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ABSTRACT

Previous research has suggested that many elementary school teachers have a firmer understanding of addition and subtraction concepts than they have of more complex topics in elementary mathematics, such as division. In this study, (n=24) prospective elementary teachers' understanding of aspects of division involving remainders was explored by examining the problems they posed. The teachers' responses revealed a clear preference for posing problems associated with real world contexts, suggesting that the term "story problem" had that particular meaning for this group. Although the vast majority of problems were situated in contexts, the problems and solutions frequently did not represent a solid connection between the mathematical and situational aspects of the problem. About two-thirds of the subjects spontaneously posed problems and proposed solutions that reflected their understanding of the relationship between the given computation and at least one member of its family of associated computations. For 20 posed problems, almost all of which were partitive division situations, subjects failed to specify the need for equal-sized groups or the number of items per group in the division situation being posed. The findings were that, although some aspects of division, such as its connection to different types of problem situations and its relationship to multiplication, were fairly well understood by most of the subjects in this study, limited or flawed understanding was also noted in many different areas. The document contains examples of the question sheets used in the study and examples of posed problems. (Contains 14 references.) (MKR)

The Posing of Division Problems by Preservice Elementary School Teachers: Conceptual Knowledge and Contextual Connections

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The Posing of Division Problems by Preservice Elementary School Teachers: Conceptual Knowledge and Contextual Connections

Previous research has suggested that many elementary school teachers have a firmer understanding of addition and subtraction concepts than they have of more complex topics in elementary mathematics, such as division (e.g., Ball, 1990). In particular, several different misunderstandings and confusions held by prospective or actual elementary school teachers have been identified in the area of division, such as those related to division by zero (Ball, 1990) and to situations in which division does not result in a quotient that is smaller than the dividend (Tirosh & Graeber, 1989, 1990). It has also been argued that teachers' misunderstandings about division may be transmitted to their students because the misconceptions and confusions interfere with their successfully teaching division concepts with a firm conceptual foundation (Graeber, Tirosh, & Glover, 1989). Thus, it seems useful to examine the understandings and misunderstandings of teachers in areas in which students are known to encounter conceptual difficulties.

The arithmetic operation of division, even division of whole numbers, is notoriously difficult for students. One reason for students' difficulty with division of whole numbers is the occurrence of remainders. During their elementary school years, students are taught to express the answer to division problems involving remainders in several different forms (e.g., 8 R4, 8 1/2, and 8.5), but neither the general conceptual aspects of remainders nor the importance of remainders in grasping subtle aspects of the relationship between multiplication and division are typically well taught to elementary school students. Moreover, it is well documented that students encounter special conceptual obstacles when a division computation involving a remainder is embedded in a story situation (Greer, 1992; Silver, Shapiro & Deutsch, 1993).

A correct solution to a story problem involving division with a remainder requires an understanding of the context and the quantities involved in the problem, in order to allow a successful interpretation of the remainder (Silver, Mukhopadhyay & Gabriele, 1992). The same symbolic representation of a division problem involving a remainder can serve as the computational model for several different problem situations, each of which derives a different answer from the division computation (Silver et al., 1992). For example, Silver et al. (1992) identified three types of questions that could be solved using the same division computation, $100 \div 40$:

> Mary has 100 brownies which she will put into containers that hold exactly 40 brownies each. (1) How many containers can she fill? (2) How many containers will she use for all the brownies? (3) After she fills as many containers as she can, how many brownies will be left over? (pp. 29-30)

The solution to the first question (i.e., 2 containers) essentially ignores the remainder and is referred to as a quotient-only problem type. The second question requires consideration of the remainder in determining an answer. The solution of 3 containers is derived by incrementing the quotient and is called an augmented-quotient problem type. The existence of the remainder leads to a solution to the third question, which essentially ignores the quotient, and is called a remainder-only problem type. In each type of question, the solution is dependent upon the wording of the problem, and an adequate representation of the situation in which the problem is embedded and the quantities involved.

Given students' difficulty with division involving remainders, it is reasonable to examine the understandings that their teachers may have of the issues related to this area. It has been argued that problem posing can provide a window through which to view mathematical thinking and the understanding of mathematical concepts (Silver, 1994). In this study, a problem-posing task is used to explore prospective elementary school teachers' understandings related to division with remainders. It seems reasonable to use a problem-posing task for such an exploration of understanding, since several studies have successfully used problem posing as a way to examine students' (Hart, 1981; Bell, Fischbein, & Greer, 1984; Greer & McCann, 1991) and teachers' (Ball, 1990; Simon, 1993) knowledge of various mathematical concepts. In some of the studies, an equation or numerical solution has been presented and the task of the subjects has been to generate problems corresponding to the equation or solution. For example, Ball asked prospective teachers to pose problems which involved settings in which division of fractions would be the appropriate computational model. Simon provided three possible solutions to a division computation and asked prospective teachers to pose problems that would lead to each of these solutions. In this study, prospective teachers' understanding of aspects of division involving remainders was also explored by examining the problems they posed. As in the work of Simon (1993), the approach used here involved requesting prospective teachers to pose division problems in response to a stimulus, but it was different in that prospective teachers were asked in this study to pose multiple problems corresponding to the computation and they were given no suggested solutions. In this way the combinations of problems and solutions together provided a basis for analysis of their understandings and misunderstandings related to division involving remainders.

Method

Subjects

The subjects were 24 prospective elementary school teachers (8 males, 16 females) enrolled in a mathematics content course at a large, urban, public university. The course was one designed for prospective elementary school teachers, and it treated the usual topics dealing with number systems, number theory, and some geometry. All students had taken at least one other college-level mathematics course (i.e., a mathematics course that fulfilled the graduation requirements of the university), and all had completed at least two years of college-preparatory mathematics at the high school level.

Task and Administration

Each subject was tested during a class meeting held during the ninth week of a 15-week course. Subjects worked individually and were given 20 minutes to complete the problem-posing task shown in Figure 1.

Insert Figure 1 about here

Results

A total of 100 responses was provided. The modal number of responses was 5. There were 21 subjects who provided three or more responses, and 15 of these subjects provided four or more responses. In general, subjects provided different solutions for each of the different problems that they posed; 18 subjects provided different solutions for all of the problems that they posed and only four subjects provided two numerically identical solutions.

Of the 100 responses, 84 involved the posing of problems that "matched" in some holistic way the given computation. That is, these 84 problems corresponded to the given computation, $540 \div 40$, or to some closely associated computation (e.g., $13.5 \times 40 = 540$) that involved the complete set of numbers in the given computation. The other 16 responses were somewhat unexpected and unusual because they did not correspond to the entire computation but rather, if they could be associated with the computation at all, only to some part of the computation. In this section of the paper, information about these 16 responses is presented briefly first. The remainder of this section reports a more extensive analysis of the other 84 responses since they provide a rich source of information about subjects' understanding of division involving remainders. The responses are analyzed with respect to what they reveal about subjects' mathematical understandings and misunderstandings concerning division and related ideas. In addition to these analyses focusing on conceptual understanding, the problems are also examined with respect to what they reveal about subjects' ability to relate the given mathematical computation to real world contexts.

Unexpected Responses

Of the 16 posed problems that did not correspond to the entire long division computation, $540 \div 40$, 11 were ones that corresponded to computations embedded within it, such as the following problem:

What is the missing number in the following operation?

$$\begin{array}{r} 13 \\ 40\overline{)\,540} \\ \underline{40} \\ A \\ \underline{120} \\ 20 \end{array}$$

Solution: 140[1]

The posing of these problems demonstrates that some subjects saw the given visual arrangement of numbers not only holistically as a long division problem with a remainder, but also analytically as a set of subproblems embedded within the larger computation. One subject posed a total of six problems, each of which was a subproblem embedded within 540 ÷ 40; two other subjects also generated at least one problem of this type.

The given arrangement of numbers was also apparently viewed by some subjects as a collection of numbers not specifically embedded in a long division computation. In fact, there were five problems posed that corresponded to a computation not associated with 540 ÷ 40, such as the following:

> I have $5.40 and I want to lend it out to you at 13.5% monthly interest. How much will you owe me in one month? *Solution*: 40 cents

There were four subjects who generated problems that corresponded to computations not associated with the given one, although these problems all involved numbers from the given long division.

As the two examples shown above illustrate, not all of the problems that corresponded to an embedded or non-associated computation were written within a story problem context. In fact, 6 of the 16 "story" problems were actually mathematical computations simply accompanied by words, similar to the first example that illustrates an embedded problem. These 16 problems are interesting in that they reveal ways in which the visual arrangement of numbers in the long division computation was viewed. However, the posing of these problems appears to be due, at least in

[1] In this paper all designated "solutions" are the actual solutions provided by subjects for their problems, regardless of correctness.

part, to subjects being challenged by the task requirement to generate more than one problem. When these unexpected responses were given, they were more likely to appear later rather than earlier in the sequence of problems posed by an individual subject. Nevertheless, since the focus of this study is the mathematical understanding of division problems involving remainders, these 16 problems are of somewhat marginal interest, and they are not considered further in the reporting of results from this study.

Recognizing Division Situations

As was expected, the majority of posed problems were ones that directly corresponded to 540 ÷ 40, such as the following examples:

> There are 540 apples in a basket. There are 40 neighborhood children who want apples. How many apples can each of the children have, ensuring that they all have the same number of apples? *Solution:* 13
>
> The forty students of the ski club have to raise $540. If they divide it up equally, how much must each person raise? *Solution:* $13.50

Of the 84 posed problems that corresponded to the entire long division computation, 61 were of this type. These problems provide direct evidence that subjects generally perceived the given visual arrangement of numbers as a long division problem with a remainder, and they were able to generate a problem situation corresponding to it. Problems corresponding directly to 540 ÷ 40 were not only the most frequently posed overall, but it was also the case that such problems were posed by almost everyone. In particular, at least one problem of this type was posed by 23 of the 24 subjects. Moreover, this was the type of problem most frequently posed as a first response (i.e., the response in the upper left hand space on the answer sheet). As their first response, 22 subjects posed a problem that directly corresponded to 540 ÷ 40. In addition, 7 of these 22 subjects posed only such problems. Since so many division story problems of this type were posed, they were subjected to further scrutiny.

Relating Division Problem Structures and Solutions

One characteristic of division problems is that they refer to partitive or quotitive division situations. A partitive division situation specifies the number of items and the number of equal-size groups, and the solution expresses the number of items per group. A quotitive division situation specifies the number of items and the number of items in each equal-size group, and the solution expresses the number of equal-size groups that can be formed. Of the 61 division problems corresponding to 540 ÷ 40, 39 were partitive and 19 were quotitive. Three problems were unable to be classified as partitive or quotitive: two were yes/no questions for mathematical computations (e.g., Can 540 be divided by 40 evenly?) and one was an incorrect reference to the Euclidean Algorithm.

Another characteristic of division story problems is that they can generally be classified as having one of four types of problem structures. Three of these problem structures were identified by Silver et al. (1992): (a) augmented-quotient problem situations (AQ), in which the correct approach is to increment the quotient if a remainder occurs in the computation; (b) quotient-only problem situations (QO), in which the correct approach is to ignore the remainder; and (c) remainder-only problem situations (RO), in which the correct approach is to give the remainder as the solution. An additional problem structure is also fairly common; a quotient-part situation (QP) for which it would be correct to include some form of the remainder with the quotient, expressing these together as a single entity. For the given computation, this could be done by expressing the solution in one of two ways: (a) 13 R20, to denote 13 with 20 items remaining, for QP-R problems involving the division of discrete rather than continuous quantities and for which it would be correct to include the remainder with the quotient; or (b) 13.5 (or 13 1/2), for QP-F problems involving the division of continuous rather than discrete quantities and for which it would be correct to include the remainder with the quotient.

To examine the frequency with which each type of problem was posed, each of the 61 problems corresponding to 540 ÷ 40 was classified according to its division structure. A total of 50 of the 61 problems was able to be classified into one of the four categories. Figure 2 illustrates an example of each of the four problem structures along with identification of each problem as either partitive or quotitive. Of the problems unable to be classified according to problem structure, six were yes/no questions for mathematical computations (e.g., Can 540 be divided by 40 evenly?); four problems required interpretation of the remainder in a way different from the manner indicative of the problem structures above (e.g., as the number of 20 passenger buses needed if 540 students were to be transported in 40 passenger buses); and one problem was an incorrect reference to the Euclidean Algorithm as mentioned above. All four types of problem structures were represented in subjects' responses, thereby providing evidence that they were generally aware that several different types of division problems could be posed corresponding to one computation. In fact, there were 15 subjects who posed problems relating to two or more problem structures, and 6 of these 15 subjects posed problems that related to three or more of the problem structures; one subject posed problems related to all four problem structures.

Insert Figure 2 about here

In order to analyze the consistency of problem structures with the proposed solutions, each of these problems was also examined for correctness. Subjects were fairly successful in supplying correct solutions for their problems. Overall, subjects provided a correct solution for 72% of the 50 posed problems. Moreover, the one subject posing problems related to all four division problem structures supplied a correct solution for all four of those problems. Nevertheless, many subjects who generated more than one type of division problem had difficulty solving them all correctly. Of the 15 subjects who posed problems related to more than one division structure, only 8 supplied a correct solution for each of their problems. Figure 3 illustrates the frequency of each problem structure and the number of times the proposed solution was correct.

Insert Figure 3 about here

A total of 12 RO problems was posed, of which 9 were partitive division problems and 3 were quotitive division problems. Subjects who posed RO problems were successful in supplying a correct solution to their problems approximately 83% of the time. Two RO problems were incorrectly matched with a proposed solution of 13 R20. Thus, the subjects included the correct solution (20) as a portion of their answer, but the entire proposed solution was inappropriate for the RO problems posed.

A total of 13 QO problems was posed, of which 10 were partitive division problems and 3 were quotitive division problems. Subjects were able to correctly relate the QO problem structure with its solution approximately 69% of the time. Three QO problems were incorrectly matched with proposed solutions of 13.5, 13 1/2, and 13 R20. The other QO problem was matched with an incorrect solution of 13.2 which appeared to reflect confusion about the remainder in relation to the quotient. This error is analyzed more fully in the next section of the paper.

A total of five AQ problems was posed, of which all were quotitive division problems. For AQ problems, subjects were successful in supplying a correct solution to their problem only 60% of the time. The following problem illustrates one of the three AQ problems that was solved correctly:

> 40 ants can fit on a leaf. If there are 540 ants trying to cross the river, how many leaves must the ants gather? *Solution*: 14

A correct solution to an AQ problem suggests a solid understanding of division, since the solution of 14 does not appear directly as part of the computation. Rather, it results from an interpretation of the computational result in light of the problem setting (Silver, et al., 1993). Thus, AQ problems may be not only difficult to solve, but also difficult to pose. This inference is supported by the findings that only three AQ problems were posed and solved correctly. Two additional AQ problems corresponding to $540 \div 40$ were posed with incorrect solutions. The following problem and solution is an illustration:

> There are 540 students in the school. How many classrooms would there have to be to contain 40 students in each one? *Solution:* 13

The proposed solution in this case fails to recognize the necessity of considering how the remainder affects the number of classrooms needed, thereby leaving 20 students without a classroom. The other incorrect solution proposed for an AQ problem was 13 R20.

A total of 20 QP problems was posed of which 16 were partitive division problems and 4 were quotitive division problems. Overall, problems posed with a QP structure were correctly related to the proposed solution 70% of the time. There are two versions of QP problems: (a) QP-F problems for which it would be correct to express the remainder as a decimal or fractional portion of the divisor (13.5 or 13 1/2) and (b) QP-R problems for which it would be correct to include the remainder with the quotient (13 R20, to denote 13 with 20 items remaining).

Of the 20 posed problems with a QP structure, 8 were QP-R problems (5 partitive and 3 quotitive) and 12 were QP-F problems (11 partitive and 1 quotitive). Subjects were more successful with QP-F problems than with QP-R problems. In particular, correct solutions were given for 75% of the QP-F problems but for only 63% of the QP-R problems. There were three incorrect solutions for QP-R problems. For one QP-R problem, the solution 13 1/2 was given incorrectly; the other two incorrect solutions appeared to be due to clerical errors. One of the incorrect solutions for QP-F problems was 13; and the other two incorrect solutions for QP-F problems involved an apparent misinterpretation of the relationship of the remainder to the divisor. This type of error, which had also been noted above for one QO problem, is discussed in more detail next.

<u>Expressing the Remainder Incorrectly as Part of the Quotient</u>

The remainder in a long division computation has a complex relationship to the numbers in the computation. As has been noted, when the computation corresponds to a contextual situation, interpretation of the remainder in the setting can influence the problem solution in important ways. Yet even in pure computation, the remainder has an important relationship both to the divisor and to the dividend. On the one hand, the remainder represents a portion of the dividend, since it expresses the number of "left overs." This is the sense in which a solution of 13 R20 is a correct answer to the given computation. On the other hand, the remainder also represents a portion of the divisor since it expresses the part of the divisor left unaccounted for in the quotient. This is the sense in which a solution of 13.5 or 13 1/2 is a correct answer to the given computation.

In general, the subjects in this study appeared to be able to determine which of the two appropriate interpretations of the remainder was correct in most cases. In particular, among the problems posed by subjects in this study, there were nine problems for which subjects correctly obtained a solution of 13.5 or 13 1/2, and there were five problems for which subjects correctly obtained a solution of 13 R20. Nevertheless, three posed problems and their corresponding proposed solutions suggested that some subjects incorrectly interpreted the remainder as also being a part of the quotient. These three problems are given below:

> At 40 mph how long would it take to travel 540 miles? *Solution*: 13 hrs. 20 min.
>
> The cost for book money is $540.00 For a class of 40 to order through a book club what was each child's cost with 40 children in the club. *Solution:* $13.20
>
> There are 540 jelly beans and 40 children. If divided equally among the children, how many jellybeans does each one get? *Solution:* 13.2

The three subjects who posed these problems revealed an apparent conceptual misunderstanding of the remainder as a portion of the quotient rather than as a fractional part of the divisor, which would have been correct in each of these cases. On the basis of these responses alone it is not possible to determine the rationale for the proposed solutions. Since each subject made only one such error, it is not the case that the error was dominant in the response set of any subject. However, two of the three subjects also exhibited other conceptual misunderstandings in other posed problems/solutions, thereby suggesting the possibility that this error reveals fundamental conceptual confusion for those two subjects rather than being an artifact of time or task constraints.

This consideration of the form in which some proposed solutions were supplied by subjects in this study revealed an interesting type of error. A more complete examination of forms in which solutions were provided by subjects in this study is undertaken next.

Forms of Proposed Solutions

Subjects' proposed solutions for each of the 50 problems that had been classified by problem structure were examined with respect to its form. Of the 50 solutions provided for the problems, 45 matched one of the five different possible solutions that were expected for problems related to the given division computation: 14, 13, 20, 13.5 (or 13 1/2) and 13 R20. Figure 4 shows the frequency of occurrence and correctness of each expected form of solution.

Insert Figure 4 about here

Solutions of 14 or 20 were proposed a total of 13 times, and they were always proposed correctly. A solution of 13 was proposed a total of 11 times and subjects were correct in proposing this solution 82% of the time. The two solutions of 13 that were inappropriately related to the structure of the posed problem involved an incorrect interpretation of the problem situation. One was proposed for an AQ problem and the other for a QP-F problem.

In general, solutions of 13.5, 13 1/2 or 13 R20 were proposed for 21 of the posed problems. However, subjects correctly matched these solutions to problems only 67% of the time, and these solutions were given incorrectly for all types of division problems. These forms of solutions are often associated with a purely computational approach to division problems, and subjects have much more experience with pure computation than with using division as a way of solving a contextual problem, so the high error rate for these forms of solution may be due to these factors.

Subjects who inappropriately proposed solutions of 13.5 or 13 1/2 failed to consider that they would be dividing discrete objects, such as puppies and children, in half. Two of these solutions appeared for QO problems and the other solution of this type was proposed for a problem in which the number of items remaining (i.e., a solution of 13 items with 20 left over) would have been appropriate. Subjects who inappropriately proposed a solution of 13 R20 included more information in the solution than was required for the problem that was posed. A solution of 13 R20 was inappropriately proposed once each for AQ and QO problems and twice for RO problems.

The five solutions that did not have one of the five expected forms fell into two categories. Two of these problems were expressed in unexpected forms because of clerical errors on the part of the subjects. The other three problems were due to subjects' conceptual misunderstandings in interpreting the remainder as a part of the divisor, as discussed in the previous section.

The Relationship between Division and Multiplication

The division computation given in the task is one member of a family of associated computations that represents the relationship among the divisor, dividend, quotient, and remainder of a division problem. In all, 23 of the posed problems were ones that corresponded to a member of this family of associated computations different from 540 ÷ 40, the one provided in the task. For example, the following two problems would correspond to 540 ÷ 13.5 and 40 x 13.5, respectively:

> After the dinner banquet, 540 pieces of candy were distributed. If 13.5 pieces were given to each person, how many people attended the banquet? *Solution*: 40

> If a product is sold for $13.50/pound, and someone purchases 40 pounds, what is the total cost? *Solution*: $540

These problems are interesting because they demonstrate that many subjects appeared to possess some understanding of the relationship between division and multiplication. In fact, 15 of the 24 subjects posed at least one problem that demonstrated an understanding of the relationship of 540 ÷ 40 to some other member of this family of associated multiplicative expressions or computations. The spontaneous generation of such problems by a majority of the subjects provides evidence that an appreciation of at least some aspect of the relationship between division and multiplication was readily available to these subjects. It is unclear whether or not the nine subjects who did not generate such problems had a similar understanding of the relationship between division and multiplication, since the task instructions did not specifically ask for such problems.

Although the 23 problems relating 540 ÷ 40 to an associated computation revealed understanding of the relationship between division and multiplication, some aspects of this relationship are complex and subtle. Therefore, further examination of the responses was undertaken to discern the nuances of understanding and misunderstanding revealed therein concerning the relationship between multiplication and division.

One subtle aspect of the relationship between division and multiplication in this case is due to the presence of the remainder in the computation. When a division of two whole numbers can be done without a remainder, there is an interchangeability between the quotient and the divisor that derives directly from the commutativity of multiplication of whole numbers. For example, the division $15 \div 3 = 5$ is related to $15 \div 5 = 3$, since $3 \times 5 = 5 \times 3 = 15$. But when a remainder occurs, the quotient and divisor can not be interchanged in this way. In the particular computation given in this task, a person who understood this aspect of the relationship between multiplication and division and this aspect of the relationship among the divisor, dividend, quotient, and remainder would avoid posing problems that treated the quotient and divisor as if they were interchangeable. Three problems were posed that corresponded to the computation, $540 \div 13$, which is obtained by interchanging the divisor and quotient in the original problem. Only one of these three problems was accompanied by a correct solution:

> If there are 540 children who need transportation from school to home and only 13 buses. How many children will go on each bus evenly? How many will be left?
> *Solution:* 41 R7

This problem together with its proposed solution reveal that this subject did not assume the quotient and divisor of the original problem would be interchangeable. The calculation appearing on her paper also indicates this subject was aware that the solution would be a number different from the divisor in the given long division computation.

The other two problems revealed that the subjects did not appreciate this subtle aspect of the relationship among the divisor, dividend, quotient, and remainder, as can be seen in the example shown here:

> There are 540 balls. How many bags are needed so that each bag will have the same number of 13 balls in the bag? *Solution:* 40

In this case, the poser treated the numbers 13 and 40 as if they were interchangeable, thereby failing to give the correct solution of 41 for the posed problem. This error is hereafter referred to as the divisor/quotient (DQ) interchange error.

In contrast to the subjects who made the DQ interchange error, another group of subjects appeared to realize that when the quotient and remainder are together expressed as a single entity, as in 13.5 or 13 1/2, then this entity is interchangeable with the divisor. In this case, the interchange of the quotient/remainder with the divisor in the original computation leads to related computations, such as 540 ÷ 13.5 and 40 x 13.5, which can each correspond to a problem. Ten subjects posed problems corresponding to 540 ÷ 40, each of which had solutions of 13.5 or 13 1/2. Producing a problem with a solution of 13.5 or 13 1/2 was apparently facilitative of the generative requirements of the task, since six of these subjects subsequently posed at least one other problem corresponding either to 540 ÷ 13.5 or to 40 x 13.5. In fact, three subjects actually generated problems corresponding to both of these computations. The sets of problems posed by these subjects provide evidence that they probably recognized that the quotient/remainder and divisor were in fact interchangeable in this case. Further evidence of their understanding important aspects of the relationship among the divisor, quotient and remainder is also derived from the fact that none of these subjects incorrectly interchanged in the manner discussed above.

These strings of closely related problems suggest that these subjects were using the family of computations related to 540 ÷ 40 = 13.5 in order to generate a set of problems that were interrelated but which each had a different solution. It is interesting to note, however, that at least some of the problems associated with this understanding of the mathematical relationship among the divisor, quotient, and remainder were somewhat deficient in other regards, such as the correspondence among the computational model, the problem context and the proposed problem solution. These aspects of the problems posed by subjects in this study are discussed more fully in the next section.

<u>Relationship to Real-World Situations</u>

The subjects in this study were directed to write story problems. In general that is exactly what they did. In fact, of the 84 problems under consideration, 95% were situated in a context and every subject was successful in generating at least one story problem. However, most subjects posed each problem within a different context. There were 17 subjects whose posed problems each had a different real-world context. Of the 7 subjects who posed two or more problems within one context, 2 of these subjects posed all of their problems within one contextual situation. Posing a story problem involving division with a remainder and then proposing its solution requires that the poser coordinate information about the problem situation or context with the results of the mathematical computation in order to determine whether the solution makes sense within the problem context. Thus, the posed problems provide an interesting data set for examination of subjects' understanding of division within a real-world situation.

In general, subjects showed little evidence of confusion or difficulty in relating the division computation, or one of its multiplicative relatives, to a real world context. In fact, 61 of the 84 problems were completely sensible in this regard. Additionally, 10 subjects generated at least one sensible problem. Some of the problems were quite clever in their coordination of mathematical requirements and real world context, as is evidenced by the following example in which a subject introduces the notion of different size buses to accommodate the remainder that appears in the computation:

> There are 540 students. You have 20 and 40 passenger buses available to take these students to the museum.
> a. How many 40 passenger buses do you need? *Solution*: 13
> b. How many 20 passenger buses? *Solution:* 1

Note that the solutions provided by this student, 13 and 1, show a clear understanding that the remainder in this problem represents students rather than buses. Of course, this problem could in fact have many solutions because the problem does not require that students would be equally distributed among the buses. Nevertheless, since the computation was present in the task, and since the computation can be seen as corresponding to the division of 540 into equal-size groups of 40, it is reasonable to assume that the constraint of equal-size groups was implicit in the task and therefore also in the posed problem.

Although most problems were contextualized in quite reasonable ways, there were 23 problems for which the context had some unreasonable aspect. This type of problem was generated by 14 subjects. One salient group of such problems involved the treatment of essentially discrete and indivisible quantities as if they were continuous and divisible, as is shown in the following posed problems:

There are 540 pencils and 40 new students. How many pencils will each student get? *Solution:* 13 1/2

12 chickens produced a total of 540 eggs in one day. The eggs are to be sent to 40 area stores in cartons. How many eggs will each store get? Will it be an even number of eggs? *Solution:* 13 1/2

Farmer Joe planted pumpkin seeds in order to make pumpkin pies. He grew a total of 540 pumpkins and had 13 1/2 pumpkins in each row. Joe had forgotten to count how many rows he had. How many rows did Joe use to grow his pumpkins? *Solution:* 40

If there are 540 puppies and 40 large cages, how many puppies will occupy each cage if each cage has the same number of puppies. *Solution:* $540 \div 40 = 13.5$

A school has a total of 540 kids in grades K to 6. There are a total of 40 teachers in the school. How many kids would each teacher be responsible for if each class was divided up evenly? *Solution*: 13 1/2

Mrs. Jones teaches a large lecture on math and wanted to put her students in 40 rows. She figured out that if she did this she would get 13 1/2 kids in each row. If she went about this, how many students would she have? *Solution*: 540

The first two problems refer to quantities that would be difficult or unusual to partition into halves (although pencils are well known for breaking!). Note also in the second problem the use of the word "even". It is unclear whether the word even is being used here to mean numbers divisible by 2 or to mean that the division results in a whole number quotient without remainder. Both uses of the word even are fairly common. If the latter meaning is the one intended, then the query about an "even" number of eggs may have reflected this subject's quandary in dealing with the notion of half-eggs reflected in his solution to the problem.

In the third problem, the notion of half-pumpkins certainly borders on the improbable. Although unusual, it is important to note that it is actually possible to construct a model for this problem situation using a rectangular array in which corner pumpkins share both a row and a column, thereby making it possible to think about half-pumpkins without actually dividing them. Nevertheless, it is unlikely that this model is what the subject posing this problem had in mind, especially since he also posed other problems that contained unrealistic half units. The fourth, fifth and sixth problems have the unfortunate characteristic that, in order to obtain the proposed solutions, one is required to divide children and puppy dogs into halves!

Not all of the unrealistic problems involved halves. In some cases, the problems simply contained implausible assumptions or conditions, such as each person eating a very large number of apples or cupcakes. The following problem illustrates a situation with such an implausible condition:

> There are 540 lockers in Joe Blow High School. The lockers must be divided equally among 40 students. After each student receives his or her share of lockers, how many lockers will be left over? *Solution*: 20

The faulty connection in this problem is not due to dividing lockers into half lockers, but rather to assigning an unreasonable number of lockers to each student. In some cases more than one form of unreasonableness was present, as in the problem given above in which hens lay many eggs in one day and those eggs are divided into halves.

These problems clearly indicate that at least some subjects posed problems and gave solutions that resulted from greater attention to the mathematical requirements than to the coordination of the mathematical constraints with those implied by the context in which the problem was posed. Thus, although contexts were commonly provided for the posed problems, the contexts were not always carefully considered when the solutions were provided for the posed problems.

Sufficiency of Information in Posed Problems

Because subjects posed problems within contexts, it was possible to examine the posed problems with respect to the adequacy of the information provided in the problem to allow obtaining the proposed solution, or any solution at all. In general, the posed problems were quite reasonable in this regard. In fact, 57 of the 84 posed problems clearly provided both the necessary and sufficient information to solve the problem and obtain the proposed solution. Of the remaining 27 problems, only one problem was so ambiguous that it could not be solved in a manner that would yield the proposed solution, but the other 26 problems were interesting because they raise a potential concern about subjects' attention to the sufficiency of information in their posed problems. In particular, for 20 problems subjects failed to specify the need for either equal-size groups or the number of items per group in the division situation being posed. Two of these problems depicted a quotitive division setting, and the other 18 were partitive division problems. One of the two quotitive problems failed to specify the number of items per group, while the other failed to state the need for equal-size subgroups. For all of the partitive division problems the need for equal-size subgroups was not specified. The following examples illustrate responses for which subjects failed to provide sufficient information. These problems illustrate a quotitive and partitive division setting, respectively:

> If 540 people are to be divided into groups (each group with the same number of people), how many extra people will there be? *Solution*: 20

> If there are 540 students enrolled in school and there are 40 classrooms that the students could be assigned to, how many students will be in each room? *Solution*: 13

Five subjects posed over two-thirds of these problems (14 out of 20), and two of these five subjects failed to specify equal groups in all of their posed problems. Subjects explicitly stated the need for equal-size subgroups for 42 problems. However, there were eight subjects who posed at least one division problem for which they recognized the necessity to specify equal-size subgroups and at least one division problem for which they failed to state equal groups. Given the circumstances under which these data were collected, however, it is not completely clear if the subjects who generated these problems actually did not understand the requirement of equal-size groups. It is possible that the subjects simply took that to be an implicit assumption that was reasonable in this setting, since equal-size groups are implied by the given division computation.

Another six problems were interesting for another reason related to information sufficiency. These problems, like the one shown here, all corresponded to the computation, 540 - [40 x 13]:

> Mom made 540 cupcakes for 40 people. If each person ate 13 cupcakes, how many would be left over? *Solution*: 20

From the perspective of sufficiency of information, one could argue that these problems actually contain excess information if they are considered division problems involving a remainder. As a division with remainder problem, it would only be necessary to specify that the number of cupcakes eaten was the same for each person; it would be unnecessary to provide the exact number since that would be obtained within the computation. It is possible that these subjects were thinking of the problem as a long division problem and simply taking advantage of the opportunity to obtain what would otherwise be a difficult solution to obtain by simply taking the answer directly from the given long division computation. However, another explanation is also plausible. In particular, if the problem is considered in relation to the computation, 540 - [40 x 13], one can see that the specification of the number of cupcakes for each person allows the problem to be solved using arithmetic operations other than long division. Since long division is complex, the posing of the problem in this way may have been a way for subjects to avoid the cognitive complexity associated with long division. Unfortunately, from the information provided along with subjects' responses, it is impossible to be completely certain of the rationale subjects had in posing these six problems.

One particularly elegant example of a problem posed in a manner that avoided long division as a requirement for solution, but which contained evidence of considerable understanding of various aspects of division is the following:

> Luke worked 9 hours for a lawn maintenance company. It takes him 40 minutes to mow a lawn, how much longer does he need to work in order to mow 14 lawns? *Solution*: 20 minutes

This problem corresponds to the following computation: (40 minutes/lawn x 14 lawns) - (9 hrs. or 540 min.) = 20 minutes. Since the number of lawns was chosen to be 14, it appears that the subject who posed this problem was aware of the solution to the augmented-quotient problem corresponding to 540 ÷ 40. The subject was then able to manipulate the numbers in the given computation to derive a problem requiring multiplication and subtraction rather than division. The substitution of 9 hours for 540 minutes is also a nice feature of this subject's posed problem. In posing this problem and proposing a correct solution, this subject revealed a strong conceptual understanding of the relationship among the dividend, divisor, quotient, and remainder.

Discussion

In this study, subjects were asked to pose problems that matched a given long division computation, and that is clearly what they tried to do. Almost all the posed problems related in some way to the given computation, including not only the majority of problems that corresponded directly to it, but also both the problems that corresponded to a member of the family of associated computations and those that corresponded to a computation that was embedded in the given long division. In all, 95 of the 100 posed problems related in some clear way to the given computation.

Since almost all subjects generated three or more problems, the large number of posed problems corresponding to the given computation in some way suggests that the task was appropriate for the subjects, who were able to generate reasonable responses.

The responses given to the task used in this study also provide some insights into prospective teachers' interpretation of terms like "story problem" and "different solutions". In particular, subjects in this study were asked to "write as many different story problems" as they could. Their responses revealed a clear preference for posing problems associated with "real world" contexts, thereby suggesting that the term story problem had that particular meaning for this group of subjects. In response to the request that they pose <u>different</u> story problems, most subjects responded by posing problems in different situational contexts. Only two subjects tried to pose all their problems within the same context. Subjects were also asked to generate problems that had different solutions. Once again, subjects appeared to have a common interpretation of this request, in that most all subjects proposed numerically different solutions to each problem in the set they posed. Only four subjects gave the same numerical solution to two problems in their sets of posed problems; in each case the numerically identical solutions could actually be considered different because they pertained to different situations and referred to different quantities.

One of the strengths of the posing task used in this study was that it called for subjects to pose multiple problems related to the given computation. The multiplicity of responses provided a rich array of responses that displayed aspects of subjects' understandings and misunderstandings that might have remained invisible if only a single response were requested and examined. Another feature of the task that appeared to be quite important was its demand that subjects pose not only the problems, but also their corresponding solutions. This feature allowed for the systematic examination of the relationship between posed problems and proposed solutions in a variety of ways, thereby revealing important aspects of the connections and disconnections in subjects' thinking about division and problem situations.

Although the vast majority of problems were situated in contexts, the problems and solutions frequently did not represent a solid connection between the mathematical and situational aspects of the problem. In particular, about 25% of the posed problems had some unreasonable aspect to them, such as having problem conditions that were highly implausible (e.g., 13 lockers assigned to each student) or proposed solutions that required taking halves of discrete objects (e.g., eggs, children, puppy dogs) not reasonably subjected to partitioning. The presence of problems with unreasonable conditions or solutions could be attributed to the lack of time available for editing tasks. It is possible that subjects would have discovered the unreasonable aspects of their tasks if they had been given an opportunity to review and edit them. Nevertheless, since some research (e.g., Silver, Shapiro & Deutsch, 1993) has shown that students often dissociate formal school mathematics from real world situations, the high frequency of problems with unreasonable aspects posed by the prospective teachers in this study warrants further investigation. The possibility that teachers' own deficiencies in this area might actually contribute in subtle or overt ways to students' unfortunate dissociation of mathematics from context should be carefully examined. Moreover, Simon (1993) also reported that prospective elementary teachers were limited in their ability to pose reasonable problems for a division with remainder computation when the solution was specified for them. The findings of the study reported here are based on responses to a different task than the one used by Simon, and the analysis of subjects' responses with respect to problem context is more extensive in this study, yet the findings of the two studies are compatible in this regard. Collectively, the findings of these studies suggest that the quality of prospective teachers' connection of mathematics to reasonable situational contexts needs to be addressed not only in research but also in teacher preparation programs.

Many, though not all, of the posed problems with unreasonable conditions or solutions involved a solution or other problem element involving 13 1/2 or 13.5. But the other form of expressing the quotient/remainder entity, 13 R20, was problematic for subjects in other ways. This form of solution was the one most frequently given incorrectly. In fact, this form of solution was correct only about half of the times that it was proposed. Subjects apparently had difficulty coordinating this solution form with problem settings and questions that would correctly lead to a solution correctly expressed as 13 R20. This difficulty is probably not surprising, since the unusual numerical form 13 R20 is typically introduced to students in elementary school as a formalism to be used primarily in expressing answers to decontextualized computation problems and is typically replaced in later years by expressions for the quotient/remainder entity that involve decimals and/or fractions. Although the subjects in this study clearly had difficulty posing problems with solutions that would correctly have this numerical form, they were able to pose a large number of other problems that were correctly solved by their proposed solution of 20. This suggests that many of the prospective teachers did not view 13 R20 only as a single entity form but were able to recognize the remainder, 20, as an independent entity as well.

A facility or lack of facility in dealing with the various forms of expressing solutions to division problems was directly related to the difficulties that many subjects appeared to have in aligning division problem structures with appropriate forms of solution. Each of the four major division problem structures (i.e., AQ, QO, RO, and QP) had associated errors, with about one-third of the solutions incorrect for three of the four major problem structures. Thus, although augmented-quotient problems appeared to present special difficulties for subjects in this study, errors were associated with other problem structures as well. The relatively low rate of posing augmented-quotient problems may have been due to either the inherent difficulty or the relative obscurity of this problem structure, since the solution does not appear directly in the given computation.

In contrast to the rather large number of apparent disconnections and difficulties encountered in connecting the given division problem to reasonable contexts and correct solutions, many subjects gave evidence of understanding some important aspects of the relationship between division and multiplication. About two out of three subjects spontaneously posed problems and proposed solutions that suggested their understanding of the relationship between the given computation, 540 ÷ 40, and at least one member of its family of associated computations, such as 40 x 13 1/2. Since subjects were not directed to pose such problems, it is encouraging that so many subjects were able to do so. Nevertheless, it was also the case that one-third of the subjects did not pose such problems. Furthermore, the spontaneous emergence of the erroneous DQ (divisor/quotient) interchange error in problems posed by two subjects suggests that a completely correct understanding of the relationship among the divisor, dividend, quotient and remainder was not possessed by all of the prospective teachers in this study. Although the DQ interchange error was evident in the problems posed by only two subjects, completely correct understanding was evident in the problems posed by only one subject; for the remaining subjects, it is unclear whether or not they possessed this misconception, since their posed problems provided neither evidence nor counter-evidence for its existence. We know of no other study that has identified this misconception for division, so the nature and persistence of the DQ interchange error certainly warrants further research.[2]

Another finding of this study that also warrants further investigation concerns the large number of posed problems, almost all of which were partitive division situations, in which subjects failed to specify the need for equal-size groups. Given the conditions under which problems were posed in this study, it is possible that subjects assumed the existence of equal-size groups because of the presence of the given computation, in which the assumption of equal-size groups is operative. Nevertheless, the relatively large number of problems posed by a fairly small number of subjects suggests that these subjects may not understand the importance of equal-size groups within division problem situations. Since this is an essential component of a complete understanding of division involving remainders, further investigation of the extent of misunderstanding about this issue among prospective and actual elementary school teachers is warranted.

In this study prospective elementary school teachers were asked to generate several different mathematics problems, each with a different solution, that corresponded to a given long division computation. The problem-posing task used in this study was successful in eliciting an array of responses that provided the basis for an extensive analysis of both the conceptual understandings and misunderstandings that prospective elementary school teachers appear to have concerning division involving remainders and the kinds of connections and disconnections in relating a

[2]The authors would be grateful if any reader of this paper can refer us to studies documenting the nature of the DQ interchange error.

division calculation to a proble.n situation that were displayed in their responses. The findings were that, although some aspects of division, such as its connection to different types of problem situations and its relationship to multiplication, were fairly well understood by most of the subjects in this study, limited or flawed understanding was also noted in many different areas. These limitations in teachers' understandings of division are especially worrisome since students are known to have difficulty learning division concepts and skills. It is possible that deficiencies in teachers' knowledge could be passed on to students during instruction. Moreover, it is almost certain that these deficiencies in knowledge will make it more difficult for these prospective teachers to be successful in teaching mathematics in accordance with the spirit of the current mathematics education reform movement, as it is represented in documents like the Professional Standards for Teaching Mathematics (NCTM, 1991). It has been argued that in order to teach in ways that emulate current reform ideas, a teacher must possess broad, flexible knowledge of content (Lampert, 1988). Therefore, deficiencies in prospective teachers' understandings of division need to be addressed in teacher preparation programs in order to ensure that the next generation of teachers understands mathematics well, thereby enabling them to help their students also understand it well.

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Figure 1

Posing task

Please work individually on the task given below.

$$
\begin{array}{r}
13 \\
40 \overline{)\,540} \\
\underline{40} \\
140 \\
\underline{120} \\
20
\end{array}
$$

In the spaces provided, write as many different story problems as you can that match the computation shown above. The story problems you propose must all have different solutions. For each story problem you propose, indicate the solution in the box provided.

Solution	Solution
Solution	Solution
Solution	Solution

Figure 2

Example of a Posed Problem for Each Division Problem Structure

Problem Structure	Posed Problem and Proposed Solution
AQ (Quotitive)	40 ants can fit on a leaf. If there are 540 ants trying to cross the river, how many leaves must the ants gather? *Solution*: 14
QO (Quotitive)	There are 540 balls. How many bags are needed so that each bag will have the same number of 40 balls in the bag? *Solution*: 13
RO (Partitive)	If you have 40 needy families and 540 cans of food and each family must receive the same amount of cans, how many cans are left over? *Solution*: 20
QP (Partitive)	The forty students of the ski club have to raise $540. If they divide it up equally, how much must each person raise? *Solution*: $13.50

Figure 3

Frequency of Posed Problems and Correctness of Proposed Solutions for Each Problem Structure

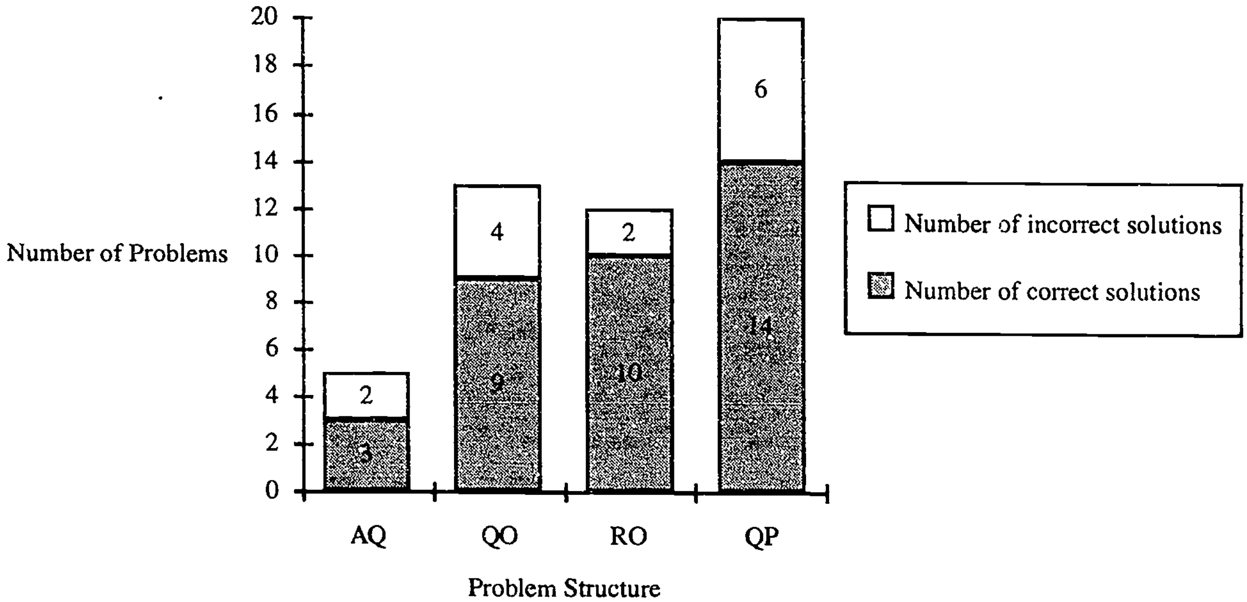

Figure 4

Frequency of Proposed Solutions and Correctness for Each Form of Solution