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ABSTRACT

The use of a visual representation in learning about proportional relations was studied, examining students' understandings of the invariance of a multiplicative relation on both sides of a proportion equation and the invariance of the structural relations that exist in different semantic types of proportion problems. Subjects were 49 high-ability sixth-grade mathematics students using the Jasper Adventure Series of problems. The first research question was whether the provided visual representation would influence student understanding of proportion. Students spontaneously transferred an extrapolation strategy to a second problem domain, suggesting that instruction facilitated a recognition of the structural equivalence in the two domains. The second question was whether the visual representation would interact differently with student reasoning on the part/whole and rate problems. Students were more successful using the representation presented during rate instruction on part/whole problems than vice versa. Differences in student reasoning tentatively suggest a sequence of instruction in which visually-mediated instruction with rate problems precedes that of part-whole problems. Three figures present study findings. (Contains 8 references.) (SLD)

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Visual manipulatives for proportional reasoning

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1. Objectives

The current experiment investigates the use of a visual representation in learning about proportional relations. In particular, we examine student understanding of two kinds of invariance that play a role in proportional reasoning: (1) the invariance of a multiplicative relation on both sides of a proportion equation; and (2) the invariance of the structural relations that exist in different semantic types of proportion problems. The goal of our work is the design of learning environments that facilitate a move from implicit to a more explicit understanding of proportionality.

2. Theoretical framework

Structural invariance under certain transformations is a central concept in mathematical thinking (Behr, Harel, Post & Lesh, 1992). For example, the question of whether 2/3 and 4/6 are equivalent is a question of invariance of a multiplicative relation (Lesh, Post & Behr, 1988). Similarly, Inhelder and Piaget (1958) claim that the shift from preproportional to proportional reasoning involves the recognition of the structural similarity on both sides of the proportion equation.

There are two types of invariance or structural equivalencies on which we focus in the study of proportional reasoning. One is the multiplicative relation that is preserved in the transformation of a ratio into an equivalent ratio. This can also be conceived of as the preservation of a structural invariant in the two ratios (Lesh et al., 1988). A second type of invariance is the structural equivalence between different types of ratio semantics such as rate and part-whole. Understanding the applicability of proportionality in multiple contexts is a prerequisite to both qualitative and quantitative proportional reasoning (Lamon, 1993).

We facilitate movement from implicit to explicit understandings of these two types of structural equivalencies through the use of a qualitative and manipulable visual representation. We use a qualitative representation because qualitative understanding precedes and is necessary for the ability to manipulate numerical proportions (Inhelder & Piaget, 1958). Moreover, rote knowledge of symbols and procedures often interferes with students' attempt to build on their informal understanding (Mack, 1990). We use a manipulable visual representation because it can emphasize the invariant relationships within and between ratios. When an appropriate visualization of a ratio is transformed into a visualization of an equivalent ratio, the spatial relationship between the two quantities remains invariant, although the overall size of the representation changes. This makes it more likely that the invariant ratio will be explicitly recognized and articulated by the student, thus becoming a mathematical object (Cobb, Yackel & Wood, 1992) for the student.

In pilot interviews we found evidence that the qualitative contrast of variant and invariant transformations in a visual representation can help sixth-graders explicitly grasp the relational structure of proportions. In the following example, Mike has drawn a bar chart to represent the number of students who will and won't buy a ticket in a sample, and he has extrapolated that relationship in a second bar chart of the population. Below, the interviewer asks Mike what is variant and invariant in his sample and population representations. Mike initially struggles to articulate what remains invariant.

Int: what changed?
Mike: the length [of a bar representing the number of students who will buy a ticket]
Int: what didn't?
Mike: the amount that would buy a ticket

The interviewer pointed out to Mike that he has indicated on his representation that the number of students who will buy a ticket in the sample and population is different, and then again asks:

Int: what changed?
Mike: the amount of people [who will buy a ticket]
Int: what didn't?
Mike: the fraction of how many people are going to buy and how many aren't

Notice in the first interchange that Mike has answered one question with a property of the representation and another in terms of the quantities being represented. In the second interchange, he focuses solely on the quantities being represented, and successfully discriminates between the magnitude of quantities and the relationship between quantities. In the current study, we examine whether a specifically designed visualization supports a shift, such as Mike's, to a more explicit understanding of proportional invariances.

3. Representation and problem types

The representation used in the current study has been designed to make the structural similarity or dissimilarity between two ratios available for inspection (Figure 1). Because of the importance of proportional reasoning in distinct semantic domains, the visualization is employed for two ratio semantics: (1) part-whole (i.e., given the part of a sample who says they will buy tickets, extrapolate to find how many will buy tickets out of the population); (2) rate (i.e., given the flow rate of a hose, extrapolate to find how long it will take to fill a large tank).

There are two differences between the part-whole and rate problems that may influence the effect of the visual representation. One is the greater complexity of the relationships between the quantities in part-whole problems. The part-whole visualization (Figure 1a) satisfies the complex set of relational constraints contained in a part-whole ratio. For example, consider comparing the part-whole ratios *a/b* and *c/d*. The *a* (e.g., people who will buy a ticket) can be meaningfully subtracted from *b* (e.g., total people) , resulting in an interpretable quantity ~*a* (e.g., people who will not buy a ticket). Consequently, ~*a* and *a* can be placed in relationship to the whole (*a/b* & ~*a/b*), as well as placed in relationship to each other (*a*/~*a* as in an odds ratio). Given the same set of relationships between *c* and *d*, there is a large space of possible relationships that the student must coordinate. In contrast, in a rate problem the *a* and *b* terms are from distinct measurement spaces, yielding a smaller space of relational combinations. For example, one does not normally think of subtracting gallons from minutes. Thus, in Figure 1b, the relative sizes of the boxes expressing a single rate is arbitrary. This reflects the unimportance of the additive relationship between numerator and denominator. The second difference is that the quantities in rate problems form an intensive quantity (Schwartz, 1988), which is itself a well-known entity. The ratio of gallons per minute refers to a concept that is familiar to these students – speed. Thus, they already have a conceptual entity to which the ratio can refer.

The current study asks two questions: (1) Will our visual representation influence student understanding of proportions? (2) Does our visual representation interact with student learning differently in the part-whole and rate domains? One hypothesis is that students will have more trouble with rate problems when using visualizations because there are less constraints on the construction of ratios. An alternative hypothesis is that students will have more trouble with part-whole representations because they have a larger space of potential relations.

4. Method

Forty-nine high-ability sixth-grade mathematics students participated. For the two crossed between-subject factors the students were counter-balanced by ability. One factor, base-domain, determined if the student learned to use the visualization in the context of rate or part-whole problems. This allowed us to assess student conceptions of proportionality and to assess the visualization in the two domains. The second factor, switch, determined if the student would get an additional day using the representation in the alternative base-domain. For example, a "switch" student in the rate base-domain spent an extra hour with an instructor using the representation in the part-whole base-domain. A "no-switch" student received no additional instruction. This manipulation tests whether instruction in two domains is better than one and whether one domain better prepares the student for the alternative domain.

We collected two types of data: (a) problem-solving and (b) visualization. (a) The problem-solving measures created an additional pair of within-subject factors: pre-post performance and problem-type. On both pre- and post-tests students solved the same rate and part-whole problems. In the rate problem, the students were given two hose rates and had to determine which would fill a tank of unknown size faster. In the part-whole problem, the students were given two participation rates (found from a survey sample) and had to determine which would yield more total participants in a population of unknown size. Because these problems were more complex than those taught in class, they tested whether students could generalize their learning. (b) Prior to instruction we asked the students to create their own visualizations of like-ratio proportion problems for rate or part-whole problems. These visualizations reveal the relations that students are able to represent in the two base-domains. After instruction, we asked the students to use our visualization to represent a proportion from the alternative base-domain. This provides a second opportunity to evaluate the relational invariances students represent in each domain.

Prior to the intervention students spent one week working on complex a video-based mathematics problem from *The Jasper Adventure Series* (Cognition and Technology Group at Vanderbilt, 1992). Embedded in the larger video problem were two sub-problems that required extrapolating a ratio to a second ratio. In one sub-problem, students used the flow rate of a hose to determine how long it would take to fill a tank of a specific size. In the other sub-problem, students determined the expected level of participation at a school given a sample participation rate (found from a survey). The Jasper adventure created the context for our intervention. Prior to instruction in the visualizations students worked on the pre-test. Then one author spent three days teaching each base-domain using the visual representation in a classroom setting. Figure 2 provides a sample of the constructions and extrapolations that the students constructed. For example, in the ratio base-domain students were given a flow rate of 10 gallons per 1 minute. They were then asked to show how to extrapolate to 6 minutes without using numbers. Then they had to prove visually why 10 gallons per 2 minutes would take longer than 10 gallons per 1 minute. After classroom instruction, structured interviews were individually conducted with the switch students from each class in the alternative base-domain. After the week needed to interview the switch students, all students took the post-test.

5. Results

Here we report the pre-post differences on the problem-solving measures, and develop an interpretation of the visualizations students constructed. The student solutions to the rate and part-whole problems were coded for accuracy and strategy use with 98% agreement. In the first analysis, the number of problems with correct solutions and methods was compared to the number of problems with either incorrect answers or methods. A four-way multivariate analysis completely crossed the within-subject factors of problem-type and pre/post-test with the between-subjects factors of base-domain and switch. Figure 3 shows that, overall, rate problems were solved more accurately on pre- and post-tests ($F(1,45)=44$, $p<.001$) suggesting that rate problems are easier. The only pre/post effect was an interaction with problem-type, base-domain, and switch ($F(1,45)=6.7$, $p<.05$). On the rate problems, the rate base-domain students improved while the sampling students' performance declined. On the sampling problems, the rate students' performance did not change, while the sampling switch students' performance declined and the non-switch sampling students improved. One interpretation is that rate problems serve as a better initial context for learning about proportional reasoning with our visualizations.

A second analysis examined whether students switched strategies. Here we discuss two strategies: a reduction strategy and an extrapolation strategy. In a reduction strategy, two unlike fractions are reduced to a common denominator (e.g., 4.2 gals. per min. vs. 6 gals. per min.). In an extrapolation strategy, two unlike fractions are extrapolated to a common numerator or denominator (e.g., 42 gals. per 10 min. vs. 42 gals. per 7 min.). The simplest pre/post effect involved the increase in the use of an extrapolation strategy, shown in Figure 4. On both problem types, students in both conditions used an extrapolation strategy more frequently after instruction ($F(6,38)=9.4$, $p<.001$). This means that the students transferred the idea of "stretching" (or, multiplying) a ratio to find a missing value (as in Figure 1), to stretching two ratios to test their comparability. There was also a complex domain by pre-post by problem-type by solution strategy interaction ($F(6,38)=12.3$, $p<.001$). On the post-test compared to the pre-test, sample base-domain students used less reduction strategies on the rate problems, while rate base-domain students used less reduction strategies on the sampling problems. This suggests an intriguing transfer across problem types for which we do not currently have an interpretation.

The students' initial visualizations provide some indication of why rate problems were solved better overall. For the part-whole problems, 57% of the students successfully represented *a/b* and *c/d*, but only 21% of the students represented any connection between the ratios. In particular, they tended to make a separate representations for each ratio – often strikingly non-proportional (Figure 5a). For the rate problems, none of the students incorporated the *a/b* or *c/d* into one representation, but 54% represented the relationship between the ratios. They tended to make representations that indicated how accumulations of water and time would lead to filling of the larger tank (Figure 5b). Thus, the rate representations tended to foreshadow the extrapolation transformation that we used in our instruction.

The final task was for the students to use the representation they learned in their base-domain to represent a proportion in the alternative base-domain. 75% of the switch students were able to do this, showing that the representation is easily transferred across domains with moderate instruction. Only 8% of the non-switch part-whole students could represent the fractional aspect of a rate. For the most part, these students created a single object that stood for both gallons and time (Figure 5c). In contrast, 43% of the non-switch rate students were able to represent the $a/b = c/d$ structure of the proportion. Interestingly, they did this by treating the numerator and denominators as separate entities of arbitrary relative size (Figure 5d). Thus, they did not attend to the part-whole relationship so much as they attended to the proportional relationship between a/b and c/d.

In the current study, we were interested in two questions. The first was whether our visual representation would influence student understanding of proportion. The key finding for this question was that students spontaneously transferred an extrapolation strategy to second problem domain . This suggests that our instruction facilitated a recognition of the structural equivalences in two domains.

The second question was whether our visual representation would interact differently with student reasoning on the part/whole and rate problems. The key finding for this question was that students were more successful at using the representation presented during rate instruction on part/whole problems than vice versa. The rate students were able to make this transfer because the relationship between the part and whole can be ignored in a successful representation, as in Figure 5c. The part/whole students, however, were unable to find a way to represent the relationship between time and water in a single box. One interpretation is that once the part/whole students had learned a representation that captured a complex part/whole structure, it was difficult for them to use the representation when the size of the bottom part of the visualization did not constrain the size of the top part. However, these difficulties were overcome with the minimal instruction provided to the switch students.

6. Educational implications

In the current research we have examined the potential of a manipulable visual representation for highlighting the structural invariances within a proportion and the proportional invariances between domains. Our preliminary analyses indicate that this approach leads to an understanding that transfers to more complex proportion problems. The differences in student reasoning with the two ratio semantics tentatively suggest a sequence of instruction in which visually-mediated instruction with rate problems precedes that of part-whole problems. This is supported by the current results, despite the authors' intuitions that part whole ratios have a more satisfactory spatial representation. We are currently implementing a computer-based learning environment which would facilitate manipulation of the visual representations, as well as integrate numerical and spatial representations.

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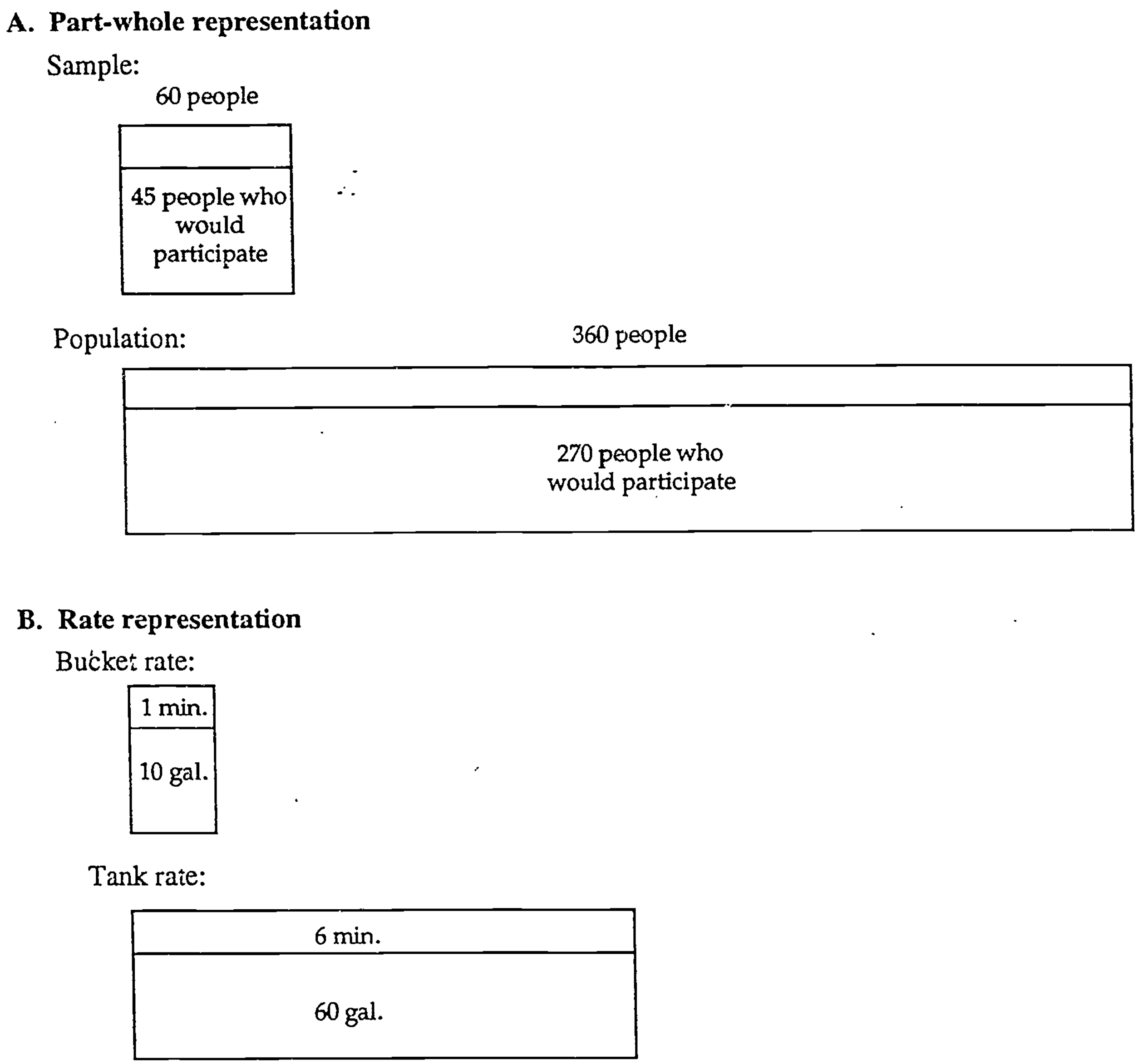


Figure 1. Visual representations of part-whole and rate proportion problems.

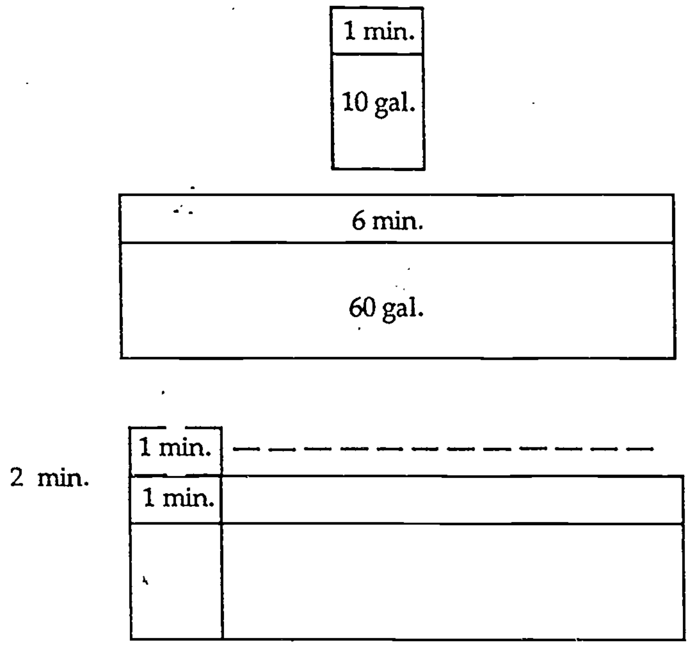
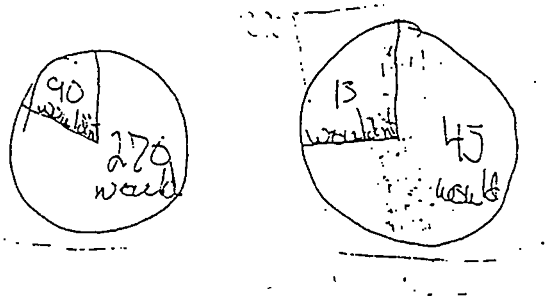
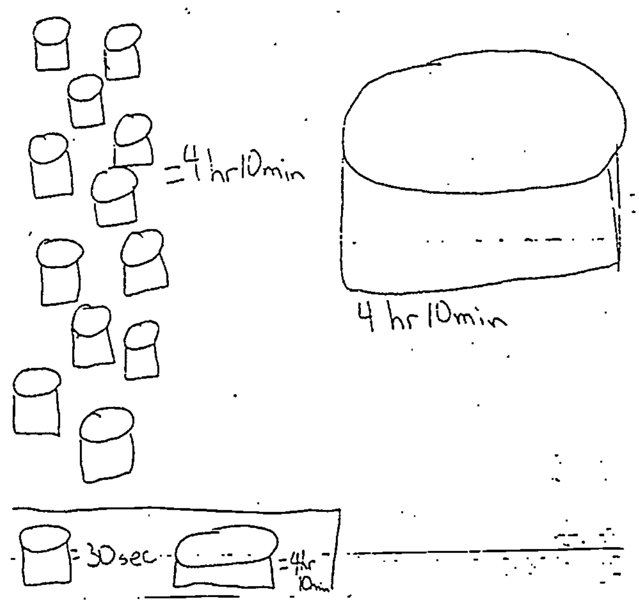


Figure 2. An example of a visual proof to demonstrate that 10 gal./2 min. is a slower rate than 10 gal./1 min.

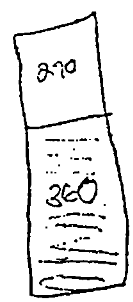
3a. An uninstructed representation for a part-whole problem.



3b. An uninstructed representation for a rate problem.



3c. A rate student's representation of a part-whole problem using the box formalism.



3d. A part-whole student's representation of a rate problem using the box formalism.

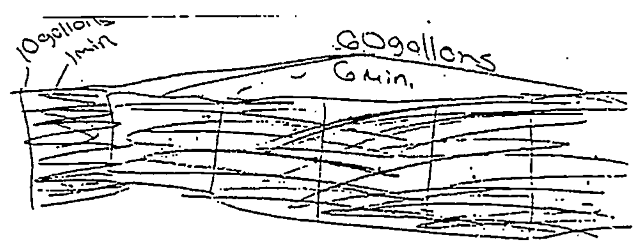


Figure 3. Sample visualizations created by students before they were taught the box representation and after they had learned the representation in a different semantic domain.