DOCUMENT RESUME

ED 382 272 JC 950 256

AUTHOR Browne, Joseph, Ed.
TITLE The AMATYC Review. 1994-1995.
INSTITUTION American Mathematical Association of Two-Year Colleges.
REPORT NO ISSN-0740-8404
PUB DATE 95
NOTE 174p.
AVAILABLE FROM AMATYC Office, State Technical Institute at Memphis, 5983 Macon Cove, Memphis, TN 38134 (two issues free with $50 membership).
PUB TYPE Collected Works - Serials (022)
JOURNAL CIT AMATYC Review; v16 n1-2 1994-1995

EDRS PRICE MF01/PC07 Plus Postage.
DESCRIPTORS *College Mathematics; Community Colleges; Computer Simulation; *Curriculum Development; Functions (Mathematics); Linear Programming; *Mathematical Applications; *Mathematical Concepts; Mathematical Formulas; *Mathematics Instruction; Mathematics Teachers; Technical Mathematics; Two Year Colleges; *Writing Across the Curriculum

ABSTRACT

Designed as an avenue of communication for mathematics educators concerned with the views, ideas, and experiences of two-year college students and teachers, this journal contains articles on mathematics exposition and education, and regular features presenting book and software reviews and math problems. In addition to regular features such as "The Chalkboard"; "Snapshots of Applications in Mathematics"; "Notes from the Mathematical Underground"; and "The Problem Section," volume 16 contains the following major articles: "Implementing Change in the Mathematics Curriculum," by Sheldon P. Gordon; "Sixty-Thousand Dollar Question," by A. Arvai Wieschenberg and Peter Shenkin: "A Conversation about Russell's Paradox," by Paul E. Bland; "The FFT (Fast Fourier Transform): Making Technology Fly," by Barry A. Cipra; "The Effect of Cooperative Learning in Remedial Freshman Level Mathematics," by Carolyn M. Keeler and Mary Voxman; "Writing To Learn Mathematics: Enhancement of Mathematical Understanding," by Aparna B. Ganguli; "The Apotheosis of the Apothem," by Steven Schwartzman; "Relaxation Functions," by Homer B. Tilton; "A Pattern for the Squares of Integers," by Kim Mai; "Using an 'n X m' Contingency Table To Determine Bayesian Probabilities: An Alternative Strategy," by Eiki Satake, William Gilligan, and Philip Amato; "An Appropriate Culminating Mathematics Course," by Bill Haver and Gwen Turbeville; and "Community College Success in an International Mathematics Competition," by John Loase and Rowan Lindley. (BCY)

The AMATYC Review

published by the

AMERICAN MATHEMATICAL ASSOCIATION
OF TWO YEAR COLLEGES

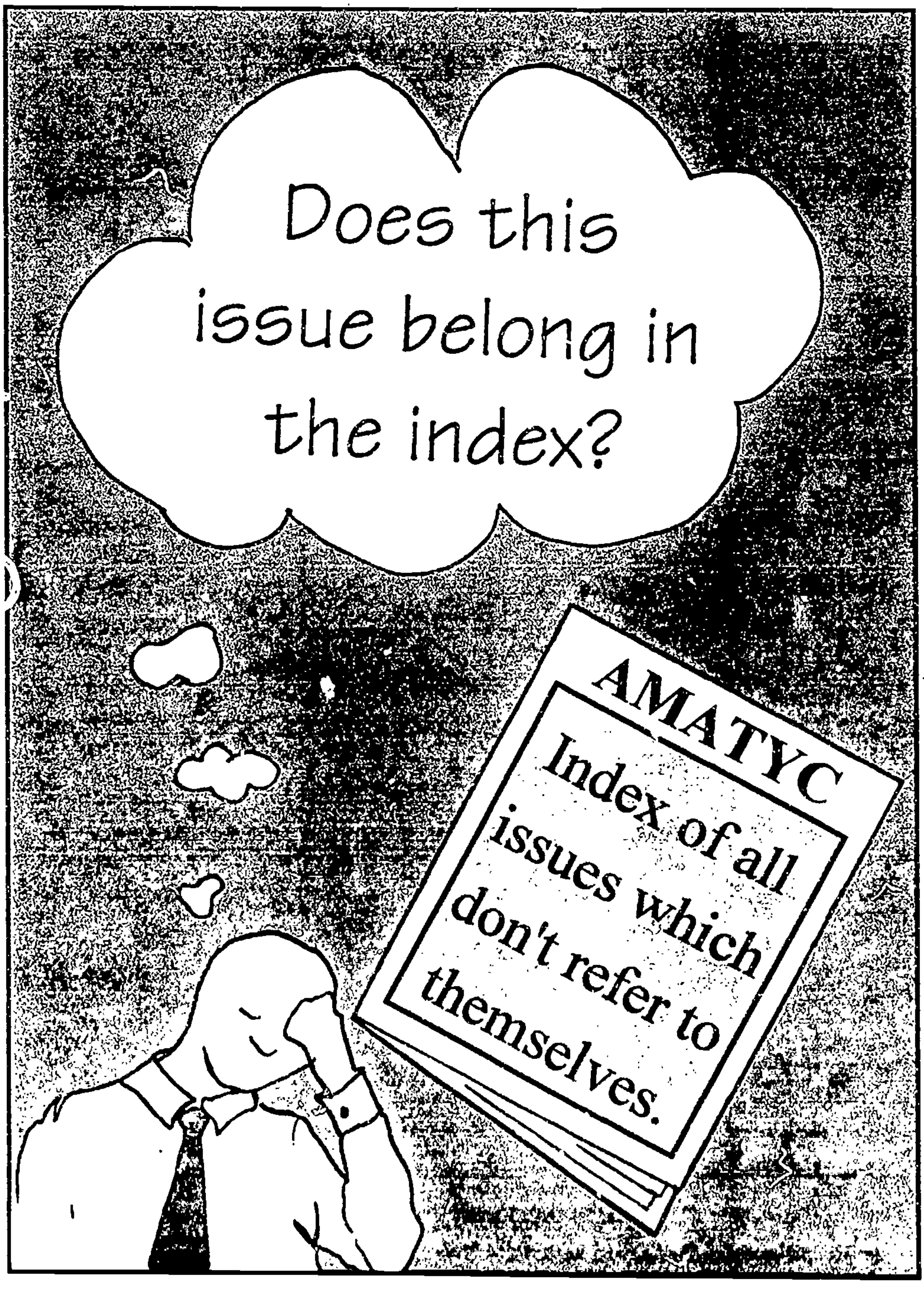
VOLUME 16, NUMBERS 1 AND 2
Fall - Spring 1994 -1995

Joseph Browne, Ed.

THE AMATYC REVIEW

Published by the
AMERICAN MATHEMATICAL ASSOCIATION OF TWO-YEAR COLLEGES

VOLUME 16, NUMBER 1 Fall 1994




TEXAS INSTRUMENTS
Calculators
CLASSROOM
TI 108
TI 12
TI 34
TI 7
CALL FOR CURRENT PRICING
GRAPHING
TI 81
TI 82
TI 85
CASIO
Calculators
GRAPHING
FX6300G
FX7700GE
FX9700GE
CLASSROOM
SL450
AZ45F
FX55
CALL FOR CURRENT PRICING
hp HEWLETT PACKARD
Graphing Calculators
HP 48G
Hundreds of built in equations. 32 KB Memory for complex problem solving. RS-232 lets you link to your computer through a serial port. Two way infrared I/O for easy data transfer. 3-D Graphics, calculus, and symbolic math functions
HP 48GX
Same as 48G with expendable port
CALL FOR CURRENT PRICING
SHARP®
Calculators
FRACTIONS
ELE300
GRAPHING
EL9200
EL9300
CALL FOR CURRENT PRICING
Overhead Calculators and Class Sets for these and other models are available!
DISTRIBUTORS, INC.
3 LOCATIONS
DALLAS
KANSAS CITY
LOS ANGELES
CALL FOR A FREE COPY OF OUR CURRENT CATALOG
1-800-234-3396

The Official Journal of the

American Mathematical Association of Two-Year Colleges

PURPOSE: AMATYC remains committed to its original purposes:

- To provide a national forum for the exchange of ideas
- To further develop and improve the mathematics education of students of two-year colleges
- To coordinate activities of affiliated organizations on a national level
- To promote the professional development and welfare of its members

The AMATYC Review provides an avenue of communication for all mathematics educators concerned with the views, ideas and experiences pertinent to two-year college teachers and students.

SUBMISSION OF MANUSCRIPTS: Manuscripts must be typed, doubled-spaced, on one side of 8-1/2" x 11" paper. They should not have been published before, nor should they be under consideration for publication elsewhere. Photographs should be black and white glossy prints. To provide for anonymous reviews, the author's name and affiliation should appear on a separate title page. The title should also appear on the first page of the exposition. Authors are advised to consult the *Publication Manual of the American Psychological Association.* A guideline for authors is available from the editor and is also printed in the Fall 1993 issue. Five copies of each manuscript should be submitted to Joseph Browne, Onondaga Community College, Syracuse, NY 13215.

ADVERTISING: For information concerning advertising rates and schedules, contact the advertising manager, Larry Lance, at the address given below.

PUBLICATION: *The AMATYC Review* is published twice a year in the Fall and Spring.

ISSN 0740-8404

The AMATYC Review Fall 1994 Volume 16, Number 1

TABLE OF CONTENTS

EDITOR'S COMMENTS ....p. 4

ABOUT THE COVER ....p. 4

LETTERS TO THE EDITOR ....p. 5

VIEWPOINT
Implementing Change in the Mathematics Curriculum ....p. 8
by Sheldon P. Gordon

MATHEMATICAL EXPOSITION
Sixty-Thousand Dollar Question ....p. 14
by A. Arvai Wieschenberg and Peter Snenkin

A Conversation About Russell's Paradox ....p. 18
by Paul E. Bland

The FFT: Making Technology Fly ....p. 25
by Barry A. Cipra

SHORT COMMUNICATIONS
The Product Function of Two Continuous Functions is Continuous (Alternate Proofs) ....p. 30
by Brian Mitchell, Jonathan Picklesimer, and Dean B. Priest

Functional Fish and Other Mathematical Menagerie Members ....p. 33
by Margaret Spurlin Willis

MATHEMATICS EDUCATION
The Effect of Cooperative Learning in Remedial Freshman Level Mathematics ....p. 37
by Carolyn M. Keeler and Mary Voxman

Writing to Learn Mathematics: Enhancement of Mathematical Understanding ....p. 45
by Aparna B. Ganguli

You Can't Do That! ....p. 53
by Richard L. Francis

REGULAR FEATURES
The Chalkboard ....p. 60
Edited by Judy Cain and Joseph Browne

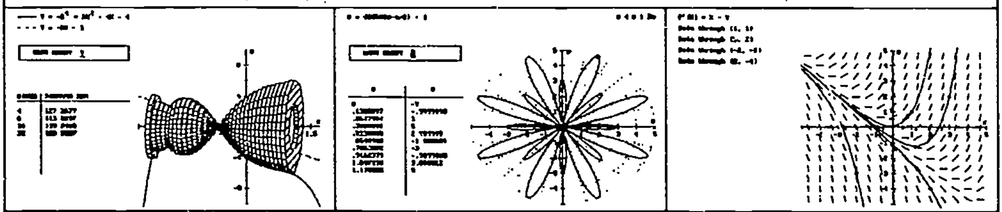
Snapshots of Applications in Mathematics ....p. 64
Edited by Dennis Callas and David J. Hildreth

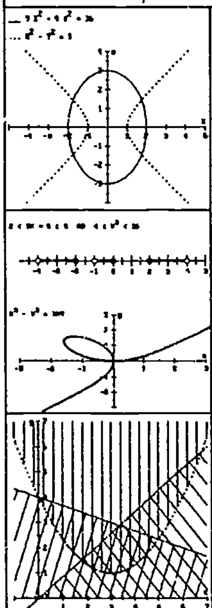
Notes from the Mathematical Underground ....p. 68
Edited by Alain Schremmer

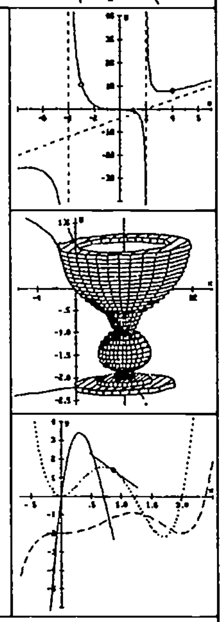
Software Reviews ....p. 72
Edited by Shao Mah

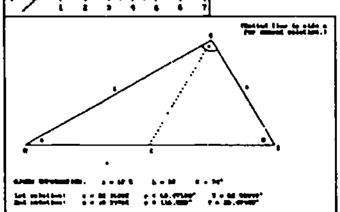
The Problem Section ....p. 74
Edited by Michael W. Ecker

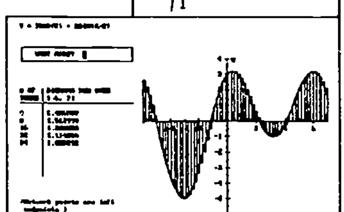
Advertiser's Index ....p. 86

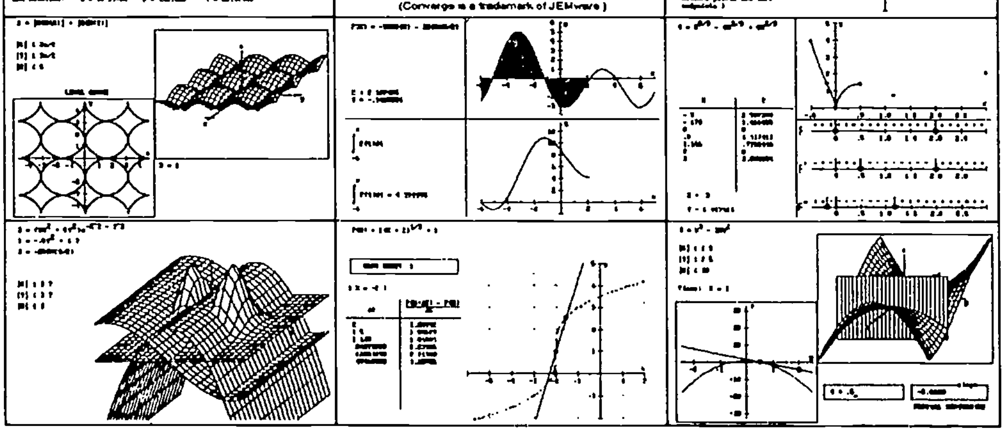












BEST COPY AVAILABLE

Editor's Comments

Help Wanted: Book Review Editor

Our Book Review Editor, John Edgell, has resigned, citing personal circumstances. We wish him well, but now the position is open. We are attempting to shift the emphasis in this department away from reviews of standard textbooks toward what might be called general interest or personal library books. We are seeking somewhat longer reviews (2–4 typed pages) than in the past, and welcome reviews which compare the book to others on the same topic. The editor will probably have to do some soliciting (twist some arms) to get reviews of this type, so we seek someone who knows both books and people. If this sounds like something you would like to do, send me a letter with some of your thoughts about the Book Review section of the journal.

US First in International Mathematical Olympiad

Congratulations to the six members of the US International Mathematical Olympiad team which not only won the nine hour competition in Hong Kong this summer, but also became the first team to ever achieve the maximum possible score. The top five teams (in order) were the USA, China, Russia, Bulgaria, and Hungary. United States team members, all of whom won individual gold medals, were Jeremy Bem, Ithaca (NY) H.S., Aleksandr L. Khazanov, Stuyvesant H.S. (New York City), Jacob A. Lurie, Montgomery Blair H.S. (Silver Spring MD), Noam M. Shazeer, Swampscott (MA) H.S., Stephen S. Wang, Illinois Mathematics and Science Academy (Aurora), and Jonathan Weinstein, Lexington (MA) H.S. AMATYC is one of the sponsoring organizations of the US team. We note that in spite of all the criticism of the United States' schools, its best students compare favorably with the best anywhere. Here is one problem from the competition:

> Show that there exists a set A of positive integers with the following property: For any infinite set S of primes there exist two positive integers m in A and n not in A each of which is a product k distinct elements of S for some k greater than 1.

About the Cover

Our cover for this issue relates to the paradox of Bertrand Russell. In his contemplations on the foundations of mathematics, Russell conceived of the following:

$$R = \{\text{all sets which are not elements of themselves}\}.$$

The paradox arises in trying to determine whether or not R itself belongs in this set. Paul Bland has written a nice interpretation of this paradox in the traditional dialog form which you will find in this issue. The cover design is by Sarah Browne.

For more on Russell's paradox and its connection to Kurt Gödel's remarkable theorem that all reasonably broad mathematical systems contain statements which are undecidable (i.e. statements which can neither be proved true nor false), see *The Emperor's New Mind* by Roger Penrose (Oxford University Press 1989). Penrose may get into more philosophy of mathematics than some will want to wade through, but he includes some superb exposition of sophisticated mathematical concepts, understandable without an advanced mathematical background.

Letters to the Editor:

Although technology is rapidly changing the mathematical highway, the fundamental goals of the mathematics experience remain constant. Key components of this experience include (1) developing problem solving skills, (2) appreciating the genius of those on whose shoulders we stand, and (3) recognizing and discovering connections between ideas.

A recent letter to this journal (see Spring, 1994, p. 5) criticized [the inclusion of] an article on partial fractions (Calculus to Algebra Connections in Partial Fraction Decomposition, Fall, 1993). The criticism focused on two premises: partial fraction decomposition is currently programmed into readily accessible software, and partial fractions are not essential for integration presumably because many cases of integration are currently programmed into existing software.

We feel the paper contributes to the goals of the mathematical experience. This paper focused on the intellectual foundations of partial fraction decomposition and quick mental calculation of the decomposition. Ability to quickly do mental calculation is an important part of estimating answers – an increasingly significant skill as electronic technology becomes pervasive. The concepts developed in this paper encourage the student to understand the mathematics behind partial fractions, see connections between concepts, and expand their personal collection of alternatives.

The paper referenced the application of partial fractions to integration – which is practically useful and historically significant. Technology does indeed provide practical alternatives to completing these tasks. The completion of the task, however, is not the essence of the mathematical experience. Furthermore, partial fractions have extensive application to the theory and practice of integral transforms, to the residue theory and contour integration, to the theory of meromorphic functions, and approximation by rational functions. The many applications of partial fractions make understanding the intellectual foundations of the concepts even more important – especially when electronic technology is elected to complete the calculations.

Joseph Wiener and Will Watkins
University of Texas-Pan American

I greatly enjoy reading *The AMATYC Review* because it is informative as well as entertaining. The fact that these two qualities do not have to be mutually exclusive is no better exemplified than in the "Lucky Larry" series. Here are a few comments.

LL#6 (Fall, 1993): It turns out Lucky Larry was not just lucky this time. He was simply making use of "Larry's Lemma" to the Pythagorean Theorem which reads "If a right triangle has hypotenuse c equal to one more than one of its legs, say a, then $a + c = b^2$."

Proof: By the Pythagorean Theorem, $b^2 = c^2 - a^2$. This implies that

$$b^2 = (c - a)(c + a) = (c + a), \text{ since } c - a = 1.$$

More generally, it can be similarly shown that if the hypotenuse c is n more than a, then $b^2 = n(a + c)$.

LL#7 (Fall, 1993): Again Larry was not really lucky. He simply applied "Larry's

Law of Sines": $\lim_{x \to 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}$.

Proof: The expression clearly satisfies L'Hôpital's Rule. Thus, the limit is equivalent to $\lim_{x \to 0} \frac{a \cos ax}{b \cos bx} = \frac{a}{b}$.

Thus, if you know what you are doing, the "canceling" may be valid; just as valid as any other memorized but seldom proven manipulation.

When an "error" produces correct results under certain measurable conditions, should we continue to classify it as such?

Michael Sawyer
Houston (TX) Community College

Here is a "Lucky Larry" example by three college professors, Barker, Rogers, and Van Dyke. They pulled the following boner in their *Intermediate Algebra* (Second Edition, Saunders College Publishing Co.). On page 273, problems #80 and #81, they use a bogus formula which gives "reasonable" but incorrect answers for the values they chose to insert, but gives absurd answers for some other values. Their formula, $v = \sqrt{v_0 + ad}$, should be $v = \sqrt{v_0^2 + 2ad}$, where v is velocity, d is distance, a is acceleration, all in consistent units.

Mike Majeske
Capita. Community-Technical College (CT)

I have some concerns about the statistical results provided in the article "Predicting Grades in Basic Algebra" by Elsie Newman (Spring, 1994). Basically, the article describes a research study which attempted to find significant correlations for the purpose of predicting grades. Although we might find any number of such studies in the literature over an extended period, there are two faults with this approach.

First, this study uses correlation techniques (with the related regression methods); the statistical premise in correlation is that the variables are interval in nature. Grades do not qualify. A 4.0 does NOT imply twice of something compared to a 2.0. This is a serious technical violation of assumptions.

Secondly, this study presumes that we ought to be able to find some variable to predict a course grade. The purpose stated is "Choosing an effective method of placing students into courses..." (page 47). Good placement is an issue of readiness for success, not predicting the level of success. Valid contemporary research – such as that advocated by the MAA Committee on Testing – assumes that a placement system results in higher pass rates, not a higher grade distribution.

At the same time, I believe the author could provide a positive contribution by sharing data in a cross-tabulation form: success rates by levels of the various placement variables. Although such non-parametric methods don't lend themselves to quick prediction equations, the use of statistics such as chi-square can allow valid interpretation of each variable's relationship to success. Comparing the strengths of each relationship allows for establishing the variables with the largest impacts on success.

Jack Rotman
Lansing Community College

VIEWPOINT

Implementing Change in the Mathematics Curriculum

by

Sheldon P. Gordon
Suffolk Community College
Selden NY 11784

Shelly Gordon is professor of mathematics at Suffolk Community College. He received his PhD from McGill University. He is co-project director of the multi-year NSF-supported Math Modeling/PreCalculus Reform Project and is a member of the Harvard Calculus Project Consortium. Gordon is the co-author of Contemporary Statistics: A Computer Approach, is the co-editor of the MAA Notes volume, Statistics for the Twenty First Century, and is the author of over 75 articles on mathematical research and mathematics education.

Introduction

Over the last few years, there has been intense discussion about the undergraduate mathematics curriculum. This has resulted in calls, both from within and without mathematics, for major changes in the way that many courses are taught. Many are due to the growing availability of technology. Other pressures have been exerted to invigorate calculus in particular and mathematics education in general by making it more modern, more interesting, more intuitive and more successful.

As a member of the Harvard Calculus Reform project (Gordon & Hughes Hallett, 1992) and co-director, with B. A. Fusaro, of the Math Modeling/PreCalculus Reform Project (Gordon, 1994), I am well aware of the problems, challenges and opportunities involved. One indication of the need for reform is that, nationally, only about 15% of the 600,000 students who take precalculus or college algebra/trigonometry courses each year ever go on to *start* calculus. Relatively few of these actually complete the calculus sequence successfully. For a broader overview of the problems associated with reforming these courses at the two year level, see (Rodi & Gordon, 1994); the interested reader should also see the variety of ideas presented in (Peressini & Albers, 1994). A major effort is also underway through AMATYC to develop a set of Curriculum Standards for the two year colleges that call for dramatic changes in both the content and delivery of all courses from arithmetic up through precalculus. The draft volume (D. Cohen, et al, 1993) should be required reading for all faculty involved in teaching such courses.

At the same time, computer scientists have requested a far greater role for discrete mathematics in the undergraduate curriculum. Statisticians have called for major changes in the focus of the introductory statistics courses to reflect both exploratory data analysis and the use of computers as educational and computational tools. They have also initiated major projects to introduce statistics in primary and secondary schools which will eventually force most colleges to adapt their own statistics offerings. See (Gordon & Gordon, 1992), (Watkins et al, 1992) and (Cobb, 1992).

The mathematics community is beset on all sides with pressures to introduce significant, far-reaching changes into our entire curriculum, a curriculum with which we have been very comfortable for our entire professional lives.

Implementing Change in Mathematics

In order for any major undergraduate mathematics curriculum reform project to be successful, the mathematics faculty and the administration must become sensitized to the problems and pressures that exist. Faculty must have ample time to think through the implications of such changes. Often change in one part of the curriculum impacts significantly on other aspects as well. For instance, changes in calculus quickly suggest changes in the successor courses, such as differential equations, as well as the precursor courses such as college algebra/trig or precalculus.

Probably the most critical need when implementing curriculum reform is for faculty training. Many of the major funded projects in calculus, precalculus, differential equations and statistics involve rather substantial changes in the content to emphasize mathematical ideas that many faculty members may never have seen or may have forgotten. Often, these changes are most keenly reflected in the very different problems that students are asked to solve; they are problems that require considerable thought; that involve applying the mathematical concepts in surprisingly different ways; that involve applications from a variety of disciplines, not just the traditional physical sciences. Most of the reform projects involve some degree of technology, such as computers or graphing calculators or statistical packages, creating the need for workshops in these areas.

The need for training workshops may be particularly essential for those institutions that use large numbers of adjunct faculty to cover these courses. There is also a need to provide additional training to the personnel in tutoring centers since they will be faced with unfamiliar questions from students taking these "new" courses. Similarly, student graders and graduate students working as TA's at large universities typically need training as well.

It is also necessary for mathematics instructors to open dialogues with faculty from client disciplines. Engineering and physics instructors, for example, have needs which should be taken into account by the mathematics faculty. At the same time, the client disciplines should be apprised of the changes being considered in the mathematics offerings to determine how such changes will affect their own courses. Often, such users are more anxious for changes than the mathematics faculty because they find that students coming out of traditional mathematics courses cannot apply their mathematics to new situations.

It is also desirable to reach out beyond the local campus. At two year colleges, it is essential to develop dialogues with the local four year colleges to smooth out the transfer arrangements. Often, such changes are welcomed by your neighboring colleges and universities and, there are many instances where universities decided to modify their own courses because of contacts with local two year colleges that had already implemented such changes.

Further, it may also be a good idea to involve local business and industry leaders so that they can describe their needs for technically trained personnel. It is also necessary to involve high school mathematics instructors to develop articulation both ways — colleges responding to changes in high school courses and high schools preparing their students for the new courses being introduced in the colleges.

In addition, curriculum reform is not something that should be conducted in a vacuum at each individual campus. Rather, people should seek to develop joint regional networks of institutions involving a variety of institutions, including high schools, two year colleges, four year colleges and universities. Such regional networks have been extremely successfu' in attracting NSF support for collaborative activities instituting curricular reform. In fact, one of the major thrusts at NSF currently is to fund projects, particularly network projects, that seek to implement previously funded and successful NSF grant projects. A special emphasis is being placed on involving two year colleges in such efforts. Such networks, particularly if they receive outside funding, can provide a variety of essential services to a curriculum reform effort. This may cover regional training workshops and sources for information and assistance in your efforts, as well as released time during the academic year to implement the new curriculum.

Perhaps equally importantly, such workshops, particularly if they cover a number of neighboring institutions, can provide a spirit of camaraderie as a group of people conduct similar innovative curricular activities.

Furthermore, while *you* may see the need for curriculum reform as an obvious, you should probably not expect all members of your department to agree enthusiastically, if at all. Too many mathematicians view their own training as the only way to teach mathematics. Such individuals can always be counted on to oppose curricular change. They dismiss observations that students

a) do poorly in existing courses,

b) do not appreciate the mathematics they are being taught,

c) cannot transfer their mathematics to other disciplines, *and*

d) retain frightening little of the material they were previously taught

as either undocumented conjecture or something that is solely the fault of the students, not the course or the way it was presented. In fact, in the minds of some faculty, high success and retention rates are cause for suspicion, not celebration and replication.

11

Specific Recommendations

Finally, and perhaps most importantly, to implement any major changes in the mathematics curriculum at your own institution, it is essential to recognize in advance the type of support that must be provided by the school. This includes full institutional support for the proposed curricular changes that goes well beyond just providing lip service to the desirability of curriculum reform. Administrators must be prepared to expend funds, as needed, and to "count" major curriculum development activities as important professional accomplishments towards promotion and tenure. Administrators must be willing to provide:

1. released time to develop new curricula.
2. time and opportunity to retrain faculty and to allow faculty to attend workshops and courses to expose them to newer mathematical philosophies, ideas and methods. Such workshops are often major components of funded projects such as the Harvard project and the author's PreCalculus Reform Project; all major professional meetings now provide a host of such workshops. Similarly, the NSF is also funding a wide variety of training workshops in fields such as differential equations and statistics/data analysis.
3. time and opportunity to become familiar with appropriate technology, both hardware and software, which is often an essential component of new curricula.
4. time and opportunity to develop new test materials which reflect the curricular changes. Tests determine what the students believe to be important; if we want to change the content of a course, we must change the tests to reflect that change. This is often an extremely time-consuming and challenging process.
5. time and opportunity to develop appropriate placement instruments to reflect the new curriculum and its effects on other aspects of the curriculum.
6. adequate computer laboratory facilities, include appropriate staffing and maintenance of these facilities, if computer usage is a significant aspect of the new curriculum.
7. adequate funds to purchase computer equipment for classroom demonstrations and to provide in-office equipment for each faculty member. This may include sophisticated calculators and their classroom technological support.
8. time, opportunity and, if necessary, funding to permit faculty to develop dialogues within their department, with other departments in their institution, with other local colleges and high schools, and with colleagues at more distant locations via attendance at conferences and special meetings.
9. access to e-mail and through it the Internet network to provide on-going contact with the rest of the mathematics community. Nodes have already been established to allow for the exchange of ideas, materials, exams and project assignments on the Harvard project in particular and calculus reform in general; another has been established for the exchange of ideas, materials, programs and sets of data for introductory statistical education.
10. time and opportunity to develop outcomes-assessment vehicles.

The payback to the institution for these expenditures can be immensely rewarding. Students will have better courses; they will be better motivated and better prepared for successor courses; and they will be more likely to continue their studies of mathematics and related courses. Faculty members will rejuvenate themselves professionally; they will learn new mathematical ideas; they will re-establish contacts with colleagues locally and at other institutions. The rewards are well worth the investment.

References

Cobb, G. (1992). *Teaching statistics.* In L. Steen (Ed.), *Heeding the call for change: Suggestions for curricular action,* MAA Notes #22 (pp. 3-43). Washington: Mathematical Association of America.

Cohen, D., et al. (1993). *Standards for curriculum and pedagogical reform in two-year college and lower division mathematics* (Draft version). Memphis: American Mathematical Association of Two-Year Colleges.

Gordon, S.P. (1994). *Functioning in the real world: Models for precalculus reform.* In A. Peressini and D. Albers (Eds.), *The bridge to calculus,* MAA Notes Volume #36. Washington: Mathematical Association of America.

Gordon, F.S. and Gordon, S.P. (Eds.). (1992). *Statistics for the twenty first century.* MAA Notes #26. Washington: Mathematical Association of America.

Gordon, S.P. and Hughes Hallett, D. (1991). The implications of calculus reform on the two year college. *The AMATYC Review, 12*(2), 50-57.

Peressini, A. & Albers, D. (Eds.). (1994). *The bridge to calculus,* MAA Notes Volume #36. Washington: Mathematical Association of America.

Rodi, S. and Gordon, S.P. (1994). *The implications of precalculus reform on the two year colleges.* In A. Peressini & D. Albers (Eds.), *The bridge to calculus.* MAA Notes Volume #36. Washington: Mathematical Association of America.

Watkins, A., Burrill, G., Landwehr, J. & Schaeffer, R. (1992). *Remedial statistics?: The implications for colleges of the changing secondary school curriculum.* In F.S. Gordon and S.P. Gordon (Eds.), *Statistics for the twenty first century.* MAA Notes #26 (pp. 45-55). Washington: Mathematical Association of America.

Acknowledgement

The author sincerely acknowledges the support provided by the Division of Undergraduate Education of the National Science Foundation under grants #USE-89-53923 to the Harvard Calculus Reform Project and #USE-91-50440 to the Math Modeling/PreCalculus Reform Project. However, the views expressed here are those of the author and do not necessarily reflect those of the National Science Foundation or the individual projects.

> To speak algebraically, Mr. M. is execrable, but Mr. G. is $(x+1)$ecrable.
>
> Edgar Allen Poe

MATHEMATICAL EXPOSITION

Sixty-Thousand Dollar Question

by

A. Arvai Wieschenberg and Peter Shenkin
John Jay College of Criminal Justice
City University of New York
New York NY 10019

Agnes Arvai Wieschenberg received her Ph.D. from Columbia University and is an assistant professor of mathematics at the John Jay College of Criminal Justice of the City University of New York. Her research interests are numerical methods and their microcomputer implementation and the integration of computer technology in the undergraduate mathematics curriculum.

Peter Shenkin received his Ph.D. from the Courant Institute of New York University and is an associate professor of mathematics at the John Jay College of Criminal Justice of the City University of New York. He is involved with writing mathematics software for undergraduate mathematics courses.

A friend of ours, Rita, took on a new academic position several weeks ago and was asked to fill out some forms. Among many others she was faced with the following question: "Would you like to receive 24 equal bi-monthly payments during the period between September 1 and August 31 of the next year, or would you prefer 20 equal bi-monthly payments between September 1 and June 30?" In either case the first payment was to be made on September 15. She was about to check off the box for 20 payments when a colleague (not from the Mathematics Department, of course) explained that the 24 payment method was better because "that way your taxes are taken out later." Rita was perplexed because she thought she was making the same amount of money either way and the 20 payment method meant she would get the money sooner. Did her colleague know what he was talking about? When she asked us what the correct choice was, our immediate reaction was "20 payments." Were we flippant in our consideration of the question? Could it be that certain tax or other considerations might prove our gut decision to be wrong?

The next logical step was to investigate several different scenarios. The **future value** of a sequence of investments (cash flows) after a fixed period of time is the total amount that the investments will grow to (shrink to) at the end of the time period. Note that this definition does not preclude the investments being negative when they are paid out instead of received.

The application of future value we give in this paper involves compound interest. If we invest an amount A and are paid an annual interest rate r compounded annually, the value of our investment at the end of the nth year is known to be $A(1 + r)^n$.

Note that successive yearly values make up a geometric progression with initial term A and with common ratio $1 + r$. If we invest amount A in each of n years then we have an **annuity**. The **value of an annuity** after n years is the sum of the future values of the individual yearly investments at the end of year n. If the investment is made at the end of the investment year, we have

Investment Year	Future Value
1	$A(1 + r)^{n-1}$
2	$A(1 + r)^{n-2}$
.	. . .
$n - 1$	$A(1 + r)$
n	A

The sum of the future values is the sum of a geometric series with n terms, first term A and common ratio $(1 + r)$. The given sum is

$$S(A, n, r) = A\ \frac{(1 + r)^n - 1}{r}.$$

It should be obvious that the restriction to yearly time periods is artificial. If r is the periodic interest rate during a single compounding period and n is the number of compounding periods, then the same formulas hold. If the future value is left invested at rate r' per period for an additional k periods (e.g. during the summer months) the resulting value is equal to:

$$S(A, n, r) \cdot (1 + r')^k.$$

With this preparation out of the way let us start to deal with Rita's problem. In order to deal with round numbers we shall choose earnings per year of \$48,000. This results in 24 payments of \$2000 each or 20 payments of \$2400 each. Our comparison scheme will be the following. Rita chooses 20 payments and invests the after tax excess each pay period in a savings account that yields 6% interest per year. The savings account is compounded bimonthly on pay day. During July and August she will need to withdraw \$2000 minus taxes from her account each pay day to keep cash flow constant for the entire year. If an overall withholding tax rate of 40% is assumed, we see that Rita will deposit (2400 – 2000)(1 – .40) = \$240 after each of her 20 paychecks and withdraw (2000)(1 – .40) = \$1200 on each of the next four pay periods.

The periodic interest rate is .06/24 = .0025 = .25%. Assuming that taxes are withheld from interest at the .40 rate at the end of each compounding period, we can use an after tax rate of $r = (.0025)(1 - .40) = .0015$ for our computations.

The value of the first 20 deposits after one year is

$$S(240, 20, .0015)(1 + .0015)^4 = \$4898.30.$$

The value of the last four withdrawals after one year is

$$S(-1200, 4, .0015) = -\$4810.81.$$

Summing the two numbers we get $87.49 as the amount left in the savings account after one year. Thus, under our assumptions, if she chooses the 20 payment method, Rita can receive the same amount each month as under the 24 payment method and still have $87.49 left over at the end of the year. (In fact, since taxes usually aren't withheld from savings accounts, she is a little better off.)

It may be easily seen that under the assumptions we have used, Rita would never be better off taking the 24 payment option. In fact, during each of the last four pay periods she will withdraw exactly five times the amount of a periodic deposit for the previous pay periods, since the money put away during the previous 20 pay periods exactly equals the excess take home pay for those periods and this is used up evenly over the last four periods. The interest earned, less taxes, is exactly the "profit." At the very worst the monthly excess using the 20 payment plan could be hidden under a mattress to be withdrawn during July and August.

Generalizing, it can be seen that the future value of Rita's "investments" for any periodic deposit A and interest rate $r > 0$ may be given by

$$S(A, 20, r)(1 + r)^4 + S(-5A, 4, r) = \frac{A}{r}[(1 + r)^{24} - 6(1 + r)^4 + 5].$$

This expression approaches 0 as r approaches 0 and is an increasing function of r for $r > 0$.

Did Rita's question make sense at all? Are there situations in which the 24 payment choice is better? Suppose that on January 1 tax rates go down. (We should be so lucky.) Let's assume that overall withholding rates decrease from 40% to 30%. In that case, after taxes our friend will get $400 (1 – .3) = $280 extra during each pay period in the new year. The effective bimonthly interest rate during the new year is (.0025)(.7) = .00175.

Then letting $r_1 = .0015$ and $r_2 = .00175$ the future value of picking the 20 payment option is

$$S(240, 8, r_1)(1 + r_2)^{16} + S(280, 12, r_2)(1 + r_2)^4 + S(-1400, 4, r_2)$$

which equals –$213.51. So, if there is going to be a large tax decrease, the 24 payment choice might be better. Note that, in this case, the person choosing the 24 payment option actually has a greater after tax take home pay of some $320.

We close with the following thought. Suppose Rita takes the $87.49 and invests that amount in a tax deferred retirement account for each of the next 45 years. Say this account compounds interest one time per year and offers a yield of 10%. How much will she have in this account after a working lifetime of 45 years? The answer is just $S(87.49, 45, .100)$ or $62896.98. That is quite a sum of money for simply saying "20" instead of "24."

20

A Conversation About Russell's Paradox

by

Paul E. Bland
Eastern Kentucky University
Richmond KY 40475

Dr. Bland received the Ph.D. in mathematics in 1969 from the University of South Carolina. He is currently Professor of Mathematics at Eastern Kentucky University. In 1990, he was a National Faculty Exchange Program Professor to Bucknell University in Lewisburg, Pennsylvania. In 1993, he was Visiting Professor of Mathematics at the University of Otago in Dunedin, New Zealand where he did research on relatively flat modules. Professor Bland has published articles of both a research and expository nature.

John is a first semester senior majoring in mathematics. He has started to think about what mathematics courses he will take during his final semester. The following conversation began shortly after the beginning of the Fall Semester in the office of Professor Straightline, a mathematics professor at the university.

PROFESSOR: Hi, John. How are you doing today?

JOHN: Not bad, Professor. I was thinking about taking a course in foundations of mathematics during my final semester. Could you tell me a little bit about the course?

PROFESSOR: You'll mostly study axiomatic set theory.

JOHN: Axiomatic set theory!

PROFESSOR: Yes.

JOHN: I don't know whether or not I would be interested in such a course. I seem to be able to handle sets O.K.

PROFESSOR: Are you sure you know what a set is?

JOHN: Sure, a set is any collection of objects.

PROFESSOR:. Well, O.K., but did you know that there are contradictions within set theory when you look at it that way?

Lord Bertrand Russell, 1872-1970, was an English philosopher/logician who discovered a contradiction in set theory as practiced in his day. This discovery, now known as Russell's Paradox, caused a revolution in mathematics, the result of which is axiomatic set theory.

JOHN: Not really.

PROFESSOR: Russell's Paradox.

JOHN: Yes, I have heard of that, but I must confess that I don't understand its significance.

At this point Professor Straightline goes to his office black board and writes:

$$\text{If } S = \{A \mid A \notin A\}, \text{ then } S \in S \text{ if and only if } S \notin S.$$

PROFESSOR: That's really the complete story.

JOHN: Just one sentence?

PROFESSOR: Yes. But what a profound sentence. It started a revolution in mathematics.

JOHN: I don't understand its meaning.

PROFESSOR: If you take the course, you'll find out what it means.

JOHN: Well, I'll think about taking the course, but I'm not sure yet what I'm going to do.

PROFESSOR: I think you would benefit from such a course in that you would gain a deeper understanding of an important part of mathematics.

JOHN: Well, O. K., I'll give it some thought.

At this point John turns to leave the professor's office. Before he can leave, however, Professor Straightline speaks to him again.

PROFESSOR: John.

JOHN: Yes, Professor.

PROFESSOR: Do you have any free time to work for me this semester?

JOHN: Ye.., what do you have in mind?

PROFESSOR: I have to do a library search for a university committee I am serving on and I need some help.

JOHN: What kind of search?

PROFESSOR: Well, it seems that the library contains several catalogues each of which contains a list of books in the library that deals with a single subject. For example, one of these catalogues is a list of books in the library that deals with American history from 1750 through 1900. Another catalogue is a list of books in the library that deals with calculus.

JOHN: You mean each of these catalogues contains nothing else?

PROFESSOR: Nothing other than a short introduction explaining what the catalogue contains.

JOHN: I see. You want me to make a list of these catalogues for you?

PROFESSOR: Well, not exactly. Some of the authors decided to list their own catalogue in the catalogue that they compiled.

JOHN: What do you mean?

PROFESSOR: The catalogue that lists books tl at deal with American history from 1750 to 1900 is named *A Cat 'ogue of American History Books. 1750-1900.* The author realizing that this catalogue itself was a book about American history from 1750 to 1900 included it in the list.

JOHN: The catalogue lists itself?

PROFESSOR: Yes.

JOHN: What about the catalogue that lists calculus books. Does it list itself?

PROFESSOR: No, it doesn't.

JOHN: You want a list of these catalogues?

PROFESSOR: Yes, I want you to make a complete list of all the catalogues that do not list themselves. Write a short introduction to the list, bind the introduction together with the list in some fashion, leave one copy in the library and bring me a copy. When you have completed this task successfully, I'll pay you $100.00.

JOHN: Sounds like a big job to me.

PROFESSOR: Not really. Mr. Spindle, the head librarian, has most, if not all, of the work already done. All you have to do is an electronic search to make sure his list is complete.

JOHN: O. K. I'll give it a shot. I hope it doesn't take too much time from my studies.

PROFESSOR: It shouldn't.

This ended the conversation and John left Professor Straightline's office and headed for the library. Thinking to himself, John says, "I'm going over to the library and see if I can't make myself a quick $100.00." Upon entering the library, he sees Mr. Spindle, the head librarian.

JOHN: Good morning. Mr. Spindle.

MR. SPINDLE: Good morning. What can I do for you?

JOHN: I understand that you have a list of books in the library each of which is a catalogue that lists books that deal with a single subject.

MR. SPINDLE: Well, yes I do. In fact, I have two such lists. I have a list of all such catalogues which list themselves and I have a list of all such catalogues which do not list themselves. Do you want both lists?

JOHN: No, I only need the list of catalogues which do not list themselves.

MR. SPINDLE: O.K., just a minute. Oh, here it is.

JOHN: Thanks. Now I have to do an electronic search to be sure your list is complete.

MR. SPINDLE: No need. I updated the list this morning and I can assure you that it is complete.

JOHN: Great, that saves me a lot of work.

MR. SPINDLE: Has Professor Straightline sent you to help with his library search?

JOHN: Well, yes. How did you know?

MR. SPINDLE: I knew that he wanted this information and that's why I completed work on the list this morning.

At this point John noticed a slight grin cross Mr. Spindle's face, but he didn't give it much thought. He was excited by the fact that he was going to make $100.00 for about 30 minutes work, money he could surely use. He went to his room, wrote a short introduction to his list, bound two copies of his introduction and the list, and returned one of these to Mr. Spindle at the library. He then headed for Professor Straightline's office. He was lucky. He found the Professor in his office.

JOHN: Well, Professor, I'm finished with the library search.

PROFESSOR: Really? Let's see what you've got.

JOHN: Here is a bound and complete list of all catalogues in the library that do not list themselves. As you instructed, I have written a short introduction and bound it with the list.

PROFESSOR: Did you put a copy of this in the library?

JOHN: Yes, I just returned from the library, where I gave Mr. Spindle a copy.

PROFESSOR: What have you named your catalogue?

JOHN: I named it *The Catalogue of Catalogues Which Do Not List Themselves.*

PROFESSOR: Fine. Let's look over what you have done.

JOHN: This is the easiest $100.00 I ever made

PROFESSOR: Wait a minute, I'm not sure you are finished.

JOHN: What do you mean?

PROFESSOR: I think your list is incomplete.

JOHN: Mr. Spindle assured me that the list he gave me was complete.

PROFESSOR: I'm sure it was at that time.

JOHN: I don't get it.

PROFESSOR: Your book that you just placed in the library is a catalogue that deals with a single subject, is it not? It deals with catalogues which do not list themselves.

JOHN: Well, yes.

PROFESSOR: Inspecting your catalogue of catalogues, I see that your catalogue does not list itself.

JOHN: That's true.

PROFESSOR: Then, if your list is to be complete, you must put the name of your catalogue on your list in *The Catalogue of Catalogues Which Do Not List Themselves.*

JOHN: O.K., I see that.

John now takes out his pen, opens The Catalogue of Catalogues Which Do Not List Themselves, turns to the list therein and writes "The Catalogue of Catalogues Which Do Not List Themselves" as the last entry in the list.

JOHN: There, Professor, I'm done. Can I have my $100.00?

PROFESSOR: Not yet.

JOHN: Why not? I've done my job.

PROFESSOR: Not quite. Your catalogue now lists itself and so its name does not belong on the list of all catalogues which do not list themselves.

JOHN: But... But, if I don't list my catalogue, then its name belongs on the list and if I list my catalogue, then its name does not belong on the list.

PROFESSOR: Exactly!

JOHN: Let's see now if I understand this correctly. I must list my catalogue if and only if I do not list my catalogue.

PROFESSOR: Correct.

JOHN: I guess I'm trapped. But wait a minute. I'm beginning to see something.

PROFESSOR: What?

JOHN: The situation you have put me in is just a non-mathematical form of Russell's Paradox.

PROFESSOR: Eureka!

JOHN: Beautiful! Bertrand Russell must have been a pretty sharp fellow.

PROFESSOR: A very sharp fellow indeed. John, you might also consider taking a history of mathematics course so that you can learn something about the great men and women of mathematics.

JOHN: That's an idea. Well, I guess I don't get the $100.00?

PROFESSOR: No, but you do get lunch. My treat.

At this point Professor Straightline and John go to the school cafeteria where they continue to discuss Russell's Paradox and its ramifications in more detail. John now better understands its significance and why one needs to take an axiomatic approach in order to eliminate this paradox from set theory. After lunch Professor Straightline and John are walking across campus when the professor indicates that he has to go to a meeting in the Administration Building. John heads back to the Mathematics Building and, on the way, he meets up with Allison, one of his fellow mathematics majors.

ALLISON: Hi, John. How's it going?

JOHN: O.K. I guess. Just had lunch with Professor Straightline.

ALLISON: I've been talking with him, also. I had lunch with him yesterday.

JOHN: What are you going to take next semester?

ALLISON: I'm thinking about taking foundations. Seems like it might be interesting.

JOHN: Do you understand Russell's Paradox?

ALLISON: Yes, I think I understand it pretty well.

JOHN: Professor Straightline explained it to me this morning in a rather unique manner.

ALLISON: (*Laughingly.*) Did you do his library search?

JOHN: Yeah!

ALLISON: Me, too.

> Bertrand Russell was most mortified
> When a box was washed up by the tide,
> For he said with regrets,
> "Why, the set of all sets
> Which belong to themselves is inside."
>
> Paul Ritger

The FFT: Making Technology Fly

by

Barry A. Cipra

Barry A. Cipra is a mathematician and writer based in Northfield, Minnesota. He is a Contributing Correspondent for the AAAS journal Science and a regular writer for SIAM News. Cipra was also author of the inaugural issue of "What's Happening in the Mathematical Sciences," a new publication of the AMS.

(Editor's Note: This article originally appeared in the "Mathematics That Counts" Series of the SIAM News (Vol. 26, Num. 3, May 1993). As part of a cooperative initiative between AMATYC and the Society for Industrial and Applied Mathematics (SIAM) we have been encouraged to reprint articles from this series. We hope you will enjoy these reports on modern uses of mathematics.)

"Work smarter, not harder" is a slogan often bandied about by business efficiency task forces. Everyone knows that a modicum of planning and organization can save a lot of wasted effort. Smart managers are constantly looking for ideas that will streamline their operations and improve the productivity of their workforces. Any edge, even one as small as one percent, is worth it.

So how about an idea that increases productivity by a thousand percent?

The "business" of scientific computing saw just such an idea back in 1965, stuffed in a suggestion box known as *Mathematics of Computation*, a journal published by the American Mathematical Society. In a five-page paper titled "An Algorithm for the Machine Calculation of Complex Fourier Series," James W. Cooley of the IBM T.J. Watson Research Center (now at the University of Rhode Island) and John W. Tukey of Princeton University and AT&T Bell Laboratories laid out a scheme that sped up one of the most common activities in scientific and engineering practice: the computation of Fourier transforms.

Their algorithm, which soon came to be called the fast Fourier transform – FFT for short – is widely credited with making many innovations in modern technology feasible. Its impact extends from biomedical engineering to the design of aerodynamically efficient aircraft. Over the last two and a half decades, Cooley and Tukey's paper has been cited in well over a thousand articles in journals ranging from *Geophysics* to *Applied Spectroscopy*. According to Gil Strang of MIT, the FFT is "the most valuable numerical algorithm in our lifetime."

Curiously, "An Algorithm for the Machine Calculation of Complex Fourier Series" came close to never being published. But more about that later. First, why is the FFT such an important algorithm?

To understand why the fast Fourier transform has had such a profound impact on technology requires some appreciation of the Fourier transform itself. Whole books have been written on the subject of Fourier analysis and its applications. Their common theme is the incredible versatility of a simple mathematical tool.

The Fourier transform stands at the center of signal processing, which encompasses everything from satellite communications to medical imaging, from acoustics to spectroscopy. Fourier analysis, in the guise of X-ray crystallography, was essential to Watson and Crick's discovery of the double helix, and it continues to be important for the study of protein and viral structures. The Fourier transform is a fundamental tool, both theoretically and computationally, in the solution of partial differential equations. As such, it's at the heart of mathematical physics, from Fourier's analytic theory of heat to the most modern treatments of quantum mechanics. Any kind of wave phenomenon, be it seismic, tidal, or electromagnetic, is a candidate for Fourier analysis. Many statistical processes, such as the removal of "noise" from data and computing correlations, are also based on working with Fourier transforms.

Fourier analysis had been around for 150 years when Cooley and Tukey's paper appeared. The theory and many of the applications had been highly developed. But the relatively primitive state of the computational aspects of Fourier analysis limited what could be done in practice. The fast Fourier transform changed all that.

Partly at fault for the primitive state of Fourier computation was the utter conceptual simplicity of the transform itself. Stripped to its mathematical essentials, the digital process of computing a Fourier transform boils down to multiplying a vector of data by a matrix consisting of roots of unity. If there are N data points, then the entry in the hth row and kth column of the $N \times N$ "Fourier matrix" is $e^{2\pi ihk/N}$. One is hard pressed to think of a more elegant or more straightforward mathematical idea.

The problem is, every entry in the Fourier matrix is non-zero, which means that, in practice, the conceptually simple task of multiplying a matrix by a vector becomes an onerous chore involving a total of N^2 multiplications. If N is small, the prospect is not so bad. But when you're trying to work with large data sets, the N^2 slowdown becomes a computational bottleneck. Even at $N = 1000$, the amount of computation – $N^2 = 1{,}000{,}000$ multiplications – becomes a strain.

The fast Fourier transform is a simple and elegant way of streamlining this laborious computation. As is often the case with simple and elegant methods, many of the ideas underlying the fast Fourier transform had been anticipated by others, including, in this case, Gauss (back in 1805 – even before Fourier), but it was Cooley and Tukey who turned the technique into a staple of scientific computing.

In essence, they realized (as others had before them) that the straightforward approach to the Fourier transform had the computer doing the exact same multiplications over and over – a process that can be likened to a store clerk making a separate trip to the stockroom for each and every item in an order. By organizing the computation in an efficient manner that took advantage of the algebraic properties of the Fourier matrix, Cooley and Tukey found that they could eliminate almost all of these redundant calculations. The saving achieved by their algorithm is staggering.

Mathematically, the fast Fourier transform is based on a factorization of the Fourier matrix into a collection of sparse matrices – matrices in which most of the entries are equal to zero. The process is easiest to explain, and implement, when N is a power of 2, such as 1024. It begins with a factorization that reduces the Fourier matrix of size N to two copies of the Fourier matrix of size $N/2$. This reduction continues for lg N steps ("lg" standing for the base-2 logarithm), until the original matrix is written as a product of 2lg N sparse matrices, half of which can be collapsed to a single permutation matrix. (The diligent reader is invited to try his or her hand at working out the details.)

The upshot of all this is that a computation that originally required N^2 multiplications can now be done with only Nlg N multiplications. (The number of additions is likewise much smaller.) For $N = 1024$, that's a hundredfold less computation. Moreover, the speedup factor actually gets better as the problem gets larger! (This is one of the ways in which mathematical improvements can outpace other approaches to efficiency.) For example, building a new computer that runs a hundred times faster than the one it replaces is fine, but then all you've got is a hundredfold speedup. Faster algorithms, on the other hand, often post ever bigger gains on ever bigger problems.

Sometimes theoretic improvements in the efficiency of an algorithm remain at the level of theory, but that hasn't been the case with the fast Fourier transform. Every computer that does numerical computation on a large scale runs some version of the Cooley-Tukey algorithm. The fast Fourier transform was seized upon by researchers interested in everything from signal detection in radar systems to the assessment of heart valve damage by biomedical instrumentation. One measure of the impact of Cooley and Tukey's algorithm can be found in the *Scientific Citation Index*; Their 1965 paper is still cited anywhere from 50 to 100 times each year.

What all the applications have in common is huge, sometimes mind-boggling amounts of data. Moreover, for many applications, the data must be dealt with in something close to real time. It doesn't do much good to detect an incoming missile if the signal is processed after the warhead detonates, nor is a cardiologist's report that arrives on the day of the wake of any avail. (Even more Fourier-intensive are such imaging techniques as CAT scans and MRI. Researchers have also applied Fourier analysis in devices that monitor blood flow by means of an ultrasonic Doppler shift, sort of like a vascular radar gun.)

"Everywhere the Fourier transform is used, the fast Fourier transform can do better," says Ingrid Daubechies, an expert on the mathematics of signal processing at AT&T Bell Laboratories and Rutgers University. (Daubechies's specialty, wavelets, may take over part of the burden now shouldered almost exclusively by the fast Fourier transform, but it won't supplant it entirely. In some regards, the potential importance of wavelet theory is built on the success of the FFT.) "Computations that would have taken years can be done in seconds" with the fast Fourier transform, she explains.

The FFT was put right to work by Lee Alsop, a geophysicist at IBM and Columbia University. Alsop analyzed the seismographic record of a 1965 earthquake in Rat Island, Alaska, using 2048 data points representing a 13.5-hour period. A conventional Fourier transform took more than 26 minutes to do the analysis; the FFT spit out the answer in 2.4 seconds. Moreover, by running tests with numerically generated data, Alsop found that the fast Fourier transform was not only faster, but also more accurate than conventional methods.

A 1966 paper, "Fast Fourier Transforms for Fun and Profit," by Morven Gentleman and Gordon Sande, put Alsop's observation on a firm footing. Gentleman and Sande showed that, on average, the fast Fourier transform error was lower by a factor of $N/\lg N$, which was also the speedup factor. Their paper explored many other aspects of the FFT as well, including the computation of convolutions, which is at the heart of digital signal processing.

Cooley takes pains to praise the Gentleman-Sande paper, as well as an earlier paper by Sande (who was a student of Tukey's) that was never published. In fact, Cooley says, the Cooley-Tukey algorithm could well have been known as the Sande-Tukey algorithm were it not for the "accident" that led to the publication of the now-famous 1965 paper. As he recounts it, the paper he co-authored with Tukey came to be written mainly because a mathematically inclined patent attorney happened to attend the seminar in which Cooley described the algorithm. The lawyer, Frank Thomas, "saw that it was a patentable idea," Cooley explains, "and the policy of IBM and their lawyers was to be sure that nobody bottled up software and algorithms by getting patents on them." A decision was quickly reached to put the fast Fourier transform in the public domain, and that meant, in part, publishing a paper.

The publication of the Cooley-Tukey algorithm brought on an explosion of activity, as researchers from all branches of science rushed to apply the new technique to their own computational problems. One of the most spectacular applications – at the time – was an FFT computation of a sequence of 512,000 data points taken from interferometer measurements. This computation, done using a program designed by Norman Brenner at MIT, enabled Janine Connes, an astronomer at the University of Paris, to calculate the infrared spectra of the planets for a book that has become a standard reference in the subject.

The main problem with Connes's computation was that the 512,000 data points exceeded what was then available in high-speed memory. Brenner's solution was to "schedule" the fast Fourier transform so that data could be moved in and out of auxiliary storage without paying too great a penalty. This technique, modified to new hardware and new computer architectures, is still very much in use.

The applications of the fast Fourier transform are continuing to expand, as computers themselves become more powerful and as researchers continue to refine the algorithm for particular problems and adapt it to new machine architectures. There are even hints of faster Fourier transforms to come. Steven Orszag of Princeton University and Eytan Barouch of Clarkson University, for example, are putting some new twists into the notion of the FFT in order to carry out the particularly large computations involved in large-scale simulations of computer chips. It would only be fitting if a fast Fourier transform designed its own efficient circuitry.

In the meantime, the fast Fourier transform is rapidly becoming part and parcel of the engineering curriculum. Textbooks on the subject are now appearing, and the technique is even being taught at the undergraduate level. Who knows? It could even show up someday in the business school syllabus. At the very least, it's a good model for what can be achieved with a little bit of organized effort. It's easy to find algorithms that work harder than the fast Fourier transform; it's hard to find any that work smarter.

SHORT COMMUNICATIONS

The Product Function of Two Continuous Functions is Continuous (Alternate Proofs)

by

Brian Mitchell
Student

Jonathan Picklesimer
Student

Dean B. Priest
Harding University
Searcy AR 72149

Dr. Dean Priest is Dean of the College of Arts and Sciences and Professor of Mathematics at Harding University, Searcy, Arkansas. He is a member of the CPR-MATYC which is a 30-member national task force for the American Mathematical Association of Two-Year Colleges. Funded by Exxon and the National Science Foundation, the task force will develop national standards for mathematics curriculum and instruction at two-year colleges.

Several years ago I restructured my Intermediate Analysis class using the the R. L. Moore method. It has been gratifying to see the number of original student proofs which have resulted, such as Gant and Priest (1975) and Baldwin and Alkire (1983).

This semester the class struggled more than usual with ε–δ proofs for sum/difference and product/quotient continuity theorems. Finally two students produced original results.

One proof provides a pedagogical short cut for teachers of elementary calculus and relies on

Lemma P. If the function f is continuous at the real number p, then f^2 is continuous at p.

Proof. Suppose f is a function continuous at the real number p. By usual, familiar lemmas, f is bounded on some neighborhood of p. So there exists a positive number δ_1 and a positive number B_f such that

$$|f(x)| < B_f \qquad \forall x, |x-p| < \delta_1.$$

Also, for $\varepsilon > 0$, there exists a δ_2 such that

$$|f(x) - f(p)| < \frac{\varepsilon}{B_f + |f(p)|} \qquad \forall x, |x-p| < \delta_2.$$

Choosing δ to be the minimum of δ_1 and δ_2 it quickly follows that

$$|f^2(x) - f^2(p)| \le (|f(x) - f(p)|)\,(|f(x)| + |f(p)|) < \varepsilon \qquad \forall x, |x-p| < \delta.$$

The second proof relies on the following rather obvious (but new to me)

Lemma M. Let ε be an arbitrary positive number. If M and N are real numbers such that $|M+N| < \varepsilon$ and $|M-N| < \varepsilon$, then $|M| < \varepsilon$.

Proof. Suppose ε is an arbitrary positive number and each of M and N is a number such that $|M+N| < \varepsilon$ and $|M-N| < \varepsilon$. The fact that $|M| < \varepsilon$ follows immediately from

$$2|M| = |2M| = |(M+N) + (M-N)| \le |M+N| + |M-N| < \varepsilon + \varepsilon = 2\varepsilon.$$

Theorem. If f and g are continuous functions at the real number p, then the product function fg is continuous at p.

Proof 1. Suppose the conditions of the theorem. By Lemma P the functions f^2 and g^2 are continuous at p. By the usual theorems on the sum and difference of continuous functions and the fact that $k \cdot f$ (k as constant) is continuous when f is continuous, the theorem easily follows using $fg = \frac{1}{2}\,[(f+g)^2 - (f^2 + g^2)]$.

Proof 2. Again, assume the conditions of the theorem and let ε be an arbitrary positive number. As in Lemma P there are δ_1 and δ_2 neighborhoods of p and bounds B_f and B_g such that

$$|f(x)| < B_f \qquad \forall x, |x-p| < \delta_1$$

and

$$|g(x)| < B_g \qquad \forall x, |x-p| < \delta_2.$$

Also there are δ_3, δ_4 neighborhoods of p such that

$$|f(x) - f(p)| < \frac{\varepsilon}{B_g + |g(p)|} \qquad \forall x, |x-p| < \delta_3$$

and

$$|g(x) - g(p)| < \frac{\varepsilon}{B_f + |f(p)|} \qquad \forall x, |x-p| < \delta_4.$$

Now let $\delta = \text{minimum}\ \{\delta_1, \delta_2, \delta_3, \delta_4\}$, $M = f(x)g(x) - f(p)g(p)$ and $N = f(x)g(p) - f(p)g(x)$. It is easy to show that

$$|M+N| < \varepsilon \text{ and } |M-N| < \varepsilon \qquad \forall x, |x-p| < \delta.$$

So using Lemma M it follows that

$$|f(x)g(x) - f(p)g(p)| = |M| < \varepsilon \qquad \forall x, |x - p| < \delta.$$

Thus, fg is continuous at p.

Finally, it follows readily from Lemma M and Proof 2 that $|N| < \varepsilon$. Thus, an interesting corollary is that there are constant multiples of f and g which can be made arbitrarily close on a neighborhood of p.

References

Baldwin, T. J. & Alkire, W. G. (1983). Two additional proofs of the countability of the rationals. *Journal of Undergraduate Mathematics, 15*(1), 9-11.

Gant, K. (student) & Priest, D. B. (1975). The countabililty of the rationals revisited. *The AMATYC Review*, *6*(3), 41-45.

Functional Fish and Other Mathematical Menagerie Members

by

Margaret Spurlin Willis
Piedmont Virginia Community College
Charlottesville VA 22902

Margaret Spurlin Willis is a professor of mathematics at Piedmont Virginia Community College where most of her teaching is in developmental mathematics. Her Ed.D. in mathematics education is from the University of Virginia. She is currently president of VMATYC.

How much license is a mathematics teacher allowed in trying to gain and keep the attention of the class as well as in trying to provide, for at least some of the students, memorable examples – examples which may provide a bit of humor but also an insight into the concept under discussion?

For my community college mathematics students, I have used seemingly silly sayings, simplifications, and nonmathematical situations in this regard. Some of these have contained references to mathematical menagerie members such as pink elephants, Rene the Bat, an invisible rabbit, or functional fish.

Dependent upon the concept being presented, these aids have been used in classes ranging from arithmetic to precalculus. Although the classes vary in content level, they possess a quality common to many community college mathematics classes – that of containing students who are in one or more of several dichotomous groupings – those who need only a review of the material/those who are having their first exposure to the material; those who have just graduated from high school/those who have been away from an educational setting for a number of years; those with a high maturity level/those with a very low maturity level. The reaction to and usefulness of these aids depend upon the combination of groupings within a class. Overall, the response to the various aids has been positive. However, even a negative response within a class can provide the opportunity for discussion of alternative ways of viewing either the particular concept or of mathematics in general.

The statement, "Pink elephants must order donuts and orange sherbet" aids students in remembering order of operations as follows:

1. By using the first letter or letters of each word in the sentence to remember specific order:

   pink: parentheses and other grouping symbols;
   elephants: exponents or radicals;

<u>m</u>ust:	multiply;
<u>o</u>rder:	or;
<u>d</u>onuts:	divide;
<u>a</u>nd:	add;
<u>o</u>range:	or;
<u>s</u>herbet:	subtract.

2. By having the <u>or</u>'s to help remember that precedence is not explicitly given multiplication over division nor addition over subtraction.
3. By having the word <u>order</u> in the sentence to remember that the saying indicates order of operations.

In the introductory sections on graphing on the rectangular coordinate plane, Rene the Bat gets to fly. Since Rene, named for Rene Descartes, likes a challenge, he will fly only horizontally or vertically, will land only if he has tried to fly horizontally before flying vertically, and will return to his cave of origin between landings. Rene is also used in naming quadrants: Rene is feeling extremely negative today. Therefore, both his horizontal and vertical moves are negative. In what quadrant will Rene land? If even a few students can better visualize the movements in the coordinate plane because of Rene the Bat, then its use is worthwhile.

When I walk into math class with a black top hat, the usual question is, "Do we get to see a white rabbit?" The answer is "Sorry, the rabbit's invisible." However, when patterned handkerchiefs are pulled from the hat, the faces of the students show expressions of disbelief – "Am I really in a math class?" or curiosity – "What's she going to do now?" Does this situation have a value other than an attention getter? I hope so, because with the hat we talk about symmetry. Dependent upon how the hat is held, is it symmetrical with respect to a vertical or horizontal orientation, or neither? The patterned handkerchiefs are also used to discuss symmetry. Are the patterns symmetrical? Why or why not? If we could erase part of a pattern, would what is left be symmetrical? After the hat and the handkerchiefs, we look for symmetry in the patterns of the students' clothing. Although these students are community college students, I believe getting their attention and trying to give them a bit of personalized mathematics is important.

Functional fish are fish used to study functions. After a formal definition of function has been discussed, the students are told '' now get to hear a fish story which it is hoped will eventually be more functional than fishy.

Visualize a fish bowl with three goldfish in it. Could each of these fish be placed in its own separate fishbowl? The answer is yes. We could place fish-one in bowl-one, fish-two in bowl-two, fish-three in bowl-three: for each fish there is one and only one bowl (for each x there is one and only one y). Could all three fish be placed in one bowl? Yes, we had this at the beginning with fish-one in bowl-one, fish-two in bowl-one, fish-three in bowl-one: for each fish there is one and only one bowl (for each x there is one and only one y). Now, what about placing a fish in more than one bowl at the same time? The bowls must be separate and one bowl must not be placed inside another. Under these conditions, could fish-one be placed in bowl-one and at the same time be placed in bowl-two or bowl-three? The answer could be yes if we dismember poor fish-one and place different parts of it in separate bowls, but would we want to? No! We want a <u>functional</u> fish. And, to have a <u>functional</u> fish we can have that fish going to only one bowl: for each fish there is one and only one bowl (for each x there is one and only one y). Notice that one bowl is allowed to have more than one fish and that the fish can stay functional (each y is allowed to have more than one x and the relationship can stay functional). According to student comments, functional fish have helped them internalize the function concept and visualize the restrictions on x in the function definition.

To all the above mathematical menagerie members it could be said, "But that's not math!" My response would be "That depends upon how they're used and if they work." If they are used in a mathematically correct manner and if they aid in gaining student attention in the mathematics classroom, furthering student understanding of mathematical concepts, and prolonging student retention of mathematical material, then my desire would be to keep expanding the mathematical menagerie.

Lucky Larry #14

Larry used a selective multiplication technique to solve this equation:

$$\frac{3x}{x-4} = 5 + \frac{10}{x-4}.$$

Larry multiplied the parts that bothered him by $x - 4$ in order to clear fractions.

$$\frac{3x}{\cancel{x-4}} = 5 + \frac{10}{\cancel{x-4}}.$$

This left him with

$$3x = 15$$

$$x = 5.$$

Submitted by Dona Boccio
Queensborough Community College
Bayside NY 11364

I find the great thing in this world is not so much where we stand as in what direction we are moving.

Oliver Wendell Holmes

MATHEMATICS EDUCATION

The Effect of Cooperative Learning in Remedial Freshman Level Mathematics

by

Carolyn M. Keeler
University of Idaho
Boise Graduate Center
Boise ID 83712

and

Mary Voxman
University of Idaho
Moscow ID 83843

Carolyn M. Keeler is Assistant Professor in the Department of Educational Administration at the University of Idaho. She has been teaching research design and computer statistical analysis at the Boise Graduate Center in Boise, Idaho for four years.

Mary Voxman is a Senior Instructor in the Mathematics and Statistics Department at the University of Idaho in Moscow, Idaho. She has been teaching first and second year math courses for twelve years.

Abstract

This study investigated whether a cooperatively structured class where college students worked together in groups would result in greater retention of students in the course and would produce higher final exam scores and grades than the traditional lecture class. The study was conducted in two sections of a remedial mathematics course, Math 50, Pre-college Algebra, taught by the regular, full time senior instructor. All standards, such as assignments, six tests, and final exam, were identical in the two sections. The subjects included both traditional and nontraditional students. The cooperative learning section performed better on the final exam and received higher final grades than the lecture section. Students completed a questionnaire of group process and generally felt that group members were cooperative and communicative and indicated a desire to work in this type of group format again. Future research using alternatives to the lecture format in college classes is needed.

Background

Cooperative learning is receiving increased coverage in educational journals. *Educational Leadership* (Special Issue, 1989) dedicated an entire issue to articles exploring the positive aspects of various models of cooperative learning and The National Council of Teachers of Mathematics has recently promoted the use of cooperative classroom structures through two Standards documents (NCTM, 1989, 1991). Some researchers investigating the use of cooperative structures in college courses have found that cooperative pair activities have resulted in improved performance in problem-solving, on quizzes, and on tests (Barrall & Axelrod, 1978; Dees, 1991; Keeler & Steinhorst, in press). Other studies have resulted in findings that students learn just as well in cooperative settings as in traditional classes and develop a more positive attitude toward mathematics (Beach, 1974; Brechting & Hirsch, 1977; Davidson & Kroll, 1991; Shaughnessy, 1977). LeGere (1991) reports that students found math less threatening, became more engaged in learning, and used more high level skills in solving problems when they worked in groups. In another study set in the college algebra classroom, students' responses on an attitude questionnaire indicated that they felt the group to be beneficial in solving homework problems which presented some difficulty and that the group experience provided them with the ability to answer related questions on the exam (DeMarois, 1991). One difficulty encountered by the instructor was reverting to a lecture format in response to a desire on the part of students to be passive in the learning process.

This propensity to teach as we have taught, or have been taught, is one of the largest barriers to changing instructional practices. Most importantly, we continue to instruct future teachers using a lecture format even when teaching alternative strategies. In contrast to this practice, a cooperative lab approach was used in a study set in one section of a college mathematics course for elementary teachers (Weissglass, 1977). A second section of the course was taught in the lecture format normally used for instruction. The increase in math skills and information in a pretest-posttest design was larger for the experimental group although the difference was not statistically significant.

In the study being reported here, the purpose was to investigate whether a cooperatively structured class where students work together in groups would result in greater retention of students in the course and would produce higher final exam scores and grades than the traditional lecture class. The study was conducted in a remedial mathematics course taught to incoming freshmen, a population at the university which includes both traditional and nontraditional students. In addition to performance data, a Student Questionnaire of Group Process was administered in the cooperative section to measure how students perceived their group to be functioning. A minimal level of teacher intervention was used with little increase in the paper work required.

Method

Design

The study was conducted using two sections of Math 50, Pre-college Algebra, a review of algebra including factoring, rational expressions, exponents, radicals, quadratic equations, and equations of lines. Both sections were taught by the same senior instructor. The students were assigned to the section based on their requested schedule. There was no announcement prior to the start of the semester about the cooperative learning model. The major source of invalidity in this nonequivalent control group design is the nonrandom selection of subjects (Campbell & Stanley, 1963). To assess initial differences and ascertain the lack of any selection bias, the two participating sections were given a pretest over preparatory math skills and course content and compared on background characteristics that could be assumed to result in differential responses to treatment conditions. The two sections did not score significantly different on the pretest and were not different as to the students' ages, gender composition, math backgrounds, or major fields of study.

Setting and Subjects

The setting of this study was a university established as a land grant research institution located in a rural area of the Northwestern United States. The subjects included all students enrolled in two sections of Math 50. The enrollment in the cooperative approach was limited to 34 by the size of the assigned room and the enrollment in the traditional lecture approach was 42. The majority of these students were freshmen taking the course to make up a math deficiency. Students who do not score high enough on the SAT, ACT, or math placement exam must pass Math 50 to get into any core math course which is part of the general undergraduate requirements.

Students taking Math 50 can be separated into three categories. The first group is made-up of returning students who have been out of college for three to twenty years or more. Generally this is a highly motivated group of people but it is common to find math anxiety and fear among these students. Nevertheless, they are usually successful as they keep up with the presentation of material, do homework assignments, and get help when needed. The second group of students are those who have recently graduated from high school and need the course as a review. These students generally have sufficient study skills to successfully finish the course. The third group consists of students who do not have the background in math or the study skills to be successful and who also suffer from math anxiety or hatred of math or both. These students often have not figured out why they are in college and consequently are the hardest students to help and comprise the largest proportion of students who fail the course. Many students in this group skip class regularly or quit coming completely; if they remain enrolled they hand in homework only sporadically, do not seek help, and often come to class only for tests.

General Procedures

The same text and materials were used, all assignments were the same and the same tests were given to each section. Students in both sections had one test score dropped before the final average was calculated. In addition, the use of a math assistance center was promoted where tables are labeled for each course and students are encouraged to study together and ask for assistance from the lab monitor.

Procedures in Traditional Lecture Section

The class was held three times a week in a 50 minute session Monday, Wednesday, and Friday. The instructor used a traditional lecture approach. The first 20 minutes of each class was spent on discussing the homework. This was followed by a lecture on the new material with many examples given to illustrate new concepts.

Procedures in Cooperative Learning Section.

Students were allowed to form pairs by self-selection, usually by friendship, proximity, or similar program of study. Groups of four students assigned to work together, called quads, were then formed by the instructor on the basis of the pretest. A student who did well on the test and one who did poorly was assigned to each quad. If they had a partner from the self-selection process, their partner was also included in the quad. In this way, the groups were made as heterogeneous as possible by math performance. The students were given a description of group activities which explained the function of the quads. Some groups were reformed at midterm on the basis of a student questionnaire rating group process. There were also a few groups of three formed.

The first five to seven minutes of each class was spent working in quads on two to four of the more challenging problems from the assigned homework. It was believed that explaining concepts and solutions to group mates would clarify information in all members' minds and better prepare students for the next concept to be presented. In the next 15 minutes, groups were chosen at random to present the solution to each of these problems on overhead transparencies prepared by the instructor. Over the term, each member of the group took a turn presenting. Discussion of the solutions and the concepts under study followed. The remainder of the session was dedicated to the presentation of new material by the instructor through lecture and the use of examples as in the lecture format class.

Quad members were expected to assist each other with questions on the homework and support the others' learning. Quads were encouraged to form group study sessions outside of class. This was not required or monitored. A group reward structure was used to promote group interaction and helping behavior among members. Research has shown that group rewards increase the motivation to assist everyone in the group to learn the material (Webb, 1991). This form of positive interdependence among quad members was created by the addition of bonus points for group performance on each exam. Bonus points were awarded to each group member on the following basis:

8 points – If at least one group member scored in the nineties, the group average was in the eighties, and no one scored below 60.

6 points – If at least one group member scored in the eighties, the group average was in the seventies, and no one scored below 60.

4 points – If the group average was in the seventies and no one scored below 60.

Assessment of Student Variables

Background information. Information on age, gender and math background was collected through an assignment given in the first session to write a math autobiography. These were collected in the second session of the course.

Group functioning. At midterm, and again at the end of the semester, students were asked to rate group processes such as communication, cooperation, and assistance in completing the homework assignments. Positive interdependence was built into the model by awarding bonus points for group performance on exams; therefore, students were asked to rate their degree of concern with how well group members would perform on each exam. Students were also asked to rate the degree to which they would like to work in this type of group format again.

Performance data. The grades of students on several classroom tasks were used as performance indicators. The five highest scores out of six tests, homework points, and final exam scores were combined to make up final course averages. The scores on the final and the final course averages were the basis of analysis.

Data Analysis

The possible inflation of final grades due to the awarding of bonus points was explored. Although only two letter grades were inflated due to the additional points, the changes in final course averages were more pervasive. Therefore, the analysis of final grades was conducted on average total points without bonuses.

Results

Comparison of Student Performance Data

An analysis of variance (ANOVA) was conducted on final exam scores and final course averages by treatment. Mean final exam scores and final course averages are presented in Table 1 for the cooperative learning section and the lecture section. The cooperative learning section performed better on the final exam and received higher final grades than the lecture section although neither difference was statistically significant. The difference in final course averages indicated a trend toward higher grades in the cooperative learning section $[F(1,59) = 2.79, p < .10]$.

Table 1. Mean Final Exam Scores and Final Course Averages by Treatment

	Final Exam Score		Final Course Average	
Treatment	Mean	S.D.	Mean	S.D.
Cooperative Section	137.2	34.5	75.5	12.0
Lecture Section	131.9	32.8	70.7	13.1

Distributions of the students' final course averages for the cooperative learning and lecture sections are shown in Figure 1. A comparison of the two distributions illustrates the proportion of students who failed the course in the lecture section and the greater number of students with higher averages in the cooperative learning section. The results in the lecture section also illustrate the large number of students with scores around 70. The key in each distribution has a count of students who did not withdraw but did not complete the course; they are listed as "missing."

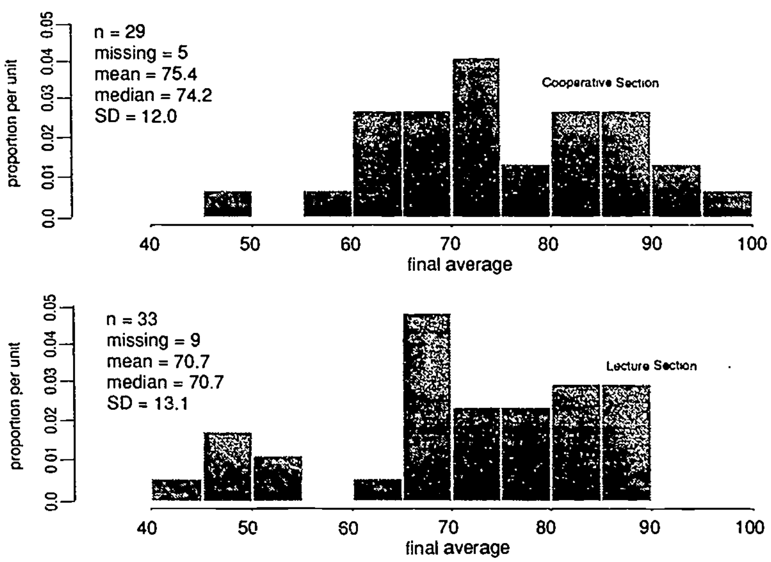


Figure 1: Final grade distribution

The most pronounced effect is the contrast between the number of students who successfully completed the course. In the lecture class, 12 students received an F, either earned or through absence, versus two in the cooperative class. The result is an overall pass rate of 93% in the cooperative section versus 69% in the lecture section. Inspection of the final grade distribution by treatment, shown in Table 2, reveals that another effect was the higher percentage of students receiving A's and B's in the cooperative learning section, 43% compared to 28% in the lecture section. The table reflects the elimination of earned bonus points from the cooperative learning group which dropped one grade from a B to a C and one grade from a C to a D.

Table 2. Grade Distribution by Treatment

	Grade						
Treatment	A	B	C	D	F_e	F_q	W
Cooperative Section (*N* = 34)	6	8	8	6	1	1	4
Cooperative w/o Bonus (*N* = 34)	6	7	8	7	1	1	4
Lecture Section (*N* = 42)	4	7	15	1	6	6	3

F_e Earned F; F_q Quit attending but did not officially withdraw, received grade of F.

Comparison of Number of Students Successfully Retained in the Course

The hypothesis whether more students would be retained in the course with the use of the cooperative structure or with the traditional lecture structure was tested by comparing the number receiving grades with the total enrollment or *N*. In the cooperative section, 85% of the students completed the course and in the lecture section 79% of the students completed the course. The data support the expectation that more students would remain in the course, continue to attend and participate in class activities, and complete the requirements if they were cooperatively involved with their peers.

Student Assessment of Group Process

Students completed an anonymous student questionnaire which was kept confidential. The midterm results were used to monitor how the groups were functioning and make some adjustments in group membership. The end-of-term results give an indication of the students' satisfaction with the use of groups.

Students generally felt that there was good communication and cooperation among the group members. Most indicated that they would like to work in this type of group format again. Student comments to the open response questions were supportive of using groups. Suggestions for improvement included having group study times at the math assistance center, encouraging more communication between group members outside of class, and allowing more time for group problem solving session in class.

Discussion

The results of this study of the effect of cooperative learning groups on college students' academic performance and attrition in remedial math supported the expectation. Working in cooperative groups resulted in higher mean final exam scores and higher final course averages. The percentage of students receiving passing grades was much higher in the cooperative group than in the lecture group. The student retention in the course indicates that students will remain enrolled in a course and have a better probability of successful completion when they are involved in working cooperatively with their peers, support each other's learning, and encourage members of the group to participate in daily activities. Students generally reported that group members were cooperative and communicative and indicated that they would like to work in groups again.

The results of this study of the effect of cooperative learning groups on college students cannot be generalized past the present population of traditional and nontraditional entering freshmen. The results do extend our knowledge of the receptivity of college students to working in groups. Future research on larger samples, selected randomly from the college population, would be beneficial. However, research in actual classrooms rather than in laboratory studies of short duration should be emphasized. The finding that learning is maintained or improved slightly by group work while attrition is reduced may provide encouragement to college instructors to research cooperative learning concepts in their courses.

References

Barrall, M.E. & Axelrod, S. (1978). The effects of learning partners and retests on pretest-posttest scores, final course grades, and student attitudes. *Research in Higher Education, 8*, 177-187.

Beach, L.R. (1974). Self-directed student groups and college learning. *Higher Education, 3*, 187-200.

Brandt, R.S. (Ed.). (1989). Cooperative learning [Special issue]. *Educational Leadership, 47*(4).

Brechting, M.C. & Hirsch, C.R. (1977). The effects of small-group discovery learning on student achievement and attitudes in calculus. *The AMATYC Review, 2*, 77-82.

Campbell, D.T. & Stanley, J.C. (1963). Experimental and quasi-experimental designs for research. Boston: Houghton Mifflin.

Davidson, N. & Kroll, D.L. (1991). An overview of research on cooperative learning related to mathematics. *Journal for Research in Mathematics Education, 22*(5), 362-365.

Dees, R.L. (1991). The role of cooperative learning in increasing problem-solving ability in a college remedial course. *Journal for Research in Mathematics Education, 22*(5), 409-421.

DeMarois, P. (1991, January). *A student centered approach to teaching college algebra.* Paper presented at the meeting of the MAA/AMS Joint Mathematics Conference, San Francisco.

Keeler, C.M. & Steinhorst, R.K., (in press). Cooperative learning in statistics. *Teaching Statistics.*

LeGere, A. (1991). Collaboration and writing in the mathematics classroom. *Mathematics Teacher, 84*(3), 166-171.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: The Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (1991). *Professional standards for teaching mathematics.* Reston, VA: The Council of Teachers of Mathematics.

Shaughnessy, J.M. (1977). Misconceptions of probability: An experiment with a small-group, activity-based, model building approach to introductory probability at the college level. *Educational Studies in Mathematics, 8*(3), 295-316.

Webb, N.M. (1991). Task-related verbal interaction and mathematics learning in small groups. *Journal for Research in Mathematics Education, 22*(5), 366-389.

Weissglass, J. (1977). Mathematics for elementary teaching: A small-group laboratory approach. *American Mathematical Monthly, 84*, 377-382.

Writing to Learn Mathematics: Enhancement of Mathematical Understanding

by

Aparna B. Ganguli
General College, University of Minnesota
Minneapolis MN 55455

Aparna B. Ganguli is an Assistant Professor of Mathematics in the School of Mathematics, Institute of Technology, University of Minnesota. Her research interests are in the areas of use of the microcomputer as a teaching tool and use of writing in teaching mathematics. She has published papers in both areas in journals such as Journal for Research in Mathematics Education, College Teaching, Collegiate Microcomputer and Educational Studies in Mathematics.

Abstract

Twenty-five college students enrolled in an intermediate algebra class were asked to complete five-to-seven-minute in-class writing assignments about mathematical concepts, procedures for solving problems, etc., along with solving algebraic problems in a ten-week lecture class. Analysis of the written responses at the beginning and at the end suggests that writing assignments helped remedial students to think mathematically.

It is well documented that the need for remediation in mathematics is the most common problem at public colleges (Albers, Anderson, & Loftsgaarden, 1987) and that the number of college students who enroll in remedial mathematics classes has increased extensively over the past decade (Wepner, 1987). These college students face severe difficulty in mathematics because of their lack of understanding of mathematical concepts. According to Howson and Wilson (1990), this is not too surprising if one considers the way so much mathematics has been taught in the past. Most generally, in their history of learning mathematics, these students were asked to work out the problems and get the right answers by manipulating numbers only. They view mathematics as a very difficult subject with rules and algorithms which are meaningless to them. According to Buxton (1981), mathematics is associated with a strong sense of failure for many students who remember mainly tests and examinations and the fear of "getting it wrong."

In traditional mathematics classes, students were seldom given the opportunity to express their thoughts and thus clarify their understanding of mathematics. One important component of learning mathematics — how to think — was never emphasized for these students. They may need an alternative teaching strategy which will foster real thinking rather than learning by rote.

Fulwiler (1982) believes that writing is a valuable learning tool and that it promotes thinking. Rose (1989) points out that writing down mathematical concepts, processes, and applications helps students to record their understanding. The Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989) states: "The simple exercise of writing an explanation of how a problem was solved not only helps clarify a student's thinking but also may provide other students fresh insights gained from viewing the problem from a new perspective" (p. 142).

The study reported here was conducted in order to introduce changes in the traditional way of teaching mathematics in a college remedial mathematics class. The purpose of the study was to examine the effect of regular writing-to-learn mathematics activities on the development of students' mathematical understanding.

Method

Subjects

Twenty-five college students enrolled in an intermediate algebra class participated in the study. The author was the instructor of the class and conducted the study.

Procedure

The study was conducted during spring quarter 1991, a ten-week instructional period at the University of Minnesota. Students met three days a week for lecture and two days for recitation. A survey questionnaire was administered at the beginning of the course to find out whether or not the students were familiar with writing-to-learn activities in mathematics.

Students were not given any writing assignments during the first twenty lectures of the quarter. However, in each of the last ten lecture sessions they were given a five-to-seven-minute in-class writing assignment. Samples of writing assignments appear in the appendix.

Students' writings were collected, graded on a ten-point scale, and returned with appropriate comments. The scale was developed using the following criteria:

no points for not completing the assignment
2 points for some answer attempted, problems noted by student (e.g., "I'm confused")
4 points for a mainly mathematical response
6 points for some answer attempted with some explanation
8 points for some explanation with correct mathematical response
10 points for very good written response with correct mathematics

The students were informed that if the number of points earned on writing assignments totalled higher than one of the two midterm scores, the lower midterm score would be replaced by the total writing assignment score. Students' understanding of algebraic concepts before and after the integration of writing assignments were compared.

Results and Discussion

Analysis of the survey questionnaire administered before the experiment indicated that 92% of the students (23 out of 25) had never had any writing assignments in their previous mathematics class, only 20% (5 out of 25) indicated that they thought writing assignments in a mathematics class were a good idea, and 100% were willing to try "writing to learn mathematics" if it might help them to get good grades.

Samples of student writings at the beginning of the experiment indicated that many basic concepts were not clear to the majority of the students. In the first assignments, only 20% of the students showed by writing that they understood what a linear equation is. Their most common mistake was to say that a linear equation has only one variable. The four students who had partially correct answers at the beginning responded as follows: "Linear equation is a line" (2 students); "Linear equation represents a line;" and "The graph of a linear equation is a line."

Although the teacher stressed in her lecture that the exponent of the variable is critical in deciding the nature of the graph of an equation, none of the students mentioned that point in the first writing assignment. Exponents of the variable were mentioned by some students, but in many cases their writings did not make any sense. Lack of understanding of mathematical terms and concepts is clearly demonstrated in the following examples of student writings: "Linear equation — an equation with one or more exponents — to the first power;" "Linear equation — its exponent is only one variable;" "Linear equation — an equation from a line with the variable equal to one."

Clearly students did not know meanings of any of the terms they had used in their writings. These responses inspired the teacher to make an extra effort in conducting this experiment. The goal was that through these writing activities "students would develop an internal method of learning and evaluating their own understanding and then finally become their own teacher" (Keith, 1991).

Students started to make sense out of the mathematical terms after integration of writing. Many students for the first time understood the proper meaning of terms such as "coefficient," "exponent," and "linear equation" as they wrote. It was well documented that after integration of writing fewer and fewer students wrote inadequate meaningless responses. At the end of the quarter, 100% of the students gave acceptable mathematical answers to the question, "What is a linear equation?", as follows: "A linear equation is an equation in which the highest exponent of the variable is one" (40% — 10 out of 25); "In a linear equation the power of the variable can be only one" (48% — 12 out of 25); "In a linear equation the power of x is not greater than one" (12% — 3 out of 25). In a later discussion it was clear that the students did not just memorize this definition but actually internalized this concept (see teacher's log).

Another observation from the first written responses was that although such concepts as intercepts, slopes, and excluded value of a rational expression were discussed at length in class and students did many homework problems on them, the majority of the students did not internalize these concepts. Students' writings, in the beginning, proved that students did not think mathematically. After introducing undefined rational expressions, a writing assignment for the purpose of brainstorming was given to the students (see Figure 1). The expectation was that the students would apply their previously learned knowledge to analyze a new problem situation. Only one student (4%) wrote about the value of x where the rational expression could be undefined, while another student (4%) wrote about the x-intercept where the numerator must be zero. Some of the students resented these "new" questions. They expected to solve only problems similar to those they had already been shown how to solve. For example, one student wrote, "I don't remember ever studying this in class. Is this a fair question? I don't think so." Another wrote, "I do not understand this assignment, material never covered, can't figure it out."

After the graded assignments were returned, discussion indicated that students

The curved lines of the following graph represent the rational expression $y = R(x) = \frac{Q(x)}{P(x)}$. The dotted lines indicate that they are *not* part of the graph of $R(x)$. That means the lines $x = 1$ and $R(x) = 1$ are excluded from the graph of $R(x)$. Write a story about $R(x)$. The story must contain five complete sentences about $R(x)$.

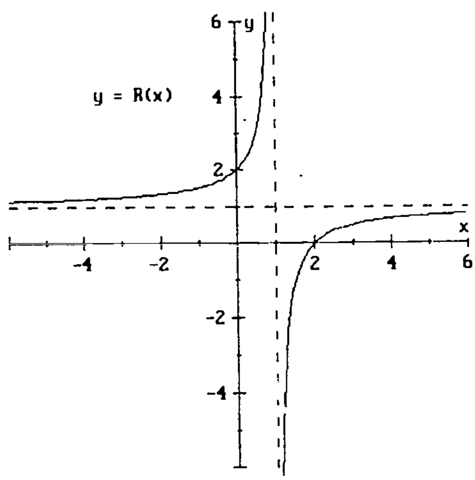


Figure 1: Sample Problem for Brainstorming

weren't attempting to apply previously learned concepts to analyzing the new problem situations. When it was pointed out that on the graph indicated, the curve passed through two points (0, 2) and (2, 0), many things became clear to the majority of the students. During the discussion, students speculated that $Q(x)$ might be equal to $x - 1$, since $x = 1$ is excluded from the graph, and that the point (2, 0) indicated the numerator $P(x)$ could be equal to $x - 2$. Students were thrilled to discover that the possible rational expression might be $\frac{x-2}{x-1}$. The teacher did not contribute a single thing after pointing out that the points (2, 0) and (0, 2) were on the curve. She listened to the students quietly and entered the following excerpt in her log book that day:

R: Oh! That means when $x = 2$, then $y = 0$.

E: That means $R(x)$ can be zero when $x = 2$.

H: Does that mean $R(2) = 0$?

B: Oh! That means $R(x)$ can be zero if $x - 2 = 0$.

R: Yeah! That means $R(x)$ is $x - 2 = 0$.

M1: No. That means $R(x)$ is $x - 2$.

R: But the graph is not a line. Power of x is 1 (if it is $x - 2$).

M2: That's right. $R(x)$ has to be a fraction.

(Teacher's comment: Nice! Seems everybody knows ... the power of x is 1 in a linear equation and the graph has to be a line if it is $x - 2$.)

W: But we don't know anything about the denominator.

F: Well, then what is the denominator?

(Silence for a while).

R: No idea.

W: If it has a denominator, then it could be undefined.

M1: $x = 1$ and $R(x) = 1$ are excluded from the graph. What do these mean?

R: Umm . . . does that mean x and y can never be 1?

F: Does that mean they are undefined at 1?

R: I know. That means $R(x)$ is undefined if $x = 1$.

F: That's right. So what is the denominator?

M1: The denominator could be $x - 1$.

R: Yeah!

H: That means $R(x) = \frac{x-2}{x-1}$.

M1: Really?

(Teacher wrote: Oh, what a discovery!)

Then the sophisticated discussion centered on why $R(x) = y = 1$ was excluded from the graph. The students thought intensively. Time ran out and the question was reassigned as an out-of-class activity. Most of the students rewrote and resubmitted their answer. Three of them — students R, M1, and W — actually explained in writing why $R(x) = y = 1$ was excluded from the graph. Many expressed informally that they had really enjoyed this assignment. The most common response was that they understood how previously learned concepts such as x-intercept, y-intercept, and undefined rational expressions could be applied in this assignment when they rewrote it.

After completing five writing activities, consulting, and then rewriting, the majority of the students demonstrated an enhancement of mathematical understanding as evidenced from the analysis of their writing assignments at the end of the quarter. Students no longer complained if the problem in the assignment had not been previously encountered.

During the last day of instruction, students were asked to respond to the question, "Did these writing assignments help you to understand mathematics better?" All 25 students answered positively. A typical response was, "It helps us understand the meanings of what we are doing and not just actually the formula. This way we sometimes can see why a formula is used."

Conclusion

Analysis of the written responses indicates that writing assignments may have helped remedial students to think mathematically. When the students were asked to write down a simple mathematical concept on paper, they recognized and realized the shortcomings of their understanding. After the writing assignments were collected, many students opened their books or lecture notes to try to find out whether or not they knew the answer.

One student reported that only after discussion of her writing assignment on solving rational equations and simplifying rational expressions did she understand her mistakes. She had always multiplied by the Lowest Common Denominator (LCD) and got rid of the denominators, not knowing the concept behind the process. Another student reported that the concept behind the process of rationalization of the denominators was not clear to her until she tried to write it down.

The same students who were resentful at first came forward later and said that they had enjoyed the assignments because they helped the students to evaluate their mathematical understandings.

It was clear from this author's experience that these students needed to be exposed to activities that challenged them to think about mathematical concepts rather than merely solve routine algebraic problems. The writing-to-learn-mathematics teaching strategy appeared to help students not only to reexamine their mathematical understandings but also to apply previously learned concepts to analyzing new problems.

References

Albers, D., Anderson, R., & Loftsgaarden, D. (1987). *Undergraduate programs in the mathematical and computer sciences: The 1985-1986 survey.* Washington, DC: Mathematical Association of America.

Buxton, L. (1981). Do you panic about mathematics? Heinemann.

Fulwiler, T. (1982). Writing as an act of cognition. In C. W. Griffin (Ed.), *New directions for teaching and learning: Teaching writing in all disciplines.* San Francisco: Jossey Bass.

Howson, G., & Wilson, B. (1986). *School mathematics in the '990s*. Cambridge University Press.

Keith, S. (1991). Review of writing to learn mathematics materials. Personal communication.

National Council of Teachers of Mathematics (1989). Mathematics curriculum and evaluation standards for school mathematics. Reston, VA.

Rose, B. (1989). Writing and mathematics: Theory and practice. In Connoly, P., & Vilardi, T. (Eds.), *Writing to learn mathematics and science* (pp. 15-30). Teachers College, Columbia University.

Wepner, G. (1987). Evaluation of a post secondary remedial mathematics program. *Journal of Developmental Education, 11*(1), 6-9.

Appendix: Samples of Writing Assignments

1. Define with examples: constant, variable, coefficient, exponents, linear equation.
2. A friend of yours enrolled in an algebra class panics, knowing that one of the requirements of the class is to simplify complex fractions. This friend wants to stay away from fractions, let alone complex fractions. Write a comforting letter to your friend explaining in detail what a complex fraction is, how negative exponents are involved in complex fractions, and how a complex fraction can be made simple. Use at least two examples in your explanation. Assume that your friend must rely on you for a complete explanation.
3. One of your friends missed the algebra class on Friday and doesn't understand synthetic division. Please write a note to this friend explaining synthetic division with appropriate example(s). Assume that your friend must rely on you for a complete explanation.
4. Describe the process of finding the numbers that make a rational expression $R(x) = \dfrac{P(x)}{Q(x)}$ undefined.
5. When can $R(x)$ be defined at all points?
6. Please look at the following problems and explain with complete sentences the steps involved to work out #1 and #2. Discuss the similarities and the differences between these two problems.

   #1. Simplify: $\dfrac{2}{x} + \dfrac{3}{x+2} - \dfrac{4}{5x}$

   #2. Solve: $\dfrac{2}{x} + \dfrac{3}{x+2} = \dfrac{4}{5x}$
7. Please write—in short, complete sentences — a definition of the most important concept we discussed today. Give examples as you discuss.

55

You Can't Do That!

by

Richard L. Francis
Southeast Missouri State University
Cape Girardeau MO 63701

Southeast
Missouri State University

Richard L. Francis is a professor of mathematics at Southeast Missouri State University, where he has taught since 1965. He received a B.S. degree from Southeast Missouri State University and master's and doctorate degrees (including post-doctoral work) from the University of Missouri (Columbia). His major scholarly interests are number theory and the history of mathematics.

Among the most frequent of words in the typical classroom hour are those of the teacher's emphatic declaration, YOU CAN'T DO THAT. Such is the response long characterizing encounters by mathematics teachers with persistent errors on the part of the struggling student. Actually, these mathematical shortcomings need not end in frustration and defeat. To the contrary, some of the most significant of classroom opportunities stem from a careful consideration of student errors. Accordingly, our task will be that of looking briefly at the clues these error types provide for effective instruction.

Though the teaching emphasis is built around correct mathematical procedure, it proves rewarding also to focus attention on just the opposite, namely, poor and totally incorrect techniques. By being an anticipator of errors, the teacher can call deliberate and specific attention to mathematical pitfalls. Greater student insight and proficiency may well be the outcome.

Errors, superficially quite diverse, are ordinarily not of a hodgepodge nature but fall instead into some mold or easily described category. The greatest emphasis in our account of frequent error types and appropriate categorizing will be that of a profound mathematical concept called the *homomorphism*. Yet, in spite of its frightening sound, a certain simplicity of pattern projecting or generalizing will emerge. Before examining this broad area of error types, others will first be mentioned and illustrated briefly with FAVORITE examples. Note in the process how seemingly diverse error types fit nicely under one description.

Improper Equation Solving Techniques

The student likely knows that if two numbers have a zero product, then at least one of these numbers is zero. This is essentially the strategy underlying the solving of equations by factoring. Still, a certain generalization, though faulty, is easily applied. Consider the equation $ab = 7$ and the equating of each of its factors to 7. All of this may seem reasonable to the beginner. It raises the question, "Can one obtain a product of 7 without using 7 as a factor?" Spending time on such an erroneous technique with the aim of enhancing student awareness appears promising. It may well head off difficulties as the student is called upon to solve equations similar to $(x - 2)(x - 3) = 12$. Once the matter is given sufficient emphasis, such extraordinary equations as $(3 - x)(x + 2) = 4$ can be explored. Here, the generally erroneous solving technique coincidentally produces a correct solution set. Likely many incorrect equation solving techniques come to mind, some involving matters more fundamental than those of the illustration above.

Illegal Cancellation

One is reminded here of Freshman's Theorem. It reads: "If two x's appear on the same piece of paper, they can be cancelled." It is easy to establish the incorrectness of replacing $\frac{2+3}{2+8}$ by $\frac{3}{8}$ as the right answer is so obviously $\frac{5}{10}$ or $\frac{1}{2}$. In algebra, the incorrectness may be hidden, as in $\frac{2x+3}{2x+8} = \frac{3}{8}$, until we check by substituting values for x. Proficiency in arithmetic is related to proficiency in algebra, but students need help in seeing the connection. The area of cancellation errors is broad. Note, for example, such easily generalized mistakes as the reduction $\frac{8!}{2!} = 4!$ or the replacement of $\frac{\log 3}{\log 5}$ by $\frac{3}{5}$.

Operations

Various categories of operational errors come to mind. These range over number groupings, radical simplification, notions of precedence, decimals, notational abuse, and exponential forms. And more! Consider the following:

$$24 \div 6 \div 2 = 8$$

$$\sqrt{2} + \sqrt{3} = \sqrt{5}$$

$$-3^2 = 9$$

$$.3^2 = .9$$

$$\frac{1}{2}x = \frac{1}{2x}$$

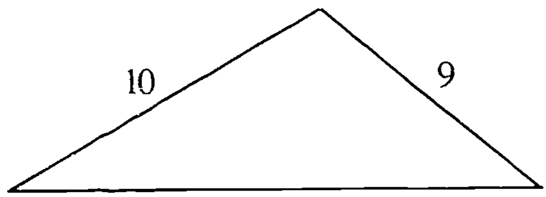
$$\frac{x^{10}}{x^5} = x^2.$$

Because of the mechanical, highly superficial features of computation, errors easily arise. These errors surface in the absence of insight as to "what's really happening" when numbers are combined.

Misapplication of Formulas

The formula is often the means by which the transition is made from arithmetic to algebra. Even so, it makes an appearance in the earlier years of formal mathematical training. Such relationships as $A = \frac{bh}{2}$ or $C = \pi d$ or $F = \frac{9}{5}C + 32$

fall into this earlier category. The matter thus proves to be a long-term stumbling block. In the more advanced setting, the Theorem of Pythagoras and the Quadratic Formula suggest areas of improper application. Assuming the relevance of the Theorem of Pythagoras to the triangle below (without actual verification one way or the other) leads to an incorrect listing of trigonometric function values of the angles. Likewise, an erroneous calculating of the area by writing $A = \frac{(9)(10)}{2} = 45$ (as opposed to the correct answer of 36 by Heron's Formula) is still another instance.



The Quadratic Formula also provides a recurring area of misapplication. Consider the solving of $x^2 + 5x = 6$ by regarding a, b, and c as 1, 5, and 6 respectively. In such a problem, not only are a, b, and c sources of difficulty, so too are the various signs. Most have witnessed the replacing of $-b$ by -2 when b itself is -2. Nor is the extent of the fraction line in the formula always dealt with properly. Other formula and definition errors [such as $(\log x)^n = n \log x$ or $|x| = x$] all come together to identify misapplication as a major area of teaching concern.

Notation

Mathematical notation permits the solver to write concisely what otherwise would be awkward and difficult. In its symbolism, mathematics seems to have found its distinguishing mark. Yet, difficulties arise in usage; that is, the message conveyed is not always the one intended. The few notational slip-ups below are elementary but appear quite frequently.

.25¢ for 25¢

$2x = 6 = 3$

$\sqrt{9} = \sqrt{3}$

$2 > x > 3$

$9^{1/2}$ versus $9\ 1/2$

$\sin x^2$ versus $\sin^2 x$

$\emptyset$ versus $\{\emptyset\}$

$\wedge$ versus $\vee$.

The mathematics teacher can easily extend this small sampling of notational errors to great length.

The calculus provides an interesting instance of careless notation. Often we abbreviate the repeated writing of the limit prefix by such forms as

$$\lim_{x \to 3} \left[\frac{x^2 - 9}{x - 3} = \frac{(x-3)(x+3)}{x-3} = x + 3 \right] = 6.$$

Yet in applying L'Hôpital's Rule, the student may wish to do much the same by writing

$$\lim_{x \to 3} \left[\frac{x^2 - 9}{x - 3} = \frac{2x}{1} \right] = 6.$$

Actually, the latter equates expressions which should not be equated, though the final answer is correct. Still, an element of incorrectness has somehow made an appearance in the solver's writing. Notational abuse appears all along the way in mathematics (both in the basic and in the advanced settings) and should be consistently guarded against.

Failure to Understand Units

Applied problems generally necessitate the use of units in expressing final answers. Such problems may also contain a variety of units in their formulation. To bring these components together correctly calls for a careful consideration of relationships.

Note the commonplace errors of which the following are typical:

1 square foot = 12 square inches, or

1 square yard = 3 square feet, or

1 cubic meter = 100 cubic centimeters.

Similar errors also arise when mixed units of length (and of areas and volumes) appear. Strategies in solving which involve formulas must also take into account restrictions placed on units. For example, the circular arc length and sector area formulas

$$s = r\theta \text{ and}$$

$$A = \frac{1}{2} r^2\theta$$

require that θ be expressed in radians. This often calls for conversion, a process easily overlooked. Conversion techniques provide a fertile field for error types. These involve such diversities as those extending from radian and sexagesimal measure to metric and English measure.

Picture Terms

The use of picture terms to describe mathematical procedures is well-established. Such words as *cancel*, *cross-multiply*, *remove parentheses*, *transpose*, and *invert* illustrate this nicely. Recall the earlier references to careless cancellation [and variations akin to $\sin x = \sin y$ implies $x = y$]. Should the student not understand the notions underlying the named processes, they become little more than mechanical, superficial activities. So, do we cross-multiply or do we multiply both members of an equation by the same number? Do we remove parentheses or do we apply a distributive law? Do we transpose ("move" numbers across the equal symbol with signs changed) or do we apply an addition-subtraction law? *The concern is not the fact that these well established picture terms are used, but that they are often misunderstood by the student;* they can create a variety of false impressions. Accordingly, such processes must be handled cautiously so as to convey proper meaning.

Another example is afforded by the word *superposition*. This word is to geometry what transpose is to algebra. Can geometric figures literally be "picked-up" and moved so as to make their corresponding parts coincide? Or, does this word, if not used cautiously, conceal the fact that geometric figures are abstractions? And then, there are mnemonic devices of which FOIL (a method of binomial multiplication) is a classic illustration. Picture processes, similar to FOIL, also have the potential of creating problems in understanding.

Homomorphisms

All algebra teachers have experienced difficulty with students who square $a + b$ by writing $a^2 + b^2$. Such warnings as "Don't forget the middle term" are likely the words of frequent reminder. The pattern underlying this error type is an easy one for students to identify and, in so many areas of mathematical study, apply improperly. Mathematicians describe this simple pattern by use of the not-so-familiar word "homomorphism."

Let $+$ and $*$ be binary operations on sets A and B. Further, let f be a rule associating each element of set A with a single element of B. We will call the rule f a *homomorphism* provided the following is satisfied for all x and y in set A:

$$f(x + y) = f(x) * f(y).$$

Since $f(x)$ is associated with x by the rule f, we will also call $f(x)$ the image of x. With a proper interpretation of the word "sum," the homomorphism statement may be read as "the image of the sum is the sum of the images." The operations $+$ and $*$ could be alike. Well known examples are those given below. Each is CORRECT, subject to a proper domain of interpretation.

$-(a + b) = -a + -b$	Additive inverse of the sum
$2(a + b) = 2a + 2b$	Double of the sum
$(ab)^2 = a^2b^2$	Square of the product
$\sqrt{ab} = \sqrt{a}\sqrt{b}$	Square root of the product
$\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	Absolute value of the product
$\log(ab) = \log a + \log b$	Logarithm of the product
$(f + g)' = f' + g'$	Derivative of the sum
$D(pq) = D(p) + D(q)$	Degree of the polynomial product
$\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$	Conjugate of the complex number sum
$(ab)^{-1} = a^{-1}b^{-1}$	Reciprocal of the product

These TRUE statements have the potential for suggesting similar relationships which are FALSE. That is, many "homomorphic" errors occur in areas of study closely related to the above. Needless to say, the student will often "discover" homomorphisms where, in fact, none are to be found. The perils of impulsive generalizing, here as elsewhere in mathematics, are obvious.

A listing of ERRORS, each homomorphic in appearance, now follows. These prove but the tip of the iceberg.

$(a+b)^2 = a^2 + b^2$	Square of the sum
$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$	Square root of the sum
$(A \cup B)' = A' \cup B'$	Complement of the union
$\log(x+y) = \log x + \log y$	Logarithm of the sum
$\tan(x+y) = \tan x + \tan y$	Tangent of the sum
$\mathrm{n}(A \cup B) = \mathrm{n}(A) + \mathrm{n}(B)$	Cardinality of the union
$\int f(x)g(x)dx = \int f(x)dx \int g(x)dx$	Integral of the product
$\lvert a+b \rvert = \lvert a \rvert + \lvert b \rvert$	Absolute value of the sum
$(a+b)! = a! + b!$	Factorial of the sum
$(fg)' = f'g'$	Derivative of the product
$(a+b)^{-1} = a^{-1} + b^{-1}$	Reciprocal of the sum
$\varnothing(p+q) = \varnothing(p) + \varnothing(q)$	Totient of the sum
$2(pq) = (2p)(2q)$	Double of the product
$\sim(p \vee q) \longleftrightarrow \sim p \vee \sim q$	Negation of the disjunction
$[\![a+b]\!] = [\![a]\!] + [\![b]\!]$	Greatest integer value of the sum

Note the range of mathematical topics over which this error type extends. Some of the topics are elementary as in the arithmetic example of "the double of the product." Some are in-between, much like the "tangent of the sum." And some are advanced, similar to the "integral of the product" or "the totient of the sum." Note too how easily and quickly mathematical techniques would unfold if these results were indeed valid. The binomial theorem would hardly be needed as the *n*th power of a sum would equal the sum of the *n*th powers. Trigonometric identities would be easily derived and remembered. Likewise, integration by parts would scarcely appear as the integral of a product would be the product of the integrals. And on and on the speculating continues. Perhaps it is by such wishful thinking and a desire for simplicity of approach that the student is further motivated to proceed homomorphically.

In each of the above error encounters, focus must also be placed on what corrective adjustments are needed (a greater adjustment in many cases than merely drawing a slash through the equal sign). It may be the changing of an operation symbol as in the "complement of the union of two sets" or the inserting of a term (or condition) as in the "cardinality of the union" statement. Counterexamples are easy to find wherewith to convince the students of their errors in the case for incorrectly stated homomorphisms.

The successful mathematics teacher builds on an anticipation of student error tendencies. Important considerations are those of

AWARENESS of stumbling blocks and precise pitfalls surrounding the many error varieties,

EMPHASIS on appropriate details and cautious wordings which will guide the students' steps aright,

ENCOURAGEMENT of discovery and generalization in the directed setting,

ATTENTION and focus not only on right procedure but also the recurring erroneous, and a

VIGILANCE which takes into account the persistent nature of error tendencies.

The learner, having mastered an area of error prone difficulty, may find it instructive to consider various solutions in which wrong methods yield right answers. These include a cancellation of digits, e. g., sixes, so as to give

$$\frac{16}{64} = \frac{1}{4}$$

or the exponential-base manipulation

$$2^5 9^2 = 2592$$

or the subtlety

$$\sin(a+b)\sin(a-b) = (\sin a + \sin b)(\sin a - \sin b) = \sin^2 a - \sin^2 b.$$

Note the erroneous homomorphism of this last example, yet the correct result it produces. The student accordingly has the feeling of being in the "know" in "making these errors." They are both deliberate and purposeful. Achievement takes the form of genuine understanding. He or she will often respond in a way which is uplifting and insightful, realizing that, as stated earlier, and, in general, YOU CAN'T DO THAT.

Lucky Larry #15

The problem was to find the roots of $f(x) = x^2e^{-x} - e^{-x}$. Lucky Larry remembered to set the expression equal to zero, but instead of factoring out e^{-x}, he *divided* it out, getting

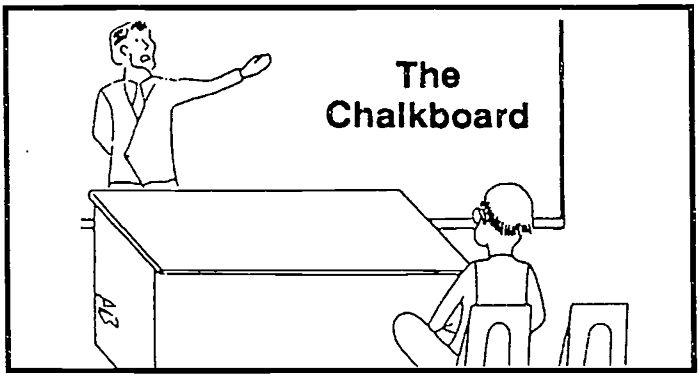
$$x^2 - 1 = 0$$

$$x = \pm 1.$$

Normally, of course, dividing out a variable expression loses solutions, but Larry lucked out because this time the expression he divided out, e^{-x}, cannot equal 0.

Submitted by Delores Anderson
Truett-McConnell College
Cleveland GA 30528

REGULAR FEATURES



Edited by

Judy Cain — Tompkins Cortland Comm. College, Dryden NY 13053

and

Joseph Browne — Onondaga Comm. College, Syracuse NY 13215

This column is intended as an idea exchange. We hope to facilitate an open exchange of ideas on classroom management, teaching techniques, tips for helping students get past the usual stumbling blocks, techniques for improving student participation, etc. We know there are lots of good ideas out there, and this is your chance to share them. Please send your contributions to Judy Cain.

Bringing Enthusiasm to your Classroom With E-Mail

Electronic mail offers a fun and unusual way to do some of the "same old stuff." If you haven't used e-mail before, this is your opportunity to learn. If you are an experienced user, this may be the time to think of classroom applications.

I am a novice user; however, I decided to try an experiment this semester that has been very exciting. One of the side benefits is that this has pushed me into becoming more accomplished myself — I am staying one step ahead of the students!

First, I required each student to get an e-mail address. (My university provides this for all enrolled students.) Next, I set up a group mailing list for each class. Then I began using the computer for various activities. Test grades, assignment grades, and final grades are all sent via e-mail. Using my mailing group, I send a message containing everyone's grade matched to his/her social security number.

E-mail is the perfect place to post messages to the entire class. Many individual questions or items of business that normally require an office visit or a telephone call can be handled via electronic mail. With e-mail, you can create an electronic discussion group. (Some students who won't speak up in class participate actively in an e-mail encounter.)

I have also used electronic mail to make writing assignments in my developmental math classes. Students must return the assignment to me through e-mail. I then grade and return it at my convenience; this can save class time as well as paper.

My students have been most enthused about this project. (I get regular notes from some of them.) For many, it is their first semester in school and their first encounter with the computer. I look forward to future semesters and more adventures.

Submitted by Sandra Villas, Doña Ana Branch Community College, Las Cruces, NM 88003-8001

Small Group Learning?
Yes, But How Do I Assign the Groups?

Recent mathematics education reform efforts, including the draft AMATYC *Standards*, call for the use of cooperative groups in instruction. The assignment of groups then becomes a question. Stable group assignments would fail with my student population, for reasons such as transportation, finances, motivation, etc.

One way to assign groups is according to where the students are sitting. This minimizes moving but fosters a certain comfort level among students who work together regularly. A second method is to allow students to choose their groups, but a negative aspect to this method is that non-native speakers tend to work only with each other. Yet another way is to pair high- and low-achieving students and use the middle of the class to round out the groups.

A favorite method of mine is a "random" one which ensures that students work with many others in the class. Take a partial deck of ordinary playing cards to class which includes a card for each student on the roll. Include all four aces, four twos, four threes, and so on; at the beginning of class, remove the highest numbered cards as necessary to account for those who are not in class. Shuffle the cards and hand them out; then ask the students to meet in groups according to the number on the card — the four aces form a group, the four twos another, and so on. The last students, if not a full group, can join other groups or work as a smaller group. If groups of three are desired, create a deck of cards of the appropriate size with only three suits.

Using a deck of cards is fast and easy, and adapts well to the number of students present on any given day. Through random assignment, students who have difficulty working cooperatively are not always placed with the same students. Group work is important because it allows students to communicate about mathematics with a variety of people. It encourages speaking, thinking, and listening skills as well as use of proper mathematics vocabulary. A student's future workplace experience, which rarely includes selecting the persons with whom one works, is simulated in the frequently changing group assignments produced by the cards. Try it!

Submitted by Susan S. Wood, J. Sargeant Reynolds Community College, Richmond VA 23285-5622

Questioning in Math Classes

Much of our teaching time is inevitably devoted to the presentation of concepts and the working of examples. One thing that sometimes gets short shrift is questioning — not the kinds of questions that are usually included on quizzes and tests, but the more thoughtful questions that test understanding of concepts and notation and that may have many answers. Teachers and textbook writers should ask more questions like those that follow. Some redundant examples are included to show alternate ways of asking for the same information.

- How do you recognize an equation when you see one?
- Why is an equation called an equation?
- Explain the difference between an equation and an expression.
- What can you do to an equation that you can't do to an expression?
- Find an untrue statement that becomes true if both sides are squared.
- How do you recognize a proportion when you see one?
- Name a "short cut" that works for a proportion but doesn't generally work for an equation.
- Is an inequality an equation?
- In what ways should an inequality be handled differently from an equation?
- What does it mean to solve an equation?
- Must grouping symbols be in a particular order?
- Find an example in which performing addition before subtraction leads to a wrong answer.
- The equation $1 + \tan^2\theta = \sec^2\theta$ is a common trigonometric identity. It is true for an infinite number of values of θ. Are there any values of θ for which the identity fails?

The advantage of asking questions like these is that students' responses indicate what they really understand; their responses may also reveal common misunderstandings that teachers may never have suspected. For example, a large majority of American students believe that brackets may not be placed inside parentheses, yet few teachers realize how widespread that misconception is. If thoughtful questions are asked in early courses, misunderstandings can be corrected. Another advantage is that students get practice at explaining things articulately, an ability that has been sadly lacking in recent years but that at least some employers will gladly pay for.

Submitted by Steven Schwartzman, Austin Community College, Austin TX

> If the English Department was as fair to our students as we are to theirs, there would be a course "Shakespeare for Non-readers." Math majors take Shakespeare with the English majors, but English majors take "Introduction to Counting."
>
> Herb Gross

D. C. HEATH

Success in Mathematics

New for 1995!

Explore, estimate, and discover graphically—verify, generalize, and determine algebraically.

INTERMEDIATE ALGEBRA: A GRAPHING APPROACH (33325-5) Hardcover

ALGEBRA: A GRAPHING APPROACH (33318-2) Hardcover
Hubbard/Robinson

Now available . . . with more real data to solve more real problems.

BRIEF CALCULUS, FOURTH EDITION (35165-2)
Larson/Hostetler/Edwards

New calculus texts with early transcendental functions:

CALCULUS: EARLY TRANSCENDENTAL FUNCTIONS (39349-5)

INTERACTIVE CALCULUS: EARLY TRANSCENDENTAL FUNCTIONS, CD-ROM FOR WINDOWS (39119-0)

CALCULUS OF A SINGLE VARIABLE: EARLY TRANSCENDENTAL FUNCTIONS (39348-7)
Larson/Hostetler/Edwards

Now with expanded pedagogy and comprehensive support packages

ESSENTIAL MATHEMATICS, THIRD EDITION (35287-X) Paper

BASIC MATHEMATICS FOR COLLEGE STUDENTS, SEVENTH EDITION (35286-1) Hardcover
Wright

Updated with a full-color design.

UNDERSTANDABLE STATISTICS, FIFTH EDITION (35513-5)
Brase/Brase

New to the market-leading Precalculus Graphing Approach Series

TRIGONOMETRY: A GRAPHING APPROACH (28296-0)
Larson/Hostetler/Edwards

Recently published:

CALCULUS, FIFTH EDITION (35335-3)
INTERACTIVE CALCULUS CD-ROM FOR WINDOWS (35712-X)

CALCULUS, ALTERNATE FIFTH EDITION (35336-1)

MULTIVARIABLE CALCULUS, FIFTH EDITION (39345-2)
Larson/Hostetler/Edwards

INTERMEDIATE ALGEBRA: GRAPHS AND FUNCTIONS (33755-2) Hardcover
Larson/Hostetler/Neptune

PRECALCULUS: FUNCTIONS AND GRAPHS, A GRAPHING APPROACH (35206-3)

PRECALCULUS WITH LIMITS: A GRAPHING APPROACH (35251-9)
Larson/Hostetler/Edwards

D. C. Heath texts supply what you and your students need for success in mathematics: a solid presentation of mathematics, clear, understandable writing, outstanding exercise sets, numerous real-life applications, sound pedagogy, and a wide range of quality supplements. So whichever course you teach, with or without technology, you can count on D. C. Heath. For more information on these or our other fine texts call 1-800-235-3565.

D. C. Heath and Company
125 Spring Street
Lexington, MA 02173

Snapshots of Applications in Mathematics

Dennis Callas
State University College of Technology
Delhi NY 13753

David J. Hildreth
State University College
Oneonta NY 13820

The purpose of this feature is to showcase applications of mathematics designed to demonstrate to students how the topics under study are used in the "real world," or are used to solve simply "charming" problems. Typically one to two pages in length, including exercises, these snapshots are "teasers" rather than complete expositions. In this way they differ from existing examples produced by UMAP and COMAP. The intent of these snapshots is to convince the student of the usefulness of the mathematics. It is hoped that the instructor can cover the applications quickly in class or assign them to students. Snapshots in this column may be adapted from interviews, journal articles, newspaper reports, textbooks, or personal experiences. Contributions from readers are welcome, and should be sent to Professor Callas.

Radio Station Selection

(to accompany integration of trigonometric functions)

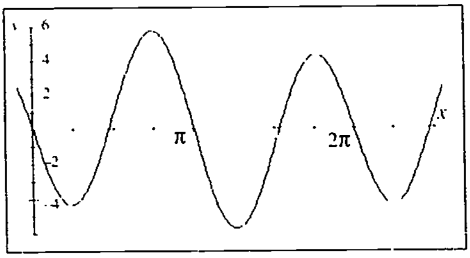
by Charles Johnson, SUNY College of Technology, Delhi NY

When we want to listen to the local Walton radio station, WDLA, "1270 on your AM dial," to find out what the latest weather forecast is, we simply turn the dial to 1270. It is clear to us that the radio is receiving signals from many radio stations, so how are we able to select the station we want to listen to? The answer to this question is not trivial as it involves physics, mathematics, and technology. Here we will only be concerned with the mathematics.

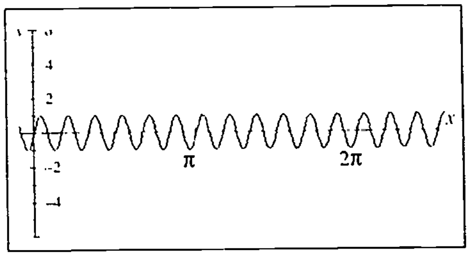
As you may suspect, radio stations are not permitted to send out just any signal. Each station is assigned a "carrier signal," a function of the form $\sin nt$, and this determines the station's frequency, which is what WDLA's "1270" refers to. The station sends out a signal of the form $g(t)\sin nt$, where the function $g(t)$, which represents the talk, music, and etc. which the station is sending out, varies much more slowly than $\sin nt$. (Cosine functions as well as sine functions are used. Also, while n does not need to be an integer, we will assume that n is an integral value.)

Since the function $g(t)$ is varying slowly in comparison to the function $\sin nt$, the graph of $g(t)\sin nt$ looks somewhat like the graph of $\sin nt$ except that the amplitude varies depending on $g(t)$. Hence this kind of signal is called an "amplitude modulated" signal, or AM signal. Consider the graphs in Figure 1. In actual practice n is rather large, near a million, and therefore $\sin nt$ oscillates very rapidly. We are considering relatively small values for n so that we can "see" what we are writing about.

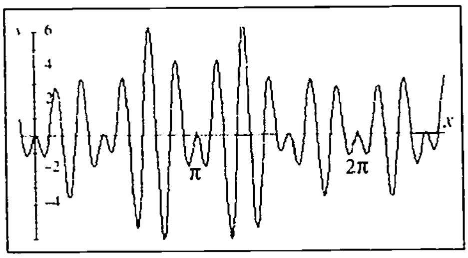
Now, if your radio received signals from just one station, there would not be a concern, for when you turned the radio on it would be "tuned" to that station.



$y = \sin t - 5\sin 2t$



$y = \sin 12t$



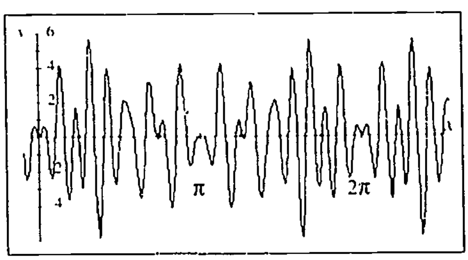
$y = (2\sin t - 5\sin 2t)\sin 12t$

Figure 1

However, the radio receives signals from many stations and these signals are superimposed on one another. For example, suppose that radio station A sends the signal $(2\sin t - 5\sin 2t)\sin 12t$, station B sends the signal $(\sin t + 2\sin 2t)\sin 15t$, and station C sends the signal $(3\sin t + 7\sin 2t)\sin 20t$. Then the signal received by the radio is

$$f(t) = (2\sin t - 5\sin 2t)\sin 12t + (\sin t + 2\sin 2t)\sin 15t + (3\sin t + 7\sin 2t)\sin 20t,$$

whose graph is given in Figure 2.



$y = (2\sin t - 5\sin 2t)\sin 12t + (\sin t + 2\sin 2t)\sin 15t + (3\sin t + 7\sin 2t)\sin 20t$

Figure 2

If you wish to receive the signal from station C, i.e., to "tune in" station C, the problem from the mathematical viewpoint is to recover the function $3\sin t + 7\sin 2t$. To solve this problem you would first remove the parentheses and rewrite

$$f(t) = 2\sin t \sin 12t - 5\sin 2t \sin 12t + \sin t \sin 15t + 2\sin 2t \sin 15t + 3\sin t \sin 20t + 7\sin 2t \sin 20t.$$

Next, using the trigonometric identity $\sin A \sin B = \frac{1}{2}\,[\cos(A - B) - \cos(A + B)]$, rewrite

$$f(t) = \frac{2}{2}\,[\cos 11t - \cos 13t] - \frac{5}{2}\,[\cos 10t - \cos 14t] + \frac{1}{2}\,[\cos 14t - \cos 16t] + \frac{2}{2}\,[\cos 13t - \cos 17t] + \frac{3}{2}\,[\cos 19t - \cos 21t] + \frac{7}{2}\,[\cos 18t - \cos 22t].$$

In this simplified model, to tune to station C we form the integral $\int_0^\pi f(t)\cos(n + k)t\,dt$, where n is the value found in the carrier signal ($\sin 20t$), and $k = 1$ and 2 (since there are two terms in $g(t)$). Therefore, in this case, we evaluate $\int_0^\pi f(t)\cos(20 + 1)t\,dt$ and $\int_0^\pi f(t)\cos(20 + 2)t\,dt$. It turns out that all but one of the terms in each of the integrals vanishes since $\int_0^\pi \cos nt \cos mt\,dt = 0$ for $n \neq \pm m$. Hence,

$$\int_0^\pi f(t)\cos 21t\,dt = -\frac{3}{2}\int_0^\pi \cos^2 21t\,dt = \left(-\frac{3}{2}\right)\left(\frac{\pi}{2}\right)$$

and

$$\int_0^\pi f(t)\cos 22t\,dt = -\frac{7}{2}\int_0^\pi \cos^2 22t\,dt = \left(-\frac{7}{2}\right)\left(\frac{\pi}{2}\right).$$

Multiplying by $-\frac{4}{\pi}$ recovers the coefficients 3 and 7 of $\sin t$ and $\sin 2t$, respectively, and thus we have recovered station C's signal. The technique used in the example will work as long as the stations have carrier signals of the form $\sin nt$ and are broadcasting signals of the form $g(t) = a_1 \sin t + a_2 \sin 2t + \ldots + a_m \sin mt$, where m is smaller than half the minimum difference between the assigned frequencies n of the different stations.

So, the next time you hear the radio announcer say, "Tune in tomorrow for the next exciting episode," remember that while it may be easy for you to turn the dial on the radio, what the radio is doing is not trivial!

Exercises

1. Show that $\int_0^\pi \cos nt \cos mt\,dt = 0$ for $n \neq \pm m$. You will need to use the trigonometric identities

$$\cos(nt - mt) = \cos nt \cos mt + \sin nt \sin mt$$
$$\cos(nt + mt) = \cos nt \cos mt - \sin nt \sin mt.$$

Add the two identities and solve for $\cos nt \cos mt$. Replace $\cos nt \cos mt$ in the integral by what you just found it to equal and then evaluate the integral.

2. Show that $\int_0^\pi \cos^2 nt\,dt = \frac{\pi}{2}$, for $n = 1, 2, \ldots$. Here you need to use the trigonometric identity $\cos^2 nt = \frac{1}{2}(1 + \cos 2nt)$.

3. Using the information given in the example in the article, and the technique illustrated to recover station C's signal, show how station B's signal would be recovered.

4. Many people listen to FM stations as well as AM stations. In fact, FM stations are generally preferable to AM stations as there is less "static" received from a FM station. Write a brief article explaining the difference between an AM signal and a FM signal. A possible reference for this article is *Receiving Systems Design* by Stephen Erst, Artech House, 1984.

This snapshot was based on an article by Clark Benson (1993) and was produced as part of a project sponsored by the State University of New York and the National Science Foundation (Division of Undergraduate Education).

Reference

Benson, C. (1993). How to tune a radio. In *Applications in Mathematics: MAA Notes, No. 29*, (pp. 126–136). Washington: Mathematical Association of America.

Lucky Larry #16

When faced with solving

$$\log(2x + 1) - \log(3x - 1) = 0$$

Larry just dropped the logs and proceeded.

$$(2x + 1) - (3x - 1) = 0$$
$$-x + 2 = 0$$
$$x = 2$$

This is actually valid when i) the right-hand-side is 0, and ii) the left-hand-side is a *difference* of logs. Change either of these and Larry had better use a more conventional method.

Submitted by Joan Page
Onondaga Community College
Syracuse NY 13215

Notes from the Mathematical Underground

Edited by

Alain Schremmer
Mathematics Department, Community College of Philadelphia
1700 Spring Garden Street
Philadelphia PA 19130

Since this is a new column, a *caveat* might be in order. As we all know, there currently is a "crisis" in mathematics education. (As has been the case, roughly, as far back as I can remember, which is, alas, quite far enough.) Since the latter involves mathematics, faculty, and students, not necessarily in that order, this is where its origin must be sought and this is what I would like to do here. By definition, however, faculty are above suspicion or, at least, will be in this column. Students being *the* given of the situation and, in any case, presumably not part of the readership, any discussion of how they *ought* to be would be futile and I will leave them mostly alone and not discuss pedagogy. This leaves mathematics and this is what I intend to discuss: No "how to do it," no sugar-coating by way of "applications" or "math history" or whatever, no "high tech" religion. But mathematics does not exist in a vacuum and my distinguished colleagues will, of necessity, have to be considered as *parameters* in the equation.

It might have occurred to the reader that I left textbooks out of this equation and it is indeed fashionable in academic circles to deplore the state of the textbook art (except among authors, of course; see, for instance, Anton (1991)) and blame it for much of the educational fiasco. As publishers, though, are fond to point out, it is faculty who design the courses, who order the textbooks, and who write them. Moreover, when they wish to be nasty, they are quite prone to listing all the non-conformist texts on which they say they lost their corporate shirt.

Which brings me, precisely, to the main issue that I intend to pursue here, that of the alternatives to the *mathematical* underpinnings of our teaching. One way, then, in which I would like to do this is to discuss textbooks that didn't make it as mainstream texts because, presumably, they were "too different." Here, I mainly think of calculus texts such as those by Levi, Keisler, Strang, Flanigan-Kazdan, Freed, etc. I would also like to discuss ideas that briefly appeared in texts but were dropped in subsequent editions—when there was one, such as Munroe's definition of variables or Gillman-McDowell's definition of the integral. There are also very simple ideas, such as Lang's treatment of the transcendental functions or that of Finney-Ostbey, that appear in more advanced texts but which, somehow, never made it to "elementary" textbooks.

To give a more general yet concrete example of what I have in mind, a glance at any Arithmetic text will show why the Metric System is not catching in this country: It is presented exactly in the same manner as the now almost defunct English system, namely as just another collection of units with conversion factors to be memorized. In other words, a system it is not. The key to the Metric System is that it complements the Decimal System by providing *units* that behave decimally. To take money as our *model* — it is, after all, the only metric entity in the U. S., we have the following *units*: **Cleveland, Franklin, Hamilton, Washington, Dime, Penny, Milly** each worth TEN of the next one. The idea is that one is not allowed to hold more than nine objects of a kind because we could not symbolize such a holding with TEN digits. This requires the notion of *exchange*.

EXAMPLE 1. Say we hold, on the one hand, **5 Hamiltons, 6 Washingtons, 4 Dimes** and, on the other hand, **7 Washingtons, 8 Dimes.** Adding the dimes, we get twelve dimes which we are not allowed to hold so that we must exchange TEN of them for ONE Washington. We now have fourteen Washingtons, TEN of which we must exchange for ONE Hamilton. Altogether, we now have: **6 Hamiltons, 4 Washingtons, 2 Dimes.**

EXAMPLE 2. Say we hold **3 Hamiltons, 0 Washingtons, 5 Dimes** out of which we must pay **7 Dimes**. Normally, to get dimes, we would exchange ONE Washington for TEN dimes but we do not have any Washingtons so that we must first exchange ONE Hamilton for TEN Washingtons and then ONE Washington for TEN dimes so that we now have fifteen dimes from which we can now pay the seven dimes that we owe and which leaves us with eight dimes. Altogether, we are left with: **2 Hamiltons, 9 Washingtons, 8 Dimes.**

EXAMPLE 3. Say three robbers stick up the bank of a one-bank town and run away with **2 Clevelands, 9 Franklins, 0 Hamiltons, 5 Washingtons**. Dividing the loot, each will only get **0 Clevelands, 3 Franklins, 0 Hamiltons, 1 Washington** since they can hardly return to the bank to *exchange* the remaining **2 Clevelands, 0 Franklins, 0 Hamiltons, 2 Washingtons** for further division. The usual algorithm is based on the—usually unstated — assumption that changing facilities are available.

Seen in this light, then, all the metric system does is to substitute Kilo$, Hecto$, Deka$, $, Deci$, Centi$, Milli$ for the above units. Observe that adhering tightly to the mathematics of the matter makes much more sense than speaking of "carry over" or "borrowing." Moreover, this allows for a much more compact notation and a great deal of flexibility: Given a holding, it is easy to describe it in terms of any unit one wishes.

EXAMPLE 4. Instead of writing 3 Kilo$, 4 Hecto$, 5 Deka$, 6 $, 7 Deci$, 8 Centi$ and assuming we wanted to think in terms of Hecto$, we would write 34.5678 Hecto$ where the purpose of the decimal point is to *point* at the digit that corresponds to the Hecto$. If we wanted to think in terms of Deci$, we would just move the point accordingly: 34567.8 Deci$.

While this is called *unit conversion*, observe that it does not entail any exchange in the above sense.

More generally, as two-year College faculty, we are caught between the rock of what our students are and the hard place of most of the curriculum being driven by four-year institutions. While most of us tend to play it safe, that is conventionally, there is an Underground that tends to rebel and do things somewhat differently and if this Journal, just like any other professional journal, must reflect what, in the main, *is*, it is to the glory of *The AMATYC Review* that it would offer some room to those of us who think differently in mathematical matters, presumably on the basis of there being much that we can learn from this Underground as to what *could be*. As for my own intentions concerning this column, then, I mainly want to bring up such specific matters as above and argue, with or against whomever cares to write me, *why* we insist on teaching the usual nonsense. I would very much hope for the Underground to become the heart of this column and this column is open not just to the Underground but also to all those who wish to argue with the Underground.

However, having once dabbled in "generalized abstract nonsense," I would like also to discuss less directly mathematical things not usually discussed in so-called professional journals but which I nevertheless deem very relevant, both to our profession and to my purpose in this column. For instance, we declare that mathematics is useful but what do we mean by that? It is often said that this country lacks mathematicians or, at least, that a good background in mathematics helps in getting a good job but, given the number of PhDs in mathematics that did *not* get a permanent position these last few years, I have my doubts as to the first saying and, on logical grounds, I wonder exactly *why* the second saying should be true at all as I shall argue in a future column.

I hold that I learned to read because I felt that reading would afford me pleasure, because it allowed me to escape the real world, just as playing chess would, and not because I needed to be able to read owner's manuals. Similarly, I did not learn what I know of mathematics because it allowed me to find the profit a farmer makes by enclosing his field at so many francs a meter of barbed wire, etc. (This traces me, doesn't it?) It was, partly in spite of my teachers, even though they taught *mathematics*, not cooking by numbers, because I fell in love with mathematics and wanted to play with it, this even though I knew very early on that this love would never be requited and that I would never be able to prove a new lemma or even a new theorem. I hold that we do a disservice to our students, to ourselves, and to mathematics, not necessarily in that order, in thinking that anybody needs a contingent reason for studying mathematics.

Reference

Anton, H. (1991). In defense of the fat calculus text. *UME Trends*, 2(6), 1.

> Some persons have contended that mathematics ought to be taught by making the illustrations obvious to the senses. Nothing can be more absurd or injurious: it ought to be our never-ceasing effort to make people think, not feel.
>
> A.T. Coleridge

74

Software Reviews

Edited by Shao Mah

Title: TEMATH (Tools for Exploring Mathematics) Version 1.5
Authors: Robert Kowalczyk/Adam Hausknecht
Distributor: Brooks/Cole Publishing Co.
Pacific Grove CA 93950
Price: $75
Computer: Macintosh Plus, SE, II or later model with one MB of memory, System 6.0 or later.

TEMATH is one of the few mathematical software packages developed for the Macintosh computers. It was designed by the developers to be a tool to explore mathematical concepts. TEMATH consists of a two dimensional grapher for rectangular, polar and parametric equations. It also contains a matrix calculator, an expression calculator and a facility for handling and manipulating data. It can be used to find roots, to construct a tangent line to a given curve at a given point, to find extrema for a function, to generate a set of random data, to evaluate a definite integral and to find arc lengths.

The window for the grapher is divided into four parts: graph window, work window, report window and tool palette. The graphs of functions are shown in the graph window. The work window is used to enter the function and to set its domain and range. The report window can display the results from various mathematical computations such as computing roots, maxima, areas, and other aspects related to the graph. The tool palette of the graph window provides many functions such as zooming a graph, finding an arc length, drawing a tangent line at a given point, finding asymptotes, evaluating a definite integral, finding a single root and locating relative maximum and minimum points.

TEMATH contains a matrix calculator. Using the matrix calculator, a user can enter matrix data, perform matrix arithmetic and matrix operations such as transpose, inverse, determinant, elementary row or column operations and solving the systems of linear equations.

TEMATH has a special feature which handles data tables. Data can be entered as discrete data or generated by any function. A data table can be graphed and fitted with a line or curve. The set of data in a data table can also be used to find the maximum, the minimum, the average value, the standard deviation and do regression analysis.

The expression calculator window contains a text area, a set of symbol buttons, and a scrollable list of predefined mathematical functions. The expression calculator is used to evaluate numerical expressions which the user enters. The expressions can either be entered by keyboard or by using a mouse.

TEMATH is a user friendly software and it has many attractive features. Firstly, the easy-to-use grapher has fast executing algorithms which makes it a very desirable software for classroom demonstrations or for a quick view of the properties of a function. Secondly, the data table is a special feature in manipulations of data. A student can either gather a set of sampling data or generate a set of random data for statistical analysis. Thirdly, the matrix calculator can perform many matrix related numerical computations. It can be used to find null space, to perform elementary operations, to calculate a determinant or an inverse, to reduce a matrix to row echelon form or reduced row echelon form and to perform matrix arithmetic. It is a very helpful tool for precalculus students or linear algebra students. Therefore, TEMATH is a piece of software worth purchasing. (SM)

Send Reviews to: Shao Mah, Editor, Software Reviews
The AMATYC Review, Red Deer College, Red Deer, AB, Canada T4N 5H5

There once lived a man
who learned how to slay dragons
and gave all he possessed
to mastering the art.

After three years he was fully prepared but,
alas, he found no opportunity
to practice his skills.

Dschuang Dsi

As a result he began
to teach how to slay dragons.

Rene Thom

The Problem Section

Dr. Michael W. Ecker
Problem Section Editor
The AMATYC Review
909 Violet Terrace
Clarks Summit PA 18411

Dr. Robert E. Stong
Solution Editor
The AMATYC Review
150 Bennington Road
Charlottesville VA 22901

Greetings, and welcome to still another Problem Section!

The AMATYC Review Problem Section seeks lively and interesting problems and their solutions from all areas of mathematics. Particularly favored are teasers, explorations, and challenges of an elementary or intermediate level that have applicability to the lives of two-year college math faculty and their students. We welcome computer-related submissions, but bear in mind that programs should supplement, not supplant, the mathematical solutions and analysis.

When submitting material for this department, please note that we have separate editors for problems and for solutions. Send **two copies** of your new **problem proposals**, preferably typed or printed neatly with separate items on separate pages, to the Problem Editor. Include **two copies** of a solution, if you have one, and any relevant comments, history, generalizations, special cases, observations, and/or improvements. Please include your name (title optional, no pseudonyms), affiliation, and address of same. Enclose a mailing label or self-addressed envelope if you'd like to be assured a reply. All **solutions** to others' proposals, *except Quickies*, should be sent directly to the Solutions Editor.

Dr. Michael W. Ecker (Pennsylvania State University—Wilkes Barre)
Dr. Robert E. Stong (University of Virginia)

Baker's Dozen

This column's problem numbers have past set Z. So, I will follow my own inclination to mimic the way spreadsheets number their columns: AA, AB, and so on. Thanks also to Philip Mahler of Middlesex Community College for independently making the same suggestion.

Quickies

Quickies are math teasers that typically take just a few minutes to an hour. Solutions usually follow the next issue, listed before the new teasers. All correspondence to this department should go to the Problem Editor.

Comments on Old Quickies

Quickie #12. If the sides of a quadrilateral are 3, 13, 12, and 4 units, find its area.

J. Sriskandarajah (the proposer) clearly intended to have you think of drawing a diagonal of length 5, thus obtaining the 3-4-5 and 5-12-13 right triangles. (Draw it for yourself!) This results in respective areas 6 and 30, with total area 36.

But is SSSS (four sides match four corresponding sides) a sufficient condition for quadrilateral congruence? Solutions Editor Bob Stong quickly pointed out that the solution is not unique: "Knowing the sides of a quadrilateral is almost never sufficient to determine it. The only case I can think of yielding uniqueness is the one in which one side equals the sum of the other three."

Bob continued by calling the quadrilateral $ABCD$ with $AB = 3$, $BC = 13$, $CD = 12$, $DA = 4$. Because $AB + BC = CD + DA = 16$, one possibility is zero area resulting from a line segment of length 16. To determine the area, you really need to know BD, the distance from B to D. The possibility $BD = 5$ gives the intended solution above. But another possibility is $BD = 7$, which makes B, A, and D collinear. This results in a triangle with sides 12, 13, and 7, whose area may be found to be about 41 by Heron's formula.

Bob checked further after first assuming that the 12-13-7 triangle gave the largest area. His computer check showed the largest area results when $BD = 6.75$, yielding a bit over 43 square units.

Quickie #13. Suppose a positive integer n is chosen in such a way that the probability of choosing $n + 1$ is $\frac{9}{10}$ that of choosing n (for all n). What is the probability of choosing 2? What is the mathematically expected (average) choice?

The sum of the probabilities of the elementary outcomes is $S = 1$ and we have an infinite geometric series with unknown first term a and common ratio $r = \frac{9}{10}$. Use the familiar equation for an infinite geometric series: $S = \frac{a}{1-r}$ to obtain $a = \frac{1}{10}$. Thus, the second term $= ar = \frac{9}{100}$.

The mathematical expectation is the sum $1a + 2ar + 3ar^2 + \ldots$. Temporarily regard this as a function of r (with a = the constant $\frac{1}{10}$), namely the derivative of the sum function: $a + ar + ar^2 + \ldots = f(r) = \frac{a}{1-r}$ with $-1 < r < 1$. Then $f'(r) = \frac{a}{(1-r)^2}$ and we want $f'(.9) = 10$.

Quickie #14: Investigate the limit of $x^{\frac{1}{\ln x}}$ as x increases without bound.

Write $y = x^{\frac{1}{\ln x}}$ and take the ln of each side. Then $\ln(y) = 1$, so $y = e$, a constant function. Now the answer is e no matter what x approaches.

Quickie #15: Find a divergent alternating series whose nth term nevertheless approaches 0.

Represent the series as $a_0 - a_1 + a_2 - a_3 + - \ldots$. We create $\langle a_n \rangle$ defined on odds and evens by $a_{2j-1} = \frac{1}{2^j}$, $a_{2j} = \frac{1}{j+1}$. This gives a formal sum

$$1-\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{8}+\frac{1}{4}-\frac{1}{16}+-\ldots,$$

which, roughly speaking, involves the harmonic series minus a geometric series.

New Quickies

Quickie #16: Proposed by the Problem Editor based on teaching Calculus I.

Consider the standard max-min problem typified by this: "You are to build a rectangular enclosure with front side made of material costing $10 per foot and with other three sides (back, left, right) of material costing $5 per foot. If the enclosure is to contain exactly 600 square feet and to be built at minimum total cost, how long should the sides be and what is the total cost?"

Note the small coincidence in that the answer involves a total cost of $600, which matches the constraint of area 600 (aside from units, of course). Question: Characterize when this happens more generally, and thus describe the relationship that exists in such problems in which the resultant minimum cost matches the given area (all in appropriate units).

Quickie #17: Proposed by the Problem Editor based on teaching Calculus I.

Choose a number x randomly from the interval $[0, \pi]$ and a second number y randomly from $[0, 1]$. What is the probability that $y < \sin x$?

Quickie #18: Proposed by Michael H. Andreoli, Miami Dade Community College.

From the interval [0, 1] choose two numbers at random. What is the probability that the resulting segments can be used to form a triangle? (Note: There may be some ambiguity here and/or in #17. Use your best judgment. In any case, proposer points out that a simple, algebraic, non-calculus solution exists for #18.)

Correction

Problem Z-1 last time had an error (my fault) in printing b instead of bx, but since the problem spoke of odd polynomials, readers should have realized that something was amiss. In any case, I re-solicit solutions.

Problem Z-1. Proposed by the Problem Editor, Pennsylvania State University, Wilkes-Barre Campus, Lehman, PA.

Characterize all invertible, odd, fifth-degree polynomials. That is, determine necessary and sufficient conditions on the real numbers a and b to make $p(x) = x^5 + ax^3 + bx$ invertible. (Non-monic versions of these polynomials are just scalar multiples of the monic $p(x)$ shown.)

New Problems

***Set AA Problems** are due for ordinary consideration April 15, 1995.* Of course, regardless of deadline, no problem is ever closed permanently, and new insights to old problems – even Quickies – are always welcome. However, our Solutions

80

Editor requests that you please not wait until the last minute if you wish to be listed or considered on a timely basis.

Problem AA-1. Proposed by Philip Mahler, Middlesex Community College, Bedford, MA 01730.

A heavy rock is dropped into a deep well. Three seconds later a splash is heard. How far down is the surface of the water?

Proposer's and Problem Editor's Comments: If there is little water in the well, then we have a good estimate of the depth of the well. This question requires some physics, but such permeates math texts. This question makes an elegant little research problem for students, who will need to catalogue their assumptions carefully to handle the purposeful ambiguities. Such include not only questions of air resistance and the speed of sound in air, but even the degree of accuracy. Once made, this question is more of a teaser/quickie, but its charm alone may suffice to elevate its status.

Problem AA-2. Proposed by Kenneth G. Boback, Pennsylvania State University, Wilkes-Barre Campus, Lehman, PA 18627.

Convert 9.3 (base 10) into its base-7 equivalent representation.

Problem Editor's Comment: My colleague Ken originally proposed this for our annual Newton Mathematical Society Competition in early 1994. It could be considered a quickie, but I am interested in seeing other methods of solution.

Problem AA-3. Proposed by Juan-Bosco Romero Marquez, Universidad de Valladoled, Valladoled, Spain.

Evaluate $F_nF_{n+p+3} - F_{n+1}F_{n+p+2}$, where the $<F_n>$ are the usual Fibonacci numbers defined here by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. (Addendum by Problem and Solution Editors: What if we use $F_0 = 1 = F_1$ instead?)

Problem AA-4. Proposed by J. Sriskandarajah, University of Wisconsin, Richland Center, WI 53581.

Around any equilateral triangle circumscribe a rectangle so that each side of the original triangle cuts off a right triangle from the rectangle. (Note: We assume that one vertex of the triangle coincides with one of the rectangle.) Prove that the sum of the areas of the two smaller right triangles equals the area of the largest one thus formed.

Problem AA-5. Proposed by Frank Flanigan, San Jose State University, San Jose, CA 95192.

Given the positive reals $a_1, a_2, \ldots, a_n$, define the products

$$A_i = a_1 \ldots a_{i-1}a_{i+1} \ldots a_n \text{ for } i = 1, 2, \ldots, n.$$

Solve for x:

$$A_1(x - a_1) + A_2(x - a_2) + \ldots + A_n(x - a_n) = 0.$$

Set Y Solutions

A Bijection

Problem Y-1. Proposed by Michael H. Andreoli, Miami Dade Community College (North), Miami, FL.

Find a one-to-one, onto function f: $[0,1] \to (0,1)$.

Solutions by Charles Ashbacher, Decision Mark Corp., Cedar Rapids, IA; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Joseph Browne, Onondaga Community College, Syracuse, NY; Mike Dellens, Austin Community College, Austin, TX; David Dobbs, University of Tennessee, Knoxville, TN; Donald Fuller, Gainesville College, Gainesville, GA; David Gillette, Chemeketa Community College, Salem, OR; Delbert Greear, Gainesville College, Gainesville, GA; Stephen Plett, Fullerton College, Fullerton, CA; David Price, Tarrant County Junior College, Fort Worth, TX; Grant Stallard, Manatee Community College, Bradenton, FL; Wesley W. Tom, Chaffey College, Alta Loma, CA; the Problem Editor; and the proposer.

One such function is given by

$$f(x) = \begin{cases} \frac{1}{2} & \text{for } x = 0, \\ \frac{1}{n+2} & \text{for } x = \frac{1}{n} \text{ with } n \geq 1, \text{ and} \\ x & \text{otherwise.} \end{cases}$$

More generally, choosing a sequence a_n of distinct terms in $[0,1]$ with $a_0 = 0$ and $a_1 = 1$, one can let $f(a_n) = a_{n+2}$ and $f(x) = x$ for all other x.

Smaller Multiple

Problem Y-2. Proposed as private communication by Jack Goldberg (Teaneck, NJ in 1985) to Michael W. Ecker (Problem Editor), Pennsylvania State University, Lehman, PA.

Consider a real number x with integer part I and decimal part $.a_1a_2...a_n$. If we multiply x by 10^n we obtain an integer, of course. Under what condition(s) is there a positive integer $m < 10^n$ such that mx is integral?

Solutions by Robert Bernstein, Mohawk Valley Community College, Utica, NY; Joseph Browne, Onondaga Community College, Syracuse, NY; Donald Fuller, Gainesville College, Gainesville, GA; Delbert Greear, Gainesville College, Gainesville, GA; Leonard L. Palmer, Southeast Missouri State University, Cape Girardeau, MO; Stephen Plett, Fullerton College, Fullerton, CA; and the proposer.

The condition under which m exists is that a_n not be 1, 3, 7, or 9. The set of m for which mx is integral is a subgroup of the integers and contains 10^n. Any smaller m must then be a divisor of 10^n. Then a_n is even if and only if m can be taken to be $\frac{10^n}{2}$, and a_n is divisible by 5 if and only if m can be taken to be $\frac{10^n}{5}$.

A Balancing Act

Problem Y-3. Proposed by Jim Africh, College of DuPage, Glen Ellyn, IL.

In triangle ABC, let D be on side AB such that $\frac{AD}{DB} = \frac{2}{3}$, E on side BC such that $\frac{BE}{EC} = \frac{1}{5}$, AE and CD meet at F, and AC and BF meet at G. Find $\frac{BF}{FG}$.

Solutions by Delbert Greear, Gainesville College, Gainesville, GA; Grant Stallard, Manatee Community College, Bradenton, FL; and the proposer.

Consider placing weights of 15 units, 10 units, and 2 units at the vertices A, B, and C. The edge AB then balances at the point D with $\frac{AD}{DB} = \frac{10}{15}$ and the edge BC balances at the point E with $\frac{BE}{EC} = \frac{2}{10}$. The center of gravity of the triangle is then located at the point F. The weights at A and C may be replaced by a single weight of 17 units at the point G, and the line BG will then balance at the point F with $\frac{BF}{FG} = \frac{17}{10}$.

Partial Fractions

Problem Y-4. Proposed by Michael H. Andreoli, Miami Dade Community College (North), Miami, FL.

Find a closed-form expression (without using a symbolic math program such as Derive) for the antiderivative of $\frac{1}{1+x^4}$.

Solutions by Charles Ashbacher, Decision Mark Corp., Cedar Rapids, IA; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Nicholas G. Belloit, Florida Community College at Jacksonville, Jacksonville, FL; Kenneth G. Boback, Penn State University – Wilkes-Barre, Lehman, PA; Prashant Dalvi, Del Mar College, Corpus Christi, TX; Donald Fuller, Gainesville College, Gainesville, GA; Dan Gallup, Pasadena City College, Pasadena, CA; Delbert Greear, Gainesville College, Gainesville, GA; Richard H. Jarnagin, Manatee Community College, Bradenton, FL; Michael J. Keller, St. Johns River Community College, Orange Park, FL; Leonard L. Palmer, Southeast Missouri State University, Cape Girardeau, MO; Stephen Plett, Fullerton College, Fullerton, CA; David Price, Tarrant County Junior College, Forth Worth, TX; J. Sriskandarajah, University of Wisconsin

Center – Richland, Richland Center, WI; Grant Stallard, Manatee Community College, Bradenton, FL; Wesley W. Tom, Chaffey College, Rancho Cucamonga, CA; and the proposer.

The method of partial fractions is the obvious technique to use for this problem. One has

$$\frac{1}{1+x^4} = \frac{Ax+B}{x^2+\sqrt{2}\,x+1} + \frac{Cx+D}{x^2-\sqrt{2}\,x+1}$$

from which $B = D = \frac{1}{2}$, $A = \frac{\sqrt{2}}{4}$ and $C = -\frac{\sqrt{2}}{4}$. Then integrating gives

$$\int \frac{dx}{1+x^4} = \frac{\sqrt{2}}{8}\ln\left|\frac{x^2+\sqrt{2}\,x+1}{x^2-\sqrt{2}\,x+1}\right| +$$

$$\frac{\sqrt{2}}{4}\left\{\arctan\left(\sqrt{2}\,x+1\right) + \arctan\left(\sqrt{2}\,x-1\right)\right\} + C.$$

A Determinant

Problem Y-5. Proposed by the Solution Editor, University of Virginia.

Let p, q, and r be fixed nonnegative integers and let $C(m, n)$ equal, as usual, the binomial coefficient representing the number of unordered selections of n items from m. What is the value of the determinant of the $(r+1)\times(r+1)$ matrix $[a_{ij}] = [C(p+i+j-2, q+j-1)]$?

No solutions were received.

Let

$$A = \det\begin{pmatrix} C(p,q) & C(p+1,q+1) & \dots & C(p+r,q+r) \\ C(p+1,q) & C(p+2,q+1) & \dots & C(p+r+1,q+r) \\ & & \dots & \\ C(p+r,q) & C(p+r+1,q+1) & \dots & C(p+2r,q+r) \end{pmatrix}.$$

Looking first at the case $q = 0$, subtract from each row the row above it, using the identity $C(a+1, b+1) = C(a, b+1) + C(a, b)$, to obtain

$$\det\begin{pmatrix} C(p,0) & C(p+1,1) & \dots & C(p+r,\ r) \\ 0 & C(p+1,0) & \dots & C(p+r,r-1) \\ & & \dots & \\ 0 & C(p+r,0) & \dots & C(p+2r-1,r-1) \end{pmatrix}.$$

from which an easy induction gives $A = 1$.

Returning to the general q, write $C(i, j) = \frac{i!}{j!\,(i-j)!}$ and observe that the columns have common factors $\frac{1}{q!}, \frac{1}{(q+1)!}, \ldots, \frac{1}{(q+r)!}$, while the rows have common factors $\frac{1}{(p-q)!}, \ldots, \frac{1}{(p+r-q)!}$. Thus

$$A = \frac{1}{q! \ldots (q+r)!\,(p-q)! \ldots (p+r-q)!} \det \begin{pmatrix} p! & (p+1)! & \ldots & (p+r)! \\ (p+1)! & (p+2)! & \ldots & (p+r+1)! \\ & & \ldots & \\ (p+r)! & (p+r+1)! & \ldots & (p+2r)! \end{pmatrix}.$$

For $q = 0$ this gives

$$1 = \frac{1}{0!1! \ldots r!p!(p+1)! \ldots (p+r)!} \det \begin{pmatrix} p! & \ldots & (p+r)! \\ & \ldots & \\ (p+r)! & \ldots & (p+2r)! \end{pmatrix}$$

which evaluates the determinant of factorials: Thus

$$A = \frac{p!\,(p+1)! \ldots (p+r)!0!1! \ldots r!}{q!\,(p+1)! \ldots (q+r)!\,(p-q)! \ldots (p+r-q)!}.$$

This may be rewritten in many forms, such as

$$A = \frac{C(p,q)\,C(p+1,q) \ldots C(p+r,q)}{C(q,q)\,C(q+1,q) \ldots C(q+r,q)}$$

or

$$A = \frac{C(p+r,q)\,C(p+r,q+1) \ldots C(p+r,q+r)}{C(p+r,0)\,C(p+r,1) \ldots C(p+r,r)}.$$

Common Birthdays

Problem Y-6. Proposed by Steve Plett, Fullerton College, Fullerton, CA.

What is the smallest number of people needed so that the probability that at least three of them share a common birthday is at least 50%? More generally, what is the probability that at least three of m people share a birthday?

Solved by Robert Bernstein, Mohawk Valley Community College, Utica, NY; and the proposer.

For a group of m people, one has

$$p = 1 - \frac{N}{365^m}$$

where N is the number of ways to assign birthdays to m people so that no more than two have a common birthday. The number N is a sum of terms, indexed by the number of pairs i of people who have a common birthday, where i ranges from 0 to $\left\lfloor \frac{m}{2} \right\rfloor$. With a given i value, one chooses $m-i$ birthdays from the set of 365 possible birthdays and chooses i of these $m-i$ days to be these used for pairs of people. If the people are chosen in order ($m!$ ways) and the first i pairs of people are assigned to the i duplicated days, one can rearrange the i pairs in $(2!)^i$ ways without changing the assignments of birthdays. Thus

$$N = \sum_{i=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \binom{365}{m-i}\binom{m-i}{i}\frac{m!}{(2!)^i}.$$

Calculations with a computer indicate that with 87 people the probability of having three or more people with a common birthday is about 49.946%, and 88 people are needed to get a probability of over 50%.

AMATYC Reviewers

Sunday A. Ajose	East Carolina University	Greenville, NC
Patricia Allaire	Queensborough C.C.	Bayside, NY
Charles Ashbacher	Kirkwood College	Hiawatha, IA
Joe Dan Austin	Rice University	Houston, TX
John W. Bailey	Clark State C.C.	Springfield, OH
Richelle Blair	Lakeland Comm. College	Mentor, OH
Barbara Bohannon	Hofstra University	Hempstead, NY
Randall Brian	Vincennes University	Vincennes, IN
Robert Decker	University of Hartford	W. Hartford, CT
John DeCoursey	Vincennes University	Vincennes, IN
David Dyer	Prince George's C.C.	Largo, MD
Joseph R. Fiedler	California State University	Bakersfield, CA
Kathleen Finch	Shoals Comm. College	Muscle Shoals, AL
Gregory D. Foley	Sam Houston State University	Huntsville, TX
Richard Francis	Southeast Missouri State University	Cape Girardeau, MO
Charles Franke	Eastern Kentucky University	Richmond, KY
Florence Gordon	New York Institute of Technology	Old Westbury, NY
Sheldon Gordon	Suffolk County C.C.	Selden, NY
Chitra Gunawardena	Univ. of Wisconsin-Fox Valley	Menasha, WI
K. L. D. Gunawardena	Univ. of Wisconsin-Oshkosh	Oshkosh, WI
Russell Gusack	Suffolk County C.C.	Selden, NY
Bruce Haney	Onondaga Community College	Syracuse, NY
Ronald Harkins	Miami University	Hamilton, OH
Peter Herron	Suffolk County C.C.	Selden, NY
Ann D. Holley		Los Gatos, CA
Larry Lance	Columbus State C.C.	Columbus, OH
Jim Langley	Northeast State Technical C.C.	Blountville, TN
Michael Lanstrum	Kent State University-Geauga	Burton, OH
Edward Laughbaum	Columbus State C.C.	Columbus, OH
Deborah Levine	Nassau C.C.	Garden City, NY
Ved P. Madan	Indiana University East	Richmond, IN
Richard F. Maruszewski	United States Naval Academy	Annapolis, MD
John Mathews	California State University	Fullerton, CA
George Matthews	Onondaga Community College	Syracuse, NY
Pamela E. Matthews		Vancouver, WA
Mary McCarty	Sullivan C.C.C.	Loch Sheldrake, NY
Art Merifield	Seneca College	Toronto, CANADA
Coreen Mett	Radford University	Radford, VA
Ted Moore	Mohawk Valley Community College	Utica, NY
Kylene Norman	Clark State C.C.	Springfield, OH
Terry A. Nyman	Univ. of Wisconsin-Fox Valley	Menasha, WI
Carol Olmsead	University of Nevada	Reno, NV
Joan Page	Onondaga Community College	Syracuse, NY
Deborah Parker	Mississippi County C.C.	Blytheville, AR
Don Pfaff	University of Nevada-Reno	Reno, NV
Stephen Plett	Fullerton College	Fullerton, CA
Ben Pollina	University of Hartford	W. Hartford, CT
Douglas Robertson	Univ. of Minnesota	Minneapolis, MN
Jack W. Rotman	Lansing C.C.	Lansing, MI
Alain Schremmer	Community College of Philadelphia	Philadelphia, PA
Gene Sellers	Sacramento City College	Sacramento, CA
Brian Smith	Dawson College	Quebec, CANADA
J. Sriskandarajah	Univ. of Wisconsin	Richland Center, WI
Leslie R. Tanner	The College of Idaho	Caldwell, ID
Marcia Weisser	New York Institute of Technology	Old Westbury, NY
John Williams	University of Hartford	W. Hartford, CT
Margaret Willis	Piedmont Virginia C.C.	Charlottesville, VA

WHY JOIN AMATYC?

The American Mathematical Association of Two-Year Colleges (AMATYC) was established in 1974 to provide a unique, national forum for two-year college mathematics educators. Today, AMATYC is the only national organization that exclusively serves the needs and purposes of this group.

AMATYC holds a national conference annually in a major city. AMATYC encourages two-year college mathematicians to assume responsible leadership positions; to interact on an equal basis with four-year university personnel; to be members on a proportional basis of national steering or policy committees; to represent at the national level the concerns of two-year college mathematics educators. *The AMATYC Review*, published twice yearly, provides an opportunity to publish articles by and for two-year college faculty. A national newsletter is published three times yearly for all members. AMATYC is a member of the National Conference Board of the Mathematical Sciences.

REGULAR MEMBERSHIP APPLICATION The American Mathematical Association of Two-Year Colleges

Mail to: AMATYC Office, State Technical Institute at Memphis, 5983 Macon Cove, Memphis, TN 38134

First Name MI Last Name Position

College Phone

College Address E-Mail

City State Zip

Residence Address Phone

City State Zip

Indicate preferred mailing address: ☐ College ☐ Residence

All payments in U.S. funds payable to AMATYC. Membership fee:

☐ $50 Yearly Regular Membership (any person interested in mathematics education at the two-year college level)

☐ $10 Yearly Associate Membership (full-time student, no voting privileges)
Name of AMATYC Member Sponsor ____________________

☐ $1000 Regular Life Membership

Membership includes *The AMATYC Review* and *The AMATYC News*. In addition, the following journals are available at extra cost:

☐ *Mathematics and Computer Education Journal* $20 for 3 issues

☐ *The College Mathematics Journal* $43 for 5 issues

Total Amount Enclosed ____________ Date Paid ____________

OPTIONAL DEMOGRAPHIC INFORMATION (Please check one in each category)

Category 1

☐ African American
☐ American Indian/Alaskan Native
☐ Asian
☐ Hispanic
☐ White, not Hispanic
☐ Other, please specify ____________

Category 2

☐ Female
☐ Male

Category 3

☐ Two-year college
☐ Four-year college
☐ Other, please specify ____________

Rev 5, 1994

LIBRARY SUBSCRIPTION

Annual Library Subscription Fee: $50 [Includes *The AMATYC Review* (Fall and Spring) and *The AMATYC News* (Fall, Winter and Spring)]

Name of Library

Name of Institution

Address

City State Zip

NOTE: This application form is your invoice for the subscription fee and payment must accompany the completed application. Retain a photocopy of this form for your records if necessary. Please make check payable in U.S. funds to AMATYC and mail to:

AMATYC Office, State Technical Institute at Memphis,
5983 Macon Cove, Memphis, TN 38134

Rev S. 1994

INSTITUTIONAL MEMBERSHIP

Colleges, publishers, and other institutions can support AMATYC by being Institutional Members. As indicated below, the annual membership fee is $250. Subscriptions to *The AMATYC Review* and *The AMATYC News* are included in this fee. Of particular importance to collegiate institutional members is the AMATYC committee for mathematics department chairpersons.

An additional benefit of institutional membership is one *complimentary* AMATYC conference early registration. Future conventions, which are held in the Fall, will be in Tulsa (1994) and Little Rock (1995). Institutional members support the excellence of the programs at these annual conferences.

INSTITUTIONAL MEMBERSHIP APPLICATION/INVOICE

Mail to: AMATYC Office, State Technical Institute at Memphis,
5983 Macon Cove, Memphis, TN 38134

AMATYC College Contact Person Position

Name of Institution

Address

City State Zip

Membership Fee – $250 in U.S. funds payable to AMATYC (includes *The AMATYC Review*, *The AMATYC News*, and one complimentary conference early registration)
Note: Institutional membership does not include any voting privileges.

Rev S. 1994

Lucky Larry #17

Larry had an ingeneous way to solve

$$(x-1)(3x+2)=4x.$$

Subtracting $4x$ from each *factor* he proceeded to get

$$(x-4x-1)(3x-4x+2)=0$$

$$(-3x-1)(-x+2)=0$$

$$x=\frac{-1}{3} \quad x=2$$

both of which check! Maybe Lucky Larry is more sophisticated than we think. Could he have realized that equation is of the form $(ax+b)(cx+d) = ex$ where $c+a=e$ and $b+d=1$, and therefore his method would work?

Submitted by Ira Rosenthal
Palm Beach Community College
Palm Beach Gardens FL 33410

Advertiser's Index

Addison-Wesley Publishing Co. .... p. 29
Arrowhead Distributors, Inc .... Inside front cover
D.C. Heath and Co. .... p. 63
H & H Publishing Co., Inc. .... p. 13
JEMware .... p. 3
John Wiley & Sons, Inc. .... p. 17
KRIEGER Publishing Co. .... p. 82
MathWare .... p. 32
McGraw-Hill, Inc. .... p. 24
Princeton University Press .... p. 7
PWS Publishing Co. .... p. 71
Saunders College Publishing .... p. 36
West Publishing Corp. .... p. 52

90

AMATYC Institutional Members

(as of May 31, 1994)

Addison-Wesley Pub. Co., Reading, MA 01867
Albuquerque Tech-Voc Inst., Albuquerque, NM 87106
Anoka Ramsey Comm. College, Coon Rapids, MN 55433
Austin Comm. College (RGC Campus), Austin, TX 78701
Baltimore City Comm. College, Baltimore, MD 21215
Bellevue Comm. College, Bellevue, WA 98007
Bermuda College, Devonshire, Bermuda DVBX
Bristol Comm. College, Fall River, MA 02720
Bucks County Comm. College, Newton, PA 18940
Burlington County College, Pemberton, NJ 08068
Butte College, Oroville, CA 95965
Cabot College of Applied Arts, St. John's, NF A1C 5P7
Charles County C. C., La Plata, MD 20646
Clark State Comm. College, Springfield, OH 45502
Clovis Comm. College, Clovis, NM 88101
College of DuPage, Glen Ellyn, IL 60137
College of Lake County, Grayslake, IL 60030
Columbus State Comm. College, Columbus, OH 43216
Comm. College of Philadelphia, Philadelphia, PA 19130
Comm. College of Southern Nevada, N. Las Vegas, NV 89030
C. S. Mott Comm. College, Flint, MI 48503
County College of Morris, Randolph, NJ 07869
Cuyahoga Comm. College, Parma, OH 44130
Dean Junior College, Franklin, MA 02038
De Kalb College South, Decatur, GA 30034
Delaware County Comm. College, Media, PA 19063
Delta College, University Center, MI 48710
East Central College, Union, MO 63084
El Paso Comm. College, El Paso, TX 79998
Erie Community College - South, Buffalo, NY 14127
F.H. Laguardia Comm. College, Long Island City, NY 11101
Florida C. C. at Jxon, Jacksonville, FL 32046
Fox Valley Tech. Inst., Appleton, WI 54913
Fullerton College, Fullerton, CA 92634
Galveston College, Galveston, TX 77550
Garland County Comm. College, Hot Springs, AR 71914
Genessee Comm. College, Batavia, NY 14020
Grant Mac Ewen College, Edmonton, AB T5J 2P2
Green River College, Auburn, WA 98002
Harding University, Searcy, AR 72149
Harrisburg Area C.C., Harrisburg, PA 17110
Heartland Comm. College, Bloomington, IL 61701
Houghton Mifflin Co., Boston, MA 02116
Houston Comm. College-Stafford, Stafford, TX 77477
Howard Comm. College, Columbia, MD 21044
Illinois Central College, East Peoria, IL 61635
Illinois State University, Normal, IL 61761
Inver Hills C. C., Inver Grove Hts., MN 55076
Joliet Jr. College, Joliet, IL 60436
Kapiolani Comm. College, Honolulu, HI 96816
Kean College of New Jersey, Union, NJ 07083
Kennebec Valley Tech College, Fairfield, ME 04937
Kennedy-King College, Chicago, IL 60621
Lane Comm. College, Eugene, OR 97405
Lassen College, Susanville, CA 96130
Madison Area Tech College, Madison, WI 53704
Marine Institute, St. John's, NF A1C 5R3
Massachusetts Bay Comm. College, Wellesly Hills, MA 02181
Massasoit Comm. College, Brockton, MA 02402
Metropolitan Comm. College, Omaha, NE 68103
Miami Dade Community College, Miami, FL 33132
Middlesex County College, Edison, NJ 08818
Minneapolis C.C., Minneapolis, MN 55403
Mohawk Valley Comm. College, Utica, NY 13501
Montgomery College, Takoma Park, MD 20912
Moraine Valley Comm. College, Palos Hills, IL 60465
Mount Royal College, Calgary, AB T3E 6K6
Mt. Wachusett Comm. College, Gardner, MA 01440
Naugatuck Valley Comm. Tech. Coll., Waterbury, CT 06708
New Mexico State University, Las Cruces, NM 88003
Normandale Comm. College, Bloomington, MN 55431
North Hennepin C. C., Minneapolis, MN 55455
North Idaho College, Coeur D'Alene, ID 83814
North Lake College, Irving, TX 75038
North Seattle Comm. College, Seattle, WA 98103
North Shore Comm. College, Danvers, MA 01923
Northern Arizona University, Flagstaff, AZ 86011
Oakton Comm. College, Des Plaines, IL 60016
Onondaga Comm. College, Syracuse, NY 13215
Oregon Inst. of Technology, Klamath Falls, OR 97601
Parkland College, Champaign, IL 61821
Pensacola Jr. College - Pensacola, Pensacola, FL 32504
Pima Comm. College, Tuscon, AZ 85703
Polk Comm. College, Winter Haven, FL 33881
Portland Comm. College, Portland, OR 97280
Prince George's Comm. College, Largo, MO 20772
Raritan Valley C. C., Somerville, NJ 08876
Richard D. Irwin Inc., St. Paul, MN 55105
Robert Morris College, Coraopolis, PA 15108
Rock Valley College, Rockford, IL 61114
San Francisco State University, San Francisco, CA 94116
San Juan College, Farmington, NM 87402
Santa Barbara City College, Santa Barbara, CA 93109
Schoolcraft College, Livonia, MI 48152
Southeast Mo. State University, Cape Girardeau, MO 63701
Springfield Tech Comm. College, Springfield, MA 01101
State Tech Inst. at Memphis, Memphis, TN 38134
St. Charles Co. Comm. Coll., St. Peters, MO 63376
St. Louis Comm College, St. Louis, MO 63135
SUNY Ag & Tech College, Alfred, NY 14802
Tarrant County Jr. College S., Ft. Worth, TX 76119
The College Board, Philadelphia, PA 19104
Three Rivers Community College, Poplar Bluff, MO 63901
Tulsa Junior College, Tulsa, OK 74133
UND - Williston, Williston, ND 58801
Univ. of Alaska-Anchorage, Anchorage, AK 99508
University of Wyoming, Laramie, WY 82071
William Rainey Harper College, Palatine, IL 60067

AMATYC at Tulsa • November 3-6, 1994

(Based at the Tulsa Marriott Southern Hills)

Are you planning on coming to TULSA for the twentieth annual AMATYC conference, November 3-6, 1994? If not, you are planning on missing a really special event. We're all geared up and ready to go! The Tulsa conference will provide the traditional AMATYC events, good friends, good entertainment, challenging sessions and workshops, and distinguished speakers. Yet each AMATYC conference is unique in its own way, and this one is no exception.

Discover the Excellence of the unique features and presentations planned for the 1994 conference:

- The conference will be highlighted by two highly regarded speakers. The Keynote Address will be given during Friday morning's opening session by noted Mathematics educator Uri Treisman of the Charles A. Dana Center for Mathematics and Science Education at the University of Texas at Austin. Treisman will be speaking on "Culture, Curriculum, and Community: Making the Reform of Collegiate Mathematics a Reality." Saturday morning's Breakfast Speaker will be Jaime Pinkham of the American Indian Science and Engineering Society who will motivate and inspire us with his presentation "Looking at a Future Rooted in History and Tradition."
- The Thursday morning offerings at the Doubletree Hotel will include a Concentration in Distance Learning from 8:30 am until 1:00 pm with sessions on the overview, the issues, and distance learning in mathematics.
- There will hands-on workshops on the Internet, first a general introduction to the Internet for people who are computer users and another using the Internet to enhance mathematics instruction.
- Application of mathematics for the classroom will include two features, presentations by professionals of understandable applications of two-year level mathematics and field trips to business and industry to view actual applications related to mathematics.
- Thursday evening the forum will address the issues in position statements proposed on "Student Learning Problems" and "Guidelines for Internships." Also, the group will consider the issues of "Adjunct Faculty" and "Delivering Technology to Students" for the purpose of developing an AMATYC position paper.
- "Standards for Introductory College Mathematics" will be the topic of concern Thursday evening at 7:30 pm in a Forum for all AMATYC members.
- Division and Department Chairs will enjoy the colloquium designed to foster communication for producing innovative curricula to provide students with the skills of the modern workplace.
- There are two special features after the Friday morning breakfast. "Professionalism of Two-Year College Mathematics Faculty" presented by Rikki Blair, Midwest Vice-President, will cover the growth and professional recognition of AMATYC since being founded in 1974. Also, Mr. Jewruh Bandeh of the Tulsa Community Action Agency will lead the session "Making Sense of Cultural Diversity: Learning to Work with People from Different Cultural Backgrounds."
- Friday evening will feature entertainment that can be enjoyed by all. All conference attendees and guests are invited to the Ballroom at the Marriott to be entertained by the Discovery Land Singers who will present excerpts from the award winning musical Oklahoma!. After a short break, the evening will continue with the nationally recognized Stonehorse Band, a versatile group with a line dance instructor so everyone can participate.
- Discover the rich heritage of Tulsa through one or more of the optional tours. The tour guides will take the groups to visit historical sites such as the Greenwood Black Heritage Center, mansions of oil millionaires, the Gilcrease and Philbrook museums, and the art-deco "skyscrapers" of downtown Tulsa which were built during the oil boom of the 1920s. Other tours will highlight the unique campus of Oral Roberts University and Woolaroc (woods, lakes, and rocks), the nature preserve established by Frank Phillips. The Saturday evening tour will begin with dinner at a restaurant in the historic district, then will proceed to Tulsa's Spotlight Theater to see "The Drunkard," a melodrama which has delighted viewers for over forty years.

For Registration Information, contact:

AMATYC • State Technical Institute at Memphis • Memphis, TN 38134 • (901) 383-4643
Fax (901) 383-4503 • Email AMATYC@STIM.TEC.TN.US

92

American Mathematical Association of Two-Year Colleges

EXECUTIVE BOARD 1993-1995

President
Marilyn Mays
North Lake College
5001 MacArthur Blvd.
Irving, TX 75038-3899
(214) 659-5238

President-Elect
Wanda Garner
Cabrillo College
6500 Soquel Drive
Aptos, CA 95003
(408) 479-6329

Treasurer
Robert Malena
CCAC-South
1750 Clairton Rd.
W. Mifflin, PA 15122
(412) 469-6228

Secretary
Martha Clutter
Piedmont Virginia C.C.
Rt. 6, Box 1
Charlottesville, VA 22902-8714
(804) 977-3900

Past President
Karen Sharp
Mott Community College
1401 E. Court Street
Flint, MI 48503
(810) 232-3980

REGIONAL VICE PRESIDENTS

Northeast
Sadie C. Bragg
Borough of Manhattan C.C.
199 Chambers Street
New York, NY 10007
(212) 346-8820

Mid-Atlantic
Susan S. Wood
J. Sargeant Reynolds C.C.
Box 85622
Richmond, VA 23285-5622
(804) 371-3027

Southeast
Linda Exley
DeKalb College
555 N. Indian Creek Dr.
Clarkston, GA 30021
(404) 299-4214

Midwest
Rikki Blair
Lakeland Comm. College
7700 Clocktower Dr.
Mentor, OH 44060
(216) 953-7341

Central
Helen Banes
Kirkwood Community College
6301 Kirkwood Blvd., SW
Cedar Rapids, IA 52406
(319) 398-5558

Southwest
Therese Jones
Amarillo Jr. College
P.O. Box 447
Amarillo, TX 79178
(806) 371-5091

Northwest
Vicky Ringen
N. Seattle Comm. College
9600 College Way N.
Seattle, WA 98103
(206) 528-4511

West
Bill D. New
Cerritos College
11110 E. Allondra
Norwalk, CA 90650
(310) 860-2451

ACADEMIC COMMITTEES

Developmental Mathematics
Jack Rotman
Lansing Comm. College
P.O. Box 40010
Lansing, MI 48901
(517) 483-1079

Student Mathematics League
Glenn Smith
Santa Fe Comm. College
3000 NW 83rd St.
Gainesville, FL 32606
(904) 395-5306

Technical Mathematics
Rob Kimball
Wake Technical College
9101 Fayetteville Rd.
Raleigh, NC 27603-5696
(919) 772-0551 Ext. 285

Technology in Mathematics Education
Brian Smith
Dawson College
3040 Sherbrooke St. W.
Montreal, Quebec,
Canada H4V2X5
(514) 931-8731 Ext. 1713

Equal Opportunity in Mathematics
Marcella Beacham
Richard J. Daley College
7500 S. Pulaski Rd.
Chicago, IL 60652
(312) 735-3000 Ext. 379

Education
Phil DeMarois
Harper College
1200 W. Algonquin Rd.
Palatine, IL 60067
(708) 397-3000 Ext. 2728

Placement and Assessment
Nancy Sattler
Terra Technical College
2830 Napoleon Rd.
Fremont, OH 43420
(419) 334-8400

Grants
Margie Hobbs
State Technical Institute
5983 Macon Cove
Memphis, TN 38134
(901) 383-4321

The AMATYC REVIEW
Onondaga Community College
Syracuse, New York 13215

Non-Profit
Organization
U.S. Postage
PAID
Permit No. 973
Syracuse, New York

1994 TULSA AMATYC CONVENTION

November 3–6, 1994

Tulsa Marriott
Southern Hills
Tulsa, Oklahoma

Conference Committee Chairperson

Audrey Rose
Tulsa Junior College, Southeast Campus
10300 East 81st Street South
Tulsa OK 74133
(918) 631-7685

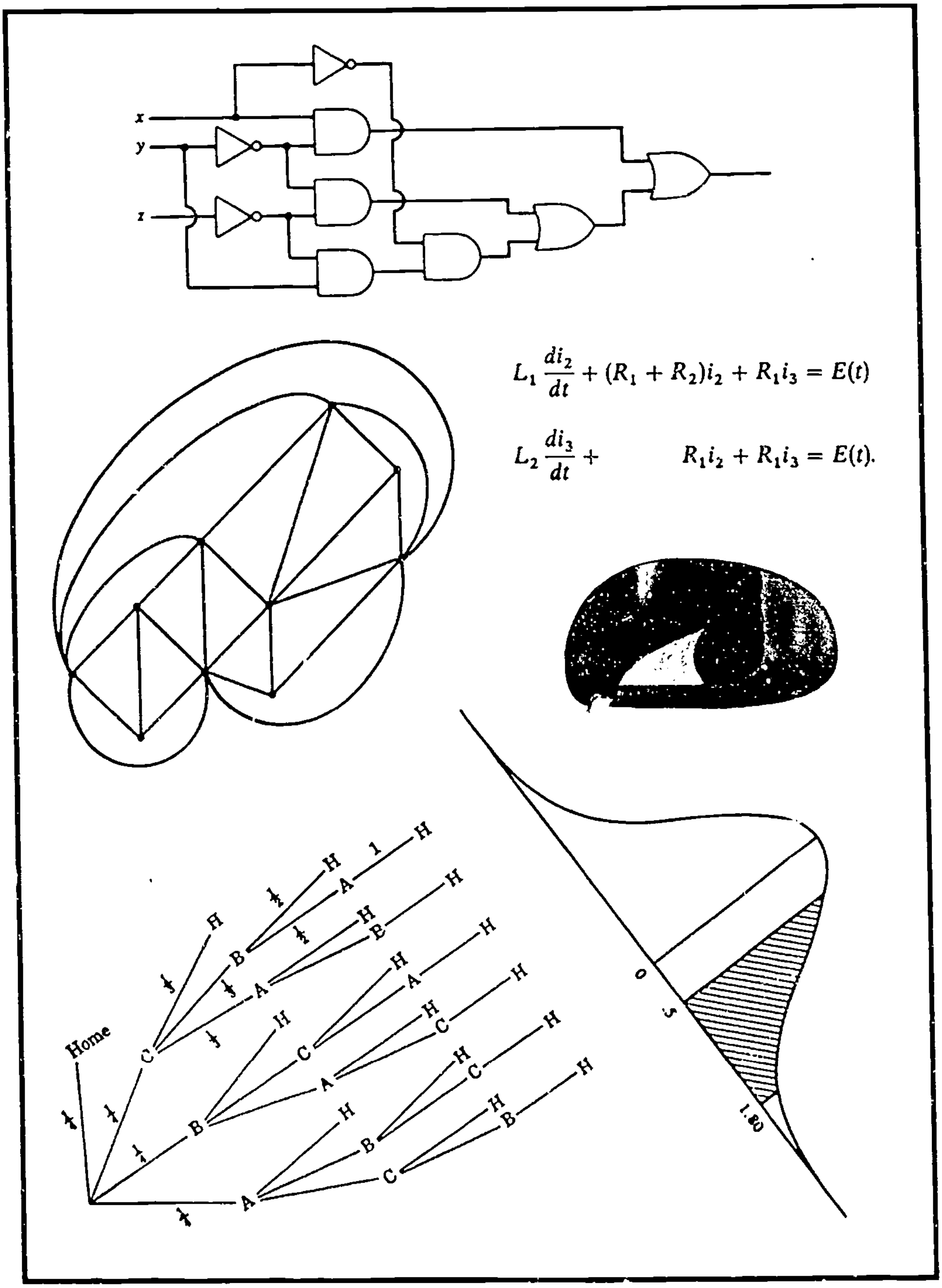
DISCOVER THE EXCELLENCE

ISSN 0740-8404

THE AMATYC REVIEW

Published by the
AMERICAN MATHEMATICAL ASSOCIATION OF TWO-YEAR COLLEGES

VOLUME 16, NUMBER 2 Spring 1995



American Mathematical Association of Two-Year Colleges

EXECUTIVE BOARD 1993-1995

President
Marilyn Mays
North Lake College
5001 MacArthur Blvd.
Irving, TX 75038-3899
(214) 659-5238

President-Elect
Wanda Garner
Cabrillo College
6500 Soquel Drive
Aptos, CA 95003
(408) 479-6329

Treasurer
Robert Malena
CCAC-South
1750 Clairton Rd.
W. Mifflin, PA 15122
(412) 469-6228

Secretary
Martha Clutter
Piedmont Virginia C.C.
Rt. 6, Box 1
Charlottesville, VA 22902-8714
(804) 961-5337

Past President
Karen Sharp
Mott Community College
1401 E. Court Street
Flint, MI 48503
(810) 232-3980

Northeast Regional Vice President
Sadie C. Bragg
Borough of Manhattan C.C.
199 Chambers Street
New York, NY 10007
(212) 346-8820

Mid-Atlantic Regional Vice President
Susan S. Wood
J. Sargeant Reynolds C.C.
Box 85622
Richmond, VA 23285-5622
(804) 371-3027

Southeast Regional Vice President
Linda Exley
DeKalb College
555 N. Indian Creek Dr.
Clarkston, GA 30021
(404) 299-4214

Midwest Regional Vice President
Rikki Blair
Lakeland Comm. College
7700 Clocktower Dr.
Kirtland, OH 44094-5198
(216) 953-7341

Central Regional Vice President
Helen Banes
Kirkwood Community College
6301 Kirkwood Blvd., SW
Cedar Rapids, IA 52406
(319) 398-5558

Southwest Regional Vice President
Therese Jones
Amarillo Jr. College
P.O. Box 447
Amarillo, TX 79178
(806) 371-5091

Northwest Regional Vice President
Vicky Ringen
N. Seattle Comm. College
9600 College Way N.
Seattle, WA 98103
(206) 528-4511

West Regional Vice President
Bill D. New
Cerritos College
11110 E. Allondra
Norwalk, CA 90650
(310) 860-2451 Ext. 2658

ACADEMIC COMMITTEES

Developmental Mathematics
Jack Rotman
Lansing Comm. College
P.O. Box 40010
Lansing, MI 48901
(517) 483-1079

Student Mathematics League
Glenn Smith
Santa Fe Comm. College
3000 NW 83rd St.
Gainesville, FL 32606
(904) 395-5297

Technical Mathematics
Rob Kimball
Wake Technical College
9101 Fayetteville Rd.
Raleigh, NC 27603-5696
(919) 772-0551 Ext. 285

Technology in Mathematics Education
Brian Smith
Dawson College
3040 Sherbrooke St. W.
Montreal, Quebec,
Canada H4V2X5
(514) 931-8731 Ext. 1714

Equal Opportunity in Mathematics
Marcella Beacham
Richard J. Daley College
7500 S. Pulaski Rd.
Chicago, IL 60652
(312) 838-7649

Education
Phil DeMarois
Harper College
1200 W. Algonquin Rd.
Palatine, IL 60067
(708) 397-3000 Ext. 2728

Placement and Assessment
Nancy Sattler
Terra Technical College
2830 Napoleon Rd.
Fremont, OH 43420
(419) 332-1002 Ext. 226

Grants
Margie Hobbs
State Technical Institute
5983 Macon Cove
Memphis, TN 38134
(901) 383-4321

AIIM

Association for Information and Image Management

1100 Wayne Avenue, Suite 1100
Silver Spring, Maryland 20910
301 587 8202

Centimeter

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 mm

1 2 3 4 5

Inches

1.0 1.1 1.25 1.4 1.6 1.8 2.0 2.2 2.5 2.8 3.2 3.6 4.0

MANUFACTURED TO AIIM STANDARDS
BY APPLIED IMAGE, INC.

The Official Journal of the

American Mathematical Association of Two-Year Colleges

MISSION OF AMATYC: Recognizing the vital importance of the first two years of collegiate mathematical education to the future of our students and the welfare of our nations, AMATYC is committed to the following:

- to positively impact the preparation of scientifically and technologically literate citizens;
- to lead the development and implementation of curricular, pedagogical, assessment and professional standards for mathematics in the first two years of college;
- to assist in the preparation and continuing professional development of a quality mathematics faculty that is diverse with respect to ethnicity and gender;
- to provide a network for communication, policy determination, and action among faculty, other professional organizations, accrediting associations, governing agencies, industries, and the public sector

The AMATYC Review provides an avenue of communication for all mathematics educators concerned with the views, ideas and experiences pertinent to two-year college teachers and students.

SUBMISSION OF MANUSCRIPTS: Manuscripts must be typed, doubled-spaced, on one side of 8-1/2" x 11" paper. They should not have been published before, nor should they be under consideration for publication elsewhere. To provide for anonymous reviews, the author's name and affiliation should appear on a separate title page. The title should also appear on the first page of the exposition. Authors are advised to consult the *Publication Manual of the American Psychological Association*. A guideline for authors is available from the editor and is also printed in the Fall 1993 issue. Five copies of each manuscript should be submitted to Joseph Browne, Onondaga Community College, Syracuse, NY 13215.

ADVERTISING: For information concerning advertising rates and schedules, contact the advertising manager, Larry Lance, at the address given below.

PUBLICATION: *The AMATYC Review* is published twice a year in the Fall and Spring.

ISSN 0740-8404

The AMATYC Review Spring 1995 Volume 16, Number 2

TABLE OF CONTENTS

EDITOR'S COMMENTS AND ABOUT THE COVER .... p. 4

LETTER TO THE EDITOR .... p. 6

MATHEMATICAL EXPOSITION

The Apotheosis of the Apothem .... p. 7
by Steven Schwartzman

Relaxation Functions .... p. 15
by Homer B. Tilton

A Pattern for the Squares of Integers .... p. 22
by Kim Mai

SHORT COMMUNICATIONS

$\lim_{n \to \infty} \left\{ \left(1 + \frac{1}{n} \right)^j \right\}_{j=0}^{j=n}$ Produces e^x .... p. 28
by Kurtis D. Fink and John H. Mathews

Lucky Larry Meets Thomas Simpson .... p. 30
by John deValcourt

MATHEMATICS EDUCATION

Using an $n \times m$ Contingency Table to Determine Bayesian Probabilities: An Alternative Strategy .... p. 34
by Eiki Satake, William Gilligan, and Philip Amato

An Appropriate Culminating Mathematics Course .... p. 45
by Bill Haver and Gwen Turbeville

Community College Success in an International Mathematics Competition .... p. 52
by John Loase and Rowan Lindley

REGULAR FEATURES

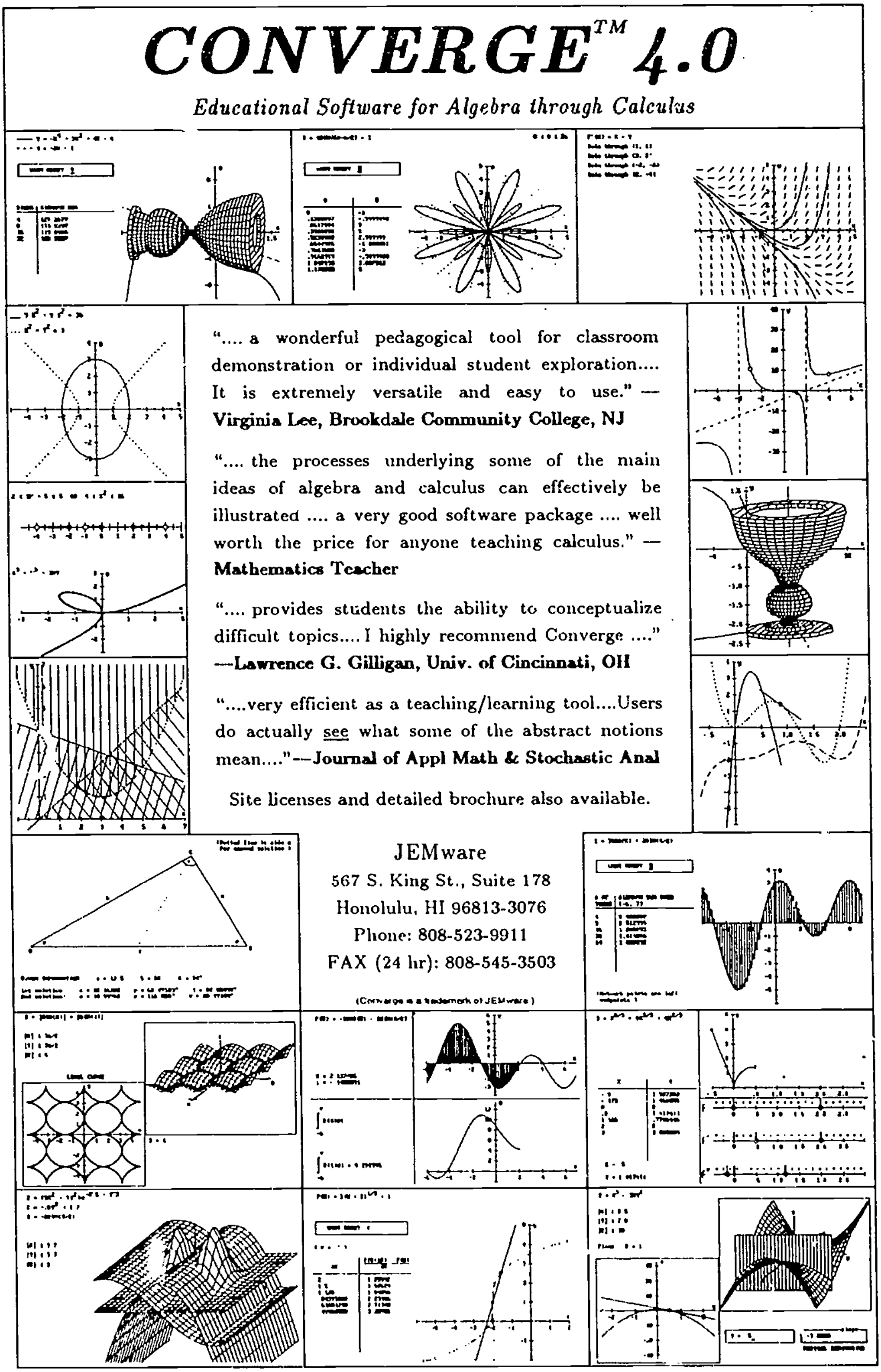
The Chalkboard .... p. 56
Edited by Judy Cain and Joseph Browne

Snapshots of Applications in Mathematics .... p. 59
Edited by Dennis Callas and David J. Hildreth

Notes from the Mathematical Underground .... p. 63
Edited by Alain Schremmer

The Problem Section .... p. 67
Edited by Michael W. Ecker

Advertiser's Index .... p. 73



CONVERGE™ 4.0
Educational Software for Algebra through Calculus
".... a wonderful pedagogical tool for classroom demonstration or individual student exploration.... It is extremely versatile and easy to use." — Virginia Lee, Brookdale Community College, NJ
".... the processes underlying some of the main ideas of algebra and calculus can effectively be illustrated .... a very good software package .... well worth the price for anyone teaching calculus." — Mathematics Teacher
".... provides students the ability to conceptualize difficult topics.... I highly recommend Converge ...." —Lawrence G. Gilligan, Univ. of Cincinnati, OH
"....very efficient as a teaching/learning tool....Users do actually see what some of the abstract notions mean...."—Journal of Appl Math & Stochastic Anal
Site licenses and detailed brochure also available.
JEMware
567 S. King St., Suite 178
Honolulu, HI 96813-3076
Phone: 808-523-9911
FAX (24 hr): 808-545-3503
(Converge is a trademark of JEMware.)

Editor's Comments and About the Cover

Can Two-Year Colleges and Four-Year Colleges Compete?

We have a pretty good women's volleyball team at my college. They're not quite national champion caliber, but frequently best in the northeast. This has given the coach an additional problem: finding challenging opponents so that the team can improve. Since long distance travel is too expensive, a natural thought was to play teams from four-year colleges. But four-year college teams don't want to play two-year college teams; it's a "no win" situation for them. If the four-year team wins, well that's just what was supposed to happen; if they lose, it looks terrible for them.

In mathematics, however, there is a place where two-year and four-year colleges can compete on equal footing: the Mathematical Contest in Modeling. This competition, now in its eleventh year, is unique in that the competitors are teams of three students working together to solve one open-ended, realistic problem (chosen from two) which calls for the application of mathematical tools and techniques. The teams have a long weekend to work on the problem, receiving it on Friday and having to get their solution postmarked on Monday. Teams may, and often do, work around the clock. They may use books, computers, libraries, anything except other humans. The team's solution must be typed and must address the assumptions made, justification of the assumptions, analysis and design of the model, results, strengths and weaknesses of the model, and references. Here is one of the problems from the 1993 competition:

> The Aspen-Boulder Coal Company runs a loading facility consisting of a single large coal tipple. When coal trains arrive, they are loaded from the tipple. The standard train takes 3 hours to load and the tipple's capacity is 1-1/2 standard trainloads of coal. Each day, the railroad sends three standard trains to the loading facility and they arrive any time between 5 AM and 8 PM local time. Each of these trains has three engines. If a train arrives and sits idle while waiting to be loaded, the railroad charges a special fee, called a *demurrage*. The fee is $5000 per engine per hour. In addition, a high-capacity train arrives once a week every Thursday between 11 AM and 1 PM. This special train has five engines and holds twice as much coal as a standard train. An empty tipple can be loaded directly from the mine to its capacity in 6 hours by a single loading crew. This crew (and its associated equipment) costs $9000 per hour. A second crew can be called out to increase the loading rate by conducting an additional tipple-loading operation at the cost of $12,000 per hour. Because of safety requirements, during tipple loading no trains can be loaded. Whenever train loading is interrupted to load the tipple, demurrage charges are in effect.
>
> The management of the Coal Company has asked you to determine the expected annual costs of this tipple's loading operations. Your analysis should include the following considerations:
>
> a) How often should the second crew be called out?

b) What are the expected monthly demurrage costs?

c) If the standard trains could be scheduled to arrive at precise times, what daily schedule would minimize loading costs?

d) Would a third tipple-loading crew at $12,000 per hour reduce annual operations costs?

e) Can this tipple support a fourth standard train every day?

The judging is "blind," meaning the judges don't know whether the paper they are looking at came from Cal-Berkeley or some place they've never heard of. Last year (1994) over three hundred teams were entered, mostly from the United States, Canada, and China, with a few from other nations. About the best third of the papers receive one of the recognitions: Outstanding, Meritorious, or Honorable Mention. Last year six "Outstandings" were awarded (one to a team from the North Carolina School of Science and Mathematics, one of two high schools participating). Twelve two-year colleges entered in 1994, up from six the year before. They earned four Honorable Mentions, up from two in 1993. On two occasions two-year colleges have earned a Meritorious designation: Westchester (NY) Community College in 1991 and Midlands Technical College (SC) in 1992.

It would seem at first that two-year colleges would be at a considerable disadvantage in this contest. After all, the four-year colleges enroll most of the brightest students and their students have had more mathematics. While this is, in general, true, there is another factor which is quite important in this competition and favors the two-year college: *maturity*. The average age of students at two-year colleges (29 at my college) is significantly higher than at four-year colleges. Our students have greater "real world" experience and tend to be more pragmatic. These are relevant qualities in mathematical modeling. The ages of the Midlands Tech team mentioned above were 34, 31, and 26. The ages of an honorable mention team I entered in 1993 were 32, 28, and 24.

One of the most consistently successful two-year colleges in this competition has been Westchester Community College. John Loase and Rowan Lindley have coached these teams. In this issue they tell how they go about selecting teams and preparing them. Other schools have used other methods, and some teams don't really do any preparation. In my own case, I believe that finding students with the right combination of interest, motivation, and ability is the major factor contributing to success. I then customize the preparation to what they are willing to do.

The competition takes place in February. Teams work wherever they wish to, sc no traveling to a specific site is involved. There is no fee for participating. Major funding comes from the National Security Agency, with additional support from the Operations Research Society of America and SIAM. The competition is conducted by the Consortium for Mathematics and Its Applications. For more information and registration materials contact COMAP, Suite 210, 57 Bedford Street, Lexington MA 02175-4496.

Letter to the Editor:

Re: *The Chalkboard* in the Spring, 1994, issue. I would like to point out the connection between Horner's method and the synthetic division algorithm. An example should suffice. Consider evaluating $2x^3 - 3x^2 + x - 4 = x(x(x(2) - 3) + 1) - 4$ for $x = -2$ by synthetic division.

−2	2	−3	1	−4
		−4	14	−30
	2	−7	15	−34

It is easily seen that this process is a tabular form for Horner's method. Also, this format is easier to deal with when some coefficients are zero.

Philip Mahler
Middlesex (MA) Community College

MATHEMATICAL EXPOSITION

The Apotheosis of the Apothem

by

Steven Schwartzman
Austin Community College
Austin TX 78701

Steven Schwartzman is the author of The Words of Mathematics, *a reference book that traces the origins of over 1500 mathematical terms. Steven currently teaches part-time at Austin Community College.*

Calculus students sometimes notice — and calculus teachers sometimes point out to them — that the derivative of the area of a circle with respect to the radius is the circumference of the circle, and the derivative of the volume of a sphere with respect to the radius is the surface area of the sphere:

Circle	Sphere
$A = \pi r^2$	$V = \dfrac{4}{3}\pi r^3$
$\dfrac{dA}{dr} = 2\pi r = C$	$\dfrac{dV}{dr} = 4\pi r^2 = S.A.$

(1)

The relationships are "interesting," "curious," or "neat," and that's usually the end of the subject. But is there more to it than that? Does the same differential relationship apply to other figures, or are circles and spheres special in this regard, as they are in so many other ways?

To answer these questions, let us begin by examining what happens with a square, which has properties that distinguish it from other quadrilaterals and even from other rectangles. If s represents the length of a side of the square, then the area of the square is given by $A = s^2$ and the perimeter of the square is given by $p = 4s$. Since $\dfrac{dA}{ds} = 2s$, it is not true that the derivative of the area of a square with respect to a side equals the perimeter. Perhaps a more favorable result can be obtained with a different choice of variable. After all, if the area and circumference of the circle had been given in terms of the diameter d, we would have had $A = \dfrac{1}{4}\pi d^2$ and $\dfrac{dA}{dd} = \dfrac{1}{2}\pi d \neq \pi d = C$, so the derivative of the area of a circle with respect to its diameter doesn't equal the circumference.

Taking our clue from (1), we ask what segment in a square is akin to the radius of a circle. One reasonable choice is the segment of length w that goes from the center of the square to any vertex, as shown in figure 1. In that case, the area of the square is given by $A = 2w^2$, and the perimeter is given by $p = 4w\sqrt{2}$, so $\frac{dA}{dw} \neq p$.

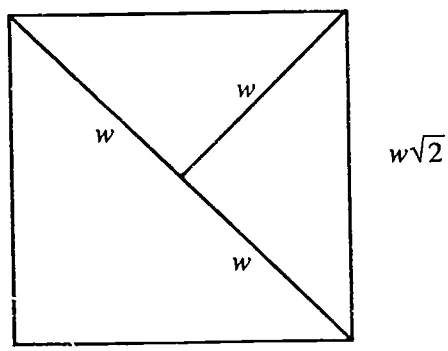


Figure 1: Half the diagonal is chosen as the basic unit.

Perhaps the differential relationship failed because the chosen segment didn't meet each side of the square perpendicularly, as every radius meets its circle perpendicularly. With that in mind, the other plausible choice for the basic unit of a square is a segment of length h dropped perpendicularly from the center of the square to the center of any side, as shown in figure 2. Such a segment in a regular polygon is known as an apothem. (The word is derived from Greek *apo* "away (from)" and the root of the verb *tithenai* "to set, to put," so an apothem is literally a segment that "sets out away from" the center of a regular polygon and goes to the nearest point on a side.) This time the area of the square is given by $A = 4h^2$, and the perimeter is given by $8h$, so $\frac{dA}{dh} = p$.

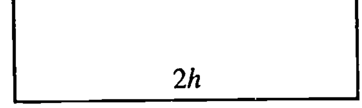


Figure 2: Half the height is chosen as the basic unit.

Having now found what seems to be the correct basic unit, we might try a rectangle less special than the square to see if the differential relationship still holds. Figure 3 shows a rectangle twice as wide as it is high. The area of that rectangle is given by $A = 8h^2$, and the perimeter is given by $p = 12h$, so $\frac{dA}{dh} \neq p$. Why did the differential relationship fail this time? In a circle all radii are equal, whereas in the oblong rectangle the two "radii" are unequal. If all basic segments in a polygon are to be equal, the figure in question must be a regular polygon, and the basic segment will be an apothem.

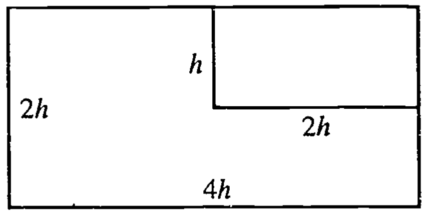


Figure 3: An oblong rectangle twice as wide as it is high.

Figure 4 shows part of a regular n-gon circumscribed about a circle of radius r. Triangle OVX is one of the n congruent triangles that make up the polygon. Central angle $\theta = \frac{\pi}{n}$, and $WX = r\tan\frac{\pi}{n}$. The area of $\Delta VOX = r \cdot WX$, so $A_{\Delta VOX} = r^2 \tan\frac{\pi}{n}$, and $A_{n\text{-gon}} = nr^2 \tan\frac{\pi}{n}$. At the same time the perimeter of the polygon is given by $p_{n\text{-gon}} = n \cdot 2 \cdot WX = 2nr\tan\frac{\pi}{n}$. As a result, for any regular polygon circumscribed about a circle, $\frac{dA}{dr} = p$.

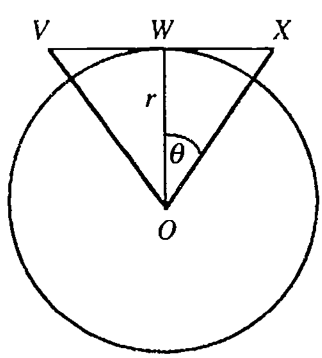


Figure 4: Part of a regular n-gon circumscribed about a circle.

Further thought reveals that the theorem is even more general. Instead of a regular polygon, imagine *any* polygon that can be circumscribed about a circle, as shown in Figure 5. This time ΔOLM is purposely drawn so as not to be congruent to ΔOLK. As before, $LM = r\tan\theta$, and the area of ΔOLM is given by

$$A_{\Delta OLM} = \frac{1}{2} r \cdot LM = \frac{1}{2} r^2 \tan\theta.$$

It is now clear that $\frac{dA_{\Delta OLM}}{dr} = LM$. The entire polygon is made up of $2n$ not-necessarily-congruent triangles like ΔOLM and the perimeter of that polygon is made up of $2n$ not-necessarily-equal segments like $\overline{LM}$.

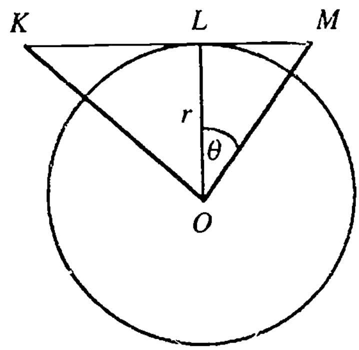


Figure 5: Part of an *n*-gon circumscribed about a circle.

Because the derivative of a sum is the sum of the derivatives, the following theorem has been proved:

Theorem 1: If a polygon is circumscribed about a circle of radius r, and if both the area A and perimeter p of the polygon are expressed in terms of r, then $\frac{dA}{dr} = p$.

The theorem can be extended into three dimensions by considering a polyhedron circumscribed about a sphere with radius r and center at O. A piece of such a polyhedron is shown in Figure 6. Triangle ZDE, which is part of one face of the polyhedron, lies in the plane tangent to the sphere at point Z. Let $\angle ZOE = \alpha$, $\angle ZOD = \beta$, and $\angle DZE = \delta$. Then $ZE = r\tan\alpha$, $ZD = r\tan\beta$, and the area of triangle ZDE is given by $A_{\Delta ZDE} = \frac{1}{2} \cdot ZD \cdot ZE \cdot \sin\delta$, or $A_{\Delta ZDE} = \frac{1}{2} r^2 \tan\alpha \tan\beta \sin\delta$.

Because of right angles OZD and OZE, the volume of tetrahedron $ZDEO$ is

$V_{ZDEO} = \frac{1}{3} r \cdot A_{\Delta ZDE}$, or $V_{ZDEO} = \frac{1}{6} r^3 \tan\alpha \tan\beta \sin\delta$. It can now be verified that $\frac{dV_{ZDEO}}{dr} = A_{\Delta ZDE}$.

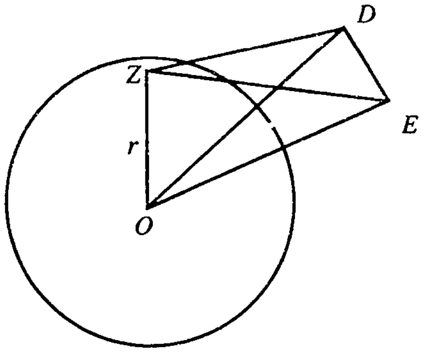


Figure 6: Part of a polyhedron circumscribed about a sphere of radius *r*.

The entire polyhedron circumscribed about the sphere can be divided into tetrahedra like *ZDEO*, and the entire surface of the larger polyhedron is made up of triangles like triangle *DZE*. As in the two-dimensional case, since the derivative of a sum is the sum of the derivatives, the following theorem has been proved:

Theorem 2: If a polyhedron is circumscribed about a sphere of radius *r*, and if both the volume *V* and surface area *S.A.* of the polyhedron are expressed in terms of *r*, then $\frac{dV}{dr} = S.A.$

Interestingly enough, although neither of the differential relationships expressed in (1) is covered by Theorem 1 or Theorem 2, both of the relationships in (1) can now be explained. The fact that the derivative of the area of a circle with respect to its radius equals its circumference corresponds to a limiting case of Theorem 1 in which the number of sides in the circumscribed polygon approaches infinity. Similarly, the fact that the derivative of the volume of a sphere with respect to its radius equals its surface area corresponds to a limiting case of Theorem 2 in which the number of faces in the circumscribed polyhedron increases without bound.

For a regular polygon, the radius of the inscribed circle was an apothem of that polygon. For a regular polyhedron, the radius of the inscribed sphere may by extension be called an apothem as well. But Theorems 1 and 2 apply to *any* circumscribed polygon or polyhedron, not just a regular one. For that reason I like to say that each of those theorems represents an apotheosis of the apothem. (The word *apotheosis* is from Greek *apo* "away (to)" and *theos* "god," so an apotheosis is literally a deification, or, figuratively speaking, the exaltation of an ideal.) I think *apotheosis* is an appropriate name for a differential relationship which can be extended not only from polygons circumscribed about circles to polyhedra circumscribed about spheres, but even, for anyone so inclined, to polytopes circumscribed about hyperspheres.

In Another Light

According to the Neoplatonic philosopher Proclus (c. 410–485), the Egyptian king Ptolemy I once asked Euclid (c. 300 B.C.) if there wasn't an easier way for him to learn geometry than by studying the thirteen books of Euclid's *Elements.* Euclid's now-famous answer was that there is no royal road to geometry. Nevertheless, there are often many routes that lead to the same mathematical truth. A reader of this article described a more calculus-oriented way to prove Theorems 1 and 2. In the alternate approach, r represents any linear dimension of a polygon whose area is A. If r is increased to $r + \Delta r$, then r has been "magnified" by a factor of $\left(1 + \frac{\Delta r}{r}\right)$, and the area of the larger, similar polygon is $A\left(1 + \frac{\Delta r}{r}\right)^2$. Setting up a derivative,

$$\Delta A = A\left(1 + \frac{\Delta r}{r}\right)^2 - A = \frac{2A}{r} \cdot \Delta r + \frac{A}{r^2} \cdot (\Delta r)^2,$$

$$\frac{\Delta A}{\Delta r} = \frac{2A}{r} + \frac{A}{r^2} \cdot (\Delta r), \text{ and finally}$$

$$\frac{dA}{dr} = \lim_{\Delta r \to 0} \frac{\Delta A}{\Delta r} = \frac{2A}{r}. \qquad (2)$$

If the polygon in question happens to be triangle OAC, as shown in Figure 5, then $A_{\Delta OKM} = \frac{1}{2} \cdot height \cdot base = \frac{1}{2} \cdot r \cdot KM$. Substituting into (2) yields

$$\frac{dA}{dr} = \frac{2A}{r} = \frac{2\left(\frac{1}{2} \cdot r \cdot KM\right)}{r} = KM. \qquad (3)$$

Theorem 1 follows from the fact that the area (and perimeter) of a circumscribed n-gon is the sum of the areas (and circumscribed segments) of triangles like triangle OKM, and from the fact that the derivative of a sum is the sum of the derivatives. As before, the theorem includes the first part of (1) as a limiting case in which the number of sides of the n-gon goes to infinity and the perimeter of that n-gon is transformed into the circumference of the inscribed circle.

A similar approach involving derivatives could be applied to the three-dimensional case represented in Figure 6, in which tetrahedron $ZDEO$ is part of an n-hedron circumscribed about a sphere. Following the reasoning just outlined,

$$\Delta V = V\left(1 + \frac{\Delta r}{r}\right)^3 - V = \frac{3V}{r} \cdot \Delta r + \frac{3V}{r^2} \cdot (\Delta r)^2 + \frac{V}{r^3} \cdot (\Delta r)^3,$$

$$\frac{\Delta V}{\Delta r} = \frac{3V}{r} + \frac{3V}{r^2} \cdot (\Delta r) + \frac{V}{r^3} \cdot (\Delta r)^2, \text{ and finally}$$

$$\frac{dV}{dr} = \lim_{\Delta r \to 0} \frac{\Delta V}{\Delta r} = \frac{3V}{r}. \qquad (4)$$

1.8

Now, the volume of tetrahedron *ZDEO* is given by the formula

$$V_{ZDEO} = \frac{1}{3} \cdot height \cdot (base\ area) = \frac{1}{3} \cdot r \cdot A_{\Delta ZDE}.$$

Substituting into (4) yields

$$\frac{dV}{dr} = \frac{3V}{r} = \frac{3\left(\frac{1}{3} \cdot r \cdot A_{\Delta ZDE}\right)}{r} = A_{\Delta ZDE}. \quad (5)$$

Theorem 2 follows from the fact that the volume (and surface area) of a circumscribed *n*-hedron is the sum of the volumes (and areas of the circumscribed faces) of tetrahedra like *ZDEO*, and from the fact that the derivative of a sum is the sum of the derivatives. As before, the theorem includes the second part of (1) as a limiting case in which the number of faces of the *n*-hedron goes to infinity and the surface area of that *n*-hedron is transformed into the surface area of the inscribed sphere.

This method can be generalized to higher dimensions. What is more, the calculus approach lends itself to the interpretation of a disk (i.e. a circle and all its interior points) as an infinite set of concentric rings of typical radius *r* and thickness *dr*. Likewise, a ball (i.e. a sphere and all its interior points) may be viewed as an infinite set of concentric spherical shells of typical radius *r* and thickness *dr*. Because of the way the rings and shells increase, layer upon layer, the calculus-oriented approach to Theorems 1 and 2 might be called a glorification of agglutination or an exaltation of exfoliation.

References

Bartlett, J. (1938). *Familiar quotations* (11th ed.). Boston: Little, Brown and Company.

Bridgwater, W. and Kurtz, S. (Eds.). (1969). *The Columbia encyclopedia* (3rd ed.). New York: Columbia University Press.

Watkins, C. (1985). *The American heritage dictionary of Indo-European roots*. Boston: Houghton-Mifflin Company.

> The death of Archimedes at the hands of a Roman soldier is symbolical of a world change of the first magnitude. The Romans were a great race, but they were cursed by the sterility which waits upon practicality. They were not dreamers enough to arrive at new points of view, which could give more fundamental control over the forces of nature. No Roman lost his life because he was absorbed in the contemplation of a mathematical diagram.
>
> Alfred North Whitehead

Discover Essential New Mathematics Titles

from Academic Press

Encyclopedia of Integer Sequences

Neil J.A. Sloane and Simon Plouffe

This encyclopedia contains more than 5,000 integer sequences, over half of which have never before been catalogued. The **Encyclopedia** gives the name, mathematical description, and citations to literature for each sequence.

March 1995, c. 587 pp., $44.95/ISBN: 0-12-558630-2

The Theory of Gambling and Statistical Logic

Revised Edition

Richard A. Epstein

Endorsement for the Revised Edition, now in Paperback

"Richard Epstein's marvelous book is now a classic account of gambling games and their underlying laws of probability. Its range of topics exceeds that of any comparable work. Clearly written, accurate, it should be on the bookshelf of anyone seriously interested in probability theory, especially in its application to recreational games."

—Martin Gardner, Former Columnist, *Scientific American*

Paperback Reprint: $29.95
February 1995, c. 465 pp./ISBN: 0-12-240761-X

A Primer of Lebesgue Integration

H.S. Bear

Prepublication Review

"This is a wonderful little book that presents Lebesgue theory on the line and the plane as a completely natural extension of the familiar Riemann theory encountered in calculus....Ideal for a semester undergraduate course or seminar, though it is equally suitable for self-study."

—Kenneth A. Ross, University of Oregon

February 1995, c. 160 pp., $19.95/ISBN: 0-12-083970-9

Linear Algebra

An Introduction

Richard Bronson

Key Features

- Begins with the concrete and computational and leads the reader to a choice of major applications (Markov chains, least squares approximation, and solution of differential equations using Jordan normal form)
- Covers matrices, vector spaces, linear transformations, as well as applications to Jordan canonical forms, differential equations, and Markov chains
- Gives computational algorithms for finding eigenvalues and eigenvectors
- Includes ample exercises with answers and hints, in addition to other learning features

Paperback: $39.95 (tentative)
May 1995, c. 414 pp./ISBN: 0-12-135245-5

New Edition Available in July!

Mathematical Methods for Physicists

Fourth Edition

George B. Arfken and Hans-Jurgen Weber

Key Features

This revised Fourth Edition includes:

- Modernized terminology
- Group theoretic methods brought together and expanded in a new chapter
- An entirely new chapter on nonlinear mathematical physics
- Significant revisions of the differential equations and complex variables chapters
- Many new problem sets
- 40 new or improved figures
- Update of computational techniques for today's tools, including microcomputers, Numerical Recipes, and *Mathematica®*, among others

July 1995, c. 952 pp., $59.95 (tentative)/ISBN: 0-12-059815-9

Order from your local bookseller or directly from

ACADEMIC PRESS, INC., Order Fulfillment Dept. DM27098
6277 Sea Harbor Drive, Orlando, FL 32887

Call Toll FREE 1-800-321-5068 • *Fax Toll FREE* 1-800-321-7377 • *e mail* ap@acad.com

BEST COPY AVAILABLE

Relaxation Functions

by

Homer B. Tilton
Pima Community College East
Tucson AZ 85709-4000

Homer B. Tilton retired from Hughes Aircraft Co. in 1989. He has been teaching algebra since that time at PCC East. He has a BS in Engineering Physics from Montana State University and has earned graduate credits in physics and engineering from UCLA and the University of Arizona.

Introduction

Early this century (circa 1926), Balthasar van der Pol coined the term relaxation oscillations to mean discontinuous waveforms (periodic functions) such as squarewaves, sawtooth waves, and pulse trains. Here, this terminology is adapted to apply to broken and finitely-discontinuous periodic and non-periodic functions, which shall henceforth be referred to as relaxation functions. By "finitely discontinuous" is meant functions having one or more (possibly an infinity of) isolated jumps of finite magnitude. Do not confuse this with that completely different subject, "relaxation methods of numerical analysis!"

The next sections deal with ways of representing such functions.

Piecewise Notation

Von Seggern (1993) in Chapter 11 deals briefly with relaxation functions where he refers to them as "Nondifferentiable and Discontinuous Functions." While von Seggern's treatment of these functions is more extensive than many writers', still that chapter consists of a mere 13 pages in a book of more than 380 total pages and 14 chapters. Many similar books do not treat these functions at all, so von Seggern has moved this area of mathematics forward some.

Traditional mathematical notations for relaxation functions seem, to this writer, to be needlessly complicated. For example, von Seggern represents a positive-going linear sawtooth wave by this piecewise expression (his #11.2.5):

$$y = c\left[\frac{2x}{a}\sum_{n=-\infty}^{\infty}\{H[x-na]-H[x-(n+1)a]\} - 2\sum_{n=-\infty}^{\infty} n\{H[x-na]-H[x-(n+1)a]\}-1\right] \quad (1)$$

where H is the Heaviside step function (previously defined by him in a piecewise manner). You'll recall that $H(x)$ is zero for negative x, 1 for positive x, and variously defined as 0 or 1 (or often undefined) for $x = 0$. Von Seggern's graph of equation (1) is like Figure 1 except for phase and scale.

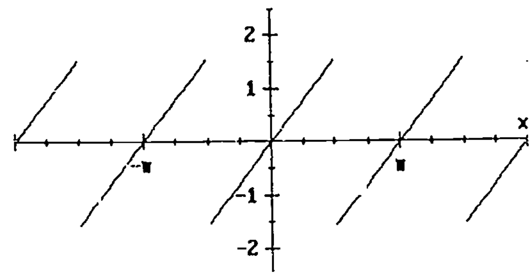


Figure 1.

Von Seggern's notation is typical and commonplace, and he is not deliberately being singled out for criticism. Indeed, it is virtually the entire mathematical community that suffers from a fixation on piecewise notation leading to an excess of mathematical symbology when it comes to representing relaxation functions.

Compact Notation

To show that equation (1) is needlessly complicated, here is an alternate compact expression for that same sawtooth wave after Tilton (1986):

$$y = B\ \mathrm{Arctan}[\tan(ax + \phi)] \qquad (2)$$

where B, a, and ϕ determine amplitude, frequency, and phase, respectively. In what follows, assumed values for those constants are $B = 1$, $a = 1$, and $\phi = 0$.

The comparative simplicity of equation (2) is obvious. But is it really a linear sawtooth wave? As quick evidence that it is, let your computer or graphing calculator plot $Y = \mathrm{Arctan}(\tan x)$. I did, and it came out looking exactly like Figure 1. A logical argument follows.

Arctan gives the principal value of the inverse tangent function. A little thought will show that equation (2) is not the straight line $y = x$ as it might first appear to be, since Arctan is limited to values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Thus $\mathrm{Arctan}(\tan x)$ will have those minimum and maximum values. Application of the Arctan operation to the tangent function serves to linearize; and since $\tan x$ is periodic with period π, $\mathrm{Arctan}(\tan x)$ will also be periodic with period π. Only the "zeroth" cycle (from $x = -\frac{\pi}{2}$ to $\frac{\pi}{2}$) will conform to the equation $y = x$.

In summary, the tangent function is a nonlinear sawtooth wave of infinite amplitude; application of the Arctangent operation to it causes it to be linearized and given an amplitude of $\frac{\pi}{2}$.

Other Relaxation Functions

Equation (1) is expressed in what is called piecewise notation. By contrast, equation (2) is in compact notation. Generally, mathematical works use piecewise notation for relaxation functions. Still, who today would represent the full-wave rectified sinewave $|\sin x|$ in piecewise form? Certainly not von Seggern. (See his #11.4.5.) So why does the mathematical community at large still insist on representing sawtooth, squarewave, pulse trains, and other relaxation functions in piecewise notation? Perhaps it is a kind of inertia rooted partly in a subconscious (or conscious) refusal to give complete legitimacy to the absolute-value and principal-value concepts.

How might one represent a squarewave in compact notation? Here are two ways:

$$Y = \operatorname{sgn}(\sin x)\,;\;\; Y = \operatorname{sgn}(\cos x). \tag{3}$$

See graphs in Figures 2 and 3. Here $\operatorname{sgn}(u) = \frac{u}{|u|}$ except possibly at $u = 0$. (But this question is resolved shortly.) The Heaviside step function in compact notation can now be written

$$H(x) = \frac{1}{2}\operatorname{sgn}(x) + \frac{1}{2}. \tag{4}$$

What happens to sgn(x) at the point of discontinuity? Sgn(0) is often defined as zero, since it appears at first glance that any other definition will destroy the desired antisymmetry of that function.* But that is not so. Consider the function $\frac{1}{x}$; it is odd and therefore antisymmetric, but its value at $x = 0$ is not zero. One might say that its "value" there is $\pm\infty$ depending on which direction it is

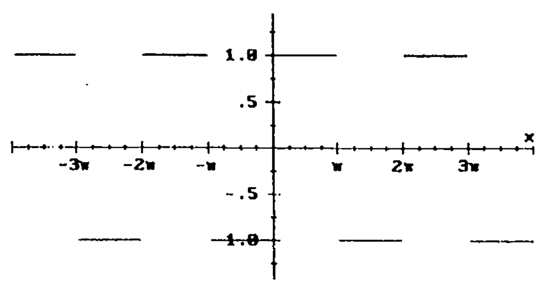


Figure 2.

* Antisymmetry, as the term is used here, is the kind of symmetry exhibited by the graph of an odd function. This is often called point symmetry or central symmetry; however, while both of those terms imply rotation about a point, they may be seen to be ambiguous as to the angle of that rotation.

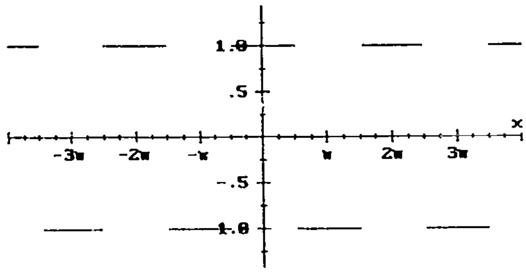


Figure 3.

approached from. This kind of "flip-flop" action in the value of a function is called infinitesimal hysteresis because it is hysteresis affecting only a single point. (See "catastrophe theory" for an illustration of normal hysteresis.)

In a similar way one can define sgn(x) at $x = 0$ as -1 for positive progression along the x-axis and as $+1$ for negative progression. Thus, with the value of sgn(x) always ± 1 and never zero, there are now the highly useful relations $\frac{1}{\text{sgn}(x)} = \text{sgn}(x)$ and $|\text{sgn}(x)| = 1$. Also, $H(x)$ as defined by equation (4) is always 0 or 1 and never $\frac{1}{2}$. All this, while at the same time insuring that sgn(x) is antisymmetric. Its antisymmetry is global, as is that of $\frac{1}{x}$, in that both directions of progression must be considered.

In a sense, $\frac{x}{x}$ and $\frac{\sin x}{x}$ at $x = 0$ are both 1, that sense being that the limits from both directions are 1, and there is no hysteresis involved. By comparison, application of that same bilateral limiting process to $\frac{x}{|x|}$ results in the hysterical behavior as described in the above paragraph. Or one can simply define sgn(x) as $\frac{x}{|x|}$ when x is not zero, and as described above otherwise. (But this author finds that definition to be artificial and therefore intuitively distasteful.)

And would you believe the following are triangular waves?

$$Y = \text{Arccos}(\cos x),\ Y = \text{Arcsin}(\sin x). \tag{5}$$

See Figures 4 and 5.

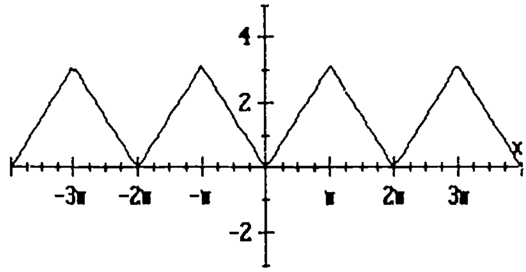


Figure 4.

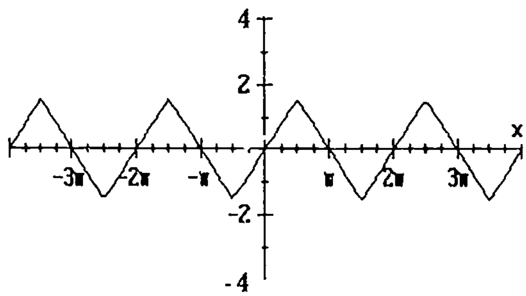


Figure 5.

One can build on these ideas to construct a large arsenal of different relaxation functions – all expressed in compact notation.

This alternate definition can now be formulated:

> A relaxation function is any function formed from a function of standard analysis by applying any combination of absolute-value and principal-value operations.

As a general rule, it is found that relaxation functions are finitely discontinuous or have a finitely discontinuous *n*th derivative.

Delta Functions

Dirac's delta function, $\delta(x)$, is defined as zero everywhere except at x where it is defined so that its definite integral over an interval including $x = 0$ is equal to 1. Intuitively, it is a pulse function having a single pulse at $x = 0$ of infinitesimal width and "infinite" (reciprocal of infinitesimal) amplitude, that pulse having an "area" or <u>content</u> of 1. The Dirac delta function is really a <u>distribution</u> not a function. Other delta functions can be synthesized from the Dirac delta function.

Delta functions are not relaxation functions, but they figure prominently in the mathematics of relaxation functions or <u>relaxation analysis</u>.

Lighthill (1958) has referred to delta functions as "generalized functions." Again, Lighthill builds trains of delta pulses (he calls them "a series of deltas") in a piecewise manner as nearly everybody else does. But these can also be represented in compact form.

For example this is an alternating delta pulse train with period 2π:

$$y = \cos x\ \delta(\sin x). \tag{6}$$

A minute's consideration should prove convincing. [Recall that δ is an even function so that $\delta(-u) = \delta(u)$.] Equation (6) can be integrated to get the squarewave $H(\sin x)$ using this indefinite integral with $u = \sin x$:

$$\int \delta(u)\, du = \frac{1}{2}\ \mathrm{sgn}(u) + C. \tag{7}$$

The Usefulness of Compact Notation

As another illustration of the usefulness of compact notation for relaxation functions, consider this integral appearing in virtually every table of integrals:

$$\int \frac{\mathrm{Arctan}\, u}{u^2 + 1}\, du = \frac{1}{2}\ \mathrm{Arctan}^2\, u + C. \tag{8}$$

Now substitute $\tan x$ for u to get the integral of the sawtooth wave of equation (2)! One would be hard-pressed indeed to do this kind of analytical manipulation using piecewise notation.

And if one extends the definition of derivative to include the relation

$$d\,[\mathrm{sgn}(u)]/dx = 2\ \delta(u) du/dx \tag{9}$$

then this opens up a further universe of things one can do using compact notation. Tilton (1986) presents an extensive treatment of these ideas, showing how to analytically synthesize, differentiate, and integrate virtually any relaxation function that can be imagined.

References

Lighthill, M.J.(Now Sir James). (1958). *An introduction to Fourier analysis and generalised functions*. Cambridge (England): University Press.

Tilton, H.B. (1986). *Waveforms: A modern guide to nonsinusoidal waves and nonlinear processes*. Englewood Cliffs, NJ: Prentice-Hall.

Von Seggern, D. (1993). *CRC standard curves and surfaces*. (2nd ed.). Boca Raton, FL: CRC Press.

> Perhaps the greatest paradox of all is that there are paradoxes in mathematics.
>
> Kasner and Newman


TEXAS INSTRUMENTS
Calculators
CLASSROOM
TI 30X
TI 35X
TI 34
TI 36XSLR
CALL FOR CURRENT PRICING
GRAPHING
TI80
TI 81
TI 82
TI 85
TICBL
CASIO
Calculators
GRAPHING
FX7300G
FX7700GE
FX9700GE
FX9800G
CLASSROOM
FX250
FX300
FX991
CALL FOR CURRENT PRICING
HEWLETT PACKARD
Graphing Calculators
HP 48G
Hundreds of built in equations. 32 KB Memory for complex problem solving. RS-232 lets you link to your computer through a serial port. Two way infrared I/O for easy data transfer. 3-D Graphics, calculus, and symbolic math functions
HP 48GX
Same as 48G with expandable port
CALL FOR CURRENT PRICING
SHARP®
Calculators
FRACTIONS
ELE300
GRAPHING
EL9300
CALL FOR CURRENT PRICING
Overhead Calculators and Class Sets for these and other models are available!
ARROWHEAD DISTRIBUTORS, INC.
3 LOCATIONS
DALLAS
KANSAS CITY
LOS ANGELES
CALL FOR A FREE COPY OF OUR CURRENT CATALOG
1-800-234-3396

A Pattern for the Squares of Integers

by

Kim Mai
Long Beach City College
Long Beach CA 90808

Kim Mai likes to work with numbers and number theory. The pattern described below was found when she was a senior in high school. Now she is a student at Long Beach City College and she p'ans to transfer to University of California next fall.

Abstract

A pattern occurring in the squares of successive integers $0^2, 1^2, 2^2, 3^2, \ldots$ is described and proven. As a result, we can write down a list of the squares of consecutive numbers without extensive multiplication or using a calculator. Also, a corresponding algorithm for k^2 is then developed which can be used to extend the precision of any calculator or other computing device.

Suppose k is any nonnegative integer which we write in the form: $k = 10t_k + u_k$ where $t_k = \left[\!\left[\frac{k}{10} \right]\!\right]$ and $u_k = k \bmod 10$. Notice that u_k is simply the "units" digit of k. We wish to write the square of this integer in a similar form:

$$k^2 = 10\, m_k + n_k, \ (0 \le n_k \le 9), \tag{1}$$

and to obtain explicit functions for m_k and n_k. The chart on the following page serves as an illustration.

The pattern for n_k is easily seen to be cyclic and equal to 0, 1, 4, 9, 6, 5, 6, 9, 4, 1 depending upon whether u_k is 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 respectively.

The pattern for m_k is apparent but more difficult to describe. Examination of the last column of the chart shows that the differences between successive m_k's increase first by 0, then by 1, then by 2, then by 3, ... The "jumps" in these successive differences $(m_{k+1} - m_k)$ occur at $k = 3, 7, 13, 17, 23, 27, \ldots$ and generally when $u_k = 3$ or $u_k = 7$. We formalize this in the following theorem:

THEOREM: Let k be any nonnegative integer and form the decompositions:

$$k = 10t_k + u_k \qquad (u_k = k \bmod 10)$$
$$k^2 = 10m_k + n_k \qquad (n_k = k^2 \bmod 10)$$

The jumps in the successive differences of m_k are then given by:

$$(m_{k+1} - m_k) - (m_k - m_{k-1}) = \begin{cases} 1 & \text{if } u_k = 3 \text{ or } 7 \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

k	k^2 m_k n_k	m_k	n_k	$m_{k+1} - m_k$	
0	0 0	0	0		
1	0 1	0	1	0	
2	0 4	0	4	0	
3	0 9	0	9	0	
4	1 6	1	6	1	4 times
5	2 5	2	5	1	
6	3 6	3	6	1	
7	4 9	4	9	1	
8	6 4	6	4	2	6 times
9	8 1	8	1	2	
10	10 0	10	0	2	
11	12 1	12	1	2	
12	14 4	14	4	2	
13	16 9	16	9	2	
14	19 6	19	6	3	4 times
15	22 5	22	5	3	
16	25 6	25	6	3	
17	28 9	28	9	3	
18	32 4	32	4	4	6 times
19	36 1	36	1	4	
20	40 0	40	0	4	
21	44 1	44	1	4	
22	48 4	48	4	4	
23	52 9	52	9	4	

We need the following lemma to establish this:

$$\text{LEMMA: } \left[\!\left[\frac{(u_k+1)^2}{10}\right]\!\right] - 2\left[\!\left[\frac{u_k^2}{10}\right]\!\right] + \left[\!\left[\frac{(u_k-1)^2}{10}\right]\!\right] = \begin{cases} 1 & \text{if } u_k = 3 \text{ or } 7 \\ 0 & \text{otherwise} \end{cases}$$

(This lemma can be easily verified by quickly examining each of its 10 specific cases.)

PROOF OF THEOREM:

$$(m_{k+1} - m_k) - (m_k - m_{k-1})$$

$$= m_{k+1} - 2m_k + m_{k-1}$$

$$= \left[\!\left[\frac{(k+1)^2}{10}\right]\!\right] - 2\left[\!\left[\frac{k^2}{10}\right]\!\right] + \left[\!\left[\frac{(k-1)^2}{10}\right]\!\right]$$

$$= \left[\!\left[\frac{k^2+2k+1}{10}\right]\!\right] - 2\left[\!\left[\frac{k^2}{10}\right]\!\right] + \left[\!\left[\frac{k^2-2k+1}{10}\right]\!\right]$$

$$= \left[\!\left[\frac{(10t_k+u_k)^2+2(10t_k+u_k)+1}{10}\right]\!\right] - 2\left[\!\left[\frac{(10t_k+u_k)^2}{10}\right]\!\right]$$
$$+ \left[\!\left[\frac{(10t_k+u_k)^2-2(10t_k+u_k)+1}{10}\right]\!\right]$$

$$= \left[\!\left[\frac{100t_k^2+20t_k u_k+20t_k}{10} + \frac{u_k^2+2u_k+1}{10}\right]\!\right] - 2\left[\!\left[\frac{100t_k^2+20t_k u_k}{10} + \frac{u_k^2}{10}\right]\!\right]$$
$$+ \left[\!\left[\frac{(100t_k^2+20t_k u_k-20t_k}{10} + \frac{u_k^2-2u_k+1}{10}\right]\!\right]$$

$$= \left[\!\left[10t_k^2+2t_k u_k+2t_k + \frac{(u_k+1)^2}{10}\right]\!\right] - 2\left[\!\left[10t_k^2+2t_k u_k + \frac{u_k^2}{10}\right]\!\right]$$
$$+ \left[\!\left[10t_k^2+2t_k u_k-2t_k + \frac{(u_k-1)^2}{10}\right]\!\right]$$

$$= 10t_k^2+2t_k u_k+2t_k + \left[\!\left[\frac{(u_k+1)^2}{10}\right]\!\right] - 20t_k^2 - 4t_k u_k - 2\left[\!\left[\frac{u_k^2}{10}\right]\!\right]$$
$$+ 10t_k^2+2t_k u_k-2t_k + \left[\!\left[\frac{(u_k-1)^2}{10}\right]\!\right]$$

$$= \left[\!\left[\frac{(u_k+1)^2}{10}\right]\!\right] - 2\left[\!\left[\frac{u_k^2}{10}\right]\!\right] + \left[\!\left[\frac{(u_k-1)^2}{10}\right]\!\right]$$

and the result now follows from the lemma. □

EXAMPLE: When $k = 167$, $(m_{168} - m_{167}) - (m_{167} - m_{166}) = (2822 - 2788) - (2788 - 2755) = 34 - 33 = 1$ since $u_{167} = 7$. □

Notice that the theorem allows us to quickly determine the square of any number k^2 if we already know the squares of the preceding two numbers $(k-1)^2$ and $(k-2)^2$. For example, if we know that $5868^2 = 34433424$ and $5869^2 = 34445161$ then $m_{5869} - m_{5868} = 3444516 - 3443342 = 1174$. Since $u_{5869} = 9$ is not 3 or 7, then we must have $m_{5870} - m_{5869} = 1174$ as well. Thus $m_{5870} = m_{5869} + 1174 = 3444516 + 1174 = 3445690$ and $n_{5870} = 0$. Hence $(5870)^2 = 10m_{5870} + n_{5870} = 10(3445690) + 0 = 34456900$.

Once the pattern for m_k has been established, it is straightforward to derive an explicit algorithm for it. Let $k = 10t_k + u_k$ and then sum the first k entries in the last column of the chart:

$$m_k = (m_1 - m_0) + (m_2 - m_1) + (m_3 - m_2) + \ldots + (m_{10t_k} - m_{10t_k - 1}) + (m_{10t_k + 1} - m_{10t_k}) + \ldots + (m_{10t_k + u_k} - m_{10t_k + u_k - 1}). \quad (3)$$

The first $10t_k$ terms of (3) are given by:

$$[3(0) + 4(1) + 3(2)] + [3(2) + 4(3) + 3(4)] + [3(4) + 4(5) + 3(6)] + \ldots + [3(2t_k - 2) + 4(2t_k - 1) + 3(2t_k)]$$

$$= \sum_{j=1}^{t_k} [3(2j-2) + 4(2j-1) + 3(2j)]$$

$$= 10t_k^2. \quad (4)$$

The remaining u_k terms of (3) are given by:

$$r_k = \begin{cases} u_k(2t_k) & \text{if } u_k = 0, 1, 2, 3 \\ 3(2t_k) + (u_k - 3)(2t_k + 1) & \text{if } u_k = 4, 5, 6, 7 \\ 3(2t_k) + 4(2t_k + 1) + (u_k - 7)(2t_k + 2) & \text{if } u_k = 8, 9 \end{cases}$$

which simplifies to

$$r_k = \begin{cases} u_k(2t_k) & \text{if } u_k = 0, 1, 2, 3 \\ u_k(2t_k + 1) - 3 & \text{if } u_k = 4, 5, 6, 7 \\ u_k(2t_k + 2) - 10 & \text{if } u_k = 8, 9 \end{cases} \quad (5)$$

Thus we arrive at the formula for m_k:

$$m_k = 10t_k^2 + r_k \quad (6)$$

where r_k is given by (5).

EXAMPLE: If $k = 4256$ then $t_k = 425$ and $u_k = 6$ from which we compute:

$$m_{4256} = 10(425)^2 + 6[2(425) + 1] - 3 = 1811353$$

$$n_{4256} = 6$$

and hence

$$(4256)^2 = 10m_{4256} + n_{4256} = 10(1811353) + 6 = 18113536. \quad \square$$

COMMENT: Formulas (2) and (6) are valid even if k is a negative integer. (In such cases, we would have $t_k < 0$.) For instance if $k = -169 = 10(-17) + 1$, then $t_k = -17$ and $u_k = 1$. Therefore

$$m_{-169} = 10(-17)^2 + 1[2(-17)] = 2856$$

$$n_{-169} = 1$$

and hence

$$(-169)^2 = 10m_{-169} + n_{-169} = 10(2856) + 1 = 28561.$$

As an application, we now show how formulas (5) and (6) can be used to extend the precision of any calculator or other computing device. Suppose a calculator has a screen which can display only eight digits. Then the value of $(85628)^2$ will be represented as 7.3321544×10^9 where scientific notation and rounding have been introduced. However if we use formulas (5) and (6) with $t_{85628} = 8562$ and $u_{85628} = 8$, then we obtain:

$$m_{85628} = 10(8562)^2 + 8(2 \times 8562 + 2) - 10 = 733215438 \tag{7}$$

$$n_{85628} = 4 \tag{8}$$

where the simple addition in (7) is performed by hand. Consequently,

$$\begin{aligned}(85628)^2 &= 10m_{85628} + n_{85628} \\ &= 10(733215438) + 4 = 7332154384\end{aligned} \tag{9}$$

exactly.

In fact, equation (9) can be used to generate even larger squares, e.g.

$$\begin{aligned}(856283)^2 &= 10[m_{856283}] + [n_{856283}] \\ &= 10[10(85628)^2 + 3(2 \times 85628)] + [9] \\ &= 10[10(7332154384) + 513768] + [9] \\ &= 10[73322057608] + [9] \\ &= 733220576089.\end{aligned}$$

By employing scientific notation formulas (5) and (6) can also be used to extend the decimal accuracy of any calculator or other computing device.

I would like to thank my family, Jim Burton, and especially Dr. David Horowitz of Golden West College for their help and encouragement.

"Obvious" is the most dangerous word in mathematics.

Eric Temple Bell

A thing is obvious mathematically after you see it.

R. D. Carmichael

SHORT COMMUNICATIONS

$\lim_{n \to \infty} \left\{ \left(1 + \frac{1}{n}\right)^j \right\}_{j=0}^{j=n}$ Produces e^x

by

Kurtis D. Fink
Northwest Missouri State University
Maryville MO 64468
and
John H. Mathews
California State University, Fullerton
Fullerton CA 92634-9480

Kurt earned his doctorate at Idaho State University, Pocatello, Idaho. He is active in mathematical biology, differential equations and numerical analysis. Teaching interests include directing student undergraduate research projects and developing pedagogical applications for computer software including computer algebra systems. Kurt is interested in tournament play in racquetball, downhill skiing and fishing.

John earned his doctorate at Michigan State University, East Lansing, Michigan. He is active in complex analysis and numerical analysis. Ongoing projects embrace the pedagogical use of computers to enhance the teaching of mathematics at the university level including undergraduate research. He is the author of Complex Variables for Mathematics and Engineering, 2nd Edition, William C. Brown Co. Pub., 1992 and Numerical Methods for Mathematics, Science and Engineering, 2nd Edition, Prentice-Hall, Inc., 1992. John is interested in the outdoors and long distance road bicycling.

The familiar limit $\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = e$ is presented in every calculus text. In addition to the single value $\left(1 + \frac{1}{n}\right)^n$, consider the additional n numbers $\left(1 + \frac{1}{n}\right)^j$, for $j = 0, 1, \ldots, n-1$. For each n held fixed, define the set of points

$$S_n = \left\{ \left(\frac{j}{n}, \left(1 + \frac{1}{n}\right)^j \right) \right\}_{j=0}^{j=n}.$$

As n increases, the set S_n coalesces and appears to fill in the curve $y = e^x$ over [0,1]. Figure 1 shows S_{10}, S_{20}, S_{40} and S_{60}, respectively.

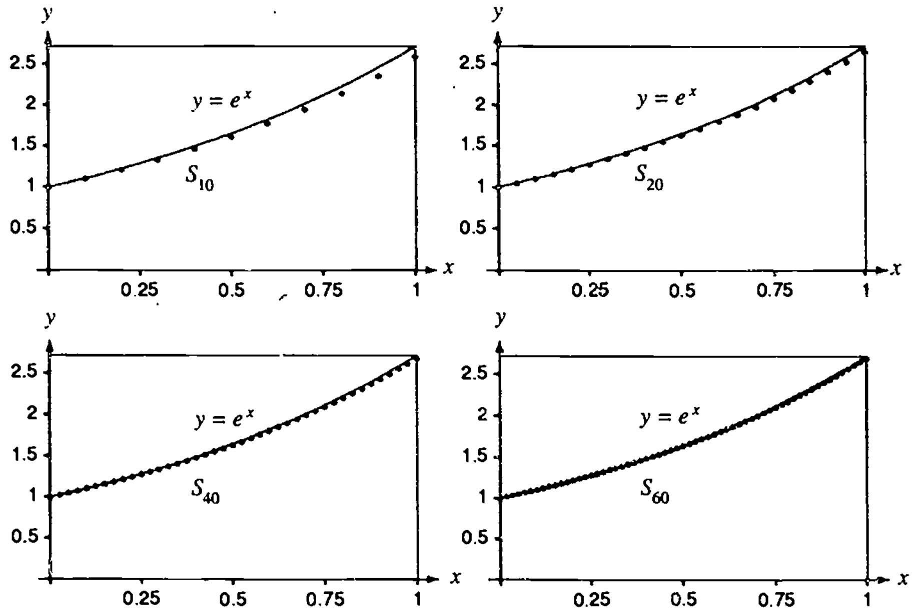


Figure 1. The sets S_{10}, S_{20}, S_{40} and S_{60} which "tend to" the curve $y = e^x$.

One interpretation of this is that S_n is the set of Euler approximations corresponding to the numerical solution of the initial value problem $y'(x) = y(x)$ with the initial condition $y(0) = 1$. As $n \to \infty$ it is known that S_n "tends to" the analytic solution $y = e^x$.

As an application, if $\left(1 + \frac{1}{n}\right)$ is replaced by $\left(1 + \frac{r}{n}\right)$ then S_n would "tend to" $y = e^{rx}$ as $n \to \infty$. For this situation r could represent the instantaneous interest rate and S_n would then be the set of n intermediate compoundings of the form $\left(1 + \frac{r}{n}\right)$ performed over one year. A series of graphs similar to Figure 1 will visually show that compound interest tends to instantaneous interest as $n \to \infty$. Note that $\left(1 + \frac{r}{n}\right)^n$ is the effective annual percentage rate (APR).

> An idea which can be used once is a trick; if it can be used twice, it is a device; if it can be used more than twice, it becomes a method.
>
> Anonymous

Lucky Larry Meets Thomas Simpson

by

John deValcourt
Cabrillo College
Aptos CA 95003

John deValcourt received his PhD in Mathematics from the University of Minnesota and has taught at the College of Santa Fe, City College of San Francisco, Cabrillo College and the University of California at Santa Cruz.

Sometimes student errors or misunderstandings can lead to new discoveries or connections to other topics. A fine example occurred in a calculus class last semester. Attempting to put the evaluation of areas under curves in a historical context, we discussed Archimedes' quadrature of the parabola. He was able to prove, without calculus, that the area within a parabolic segment is $\frac{2}{3}$ the area of the circumscribing rectangle (or parallelogram). We also talked about how his method of exhaustion did not easily generalize to other less simple cases, and how the world had to wait for the development of calculus to find areas under more complicated curves. (For an excellent account of Archimedes' quadrature of the parabola, as well as many other stories to enrich your calculus class, see Simmons (1992).)

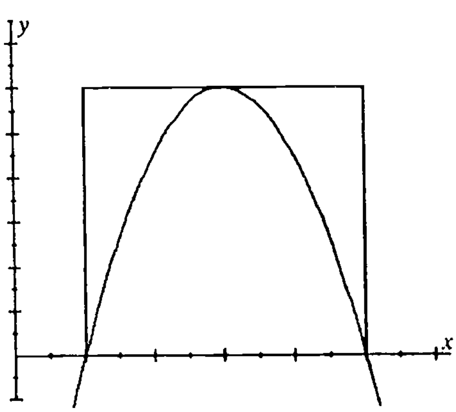


Figure 1. Archimedes' Quadrature: The area under a parabolic arch is $\frac{2}{3}$ the area of the circumscribed rectangle.

After this, the students continued to find the area under a parabolic arch by simply taking $\frac{2}{3}$ the area of a rectangle whose height is the value of the quadratic at the midpoint of the interval (see Figure 1). They found this to be quicker than using the fundamental theorem of calculus to evaluate the definite integral. One student, however, continued to use this method to find areas under an arch of a *cubic* polynomial. That is, she just used $\frac{2}{3}$ times the base of the arch times the the height at the midpoint of the base (see Figure 2), and she seemed to be getting correct answers. When I discovered what she was doing, my first reaction was, "You were just lucky, that won't work every time."

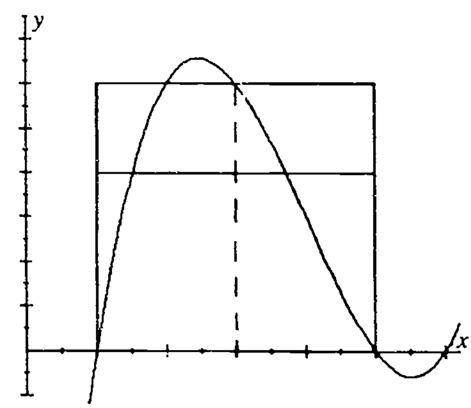


Figure 2. Student's "Misconception": The area under a cubic arch is $\int_a^b f(x)\,dx = \frac{2}{3}(b-a)f\left(\frac{a+b}{2}\right)$.

The truth is, however, that her method *will* work every time. It is equivalent to using Simpson's Rule with $n = 2$. Her method correctly finds the area of a parabola which coincides with the cubic at the midpoint and the two endpoints of the interval, which is precisely what Simpson's Rule does. Since the error of approximation for Simpson's Rule is proportional to the fourth derivative of the function, and the fourth derivative of a cubic is zero, her method is exactly correct.

We can generalize the result as follows (see Figure 3):

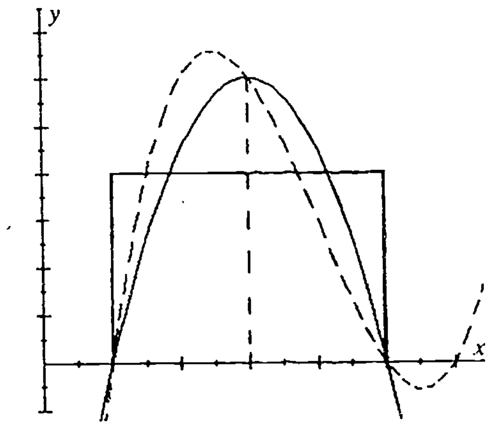


Figure 3. The average value of either a quadratic or cubic function between two zeros is $\frac{2}{3}$ the value of the function at the midpoint.

THEOREM: Let $f(x)$ and $g(x)$ be cubic and quadratic polynomials, respectively, such that both have zeros (roots) at a and b, and for which $f\left(\frac{a+b}{2}\right) = g\left(\frac{a+b}{2}\right)$. Then

$$\int_a^b f(x)dx = \int_a^b g(x)dx = \frac{2}{3}(b-a)f\left(\frac{a+b}{2}\right).$$

COROLLARY: The average value of a cubic polynomial on the interval between two zeros is $\frac{2}{3}$ the value of the cubic at the midpoint of the interval.

Reference

Simmons, G. F. (1992). *Calculus gems*. New York: McGraw-Hill Book Company.

Mathematics is the art of giving the same name to different things.

Henri Poincare

Poetry is the art of giving different names to the same thing.

Anonymous Poet

Using an $n \times m$ Contingency Table to Determine Bayesian Probabilities: An Alternative Strategy

by

Eiki Satake
William Gilligan
Philip Amato
Emerson College
Boston MA 02116

Eiki Satake is Associate Professor, Division of Computer Applications & Mathematics at Emerson College.

EMERSON COLLEGE

William Gilligan is Director of Information Technology at Emerson College.

Philip Amato is Vice President & Academic Dean at Emerson College. He was formerly Professor & Chair of the Division of Computer Applications & Mathematics.

Introduction

Although statistics education has long advocated the teaching of conditional probability in statistics courses at the college level, our observations suggest that this concern is not reflected in the treatment of Bayes' theorem as much as other topics for such courses. One of the main reasons is that the use of Bayes' theorem is still foreign to most of the instructors at the introductory level. Bayes' theorem is an extremely useful theorem, particularly for the following reasons:

a) In medicine, Bayes' theorem is used to determine the posterior probability that a patient has a disease, given two conditional probabilities (sensitivity and specificity) and one absolute probability (prevalence).

b) It enables us to update probabilities as new information becomes available. For instance, in business, the Bayes' theorem can be frequently used in decision making-analysis. It tells the decision maker to update the information and to select the appropriate action which maximizes expected profit or minimizes expected loss.

c) In general, its use often enables us to determine the probability of an event before the experiment (a priori) from the probability after the experiment (a posteriori). Examining the possible reason from its consequence can be seen very frequently in real life situations.

d) Compared to users of classical statistics, it is quite clear that one can derive probability statements about the unknown parameter values through Bayesian statistics. Iversen (1984) stated that the major difficulty with classical statistics is the way they interpret a confidence interval since it does not permit a probability statement about an unknown population parameter.

e) We can apply Bayesian methods in situations where objective methods are needed, by using subjective prior knowledge. Our earlier practical experience can be taken into account explicitly.

Traditionally, in teaching Bayes' theorem as a part of introductory probability and statistics courses at the college level, instructors use either a formula method or a tree diagram method which most of the widely used textbooks (such as Mosteller, Robert and Thomas; Strait; Mendenhall and Reinmuth; Freund and Smith) present. Although both methods are usually taught by instructors in the classroom, Polya (1968) stated that there should be a place for guessing in the teaching of mathematics and instruction should prepare for, or at least give a little taste of, invention. Tversky and Kahneman (1974, 1983) also stated that the reasoning of the statistical novice was fundamentally different from that of the expert. They used their own intuitions that guide everyday reasoning under uncertainty, and often learned statistical concepts more effectively through a simple, logical, and convenient method rather than a rigorous formula method.

The method we present here describes a way to determine Bayesian probabilities through an alternative method called "An $n \times m$ Contingency Table Method." It gives us plenty of mathematical intuitive thinking to describe the relationships of the two joint probabilities, namely a priori and a posteriori, in all cases. It also gives us a simple relationship between the data probabilities and posterior probabilities.

First, we present a typical conditional probability problem, and subsequently solved using three different methods such as (A) the traditional formula approach, (B) the use of a tree diagram, and (C) the use of an $n \times m$ contingency table.

Example: An Application to Communication Studies

A local television station has three video tape players for recording network news satellite transmissions. Recorder A is used 60% of the time; recorder B is used 30% of the time; and recorder C, 10% of the time. The probabilities that the recorders will fail to record properly are .05, .08, and .09, respectively. In preparation for the evening newscast, a taped network segment is found to have been improperly recorded. The news director does not know what recorder had been used to record that segment. What is the probability that the recording was made on recorder A? recorder B? recorder C?

Solution

A. Using the traditional formula method

The generalized Bayes' formula is defined as follows:

Let $A_1, A_2, \ldots, A_k$ be a set of mutually exclusive events such that at least one of the events must occur and no two events can occur at the same time. Let B represent a set of events. Then,

$$P(A_i|B) = \frac{P(B|A_i) \cdot P(A_i)}{\sum_{j=1}^{k} P(B|A_j) \cdot P(A_j)} \quad \text{where } i, j = 1, \ldots, k.$$

Given:

$P(A)$ = Probability that a segment is recorded by recorder A = .6
$P(B)$ = Probability that a segment is recorded by recorder B = .3
$P(C)$ = Probability that a segment is recorded by recorder C = .1.

Let D represent the event that a network news segment is improperly recorded. D' will denote the complement of the event D.

$P(D|A) = P$ (recorder A will fail to record properly) = .05.
$P(D|B) = P$ (recorder B will fail to record properly) = .08.
$P(D|C) = P$ (recorder C will fail to record properly) = .09.

The desired probabilities that an improperly recorded tape had been recorded by video tape recorder A, recorder B, and recorder C are denoted by $P(A|D)$, $P(B|D)$, and $P(C|D)$, respectively.

$$P(A|D) = \frac{P(D|A) \cdot P(A)}{P(D|A) \cdot P(A) + P(D|B) \cdot P(B) + P(D|C) \cdot P(C)}$$

$$= \frac{(.05)(.6)}{(.05)(.6) + (.08)(.3) + (.09)(.1)}$$

$$= \frac{30}{63} \text{ or } \frac{10}{21}.$$

Similarly, $P(B|D) = \frac{8}{21}$, and $P(C|D) = \frac{3}{21}$.

B. Using a tree diagram

The tree diagram as shown in Figure 1 gives a graphic representation of the relationships among the prior, conditional, and posterior probabilities related to the problem. Calculating the desired probabilities from the tree diagram follows in the same manner as with the traditional formula method.

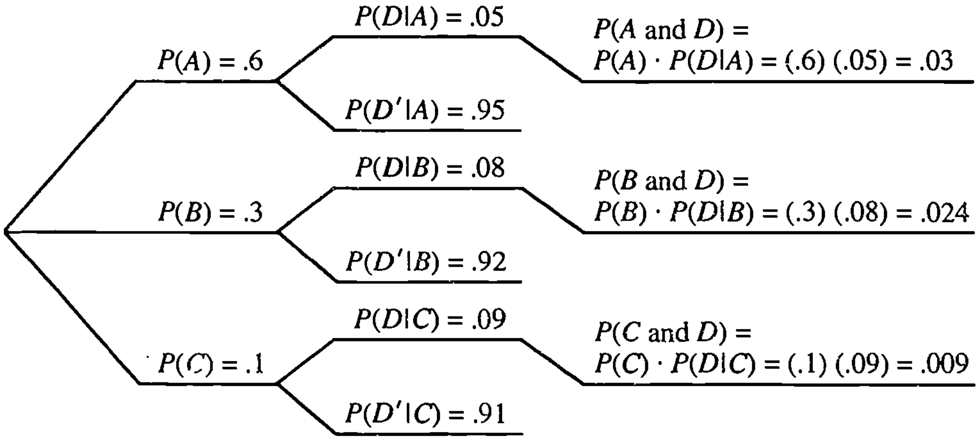


Figure 1.

The advantage which the tree diagram has with respect to understanding comes mostly from the vertical alignment of additive elements, not from the branches of the tree per se. For this reason, we believe that the tree itself is unnecessary and that a tabular approach is more effective in the classroom situation. From the figure above, $P(A|D)$ is given by $\frac{.03}{.03 + .024 + .009} = \frac{.03}{.063} = \frac{30}{63} = \frac{10}{21}$. Similarly, $P(B|D) = \frac{8}{21}$ and $P(C|D) = \frac{3}{21}$.

C. Using an $n \times m$ contingency table

An alternative means of determining the probabilities of transmission utilizing an $n \times m$ contingency table where n is the number of rows and m is the number of columns is as follows:

The contingency table has two rows and three columns: taping event (rows) and recorder (columns). The empty table is as shown in Table 1.

1. Start by recording the probabilities of events A, B, and C as column totals.

	A	*B*	*C*	Row Totals
D				
D'				
Column Totals	.600	.300	.100	1.0

Table 1

2. Given the values $P(D|A) = .05$, $P(D|B) = .08$, and $P(D|C) = .09$, we can complete the first row by multiplying the probabilities of A, B, and C by the conditional probabilities $P(D|A)$, $P(D|B)$, and $P(D|C)$, respectively, since $P(A \text{ and } D) = P(A) \cdot P(D|A)$. See Table 2.

	A	B	C	Row Totals
D	.030	.024	.009	
D′				
Column Totals	.600	.300	.100	

Table 2

The cell in the Ath column and Dth row contains the outcome for A and D; that is, recorded by A and defective. In the cell we record the value of $P(A \text{ and } D)$. Because $P(A \text{ and } D) = P(A) \cdot P(D|A)$, we record .030 in this cell.

$P(A \text{ and } D) = P(A) \cdot P(D|A) = (.6)(.05) = .030$ since 5% of all tapings by recorder A will be faulty.

Similarly, $P(B \text{ and } D) = P(B) \cdot P(D|B) = (.3)(.08) = .024$ since 8% of all tapings by recorder B will be faulty.

$P(C \text{ and } D) = P(C) \cdot P(D|C) = (.1)(.09) = .009$ since 9% of all tapings by recorder C will be faulty.

3. The second row is completed by subtracting the first row from the column totals since $P(A \text{ and } D) + P(A \text{ and } D') = P(A)$. See Table 3.

	A	B	C	Row Totals
D	.030	.024	.009	
D′	.570	.276	.091	
Column Totals	.600	.300	.100	

Table 3

The row totals complete the contingency table, giving us the probabilities of D and D'. See Table 4.

	A	B	C	Row Totals
D	.030	.024	.009	.063
D'	.570	.276	.091	.937
Column Totals	.600	.300	.100	1.000

Table 4

4. The new row corresponding to $P(D)$, (highlighted in Table 5) becomes the universe of discourse for the problem we are solving, since it is given that a faulty taping has occurred.

	A	B	C	Row Totals
D	.030	.024	.009	.063
D'	.570	.276	.091	.937
Column Totals	.600	.300	.100	1.000

Table 5

The row total is a "whole" $P(D) = .063$ comprised of three "parts," $P(A \text{ and } D) = .030$, $P(B \text{ and } D) = .024$, and $P(C \text{ and } D) = .009$. Each solution is then a ratio of "part" to "whole." So

$$P(A|D) = \frac{P(A \text{ and } D)}{P(D)} = \frac{.030}{.063} = \frac{10}{21},$$

$$P(B|D) = \frac{P(B \text{ and } D)}{P(D)} = \frac{.024}{.063} = \frac{8}{21}, \text{ and}$$

$$P(C|D) = \frac{P(C \text{ and } D)}{P(D)} = \frac{.009}{.063} = \frac{3}{21}.$$

In general, the values in the contingency table are shown in Table 6.

	A	*B*	*C*	Row Totals
D	.030 $P(A \text{ and } D)$	.024 $P(B \text{ and } D)$	.009 $P(C \text{ and } D)$	.063 $P(D)$
D′	.570 $P(A \text{ and } D')$	.276 $P(B \text{ and } D')$	.091 $P(C \text{ and } D')$	.937 $P(D')$
Column Totals	.600 $P(A)$	.300 $P(B)$	.100 $P(C)$	1.000

Table 6

The next problem illustrates the application of Bayes' rule to medical science. Bayes' rule is very effective in utilizing information about accuracy of a medical screening test to help the physician diagnose the patient's illness. We now present a typical health science-oriented problem, and subsequently solved using the contingency table method.

Example: An Application to Medical Science

A test has been developed for determining the presence of a certain disease. The test gives a positive reaction in 95% of the cases where a person actually carries the disease. However, the test also gives a positive reaction in 5% of the cases where a person does not carry the disease. From the previous clinical experience, it is believed that 2% of the population carries the disease. The interesting questions now are (1) what is the probability that a person actually carries the disease given that he or she has a positive test result? (2) what is the probability that a person does not carry the disease given that he or she has a negative test result?

<u>**Solution**</u>

The way to summarize test results in people with and without the disease lays out the data in a 2×2 contingency table. The basic table consists of four cells arranged in two columns and two rows. The column, row, and grand totals which appear at the end of each column or row are called column totals and row totals, respectively.

The column factor, disease D, separates people by presence, D^+, or absence, D^-, of disease. The row separates people by positive test result, T^+, or negative test result, T^-. The 2×2 contingency table provides an easy format for computing the probability of people with any particular disease and/or test result. Our 2×2 table is as follows.

	D^+	D^-	Row Totals
T^+	$P(D^+ \text{ and } T^+)$	$P(D^- \text{ and } T^+)$	$P(T^+)$
T^-	$P(D^+ \text{ and } T^-)$	$P(D^- \text{ and } T^-)$	$P(T^-)$
Column Totals	$P(D^+)$	$P(D^-)$	1.000

Table 7

Since 2% of the population has this disease, we can conclude $P(D^+) = .02$ and $P(D^-) = 1.00 - .02 = .98$.

Furthermore, $P(T^+|D^+) = .95$ and $P(T^+|D^-) = .05$ are given. We can complete the first row by multiplying the probabilities of D^+ and D^- by the conditional probabilities $P(T^+|D^+)$ and $P(T^+|D^-)$, respectively, since $P(T^+ \text{ and } D^+) = P(D^+) \cdot P(T^+|D^+)$ and $P(T^+ \text{ and } D^-) = P(D^-) \cdot P(T^+|D^-)$. The results are as follows:

The probability that a person has a positive test and is diseased is $P(T^+ \text{ and } D^+) = P(D^+) \cdot P(T^+|D^+) = (.02)(.95) = .019$.

The probability that a person has a positive test and is not diseased is $P(T^+ \text{ and } D^-) = P(D^-) \cdot P(T^+|D^-) = (.98)(.05) = .049$. See Table 8.

	D^+	D^-	Row Totals
T^+	.019	.049	
T^-			
Column Totals	.020	.980	1.000

Table 8

The marginal row totals are completed by adding two probabilities, .019 and .049, that is to say, the probability that a person has a positive test regardless of the disease status is $P(T^+) = .019 + .049 = .068$. The second row is completed by subtracting values in the first row from the corresponding values in the column totals, which gives us $P(D^+ \text{ and } T^-) = .020 - .019 = .001$. See Table 9.

	D^+	D^-	Row Totals
T^+	.019	.049	.068
T^-	.001	.931	.932
Column Totals	.020	.980	1.000

Table 9

Now, what are the probabilities $P(D^+|T^+)$ and $P(D^-|T^-)$? The first row total is a "whole" $P(T^+) = .068$ comprised of two "parts," $P(T^+ \text{ and } D^+) = .019$ and $P(T^+ \text{ and } D^-) = .049$. The second row total is also a "whole" $P(T^-) = .932$ comprised of two "parts," $P(T^- \text{ and } D^+) = .001$ and $P(T^- \text{ and } D^-) = .931$. Each solution is then a ratio of "part" to "whole." So

$$(1)\quad P(D^+|T^+) = \frac{P(D^+ \text{ and } T^+)}{P(T^+)} = \frac{.019}{.068} = \frac{19}{68} \approx .279 \text{ or } 27.9\%.$$

$$(2)\quad P(D^-|T^-) = \frac{P(D^- \text{ and } T^-)}{P(T^-)} = \frac{.931}{.932} = \frac{931}{932} \approx .999 \text{ or } 99.9\%.$$

$$(3)\quad P(D^-|T^+) = \frac{P(D^- \text{ and } T^+)}{P(T^+)} = \frac{.049}{.068} = \frac{49}{68} \approx .721 \text{ or } 72.1\%.$$

We obtain the somewhat surprising probability of 27.9% that a person who reacts positively to the test actually has the disease. It is more surprising that a person who reacts positively to the test actually has no disease is 72.1%. The screening test is not quite reliable to those who reacted positively to the test since the positive test result detects the disease with such a small probability.

Conclusion

The use of an $n \times m$ contingency table in teaching Bayes' theorem has several advantages. It is easy to construct and does not require any sophistication in mathematics. Yet, it provides the student with an opportunity to develop a basic understanding of conditional probability and the insights to expand that knowledge to higher cognitive levels. It helps students learn how to identify the meaning of each cell in the model, which is quite difficult in the tree diagram or formula method. The fact that both the row-sums and column-sums add up to 1 ties in nicely with the concept that the probabilities of mutually exclusive, collectively exhaustive, events add up to 1. The table also acts as a constant reminder of the exact relationships among the probabilities and the complementary conditional probabilities which comprise them. It allows students who are less able to understand traditional set notation and summation notation easier access to the concepts of Bayes' theorem. They do not have to engage in meaningless memorization of the formula; they can derive it. It also allows students in a more rigorous environment, where solid understanding of the notation is a goal, to derive Bayes' formula as a general case of the contingency table problem solution method by replacing numeric probabilities with their notational equivalents. In either case, the contingency table method, not only helps students understand the concepts of conditional probabilities in two or more events – it helps them develop "statistical heuristic."

References

Freund, J.E & Smith, R.M. (1986). *Statistics: A first course.* (4th ed.). Englewood Cliffs, NJ: Prentice Hall.

Iversen, G.R. (1984). *Bayesian statistical inference.* Newbury Park, CA: a SAGE University paper.

McClave, J.T., & Dietrich, F.H. (1994). *Statistics.* (6th ed.). New York: Macmillan College Publishing Company.

Mendenhall, W. (1993). *Beginning statistics: A to Z.* Boston: Duxbury Press.

Mendenhall, W., & Beaver, R.J. (1991). *Introduction to probability and statistics.* (8th ed.). Boston: PWS-Kent Publishing Company.

Mendenhall, W., and Reinmuth, J.E. (1986). *Statistics for management and economics.* (5th ed.). Boston: Duxbury Press.

National Council of Teachers of Mathematics 1981 Yearbook. (1981). *Teaching statistics and probability.* Reston, VA: NCTM Inc.

Polya, G. (1968). *Mathematics and plausible reasoning.* (2nd ed.). Princeton, NJ: Princeton University Press.

Strait, P.T. (1989). *A first course in probability and statistics with applications.* (3rd ed.). New York: Harcourt Brace Jovanovich.

Triola, M.F. (1986). *Elementary statistics.* (3rd ed.). Menlo Park, CA: The Benjamin/Cummings Publishing Company.

Tversky, A., & Kahneman, D. (1974). Judgement under uncertainty: Heuristics and biases. *Science, 185*, 1124–1131.

Tversky, A., & Kahneman, D. (1983). Extensional versus intuitive reasoning: The conjunction fallacy in probability judgement. *Psychological Review, 90*, 293–315.

Mathematics can never tell us whether any alleged fact is true or false. Mathematics, in short, has no more to do with truth than logic has. To say something is mathematically proved is tantamount to saying that it cannot possibly be true.

E. V. Huntington

Fourth Conference on the Teaching of Mathematics and TICAP Conference

June 23-25, 1995

The Calculus Consortium, based at Harvard University, in conjunction with the National Science Foundation (NSF) and John Wiley and Sons, Inc. announces the Fourth Conference on the Teaching of Mathematics on June 23-25, 1995 in California. This year's conference will broaden its focus from calculus to include other courses in undergraduate mathematics. A program of invited speakers, panels, and contributed papers will provide something of interest for everyone involved in the way mathematics is taught. Two and four-year college, university and secondary school faculty are welcome. Attendance will be limited. A conference for Advanced Placement calculus teachers entitled "AP's Next Generation" will follow on June 25, 1995. It is supported partially by the project Technology Intensive Calculus for AP (TICAP).

WILEY

For more information please contact:

Kim Lemmonds
John Wiley and Sons, Inc. Publishers
605 Third Avenue
New York, NY 10158
Fax: 212-850-6118
E-Mail: math@jwiley.com

An Appropriate Culminating Mathematics Course

by

Bill Haver
Virginia Commonwealth University
Richmond VA 23284-2014

Gwen Turbeville
J. Sargeant Reynolds Community College
Richmond VA 23285-5622

Bill Haver is a Professor of Mathematical Sciences at Virginia Commonwealth University, where he has served on the faculty sinced 1977. He received his B.S. from Bates College, his M.S. from Rutgers University and his Ph.D. from SUNY Binghamton, all in mathematics. He recently completed a term as a program officer in the Division of Undergraduate Education at the National Science Foundation.

Gwen Turbeville received a B.A. and M.A. from the University of North Carolina at Greensboro. She has taught at J. Sargeant Reynolds for eight years. She is currently on the Executive Board of VMATYC.

There is growing sentiment that College Algebra is an inappropriate culminating mathematics course. After completing their formal mathematics education, non-science students may study Statistics, may make use of mathematics in social science classes and may encounter quantitative issues in their workplace environment. In the longer term, they will form a maj - part of our society in such roles as civic and government leaders, journalists, elementary school teachers, and parents. The mathematical attitudes that they bring from their last mathematics course will probably not change significantly throughout their lifetime. The traditional College Algebra course, which is primarily designed to prepare students for the study of Calculus, is not the appropriate mathematical preparation for these students. Consequently, we believe that it is important that alternatives to the traditional College Algebra course be available for those students who are culminating their formal mathematics education. This view is, of course, shared widely and is discussed in reports from the Committee on Mathematical Education of Teachers (1991) and the National Research Council (1991).

Recently, the co-authors developed and team taught, during two different semesters, a freshmen level mathematics course. Support for the activity was provided by a Ford Foundation grant to Virginia Commonwealth University (VCU) and J. Sargeant Reynolds Community College (JSRCC). The purpose of the grant was to enhance transfer opportunities between the two institutions and to develop a new freshman level mathematics course to be offered at both institutions. The course has the same prerequisite structure as College Algebra, but is designed for non-science students whose primary reason for taking the course is to fulfill a minimum mathematics requirement for graduation. The goals of the course are to develop, as fully as possible, the mathematical and quantitative capabilities of the students; to enable them to understand a variety of the applications of mathematics; to prepare them to think logically in subsequent courses and situations in which mathematics occurs; and to increase their confidence in their ability to reason mathematically.

We needed a format that could be used in multiple sections by many different faculty members (including part-time instructors and/or graduate assistants). We consider ourselves successful: the course actively engaged students in doing mathematics; promoted oral and written communication of mathematics; and enabled students to actually enjoy mathematics.

We hope to encourage faculty and departments in the two-year and four-year colleges to reconsider the mathematical content and instructional methodology appropriate for non-science students who are completing their formal mathematics education. We did not actually break new ground in either mathematical content or teaching approaches. In fact, what we did was neither particularly difficult, nor terribly time consuming. And that is the point! Our experience demonstrated that it is possible to teach a non traditional course in a way that will actively engage the large percentage of our nation's students who find mathematics boring, unnecessary, and intimidating.

Mathematical Content

We based our course around the materials first developed by COMAP (1988) in *For All Practical Practices*. The actual course text was *Excursions in Modern Mathematics* by Tannenbaum and Arnold (1992). The topics chosen emphasized that mathematics consists of taking a real-world situation, converting this situation to an abstract mathematical problem, solving the abstract problem and applying what is learned to the original situation:

i) Optimization Problems. The traveling salesman problem; Euler graphs; minimum spanning trees; and scheduling problems.

ii) Handling Data. Design of surveys and the understanding of error due to chance and error due to sample bias; basic ideas of probability; and the representation and communication of data.

iii) Mathematics of Growth. Exponential growth using examples from finance and ecology; Fibonacci numbers and the golden ratio; and an intuitive notion of sequences, series, and limits as mathematical tools.

iv) Social Choice. Employing mathematics to answer questions concerning fair division; the mathematics of a preference election including Arrow's theorem; and weighted voting systems.

Student Activities and Assessment

Lecture time was held to a minimum. An instructor was in front of the class about 30% of the time, giving short lectures, leading discussions, and describing project requirements. The rest of the time was dedicated to in-class work on projects, group problem solving activities, presentations by students, and assessment of students. We unabashedly required and obtained student involvement by assigning grades to many activities, collecting about 20 different grades each semester. Course grades were based on student performance in the following categories: tests (30%), quizzes (20%), small projects (20%), and large projects (30%). The traditional tests covered several chapters of material with calculators required. Students were expected to demonstrate an understanding of the type of mathematical problems studied and algorithms developed to solve these problems. The emphasis was not on computation, but students needed to use basic mathematical tools such as the Pythagorean Theorem, quadratic formula, summation notation, percentages and exponential equations. Frequent quizzes were given to ensure that students were keeping up between tests. In addition to traditional 20 minute closed book quizzes, we used open book and/or group quizzes.

Small Projects

Approximately twelve small projects were assigned both as individual and as group efforts. Sometimes more difficult problems from the text were assigned to re-enforce skills or to provide preparation for future course topics. There was an emphasis on reporting results and processes used to generate solutions. Following is an example of a typical "small" group project:

■ ***Sample of A Small Group Project*** (abbreviated form)

Our text describes a number of different algorithms (or schemes) which make use of mathematical ideas and are answers to the following question: **How can an object or a set of objects be divided among a set of participants in such a way that ensures that each is satisfied that he or she has received a fair share of the total?**

Formation of your group: We will assign each member of the class to a group of five students. Then each group will be assigned a different "fair division scheme."

Your assignment: Your assignment is to first understand your algorithm and then to prepare a 25 minute presentation teaching the entire class how to use your algorithm. We will give you two class periods to work on this assignment, so your group will not need to meet outside of class.

Your Grade: Each member of your group will receive the same grade. Grades will be based upon: accuracy of your lesson; demonstration that all members of your group understand the algorithm; creativity; extent to which the class was actively engaged and understood your algorithm; and the extent to which the class enjoyed your lesson.

Follow-up activity: You will be given an out of class assignment to turn in which will require that you make use of each of the "fair division" algorithms presented.

Large Projects

Three "large" projects were assigned during the semester. An abbreviated version of the assignment follows:

■ ***Sample of A Large Individual Project*** (abbreviated form)

The assignment: For this project you are to write a newspaper article which is related to the mathematics that we are studying this semester. Possible types of newspaper articles include: news articles, obituaries, book reviews, feature articles, columns. The article should be 2-4 pages in length and should include graphics, headlines, and have a clear mathematical content.

Audience: Your audience for the paper is a reader of the *New York Times*. This individual is generally well-educated and is interested in a breadth of topics, but has no technical mathematical knowledge.

Grading Criteria: The criteria for grading are listed in a descending order of importance:

- **quality of ideas:** Would the reader find the article interesting?...
- **supporting materials:** Are the ideas in this piece well developed?...
- **organization:** Is the article well organized?...
- **style:** Is the language clear and appropriate?...
- **mechanics:** Are punctuation, spelling, and grammar rules followed?...

Another project was a group effort requiring out of class time to prepare written reports and an oral presentation:

■ ***Sample of A Large Group Project*** (abbreviated form)

Formation of "Polling Companies": The class should organize itself into groups of size 3–4; each group will constitute a "Polling Company." You will need to meet as a group outside of class, so make sure all members of the group are available at the same time.

Getting Started: Give your company a name, develop a logo for your company.

First Assignment: There are approximately 6,000 red and white beans in three bags. You know that 1/2 of the beans are in bag A, 1/4 in bag B and 1/4 are in bag C. The first assignment for your company is to estimate what percent of the beans are red. You are permitted to sample 144 beans. Your first report will include a description of the procedures you used to make the estimate and a clear description of how confident you are of the accuracy of your estimate. The report should incorporate all of the key words found at the end of the textbook chapter concerning survey procedures. Due in 10 days.

Second Assignment: The second assignment is to prepare a written (8–10 typed pages) proposal, supplemented by an oral presentation, requesting support to conduct a survey from any mythical organization you choose. First you need to choose a question of interest, then you need to consider the various issues surrounding surveys, then you should conduct a pilot survey of at least 20 individuals. Finally you need to prepare your written and oral report. This assignment is due 2 weeks after the due date for the first assignment. The report should include results of your pilot survey and provide a plan for conducting the full survey which addresses the following issues: bias in the wording of the question, description of the "population," sample size, stratification of sample, selection of sample, nonresponse rate, and confidence in the results to be obtained by the full survey.

Grading: Separate grades will be assigned for the first and second assignment. Grades will be based on: accuracy; demonstration that you understand the issues associated with conducting surveys; use of graphics; quality of writing and oral presentation; demonstration that all members of the company participated actively and understand the assignments; extent to which the rest of the class understood and enjoyed your presentation.

Problems We Had Teaching the Course

Most of the students who registered for this class arrived with an active "hate" for mathematics and mathematical ideas. Since the course is designed to encourage mathematical thinking and to enable students to communicate mathematics in writing and orally, student "anger" did not remain hidden as it did in the traditional College Algebra classes that we have taught. We attempted to handle this problem by talking openly about this hostility and math anxiety and by having students write about their previous experiences with mathematics. But, we still needed to put up with some good-natured hostility on occasion.

We had some of the normal problems associated with group work. As is the case at most two-year institutions and urban universities, nearly all of the students live off campus and nearly all of the students have jobs. Thus, finding time for the groups to meet is a problem. We were able to solve this problem, at least partially, by giving students time in class to form the groups and instructing them to check schedules before they finalize their groups. We used about four different sets of groups each semester and on one occasion a group developed serious internal problems. We attempted to solve this problem by taking control of the group for a period of time and helping the members to develop a plan of action. We also reassured them that for the next project the same individuals would not have to work together. (Actually the major antagonists became reasonably tolerant of each other.)

Overall we had excellent attendance; however, we found that attendance fell off near the end of the semester and that some of the better students decided that the course was easy and stopped doing as much work. Consequently, the first semester there were no A's in the course and even in the second semester we needed to repeat the goals and expectations of the course on a couple of occasions.

Things that Went Well

In our opinion, we were very successful in increasing the students' ability to communicate concerning quantitative matters. The oral presentations were of very high quality: good handouts were prepared by the students, they often would dress in a manner appropriate for their role, audience involvement was expected and the students (both those presenting and those in the audience) were usually in good spirits. Similarly, the written materials were well-prepared. Clearly, the assignments were taken seriously and were, surprisingly, interesting to read. As evidenced by test performance, by class discussions, and by oral and written presentations, the students thought about mathematics. Since the course is about mathematical ideas and uses of mathematics, it is not possible for the students to get into the mode of finding a template for the solution of the problem and to thereby avoid thinking.

We are convinced that students obtained and will retain some important mathematical ideas and the ability to consider problems that require a mathematical approach. Most of the students and both instructors enjoyed themselves. Students got to know each other better than in most introductory courses. In commuter situations, this opportunity for students to develop relationships that contain an academic component is extremely valuable.

Another measure of the success of this approach is the response of our colleagues in our departments and in other departments. This course has been officially added to the curriculum at VCU to serve as an alternate course to meet graduation requirements for students majoring in the Social

Sciences, Mass Communication, and the Humanities. It is recommended for all students preparing to be elementary school teachers. In addition, J. Sargeant Reynolds Community College has begun the process of recommending this course for addition to the Virginia Community College System offerings.

In order for this course to be the standard offering for all non-science students a large number of faculty need to be involved. As of the Fall of 1994, the course has now been taught by a total of six different individuals. Two three-hour workshops and several planning meetings are held each semester to prepare adjunct faculty, graduate students and full-time faculty to teach these mathematical ideas in the described manner.

In summary, we made use of existing materials to develop and teach a course that introduces students to interesting, contemporary mathematical approaches, in a way that involves active student participation and encourages mathematical thinking. It wasn't that hard to do, and as instructors we enjoyed the experience. We are convinced that the students significantly increased their capability to use mathematics and to communicate mathematics both orally and in writing. Our experience has strengthened our belief that this approach to mathematics provides a much more appropriate culminating mathematics experience than College Algebra for future journalists, elementary school teachers, civic and business leaders, and parents.

References

Committee on Mathematical Education of Teachers. (1991). *A call for change: Recommendations for the mathematical preparation of teachers of mathematics*. Washington, DC: Mathematical Association of America.

Consortium For Mathematics and its Applications. (1988). *For all practical purposes: Introduction to contemporary mathematics*. New York: W. H. Freeman.

National Research Council. (1991). *Moving beyond myths: Revitalizing undergraduate mathematics*. Washington, DC: National Academy Press.

Tannenbaum, P. and Arnold, R. (1992). *Excursions in modern mathematics*. Englewood Cliffs, NJ: Prentice Hall.

> It is true that mathematics, owing to the fact that its whole content is built up by means of purely logical deduction from a small number of universally comprehended principles, has not unfittingly been designated as the science of the self-evident. Experience, however, shows that for the majority of the cultured, even of scientists, mathematics remains the science of the incomprehensible.
>
> Alfred Pringsheim

Community College Success in an International Mathematics Competition

by

John Loase and Rowan Lindley
Westchester Community College
Valhalla NY 10595-1698

John Loase is Associate Professor of Mathematics at Westchester Community College. He has a doctorate in mathematics and psychology and has written articles in both areas.

Rowan Lindley is Assistant Professor of Mathematics and Computer Science at Westchester Community College. She has taught Mathematics in England and Kenya, and in the United States for eleven years.

In 1991 SUNY-Westchester Community College fielded a team in the international Mathematical Contest in Modeling that tied with the University of California-Berkeley, Fudan University (China), and University College-Galway (Ireland) among 16 teams receiving a meritorious rating. No Ivy League team exceeded or even equalled this. Our 1992 team tied for honorable mention with Harvard, Stanford and the University of Beijing. In 1994, Westchester Community College entered two teams, both of which received honorable mention, higher than teams from Berkeley, Cornell and Columbia. Results like these can educate the public about the high academic achievement of many students at community colleges throughout the nation.

Community Colleges often receive recognition for excellent teaching. But status in academia is derived mainly from scholarship, articles in key journals and plaudits from fellow researchers who understand highly specialized work, areas in which community colleges are not traditionally recognized. We welcome the opportunity to show that community college students can achieve competitive results in a scholarly competition.

As teachers at a community college, we have always believed that a sizeable portion of our students could compete academically with students from the Ivy League. One of us taught for a year at a select high school that fed the Ivy League: five graduates that year were accepted into Harvard. The parents paid a lot of money for the name of the high school and the rigor of the instruction to get a place for their child at the Ivy League school of their choice. Though the students were extremely well-prepared and generally conscientious, the economic advantage and the highly demanding expectations of the students by their parents were factors greatly contributing to their educational success.

At Westchester Community College, as is typical of community colleges, students are generally less affluent and receive less encouragement from their families. In addition, many work long hours outside the campus and/or have family and other responsibilities and pressures that distract from their educational progress. This does not mean that they are less able, only that their home environments are not gearing them for the highest academic achievement. One of our tasks as community college faculty is to give our students opportunities to shine so that they will believe themselves to be capable of high levels of academic excellence.

The Mathematical Contest in Modeling, sponsored by the Consortium for Mathematics and Its Applications (COMAP), is a good vehicle for giving our students one of these motivating opportunities. The contest requires a team of three students to solve a real world problem in three days. It is the only international mathematics contest in which students work together to find a solution; this is beneficial for community college students who are likely to have widely differing backgrounds and skills. The open ended nature of the problem to be solved and the knowledge that there is no one right answer and that partial solutions are acceptable also gave our teams confidence that they might be able to find a good approach. These factors were in our students' favor. However, there were also factors working against them which needed to be addressed.

Mathematical modeling is an emerging discipline which uses calculus based probability and statistics, graph theory and computer science to solve applied real world problems. Modeling can attack problems in fields as diverse as engineering, biology, operations research and economics, to name just a few. In order to compete successfully, our math teams had to learn materials which are normally taught in the junior and senior years of college.

Our teams were recruited from the Calculus III and Differential Equations classes in the fall semester. These students have demonstrated ability and interest in mathematics and have a strong background. The nature of the competition was described to them and they were encouraged to attend coaching sessions which were held for two hours every two weeks in the fall and as often as we could get together (approximately two hours a week) in the spring. The competition is held towards the end of February. The students were taught calculus based statistics and graph theory. In the spring they were given past contest problems (published by COMAP in their *UMAP Journal*) to work on, and the solutions were discussed. Winning solutions from past problems are very helpful. For example, the students learned that they should clearly state the assumptions that they are making, and that, within reason, they can adapt these assumptions to fit the proposed model. They also learned the importance of being able to write clearly about mathematics. We have been allocated a small budget by student activities which we use to buy books and journals for the students (see references for details).

Since 1991 we have been able to use that year's success as a good recruiting tool. Several students normally express interest. Their major concern is the work load; many are carrying loads of 17 or more credits and/or have heavy responsibilities outside the classroom. We tell them that they can do as much or as little as they have time for. The mid-winter break is a good time for the students to get together and start working cooperatively, discovering each others' areas of strength. It is helpful if the three person team(s) (a department may enter one or two teams) are formed before the break. The 1991 team did most of their preparation during this time. However, the 1994 teams were not formed until just before the competition; at that point there were five students who had been working hard on the contest material. While we were discussing this problem in the corridor one day, one of the students spied a friend whom she knew to be mathematically able. We invited him to join the teams, which he did; the six students were than divided into two teams so that the statistical and computer knowledge was distributed equally.

There are two contest problems of which the students must choose one. The faculty advisor gives them the sealed problems any time after midnight on a Thursday night and the students' solution must be postmarked on the following Monday. During the intervening time the students may consult any inanimate source, so research techniques and library access were discussed during training. Access to a computer is also necessary (computers do not have to be used in the solution but a word processor is essential). The 1991 award winning team said that they worked for three days with hardly any sleep on their problem. They drove all over New York to obtain recent research articles on graph theory. The 1994 teams had some other commitments over the weekend (a chemistry class, an important hospital appointment) but by allocating tasks and time efficiently they still came up with winning solutions (and quite different ones as the two teams may not talk to each other).

Our winning students have very varied backgrounds. Several started at our college in trigonometry and precalculus classes. Two came from India. Some came to Westchester Community College directly from high school; some are students returning to education. At least one did not graduate from high school. Three are married, one is a member of a rap group, one was a travelling musician for ten years, another was an actor. Most are struggling financially. They may not have taken as many math courses as students at four year schools, but their widely varying skills and experience in coping with life's vicissitudes may give them a different kind of advantage.

The students' victories have always been widely published on campus, and carried in the local press (the 1991 team's success was also covered in the *New York Times*). All of our students can be proud that their college is able to compete successfully in what many students regard as the most difficult academic field. They may start at a community college because they are poor, or because they did not do well in high school, or because they recently arrived in this country, but they can continue there knowing that they can obtain the best education of which they are capable, and that many doors are open to them.

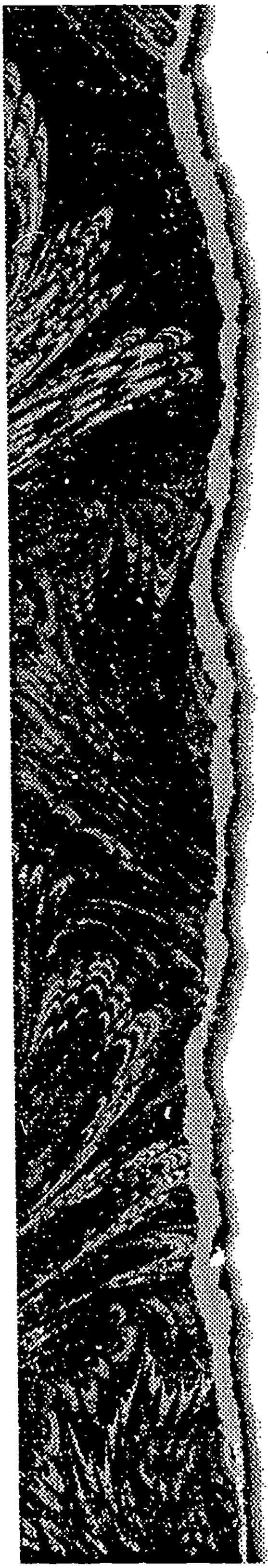
References

Chartrand, G. (1977). *Introductory graph theory.* Mineola: Dover Publications.

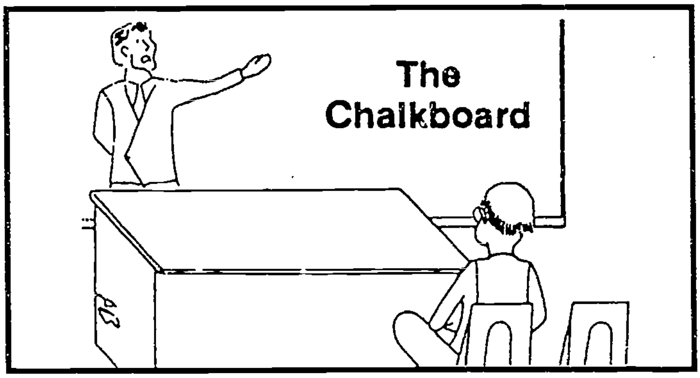
Loase, J.F. (in preparation). *An introduction to mathematical modeling.*

Giordana, F. Results of the mathematical contest in modeling. *The UMAP Journal, 15*(3), 187–206.

Volker, D. & Zorton, P. (1993). *Cliff's quick review, statistics.* Lincoln: Cliff's Notes.



REGULAR FEATURES



Edited by

Judy Cain and Joseph Browne

Tompkins Cortland Comm. College, Dryden NY 13053

Onondaga Comm. College, Syracuse NY 13215

This column is intended as an idea exchange. We hope to facilitate an open exchange of ideas on classroom management, teaching techniques, tips for helping students get past the usual stumbling blocks, techniques for improving student participation, etc. We know there are lots of good ideas out there, and this is your chance to share them. Please send your contributions to Judy Cain.

Hands-On Activities for College Math Students

We often believe that college mathematics students are too dignified and sophisticated for manipulative and hands-on activities. I discovered the fallacy of this belief when I used a hands-on activity to introduce the concept of correlation to my statistics class. I paired the students in the class and handed a ruler to each pair of students with instructions for one student in each pair to measure the other's hand-span. My plan was to generate a few ordered pairs in which the first coordinates would be the students' heights and the second coordinates would be the same students' hand-spans. I planned to construct a scatter plot, find the equation of the least squares line, and plot the least squares line on the scatter plot.

Each student in the class insisted on having his or her hand-span measured, and each student wanted his or her height and hand-span included in the scatter plot. Hands were waving in the air with requests for inclusion, a scene you might expect to see in an elementary school classroom. This simple activity generated more interest and excitement than I would ever have dreamed possible.

We are often disenchanted by our students' lack of excitement about and lack of interest in mathematics. A few simple hands-on activities might be just the stratagem needed to generate that excitement and interest. I will certainly strive to include more such activities in my mathematics classes.

Submitted by Frank W. Caldwell, Jr., York Technical College, Rock Hill, SC 29730

Euclid, the LCM, and the GCD

Many prealgebra and beginning algebra texts discuss various methods of finding the least common multiple (LCM) of two positive integers. However, methods of finding the greatest common divisor (GCD) are sometimes not covered. One method, the Euclidean Algorithm, fills this gap and provides an opportunity for student exploration as illustrated in the following project.

Use the Euclidean Algorithm to find the GCD of a variety of pairs of integers. Also find the LCM of the same pairs of integers. Then, for each pair of integers, multiply the GCD and the LCM. What do you notice about the product? (Good pairs of integers to begin with are those for which the answer is easy to see, such as 10 and 36, or 5 and 8, etc.)

Now the students can use the results to devise a method for finding the LCM of two integers using the GCD, or in fact for finding either the LCM or the GCD using the other. One group of my students did "research" to determine whether the same thing was true for a set of three integers.

Submitted by Rosemary Hirschfelder, The University of Puget Sound, Tacoma, WA 98416

More on Involving Every Student

To involve non-participatory students, I make it a practice to arrive at a lecture section early, if possible. This eventually leads to chatting with the students who sit near the front, about things other than mathematics. Students, including the non-participants (in the back, of course), start to see me as a person, not just a mathematics teacher. This alone has some effect. Additionally, current events, politics, sports subjects, controversy on campus, or whatever will often draw in a non-participant. Everyone has <u>some</u> area of interest, and in that domain, he or she is a participant. The hard part is breaking the defensive barriers; once that is accomplished, the student often becomes a mathematics participant too.

Submitted by Philip Mahler, Middlesex Community College, Bedford, MA 01730

More on Small Group Learning and Assigning the Groups

A useful way to form groups in class is to write mathematics terms (i.e., vocabulary words) dealing with the current topic of discussion on small slips of paper. Write each term on as many slips as the number of people that you wish to have in each group. Distribute the slips randomly to the class. Then tell the students that they are to form groups by finding all the people who have the same term on their papers.

The first task for each group is to discuss the term that formed the group; then the group can go on to the task of the day. If a group must report to the class at the end of the group session, their term must be defined or discussed in addition to the other topic. Not only is this an interesting way to form groups, but it also helps reinforce mathematical vocabulary.

Submitted by Maryann Justinger, Erie Community College - South, Orchard Park, NY 14127

THE MATHEMATICAL ASSOCATION OF AMERICA

MAA PLACEMENT TESTS...

can help solve your college's mathematics placement problems

- ◆ Do the members of your mathematics department complain that the students in their classes are not well prepared for their courses?
- ◆ Do your students complain that their mathematics courses are too hard or too easy?
- ◆ Do your faculty advisors rely primarily upon scores on aptitude tests or on high school mathematics courses taken in making recommendations to students about their initial college mathematics courses?

If your answers to these questions are YES, you should know about the PLACEMENT TEST PROGRAM of the Mathematical Association of America. This program assists two- and four-year colleges and universities with the development of on-campus placement programs to assess the mathematical skills of new students objectively and fairly when they enter college.

Colleges and universities pay an annual fee to subscribe to PTP. This progam provides up-to-date tests and placement testing information to its subscribers.

The MAA's Committee on Testing also offers free consulting services to any college mathematics d[illegible] planning to initiate a local mathematics [illegible] program or seekin[illegible] program.

START AN MAA STUDENT CHAPTER NOW!!!

FACULTY HELP YOUR STUDENTS BECOME PART OF THE WORLD'S LARGEST PROFESSIONAL ORGANIZATION DEVOTED TO THE INTERESTS OF COLLEGE MATHEMATICS. INCREASE INTEREST AND EXCITEMENT AMONG YOUR MATHEMATICS STUDENTS. ENCOURAGE MORE STUDENTS TO CONSIDER CAREERS IN THE MATHEMATICAL SCIENCES. MAA STUDENT CHAPTERS ARE DEVOTED TO INCREASING STUDENT INVOLVEMENT WITH MATHEMATICS CAREERS AND MATHEMATICIANS AT THE NATIONAL AND LOCAL LEVEL.

STUDENT CHAPTER MEMBERS:

- ▲ join the MAA at reduced rates;
- ▲ receive travel support for paper presentations at the summer mathematics meetings;
- ▲ receive notification of student activities at the national meetings which include a Student Lecture, a Student Workshop and the Student Hospitality Center;
- ▲ receive career information;
- ▲ automatically become members in one of the MAA's 29 regional Sections, which sponsor student workshops and banquets, career and environmental mathematics conferences and student paper sessions

WHAT SOME CHAPTERS DO:

- ▲ organize scavenger hunts for high school students;
- ▲ publish newsletters distributed to math majors and alumni math majors;
- ▲ write articles about mathematics for the school newspaper, and a puzzle contest;
- ▲ present a campus-wide evening of mathematics for the non-technical major, and, of course;
- ▲ arrange for speakers with topics from "Bernoulli Boys and the Calculus" to "Mathematics in Congress."

TO RECEIVE MORE INFORMATION ON THE PLACEMENT TEST PROGRAM, OR STUDENT CHAPTERS CONTACT: JANE HECKLER, MAA, 1529 18TH STREET, NW, WASHINGTON, DC 20036. E-MAIL: JHECKLER@MAA.ORG OR CALL 1-800-331-1MAA.

Snapshots of Applications in Mathematics

Dennis Callas
State University College of Technology
Delhi NY 13753

David J. Hildreth
State University College
Oneonta NY 13820

The purpose of this feature is to showcase applications of mathematics designed to demonstrate to students how the topics under study are used in the "real world," or are used to solve simply "charming" problems. Typically one to two pages in length, including exercises, these snapshots are "teasers" rather than complete expositions. In this way they differ from existing examples produced by UMAP and COMAP. The intent of these snapshots is to convince the student of the usefulness of the mathematics. It is hoped that the instructor can cover the applications quickly in class or assign them to students. Snapshots in this column may be adapted from interviews, journal articles, newspaper reports, textbooks, or personal experiences. Contributions from readers are welcome, and should be sent to Professor Callas.

Who Done It, And When?

(to accompany exponential functions or differential equations)

by Charles Johnson and Dennis Callas, SUNY College of Technology, Delhi, NY

Occasionally one reads in the newspaper, or hears in a news broadcast, that a dead body has been discovered and that the investigating police officer says that the body has been dead for approximately so many hours. How do you suppose that the length of time that the body has been dead is determined?

Mr. Robert Peet, a licensed mortician, states that there are many variables that help determine the time of death. For instance, rigormortis (the progressive stiffening of the muscles after death as a result of the coagulation of the muscle protein) usually sets in a few hours after death and then disappears within 24 to 48 hours. After 48 hours, the eyes will dehydrate and appear foggy and lack their normal smoothness.

Other considerations used to determine the time of death include age, body weight, the temperature of the surrounding medium (usually air or liquid), alcohol and drug levels found in the blood, etc. Even whether the deceased had a big meal prior to death can be of some help.

So what does all this have to do with mathematics? Well, provided the body was found where the temperature of the surrounding medium was fairly constant, and 48 hours had not passed since death (this second condition is necessary since the internal organs reach the temperature of the surrounding medium in approximately 48 hours), Newton's Law of Cooling can approximate the time of death.

The formula for Newton's Law of Cooling is $T = ce^{kt} + T_m$, where T is the temperature of the body, c and k are constants, t is the time since death, and T_m is the temperature of the surrounding medium.

Mr. Peet acknowledges that no coroner or detective is about to solve the above equation. What they will do, however, is take the temperature of the deceased, and from charts based on Newton's Law, determine the approximate time of death.

The following exercises vary in difficulty and should be assigned accordingly.

Exercises

1. Recent accounts of the O.J. Simpson case underscore the importance of knowing the exact time of death. If the exact moment (or a good approximation) of death is known, and O.J. has no witness to verify his whereabouts at the time of death, then he remains a prime suspect. If on the other hand the exact moment of death cannot be determined, and O.J. can account for most of his time, then it will be harder to prosecute him.

   Because there was a 10-hour lapse of time before a coroner's investigator examined the bodies, Dr. Irwin Golden, Deputy Los Angeles County Medical Examiner, testified that he could only place the time of death between 9 p.m. and midnight. Dr. Golden was widely criticized in the press for the delay, which resulted in a poor estimate of the time of death. Had the temperatures of the two deceased been determined earlier, "the more accurately a time of death can be fixed by calculating the drop in temperature" (AP release, dated July 13, 1994, quoting Wayne N. Hill, an Illinois forensic consultant).

   Explain why Mr. Hill's statement is true.

2. Suppose that the air temperature surrounding a body remains at a constant 10°F, $c = 88.6$, and $k = -0.24$. Then the formula for the temperature, T, at any time, t (in hours), will be

$$T = 88.6e^{-0.24t} + 10.$$

   a) Sketch a graph of the temperature, T, as a function of time, t, where $0 \le t \le 30$.

   b) When does the temperature of the body decrease more rapidly; just after death, or much later? How do you know this?

   c) What will the temperature of the body be after 4 hours?

   d) How long will it take for the body temperature to reach 40°?

   e) Approximately how long will it take the body temperature to be within 0.01° of the temperature of the surrounding air?

3. In the 1992 movie *Basic Instinct*, starring Michael Douglas and Sharon Stone, a doctor determines the time of death of Sharon Stone's lover. We hear a detective ask, "Do we have a time of death?" As the doctor takes the temperature of the deceased, we hear the same detective ask, "What is it, Doc?" The doctor replies, "92°, about 6 hours – puts time of death around 2 a.m., plus or minus."

   Verify the doctor's estimate of six hours by assuming that the room temperature was 70°, that the temperature of a living person is 98.6°, that when the doctor arrived at the scene of the murder he immediately recorded the body temperature as 93°, and that one hour later he recorded the body temperature as 92° (this is when we see the doctor and the detectives in the movie).

4. Newton's Law of Cooling states that the rate at which the temperature, T, changes in a cooling body is proportional to the difference between the temperature in the body and the constant temperature, T_m, of the surrounding medium (usually air or liquid). This can be modeled by the differential equation $\frac{dT}{dt} = k(T - T_m)$.

   a) Solve the differential equation, being sure to determine any parameters (constants of integration) which you encounter - there should be two! Assume that the temperature of the body at death is 98.6°, the temperature of the surrounding air is 68°, and that at the end of one hour the body temperature is 90°.

   b) What is the temperature of the body after 2 hours?

   c) When will the temperature of the body be 75°?

   d) Approximately when will the temperature of the body be within 0.01° of the surrounding air?

This snapshot was produced as part of a project sponsored by the State University of New York and the National Science Foundation (Division of Undergraduate Education).

The Devil and the Mathematician

[This story has been around for many years, but somehow seems timely again. ED]

The Devil boasted to the mathematician, "Set me a task I can't carry out, and I'll give you anything you ask for."

The mathematician replied, "Fair enough, prove that for n greater than 2, the equation $a^n + b^n = c^n$ has no non-trivial solution in integers."

They agreed on a three day period for the labor, and the Devil disappeared. At the end of the three days the Devil returned, haggard, jumpy, and biting his lip. The mathematician asked, "Well, how did you do with my task? Did you prove the theorem?"

"Eh, No," answered the Devil with a curse under his breath, "No, I haven't proved it."

"Then I can have whatever I ask for!" exclaimed the mathematician. "Wealth and fame."

"What? Oh, *that* — of course. But look here," beckoned the Devil. "If we could just prove this lemma...."

Little Rock
full of surprises

American Mathematical Association of Two-Year Colleges

21ST ANNUAL CONFERENCE

NOVEMBER 9-12

Excelsior Hotel (Headquarters)
Camelot Hotel
Capital Hotel
Holiday Inn – City Center

STANDARDS:
"The Key to Success"

- Empowerment of Students
- Technology-Based Delivery Systems
- Critical Thinking Skills and Problem Solving
- Professional Development
- Alternative Teaching Strategies
- Curricular Reforms: Calculus, Pre-Calculus, Developmental Math

Arkansas' diverse physical and cultural features, which exist nowhere else, created the "Natural State." Scenic landscapes and outdoor recreational activities abound year-round. From the breathtaking Ozarks, with their wonderful mountain heritage, to the Delta, featuring its traditional southern accent and meandering river, friendly Arkansas will greet you warmly.

In addition to activities planned for professional growth, there will be many exciting things for participants and their guests to do.

- Riverfront Park is located near the Excelsior Hotel
- Museums are within walking distance
- A family lounge overlooking the Arkansas River will be available

Notes from the Mathematical Underground

Edited by

Alain Schremmer
Mathematics Department, Community College of Philadelphia
1700 Spring Garden Street
Philadelphia PA 19130

While this column purports to criticize the mathematics we teach as a matter of course, my friends keep reminding me that being critical doesn't go over too well in academic circles. As a result, rather than to criticize, I shall often prefer to appear "constructive" by offering alternatives. The danger however, as the same friends keep warning me with glee, is that these notes could well turn into some Gospel According To Schremmer. So, once again, let me remind the reader that this column wants to be a "forum" where to argue about what's taken to be given.

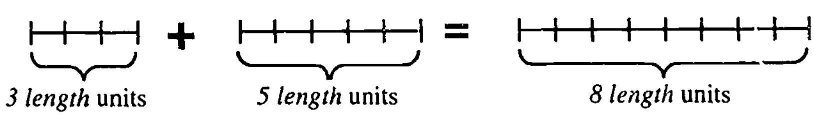
In the meantime, let me start with how we choose textbooks. As any publisher will tell you, we just open the book at three or four places to see how the author handles certain pet items of ours and, if s/he does it our way, we like the book! At this point, I find it very difficult not to indulge in heavy sarcasm. What this says though is that we take the contents of any given course for granted, if possibly up to a permutation and with a few twists and variations allowed here and there. In this, by the way, our view of mathematics is not overwhelmingly unlike that of non-mathematicians: a certain number of problems need to be covered and learning mathematics means learning the corresponding recipes. Appropriately wrapped, of course, in "theory." But just look at the exercises and/or, even better, at the exams: As has often been observed, each and every question corresponds to a key on a calculator!

However, even if I were to agree that a course consists only of teaching how to handle a given number of problems, I would nevertheless argue that the recipes that we use are not the only possible ones and that, in fact, they are not the most efficient ones. When I do so though, my interlocutors seem to fall prey to the acute discomfort usually associated with having to deal with the citizens released a few years ago from State Hospitals into the Community, i.e. onto city sidewalks. So, instead, I will sketch an alternative: to use an analogy, and if we compare the usual set of recipes to a Complex Instruction Set Computing (CISC) processor, what I will advocate here corresponds to a Reduced Instruction Set Computing (RISC) processor. In other words, instead of having a complex set of instructions, essentially one for each problem, I will propose here a reduced set of instructions which can be used over and over again, in various assemblies, to deal with a large set of problems.

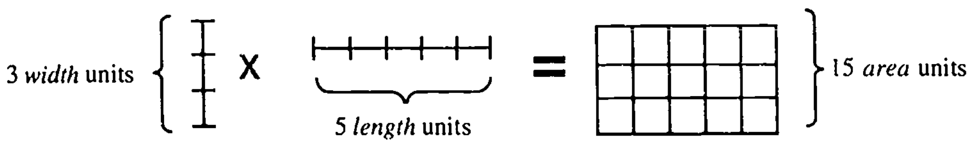
To take again basic arithmetic as my example, start with the idea that numbers should never be used without units: Never say 3 but 3 which we define of course as with the important proviso that 1 is the same as —*one* apple is the same as *an* apple. (How many times have you had to say that $x = 1x$?) Then, be sure to call 2 apples + 3 oranges anything but a *sum*. It is bad enough that we usually cannot avoid using the symbol +. I will use the term *combination*. Thus, since in a combination + cannot be computed, + is not to be read as "plus" but as "and." In contradistinction, the *sum* 2 apples + 3 apples computes to 5 apples. The main question then is: When can we turn a combination into a sum? The answer is that, *if* we can exchange, for example, 1 apple for 25 strawberries and 1 orange for 12 strawberries, *then* we can compute 2 apples + 3 oranges = 2 · 25 strawberries + 3 · 12 strawberries = 50 strawberries + 36 strawberries = 86 strawberries! This notion of *exchange*, which we already encountered in the case of the monetary combinations I discussed last Fall, namely such as **5 Franklins, 2 Hamiltons, 7 Washingtons**, is indeed an ubiquitous one.

Next, recall that the top of a fraction *numerates* the unit that its bottom *denominates* so that we should think of $\frac{2}{3}$ 🍎 as $2\,\frac{\text{🍎}}{3}$, that is as 2 third-of-an-apple. Naturally, we can exchange $\frac{\text{🍎}}{3}$ for $2\,\frac{\text{🍎}}{6}$ so that, while $\frac{2}{5}$ 🍎 + $\frac{3}{7}$ 🍎, that is $2\,\frac{\text{🍎}}{5} + 3\,\frac{\text{🍎}}{7}$, is a combination and cannot be computed, it can be *exchanged* for the sum $14\,\frac{\text{🍎}}{35} + 15\,\frac{\text{🍎}}{35}$ which computes to $29\,\frac{\text{🍎}}{35}$. Of course, sooner or later we must actually cut the apple up and make the connection with division: $2\,\frac{\text{🍎}}{3}$ is what each one of 3 persons gets when dividing up 2🍎 even if this does not always work, as, for instance, with eggs.

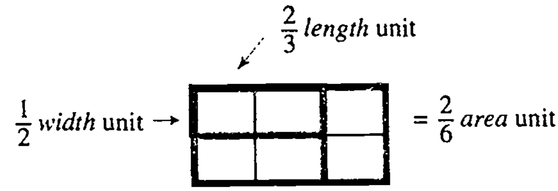
We then need to introduce multiplication but certainly not as "repeated addition" because how would we then define $\frac{1}{2} \cdot \frac{1}{3}$? Note that when we *add* two numbers, we think of them as *lengths* and we visualize the addition as



So when we *multiply* the same two numbers, we should visualize the product as the cardinal of a cartesian product:



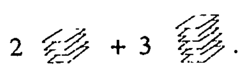
(Shades of matrix multiplication!) But then,



Eventually of course, multiplication by whole numbers must be identified with additive powers.

We now turn to multiplicative powers. By ax^n, we mean a stacks of n copies of x with the natural proviso that $ax^1 = ax$. That $ax^0 = a$ is not yet natural. Of course, as with 1🍎 = 🍎, we have $x^n = 1x^n$. Clearly, $2x^4 + 3x^4 = 5x^4$:

while $2x^4 + 3x^6$ is just a combination:



In arithmetic, the identification of additive powers with multiplication gives that $5 \cdot 3^4$ must be read as 5 *multiplied* by 4 copies of 3 that is as $5 \cdot 3 \cdot 3 \cdot 3 \cdot 3$. If nothing else, this prevents students having to compute 3^4 from multiplying 3 by 4. Now, $5 \cdot 3^0$ is equal to 5 since it means 5 multiplied by 0 copy of 3. Similarly, we define ax^{-n} as $\frac{a}{x^n}$, that is as *a divided* by *n* copies of *x*. This time however, we cannot omit the 1 in $1x^{-n} = \frac{1}{x^n}$. The key to manipulating powers then resides in the fact that we can denote division with a negative exponent as well as with a fraction bar. Either way, we use the instruction "instead of dividing, multiply by the reciprocal" (as in "instead of subtracting, add the opposite").

And that's it! We are now equipped to justify just about any computation in Arithmetic and/or Basic Algebra. (Excluding "factoring by inspection" of course! But that's quite another story.) For instance, we get in this manner Laurent polynomials as well as the exponential notation for decimal numbers. By the way, the fact that evaluating the polynomial $5x^2 + 2x + 7 + 4x^{-1} + 3x^{-2}$ at 10 yields the number 527.43 points directly at the very heart of the Lagrangian differential calculus.

Observe that most of these "instructions" are in reality semantic definitions in that they dictate how to interpret certain terms and that, under these interpretations, the usual rules and recipes become quite natural, by which I mean that they come from the students' own non-school experience. In other words, we are operating here in a *model-theoretic* mode. Of course, RISC computations tend to result in longer computations than CISC ones but even this can be seen as an advantage since it forces the students to concentrate on the computational strategy to be followed rather than on worrying whether they are using the right recipe. I would suggest that the reader make a "reduced set of instructions" in the subject of her/his interest and see what s/he can get out of it and I would be most interested in discussing here such attempts.

While none of the above has any originality whatsoever, this is certainly *not* how so-called elementary textbooks present these matters and I hope that it also illustrates how the way we cut up courses, e.g. separate arithmetic from basic algebra, affects the way we think about their contents. More generally, I would like to argue in some future column that the main problem students have with mathematics, and what drives them to memorization and, eventually, to distraction, is that the way things are presented in the conventional curriculum makes it completely impossible for them to see the broader picture, the overall architecture according to which these things hang together.

In my first column, I had promised to discuss books you are unlikely to get from your friendly Publisher Representative and, indeed, Publish or Perish, the publisher of Susan Bassein's *An Infinite Series Approach to Calculus*, is not likely to have reps. However, because of space limitations, I can only hope here to entice someone to review it for *The AMATYC Review* and will limit myself to a quotation from the author's preface to the Instructor:

> In the usual development of the derivative functions defined by infinite series, the operation of finding the limit of the difference quotient $\dfrac{f(a+h)-f(a)}{h}$ as $h \to 0$ is *unnecessary*: the correct answer for the derivative of all polynomials, rational and algebraic functions (and, in fact, all real analytic functions, that is functions that can be expressed as convergent power series) can be found by performing the algebra necessary to cancel the h in the denominator and then setting $h = 0$ because the difference quotient of such functions has only a removable discontinuity at $h = 0$.

This is very close to Howard Levi's approach in *Polynomials, Power Series and Calculus* which was published in 1968 by Van Nostrand, now defunct but whose catalogue was picked up by Springer Verlag, and which might therefore reappear someday. The two treatments however could not be more different. Finally, a caveat: the approach by infinite series is not to be confused, repeat, not to be confused, with Lagrange's approach which is by way of asymptotic expansions, a version of which we propounded in this *Review* a few years back.

> Mathematics is an interesting intellectual sport but it should not be allowed to stand in the way of obtaining sensible information about physical processes.
>
> Richard Hamming

The Problem Section

Dr. Michael W. Ecker
Problem Section Editor
The AMATYC Review
909 Violet Terrace
Clarks Summit PA 18411

Dr. Robert E. Stong
Solution Editor
The AMATYC Review
150 Bennington Road
Charlottesville VA 22901

Greetings, and welcome to still another Problem Section!

The AMATYC Review Problem Section seeks lively and interesting problems and their solutions from all areas of mathematics. Particularly favored are teasers, explorations, and challenges of an elementary or intermediate level that have applicability to the lives of two-year college math faculty and their students. We welcome computer-related submissions, but bear in mind that programs should supplement, not supplant, the mathematical solutions and analysis.

When submitting material for this department, please note that we have separate editors for problems and for solutions. Send **two copies** of your new **problem proposals**, preferably typed or printed neatly with separate items on separate pages, to the Problem Editor. Include **two copies** of a solution, if you have one, and any relevant comments, history, generalizations, special cases, observations, and/or improvements. Please include your name (title optional, no pseudonyms), affiliation, and address of same. Enclose a mailing label or self-addressed envelope if you'd like to be assured a reply. All **solutions** to others' proposals, *except Quickies*, should be sent directly to the Solutions Editor.

Right now the greatest need is for good new problems.

Dr. Michael W. Ecker (Pennsylvania State University—Wilkes Barre)
Dr. Robert E. Stong (University of Virginia)

Quickies

Quickies are math teasers that typically take just a few minutes to an hour. Solutions usually follow the next issue, listed before the new teasers. All correspondence to this department should go to the Problem Editor.

Comments on Old Quickies

Quickie #16. Consider the standard max-min problem typified by this: "You are to build a rectangular enclosure with front side made of material costing \$10 per foot and with other three sides (back, left, right) of material costing \$5 per foot. If the enclosure is to contain exactly 600 square feet and to be built at minimum total cost, how long should the sides be and what is the total cost?"

Note the small coincidence in that the answer involves a total cost of \$600, which matches the constraint of area 600 (aside from units, of course). Question: Characterize when this happens more generally, and thus describe the relationship that exists in such problems in which the resultant minimum cost matches the given area (all in appropriate units).

Outline of Solution: Consider the objective given by cost $C = ax + by$ for some numbers a and b, with C to be minimized. In the example above, for instance, front and back sides yield $a = 10 + 5 = 15$ dollars per sq. ft., whereas left and right yield $b = 5 + 5 = 10$. The area constraint is $xy = c$, the given constant area that will later be set equal to minimum cost. Solve for y in terms of x (and constants a, b) to obtain the objective function to be optimized: $C = C(x) = ax + \frac{bc}{x}$. This is well-known to have an absolute minimum over the set of positive reals. Differentiate, equate to zero, and solve for x and y. I leave the easy, fun details to you, but note the interesting answer obtained for x and y in terms of a and b. Substitute into C and equate the resulting minimum cost to the given area c to obtain the necessary and sufficient condition: $c = 4ab$. (For the example given above, $600 = 4 \times 15 \times 10$.)

Quickie #17. Choose a number x randomly from the interval $[0, \pi]$ and a second number y randomly from $[0, 1]$. What is the probability that $y < \sin x$?

$2/\pi$, from the average value of the sine function on $[0,\pi]$.

Quickie #18 from the last issue is not solved here because it has been upgraded to regular problem status; solutions are solicited. (See New Problem AB-6.)

New Quickies

Quickie #19: Proposed by the Problem Editor but similar to what has appeared elsewhere (e.g., Millersville State University math contest circa 1980).

Calculate $\log_{10}(11) * \log_{11}(12) * \log_{12}(13) * \ldots * \log_{98}(99) * \log_{99}(100)$.

Quickie #20: A howler passed on by Michael Andreoli, Miami-Dade Comm. College.

A 30-ft. ladder is leaning against a wall when the bottom of the ladder is pulled away from the wall horizontally at a constant 1 ft. per sec. Find the height of the top of the ladder (above ground level) when the top of the ladder is falling at twice the speed of light.

Comment: Share this with your calculus students after introducing related rates. Look at the geometry and the physics. Have fun! (There are many levels from which to approach this.)

New Problems

Set AB Problems are due for ordinary consideration October 15, 1995. Of course, regardless of deadline, no problem is ever closed permanently, and new insights to old problems – even Quickies – are always welcome. However, our Solutions Editor requests that you please not wait until the last minute if you wish to be listed or considered on a timely basis. Please note again that we desperately need good new problem proposals.

Problem AB-1. Proposed by the Problem Editor (Michael W. Ecker, Pennsylvania State University, Wilkes-Barre Campus).

Let a be a suitably small positive constant and define:

$$x_n = \cos a + \cos 2a + \ldots + \cos na$$

$$y_n = \sin a + \sin 2a + \ldots + \sin na.$$

Prove that the points (x_n, y_n) lie on a circle, give the center and radius in terms of a, and find any restrictions on the parameter a.

Comments: This makes for nice, simple graphing using humble BASIC and the PSET graphing command (with any Screen). If you change the typical series terms from $\cos ka$ and $\sin ka$ to $\cos ka^d$ and $\sin ka^d$ for suitable values of the parameter d (e.g., 1.1 to 1.99), you get stunning curlicue graphics. One simple program appears in the Fall 1994 double-issue of *Recreational & Educational Computing*. I'll send pictures plus program listings (or programs on disk) to anybody who sends a self-addressed envelope with sufficient postage affixed (two ounces' worth for paper listings, or three ounces' worth plus a disk if you want programs on disk). Please mention that you wish the curlicue graphics.

Problem AB-2. Proposed by Stanley Rabinowitz, Westford, MA.

Let a, b, and c be real numbers with $a > 0$ and $b^2 - 4ac < 0$. Express the quartic polynomial $ax^4 + bx^2 + c$ explicitly as the product of two quadratic polynomials with real coefficients. (The square root of a negative number or a complex number must not appear anywhere as a sub-expression within your answer.)

Problem AB-3. Proposed by Michael H. Andreoli, Miami-Dade Community College.

A fair coin is tossed repeatedly and you observe heads (H) or tails (T). Which pattern has a longer expected time to occur: HH or HT?

Proposer Comment: "This is one of the most counter-intuitive problems I know."

Problem AB-4. Proposed by J. Sriskandarajah, University of Wisconsin-Richland.

The adjacent pairs of angle-trisectors of equilateral triangle ABC meet in a triangle $A'B'C'$. Find the ratio of the area of $A'B'C'$ to the area of ABC.

Problem AB-5. Proposed by Frank Flanigan, San Jose State University.

Given distinct primes $p < q < \ldots < r$, choose a rational number b and form the following polynomial of degree r over the rationals:

$$P(x) = bx + \frac{x^p}{p} + \frac{x^q}{q} + \ldots + \frac{x^r}{r}.$$

Are there choices of b with the property that $P(n)$ is integral whenever n is? (Give necessary and sufficient conditions.)

Problem AB-6. Proposed by Michael H. Andreoli, Miami-Dade Community College.

(This was Quickie #18 last issue. I have decided to upgrade its status to a regular problem and solicit solutions to be mailed to Solutions Editor.)

From the interval [0, 1] choose two numbers at random. What is the probability that the resulting segments can be used to form a triangle?

Comment: There may be some ambiguity here; use your best judgment as to intent and meaning. Proposer points out that a simple, algebraic, non-calculus solution does exist, however.

Problem AB-7. Proposed by Stephen Plett, Fullerton College, CA.

Produce a smooth function $P(x)$ defined on $\{x: x \geq 2\}$ that is polynomial on each interval $[k, k + 1]$ with k zeros equally spaced, including the endpoints. (For instance, on [3, 4], zeros must occur at 3, 3.5, and 4.)

Comment: There are two interpretations:

a) The zeros are simple, so on [3, 4] there are exactly three zeros. In other words, on each interval $[k, k + 1]$, $P(x)$'s restriction is polynomial of degree k.

b) The zeros need not be simple, so multiple zeros are allowed on [3, 4] as long as they are all at 3, 3.5, and 4.

Steve's original intent was a), but I include b) because Bob Stong has such a nice approach if we drop the requirement of $P(x)$ being of degree k on $[k, k + 1]$.

Set Z Solutions

Odd Odd Polynomials

Problem Z-1. Proposed by the Problem Editor, Pennsylvania State University, Wilkes-Barre Campus.

Characterize all invertible, odd, fifth-degree polynomials. That is, determine necessary and sufficient conditions on the real numbers a and b to make $p(x) = x^5 + ax^3 + b$ invertible. (Non-monic versions of these polynomials are just scalar multiples of the monic $p(x)$ shown.)

Solutions (with the given $p(x)$) by Charles Ashbacher, DecisionMark, Cedar Rapids, IA; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Peter Collinge, Monroe Community College, Rochester, NY; Steve Kahn, Anne Arundel Community College, Arnold, MD; and Joseph Wiener and Miguel Paredes, University of Texas-Pan American, Edinburg, TX.

Solutions (with the intended $p(x)$) by Mike Dellens, Austin Community College, Austin, TX; Stephen Plett, Fullerton College, Fullerton, CA; Grant Stallard, Manatee Community College, Bradenton, FL; Nhu Thuy Tran (student), Onondaga Community College, Syracuse, NY; and the proposer.

The polynomial $p(x) = x^5 + ax^3 + b$ is invertible if and only if it is monotone. From the coefficient of x^5, $p(x)$ must be monotone increasing, and the condition for this is that

$$p'(x) = x^2(5x^2 + 3a) \geq 0.$$

This holds if and only if $a \geq 0$.

Note: The solution with the intended $p(x) = x^5 + ax^3 + bx$ will be given in the next issue.

Prime Triple Play

Problem Z-2. Proposed by Leonard Palmer, Southeast Missouri State University, Cape Girardeau, MO.

If you look at a table of primes you will be able to pick out pairs of primes that differ by 10. Let's call these ten-primes.

Find all triples of consecutive ten-primes $(p,\ p+10,\ p+20)$.

Solutions by Shiv Kumar Aggarwal, Embry Riddle Aeronautical University, Daytona Beach, FL; Charles Ashbacher, DecisionMark, Cedar Rapids, IA; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Peter Collinge, Monroe Community College, Rochester, NY; Bill Fox, Moberly Area Community College, Moberly, MO; Carl Libis, Idaho State University, Pocatello, ID; Stephen Plett, Fullerton College, Fullerton, CA; Steve Kahn, Anne Arundel Community College, Arnold, MD; Joseph Wiener and Miguel Paredes, University of Texas-Pan American, Edinburg, TX; and the proposer.

Since p, $p+10$, and $p+20$ are in three different congruence classes modulo 3, one of these must be divisible by 3. Since these are primes, only 3 itself is possible, and $p = 3$, giving the triple (3, 13, 23).

Comment: The problem depends on the numbers allowed. In the ring of integers, x is prime if and only if $-x$ is prime. There are then six solutions, where any one of p, $p+10$, or $p+20$ is either 3 or -3.

Linear Non-Linearity

Problem Z-3. Passed on by Charles Ashbacher, proposed by Ruiqing Sun, Beijing Normal University, Beijing, People's Republic of China, and translated by Leonard Lin, Mount Mercy College, Cedar Rapids, IA.

Given a, b, x, and y (x and y distinct) such that:

1) $ax + by = 3$
2) $ax^2 + by^2 = 7$
3) $ax^3 + by^3 = 16$
4) $ax^4 + by^4 = 42$

determine the value of $ax^5 + by^5$.

Solutions by Shiv Kumar Aggarwal, Embry Riddle Aeronautical University, Daytona Beach, FL; Robert Bernstein, Mohawk Valley Community College, Utica, NY; Bill Fox, Moberly Area Community College, Moberly, MO; Carl Libis, Idaho State University, Pocatello, ID; Stephen Plett, Fullerton College, Fullerton, CA; Grant Stallard, Manatee Community College, Bradenton, FL; Bella Wiener, University of Texas-Pan American, Edinburg, TX; and the passer.

From the identity

$$ax^n + by^n = (ax^{n-1} + by^{n-1})(x + y) - (ax^{n-2} + by^{n-2})(xy)$$

one has

$$7(x + y) - 3xy = 16 \text{ and } 16(x + y) - 7xy = 42$$

so that $x + y = -14$ and $xy = -38$. Then

$$ax^5 + by^5 = 42(-14) + 16(38) = 20.$$

Comment: The Problem Editor asked about the possibility of $x = y$. From the solution, it is clear that x and y must be different.

Fee, Fie, Fo, Fum

Problem Z-4. Proposed by Charles Ashbacher, DecisionMark, Cedar Rapids, IA.

Let phi(n) be the Euler phi function. Prove that there exist infinitely many natural numbers n such that phi(n) < phi(n – phi(n)).

Solutions by Shiv Kumar Aggarwal, Embry Riddle Aeronautical University, Daytona Beach, FL; Bill Fox, Moberly Area Community College, Moberly, MO; and the proposer.

Solution 1: Let $n = 2^r \times 3 \times 5$. Then phi($n$) $= 2^{r-1} \times 2 \times 4 = 2^{r+2}$ and n – phi(n) = $2^r \times 11$. Finally, phi(n – phi(n)) $= 2^{r-1} \times 10 >$ phi(n).

Solution 2: Let n be a prime of the form $6k - 1$ greater than 5. Then phi(n) = 2, and $n - 2$ is divisible by 3, so composite. Thus phi($n - 2$) > 2.

A Penny Spent

Problem Z-5. Proposed by the Solution Editor, University of Virginia.

Suppose that you make a purchase at the store for which the price is some number of dollars and k cents. In your pocket or purse you have p ($p \geq k$) pennies and n other coins (not pennies). Desiring to get rid of as many pennies as possible, you randomly draw coins until you get the needed k pennies. How many coins do you expect to have to draw?

Solutions by Charles Ashbacher, DecisionMark, Cedar Rapids, IA; Robert Bernstein, Mohawk Valley Community College, Utica, NY; and William T. Long, Broward Community College, Coral Springs, FL.

Let $E(k, p, n)$ be the expected number of coins drawn. Clearly, $E(0, p, n) = 0$ since no coins are required. For k positive, one has

$$E(k, p, n) = 1 + \frac{p}{n+p} E(k-1, p-1, n) + \frac{n}{n+p} E(k, p, n-1),$$

since, after drawing one coin, you will need $k - 1$ pennies if a penny was drawn or will still need k pennies if you got a non-penny. One may readily verify that

$$E(k, p, n) = k + \frac{kn}{p+1}$$

satisfies these relations, and hence is the expected number of coins drawn.

Advertiser's Index

Academic Press, Inc. .... p. 14

Addison-Wesley Publishing Co. .... p. 33

AMATYC Annual Conference .... p. 62

Arrowhead Distributors, Inc. .... p. 21

Calculus Consortium at Harvard University .... p. 44

JEMware .... p. 3

John Wiley & Sons, Inc. .... p. 27, 44

The Mathematical Association of America .... p. 58

MathWare .... p. 6

West Publishing Corp. .... p. 51

Wm. C. Brown Publishers .... p. 55

> It is a profoundly erroneous truism, repeated by all copy books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them.
>
> Alfred North Whitehead

LIBRARY SUBSCRIPTION

Annual Library Subscription Fee: $50 [Includes *The AMATYC Review* (Fall and Spring) and *The AMATYC News* (Fall, Winter and Spring)]

Name of Library

Name of Institution

Address

City State Zip

NOTE: This application form is your invoice for the subscription fee and payment must accompany the completed application. Retain a photocopy of this form for your records if necessary. Please make check payable in U.S. funds to AMATYC and mail to:

AMATYC Office, State Technical Institute at Memphis,
5983 Macon Cove, Memphis, TN 38134

Rev S. 1994

INSTITUTIONAL MEMBERSHIP

Colleges, publishers, and other institutions can support AMATYC by being Institutional Members. As indicated below, the annual membership fee is $250. Subscriptions to *The AMATYC Review* and *The AMATYC News* are included in this fee. Of particular importance to collegiate institutional members is the AMATYC committee for mathematics department chairpersons.

An additional benefit of institutional membership is one *complimentary* AMATYC conference early registration. Future conventions, which are held in the Fall, will be in Tulsa (1994) and Little Rock (1995). Institutional members support the excellence of the programs at these annual conferences.

INSTITUTIONAL MEMBERSHIP APPLICATION/INVOICE

Mail to: AMATYC Office, State Technical Institute at Memphis,
5983 Macon Cove, Memphis, TN 38134

AMATYC College Contact Person Position

Name of Institution

Address

City State Zip

Membership Fee – $250 in U.S. funds payable to AMATYC (includes *The AMATYC Review*, *The AMATYC News*, membership in the Student Math League and one complimentary conference early registration)

Note: Institutional membership does not include any voting privileges.

Rev S. 1994

WHY JOIN AMATYC?

The American Mathematical Association of Two-Year Colleges (AMATYC) was established in 1974 to provide a unique, national forum for two-year college mathematics educators. Today, AMATYC is the only national organization that exclusively serves the needs and purposes of this group.

AMATYC holds a national conference annually in a major city. AMATYC encourages two-year college mathematicians to assume responsible leadership positions; to interact on an equal basis with four-year university personnel; to be members on a proportional basis of national steering or policy committees; to represent at the national level the concerns of two-year college mathematics educators. *The AMATYC Review*, published twice yearly, provides an opportunity to publish articles by and for two-year college faculty. A national newsletter is published three times yearly for all members. AMATYC is a member of the National Conference Board of the Mathematical Sciences.

REGULAR MEMBERSHIP APPLICATION The American Mathematical Association of Two-Year Colleges

Mail to: AMATYC Office, State Technical Institute at Memphis, 5983 Macon Cove, Memphis, TN 38134

First Name	MI	Last Name	Position
College			Phone
College Address			E-Mail
City		State	Zip
Residence Address			Phone
City		State	Zip

Indicate preferred mailing address: ☐ College ☐ Residence

All payments in U.S. funds payable to AMATYC. Membership fee:

- ☐ $50 Yearly Regular Membership (any person interested in mathematics education at the two-year college level)
- ☐ $10 Yearly Associate Membership (full-time student, no voting privileges)
  Name of AMATYC Member Sponsor ______________________________
- ☐ $1000 Regular Life Membership

Membership includes *The AMATYC Review* and *The AMATYC News*. In addition, the following journals are available at extra cost:

- ☐ *Mathematics and Computer Education Journal* $20 for 3 issues
- ☐ *The College Mathematics Journal* $43 for 5 issues

Total Amount Enclosed ________________ Date Paid ______________

OPTIONAL DEMOGRAPHIC INFORMATION (Please check one in each category)

Category 1	Category 2	Category 3
☐ African American	☐ Female	☐ Two-year college
☐ American Indian/Alaskan Native	☐ Male	☐ Four-year college
☐ Asian		☐ Other, please specify ____________
☐ Hispanic		
☐ White, not Hispanic		
☐ Other, please specify ______________________________		

Rev S. 1994

AMATYC Institutional Members

(as of November 22, 1994)

Addison-Wesley Pub. Co., Reading, MA 01867
Albuquerque Tech -Voc Inst., Albuquerque, NM 87106
Anoka Ramsey Comm. College, Coon Rapids, MN 55433
Austin Comm. College (RGC Campus), Austin, TX 78701
Ball State University/Math Dept., Muncie, IN 47306
Bellevue Comm. College, Bellevue, WA 98007
Bermuda College, Devonshire, Bermuda DVBX
Bristol Comm. College, Fall River, MA 02720
Bucks County Comm. College, Newton, PA 18940
Burlington County College, Pemberton, NJ 08068
Butte College, Oroville, CA 95965
Cabot College of Applied Arts, St. John's, NF A1C 5P7
Cabrillo College, Aptos, CA 95003
Charles County C. C., La Plata, MD 20646
Clark State Comm. College, Springfield, OH 45502
Clovis Comm. College, Clovis, NM 88101
College of DuPage, Glen Ellyn, IL 60137
College of Lake County, Grayslake, IL 60030
Columbus State Comm. College, Columbus, OH 43216
Comm. College of Philadelphia, Philadelphia, PA 19130
Comm. College of Southern Nevada, N. Las Vegas, NV 89030
C. S. Mott Comm. College, Flint, MI 48503
County College of Morris, Randolph, NJ 07869
Cuyahoga Comm. College, Parma, OH 44130
Dean Junior College, Franklin, MA 02038
DeKalb College-North Campus, Dunwoody, GA 30338
Delaware County Comm. College, Media, PA 19063
El Paso Comm. College, El Paso, TX 79998
Florida C. C. at Jxon, Jacksonville, FL 32046
Fox Valley Tech. Inst., Appleton, WI 54913
Fullerton College, Fullerton, CA 92634
Galveston College, Galveston, TX 77550
Garland County Comm. College, Hot Springs, AR 71914
Genessee Comm. College, Batavia, NY 14020
Grant Mac Ewen College, Edmonton, AB T5J 2P2
Green River College, Auburn, WA 98002
Harding University, Searcy, AR 72149
Harrisburg Area C.C., Harrisburg, PA 17110
Heartland Comm. College, Bloomington, IL 61701
Honolulu Comm. College, Honolulu, HI 96817
Houghton Mifflin Co., Boston, MA 02116
Houston Comm. College-Stafford, Stafford, TX 77477
Howard Comm. College, Columbia, MD 21044
Illinois Central College, East Peoria, IL 61635
Illinois State University, Normal, IL 61761
Inver Hills C. C., Inver Grove Hts., MN 55076
Joliet Jr. College, Joliet, IL 60436
Kapiolani Comm. College, Honolulu, HI 96816
Kean College of New Jersey, Union, NJ 07083
Kennebec Valley Tech College, Fairfield, MI 04937
Kennedy-King College, Chicago, IL 60621
Lane Comm. College, Eugene, OR 97405
Lassen College, Susanville, CA 96130
Lorain Comm. College, Elyria, OH 44035
Madison Area Tech College, Madison, WI 53704
Massachusetts Bay Comm. College, Wellesly Hills, MA 02181
Massasoit Comm. College, Brockton, MA 02402
Metropolitan Comm. College, Omaha, NE 68103
Middlesex County College, Edison, NJ 08818
Mohawk Valley Comm. College, Utica, NY 13501
Montgomery College, Takoma Park, MD 20912
Moraine Valley Comm. College, Palos Hills, IL 60465
Mt. Wachusett Comm. College, Gardner, MA 01440
Naugatuck Valley Comm. Tech. Coll., Waterbury, CT 06708
New Mexico State University, Las Cruces, NM 88003
Normandale Comm. College, Bloomington, MN 55431
Northeast State Tech. Comm. College, Blountville, TN 37617
North Hennepin C. C., Minneapolis, MN 55455
North Idaho College, Coeur D'Alene, ID 83814
North Lake College, Irving, TX 75038
North Seattle Comm. College, Seattle, WA 98103
North Shore Comm. College, Danvers, MA 01923
Oakton Comm. College, Des Plaines, IL 60016
Onondaga Comm. College, Syracuse, NY 13215
Oregon Inst. of Technology, Klamath Falls, OR 97601
Palomar College, San Marcos, CA 92069
Parkland College, Champaign, IL 61821
Penn Valley Comm. College, Kansas City, MO 64111
Pima Comm. College, Tuscon, AZ 85703
Polk Comm. College, Winter Haven, FL 33881
Portland Comm. College, Portland, OR 97280
Prince George's Comm. College, Largo, MO 20772
Raritan Valley C. C., Somerville, NJ 08876
Robert Morris College, Coraopolis, PA 15108
Rock Valley College, Rockford, IL 61114
San Francisco State University, San Francisco, CA 94116
San Juan College, Farmington, NM 87402
Santa Barbara City College, Santa Barbara, CA 93109
Schoolcraft College, Livonia, MI 48152
Southeast Mo. State University, Cape Girardeau, MO 63701
Springfield Tech Comm. College, Springfield, MA 01101
State Tech Inst. at Memphis, Memphis, TN 38134
St. Charles Co. Comm. Coll., St. Peters, MO 63376
St. Louis Comm College, St. Louis, MO 63135
SUNY Ag & Tech College, Alfred, NY 14802
Tarrant County Jr. College S., Ft. Worth, TX 76119
The College Board, Philadelphia, PA 19104
Three Rivers Community College, Poplar Bluff, MO 63901
Tulsa Junior College, Tulsa, OK 74133
Turtle Mountain Comm. College, Belcourt, ND 58316
UND - Williston, Williston, ND 58801
Univ. of Alaska-Anchorage, Anchorage, AK 99508
University of Wyoming, Laramie, WY 82071
UNM-Valencia Campus, Los Lunas, NM 87031
Utah Valley Comm College, Orem, UT 84058
William Rainey Harper College, Palatine, IL 60067

BEST COPY AVAILABLE

AMATYC Reviewers

Sunday A. Ajose	East Carolina University	Greenville, NC
Patricia Allaire	Queensborough C.C.	Bayside, NY
Charles Ashbacher	Kirkwood College	Hiawatha, IA
Joe Dan Austin	Rice University	Houston, TX
John W. Bailey	Clark State C.C.	Springfield, OH
Richelle Blair	Lakeland Comm. College	Mentor, OH
Barbara Bohannon	Hofstra University	Hempstead, NY
Randall Brian	Vincennes University	Vincennes, IN
Robert Decker	University of Hartford	W. Hartford, CT
John DeCoursey	Vincennes University	Vincennes, IN
David Dyer	Prince George's C.C.	Largo, MD
Joseph R. Fiedler	California State University	Bakersfield, CA
Kathleen Finch	Shoals Comm. College	Muscle Shoals, AL
Gregory D. Foley	Sam Houston State University	Huntsville, TX
Richard Francis	Southeast Missouri State University	Cape Girardeau, MO
Charles Franke	Eastern Kentucky University	Richmond, KY
Florence Gordon	New York Institute of Technology	Old Westbury, NY
Sheldon Gordon	Suffolk County C.C.	Selden, NY
Chitra Gunawardena	Univ. of Wisconsin-Fox Valley	Menasha, WI
K. L. D. Gunawardena	Univ. of Wisconsin-Oshkosh	Oshkosh, WI
Russell Gusack	Suffolk County C.C.	Selden, NY
Bruce Haney	Onondaga Community College	Syracuse, NY
Ronald Harkins	Miami University	Hamilton, OH
Peter Herron	Suffolk County C.C.	Selden, NY
Ann D. Holley		Los Gatos, CA
Larry Lance	Columbus State C.C.	Columbus, OH
Jim Langley	Northeast State Technical C.C.	Blountville, TN
Michael Lanstrum	Kent State University-Geauga	Burton, OH
Edward Laughbaum	Columbus State C C.	Columbus, OH
Deborah Levine	Nassau C.C.	Garden City, NY
Ved P. Madan	Indiana University East	Richmond, IN
Richard F. Maruszewski	United States Naval Academy	Annapolis, MD
John Mathews	California State University	Fullerton, CA
George Matthews	Onondaga Community College	Syracuse, NY
Pamela E. Matthews		Vancouver, WA
Mary McCarty	Sullivan C.C.C.	Loch Sheldrake, NY
Art Merifield	Seneca College	Toronto, CANADA
Coreen Mett	Radford University	Radford, VA
Ted Moore	Mohawk Valley Community College	Utica, NY
Kylene Norman	Clark State C.C.	Springfield, OH
Terry A. Nyman	Univ. of Wisconsin-Fox Valley	Menasha, WI
Carol Olmsead	University of Nevada	Reno, NV
Joan Page	Onondaga Community College	Syracuse, NY
Deborah Parker	Mississippi County C.C.	Blytheville, AR
Don Pfaff	University of Nevada-Reno	Reno, NV
Stephen Plett	Fullerton College	Fullerton, CA
Ben Pollina	University of Hartford	W. Hartford, CT
Douglas Robertson	Univ. of Minnesota	Minneapolis, MN
Jack W. Rotman	Lansing C.C.	Lansing, MI
Alain Schremmer	Community College of Philadelphia	Philadelphia, PA
Gene Sellers	Sacramento City College	Sacramento, CA
Brian Smith	Dawson College	Quebec, CANADA
J. Sriskandarajah	Univ. of Wisconsin	Richland Center, WI
Leslie R. Tanner	The College of Idaho	Caldwell, ID
Marcia Weisser	New York Institute of Technology	Old Westbury, NY
John Williams	University of Hartford	W. Hartford, CT
Margaret Willis	Piedmont Virginia C.C.	Charlottesville, VA

The AMATYC REVIEW
Onondaga Community College
Syracuse, New York 13215

Non-Profit
Organization
U.S. Postage
PAID
Permit No. 973
Syracuse, New York

ISSN 0740-8404