permis de repérer quels changements profonds chez l'enseignant conditionnent le changement de contrat.

Par ailleurs l'ensemble de ces observations a formé la base d'une étude empi de la méthode du Débat Scientifique: typologie des débats, mise en évidence de techniques susceptibles d'être transmises pour provoquer le changement de contrat ([1]).

En ce qui concerne l'étude des changements dans la relation de l'étudiant au savoir, accompagnant ce changement de contrat didactique, dans le cadre de la problématique énoncée plus haut, nous avons mené nos recherches sur divers points:

-l'analyse des enregistrements et des observations nous a permis de préciser quelle proportion d'étudiants participent au débat, de quel niveau sont les arguments échangés dans l'amphithéâtre, si les connaissances mathématiques institutionnalisées sont des productions collectives des étudiants ou de l'enseignant seul, et quelle est leur nature.

-l'analyse des réponses à un questionnaire, passé après plusieurs mois de pratique du nouveau contrat didactique, portant d'une part sur les réactions des étudiants face à une conjecture, d'autre part sur leur compréhension d'un concept selon qu'il a été introduit dans un cours "clair et bien ordonné" ou dans des séances de débat. Cette analyse a examiné (1)le jugement porté globalement sur le changement de contrat didactique (2) leur conception de l'apprentissage des mathématiques (3) leur conception du savoir mathématique (4) leur attitude face à l'incertitude, en particulier en ce qu'elle induit une recherche de preuve .Nous avons également utilisé un questionnaire de bilan proposé en fin d'année à l'ensemble des sections du Deug A1 (450 étudiants).

-les contrôles des connaissances périodiquement organisés sont étudiés quant aux manifestations d'une prise de distance réflexive des étudiants vis-à-vis des questions posées et aux types de preuves développées:ils portent témoignage d'une évolution positive de la tendance au formel et à l'algorithmique vers un souci de la signification des argumentations.

S4 Premiers résultats et perspectives:

En ce qui concerne la participation effective des étudiants au débat, on observe fréquemment qu'un tiers des présents prennent la parole au cours d'une séance donnée, souvent après concertation avec leurs voisins immédiats ce qui permet de conclure à une implication forte de la collectivité dans l'activité proposée: les enjeux sont ressentis comme suffisants pour provoquer l'entrée des étudiants dans une véritable démarche reflexive. C'est là un premier résultat en contradiction avec l'avis généralement porté sur les étudiants à ce niveau. Les énoncés proposés et débattus au cours de ces séances sont loin d'être triviaux et ils permettent d'aborder les problèmes reellement posés sur les concepts enseignés: Les étudiants entrent dans un rapport de producteurs de leurs connaissances mathématiques.

Dans le premier questionnaire nous relevons

(1) chez 60% des étudiants une réponse comparative argumentée entre les deux méthodes, donnant largement (75%) la préférence au débat, sans exclure toujours les parties plus traditionnelles.

(2) en ce qui concerne les connaissances acquises, de nombreux étudiants soulignent que le débat sur des conjectures leur permet de saisir quels sont les problèmes que les connaissances mathématiques nouvelles visaient à résoudre, quelles sont les erreurs que l'on peut faire à leur propos. Pour eux, il s'agit d'une approche approfondie du concept.

(3) cela révèle une réflexion sur l'apprentissage des mathématiques prenant en compte l'importance d'une certaine épistémologie. Certains notent que seule la recherche leur permet d'assimiler un concept. L'exposé traditionnel est apprécié (30%) en situation d'institutionnalisation .Il faut toutefois noter que 10% de l'effectif rejette la méthode comme inabordable et trop peu ordonnée. En particulier ces étudiants ne se sentent pas concernés par un énoncé conjectural, souvent sans idée.

(4) face à l'incertitude d'une conjecture, nombreux sont ceux qui relèvent la difficulté du type de problème: entrer dans la conjecture, bâtir une démonstration, formuler les idées. Ils entrent donc en plein dans le type d'apprentissage visé par la méthode. Malgré ces difficultés ils se sentent concernés et intéressés: la curiosité est souvent invoquée, le débat avec les autres étudiants également. Ces réflexions témoignent d'un rapport nouveau avec leur milieu.

Le questionnaire de bilan a permis de relever de grandes différences entre les réponses des diverses sections: les considérations très scolaires observées en général sont remplacées dans la section expérimentale par des réflexions sur la compréhension et l'autonomie d'apprentissage.

Il reste à évoquer quelles sont les conséquences pour l'enseignant de ce changement de contrat didactique: elles ne sont pas négligeables. En premier lieu son rapport au savoir enseigné change: si les choix didactiques fondamentaux lui incombent toujours, il devra toutefois apprendre à ne plus être celui qui doit en toutes occasions fournir les réponses aux questions posées par les étudiants, mais au contraire en faire la dévolution à la collectivité et jouer un rôle fondamental d'organisateur du débat entre les étudiants. La collaboration d'un groupe, même réduit, pour analyser les séquences à proposer, et en observer la réalisation, est également un facteur très important. Un certain nombre de techniques de gestion du débat doivent être utilisées, dont certaines sont maintenant assez bien éprouvées, et d'autres doivent encore faire l'objet de reflexions ([1]).

Le problème de la transmissibilité de cette expérience à d'autres enseignants se pose à nous maintenant, pour une expérimentation plus large: le côté "ingéniérie" doit être développé, mais nous pensons qu'il est simultanément indispensable d'arriver à un approfondissement théorique de la méthode du débat scientifique, pour y repérer ce qui est fondamental, ou au contraire plus accessoire, et justifier scientifiquement un certain nombre de choix empiriques: un premier modèle est en cours d'épreuve, autour de la notion de situation codidactique.

Par ailleurs nous avons débuté une expérience de collaboration interdisciplinaire pour l'apprentissage par les étudiants de la démarche scientifique, en particulier dans ses aspects de mise en équation, modèlisation, visant à atténuer la disjonction souvent observée entre les mathématiques et les sciences physiques même lorsqu'elles enseignent et utilisent les mêmes outils.

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DIALECTIQUE ET PENSEE MATHEMATIQUE

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ABSTRACT

Mathematical thinking in its achieved form seems to be a linear, straightforward, thinking, without feedback from intermediate results that may interfere with the primitive assumptions. This is nothing but an utterly appearance. In building up a theory, the mind of a working mathematician goes through a series of loops, alternatively bringing him closer to the solution of his problem or taking him away from it. The pattern behind this schema is : he coins a plausible conjecture (thesis) which is submitted to an obstacle (antithesis) and now his mathematical creativity is in charge of finding an answer (synthesis) that offers a way out from the conflict. In the mathematical literature, very few papers give us clear insight into this process, we found one by D. Tall relating extensively all the troubles involved in the creation of a deductive theory. This example will be discussed below.

ENONCE D'UNE THESE EPISTEMOLOGIQUE

La pensée mathématique se présente habituellement sous une forme purement déductive. Il suffit de regarder les manuels - y compris les manuels du secondaire - et les articles dans les périodiques de recherche pour constater que, quasi invariablement, la matière défile sous forme d'un enchaînement d'axiomes, définitions, lemmes, théorèmes, corollaires et (parfois) applications; puis, le paragraphe achevé, on recommence ce rituel après avoir adapté les hypothèses initiales.

Il s'ensuit que, d'après les documents écrits disponibles, le développement de la pensée mathématique semble témoigner d'une linéarité surprenante, évitant tout retour sur elle-même et ayant dès ses débuts une vision précise des objectifs à atteindre. Une telle efficacité extrême ne se manifeste pas dans d'autres sciences, ni dans le comportement humain en général.

S'interrogeant sur les causes de ce phénomène extraordinaire, il convient d'envisager que ces écrits ne présentent la mathématique que sous sa forme achevée. La structure déductive de celle-ci est appelée à servir un triple but : (1) de vérification : le mathématicien désire s'assurer en fin de travail si ses assertions sont bien fondées, s'appuient sur les hypothèses (axiomes) initialement énoncées et, sûrtout, s'il y a absence de contradiction; (2) de communication : il est important que les idées de l'auteur puissent se transmettre à des (même sans contact personnel) collègues-mathématiciens sans perte de précision et sans ambiguité; ce but rend impératif l'emploi d'un langage formalisé à l'intérieur d'une structure déductive; (3) de conservation : il faut que les résultats de la pensée mathématique puissent résister à l'usure du temps et restent accessibles aux générations à venir, sans qu'il y ait doute sur les intentions de l'auteur. Il est important d'accentuer que l'on désire préserver les résultats et non pas la méthode ni les avatars et les détours de la recherche; cette dernière est très personnelle et très périssable dans sa façon d'agir, on saurait prétendre qu'il est impossible de la rédupliquer en détail d'un chercheur à l'autre.

Sous sa forme opératoire et créative, la mathématique n'est pas plus linéaire que d'autres sciences et d'autres activités cognitives humaines. Mis en route, le cheminement de la pensée du mathématicien prend l'aspect d'une hélice avec de multiples retours en arrière sous l'influence d'obstacles dont la nature peut être très variée. Devant une difficulté la pensée s'arrête et se rétrécit sur elle-même comme pour raffermir ses bases. Puis une impulsion, une illumination soudaine, fait redémarrer l'enchaînement des idées jusqu'à heurter un nouvel obstacle; puis un nouveau cycle commence.

D'un point de vue plus abstrait, ces constatations se traduisent sous forme du schéma général suivant. Confronté à un problème, le chercheur avance une thèse (conjecture), qui peut être vraie ou fausse, et qui subit, souvent spontanément, l'opposition d'une antithèse de nature diverse, p.ex. elle peut découler tout simplement d'une lacune ou d'un manque de compréhension dans la masse des connaissances acquises antérieurement. L'une des apparences sous laquelle l'antithèse peut se présenter, est bien connue des mathématiciens : c'est le "contre-exemple", dont l'apparition entraîne la menace de contradiction interne et par conséquent arrête définitivement un train d'idées projeté, mais visiblement non réalisable.

La simultanéité des deux, thèse-antithèse, provoque un conflit intérieur qui souvent reste dissimulé mais dont l'ampleur peut être telle qu'il entraîne un blocage complet, parfois temporaire, parfois permanent, du processus cognitif. C'est à ce moment que doit se manifester la faculté de créativité mathématique, en forgeant une synthèse capable de résoudre momentanément le conflit et de faire avancer d'un cran le mécanisme de la construction théorique.

Nous retrouvons ici l'idée de l'existence de "cognitive conflict factors" dans l'apprentissage des mathématiques, enoncée et décrite dans [6]. En effet, on peut avancer la thèse que les difficultés que les chercheurs ont dû vaincre, réapparaissent dans l'esprit de l'étudiant qui veut assimiler ultérieurement la même théorie. Nous renvoyons à la référence citée pour des exemples concrets de telles situations.

Le processus dialectique décrit ci-dessus passe le plus souvent inaperçu et reste dissimulé dans le subconscient du chercheur qui n'y prête pas attention et l'écarte de son champs de vision comme étant non pertinent pour le développement de la théorie en voie d'élaboration. Il est pourtant extrêmement important de se rendre compte de l'existence et du fonctionnement de ce processus pour comprendre la pensée du mathématicien. La littérature sur le sujet semble être très clairsemée, il n'y a que, à notre connaissance, l'école française de pensée mathématique qui y a consacré l'attention nécessaire dans les travaux de G. Bachelard, J. Cavaillès, A. Lautman e.a. (C'est pour rendre hommage à cette école que cet article à été rédigé en français). Citons quelques extraits de la thèse Essai sur la connaissance approchée (1928) de G. Bachelard : "La déduction est tout au plus une méthode d'exposition" (p. 178); "La construction progressive obéit à une véritable dialectique ... car la dialectique incline sans opprimer" (p. 181); "En mathématiques, l'enrichissement le plus décisif s'accomplit en absorbant l'antithèse dans l'hypothèse" (p. 242).

ANALYSE D'UN EXEMPLE

Rares sont les contributions dans la littérature mathématque qui nous fournissent un reportage sur le processus de découverte en mettant en lumière la va-et-vient de la pensée, les tentatives réussies et infructueuses, l'approche pénible vers la "linéarité".

Il y a le livre "Preuves et réfutations" de I. Lakatos, ouvrage splendide et original, dans lequel le lecteur est invité à suivre sur le vif les tâtonnements de la raison, cherchant à établir aussi bien l'énoncé précis que la démonstration correcte d'un théorème géométrique. Mais il faut reconnaître qu'il s'agit là d'un exemple imaginé, aussi brillant que soit le récit des hésitations de la pensée mathématique. Quant à trouver un cas réel, nous avons été heureux de rencontrer un article de David Tall qui relate amplement les sinuosités inhérentes à un morceau de recherche. L'auteur désire élaborer une théorie de nombres superréels qui contient des infiniments petits et qui est pourtant plus simple que l'analyse non-standard de A. Robinson. La technique purement mathématique utilisée ne joue aucun rôle ici. Le récit marque clairement les confrontations répétées de thèses avec d'antithèses qui amortissent, voire entravent, la réalisation graduelle de l'objectif que l'auteur s'impose. Une analyse détaillée nous a révélé l'apparition d'au moins une douzaine de cas de l'espèce, qui nécessitent autant d'interventions de la part de l'auteur afin de créer une synthèse provisoire et d'avancer d'un pas vers la réponse finale.

Citons quelques exemples d'incidents qui se produisent en cours de route et qui permettent de suivre en détail la pensée créatrice de l'auteur :

(a) Au début il y a un problème de classification, une indécision quant à ranger le problème dans la théorie des nombres transfinis ou dans la théorie de la mesure; le conflit est levé en décidant de créer une théorie ad hoc de nombres infinitésimaux;

(b) Un autre conflit surgit lorsqu'il faut postuler la nature des futurs infinitésimaux : au premier abord l'auteur est tenté de les assimiler à des fonctions rationnelles - ce ne seraient donc pas des nombres ou des points (concept traditionnel); un compromis est formulé en admettant que les infinitésimaux seront tout simplement des éléments d'un ensemble, dont la nature reste à déterminer;

(c) Plusieurs fois, une intuition qui entraîne l'introduction d'un concept nouveau fait redémarrer l'enchaînement des idées et l'élaboration de la théorie avance d'un pas décisif; ce phénomène se présente p.ex. lorsque l'auteur prend conscience du fait que la fonction $f(t) = t^{-1}$ peut jouer le rôle d'infinitésimal canonique; aussi lorsqu'il introduit une droite à l'infini;

(d) Des obstacles de nature diverse peuvent surgir et entraver une série de déductions; p.ex. en se réalisant que la théorie projetée ne sera pas applicable à des fonctions tout à fait arbitraires, d'où la résolution de ne considérer que des fonctions analytiques; autre exemple : la réticence d'un interlocuteur à accepter un point de vue qui s'éloigne trop des idées traditionnellement admises (obstacle subjectif, situé tout à fait en dehors de la structure mathématique formelle);

(e) La similitude apparente avec une théorie bien fondée (développement en série de Taylor) raffermit la confiance en la méthode suivie; cet espoir est modéré parfois par un vague sentiment de désagrément ou une menace latente d'incohérence (problèmes de convergence);

(f) L'intention d'inclure dans la nouvelle structure un concept préconçu (ex. les nombres super-entiers) peut avoir ses fondements dans l'inconscient et/ou le subjectif personnel ("whisful thinking"); lorsque la réalisation du désir s'avère impossible, l'abandon de celui-ci se heurte à une résistance psychologique : uniquement sous le poids des contre-indications formels (et donc irréfutables) l'auteur se résigne à abandonner le concept tant désiré : il n'y a pas de super-entiers dans sa théorie;

(g) A plusieurs reprises l'auteur ressent le besoin de rédiger dès le début la partie apparemment consolidée de sa théorie. Ces rédactions fréquentes sont l'extériorisation du besoin de raffermir ses bases et du souhait d'écarter de vagues doutes sur l'utilité ou la cohérence de l'édifice, en passant en revue minutieusement toutes les prémisses et conclusions dont l'enchaînement constitue la force logique et déductive de sa construction. Il est bien connu qu'une façon de contrôler la validité d'un raisonnement mathématique consiste à le répéter en l'écrivant ligne par ligne, surveillant à chaque pas l'application justifiée des règles de déduction. En faisant ceci, l'auteur désire augmenter sa confiance en la valeur et l'exactitude de ses démonstrations (qui, comme on le sait fort bien, ne seront considérées exactes que lorsqu'ils sont acceptées par la communauté des mathématiciens).

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INTERPRETATION D'ENONCES IMPLICATIFS ET TRAITEMENTS LOGIQUES

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Une experience a été menée avec des lycéens (16-18 ans) sur la recherche des obstacles qui empêchent la distinction entre condition nécessaire et condition suffisante. On met en évidence l'existence de quatre traitements distincts de l'implication. L'analyse qu'on présente de ces traitements peut permettre de comprendre pourquoi un cours traditionnel de logique n'est souvent pas suivi de succès, ouvrant ainsi des perspectives pour envisager un enseignement adéquat de la logique dans les lycées.

Les travaux d'O'Brien, Shapiro, Reali (O1), (O2), (O3), (S1), ainsi que d'autres chercheurs, ont montré dès le début des années 70 que lorsqu'on confronte des adolescents, non plus comme chez Piaget, à des experiences physiques où l'on peut manipuler les éléments qui interviennent dans un problème donné afin de dégager des relations de cause à effet, mais à des énoncés implicatifs, ces adolescents -qui étaient censés avoir atteint le stade des opérations formelles- mettent en oeuvre des procédures qui sont loin d'être en accord avec le raisonnement logique. Ces procédures non logiques ont été attribuées à des mauvaises performances des sujets interrogés à raisonner logiquement. Toutefois, des linguistes dont O. Ducrot (D1) ont mis en évidence des différences entre l'implication (materielle) de la logique et les énoncés "Si A alors B" ou "Si A,B" du langage courant,(1) jettant ainsi une ombre de doute sur les conclusions d'O'Brien et les autres, car ils avaient utilisé des énoncés implicatifs

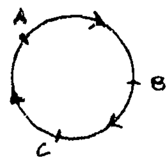
(1) "Si vous avez soif, il y a du whisky dans le réfrigérateur" est équivalent, d'après le calcul des propositions, à "S'il n'y a pas de whisky dans le réfrigérateur, alors vous n'avez pas soif", bien que ces deux expressions soient différentes dans toute langue naturelle.

du genre "If the bike is blue, it is not old", etc. B. Dumont a montré (D2), (D3) d'une facon nette l'inadéquation de ce type d'énoncés pour aborder l'étude du raisonnement logique.

Dans le but de mieux comprendre les obstacles qui empêchent la distinction entre condition nécessaire et suffisante, nous avons mis en place en 1983-85 à l'IREM de Strasbourg, France, une experience centrée sur un questionnaire et sur des entretiens cliniques. Les questions ont été choisies en termes d'un contexte algorithmique. Un tel contexte eviterait, pensions nous, et le déroulement de l'experience nous a donné raison, les problèmes linguistiques observés chez nos prédécesseurs. La population étudiée a éte constituée de plus de 300 éleves du Baccalauréat Français (classes de première: 16-18 ans).

Le type d'énoncé implicatif utilisé peut se comprendre sur l'exemple suivant:

Un circuit électique intermittent comporte trois lampes A,B,C. Chacune s'allume et s'éteint dans l'ordre A,B,C,A,B,C,A,etc. Pendant l'allumage chaque lampe émet soit une lumière rouge, soit une lumière bleue. Le fonctionnement du circuit est régi par un ordinateur dont on ne connaît pas le programme.



Un observateur a suivi pendant un long temps le déroulement du circuit et il a dégagé la règle que voici:

Si C est rouge, alors à l'instant suivant A sera bleue.

Le type de question posée était:

Sachant qu'à un certain moment la lampe C n'est pas rouge, à l'instant suivant la lampe A sera:

☐ rouge	☐ bleue	☐ opps (=on ne peut pas savoir)

Nous avons étudié trois genres de questions, qu'on peut résumer ainsi:

$$d_1 \left\{ \begin{array}{l} A \Rightarrow B \\ \underline{\text{non}A \qquad} \\ \qquad B? \end{array} \right. \qquad d_2 \left\{ \begin{array}{l} A \rightarrow B \\ \underline{\qquad\quad B} \\ A? \end{array} \right. \qquad d_3 \left\{ \begin{array}{l} A \rightarrow B \\ \underline{\qquad \text{non}B} \\ A? \end{array} \right.$$

Par exemple d_1 se lit: Si A alors B. Sachant qu'on a non A, que peut-on dire de B? On reconnaîtra en d_3 une contraposée.

TRAITEMENTS D'ENONCES IMPLICATIFS

GROUPES 1 ET 2: Dans ces groupes, qui s'étendent à environ 25% de la population chacun, les individus ne font pas de distinction entre cause et effet. Dans le groupe 1, l'énoncé implicatif "Si A alors B" apparaît comme une simple association entre deux choses: ou A et B se produisent ou aucun d'eux ne se produit. A la question: Si C est rouge alors au moment suivant A sera bleue; sachant que C est non rouge, quelle sera la couleur de A après?, une élève répond: "puisque si C est rouge A est bleue, si C est bleue A est rouge". Cette élève associe, d'apres la règle, C rouge à A bleue et C bleue à A rouge. Il n'y a pas une combinatoire ou analyse des cas possibles ici. Rien n'est hypothétique. On pensait jusqu'à maintenant que ces individus confondaient l'implication et l'équivalence. Il est clair que cela ne peut pas être vrai, vu que l'individu ne se place pas ici dans un contexte logique, et que l'équivalence (logique), qui s'exprime comme la conjonction de deux implications réciproques, ne peut apparaître qu'après avoir vraiment acquis la notion d'implication.

Pour le groupe 2, ce qui est important est ce qui figure ou ne figure pas sur l'énoncé implicatif. Dans la question (que nous schématisons): Si P est non jaune alors H sera verte. Sachant que la lampe P était jaune, quelle sera la couleur de H?, un élève dit: "on ne peut pas savoir (...) oui, parce que P est verte. Tu sais pas quelle est l'équation si P est jaune. Donc tu peux pas savoir." L'élève parle d'"équation" pour dire règle. Il justifie "tu peux pas savoir" par le fait qu'il n'y a pas de règle mentionnant le cas ou P est jaune.

En désignant la regle par $A \Rightarrow B$, les réponses typiques de ces groupes sont:

	d_1	d_2	d_3
Groupe 1	non B	A	non A
Groupe 2	opps	A	opps

GROUPES 3 ET 4: Dans ces groupes, qui s'étendent à environ 20% et 3% de la population, respectivement, il apparaît pour la première fois une analyse logique du problème. Le conditionnel "si" renvoie à une situation hypothétique (cf. l'exemple ci-apres). L'individu s'interroge sur le rapport entre les objets, débouchant ainsi sur une combinatoire. A la question: Si C est rouge, alors au moment suivant A sera bleue; sachant qu'a un moment A est non bleue, quelle était la couleur de C au moment précédent?, une élève dit: "moi j'ai fait, je suis partie de ça,[1] j'ai dit

(1) "ça" désigne "C est rouge".

si C est rouge, j'ai regardé si ça marchait". Plus loin, une autre élève arrive à la conclusion: "C forcément n'est pas rouge (...) parce que si C est rouge, c'est sûr t'as A bleue", et on sait qu'une des données du problème est justement que A était non bleue. Cette démonstration par l'absurde est loin d'être évidente: elle demande que l'élève se place dans une situation hypothétique et, comme nous l'avons vu pour les groupes 1 et 2, cela ne va pas de soi. Bien que pour les élèves du groupe 3 la résolution d'un probleme implicatif se fasse dans un contexte hypothétique, ces élèves n'arrivent pas à répondre correctement les questions de type d_2, mettant en évidence que l'apparition d'une combinatoire ne saurait rendre compte de l'intensité de la démarche logique deployée.

Les élèves du groupe 4 sont ceux qui répondent correctement toutes les questions.

En désignant la règle implicative par $A \Rightarrow B$, les réponses typiques de ces groupes sont les suivantes:

	d_1	d_2	d_3
Groupe 3	opps	A	non A
Groupe 4	opps	opps	non A

Le problème qui se présente aujourd'hui, et qui conditionnerait dans une large mesure la forme qui devrait prendre un enseignement adéquat de la logique, est celui de savoir si un individu traverse successivement les groupes que nous venons de décrire ou si, par contre, il peut rester "plafonné" dans un de ces groupes. Ces deux situations semblent également plausibles. En effet, nous avons pu observer des individus du groupe 1 qui n'acceptaient pas les raisonnements des individus du groupe 3. Mais nous avons vu des individus du groupe 2 accepter des raisonnements plus complexes. Au marge de ces deux possibilités, notre travail -croyons nous- met en évidence que la difficulté à raisonner logiquement, et en particulier à distinguer entre cause et effet, est très liée a la possibilité de l'individu à se mettre dans une situation hypothétique, et ceci ne va pas de soi. Un but de l'enseignement de la logique serait donc de proposer aux élèves des activités qui leur conduisent a des situations hypothétiques où s'exercer. Et cela ne peut pas se faire dans une logique depourvue de contenu, c'est-a-dire dans une logique formelle. C'est d'ailleurs cet aspect qui nous semble être à la base de l'échec des cours traditionnels de logique (S2), que ce soit sous forme de tables de vérité ou de théorie axiomatique. Une solution peut se trouver dans ce que nous avons appelé logique semiformelle (R1). Celle-ci se distingue de la logique formelle en ce que les propositions ont un contenu et s'éloigne de la logique du langage ou du discours en ce que, comme la logique formelle, l'univers du discours est fixé d'avance. C'est ici que nous plaçons les raisonnements mathématiques (la phrase "soit f une fonction continue" n'est pas traitée, comme en logique formelle, à la maniere d'une proposition p quelconque). Certains jeux, comme les échecs, se déroulent ici aussi. C'est peut-être là qu'il faudrait envisager l'enseignement de la logique dans les lycées. Probablement les études que nous menons actuellement au Guatemala, pourront apporter quelques réponses aux problemes encore ouverts de la logique et son enseignement.

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RESEARCH AGENDA
PROJECT PAPERS

The Research Agenda Project

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The Research Agenda Project focuses on research areas where previous work indicates that some conceptual and methodological consensus seems likely, providing that a vehicle is furnished for this purpose. Accordingly, four conferences have been held, in the areas of problem solving, effective mathematics teaching, algebra, and middle school mathematics. Conference proceedings will be published by the National Council of Teachers of Mathematics.

The Research Advisory Committee of the National Council of Teachers of Mathematics in 1985 proposed to the National Science Foundation that funding be provided for the purpose of establishing a research agenda in mathematics education. We believed that such a project was needed at this time for two reasons: first, to direct research efforts toward important questions, and second, to indicate potential support mechanisms essential to collaborative chains of inquiry.

There are many important unanswered questions about learning and teaching mathematics for which real solutions will be found only through scholarly inquiry. We agree with the critics that, in some areas, past research on the learning and teaching of mathematics has been inadequate. Many past research studies can be characterized as piecemeal inquiries rather than sets of studies reflecting conceptual coherence. Many studies have been based on inadequate conceptualization of the problem being investigated and have employed less than adequate instrumentation and methodology. Such weaknesses are typical of emerging fields of inquiry. Kuhn (1970) has argued that in the early stages of the development of any science; different scholars confronting different portions of the same phenomenon arrive at different descriptions and intepretations. Real research progress occurs only when a group of investigators agree on a specific area of specialization, arrive at a consensus on a common framework or paradigm to guide their investigations, and accept the methodologies associated with that paradigm as a means of communicating questions and results among the group's members.

We clearly know more today about teaching and learning mathematics than we did twenty years ago, before there was substantial financial support for educational research. In particular, we can point to several significant sets of studies now being carried out which are based on emerging theoretical frameworks. For example, young children's early number learning and older children's understanding of rational numers have been the subject of several recent or current studies. Most researchers would agree that single, isolated studies are rarely of much value and profitable research proceeds by a series of small steps taken within the same framework. A conference on the acquisition of number concepts and skills (Carpenter, Moser, and Romberg, 1982) has served as an example of the role such a meeting can play in providing a vehicle for increased communication, synthesis, summary, and cross-disciplinary fertilization among researchers working within a specialized area of mathematical learning.

Other such specializations have emerged which could also benefit from such collaborative efforts. We believed that the most effective way of setting a research agenda would be to focus on areas where conceptual and methodological consensus was likely. Four such specialized areas were selected for this project: the teaching and learning of algebra; the teaching and assessment of problem solving; the teaching and learning of middle school mathematics concepts; and effective mathematics teaching.

The plan for the project included a working group conference in each of the four areas, with monographs of conference proceedings to be published by NCTM, A fifth overview monograph, written by advisory board members, is also planned. The NCTM Research Advisory Committee asked Judith Sowder to prepare the formal proposal and direct the project, and suggested names for the advisory board and for conference directors and monograph editors. The advisory board for the project consists of eight people: Joseph Crosswhite, James Greeno, Jeremy Kilpatrick, Douglas McLeod, Thomas Romberg, George Springer, James Stigler, and Jane Swafford. For each of the four selected areas, two researchers serve as conference co-directors and an coeditors of the monograph of conference proceedings. These pairs are Sigrid Wagner and Carolyn Kieran for learning and teaching algebra; Edward Silver and Randall Charles for teaching and assessing problem solving; Merlyn Behr and James Hiebert for middle school number learning; and Douglas Grouws and Thomas Cooney for effective mathematics teaching.

The project began with a planning conference of advisory board members and conference directors. Besides identifying issues to be addressed and possible paper topics for each conference, tentative lists of invitees for each conference were drawn up to include representation from mathematics education researchers, both established and new to the field, both U.S. and foreign, researchers from relevant fields of psychology and from mathematics, and practitioners. The concept of working group conferences funded for 25 people precluded expanding the conference to all interested persons. We therefore decided to invite people to attend only one conference, thus maximizing the number of people involved in setting a research agenda.

The four conferences were scheduled for the spring of 1987, to allow adequate time in the fall for invitees to prepare conference papers. The conference on teaching and evaluating problem solving was held in January in San Diego, California; the conference on effective mathematics teaching was held in early March in Columbia, Missouri; the conference on the teaching and learning of algebra was held in late March in Athens, Georgia; and the conference on middle school number learning was held in May in DeKalb, Illinois. Each of the conferences was indeed a working conference. Difficult questions were addressed, discussions were lively and intense, and new understandings were reached among individual researchers across the discipline and viewpoints represented.

A second meeting of the advisory board is scheduled for June of 1987, for the purpose of evaluating the conferences and planning an overview monograph. At the PME session associated with this paper, I will discuss the contents planned for the overview monograph. All five monographs are scheduled for publication in early 1988.

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LEARNING IN MIDDLE SCHOOL NUMBER CONCEPTS

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The working group for middle school number concepts, part of the Research Agenda Project of the National Council of Teachers of Mathematics and the National Science Foundation (Grant No. MDR 8550614), met in DeKalb, Illinois, May 12-15, 1987. Because of the deadlines for printing these Proceedings, the following summary was prepared before the conference. The comments are based on first drafts of the papers written for the conference. We would like to acknowledge the authors of the papers for generating the basis for the observations recorded here, but we do not claim that this brief summary does justice to their ideas. The authors are: Merlyn Behr, Robbie Case, Kathleen Hart, James Hiebert, Thomas Kieren, Richard Lesh, Perla Nesher, Stellan Ohlsson, Thomas Post, Robert Sandieson, Judah Schwartz, Judith Sowder, Leslie Steffe, Gerard Vergnaud, and Diana Wearne. Discussants of the papers at the conference were: Ferdinando Arzarello, Alan Bell, Brian Greer, Magdalene Lampert, Glenda Lappan, Richard Lesh, Jack Lochhead, Douglas McLeod, Joseph Payne, Robert Reys, and Ipke Wachsmuth. Robbie Case and Karen Fuson served as summarizers during the final conference session.

The past decade has witnessed a significant increase in research on children's knowledge for and learning of middle school mathematics (about ages 8-13). Much of this research has focused on rational number learning, including the development of proportional reasoning skills. More recently, attention also has been directed toward multiplication and division, with emphasis on whole numbers. The aim of the conference was to reflect on the variety of individual research programs that are flourishing in this domain and to search for common themes and theories that could serve as avenues of communication. These points of contact would not only inform the participants' own work, but also would provide a means for relating with other significant work in the field, and would provide an appropriate foundation for setting a research agenda in this domain.

Middle School Number Concepts - New Complexities

Perhaps the most striking theme of the papers presented at the conference is that the mathematics students encounter in middle school is dramatically different than the early number and addition/subtraction concepts and skills of the earlier years of school. The complexities increase so markedly that the mathematics appears to be qualitatively different than anything students have encountered before.

There are two primary sources for the new difficulties of middle school mathematics. One is that the situations that generate mathematics are mostly multiplicative rather than additive. For example, how many outfits can be created from four different skirts and three different blouses? Or, how many pizzas are needed for 20 people, if three pizzas are just right for seven people? To handle even the simplest multiplication situation at an appropriate level, students must develop the notion that a unit can be composed of more than one object. This achievement requires a significant shift in their thinking, a shift that occurs over a prolonged period. The redefinition of unit is essential in multiplicative situations which are not reducible to additive ones. New concepts emerge, such as intensive quantity (e.g., represented by "per" in 25 miles per gallon), which have no appropriate analogue in addition. Using additive strategies often is inadequate and, further, may inhibit appropriate multiplicative strategies from developing. So the new complexities of multiplicative situations demand a significant extension and reorientation in children's cognitive structures for mathematics.

A second source of difficulty for students is that the number system introduced to handle multiplicative structures brings with it a host of new complexities. The system of rational numbers, although powerful and elegant, is far more complex than the whole numbers encountered to this point. Rational numbers can be represented with two very different-looking symbol systems--common fractions and decimal fractions. Each of these systems introduces complexities that move well beyond the multiplicative situations that motivated their appearance. Procedures for handling the symbols, such as arithmetic operations for fractions and decimals, generating equivalent fractions, and shifting between common and decimal fraction form, become objects of study in their own right. Many of the procedures have subtle connections with well-practiced procedures on whole numbers, connections that are important but not straightforward. Sorting out the semantic and syntactic similarities and differences between symbol systems presents a significant challenge for students.

Current Status-Descriptive Analyses of Subject Matter

Because most of the work on students' learning of middle school number concepts and skills is relatively new, research efforts generally are directed toward describing the phenomena in some detail. Careful analyses of subject matter, and descriptions of tasks designed to tap students' knowledge of the subject matter, and of their responses on such tasks characterize much of the current work in the field.

It is apparent that the analysis of subject matter is especially important in a domain with the mathematical complexities noted above. It is difficult to imagine making progress in research on children's learning of these substantive mathematical topics without a thorough foundational understanding (by the researchers) of the mathematics to be learned. Although there now exists a wealth of information on the children's knowledge and learning of beginning addition and subtraction concepts and skills, very little of the information has helped to anticipate the complexities of middle school number learning. Much of the information on children's early number skills seems almost irrelevant. Of course, there may be important relationships between the domains that have yet to be uncovered. There does exist evidence to suggest that some primative models which children have for whole number concepts interfer with the acquisition of concepts for multiplicative situations. It is clear, however, that current theories of learning middle school level mathematics are domain specific, and that the analysis of this subject matter plays a crucial role in the development of the theories.

Consequences for Instruction

Instructional programs in middle school mathematics ordinarily present a limited and overly simplified, view of the topics discussed above. Multiplication is treated as repeated addition; fractions are treated as parts-of-a-whole. The more fundamental conceptual bases for these quantitative notions rarely are treated in a systematic way. Further, the symbol systems of common and decimal fractions are handled by sets of syntactic rules. The semantics of the systems receive relatively little attention.

Although results from descriptive research cannot prescribe specific instructional programs, there are some important implications emerging from the rich descriptive work in this domain and form several recent instructional projects. Two points seem especially significant. First, there may be a significant cost to students' understanding and subsequent learning, of emphasizing, at an early point in the curriculum, (1) simplified, limited versions of concepts and (2) syntactic rules for manipulating symbols. It appears that in addition to providing only a limited view of topics such as multiplication of whole numbers and operations with fractions, an emphasis on overly simplistic concepts and strategies interferes with later efforts to acquire more complete notions of the target concepts. Students often hold on to primitive strategies, even when they are inadequate or inappropriate. Similarly, early routinization of symbol manipulation rules may inhibit the development of more conceptual, flexible solution strategies.

A second implication emerging from current research is that to improve instructional programs in middle school mathematics it will be necessary to identify the central conceptual content that underlies the various topics, and the cognitive processes that are essential for making the appropriate connections. Identifying the central concepts and processes is an important first step for several reasons. First, learning may be enhanced through instructional programs that emphasize the central concepts initially. These provide a context within which the more limiting aspects of the concepts can be treated in a meaningful way. Second, several key cognitive processes seem to be responsible for connecting the conceptual content with its symbolic representation. Because most of middle school mathematics is conducted and communicated with written symbols, instructional programs must encourage students to engage the cognitive processes that promote appropriate meanings for symbols. Third, central concepts and processes may hold the key to the problem of transfer. There is an overwhelming amount of knowledge presented in middle school mathematics programs. It is impossible for students to develop competence by acquiring each piece of information separately. They must be able to generalize acquired concepts to new situations. The central, foundational concepts show the most promise for generalizing appropriately, and the key cognitive processes support such transfer by establishing connections for the learner between related concepts and between concepts and their representations.

The current state of research in middle school mathematics suggests that the identification of central concepts and processes is not yet complete. There is a convergence of views, but not yet a consensus. In a sense, the conference could be viewed as the first conscious, collective step in this field of research toward such a consensus.

RESEARCH AGENDA CONFERENCE ON EFFECTIVE MATHEMATICS TEACHING

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Research on mathematics teaching was broadly interpreted for the conference to include classroom studies, policy studies, studies of teachers, conceptual work, and philosophical studies. Discussion at the conference centered on a wide range of topics with particular attention given to the issues addressed by the ten invited papers. To give a feel for the conference we briefly summarize each of the presented papers and then close with a short description of some of the most important themes that seemed to emerge from the discussions.

Teaching for Higher-Order Thinking in Mathematics (Penelope Peterson, University of Wisconsin)

The focus of the paper was on determining factors related to higher order thinking by elementary students and the means by which teachers can promote such thinking. The author characterizes the teaching of elementary school mathematics in the following way. It consists of essentially two activities: whole group instruction and seat work. In either case, the interaction among students or between teachers and students is minimal. Teachers seem to hold the view that problem solving can not begin until reasonable competence with computational skills is acquired. Further, most of the instructional time involves the teaching and learning of lower order skills. In the main, elementary teachers do not facilitate students' higher order thinking in mathematics.

The issue then becomes one of determining what factors contribute to higher order thinking and how teachers can be trained to encourage and promote those factors. The author describes three examples of ways to promote students' higher order thinking: (a) using small-group cooperative learning techniques to foster students' autonomy and independence as well as students' learning of cognitive strategies and metacognitive strategies; (b) explicit teaching of cognitive strategies and metacognitive strategies to elementary students; and (c) teaching first-grade teachers about results of recent cognitive science research in mathematics to enable them to devise appropriate curricula and instructional strategies.

Central to much of the research is the notion of "cognitively guided instruction". Such instruction is based on a teachers' sound knowledge of mathematics and relevant research findings, the teachers' ability to assess whether and how students solve problems, and the ability of the teacher to use this information to develop effective teaching strategies. A fundamental question associated with this research is the extent to which a cause-effect relationship exists between the teaching process and students' higher order achievement.

Interaction, Construction, and Knowledge: Alternative perspectives for mathematics education (Heinrich Bauersfeld, Universitat Bielefeld)

Bauersfeld calls into question the universality of research findings and argues for the necessity of looking at research in mathematics education from perspectives outside the field of mathematics education. He reflected upon his own experience with an extensive project in which teachers were trying new curricula and analyses involved analytical statistical designs. The researchers concluded that "softer" methods were needed to understand the complexities associated with teachers trying new curricula, the nature of the teachers' realities, and, in particular, the nature of students' errors. In general, the author saw the researchers shifting their attention away from subject matter structures and related student achievement and toward a perspective of social interaction and construction. The issue became one of understanding a teacher's reality and underlying structures of the teaching/learning process rather than searching for variables to determine the effectiveness of a training program. Thus classroom interaction between teachers and students is better conceived as a matter of social interaction investigated through the lens of symbolic interactionalism.

Bauersfeld argues for the importance of understanding meanings teachers hold and questions the applications of (and even the existence of) general knowledge that applies universally to classroom situations. Knowledge is not content free, it is always determined and defined by the social context in which the researcher works. Reality of teachers, students,/and/researchers is a product of constructions by each of the parties involved. Thus research is not a matter of "discovering" some objective reality which can be revealed by carefully defined experimental analyses; reality is a matter of individual construction.

Bauersfeld's position is based on the notion of fundamental relativism. From this perspective, "the usual research game of turning disturbing interventions into main objects of follow-up investigations and in effect of extending theory this way is doomed to circularity and failure."

Expertise in Instructional Lessons: An example from fractions (Gaea Leinhardt, University of Pittsburgh)

This paper deals with expertise in the teaching of elementary school mathematics, with an emphasis on fractions. It draws on three frames of reference: classroom processes and effectiveness research, mathematical education, and cognitive psychology. The context for the research is the public school classroom and focuses on the teaching of "expert" and "novice" teachers. Lesson segments, routines, scripts, and agendas were of particular interest in comparing experts and novices. Effective teaching is seen as "carefully crafted patterns consisting of short segments, each of which accomplishes a different goal and encompasses a different set of student and teacher actions."

The work reflects a micro-analysis of lessons in which segments and routines are analyzed in detail. Expert teachers tend to have an efficient set of basic routines and seem to have "scripts" which enable them to teach a lesson in which adjustment can be made for the individual students but, in the main, enables them to move through the lesson in an efficient way. For the novice who does not possess such scripts, there are the problems of wasting time and failing to anticipate the crucial features of a given lesson. Thus, a task for mathematics education researchers is to build a rich taxonomy of lesson scripts that are known to be successful. Too, expert teachers have agendas that are accumulations of considerable experience and knowledge that help define goals. Overall experts have routines, patterns of lesson segments, and scripts that all work more smoothly than those of novices. Experts also have richer agendas at their fingertips. Generally, novices have an absence of useable plans.

With respect to analyses of experts and novices in the teaching of fractions, experts were able to coordinate numerical and concrete demonstrations in the successful completion of a lesson while novices failed to complete even rudimentary aspects of the lesson. A central issue is how we can assist teachers to make their patterns more complex, elaborate, and useful, thereby enriching students' mathematical experiences.

<u>Implications of Research on Pedagogical Expertise and Experience for Mathematics Teaching</u> (David Berliner, University of Arizona)

This paper also dealt with differences between expert and novice teachers. Berliner gives four reasons for studying expert teachers. First, is the question of whether expert teachers resemble experts in other fields. Second, is the purpose of getting access to the cognitions that accompany behaviors that research has associated with effective teaching. Third, is to influence state and local school district policies regarding master teachers. Fourth, is to influence current policies in certain states.

Research suggests that expert teachers have more sophisticated means of analyzing instructional problems. Too, experts are more deliberate in analyzing classroom scenarios and seem to have a richer problem representation than do novice mathematics teachers.

In a study involving experts, novices, and postulants (scientists and engineers from business who were interested in teaching careers but who had no formal training or experience as teachers) in science and mathematics at the secondary level, it was found that both experts and postulants could provide a more detailed label for a given problem than could novices. Further, the experts could provide a detailed task analysis in a way that the postulants could not. Experts were also more sophisticated than postulants and novices in terms of their knowledge about the way students thought about mathematics and science and were better able to identify incorrect algorithms.

When shown slides depicting classroom situations, the postulants and novices provided literal descriptions that were accurate but the experts responded with inferences about what they saw. The experts were able to identify information that was deemed important; the novices and postulants were much less discerning. Too, experts tended to focus on the more dynamic aspects of teaching, e.g., "students working independently" while novices and postulants tended to focus on the more static aspects of teaching and generally did not describe students' activities any more than the physical aspects of the room. Experts could describe what is and is not going on; novices' descriptions tended to be step by step accounts with fewer inferences about what was happening. In general, the experts were better able to agree on what is important in classrooms, provide richer analyses of students' conceptual problems, and had an image of what is to be expected.

Content Determinants In Elementary School Mathematics (Andrew Porter, Robert Floden, Donald Freeman, William Schmidt, John Schwille, Michigan State University)

The authors distinguish content from method in an effort to focus on factors that affect what is taught. In light of competing notions of what mathematics should be taught and the limited amount of time given to teaching mathematics, it is essential that we understand factors that affect the selection of content. In some sense content decisions for elementary teachers are secondary to the selection of activities or concerns about such things as citizenship. Generally, teachers will teach that content with which they feel comfortable.

The authors used five different studies to reach various conclusions. They bring into question whether or not, and the extent to which, teachers follow the textbook. They conclude that first year teachers are more likely to follow the text but that more experienced teachers use the text more as a source book from which content is selected. It also appears that teachers are not influenced very much by standardized tests given once a year. There are, however, important influences of testing on teaching when the tests are tied to curriculum, e.g., tests involved in a management system.

Students also influence what is taught. The authors found considerable variance regarding content covered within classes. Slower students spent more time on computational skills than did their brighter counterparts. There also seemed to be some gender differences. For example, girls encountered more topics than boys; boys tend to encounter more conceptual and application oriented content than do girls.

Principals were not significant factors in determining content. Teachers' convictions did have some impact in content decisions. While teachers uniformly felt computational skills constituted the most important content, there was considerable variance among teachers in the time devoted to computational skills. Their convictions, however, seem to be generally unexamined by both themselves and others.

From Fragmentation to Synthesis: An integrated approach to research on the teaching of mathematics (Celia Hoyles, University of London Institute of Education)

The author reviews some major research themes relevant to mathematics teaching. Included are discussions of the ORACLE research programme in the United Kingdom., as well as other studies of teaching. An omission in much of this research is that there is no real consideration to the content being taught. The author argues that research on teaching more generally is not likely to improve the teaching of mathematics.

The Mathematics Teaching Project in England is described which takes account of both pupils' and teachers' perspectives. Pupils were asked to react to ten different factors that might be associated with desirable characteristics of a mathematics teacher. Teachers' perspectives included their expectations about pupil performance, e.g., teachers tend to ignore responses or questions from less able students. Gender also plays a role: boys tend to monopolize the teacher's time through questions asked and initiated. Teachers' conceptions about mathematics are additional factors that influence what happens in the teaching of mathematics.

The author provides these central recommendations: the need to preserve the complexity of the research, the need to work with teachers as research collaborators, and the need to acknowledge that teacher and pupil behavior in the context of a classroom cannot be understood through analysis of mathematical content or its related psychological complexity alone. The recontextualisation of the mathematics in order for it to be socially enacted in schools and the constraints of classroom relationships must also be considered. The use of microcomputers as a special learning environment within the theoretical model proposed is described. The author argues that researchers should resist the temptation to cope with the complexity of mathematics teaching by concentrating on only a single aspect of teaching, e.g., cognitive aspects, but rather to consider other aspects as well, e.g., attitudinal considerations and the milieu of the classroom.

Computer Usage in the Teaching of Mathematics: Issues which need answers (Janet Schofield, David Verban, University of Pittsburgh)

The study which served as a basis for the paper occurred in a large, four-year high school located in an urban setting with approximately 1300 students. The research was primarily enthnographic in nature.

The authors found that, the exception of a field test of the computer-based geometry tutor, computers were rarely, if ever, used in teaching mathematics. In general, the "mathematics teachers appeared to have little conception of what parts of the curriculum might well be taught using computers, and when and how they should be used for drill and practice, for simulations, for their graphic capabilities, or the like."

The authors identified several impediments to the utilization of computer technology in mathematics education. Lack of familiarity with computers was considered to be a major barrier. Potential embarrassment which undermined the teachers' sense of competence was one fallout from a lack of familiarity. More recently trained teachers may solve part of the problem but rapid technological advances and the general aging of the teaching profession, suggest that research is needed on how best to overcome the problem. Another barrier is the work "overload" of knowledge teachers. Since computer knowledge rested with only a select few teachers, those few had responsibilities above and beyond those of their less computer-wise counterparts.

While providing information may solve part of the problem, attitudinal factors need to be addressed as well. That is, remedies will be effective only if teachers want to use computers. It may be the case that computers pose a threat to the teachers' autonomy and to predictability in teaching. There was some concern that computers might somehow replace teachers.

Other barriers that the authors identified were logistical and practical impediments to computer usage. The use of computers requires extra preparation time and sometimes presents logistical problems in terms of moving the class to a laboratory setting.

A problem that needs to be considered is the availability of computers to minorities and females. This will become more critical as the population in the United States becomes proportionally more minority. As is the case with mathematics, computer science is seen as a male dominated field.

The authors suggest several areas worthy of consideration for promoting computer usage. One of these is a greater emphasis on individualization and another is consideration of alternative evaluation schemes--for both students and teachers.

The authors conclude that research on computers and mathematics instruction needs to be grounded in a conception of why an how computers can be effectively used.

Teacher Thought and Teacher Behavior in Mathematics Instruction: The need for observational resources (Tom Good, Bruce Biddle, University of Missouri)

Good and Biddle argue that observational research in classrooms has not been utilized sufficiently to improve mathematics education. They contend that the expansion of observational research can yield better theories for understanding the learning of mathematics and can produce more adequate models for improving mathematics teaching.

The authors discuss various reform efforts in American public education when sweeping recommendations for educational reform were made. Unfortunately, these often involved simplistic ideas about schools or curriculum problems and included little, if any, documentation of the classroom problems that reforms were intended to address. Too often reformers claimed that all teaching was similar and that all practice needed to be reformed in a simple way. Such claims are wrong and are demeaning and unfair to excellent teachers. Many of these reforms seemed to reflect an unwillingness to view teaching as a complex, challenging, multifaceted process that still inadequately understood. Rather, it was assumed that both the problems and solutions for improving American education were obvious. In addition, many of these reforms were eventually judged to have failed, but since little, if any, observational data had been collected to examine their effects in classrooms, it was difficult to say why the reforms failed.

To illustrate the problems associated with past reforms, the authors discuss two reform efforts: discovery learning, and the curricular reforms of the School Mathematics Study Group. They also note that the cycle of reform continues today and curriculum changes are still advocated with little attention to the effects of these changes on classroom practice.

Observational studies of classrooms find consistently that educational problems vary among schools and classroom, thus observational research conducted before curricular reform is undertaken can lead to improved understanding and the testing of alternative solutions to problems. As well, observational research is needed during periods of curricular reform to establish the effects, if any, of those reform efforts. To illustrate the values of observational studies, the authors describe two programs of classroom research: one that focuses on teacher expectations; the other, the studies that produced the Missouri Mathematics Program.

Can Teachers Be Professionals? (Tom Romberg, University of Wisconsin)

Romberg claims that today's mathematics teachers are not professionals, but neither are they given the opportunity to be professional. He suggests that in order for teachers to become professionals--and he firmly believes that they can--there must be a reconceptualization of the teacher's role and a radically different work environment. The author offers a definition of professionals as those who, through education and experience, have a "professed" knowledge that sets them apart from others; professionals also make use of this special knowledge when making judgments and decisions in their occupations.

Romberg suggests that current routinized and textbook-based classroom teaching does not require teachers to make use of their professional knowledge. He claims that a feeling of collegiality is vital if teachers are to become professional; however, there are many impediments to the establishment of this collegial relationship. Among these are the constraints of time, scheduling, and the conception of the teacher's role as being primarily managerial.

Three projects are described which represent the kind of work needed if teachers are to become professional. The first is the Urban Collaborative Projects sponsored by the Ford Foundation. The Ford Foundation projects are intended to provide a framework for enhanced teacher professional activities, primarily through the encouragement of collegial networks. A second project reported is the Mathematics Curriculum and Teaching Project, sponsored by the Australian federal government. The government has endeavored to provide the best possible illustrations of classroom practice and to encourage and support teachers in their efforts to implement materials and reflect on their use. The third project cited is the University of Wisconsin's Cognitively Guided Instruction Project. The aim of this project is to enrich the knowledge base of teachers so that they can make better (professional) judgments and decisions in their classrooms.

The author closes by offering suggestions that would make teaching an occupation for professionals. This requires a new conception of mathematical literacy which sees as its primary goal long-term learning. It must include the creation of epistemic and generative situations in which children explore problems, create structures, and generate questions and reflect on patterns. This requires mathematics teachers to have flexible approach and to value informal and multiple representations. The vision also requires a new school organization. This means that the public must offer teachers a professional work environment. The author offers guidelines for such an environment.

Cross-cultural Studies of Mathematics Teaching and Learning: Recent findings and new directions (Jim Stigler, Marcia Perry, University of Chicago)

The authors describe the cross-cultural studies done by the International Association for the Evaluation of Educational Achievement

(IEA) which compared mathematics achievement in numerous countries and two studies done by the Center for Human Growth and Development at the University of Michigan which compared the teaching of mathematics in the United States, Japan, and Taiwan. The authors suggest that there have been too few cross-cultural studies and that past studies have had too narrow a scope. The IEA studies have been ground-breaking in their analyses of curriculum and achievement, but have not pursued some of the more important cultural factors that surround the teaching of mathematics, nor have they examined student outcomes other than achievement.

The authors feel that more students are needed that focus particularly on how mathematics is taught in classrooms in different cultures. They describe in some detail the Michigan studies' attempt to do this, reporting on the varying amounts of time available for classroom instruction, the organization of that time, the coherence of the lesson and the amount of reflectivity promoted by teachers in different cultures. The authors claim that many aspects of culture are brought to bear in the teaching and learning of mathematics, including beliefs, attitudes, practices, tools, and traditions. They also claim that there can be no doubt that what happens in the classroom is in some sense a reflection of the wider society within which the classroom exists. Nevertheless, if reform of mathematics teaching is a goal, it seems that the classroom is good place to start. Although it is difficult to change what happens in classrooms, it is far more difficult to change broader aspects of the culture.

Included in the paper are preliminary analyses of narrative observations of first-grade mathematics classrooms in Japan, Taiwan, and the United States. The authors suggest that there is a great deal that we can learn about ourselves by carefully observing others in cross-cultural studies.

Research Issues and Themes

Discussion of the preceding papers raised many issues and important research questions. It is impossible to list them all, so we now summarize some of the more important and most frequently mentioned ideas.

1. While current research paradigms have helped establish much of our contemporary knowledge base about effective mathematics teaching, future research should not be restricted to current designs. In fact, there is a great need for more theory building research and studies that approach questions from a variety of philosophical perspectives.

2. The study of expert teachers of mathematics is becoming more prevalent. Future research of this type must carefully add ss the mathematics content taught and the sample of expert teachers studied must be carefully defined and described. Within this research area attention to special need situations such as mathematics learning in minority and English as a second language classrooms might be particularly appropriate.

3. There is a need for observation as an integral part of many research studies. This may help characterize the effects of reform movements, document the nature of instructional treatments, and assist in understanding why treatments work or fail to work.

4. Studies of teacher knowledge and teacher beliefs and especially how they moderate teaching behavior and student learning are needed. It may be particularly useful if such studies concentrate on particular teacher knowledge (e.g., knowledge of how students learn basic addition and subtraction problem solving strategies) or beliefs about specific things (e.g., the nature of mathematics).

5. The professional life of teachers in all its many aspects both

inside and outside the classroom needs further study. The effects of cultural, financial, and societal factors on what is taught, how it is taught, and what mathematics learning takes place is needed.

6. Additional cross cultural studies of student mathematics learning should be conducted and concomitantly there needs to be a better understanding of the factors associated with the differences uncovered in such studies.

7. Finally, just as teacher development and teacher education interact in important ways with mathematics teaching, so to does research in these areas. For theoretical and practical reasons, it is essential that our knowledge of teacher education move forward hand-in-hand with advances in our understanding of mathematics teaching.

In closing, it is important to point out that there is a need for much more high quality research on mathematics teaching. Research along the themes just mentioned is important, at the same time there is always a need for innovative methods and new ways to look at familiar problems.

R.A.P. CONFERENCE ON THE TEACHING AND LEARNING OF ALGEBRA

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This paper summarizes the aims, activities, and results of a four-day working conference on the teaching and learning of algebra. This conference was part of the Research Agenda Project (R.A.P.), a two-year project (1986-1988) conducted by the National Council of Teachers of Mathematics with funding provided by the National Science Foundation. The aim of the project is to develop conceptual frameworks and research agendas in four critical areas of mathematics education research -- namely, middle school number concepts, the teaching and learning of algebra, the teaching and evaluation of problem solving, and effective mathematics teaching. To achieve these goals, four different working groups, each with about 25 participants, met early this year to consider the significant issues in each area of research. The task of these working meetings was to synthesize the current knowledge base in the given area and to identify important directions for future research. Five monographs are to be produced -- the proceedings from each working group conference, plus an overview monograph to be written by members of the project advisory board.

The algebra working group met in March at the University of Georgia in Athens. It brought together mathematicians, psychologists, technologists, mathematics educators, researchers, teachers, and curriculum developers. Some of the more general questions and issues underlying the structure of the planned program were: (1) What is algebra? (2) What does research say about the learning and teaching of algebra? (3) What is algebraic thinking and how does it relate to general mathematical thinking? (4) What is the significance of representation in algebra? (5) What should algebra be, particularly in view of continuing technological advances.

The conference opened with a mathematician's perspective on the nature of algebra and on the factors involved in the learning of algebra at the school level. Several topics were examined for potential inclusion in a modified curriculum using the three criteria of intrinsic value, pedagogical value, and intrinsic excitement. The concept of function, distance formula, percentages, graphs, probability and statistics were all suggested as examples of topics with clear intrinsic value, that is, they are or will be important in the lives of the students. Topics considered to be important because of their pedagogical value, that is, not for their own utility but rather because they form a foundation for some other topic with intrinsic value, include the technique for completing the square. Certain topics were proposed because they are just so interesting and exciting that their inclusion in the curriculum does not require any other justification, for example, exponential growth and decay, the ideas of chaotic dynamics, and tomography (the science of reconstructing images of the interior of an object from shadow images, such as, those obtained from CAT scans). Standard items in the secondary school algebra curriculum were then examined from the perspective of these three criteria. A case was made for using both a function approach to the teaching of algebra and the decimal representation of real numbers.

In order to provide the group with another mathematician's perspective, a reaction to the opening session was given by a second mathematician. This presenter also underlined the need to change our approach to teaching algebra. It was argued that we must make essential concepts more understandable to students and strive to give them access to topics that otherwise would be delayed until late in the curriculum. The availability of calculators and computers permits a numerical approach to algebra quite different from the formal axiomatic approach of recent years. Furthermore, experience with computing functional values in concrete situations can provide students with a foundation for understanding functions which are represented by expressions involving variables. It was also suggested that, in order to make the teaching of mathematics exciting and interesting, teachers should be constantly on the lookout for examples of new developments in mathematics, and for applications that make sense to students.

These two presentations were followed by two reviews of the research literature by mathematics education researchers, one covering the early learning of algebra (an introduction on pre-algebra, followed by the literature on literal terms, algebraic expressions, equations and equation solving, and algebra word problems), and the other dealing with the later learning of algebra (equations in two variables, graphs, and the concept of function). The first research review traced the experience of elementary school children with simple algebra-like equations, and pointed out that children rarely use equations as a tool for solving word problems. When students are introduced to algebraic representations and procedures in secondary school, their orientation toward finding answers makes them unreceptive to the task of expressing mathematical relationships with variables and algebraic expressions.

Other negative effects of their arithmetic knowledge concern the use of the equal sign (a signal to find an answer rather than a symbol for equivalence) and their understanding of the principle of concatenation (4c meaning 4 + c rather than 4 * c). It was also reported that when learning procedures for solving equations, some students show a preference for arithmetic methods such as substitution and that others prefer algebraic methods such as transposing. There is an indication that those who prefer arithmetic solving methods may have a better sense of the left-right balance structure of an equation and may be more able to make sense of the symmetric solving procedure of performing the same operation on both sides of the equation than are those who prefer the transposing solving method.

However, the difficulties encountered by students in the early learning of algebra are not all rooted in their previous arithmetic experience. Research shows that beginning algebra students lack knowledge of the structure underlying algebraic expressions and equations. For example, they will evaluate a given expression one way on one occasion and do it another way on the next. Without a knowledge of structure, beginning algebra students cannot be consistent in their approach to testing conditions before performing some operation, nor with the process of performing the operations.

The next review that dealt with the mathematics education research literature on the later learning of algebra focused on the "cognitive obstacles" involved in learning algebra. Cognitive obstacles were characterized as difficulties that arise when (1) the learner attempts to use a mental structure that is not appropriate for the algebraic material to be learned, but is valid in another domain such as arithmetic or natural language; or (2) there is an inherent historical-epistemological obstacle in the material to be presented for which the learner has no structure to which it can be assimilated. Examples of various cognitive obstacles were presented. Students' misuse of literal terms in the context of two-variable equations was found to be based on their use of symbols and abbreviations in everyday language situations. On the other hand, the graphing of equations in two variables involves grasping the notion of continuity, a concept requiring a major epistemological jump. Finally, students' difficulties with functions appear to be related more to the inadequacies of a formal approach to teaching functions rather than to the nature of students' intuitions of the concept of function.

The conference program continued with two presentations from cognitive psychologists, one of whom spoke on the research literature related to algebra word problem solving. An aim of this report was to describe the perspective that psychologists use to study algebra problem solving. In general, cognitive research done by psychologists has not examined the question of what is to be counted as "algebra". Most of the studies have taken textbook problems as the definition of algebra. It was also pointed out that their algebra research has often proceeded in terms of what can be done, as opposed to what is needed. Thus, the study of the cognitive aspects of problem solving has illuminated the kinds of intellectual problems that students face in solving these problems, but it has not addressed the pedagogical problems of presenting the material so that students are interested in it, or of decomposing the instruction into manageable units. It was suggested that some sensitivity will be required on the part of mathematics educators who are interested in using these research results to guide their practice.

The second presentation by a psychologist focused on a cognitive model of algebraic reasoning and included discussion of general cognitive science techniques for modelling complex human behavior and a proposal for testing the model by using it to design instruction. The cognitive model was a two-tiered one. It involved a representation-building part which acts on the data and builds a richer data structure, and a problem-solving part which acts on the enriched data structure. The errors which beginning algebra students make were hypothesized to be due to their construction of a rather different data structure (a string structure) than that which experienced students build (a hierarchical tree structure). A specific instructional display that makes explicit the hierarchical structure of an equation was proposed as a possibly powerful means of helping algebra students to construct more adequate data strucures.

The potential role and impact of technology was taken up in subsequent presentations. One of these considered recent developments in intelligent computer-assisted instruction (ICAI) and cautiously described the power of ICAI for task and concept analysis, along with the potential of such systems for presenting algebraic content in substantially new ways. It was pointed out that, in the ICAI literature, competence in algebra is depicted as the possession of a set of correct algebraic rules; thus, errors are manifestations of incorrectly learned rules. However, this view of competence fails to look at incompetence as stemming from impoverished conceptual knowledge. Recent non-ICAI work suggests that if algebra instruction were to emphasize the concept of an expression as an entity having an internal structure, and if 'rules' were proposed as structure-modifying transformations which leave some aspect of an expression invariant, then the common errors which have been reported in the ICAI literature may not be so common.

Another presentation of this segment of the program focused on the issue of representation in algebra, in particular, on the use of technology to support simultaneous multiple representations (e.g., tables, graphs, equations) of algebraic subject matter. It was hypothesized that appropriate experience in a multiple, linked representation environment can provide the referential meaning missing from much of school mathematics, and also help the learner to generate the cognitive control structures required to move from one representation to another. Though some inroads have been made in developing software that embodies these ideas, it was pointed out that research has not been able to keep step with these advances. We need to address the issue of the kinds of learning which take place in representation-rich environments.

The last presentation of the program provided a mathematics educator's perspective on the impact that recent and future developments in computer technology should have on our conception of algebra, and suggested dramatic ways of modifying the curriculum, based on an assumption of universal access to the new technology. One suggestion was to rebalance the relation among skill, understanding, and problem-solving objectives in algebra. Computer tool software allows us to modify our skill-dominated conception of school algebra. Another suggestion focused on the power of literal symbols when used as <u>variables</u> in realistic problem situations (as opposed to viewing literal symbols merely as letters that represent numbers or as symbols that stand for any one of the elements of a given set). The kinds of problem situations chosen are those where one or more input variables are used to predict one output variable; in other words, the output is a <u>function</u>

of the input. However, it was pointed out that this computer-based problem-solving approach to algebra makes use of a "guess-and-test" solving strategy which may be quite unsettling to many who believe that algebra is a subject where there is always a systematic method. The presentation concluded with a challenge to generate a rich agenda of new research questions based on the opportunities provided by impressive technological capabilities.

Interspersed among the preceding presentations were both large and small group discussions. The large group discussions, led by a discussant, generally focused on the presentation which had just taken place and provided a forum for participants to state how the presented ideas informed their own research. The small group discussions were an attempt to draw out the commonalities among the various presentations and to pull together and synthesize the ideas which had been presented. Some of the questions dealt with by the small groups were: (1) What is algebra, as reflected in the papers? (2) What learning/teaching theories guide our research? (3) What research has/has not been done from a content perspective? (4) What research has/has not been done from a learning theory perspective?

The above paper presentations and discussion sessions constituted the first part of the program. They were followed by the second part -- the generation of issues to be considered for a research agenda in algebra. This proved to be a rather difficult exercise for several reasons. To begin with, it was clear that we could only predict what it would be useful to know on the basis of what we knew already. Secondly, the task was even more complex in that we were attempting to predict for a situation in which changes can happen so fast. The potential relationship between school algebra and technology made it seem impossible to be able to design a long-term research program that would retain its precise relevance throughout an extended period of time. Thus, it was considered unrealistic to expect to generate a sequential plan for a research agenda in algebra; all that could be hoped for was a grouping together of issues which were considered by individual participants to be important not just for the present but for the future too.

In generating issues for the research agenda, four broad categories emerged -- content, learning, instruction, and representation. In brief, the content dimension included issues such as the interconnections between symbolic manipulation and conceptual understanding in algebra, the appropriateness of certain algebraic ideas for different populations of students at different points in their school careers, and the cognitive and affective outcomes of different approaches to algebra. The learning dimension included issues such as the characterization and development of algebraic thinking, levels of understanding in algebraic thinking with respect to specific concepts and processes, and the identification of difficulties inherent in the learning of algebra. The instruction dimension included issues such as the effectiveness of alternative modes of instruction and novel technological approaches, the need for improved theories of learning in order to have a possible effect on algebra instruction, the study of the interaction between methods of evaluation and instruction, and the identification of expert algebra teachers' knowledge and skill. The representation dimension included issues such as the study of how students learn to use and coordinate multiple representations of a situation, and the extent to which dynamically-linked representations enhance or inhibit metacognitive processes.

The small-group sessions devoted to generating these items for the research agenda came to an end well before conference participants had had time to reflect on them or to react to them as a coherent set. This is currently being done by means of correspondence between the conference organizers and participants. Despite the fact that the agenda was far from being in its final form by the time of the closing sessions, the ideas which had been generated served as a basis for the two closing panels. One of these panel sessions provided four different perspectives on the embryonic research agenda -- that of a mathematics educator, a curriculum developer, a cognitive scientist, and a school board person. Some of the main issues which panelists tended to emphasize were implementation issues. For example, it was pointed out that a primary goal of mathematics education research has always been to have an impact on classroom practice. If research is to have an impact, teachers must be brought into the research process, into both the conceptualizing and designing phases of research. It was also suggested that research issues and results be incorporated into preservice and inservice teacher education courses. But it was pointed out that more than research studies and results are needed. It is necessary to have curriculum and assessment materials that reflect the implications of research. Other issues raised by panelists concerned: (1) the kinds of algebraic skills and understandings required by today's students, and (2) what teachers can and should do with technology.

The other closing panel session of the four-day conference focused on the issue of theoretical and conceptual frameworks. Three panelists from different research traditions discussed the existence (or non-existence) and desirability of theory in algebra research. Some of the ideas emerging from this exchange concerned the two-way relationship between theory and practice and the need to build theories based on the experience of practitioners. It was also suggested that theories need to be constructed in order to attempt to tie together the results of researchers and to be able to predict how algebra learning might take place.

In closing, it needs to be mentioned that this paper has merely skimmed the surface of the richness of the presentations and discussions which took place at the RAP algebra working conference. It is hoped that the monograph of the conference proceedings which is currently in preparation will be able to convey more profoundly the contributions of all the conference participants.

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