DOCUMENT RESUME

ED 380 270 SE 053 486

AUTHOR	Sharma, Mahesh C.
TITLE	Place Value Concept: How Children Learn It and How To Teach It.
INSTITUTION	Center for Teaching/Learning of Mathematics, Framingham, MA.
PUB DATE	Jan 93
NOTE	26p.
AVAILABLE FROM	Center for Teaching/Learning of Mathematics, P.O. Box 3149, Framingham, MA 01701 (single issue, $2).
PUB TYPE	Collected Works - Serials (022) -- Guides - Classroom Use - Teaching Guides (For Teacher) (052)
JOURNAL CIT	Math Notebook; v10 n1-2 Jan-Feb 1993
EDRS PRICE	MF01/PC02 Plus Postage.
DESCRIPTORS	Arithmetic; Cognitive Development; *Concept Formation; Elementary Education; Evaluation Methods; *Learning Activities; *Manipulative Materials; *Mathematics Instruction; Number Concepts; *Place Value; Teaching Methods
IDENTIFIERS	Cuisenaire Materials; *Mathematics History

ABSTRACT

The development of the place value system and its universal use demonstrate the elegance and efficiency of mathematics. This paper examines the concept of place value by: (1) presenting the historical development of the concept of place value in the Hindu-Arabic system; (2) considering the evolution of the number zero and its role in place value; (3) discussing the difficulties children have in understanding place value; (4) describing strategies to teach place value and 12 teaching activities that utilize manipulative materials and hands-on approaches; (5) discussing how the suggested activities make the transition from concrete experiences to symbolic understanding of place value; and (6) suggesting ways to assess children's understanding of place value. (Contains 16 references.) (MDH)

Place Value Concept: How Children Learn It and How to Teach It

by
Mahesh C. Sharma

featured in
Math Notebook
v.10 n.1&2
Jan-Feb 1993

MATH NOTEBOOK

From Theory to Practice

Information For Teachers/Parents of Children with Learning Problems in Mathematics

A Publication of
The CENTER FOR TEACHING/LEARNING OF MATHEMATICS
P.O. Box 3149, Framingham, Ma 01701
(508) 877-7895 or (617) 235-7200

Volume 10, Numbers 1 & 2
Annual Subscription: Individual $15
Institutional $20

Editor: Mahesh C. Sharma
Assistant: Lillian E. Travaglini

Jan. & Feb. 1993
Single issue $2.00
Double issue $3.00

PLACE VALUE CONCEPT

How Children Learn It and How to Teach It.

Mahesh C. Sharma

One of the most important arithmetical concepts to be learned by children, in the early elementary grades, is that of place value. This means the mastery of the Hindu-Arabic numeral system and its applications in solving problems. This number representation system, the only one in common use throughout the world, forms the basis of understanding the concept of number and its use. Understanding the concept of number and its use, really, means learning the language of mathematics, for mathematics is a language, a true symbol system (Sharma, 1985). The place value concept provides a very good and well-known example of this symbol system. The place value concept quite clearly demonstrates that this symbol system is very compact and efficient. Because of the compactness and efficiency of place value, a student, in order to use it effectively, must know that the value of a digit in a numeral is dependent on its position (i.e., the place it occupies in that numeral), for example, whether that numeral represents sets of 100s, 10s or sets of units (ones). This calls for, among other skills, a good mastery of the skill of spatial orientation and space organization (identifying the relative and absolute positions of objects in the environment - in this case the position of each numeral in the whole mumber.

Mathematics is a man-made science and the development of the place value system is truly a great achievement of human ingenuity. It is important to realize that the long and strenuous work of the most gifted minds was necessary to provide us with the simple and expressive notation of place value, which, in nearly all areas of mathematics, enables us to reproduce theorems about numbers which needed great genius to discover. It has brought, if not higher mathematics, at least arithmetic, within the reach of everyone. Each improvement in notation of number seems, to the uninitiated, a small thing, and yet, in numerical calculations it has provided a system which makes working with numbers enormously simple and useful.

Place value representation truly demonstrates the key characteristics of mathematical thinking: efficiency, elegance, and exactness. Our notation is an instance of that great spirit of economy in thought which spares waste of mental labor on what is already systematized, so that all our cognitive strength can be concentrated either upon what is known but unsystematized, or upon what is unknown.

> "1, 2, 3, 4, 5, 6, 7, 8, 9 and 0 --these ten symbols which today all peoples use to record numbers, symbolize the worldwide victory of an idea. There are few things on earth that are universal, and the universal customs which man has successfully established are fewer still. But this is one boast he can make: The new Indian numerals [and the place value that governs their use] are indeed universal." (Menninger, 1969)

No mathematics educator questions the

importance of children having a thorough understanding of the Hindu-Arabic numeration system and the place value concept inherent in it. Every one believes that unless a child has a complete understanding of this concept, arithmetic becomes a collection of procedures carried out without any meaning. Many children can get answers to algorithmic procedures without a proper understanding of this concept, but these answers do not mean much to them. Though our place value system is a marvelous one, allowing us to represent any number with just ten digits, this system is not that simple for children to understand and master. They must learn that digits have different values depending on their position in numbers (Burns & Tank, 1988, p. 61). Many studies point to the relation between poor performance on arithmetic algorithms and poor understanding of the place value principles. For example, a child showed the following procedure to solve a subtraction problem:

$$\begin{array}{r} 6 \\ \not{8}^{1}3 \\ -\ 26 \\ \hline 47 \end{array}$$

This example shows that the child did not `borrow' as he did not understand the place value concept. The procedure used by the student is quite bizarre in the sense that elements of the correct procedure are `scrambled'.

Examples such as these abound in children's work and show that the knowledge of base ten (decimal) place value system is fundamental to understanding all arithmetical algorithms. According to several studies, this is a difficult and a poorly taught concept.

> "...[T]extbooks lead teachers to systematically misteach place value. ...[A]s a consequence of limitations of the textbook approach, the profound difficulties that children experience when they are introduced to two-digit addition and subtraction with regrouping [are] entirely predictable. The source of these difficulties can be squarely located on the failure of the textbook approach to take account of children's mathematics. An analysis of place value that seems reasonable to an adult in terms of his or her own relatively sophisticated understanding of place value is no substitute for a conceptual analysis of children's mathematics." (Cobb & Wheatley, 1988)

The mastery of the place value system is an important milestone in a child's journey on the way to the acquisition of other mathematical concepts. A child's facility with place value helps him in understanding other mathematical concepts: divisibility principles, prime numbers, scientific notation, exponents, and of course mathematical algorithms. In other words, any concept that is dependent on number is dependent on place value.

I. Historical development of the concept of place value.

The development of the concept of place value has been quite slow. It has passed through many stages in its development. It took a long time for the human race to fully develop this system. It is a major milestone in the history of the evolution of mathematics systems. As the concept has been a key element in the development of mathematics by the human race, the concept is also a milestone in each child's journey on the way to the acquisition of mathematical concepts. We find that algebra was hardly touched by the Greeks who made geometry such an important science, partly, perhaps, because the almost universal use of the abacus rendered it easy for them to add and subtract without any knowledge or development of theoretical arithmetic. They did not see any need for place value. But, once the concept of place value was introduced it made the development of mathematical concepts rather quick and easy. The facility with place value helped the human race quite adept in understating the later mathematical concepts. Therefore, just after the advent of the concept of place value, we see a rapid advancement in the development of arithmetic and algebra.

A. Egyptian approach

The Egyptian hieroglyphic system has been traced back to 3300 BC and is found mainly on monuments of stone, wood or metal. But their number system did not have a well defined place value concept. For example, each quantity from one to nine was represented by the appropriate number of vertical strokes. These were usually grouped in threes or fours, thus `five' would be represented by a row of three on top of a row of two, and eight by two rows of four. The Egyptians had separate symbols for `10', `100', `1000', `10,000', `100,000', and `1,000,000'. The symbol for `10' was an inverted V-shape, so that `59' would be written

^^ !!!
!!!

A similar system of notation was developed by the Babylonians. Their medium was clay: a wedge-shaped stylus was impressed on soft clay tablets which were then baked hard in ovens or in the heat of the sun. These were surprisingly enduring, and many have survived from almost 2000 BC to the present time. Like the Egyptians, the Babylonians had separate symbols for tens and units. A unit was usually represented by a thin vertical wedge-shape, while ten was represented by a broad sideway wedge. Thus, `59' would be represented in the Babylonian system as

>> !!!
>>> !!!
!!!

At the number 60, however, the Babylonian and Egyptian systems diverged, with the Babylonians employing the rudiments of a place value system. The Babylonians used 60 as the base number. The system they employed was the beginning of a place value system. It exploited the fact that the same symbol can represent different numbers if one takes account of its position in a sequence. In the Babylonian system the symbol '!', if written to the left of other symbols, would represent `60': thus

! > > !

would now mean `81.' The Indian astronomical system's use of 60 as the base in its calculations shows that an early migration of ideas from the Asia minor to the Gangetic plain and from India to the Middle East took place and provides some evidence that each other's systems might have influenced the development of the systems of numeration.

B. Roman contribution to place value system

Despite their many achievements, the Romans did not develop the idea of place value as far as the Babylonians had done. As it is well known that, the Roman system of numbers was also based on a one-to-one correspondence of vertical strokes for the numbers one to four (I, II, III, IIII). It is possible,--as has frequently been suggested,--that these Roman numerals originally represented fingers. Certainly the Latin word `digitus'--from which our modern word `digit' was derived--means `finger'. It has also been suggested that the Roman symbol for five, `V', derives from the shape of an open hand, with the thumb on one side and the four fingers grouped together on the other, and that the symbol for ten, `X', comes from juxtaposing two `V's. Later the Romans developed the idea of using order to simplify their system. Thus, writing a smaller number involved beginning with a larger number and then placing a smaller number to the left of the larger one (as in `IX') meaning that the smaller number had to be subtracted from the larger. Whereas writing a smaller number to the right of the larger number (as in `XI' or `XV') meant that it had to be added. Although this made it simple to write down numbers, it also meant that the value of the number was not so immediately obvious. Thus `59' in the Roman system would be written as `LIX,' which is much more opaque than its counterpart in either the Egyptian or the Babylonian system. But that's where it remained until it was influenced by the Hindu-Arabic numeral system. Upto then it was unwieldy and complicated, as anyone who has ever tried to use it knows. For large numbers, the Roman numerals take on the aspect of a formidable puzzle. For example, the Roman representation for one thousand, nine hundred eighty three is MCMLXXXIII--compare this with much simpler Hindu 1983! Once place value was understood by Romans they made great contributions to the development of arithmetic.

C. Other approaches to place value

Some ancient number systems have a base twenty; probably their primitive originators counted on their toes as well as on their fingers. Most such systems are now obsolete. In early English literature, interestingly enough, there are frequent references to an early counting system of base twenty; note, for example, "three score and ten." The base sixty was employed in antiquity by several influential peoples of the Near East. A residue of a number system of base sixty is still found in our common method of measuring time and in the manner in which we divide a circle into degrees, minutes, and seconds. Other cultures--Mayan, Aztec, Etruscan--also developed similar place value systems involving numbers such as twelve, sixty, thirty, twenty, etc. as the base for their place value system. Our number system shows remnants of these bases. The measurement of time is based on sixty (1 hour = 60 minutes, 1 minute = 60 seconds). Quantity representation based on base twelve (1dozen = 12, twelve months in a year). Division of a month into thirty days indicates that the base 30 might be involved. There are many such examples to show that many attempts were made to denote numbers in manageable place value form.

D. Chinese experience with place value

The Chinese made great progress in the concept of place value. The greatest contribution of the Chinese in place value is to combine symbols for ranks and how many in that rank. China had separate symbols for units and for the ranks or gradations. For example, the Chinese--here we use Indian and Roman numerals as substitutes for ideogram--would write not CCCCCXXXIIII (as Egyptians or Babylonians did), but 5C 3X4 (which is closer to the Hindu numeration system).

> Thus the two systems of written numerals, the Indian and the Chinese, are in essence no different: both arrange numbers by gradations, and clearly differentiate the units from the numerical ranks. Both are forms gradational or 'place value' notation, if we consider a numeral's place as just another way of indicating its rank. The Chinese is a 'named,' the Indian an abstract place value notation; the former specifies, or 'names,' the ranks (=gradations), the latter does not. These two varieties of developed number systems, as we have said, differ fundamentally from ranks or gradations (=groupings) and indicated the number of these not by placing a numeral before or behind them, but merely lined them up in order to any desired number (or to the next higher grouping). (Menninger, 1969).

E. Indian (Hindu-Arabic) numeral and place value system

Our place value notation is not even two thousand years old. We can trace the origins of this useful system of notation to an early Hindu numerical scheme. The Hindu number system has the elegant place value system that we use in our decimal system. The fact that the base of the system is ten is apparent from even a cursory examination of the array that follows:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10
11, 12, 13, 14, 15, 16, 17, 18, 19, 20
21, 22, 23, 24, 25, 26, 27, 28, 29, 30
....
....

One must know the base of a numeral system to understand the meaning of the number symbolism that is employed. Thus, since the base of the common system is ten, 11 is an abbreviation for ten + 1, 12 is ten + 2, and so on; 21 is an abbreviation for 2 tens + 1, 22 is 2 tens + 2, and so on. Likewise, 4273 is an abbreviation for 4 thousand plus 2 hundreds plus 7 tens plus 3, which may be written

$$4(10 \times 10 \times 10) + 2(10 \times 10) + 7(10) + 3$$

or

$$4(10)^3 + 2(10)^2 + 7(10) + 3.$$

Now suppose we want to write the symbol for three *tens*. We put a 3 into the *second* column from the right. But we won't recognize it as the second column unless we write something down in the first column. This makes it necessary to think of three tens as three tens plus *no ones*, and to introduce a symbol to represent the absence of ones. We use the symbol 0 for this purpose, and call it *zero*. The concept of a number representing *none* was also first conceived by the Hindus, and was first represented by a dot (·) and known as *shunya*. Only later on it took the form of the current symbol: 0. Zero became a new number in the natural number system, and had to be incorporated into the addition and multiplication tables in a way which is consistent with the rest of the tables.

There were several other kinds of number systems being used at the time the Hindus came out with theirs, but none of them used zero and none of them made it easy to express numbers. Although, in the Hindu oral tradition, the use of place value is quite old, but evidence of it in writing did not come until much later. The Hindu numerical system incorporated the idea of place value in the number system: this is first mentioned in the Hindu scriptures, the *Vedas*, which have been traditionally handed down orally until today from one generation to the other. They were put into written form only about two thousand years ago. The number concept and its notations in the written form, it is believed, first occurred around 1500 BC in the *–Puranas* (Hindu Epics of *Mahabharata* and *Ramayana*). The outside world, particularly the Arabs and then the West, came to know about it around 600 AD, but it was another 200 years before they were able to incorporate the Hindu symbol for zero (.) as evidence of a separate symbol for zero. Undoubtedly, the Hindu-Arabic place value system, which is employed today in most of the civilized world, had its origin in an early counting procedure that was based on the utilization of the fingers of both hands rather than one. Therefore, it involves base ten. Thus it is also known as a decimal system--from the Latin *decem* (ten) or from the Indian language Sanskrit (an ancestor to Latin and other Indo-European languages) *dasham* (ten). The basic symbols of this are called digits, implying again a historical relationship with the fingers (from the Latin word digits for fingers). In almost all cultures fingers have constituted the first system to be used as a means of counting. Incidentally, it is not surprising that all children begin counting on their fingers.

The Hindu number system became part of the Western culture when, during the Middle Ages, it was transmitted by the Arabs to Western Europe. Therefore, the name *Hindu-Arabic* number system. It appears that the Arabs, clever merchants and traders of the early Middle Ages, had adopted the Hindu scheme of numbers and the symbol for zero in preference to others then in use because it seemed to facilitate the arithmetic of their commercial transactions.

> "...[T]he fact that it took place in India is not disputed. But Indian researchers believe that the place value notation arose sometime around 200 B.C. in India without further stimulus from outside, while non-Indian scholars have substantial grounds for seeing an external stimulus behind its development. There is no conclusive evidence for another view, and there probably never will be." (Menninger, 1969)

In the Hindu numeral system the structure of the written numerals finally corresponded to that of the spoken numbers, in which there are two kinds of `numbers' --units and ranks. It is the units which impart number to the ranks.

> Around A.D. 600 a system of numerals appeared which used only the first nine digits of the Brahmi numerals--that is, only the Brahmi digits for the units. In this new system `nine hundred thirty-three' was no longer written 900' 30' 3 as in the Brahmi manner, but now only with the units in a place value notation. With this step the transition to an abstract place value was complete, and we now have in the first nine Brahmi numerals the oldest ancestors of our own digits. From now on the only differences were to be the inevitable changes in form which resulted from the fact that these numerals passed through many hands--Indian, Arabic, and Western--before they finally took on the appearance which they have today. (Menninger, 1969).

The actual symbols of our common system of notation as we know them today, however, did not become standardized until several centuries after their preliminary adoption in Europe--after printed materials had received comparatively wide circulation. The Hindu-Arabic numerals ultimately displaced others because of their great convenience. In such a system the quantity being represented by a particular symbol depends on the position of the symbol in a sequence. They are most convenient to use because they give us a way of writing an indefinite amount of numbers while using only a small number of symbols. As a result, we have numbers greater than 9 represented by numerals of several digits. (A digit is a single-figure numeral, such as 0, 1, 2, ...,9.). The feat is accomplished by attaching different meanings to the same digit.

> The springboard for the positional principle was not ordering and grouping but `encipherment,' i.e., the assigning of an individual digit to each of the first numbers. The Indian and the Egyptian numerals both overshot this target. But whereas the Egyptian system came to a premature halt because the hieroglyphs which preceded it already contained as a set of numerals that served as a model, the Indian system was merely an early stage, a first growth capable of developing further. One hundred was already a `high' number, and here the governing principle of the number system changed to a place value; at 100, too, the numerical rank clearly made its appearance. As soon as this reverted back only a level, to the rank represented by 10, a `named' place- value notation was born. (Menninger, 1969)

F. Computer and place value

Modern computing machine techniques frequently require the use of binary system of notation; that is, the base is two. If 0 and 1 are chosen as the formative symbols of a binary system--only two symbols are required--11011 would denote the number twenty seven. This follows from the fact that, in the binary system, 11011 means $1(2)^4 + 1(2)^3 + 0(2)^2 + 1(2)^1 + 1$, which is $16 + 8 + 0 + 2 + 1$ or twenty-seven.

It should be evident from these comments that the base of a number system, like many aspects of mathematics, is quite arbitrary; certainly there is nothing "natural" about base ten except that it is the most commonly used system now.

II. Evolution of zero and its role in the place value system

Zero is important both as a numeral and as a number. Zero is the first of ten symbols--the digits--with which we are able to represent any of an infinitude of numbers. Zero is also the first of the numbers which we must represent. Yet zero, first of the digits, was the last to be invented; and zero, first of the numbers, was the last to be discovered. The invention of zero preceded its discovery by centuries. Children's representation of zero and those of early cultures is relatively late in the evolution of number systems, and the concept itself has a reputation even in our culture for being hard to grasp. It is not difficult for children to represent `nothing' by zero, but it is much more difficult to use zero in a place value system. Children's difficulty and reluctance to accept zero in place-value parallels the experience of the human race. It took along time to incorporate zero in place value. It was India that gave the world zero, and with it a practical system of arithmetic notation. Incidentally, it was India that gave the concept of infinity also. It is probably the Hindu philosophy and religion that prepared the Hindus to conceptualize both zero and infinity.

In the early Hindu system, one was faced with a problem: how to cope with a number like `three hundred and four, where the `tens' column is empty. The Babylonians and Hindus first of all solved this problem by leaving an empty space. This sometimes led to confusion. For instance, the following symbol, ^ ^ ^, might have meant either 2(60) + 1, or 2(60x60) + 1, bearing in mind that the Babylonian system was based around 60. Whereas, the Hindus wrote the numbers in the expanded form (5 thousand six hundred five) using place value based ten. Then for the longest time, the Hindus used a dot `.' to denote the zero. The symbol was originally a dot, which is one of the zero symbols children use when they invent the symbol zero (Hughes, 1986). Only later on they invented the symbol `0' the one used today. The logic of making as small a mark as possible to represent zero has apparently been followed, not only by the Hindus, but also by several contemporary African tribes (Zaslavsky, 1973).

It seems that the early Babylonians (from 2000 BC to 1700 BC) must have relied on context to distinguish which meaning was intended. Records from the more recent Seleucid period (around 300 BC) suggest that the Babylonians later developed a zero symbol consisting of two small wedges placed diagonally (//). This symbol was used as a `place holder', to show that a column was empty. Thus symbol ^^//^ would represent 2(60x60) + 0(60) + 1. This `zero' was not, however, used on the right hand end of a number to show that the units column was empty. Again, we have to assume that the Babylonians relied on context to distinguish between `60' and `1'.

Cajori suggests that the Mayan civilization of south America was probably the first to use both place value and zero in a rigorously systematic fashion: this was achieved around the first century AD. The Mayan used various symbols for zero, of which one resembles a half-closed eye. A more recent article in *Science* by H.R. Harvey and B.J. Williams(1980) suggests that the Aztec Indians also had a system which used both place value and zero. These developments of the symbol zero suggest that one reason for the late appearance of zero was that it was only in a place value system that a clear need for it arose.

> It must be understood that the dot shunya which the Hindu invented was not the number zero. It was merely a mechanical device to indicate an empty space, and that was what the word itself meant--empty. ...With shunya, the symbol zero had been invented, but the number zero was yet to be discovered. (Reid,)

However, it still seems that the use of zero in such a system can cause difficulty. Flegg (1984, p.72) argues that the adoption of the Hindu-Arabic system in Western Europe was slow because the zero presented problems of comprehension: People found it hard to understand how it was that a symbol which stood for nothing could, when put next to a natural, suddenly multiply its value ten fold. It is the same type of problem that children have about the role of zero within a place value system. But it is the invention of zero that provided the real development of the place value system.

> "With the adoption of the zero, India finally attained the abstract place value notation which was about to begin its journey of conquest through the world as the most mature and highly developed form of numerals." (Menninger, 1969)

The journey of shunya (zero) from East to West is an interesting one. It went through transformations and development. The following chart summarizes the development of the concept and the word zero as it travelled the western world.

Sanskrit (from 200 B.C. to 6th to 8th cent.)
Shunya (=empty)
Arabic (9th cent.) *a-sifr* (=empty)
Latin (13th cent.) *cifra* *zefirum*
French (14th cent.) *chiffre* *zefiro-zevero-zero*
Italian/German (15th cent.) *Ziffer*
French, English *zero*

Zero was not an instant success. It was received with a great deal of skepticism. In the west, in the Middle Ages, "it was often regarded as the creation of the Devil. "Immensely superior as it was, it was not immediately accepted. Merchants recognized its usefulness while the more conservative class of the universities hung on to the numerals of the Romans and the system of the abacus. In 1300 the use of the new numerals was forbidden in commercial papers because they should be more easily forged than the Roman numerals. It was not until 1800 that they were completely accepted all over Europe. (Reid,)

> What kind of crazy symbol is this, which means nothing at all? Is it a digit, or isn't? 1, 2, 3, 4, 5, 6, 7, 8, and 9 all stand for numbers one can understand and grasp -- but 0? If it is nothing, then it should be nothing. But sometimes it is nothing, and then at other times it is something: 3 + 0 = 3 and 3 - 0 = 3, so here the zero is nothing, it is not expressed, and when it is placed in front of a number it does not change it: 03 = 3, so the zero is still nothing, *nulla figura!* But write the zero after a number, and it suddenly multiplies the number by ten: 30 = 3 x 10. So now it is something incomprehensible but powerful. If a few `nothings' can raise a small number to an immeasurably vast magnitude, who could understand such a thing? And the old and simple one-place number with its long tail of `nothings' --in short, the zero is nothing but `a sign which creates confusion and difficulties,' as a French writer of the 15th century put it -- -- *une chiffre donnant umbre et encombre.* (Menninger, 1969)

Thus the resistance to place value and the value of zero is not limited to children who encounter it for the first time but to all those who had encountered it in the past. There are pedagogical implications of this idea: the concept of zero should be introduced with care and should be accompanied with a lot of developmental concrete activities.

III. The difficulty with place value

Children show in a variety of ways that many of them do not understand the concept of place value. For example, digits are reversed when writing numbers both in formal and informal settings (writing scores while playing with dominos); inability to realize that 10 + 7 is 17; even amongst those children who are able to read numbers such as 34 as 30 + 4 many experience difficulty in reading multidigit number. This confusion about place value is more pronounced when we ask children to write 48 as 30 + ____ .

Many children frequently make mistakes which show they are unable to grasp the significance of place value. There are two types of problems:

1. Misreading numbers like 17 and 71 or 69 and 96, and understanding them as identical in meaning -- they obviously do not take place value of the digits into cconsideration. A three digit number (368, for example) is read as three isolated one digit numbers (3, 6, 8,) without realizing what these digits really mean.

2. Students should also understand the relationship between the standard or compact form and the expanded notation of any numeral: that is, that 367 = 300 + 60 + 7. Amultidigit number having digits with a large immediate numerical value (such as 6, 7, 8, or 9), begins to be evaluated as a large number, independently of the place value that these constituent digits occupy in the general structure of the number. The student may consider that 489 is larger than 701, or that 1897 is larger than 3002, although he is not in a condition to read the precise value of a multidigit number. (Luria, 1969)

The difficulty with place value becomes much morepronounced when the student is asked to write a dictatedmultidigit number, particularly

> "when the name of a multidigit number does not coincide with its digital structure (as happens whenever the final digits are not zero or whenever zeroes are not designated in speech). ...[T]he number `one thousand twenty-eight' was written as `128' (with the omission of the zero in the hundreds place) or as `100028' (corresponding to

the naive designation of the digit- order as expressed in words), and 'one thousand three' was written as 10003" (Luria 1969, p.46).

This is supported by research observations by Kamii and Lewis (1991).

> ...[I]n one test item of the place value c.uster, pupils are shown 50019, 5019, 519, and 590 and asked to mark the one that says 'five hundred nineteen.'...In another item, pupils are shown an expanded form of a number, such as 500 + 20 + 0, and the possible answers of 700, 520, 50020, and 5200. Second graders can manipulate written symbols to answer these kinds of questions without understanding the numerical value of each digit. The results from the interviews concerning pupils' understanding ... lend further support to this statement.

Although many researchers (Fuson & Briars, 1990) believe that most of the difficulties that children encounter are due to the difference in the written and spoken form of our number system and lack of adequate preparation of children in place value concepts.

The English spoken system of number words is a named- value system for the values of hundred, thousand, and higher, a number word is said and then value of that number word is named. For example, with five thousand seven hundred twelve, the 'thousand' names the value of the 'five' to clarify that it is not five ones (=five) but it is five thousands. In contrast, the system of written multidigit number marks is a positional base- ten system in which the values are implicit and are indicated only by the relative positions of the number marks. In order to understand these systems of English words and written number marks for large multidigit numbers, children must construct named-value and positional base-ten conceptual structures for the words and the marks and relate these conceptual structure to each other and to the words and the marks.

Another reason for children's place value difficulty is the irregular nature of English words for two-digit numbers and lack of relationship between the named value and written place value. For example, 'eleven,' 'twelve,' 'thirteen,''fourteen,' etc. have no relationship with named number and the written form of the number. The names are quite arbitrary. As a result, English-speaking children use for a long time unitary conceptual structures for two-digit numbers as counted collections of single objects or as collections of spoken words (Fuson, Richard, & Briars, 1982; Fuson, 1988a; Steffe, von Glaserfeld, Richards, & Cobb, 1983; Steffe & Cobb, 1988; Fuson & Briar, 1990).

Steffe et al. (Cobb & Wheatley, 1988) identified three increasingly sophisticated concepts of ten. They have identified them as: ten as a numerical composite, ten as an abstract composite unit, and ten as an iterable unit. According to these authors, it is only with the last of these units that the child uses the increments and decrements often. With this increased understanding of ten as an iterable unit that children begin to understand the concept of place value.

The accounts of children's difficulty with place value given above are true. But the question not asked in this research is: Do some of the difficulties that children encounter and demonstrate in understanding the idea of ten exist as artifacts of teaching methodologies (conceptual models) and materials (instrumental models) in teaching the concept of place value? Because we believe that some difficulties that children demonstrate with place value are due to the teaching methodologies and materials used. In our classroom teaching and clinical teaching in tutorial settings we have found these factors to be contributing factors. We need to takeinto account the teaching methodologies and models used for teaching these concepts before we can say that all these difficulties are inherently present in learning this concept.

On analysis of textbooks, supplementary materials, and research studies we find that in most teaching situations children are taught place value using sequential materials and procedures (Fuson & Briars, 1990) with a heavy emphasis on sequential counting. These experiences result in many children constructing conceptual structures for multidigit numbers as extensions of single digit numbers.

Our observation of children's work regarding this concept suggests that short term initial difficulties of these kinds are quite common in first and second grade children because (1) not enough experience with concrete materials was provided to children. (2) teachers use inefficient instructional materials and models in introducing the concept of place value. When children are introduced to the place value concept with the help of inefficient activities such as sequential counting and premature paper pencil activities instead of appropriate concrete materials, children learn to manipulate just the symbols rather than having an understanding of the concept. When we teach these children using appropriate concrete models of these concepts their difficulties are quite easily removed. This suggests that these kinds of difficulties are not actually learning difficulties. These are examples of inefficient teaching.

On the other hand,when even after detailed explanations have been given using appropriate concrete models the student continues to have difficulty, then that signifies a "breakdown of the number system" and that situation may be an indication of the presence of acalculia or dyscalculia.

IV. Teaching activities: Learning place value through number games

In this section we look at ways of introducing young children to formal place value symbolism, ways which avoid some of the problems that children encounter. We will suggest simple concrete activities and number games which can be played with such objects as dice, magnetic numerals, Cuisenaire rods, unifix cubes, and powers of ten blocks in order to help children learn the concept of place value.

These strategies are intended to show that we can help children

1. build meaningful links between the world of written number symbols and the world of concrete reality, and

2. provide examples of linking the diverse number models from discrete (counting) to continuous (visual-spatial, pattern oriented) of number relationships and place value.

We are convinced that playing simple games and using concrete activities is an ideal way to stimulate and motivate young children to have an interest in learning mathematics. We also believe that it is only when they are stimulated and motivated that children will realize their full potential.

In order to use and understand English words and base-ten written marks, children need to link the words and the written marks to each other and need to give meaning to both the words and the marks Mastering the formal code of place-value system, therefore, involves negotiating a complex of subtle and interrelated transitions. Some of these transitions can be distinguished: from actual to hypothetical situations, from concrete to abstract formulations, from spoken to written notation, symbol, and language, from embedded to disembedded thought, from the informal understanding to the formal conceptualization. Both the concrete and the formal are important, and the child who has one without the other is at a serious disadvantage. The ability to translate fluently between different modes of representation is thus of paramount importance. But for many young children the process of translation is a constant source of difficulty. The key stumbling block is in translating from concrete to formal representations. Many children stay on the concrete for unproductively long time. They need help in freeing their thinking from the concrete, and formalization is essential in the process. These translations from one form to another are better facilitated through the following:

(a) A great deal of concrete manipulative activities;

(b) Immediate recording of those concrete activities by the child facilitated by the teacher, and

(c) the teacher asking a great deal of hypothetical questions related to the concept and the activity.

Asking hypothetical questions facilitates students' understanding of ideas from concrete to hypothetical-concrete to formal mathematical symbolism—which in turn is responsible for helping children think about the concepts and retain them. At the same time, there is little virtue in asking children to master the symbolism of formal mathematics/concepts if the related concrete understanding is lacking.

The use of manipulative materials is the best way to develop the prerequisite skills necessary for the learning place value. Some of the most useful concrete materials used in teaching the place value concept are: counting materials such as counting blocks; unifix cubes, Cuisenaire rods, converted egg cartons, place value sheets, powers of ten sets, multi-base materials, chip trading materials, dominos, dice of different sizes and different number of faces, abacus. It is not the use materials that is crucial but the emphasis on establishing the links between the concrete and the formal and ease of conceptualizing the idea of place value. For example. the chip trading activities and similar games are good starters to help children acquire the concept of grouping and later on of place value. These activities help the children to experience grouping, establishment, and reinforcement of the ideas of place value. A few cautions and suggestions are in order for the optimization of the experiences.

(1) The concrete activities must link concrete models/ materials-to-pictorial-abstract/written strongly and tightly.

(2) The models to be used must be both quantitative (sequential) and qualitative (visual/spatial) in nature to meet the needs of the concept representation and students with different learning personalities.

(3) The concrete experience should be recorded immediately both in representational and written form after or during the concrete activity.

A. Discrete vs continuous models of teaching place value

Young children start school with a range of mathematical skills: they can, for example, work out simple additions and subtractions involving concrete objects, and they can invent meaningful ways of representing small quantities on paper. But understanding large number and representing them is another matter. Many of them involve quite interesting strategies. But most of them may not have the efficiency needed for generalization. They must learn, among other things, how to connect their strategies with formal ones and to use the conventional written symbols of arithmetic. Many children fail to establish connections- or translate-between this new kind of symbolism and their concrete understanding.

Ten as a numeral composite is structurally no different from the meaning given to other number words by children when they first attain the abstract. Only in the context of the place value, does ten become important. Bundling of objects in groups of tens or hundreds (making bundles of ten coffee stirrers or making tens by glueing ten objects. For example, ten beans on a wooden sticks, etc. is a common activity in classrooms to teach the concept of place value. This activity is quite successful in teaching place value for most children. But from the conceptual point of view it is not an optimal activity for many children, particularly for children with qualitative mathematics learning personalities (See Sharma, 1989). In these cases, the childrens' focus is on the constituent elements of the composite — the individual ones that make it up— rather than the on the composite itself as a single entity (Cobb and Wheatley, 1988).

Most children by overlearning (even the quantitative mathematics learning personality children) are able to learn ten as a single entity and effectively use it at the third stage described by Steffe et al. But for many other children it does not provide an appropriate conceptual model for reinforcing the concept of place value and then place value continues to be a difficult concept. It is a good introductory activity for place value but not a strong conceptual activity. It involves a heavy emphasis on counting and, therefore, reinforces the counting skill which is inefficient and ineffective in dealing with place value or any of other arithmetic operations. In order to develop the place value concept effectively one should use materials that help the students to see:

* Ten as a collection of ten individual objects (ten as a numerical composite), ten as a representation of ten objects, and ten as a numerical composite without losing its tenness (a student can coordinate count by tens and units in the same problem), and ten as an iterative unit for counting and representing large numbers;

* The need to represent large numbers easily and effectively;

* The idea that it is easier to represent the number in a place value system;

* The need to see the relative sizes of ones, tens, hundreds, and thousands; and

* That it is possible to have representations of large numbers which is free of counting processes that uses only unit counts.

In other words, we need to create almost the similar problem situations for children as were faced by the human race which resulted in the development of the place value concept. We have an advantage, though, we can anticipate where we are going.

Activities involving counting blocks, number line, bundling, etc. are examples of sequential models, whereas, Cuisenaire rods, unifix cubes, powers of ten materials, chiptrading, etc. are examples of continuous models for teaching place value. Teaching activities should, therefore, aim at developing place value using both types of models and teachers should try to interrelate the different models.

B. Prerequisite skills for place value

Understanding the concept of place value depends on all of the following important mathematical and non-mathematical concepts:

* natural numbers
* order
* counting
* unit object
* sets of objects
* sets of single entities
* sets of sets
* numerals, and the distinction between numerals and numbers
* numeration
* bases for numeration
* spatial orientation and space organization
* representation of information

Since place value is a secondary concept, it depends on several of these primary concepts, some of these being of quite a high order of abstraction (Skemp, 1989, 61-62).

C. Visual clustering, grouping and place value

Visual clustering simply means a child's ability to identify a number by looking at a cluster (arrangement) of objects. For example, when a child sees

```
* *
* *
```

(a cluster of four objects) he automatically responses with the number 4. When the early human changed the collection of five tally marks representing five objects ///// by a more compact and visually better representation in the form ~~////~~ he(she) invented the concept of grouping and representation and when he changed

```
* * * * *    to    *   *
                     *
                   *   *
```

he invented the concept of visual clustering. Both of these examples are important developments in the direction of place value. They are extremely useful ideas because they free us from sequential counting: We do not have to count five tallies or five stars as individual objects as we can see the new arrangements in one gestalt and conceive the number five. This is the forerunner of the concept of place value. Essentially over a period of time these concepts eventually developed into the place value concept: where any number, however large, can be represented by a relatively small set of numerals.

D. Idea of exchange

The concept of place value as a representation of number into a compact symbolic notation is an example of exchange. Exchange of ten units for 1 group of ten, group of ten tens into a hundred. Therefore, before children are formally taught the concept of place value, it is important to provide them experiences in grouping. Activities involving grouping such as three green marbles can be traded with one red marble, a group of three children will be called a team, a collection of ten pages will be called a book. Activities and materials such as chip trading provide an excellent example of this preparation.

E. Place value and one-to-one correspondence

The principle of one-to-one correspondence forms the unifying link between the use of fingers, children's early attempts to represent number and the idea of tallying. It also forms the basis of many of the earliest number systems, such as the Egyptian hieroglyphic system, the Babylonian cuneiform system and the more familiar Roman system and Hindu Arabic system. To invent place value required the development of one-to-one correspondence between concrete objects to their representation, between the spoken number and the written number, and between a group and its representation. Experiential activities that help children master the concept of one-to-one correspondence are essential for the understanding of place value system.

F. Spatial orientation and space organization and place value

Each numeral represents a number which may have several values depending on its place in the number (or numeral).For example, one value is determined by its position on the number line, as the numeral six comes after five and before seven and therefore it has the value six. But in the numbers: 5678, 4567, and 34567, although it appears after five and before seven it has values other than six. It represents different values in each number. The value of the numeral is determined by the place it occupies in the multidigit number. This refers to the relative aspects of the number. The place of the numeral in a given multidigit number then determines whether the numeral is representing sets of 10 or sets of units (ones) or some other higher value. The values are dependent on its position in each number. In one it represents six hundreds, in another it represents six tens. Thus six has an absolute position on the number line therefore a fixed value of six on the number line but many other relative values. Most young children focus their attention on the absolute position of the number and they have difficulty in focusing on the relative positions of the number. Because of this confusion, place value is a difficult concept for most children. Since the human race took a very long time and many concrete experiences to discover and understand this complex yet simple concept, similarly the child takes a long time and extensive explorations to understand the place value concept.

Left-right orientation

In first grade, we introduce the number line and tell children that numbers increase as we move to the right. A few days later we tell them the place values increase to the left. To understand this the child associates the place value with left-right orientation. The day the child identifies his left from right correctly he gives correct answers to place value equations, but the next day if he mixes identifying his left from his right he gives incorrect answers to place value equations. The understanding of the place value concept, therefore, is dependent on the mastery of left-right orientation. Instruction in teaching place value, therefore, must include not only the use of manipulatives and active involvement with objects, pictures and symbols but also developing the left-right orientation.

G. Description of materials

There is a variety of materials that are already in use in classrooms and many more have the potential of becoming effective tools for teaching the place value concept. Here we describe the materials that we have found to be very useful in the development of the place value concept.

(a) *Cuisenaire rods:* It is a collection of colored rods of varying lengths from one centimeter to ten centimeters. *White* (one centimeter cube), *red* (two cm x one cm x one cm), *light green* (three cm x one cm x one cm), *purple* (four cm x one cm x one cm), *yellow* (five cm x one cm x one cm), *dark green* (six cm x one cm x one cm), *black* (seven cm x one cm x one cm), *brown* (eight cm x one cm x one cm), *blue* (nine cm xone cm x one cm), and *orange* (ten cm x one cm x one cm). One tray for two children consisting of at least ten each of all the ten colors.

(b) *Base ten blocks*: (Also known as Powers of Ten Blocks or Dienes' Blocks; the Powers of Tens blocks are colored and demarcated, whereas Dienes' blocks are not colored but demarcated with unit squares on them.) It is a collection of three dimensional blocks consisting of

units (1cm x 1cm x 1cm)
longs (1cm x 1cm x 10cm)
flats (1cm x 10cm x 10cm)
blocks (10cm x 10cm x 10cm)

c) *Unifix or multi-link cubes*. These are one inch cubes of different colors which can be linked to form long links of different sizes.

(d) *Centicubes*. These are one centimeter cubes of different colors which can be linked with each other to form links. The only difference between Unifix cubes and centicubes is in size.

(e) *The Abacus:* All civilizations have used the beads on a counting board, popularly known as an ABACUS, for solving arithmetic problem. It is a very good example of the introduction of a place value system at the concrete level. Greeks and Egyptians used the abacus for counting without really knowing the place value system. It never occurred to any of them that in these same beads were the essentials of the most efficient method of number representation the world was to develop in the next two thousand years. The abacus, although it took various forms and names in various civilizations, is basically a frame divided into parallel columns. Each column has the value of a power of ten, the number of times that a particular power occurred in a total being represented by markers of some sort, usually beads. All of the beads are identical in appearance and all stand for one unit. The value of the unit, however, varies with the column. A bead in the first column has the value of one (100); a bead in the second, of ten (101); in the third, of one hundred (102); and so on. We can clearly see the resemblance between this instrument's representing numbers on the counting board and our method of representing them in writing numbers, today. These beads provide children a very good correspondence between our numbers and the representation on the abacus. In short, our place value system, where each digit has a varying value depending upon its position in the representation of number, is simply the notation of the abacus made permanent. All that is needed to transfer a number from the board to paper is ten different symbols; for there can be only one of ten possible totals in a column: one, two, three, four, five six, seven, eight, or nine beads, or no beads at all. The column can be empty, and the tenth symbol must of necessity be a symbol for such an empty column.

H. Activities to teach place value

The following activities are designed to develop the concept of place value using different types of models and materials. These are arranged in accordance with the sequential (developmental) steps necessary for the development of the concept. These activities and steps demonstrate the use of manipulative materials included above. These activities also demonstrate how to take a child from concrete to abstract.

Activity One: To introduce and develop the concept of place value. (Level: from kindergarten to third grade).

Step 1: Ask children to arrange the Cuisenaire rods in a staircase form (taking one of each color from the tray) from largest to smallest or smallest to largest. Ask them to explore and see the relations between them. Begin with identifying the white rod with number one. Then ask them to discover the value of each colored rod in terms of the white rod. For example, ask them, "How many white rods are needed to make a rod as long as the red rod?" As children answer the question, demonstrate by actually taken a red rod and then white rods next to it and then ask them to show you that fact. By this process children discover that red rod is as long as two white ones, light green is as long as three, and so on. Most children will discover the relationships quite easily. The ones who might be having difficulty in discovering these relationships you can lead to this discovery. For example, you may ask them to put two white rods next to the red one and see if that will make a train as long as the red rod. Repeat the question for other colors. For example, which one color rod will fit the train as long as three white rods, and so on. Finally help them discover that the orange rod is as long as ten white ones. After this discovery you can say that we will identify the white rod by number one, red by number two, light green by number three, and so on and orange one by number ten. They can also discover other relationships between them such as two yellow ones put together are as long as an orange.

Step 2: Ask them to identify numbers from their colors and colors for their numbers. For example:

Teacher: Show me the rod that represents number three.
Children: Light green. (Children hold the light green rod in their hands)
Teacher: Show me the rod that represents number nine.
Children: Blue. (Children hold blue rod in their hands)
Teacher: What number does this rod represent? (Holds a yellow rod in her hand.)
Children: Five.

This activity should be repeated dozens and dozens of times. When they are able to recognize the colors with numbers and numbers with colors correctly with consistent accuracy you can transfer the activity to recording and abstract levels.

Step 3: Now we can introduce children to making larger numbers with the help of Cuisenaire rods.

Teacher: Now children show me fifteen by the help of your rods.

Most children will say they do not have fifteen. Some children will make fifteen by using fifteen white rods. One or two children from the group will be able to make it by showing an orange rod and a yellow rod. Ask these children to show this to other children. Now repeat this for dozens and dozens of times for the other two digit numbers. Now by the help of the orange rods and white rods make many more numbers in two digit form. For example: Use two orange rods and three white ones to show that it represents 23. Place two orange rods under tens and three white ones or a light green one on the place value placemat. After this record this fact on the chalk-board in terms of its pictorial (draw two orange rods and a light green rod with the appropriate lengths) form and in the form of two-digit number representation (23). Ask them to represent a two digit number given to them by using Cuisenaire rods and ask them to write the number in abstract form when it is shown to them through Cuisenaire rods. This two way representation of numbers from spoken number (twenty three) to concrete representation (two orange rods and three white ones or two orange rods and one light green rod) to abstract form (23) and from abstract (represent 25) to concrete (two orange rods and yellow rod) should be done dozens of time.

Step 4: Show a number by the help of Cuisenaire rods (plastic) on the overhead projector or using jumbo Cuisenaire rods and ask the children to tell how many tens are there in the represented number, how many ones are there in the number, etc. Ask each child to answer at least one question of this type. All of this activity should be done in the oral form.

Step 5: Show a number by the help of Cuisenaire rods on the overhead projector and ask the child to read the number and then ask them to write it in the compact and expanded form. For example:

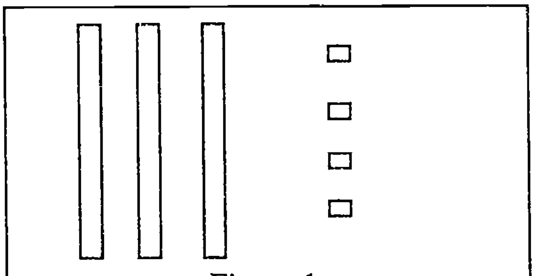

Figure 1

= 34 (three tens and 4 ones)
= 3 tens + 4 ones
= 3 x 10 + 4
= 30 + 4
= 34.

Step 6: Introduce the Cuisenaire flat (10 cm x 10 cm x 1 cm) to the class by showing it to them and then distributing at least one to each child. Ask the children: How many orange rods may fit on this flat piece. Children will have different answers. Even if one child is able to show the correct representation ask him or her to demonstrate it to other children. If not, direct them to explore the relationship. If not successful, show the children that ten orange rods or 100 white rods will make one Cuisenaire flat. Using a Cuisenaire flat, an orange rod, and ones form a three digit number and ask the children to read the number.

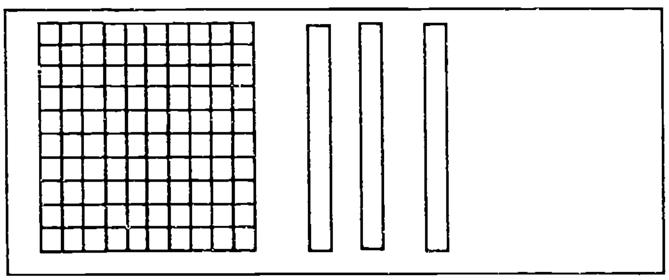

Figure 2

= 134 (one flats, three long orange rods, and 4 ones)
= 1 hundred + 3 tens + 4 ones
= 1 x 100 + 3 x 10 + 4 x 1
= 100 + 30 + 4
= 134.

Step 7: When enough practice has taken place and children are able to identify the two and three digit numbers by the help of Cuisenaire rods and place value cubes (powers of ten) and are able to give a concrete representation of a given number (by the help of Cuisenaire rods and or powers of ten), then introduce the numbers through another concrete representation (preferably a different model). We suggest the use of unifix cubes.

Show your right palm to the children and ask them "what these things are called (pointing to fingers of your right hand)?" Children will say: "fingers". You say, "yes." Then you say: "But there is another name for these fingers. In Latin, they are called digits. Therefore, we will also call them digits." Ask them to repeat the word digit. Then tell them that we will designate the thumb as one's place or one's digit. Now place a white Unifix cube on your thumb and ask them to identify the number. You can reinforce the idea that this represents number one. Put three white cubes (fastened with each other) on your thumb and ask them to read the number. Ask them how many white Unifix cubes should you place in order to represent number five. Then put several white cubes on your thumb and ask them to read the number.

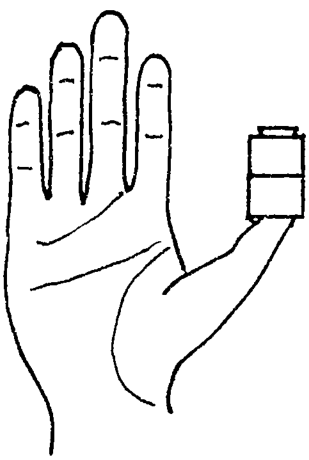

Figure 3

When they have acquired facility in this, then introduce the idea that the index finger will represent number ten (point to the appropriate finger-digit). Put an orange Unifix cube on your index finger and ask the children to read it. (All along your palm should be facing the children. This is extremely important because this will give them the correct idea of the place value as we go along.) Some children might read it as one at this point. Remind them that the index finger will represent tens and also tell them that the orange rod earlier had represented tens so we are going to use an orange cube to represent ten.

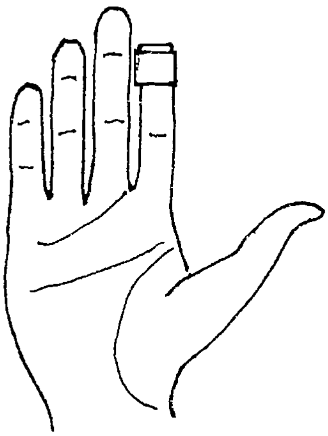

Figure 4

Teacher: I have one orange unifix cubes on my index finger (tens digit). So the correct reading is one ten or just ten. (Point to the cube and ask them to read it.)
Students: One ten.

Now put two orange unifix cubes on the index finger and read in succession: "one ten, two ten."

Teacher: What is two tens called?
Students: Twenty.
Teacher: So, two orange cubes on my tens place will represent.
Students: Twenty.

Now put several orange cubes on top of each other on the index finger and ask them to read the number represented by these cubes.

After they have acquired facility in reading tens correctly, put an orange cube on the index finger and one white cube on the thumb and ask the children to read it. If they are not able to read it correctly remind them about their individual values and tell them that the correct reading is 11 = one ten and one. Represent several numbers in a similar manner: Few numbers (one orange and few white cubes) in the tens and few numbers (two orange and a few white cubes) in the twenties, etc. Ask the children to read these numbers and ask them to help you write these numbers. This should be done for dozens of numbers and each child should get a turn in reading these numbers.

Step 8: Now you give a number and ask the children to help you make the number by the help of Unifix cubes. Again take many examples to help demonstrate the competence on the part of children.

Step 9: Now show a number on your fingers by the help of unifix cubes and ask the children to tell you how many tens and how many ones are there in the number.

For example:

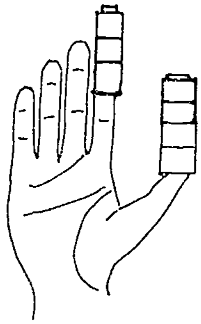

figure here

= 3 tens and 4 ones.
= 3 x 10 + 4 x 1
= 30 + 4
= 34.

Again repeat this with many examples.

Step 10: Take another color cube and put it on the middle finger (the longest finger in the hand) and ask the children to read it. Few children will discern that it is hundreds. If they do not come up with the right answer you introduce the hundreds term and ask them to read the number. Now put several cubes of the same color on the middle finger and ask them to read the number. Do it with several examples.

Step 11: Now put a few white cubes on your right thumb (with palm outstretched and facing children), a few orange cubes on your index finger, and a few red cubes on the middle finger and ask children to read the number (named-value) and then write the number (position-value).

For example:

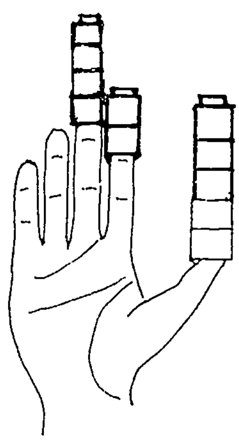

Figure 5

Teacher: Tell me the different number of cubes.
Students: 4 red cubes + 2 orange cubes + 5 white cubes.
Teacher: What number does a white cube represent?
Children: One.
Teacher: What number do five white cubes represent?
Children: Five ones .
Teacher: What number does an orange cube represent?
Children: One ten.
Teacher: What number do two orange cubes represent?
Children: Two tens.
Teacher: Or?
Children: Twenty.
Teacher: What number does a red cube represent?
Children: One hundred.
Teacher: What number do four red cubes represent?
Children: Four hundreds.
Teacher: Please read the whole number.
Children: Four hundreds, twenty, and five.
Teacher: Please write the number indicating different place values.
Student: 4 hundreds and 2 tens and 5 ones.
Teacher: Can you write it in another form?
Student: 4 x 100 + 2 x 10 + 5 x 1
Student: 400 + 20 + 5
Teacher: Another form?
Student: 425.

The teacher should show several examples of this kind: Give a number with the help of the unifix cubes and then ask the students to read it in different number forms. Then give a number and ask them to tell you how to make it using unifix cubes. This should be done with lots of examples. The best thing is to ask at least one question from each child.

Step 12: Now show a number with a few unifix cubes on the middle finger and few on the thumb but none on the index finger. For example:

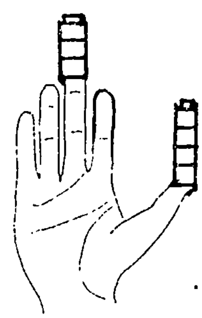

Figure 6

= 3 hundreds, no tens and 4 ones.
= 3 x 100 + 0 x 10 + 4 x 1
= 300 + 0 + 4
= 304.

Use the unfix representation to help them write the numbers.

[We have extended this activity to higher grades to demonstrate the numbers in thousands, ten thousands and later on to show tenths, hundredths, thousandths.]

Step 13: Display a few cubes on the middle finger, a few on the index finger and a few on the thumb. Now ask them how many hundreds are there, how many tens are there, how many ones are there. Then ask them what I should do to the display if I want to make it one more. Most children will be able to tell you that you need to add one more white on the thumb. Now ask if I want to make it ten more what should I add. The answer will be add one more orange cube. Do the same thing with hundreds. Ask them if I want to make it twenty more what should I add, etc.

Step 14: Display a few cubes on the middle finger, a few on the index finger and a few on the thumb. Now ask them what is the number that is one more than the number displayed. What is the number that is ten more than the number displayed? Etc.

Step 15: Display a few cubes on the middle finger, a few on the index finger and a few on the thumb. Ask them what should I take away if I want one less than the number displayed, or ten less, or hundred less. Ask them to read the number by visualizing the number in their heads after the required operation as you are displaying the number. Repeat this process with lots of other numbers where you are adding or subtracting multiples of ones or ten or hundreds or some other easier combinations.

The models and materials used in this activity provide an opportunity to construct the necessary meanings by using physical embodiment that can direct their attention to crucial meanings and help to constrain their actions with the embodiments to those consistent with the mathematical feature of the systems. The above method is more powerful than other methods including the one used by Fuson and Briars (1990) in teaching children the concept of place value.

Activity 2: Take ten pebbles. Arrange them in sets of 2, 3, 4, or 5. Ask the child: How many pebbles are there? Give me the answer in groups. What is your score? (For example: I've five sets of 2 and zero ones, etc.) Now change the initial arrangement of number of pebbles in sets and ask the same type questions again.

Activity 3: Use a converted egg carton with two positions. One child puts his/her pebbles in the units compartments on the right. How many sets of 2 can you make? As you make each set, where will you put it? (Move it to an egg space to the left.) What is your score? Repeat, counting in threes, fours and fives.

Activity 4: Extension of place value.
Materials needed: multi-base arithmetic blocks, dice.

Instructions for the activity: The game involves playing in pairs. The beginning activity may take place using base four material. Children throw a die in turn.

(a) At each throw your opponent tells you what to do. At each throw, take as many unit cubes as the score on the die. When you have four unit cubes, what should your opponent tell you to do? (Change four units for one long one.) If your opponent forgets to tell you to make a change, you can claim an extra turn. The first to complete a big cube is the winner. (When a child says: 'Take one long and one unit for a score of five', ask him to explain what he has said.)

(b) Each of you starts with a big cube. This time you are going to subtract each die score. Remember to wait for your opponent to tell you what to do. Suppose you throw a score of four, what should your opponent tell you to do? (Change the big cube for four squares. Change one square for four long ones. Remove one long one which is four units.) The first to reach exactly zero is the winner.

(c) Take a pile of material. Can you make a big cube exactly?

Activity 5: *Materials needed for the activity:* Interlocking cubes, dice, place value sheets.

Instructions for the game: Children play in pairs. They throw the die in turn. Children count their score in tens

(a) Throw the die in turn. At each throw one child takes as many cubes as his score on the die. When he/she reaches 10 (or more) he/she moves one set of 10 to the left. The first to score 30 or more is the winner. The child records his/her total in two ways (e.g., 30 + 2 = 32).

(b) In this game children start with a score of three tens. This time the child subtracts his/her score on the die at each throw. The first child to throw the dice waits the other child to tell him/her what to do. (E.g., move one ten-stick to the right and break it into units. Remove-units.) The first to reach exactly zero is the winner.

(c) This activity involves a two-colored place value board and a basket of Unifix cubes for each child. Prior to the game children have been told that 10 is the *magic number*. The only time the cubes can be snapped together is when there are ten. On a prearranged signal (a clap or a xylophone tone) each child places 1 cube on the right side of the board. Together or individually some one will read the board, "zero tens and one."

This procedure continues through many examples.

Variations are to subtract 1 from the board at each signal. This will involve breaking a ten into 10 single cubes before subtracting. One can end the activity by asking for a specific number. "Show me 53. What will 4 tens and 2 look like? What happens when you add 1 to 20 + 9? 70 plus 5 has how many tens? What is the value of the 8 in 81?

This activity can be used with any number of materials: such as one day it may be with beans and cups, on another day with the powers of ten rods. This activity can evolve into banking games involving power of tens materials and dice. Unit cubes are added or subtracted by a roll of the die. Children collect the appropriate number of blocks from the banker. Children may need to trade in 10 rods for a flat. A child who rolls a 12 may take his turn by asking for 12 unit cubes and then make the appropriate trade, a second child may do the trading mentally and ask for his rods as an orange and two whites.

Teacher directed discussion follows this activity giving the children time to reflect on their discoveries. Questions such as: What did you notice as you added cubes/beans/blocks? Did you know ahead of time when it was time to make a ten? How did you show this? Was there a pattern to the way you recorded your numbers?

Activity 6: *Materials needed for the activity*: small pebbles or other counting material.

(a) Take a collection of pebbles and estimate the number you have taken. Pattern your collection in tens and units so that I can see, at a glance, how many you have taken. How near was your estimate?

(b) Take a collection which you estimate to be 30 (40, 50, etc.). Pattern your collection in tens an units. How near were you?

Activity 7: *Material:* An abacus (two-spike) and beads of two colors to fit them, or interlocking cubes of two colors and a place value sheet, dice.

(a) Play in pairs. Throw the die in turn. Your opponent will tell you what to do. This time you exchange ten unit beads for one bead of another color (one ten-bead) before you put it on the next spike to the left. The first to get three ten-beads on the ten-spike to the left is the winner.

(b) Start with three ten-beads. This time you are going to subtract your die score in units at each throw. Wait for your opponent to tell you what to do. The first to reach exactly zero is the winner. (Later on, this game should be repeated, using beads of one color.) Record your final score.

Activity 8: *Materials:* Three dice of different sizes.

(a) Use two dice of different sizes. When you throw both dice, the larger die gives you the score in tens, the smaller die the score in units. Throw the dice six times. Each time record your score in tens and units. Then arrange the scores in order from highest to lowest, in two columns. You score one point for each ten What is your total? The highest possible.total? The lowest possible total?

(b) Extend to three dice of different sizes when the child is ready. Ask him to explain how he put the numbers in order. This time he will score one point for every hundred. The number on the third die will give number of hundreds.

Activity 9: *Materials:* strips of centimeter squared paper 100 centimeters long, two centimeters wide and colored pens.

Work in pairs. Use your number lines to measure, and to record, the following body lengths (to the nearest centimeter): the perimeters of your head, face, foot, waist, neck, wrist, and ankle. Arrange these lengths in order from longest to shortest. Work out the differences between each pair of these lengths. (This can be done by using the number lines.) Discuss your findings.

In preparation for place value notation, it is important for children to have plenty of practice in associating the written symbols and their locations with visible embodiments of tens and units (later hundreds...) and in associating both of these with the spoken words. In this topic `location'means `headed column'; later, in place value notation where there are no columns, it will mean `relative position'.

Activity 10: Its purpose is to link the spoken number words with the corresponding written numerals.

This is a game for as many children as can sit around a tray so that they can all see the tray right way up; minimum 3.

Materials needed:

Tens and unit card
Target cards
Pencil and headed paper (Place value mats) for each child
Powers of ten or base ten material, tens and units

Instructions: Rules of play

1. The target cards are shuffled and put face down.
2. In turn, each child takes the top card from the pile.
   He looks at this, but does not let the others see it.
3. Before play begins, 2 tens are put into the tray. (this is to start the game at 20.)
4. The objective of each player is to have in the tray his target number of tens and units
5. Each player in turn may put in or take out a ten or a unit.
6. Having done this, he writes on his paper the corresponding numerals and speaks them aloud in two ways. For example, he writes: 46.
7. In the above example, if a player holding a 47 target card had the next turn, he would win by putting down one more. He had achieved his target.
8. Since players do not know each other's targets they may unknowingly achieve someone else's target for them. In this the lucky player may immediately reveal his target card, whether it is his turn next or not.
9. When a player has achieved a target, he then takes a new target card from the top of the pile, and play continues.

Notes:

(a) If one side of the card is empty, a corresponding zero must be written and spoken; e.g. in figure ...

4	0

He writes 40, and speaks `four tens, zero units; forty.'

0	7

He writes 07, and speaks `zero tens, seven units: seven.'

(b) Players are only required to write the numbers they themselves make. It would be a good practice for them to write every number.

Variation

It makes the game more interesting if, at step 5, a player is allowed two moves. For example, he may put 2 tens, or put 2 units, or put 1 ten and make 1 unit, etc. This may also be used if no one is able to reach his target.

Activity 11. Relating expanded form of number to the standard form.

Materials needed

* A calculator for every two students.

* A set of fifteen to twenty cards with two- and three- digit numerals written in the expanded form.

For example:

40 + 6 = _____

70 + 9 = _____

300 + 50 + 7 = ______

As children become more confident, more challenging exercise can be given. For example:

400 + 7 = ______

30000 + 40 = _______

Instructions:

Step 1: Shuffle the cards and place them face down.

Step 2: Turn over a card and lay it face up where both students can see it.

Step 3: Both students try to write the standard form or compact name for the number represented on the card. Player one must use the calculator to find the indicated sum. Player two must use paper and pencil, but not the calculator.

As children do exercises like these, they should also read some of the numbers out loud, to be sure that they see the relationship between the expanded form and the way in which the numerals are read. For example, the sum of 300 +40 + 5 is read "three hundred forty-five." Both players keep their score. For every correct answer, the player using paper pencil gets two points and the player using the calculator gets one point. They take turns using the calculator. You can decide the total to be reached for winning the game. The one who reaches the total first is the winner.

Activity 12:

Materials needed. A calculator for every student. A set of twenty to thirty cards with examples like the following:

50 + 7 = ______

25 = _____+ _____

200 + 50 + 5 = ______

724 = ____+ _____+_____

The exercise in this set go from the standard notation to the expanded form, and from the expanded form to the standard notation.

Instructions

Step 1: Shuffle the cards and place them face down.

Step 2: Turn over a card and write the answer to the problem.

Step 3: Check your answer, using the calculator.

These activities can easily be extended to work on the place value involving decimal numbers.

V. Discussion of instructional activities

In preparation for place value notation, it is important for children to have plenty of practice in associating the written symbols and their locations with visible embodiments of tens and units (later hundreds...) and in associating both of these with the spoken words. In the beginning the location of numbers can be determinant for place value of numbers such as `headed column'; later, in place value notation where there are no identified columns, it will mean `relative position.'

These activities, described above, use concept building by physical and social experiences. The concrete component of activities use concept building by physical experience (emphasizing concrete level of knowing). These concrete experiences play an important part in the development of the concept. Drawing and recording of these activities helps students to move to representational and abstract level of knowing this concept. The social context provided by a game links these concepts with communication using both written and spoken symbols (communications level of knowing).(Sharma, 1990)

How a child is introduced to the concept and what types of materials are used to teach the concept determines the nature of child's understanding of the concept. This is quite evident by the following example: Two groups of students were presented with a card on which the following problem was written:

```
  16
+ 17
----
```

The interviewer asked the pupil to "add these numbers in your head" and to explain how he or she got the answer. Almost all pupils in both groups [children who were taught using constructivist approach and the control group] gave the answer 33. All the traditionally instructed pupils explained that they added the 6 and the 7 first, got 13, "put the `3' down here and carried the `1' there," and so on. By contrast, almost all the pupils in the constructivist group added the tens first and then the ones as follows:

$10 + 10 = 20$
$7 + 6 = 13$
$20 + 10 = 30$
$30 + 3 = 33$

The interviewer then asked the pupils to count out 17 chips. When the pupil finished, the interviewer pointed out the correspondence between the numeral 16 on the card and the 16 chips and between the numeral 17 on the card and the 17 chips. The next request was that the pupil explain, using chips, the procedure he or she had just described.[Kamii & Lewis, 1991]

According to the researchers, 83 percent of the constructivist group correctly explained regrouping with chips, but only 23 percent of the traditionally instructed group did so. The traditional group showed confusion not only in the concept formation but also in representation of their understanding of the concept. We have already pointed out that it is important to present mathematical concepts using both kinds of materials: discrete, discontinuous materials (such as: number lines, chips, counting blocks, unifix cubes, etc.) and continuous, visual-spatial materials (such as: Cuisenaire rods, unifixcubes, Dienes' blocks, powers of ten, etc.). In the Kamii and Lewis (1991) study, it seems, most of the teaching relating to place value was done using chips (a discontinuous discrete model for the concept, heavily dependent on counting). This could, therefore, be the reason for most of the children in the constructivist group could not answer place value questions correctly on standardized tests.

In the final part of the interview, the interviewer handed a sheet of paper to the pupil on which the following misaligned columns appeared:

```
 4
 35
+ 24
----
```

The interviewer asked him or her to read the answer aloud and then inquired, "Does that sound right?" On this task, the constructivist group did poorly. Only 11 percent of the constructivist group and 79 percent of the traditionally instructed group did it correctly. The result was `not surprising' to the researchers. According to them he traditionally instructed second graders did not understand place value." Our observations through our work with children using different types of models and materials shows that it is possible to have both understanding of the concept and mastery of the procedure if appropriate materials and models are used in teaching.

Place value activities with children in grades one through three using different types of models and materials should be on-going throughout the year. Each day a teacher could include calendar activities which include writing the date on an empty calendar. Children can begin to see the sequential nature of our number system. This concepts can be made clear by a daily activity of counting and recording the number of days of school that have occurred. One large can of popsicle sticks may represent the total number (180 for school days or 365 for number of days in the year) of days on the school calendar. Each school day 1 stick is removed and placed into a set of three boxes marked hundreds, tens, and ones. Each day the sticks are counted and the number recorded. On every tenth day the sticks are bundled and moved to the left from the ones to the tens box. Eventually the groups of ten will become a group of one hundred. This brief daily activity is rich with language as the names 46, 4 tens and 6, 40 plus 6, etc. are used interchangeably. Questions of the type: "Why are you writing a 4 and 6?" "Can you write a 6 and a 4 instead?" "Why not?" serve to reinforce the concept while allowing the student to use the vocabulary and make it a part of his own experiences.

Several times a week children could also be involved in estimating activities. On these occasions they can estimate a quantity of objects, verify the estimate by grouping objects by tens and hundreds, etc. This helps the child to begin thinking of ten objects as one entity. "The most basic concept children must confront is that our number system is based on the formation of groups of ten...When dealing with numbers above ten, they are required to count groups as though they were counting individual objects." (Richardson, 1984, p.133) Again, this activity provides opportunity for recording and practicing the language. The teacher may ask the students to read their estimates in more than one way: 76 =70 + 6 = 7 tens and 6.

The teacher may display a completed and an incomplete number chart (see number grid). Children use this independently as well as a teacher-directed activity. Some will use it for counting, while others will point out patterns. The incomplete chart is theirs to complete.

1	11	21	31			61			91	101	111
2											
3											
4				44		64					
5	15	25			55				95		115
6						66	76				
7											
8					58			88			
9											119
10	20	30	40	50	60	70	80	90	100		

Oral work on the Number Grid
Ask questions such as:

* `Find number three.'
* `Find the number that is ten more than three.'
* `What number will we come to if we count on ten more?'
* `and ten more.'
* `What can you tell me about the numbers?'
* `When we counted by ten more which number changed each time?'
* `If we counted on only one more, which numeral would change?'

As a final activity of the day, the teacher may give each child a completed number chart and a manila envelops. Cutting only on the lines, each child cuts his chart into 5-7 pieces. He then attempts to put his number chart back together. Puzzles are put into the envelopes, traded and tried by a friend. Eventually these number puzzles find a home in the Math Center.

Practice at the pictorial level of knowing of place value can be provided through commercial work sheets. An activity modeling quantitative aspects of representation involves each child counting how many pockets his clothing has on that day. Children are then totaled collectively, grouped by tens and counted. A qualitative activity involves pairs of colored dice with the same colored unifix cubes. Each color is given a value (e.g., yellow = 1000, blue = 100, red = 10, green = 1). The dice are rolled and the number represented by the cubes. This game is very successful in pairs. A sample roll of the dice follows:

Numbers on dice	numbers represented on the hand using unifix cubes

A place value lesson should include several activities designed to give lots of examples leading to the generalization of the place value concept followed by more examples.

Practical work here could include working on questions and activities that interest children and derived form their lives outside the clasroom.

VI. Assessment of place value concept

If we want to improve children's understanding of place value we need to change the current method of evaluation that depends on standardized achievement tests only. Such a setting tests only lower order thinking, these tests give misleading information. To find out how the pupil's scores on the achievement test correspond to their understanding, Kaami and Lewis (1991) interviewed them on their understanding of the concept. Each pupil was first shown a card with the numeral 16 written on it and asked to count out sixteen chips. The interviewer then drew an imaginary circle around the 6 in the 16 with the blunt end of a pen and asked,"What does this part [the 6] mean?" " Could you show me with the chips what this part [the 6] means?"....The interviewer then circled 1 in the 16 and asked, "What about this part [the 1]? could you show me with the chips what this part [the1] means?"

The analysis of these interviews showed that children who were taught using very little concrete manipulative materials demonstrated discrepancies between their answers on standardized tests (higher scores) and understanding using concrete materials (lower scores). Kaami and Lewis explain the "reason for the contradictory finding is that achievement tests tap mainly knowledge of symbols.

In the final part of the interview, they handed a sheet of paper to the pupil on which the following misaligned columns appeared:

```
  4
 35
+ 24
-----
```

The interviewer asked the pupil to ' read these numbers' and then to write the answer. When the pupil finished, the interviewer asked him or her to read the answer aloud and then inquired, `Does that sound right?'According to the researchers, 79 percent of traditionally instructed students and only 11 percent of students who were instructed using `constructivist approach' wrote 99 by `mechanically following the rule of adding each column. 'Kamii and Lewis reason that this`discrepancy between the findings of the achievement test and those of the interviewers is that the achievement test evaluates only if pupils can solve problems presented in the conventional form that have been taught. Achievement tests do not ask if pupils understand why their answers are correct, nor do they ask what pupils do when a problem is presented in an incorrect form.

We believe, this problem is present in both cases: Children taught in traditional ways and constructivist approach. The key element is effective models. We need to use both models: discrete and continuous to reflect children's mathematics learning personalities (Sharma, 1979). For example, the above explanation is not only true in this case of traditionally taught students but it is true in the case of students who were taught using constructivist approach. These students were taught place value concept using only one concrete model (counting model). The materials used are: number line, counting chips, etc. The number line and counting materials have been a useful, if cumbersome tool, but they do not help children translate the concrete to abstract representation easily. As a result, students continue to use the counting model (counting chips or counting steps on the number line) to arriving at answers to the place value problems. Had they been exposed to other models using materials such as: Cuisenaire rods, unifix cubes, etc. (The exercises described above.) They would have responded much better on these tests and other activities relating to place value questions. □

References

Biggs, E. (1989). *Teaching mathematics 7-13: Slow learning and able pupils.* Windsor, England: The NFER-NELSON Publishing Co.

Burns, M., & Tank, B. (1988). *A Collection of Math Lessons.* New Rochelle, NY: The Math Solution Publications.

Cajori, F. (1982). *A History of Mathematical Notations.* Volume 1: Notations in Elementary Mathematics. La Salle, Il: Open Court.

Cobb, P., & Wheatley, G. (1988). Children's initial understanding of ten. **Focus on Learning Problems of Mathematics**, 10(3), 1-28.

Flegg, G. (1984). *Numbers: Their History and Meaning.* Harmondsworth: Penguin.

Harvey, R. (1982). Learning in mathematics. In Torbe, M. (Ed.), *Language, Teaching and Learning, Volume 6,* Mathematics. London: Ward Lock.

Kamii, C., & Lewis, B.A. (1991). Achievement tests in primary mathematics: Perpetuating lower-order thinking. **Arithmetic Teacher**, 38 (9), 4-8.

Luria, A.F. (1969). On the pathology of computational operations. In J. Kilpatrick & I. Wirszup (Eds.), *Soviet **Studies in the Psychology of Learning and Teaching Mathematics. The Learning Mathematics Concepts, Volume I.*** School Mathematics Study Group, Stanford University and Survey of Recent East European Mathematics Literature. Chicago: University of Chicago.

Menninger, K. (1969). *Number words and number symbols*: A cultural history of numbers. Cambridge, MA: MIT Press.

Ore, O. (1948). *Number theory and its history.* New York: McGraw-Hill Book Company.

Reid, C. ( ). *From zero to infinity.*

Sharma, M.C. (1989). Mathematics learning personality. **Math Notebook**, , .

Sharma, M.C. (1990). Levels of knowing. Math Notebook, ,

Skemp, R. R. (1989). *Mathematics in the primary school.* London, UK: Routledge.

US Department of Education. (1986) *What Works: Research about teaching and learning.*

Zaslavsky, C. (1973). *Africa Counts.* Boston, Mass: Prindle, Weber and Schmidt.

FOCUS *ON LEARNING PROBLEMS IN MATHEMATICS*

FOCUS on Learning problems in Mathematics is an interdisciplinary journal, edited jointly by the **Center for Teaching/Learning of Mathematics (CT/LM)** and the **Research Council for Diagnostic and Prescriptive Mathematics (RCDPM).** The objective of FOCUS is to make available the current research, methods of identification, diagnosis, and remediation of learning problems in mathematics. Contributions from the fields of education, psychology, mathematics and medicine having the potential for impact on classroom or clinical practice are valued. Specifically, the types of manuscripts sought for FOCUS include reports of research on processes, techniques, tools and procedures useful for addressing problems in mathematics teaching and learning: descriptions of methodologies for conducting, and reporting and interpreting the results of various types of research, research-based discussions of promising techniques or novel programs; and scholarly works such as literature reviews, philosophical statement or critiques.

Published four times a year: Winter, Spring, Summer, Fall. For editorial, advertising, publications and subscription information write to: CENTER FOR TEACHING/LEARNING OF MATHEMATICS, P.O. Box 3149, Framingham, Massachusetts 01701. (508)877-7895 or (617) 235-7200. Fax Number (508) 788-3600.

Subscription rates:	Individual	Institutional
1 year	$25.00	$40.00
2 years	$40.00	$55.00
3 years	$52.00	$67.00

Back issues are available at $8.00 per issue. Double issues $12.00 . Outside the North American Continent, add $8.00 extra for mailing a subscription. Checks or international money orders in U.S. funds drawn on U.S. Banks only. Others will be assessed an extra conversion ana handling charge.

Manuscripts for publication should follow the guidelines in the Publication Manual of the American Psychological Association (Third Edition) and be submitted in triplicate to the editor: Editor, FOCUS ON LEARNING PROBLEMS IN MATHEMATICS, P.O.. BOX 3149, FRAMINGHAM, MA 01701. Manuscripts determined to be appropriate will be sent, with author identification removed, to at least two appropriate reviewers. Manuscripts will be reviewed for their uniqueness, significance, accuracy and potential for impact.

CENTER FOR TEACHING/LEARNING OF MATHEMATICS
P.O. BOX 3149
FRAMINGHAM, MA 01701.