DOCUMENT RESUME

ED 374 988 SE 054 944

AUTHOR Chiu, Ming Ming
TITLE Metaphorical Reasoning in Mathematics: Experts and Novices Solving Negative Number Problems.
PUB DATE Apr 94
NOTE 67p.; Paper presented at the Annual Meeting of the American Educational Research Association (New Orleans, LA, April 4-8, 1994). Some hand written figures may not copy clearly.
PUB TYPE Reports - Research/Technical (143) -- Speeches/Conference Papers (150)

EDRS PRICE MF01/PC03 Plus Postage.
DESCRIPTORS Diagrams; Grade 7; *Graduate Students; Integers; Interviews; Junior High Schools; *Junior High School Students; Mathematics Education; *Metaphors; Middle Schools; *Problem Solving; *Thinking Skills
IDENTIFIERS *Metaphorical Thought; *Negative Numbers

ABSTRACT

Metaphorical reasoning explains how people can interpret abstract representations through a complex activity and then apply them to new problems. In particular, metaphors can facilitate both conceptual understanding and problem solving by: (1) intuitively justifying mathematical operations, (2) integrating mathematical knowledge, (3) enhancing the computational environment, and (4) improving recall. In this study audiotaped interviews of (n=12) novice middle school students and (n=5) expert master's graduates solving three tasks involving negative numbers were analyzed. Through a variety of spatial and quantitative metaphors, these students reasoned metaphorically, not only to understand and solve these problems, but also to evaluate and justify their solutions. Experts articulated more metaphors and reasoned with them selectively. In contrast, novices employed metaphorical reasoning less skillfully, but they used it more frequently. Appendices include: Arithmetic is Motion Along a Linear Path metaphor, Stock Market problem, Ordering problem, and Images of Arithmetic Expressions. Contains 85 references and 12 figures. (MKR)

Metaphorical Reasoning in Mathematics:

Experts and novices solving negative number problems

Ming Ming Chiu

Education, Math, Science, and Technology

University of California - Berkeley

Running Head: Metaphorical Reasoning in Mathematics

I would like to thank Andy diSessa, George Lakoff and Alan Schoenfeld for helpful discussions. I also appreciate the comments on earlier drafts provided by Joshua Gutwill, SungWook Joh, Bruce Sherin, and Emily van Zee.

Abstract

Metaphorical reasoning explains how people can interpret abstract representations through a complex activity and then apply them to new problems. In particular, metaphors can facilitate both conceptual understanding and problem solving by a) intuitively justifying mathematical operations, b) integrating mathematical knowledge, c) enhancing the computational environment, and d) improving recall. My empirical analysis examined audiotaped interviews of twelve novice middle school students and five expert masters graduates solving three tasks involving negative numbers. Through a variety of spatial and quantity metaphors, these students reasoned metaphorically not only to understand and solve these problems but also to evaluate and justify their solutions. Experts articulated more metaphors and reasoned with them selectively. In contrast, novices employed metaphorical reasoning less skillfully, but they used it more frequently.

Metaphorical Reasoning in Mathematics: Experts and Novices Solving Problems with Negative Numbers

Students who believe mathematics is full of meaningless symbols unrelated to daily life have a historical precedent set by the distinguished logician, Augustus de Morgan (1898). He wrote:

> The imaginary expression $\sqrt{-a}$ and the negative expression (-b) have this resemblance, that either of them occurring as the solution of a problem indicates some inconsistency or absurdity. As far as real meaning is concerned, both are equally imaginary, since 0-b is as inconceivable as $\sqrt{-a}$ (p. 73).

de Morgan and his contemporaries had a consistent set of rules for arithmetic operations with negative numbers, so he had the necessary algorithmic knowledge to solve problems (Boyer, 1991; Crowley & Dunn, 1985). However, de Morgan lacked an interpretive framework in which negative numbers were meaningful. Many researchers have focused on the algorithmic rules and the notation used for manipulating negative numbers (Musser, 1972; Tatsuoka, 1984; Tatsuoka, Birenbaum & Arnold, 1989; van Engen, 1972), but few researchers have analyzed subjects' conceptual understanding.[1] In contrast, this study examines, in fine grained detail, experts and novices solving a variety of problems by metaphorically imposing meaning on negative numbers.

This study addresses three central questions. What contribution does metaphorical reasoning offer to mathematical development? What types of metaphors do people employ to solve problems involving negative numbers? And how do subjects reason metaphorically?

This paper begins with a theoretical discussion of metaphorical reasoning. Next, the methodology is described, followed by examples of the subjects' metaphors. Then, I compare novice and expert metaphorical reasoning. Finally, this paper concludes with a discussion of the results and their implications.

Theoretical Framework

This section examines mathematical thinking both as cutting down to the bare essentials of a problem (through abstract representations) and as building up and using additional resources (in complex situations). Then, through the mediation of metaphorical reasoning, I explore how people learn to manipulate abstract representations through a complex activity and apply them to new problems.

Abstract Representations

The standard translation view of applying mathematics (see Figure 1) generally includes four stages: a) selecting the essential features for abstract representation, b) applying mathematical rules to produce a result, c) translating the result back to produce a meaningful solution, and d) verifying the solution within the situation's constraints (Davis & Hersh, 1981).

Insert Figure 1 about here

Abstract representations are symbols or displays that facilitate problem solving because they a) index key features of the situation with a compact form and b) can be transformed by a set of mathematical rules applicable in many situations. The compact forms facilitate human processing by reducing the memory and attention needed to apply the mathematical rules. Consider the following situation: Is $500 enough to purchase six chairs, each costing $83? Figure 2 illustrates two solutions showing that $500 is sufficient.

Insert Figure 2 about here

In contrast, using only English words complicates not only the solution but also subsequent verification attempts:

> Eighty-three multiplied by six is eighty times six plus three times six. Eighty times six is four hundred and eighty. Three times six is eighteen. Four hundred and eighty plus eighteen is four hundred ninety-eight.

Furthermore, the mathematical rules apply across a diverse range of situations, so that changing the particulars (chairs) typically does not alter the computation. For example, a virtually identical process can be employed for calculating whether six gardens, each with perimeters of 83 feet, can be enclosed with 500 yards of picket fencing. Thus, problem solvers can use a common set of mathematical rules to manipulate compact, abstract forms across diverse situations.

To facilitate students' manipulation of these abstract forms, several researchers have designed new mathematical representations (Reussner, 1993; Shalin & Bee, 1985). For example, Shalin and Bee (1985) advocate using symbolically marked boxes to represent different types of quantities such as extensive (e.g. 5 zebras, 3 quarts of milk) and intensive (e.g. 90 km/hr, $1 /gallon). To represent a problem, a person creates a network of individual quantities and their arithmetic relations (See Figure 3).

Insert Figure 3 about here

The few viable combinations of quantity types and arithmetic operations provide additional constraints to steer users away from errors. These researchers generally advocate teaching optimal abstract representations to improve problem solving proficiency. Upon mastery of these compact forms and their accompanying rules, students can then apply them to a diverse range of situations.

Complex Situations

Educators and researchers who focus on the manipulation of abstract representations risk minimizing the importance of complex problem situations. By applying rules to abstract forms in isolation, students may not know how to cut down to the essential features of the situation. They may ignore critical aspects of the problem or apply abstract forms in inappropriate situations. Even worse, the translation view does not characterize the solution of complex problems such as carpeting an entire house. Creating and specifying problems, prioritizing different goals, and exploiting multiple perspectives are missing from the translation view. In contrast, researchers examining complex situations examine how people can improve their problem solving by building up and using additional resources. This section discusses how added complexity can facilitate both conceptual development and problem solving.

Facilitating Conceptual Development. A complex situation provides valuable resources for comprehension of abstract representations. During an activity (e.g. Saxe's (1991) Treasure Hunt game), teachers and students begin with common reference points (gold, ships) and agree upon common goals (search for gold, accumulate wealth). Moreover, they can solve problems that arise through experiential analogs of mathematical representations (computing change for a purchase). By executing actions that correspond to arithmetic operations (exchanging money for goods) and documenting them with mathematical representations (displaying the current assets e.g. "3024"), students may make sense of the mathematics through the activity. By providing students with common reference points and motivating them with meaningful goals, activities encourage them to interpret novel abstract forms through meaningful actions.

In many complex problems, one must choose the appropriate representation(s). Standard textbook exercises primarily drill manipulations of isolated mathematical representations (e.g., Dolciani, Sorgenfrey, & Graham, 1985; Graham & Sorgenfrey, 1983; Saxon, 1985; Sobel, Maletsky, Golden, Lerner, & Cohen, 1986; Yanker, Vannatta, & Crosswhite, 1981). Applications (in the form of word problems) typically provide protruding key features that cue the mathematical representation covered in the current chapter. When faced with a complex problem however, students may find that the appropriate representation is no longer transparent (Cobb, Yackel & Wood, 1992). Instead, they must identify both the critical aspects of the problem and the mathematical representation(s) that will yield productive manipulations. By beginning with a complex problem that embeds representations within the activity, students are more likely to identify other situations in which these representations can be applied productively (Saxe, 1991; Cobb et al., 1992; Williams & Kamii, 1986). Additional complex problems can help students refine these application conditions. For example, the exchange rate of dollars to dimes remains fixed but not dollars to British pounds. Consequently, instruction within a complex activity may help students identify application conditions for particular mathematical representations.

Finally, regular use of complex situations helps students extend both the form and the content of their mathematical competencies. Instead of simple computational exercises or skeletal word problems, complex problems provide students with the opportunity to create integrated mathematical views. In addition, they can incorporate mathematics learned in previous units, thereby utilizing a broader range of mathematical skills and concepts. Complex problems enable students to practice and to evaluate a wider range of their mathematical abilities in more diverse situations.

Facilitating Problem Solving. Complex situations can also provide useful resources for problem solving. Instruction that emphasizes manipulations of abstract representations risk encouraging students to choose a mathematical form prematurely. Although word problems can be quickly translated into a mathematical representation, tackling a complex problem may require elaboration and exploitation of the setting's resources.

Several researchers have shown that people solve problems concerning concrete, familiar situations more easily than their isomorphic, abstract counterparts (Cummins, Kintsch, Reusser & Weimer, 1988; Hudson, 1983; Reusser, 1985). Moreover, van Dijk & Kintsch (1983) and Johnson-Laird (1983) argue through situation models and mental models respectively that this effect stems from people imagining the situation holistically to facilitate their problem solving. In particular, Hall, Kibler, Wenger, & Truxaw (1989) show that many successful students solving algebra word problems (e.g. where do two trains moving at different speeds towards each other meet?) created additional resources (e.g., drawings and narratives) to reason directly within the situation. Hall et al. (1989) call this phenomenon model-based reasoning, and characterize four uses of it: preparatory problem comprehension, problem solving, evidence gathering, and error recovery. Thus, elaborating the situation facilitates problem solving by providing additional constraints and resources.

People may also exploit materials in their immediate surroundings. In Lobato (1991), a subject explained that if he had single dollar bills, he would place them on particular stationary store items to determine how much he could afford. Thus, he reduced a potential arithmetic problem to a correspondence problem. (See Carraher (1986), Greeno (1991), Lave (1988), Scribner (1986), and Williams & Kamii (1986) for additional examples). Capitalizing on available resources can reduce the complexity of the problem.

In short, examining complex problems can facilitate both mathematical development and problem solving. Activities can provide common reference points for students and teachers, often motivating students to learn abstract representations through meaningful goals and actions. Furthermore, students learn to identify application conditions for each mathematical representation. By solving complex problems regularly, students can both practice and evaluate a wider range of their mathematical abilities in more diverse situations. Students can also elaborate the problem situation to create additional constraints and resources. Finally, they can exploit available resources in the problem setting to reduce the complexity of the problem.

Metaphorical Reasoning

Activities provide meaningful complex situations for students to build understanding of abstract representation manipulation, but the connection between meaningful action and abstract form remains unclear. In particular, examining complex situations can not explain the successful use of simple abstract forms in future problems. Metaphorical reasoning explains how mathematical understanding is both initially built up from complex situations and then eventually trimmed down for diverse application.

This section begins with an introduction to metaphorical reasoning. Then, I specify several dimensions along which to organize different metaphors, followed by a discussion of the benefits and limitations of metaphorical reasoning. Finally, I contrast metaphorical reasoning against other types of reasoning.

What is Metaphorical Reasoning? Reasoning through a metaphor, such as VARIABLES ARE BOXES WITH NUMBERS INSIDE,[2] views a less familiar target situation (variables) through the lens of a familiar source situation (boxes) (Black, 1979; Lakoff & Johnson, 1980; Johnson, 1987; Lakoff 1987; Pimm, 1987).[3] In general, metaphors are a) unidirectional, b) alignments (not similarities), and c) imaginative constructions. Since people do not use the unknown to make sense of the known (people do not use variables to understand boxes), metaphors are unidirectional. Bi-directional perspectives that view the connection as the locus of meaning (Lesh, Landau, & Hamilton, 1983) fail to account for the absence of metaphors such as UP IS FUTURE, CONTAINERS ARE MINDS, etc. (Lakoff & Johnson, 1980; Lakoff & Turner, 1991). Furthermore, the source and target need not share any common attributes that facilitate the person's metaphorical understanding. On the contrary, comprehending a metaphor may engender creativity and imagination on the part of the learner. To make sense of the VARIABLES ARE BOXES metaphor, a student (let's call her Ana) must impose her understanding of boxes on to variables, namely that the contents of the boxes are a) not always the same, b) visible if the value of the variable is known and c) hidden from view if the variable's value is unknown. This relationship is not an inherent and transparent similarity between boxes and variables, but a constructed alignment of the two situations that enables the person to make sense of the target (variables).[4] Consequently, Ana may engage in a process of negotiation with the teacher (and other students) to determine an appropriate alignment between the source and the target. Then, she imposes her understanding of the source (boxes) on to the target (variables) metaphorically to understand and to solve algebraic problems.[5] In general, metaphorical reasoning builds understanding of an unfamiliar target phenomenon by imposing the identity of a familiar source on to it.[6]

General Dimensions of Metaphorical Reasoning. Metaphors are personal constructions, typically with idiosyncratic understandings. Consequently, they do not easily lend themselves to strict classifications. Nevertheless, one can contrast them along six dimensions, three concerning potential understanding (source comprehension, target comprehension, and systematicity) and three regarding actual use (operational level, detail, and automaticity).

A person's metaphorical comprehension depends upon the degree of prior comprehension of both the source and the target as well as the systematicity of connections between them. Upon hearing the metaphor PRIME NUMBERS ARE PRIMARY COLORS (Noldor, 1991), a layperson may simply view prime numbers as less complex than other numbers. If he does not recognize the phrase "primary colors," he may not understand the metaphor at all. In contrast, a person who understands how combining primary colors can generate other colors can build a metaphorical understanding of prime numbers as generators of composite numbers. Therefore, person's potential for metaphorical understanding of the target depends on his comprehension of the source.

People's prior target comprehension also affects their metaphorical reasoning. In the absence of any target knowledge, a person imposes the identity of the source situation on to the target, so that the target has all the attributes of the source.[7] For example, a novice reasoning through PRIME NUMBERS ARE PRIMARY COLORS may believe that because the number of primary colors is finite, the number of primes is also finite. However, mathematicians know that there are an infinite number of primes.[8] Thus, a person can use their target comprehension to recognize the limitations of a particular metaphor. Moreover, existing target knowledge can compete with the metaphor and discourage its use altogether. In Resnick & Omanson (1987), several students learned an exchange metaphor for multi-digit subtraction, but used their previous faulty algorithms to solve problems. Existing target knowledge curtails metaphorical reasoning through a particular source.

Metaphorical reasoning is also a function of the metaphor's systematicity, the number of correspondences between the source and the target. Intricate links may connect a metaphor's source and target as in AN EQUATION IS A BALANCED SCALE.[9] Both sides of the equation (scale) must have the same value (weight). Adding (placing) or subtracting (removing) the same values (weights) from both sides does not change the equality (balance). Reflexivity, symmetry, and transitivity of values (weights) among other correspondences also apply.[10] In contrast, PROBABILISTIC OUTCOMES ARE PATH BRANCHES has relatively few correspondences (at least for the author). Of the many possibilities (destinations) for a given event (region), only one outcome will occur (one path will be traversed) with a particular probability (path width). In short, the degree of metaphorical reasoning depends on the intricacy of correspondences between the source and the target.

Actual use of metaphorical reasoning also falls along three dimensions: operational level, detail, and automaticity. Operational metaphors yield specific actions that correspond to mathematical thinking. Consider how Ana can metaphorically reason via GEOMETRIC FIGURES

ARE PATHS to find the perimeter of a polygon. After choosing a starting point, Ana can walk along the sides of the figure, adding each length, until she returns to the starting point. In contrast, FUNCTIONS ARE MACHINES does not specify actions and does not facilitate mathematical operations. This is probably the most critical dimension for students since the operational level of a metaphor describes the number of productive metaphorical actions, and hence the metaphor's utility.

Metaphorical reasoning occurs in different degrees of detail. Ana's initial reasoning through the AN EQUATION IS A BALANCED SCALE metaphor may include working on an actual balance to solve algebraic problems. As Ana becomes more proficient with this metaphor, she recognizes that she uses some aspects more than others. Although she frequently performs the same operation on both sides of the equation (balanced scale), she rarely utilizes her understanding of the "=" as a fulcrum. Consequently, we would expect a person learning a novel metaphor to display elaborate metaphorical reasoning, but an experienced user to exhibit less detail while solving the same problem.

Metaphorical reasoning may also be automated through frequent use to create autonomous target reasoning. By recalling only the results in the target, the person does not reason metaphorically through the source. Consider SIMILAR POLYGONS ARE UNIFORM STRETCHES. By reasoning through this metaphor, Ana learns that growing (or shrinking) a triangle by a constant factor creates a perfect match for any mathematically similar triangle, and also yields proportional corresponding sides. Later, she may use the proportional sides result to solve a word problem without imagining a growing (or shrinking) triangle. Strictly speaking, she is no longer reasoning metaphorically. Unlike other forms of automation (such as Anderson's (1985) knowledge compilation) in which the original relationships are lost however, the familiarity of the source allows people to recreate its metaphorical alignment with the target. [11] People can both automate their metaphorical reasoning and retain access to it.

In short, metaphors are personal constructions that can be compared and contrasted along dimensions concerning potential understanding (source comprehension, target comprehension, and systematicity) and dimensions regarding actual use (operational level, detail, and automaticity). During metaphorical reasoning, the person frames his understanding of the target through a known and understood source. While greater source comprehension increases a person's potential metaphorical reasoning, greater prior target comprehension tends to curtail it. The systematicity dimension describes the intricacy of the connections between the source and the target, thereby capturing the extent of metaphorical reasoning. Meanwhile, the operational level of a metaphor characterizes its utility through the number of productive metaphorical actions. Through experience, a person learns to adapt the level of detail to the demands of the problem. Finally, a person may automate his metaphorical reasoning to create autonomous target reasoning, but he retains access to the metaphor through a familiar source.

Benefits of Metaphorical Reasoning. As productive pedagogical tools, metaphors can a) build upon intuitions, b) enhance the computational environment, c) integrate mathematical entities, d) facilitate memory recall, e) form composite models and f) provide a form of limited assessment. To demonstrate each of these benefits, I shall use an extension of the "number line" metaphor which I have named ARITHMETIC IS MOTION ALONG A LINEAR PATH (see Appendix A).[12] In this version, the default direction is to face the positive direction. A subtraction operation indicates that the person should turn around, and a negative number indicates that the person should walk backwards.

Consider how the ARITHMETIC IS MOTION metaphor helps students make sense of "7 - (-2)" by building on their spatial intuitions. Typically, students learn this by rote (subtracting a negative is adding a positive). Building on the familiar experience of walking however, this metaphor tells Ana to walk forward seven steps (to location "7"), turn around, and walk backwards two steps (to location "9"). By turning around and walking backwards, Ana moves as if she were walking forward. Note how a person reasoning through this metaphor can both distinguish the minus sign from the negative sign and understand their relationship. Ana's capacity for metaphorical reasoning through ARITHMETIC IS MOTION depends on her understanding of the source, motion. In general, people can readily build metaphors upon their intuitive understanding of familiar sources.

Reasoning through the ARITHMETIC IS MOTION metaphor also facilitates problem solving by enhancing the computational environment. Firstly, a student can metaphorically compute arithmetic expressions by counting steps. As discussed in the prior paragraph, Ana can compute by taking metaphorical steps. In addition, a student can also evaluate solutions through this metaphor's spatial constraints. Any negative answer to the problem "7 - (-2)," such as "-5," must be incorrect because a person can not take two steps from "7" and reach a location in the negative region. Therefore, students can solve problems through metaphorical computations and evaluate their solutions through metaphorical constraints. Moreover, these actions and constraints determine the operational level of a person's particular metaphor, while the detail and automaticity dimensions capture the efficiency of a person's metaphorical reasoning.

People can also integrate target (mathematical) entities metaphorically through the source situation (motion) if they understand the source sufficiently and develop adequate systematicity across the source and the target. Consider multiplication and division via ARITHMETIC IS MOTION. For any multiplication, M x N (such as -7 x 4), the first number indicates the number of steps and the second indicates the step size. Hence, multiplication is repeated addition (repeated steps). If M is negative (-7), then the person must turn around initially. So, -7 x 4 can be computed by turning around to face the negative direction and taking seven steps, each 4 units long, to reach location "-28." If N were negative, the person walks backward. A division problem (P/Q) asks how many steps of size Q a person should take to go to P. Therefore, division is the opposite of multiplication, as addition and subtraction are directional opposites. Note how this metaphor can help students understand the distinction between 4/0 and 0/0.[13] For 4/0, the student must go to location "4" with steps of size 0, which is impossible. For 0/0, the student can go to "0" with any number of steps of size 0. Since there are an infinite number of answers, not a unique one, the expression "0/0" is indeterminate. Thus, metaphorical reasoning can integrate arithmetic operations (+, -, x, /) and mathematical notions of impossibility and indeterminateness through actions and relationships in the source.

The coherent integration of mathematical entities through the source situation and increased systematicity also facilitates memory recall (Reynolds and Schwartz, 1983). Consider the difference between a teacher telling Ana that 0/0 is indeterminate and Ana engaging in a motion metaphor-based activity. When told by her teacher, Ana memorizes the fact. In the motion metaphor activity, she can access additional methods such as recalling the activity or the metaphorical actions to regenerate the result. Metaphors built on intuitions differ from other forms of automated knowledge because people can readily access familiar experiences to regenerate the alignment between the source and target. People can use metaphors to improve their recall of both facts and procedures through their memory of richer meaningful experiences and through re-derivation by metaphorical operations.

People can employ multiple metaphors either simultaneously or sequentially. Consider how Ana uses several metaphors to solve the problem X + 6 = 4. She may think about this problem as a balance with a box and six weights on one side and four weights on the other, thereby reasoning through both the VARIABLES ARE BOXES WITH NUMBERS INSIDE and the EQUATION IS A BALANCED SCALE metaphors simultaneously. In this <u>composite metaphor</u>, the box has weights inside rather than simply numbers.[14] In general, composite metaphors tend to have greater systematicity. To solve the problem, she must isolate the variable (box) and move all the numbers (weights) to the other side of the equation (scale). Since Ana knows that subtracting (removing) the same number (weights) from both sides does not change the equality (balance), she subtracts (removes) six from both sides. Subtracting (removing) six (weights) from the left side leaves the variable X (box), but how does she subtract six (weights) from two (weights)? She may employ the ARITHMETIC IS MOTION metaphor. By drawing a number line, Ana metaphorically solves "4 - 6" by moving forward four steps, turning around, and taking six steps to reach "-2" (see Figure 4).

Insert Figure 4 about here

(Note that the EQUATION IS A BALANCED SCALE metaphor treats number as a weight, but the ARITHMETIC IS MOTION metaphor treats number as a displacement.) People may reason through multiple metaphors both simultaneously (forming composite metaphors) and sequentially.

Finally, students may employ metaphorical reasoning to assess their new understanding of the target by creating metaphors with different sources. They can ask themselves the following questions: What is the most systematic and detailed metaphor I can create using matter and anti-matter as a source domain for arithmetic? What are the limitations of the metaphor? In general, a student can assess his understanding of a mathematical topic by generating new, systematic metaphors and recognizing their limitations.

In short, students can benefit from metaphorical reasoning in at least six ways. Teachers can help students build metaphors upon their intuitive understanding of familiar experiences. Then, students can solve problems through metaphorical computations and evaluate their solutions through metaphorical constraints. By exploiting the coherence of the familiar source experience(s), students can integrate target mathematical relationships. This integration of mathematics also facilitates recall through the readily accessible source. Students can also reason through multiple metaphors both simultaneously and sequentially. Finally, students can generate additional metaphors to assess their mathematical comprehension.

Limitations of Metaphorical Reasoning. Metaphors can obscure, omit, and contradict standard mathematics because they build on sources that fundamentally differ from the target mathematics. Firstly, metaphors do not provide equal attention to all relevant mathematical relations. Consider multiplicative inverses in the ARITHMETIC IS MOTION metaphor. To compute the multiplicative inverse (2/3) in traditional mathematics, Ana takes its reciprocal (3/2). In contrast, metaphorically computing the multiplicative inverse does not highlight its reciprocal nature (how many steps of size 2/3 should you take to reach 1?). Furthermore, this metaphor omits the mathematical notion of multiple levels of infinities. At best, positive infinity and negative infinity are the locations furthest from the origin in each direction. Finally, metaphors can contradict traditional mathematics. As discussed earlier, PRIME NUMBERS ARE PRIME COLORS suggests that there are only a finite number of primes. Consequently, students must construct additional target knowledge (through other metaphors or other means) to compensate for the shortcomings of reasoning through a particular metaphor. As discussed earlier, prior target comprehension can curtail metaphorical reasoning by overruling inappropriate metaphorical entailments. Since metaphors can obscure, omit, and even contradict standard mathematics, treating them as absolute rules rather than tools for inquiry leaves one vulnerable to these potential pitfalls.

Summary of Metaphorical Reasoning. Metaphorical reasoning explains how abstract representations can be understood through prior experiences. In general, metaphors are imaginative interpretations of a target situation using selected aspects of an alignment between the source and the target. Furthermore, people's metaphors fall along several dimensions: source comprehension, target comprehension, systematicity, operational level, detail, and automaticity. In the course of learning and problem solving, students can reason metaphorically to build upon intuitions, enhance the computational environment, integrate mathematical entities, facilitate memory recall, form composite metaphors and assess their mathematical understanding. Since metaphors build on sources that fundamentally differ from the targets however, metaphorical reasoning can downplay, omit, and contradict aspects of traditional mathematics.

Abstract Representations, Complex Situations, and Metaphorical Reasoning

Students need not isolate abstract representations, complex situations and metaphorical reasoning from one another. Although instruction can employ each one individually, educators can coordinate them to maximize students' mathematical understanding. Abstract representations, such as mathematical symbols, serve as easily manipulable compact forms that index more complex information. Acting as place holders, they allow people to mark important information and direct their attention to other aspects of the problem. Meanwhile activities provide students with important source experiences. By directing students toward particular relationships, teachers can highlight source knowledge amenable to metaphorical reasoning in the target. Let us begin with an activity in a complex situation such as marching in a parade. By providing common reference points (motion, direction) and by motivating students with meaningful goals (preparing for a parade), an activity enables students to perform meaningful actions (marching in step). Abstract representations such as mathematical symbols (e.g., 3 - 5) are embedded within the activity to index particular actions (walking backwards). Then, through metaphorical reasoning, students interpret the manipulation of mathematical representations (target) through the meaningful experiences of the activity (source). In the previous example, "3 - 5" is understood as walking forward three steps, then turning around and walking five steps. When faced with a novel situation, people may solve the problem metaphorically through the parade activity. Consider a group of children acting as bank tellers and customers. How much money remains in a person's account after a series of financial transactions? Although the students know that arithmetic operations on negative numbers are necessary, they are not yet familiar with the algorithmic process. Therefore, they turn to the parade's marching actions to guide them through the steps. By forming a transitive metaphor, intuitive understanding of earlier activity -> mathematical symbol use -> manipulation in new problem, the students rely on their understanding of the earlier activity to perform mathematical calculations to solve the current problem (see Figure 5).

Insert Figure 5 about here

In this view, every transfer to a new problem entails a metaphorical mapping (e.g., mathematics -> new problem), so that the application of a particular mathematical algorithm to a new situation can not be assumed. The solution process relies in part on the ease with which the problem solver constructs the metaphor and reasons within it. As students automate their metaphorical reasoning through frequent use, they rely less on the original activity and can use the results quickly as facts. However, students still retain access to their metaphorical understanding through their familiar source experiences. Later on, the limitations of their metaphorical must be addressed as well (perhaps through another metaphor). Of course, choosing appropriate activities, metaphors and representations are each challenging tasks. Nevertheless, integration of all three holds out the promise of both intuitive understanding and proficient problem solving.

<u>Contrast with other Types of Reasoning</u>

To further clarify metaphorical reasoning, I contrast it with three other types of reasoning: a) situated reasoning b) example-based reasoning, and c) symbolic mnemonics. In addition, I demonstrate how the analogical reasoning view fails to account for the phenomena I have described as metaphorical reasoning.

Situated reasoning exploits the immediate environment directly. Gibson (1966, 1979) argues that a person or an animal's environment encourages or <u>affords</u> particular behaviors. Moreover, these affordances are relational and dependent on the features of the person that engages in the activity. For example, a sturdy, knee-high rock affords sitting more so than a shoulder-high boulder. Several researchers (Brown, Collins & Duguid, 1989; Lave, 1988; Lobato, 1991; Scribner, 1986) emphasize the frequent utilization of their environment to solve math problems. In Lave (1988), a person obtains 3/4 of 2/3 of a cup of cottage cheese without any arithmetic computation. Using a measuring cup, he scoops out 2/3 of a cup of cottage cheese. Then he divides it into four sections with a knife and removes one of them. Like situated reasoning, metaphorical reasoning also relies on human experience but invokes a prior situation to frame the current problem.

Reasoning through examples (Neves, 1981; Rissland, 1985), like metaphorical reasoning, also uses a prior situation. However, an example-based reasoner searches for a prior source that is virtually <u>identical</u> to the current problem situation, unlike metaphorical reasoning which acknowledges a non-isomorphic source. After mapping the appropriate entities from the source to the target, the example-based reasoner tries to repeat his actions in the prior solution. For example, consider a textbook problem with a given solution:

> A train must travel 300 miles to Chicago. If it moves at 75 mph, how much time will pass before it reaches Chicago?
>
> 300 miles/ 75 mph = 4 hours.

Example-based reasoning succeeds in word problems such as:

Ana is driving at 60 mph. How much time will Ana need to reach her aunt's home which is 240 miles away?

At a strictly computational level, the student can simply replace numbers. For example, 75 has "mph" next to it and so does 60, so replace 75 with 60. Similarly, replace 300 miles with 240 miles and divide (240/60 = 4 hours). However, problems such as the following are more difficult:

Pedro must travel 300 miles to reach his uncle's house from work, and he drives at 60 mph. If he has already driven for an hour, how much longer will it take for him to reach his uncle's house?

Instead of incorporating the hour already traveled, the example-based reasoner substitutes the numbers as before to obtain 300/60 = 5 hours rather than 4 hours. Since example-based reasoning simply replicates a prior solution, the problem solver is vulnerable to differences between the source and the target.

Symbolic mnemonics rely on the serendipity of the notation to encode a memory aid. They are independent of the underlying conceptual meaning and depend solely on perceptual cues and transformations. Alliteration (square's sides same) and rhymes (Yo! five times five/that's twenty-five) can help students remember mathematical relationships. Or, students may learn that 7 - (-2) = 7+2 by imagining that one of the -'s rotates itself ninety degrees and moves on to the other "-" to form a "+." Like metaphors, symbolic mnemonics depend on external aids outside of the current situation. However, they employ serendipitous notation and language, not familiar experiences.

Advocates of analogical reasoning claim that a person learns through analogical reasoning to view the target in a more abstract way, namely through an abstraction of common relations of the source and the target (Gentner, 1989; Gick & Holyoak, 1980, 1983; Holland, Holyoak, Nisbett, & Thagard, 1986; Holyoak & Thagard, 1989). This claim has both theoretical and empirical shortcomings. This abstraction view fails to account for numerous metaphors and for the genesis of higher order relations in the target. Lakoff & Johnson (1980) and Lakoff & Turner (1989) argue that this abstraction view fails to explain spatial and structural metaphors that have no common intersection such as MORE IS UP, HAPPY IS UP, LOVE IS A JOURNEY, SEEING IS TOUCHING, etc. Moreover, to recognize the common higher order relations, a person must develop them in the target before analogical reasoning can occur. If a person does not know much about the target (or has a non-normative understanding of it), he can not reason analogically. Therefore, analogical reasoning requires significant, prior, normative understanding of the target as well as the source. Empirically, Gick & Holyoak (1983) demonstrate that their abstractions (principles, summaries and diagrams) facilitate, but do not necessitate problem solving in limited situations. Moreover, at least 50% of the subjects in each experiment solved the problems analogously without abstractions, whether produced by the researchers or by themselves. Abstractions may help, but they are neither necessary nor sufficient for analogical reasoning. In short, the abstraction view fails to explain entire classes of metaphors, requires significant prior understanding of the target, and provides reasoning tools that are neither necessary nor sufficient for analogical reasoning.

In summary, metaphorical reasoning differs from situated reasoning, example-based reasoning, symbolic mnemonics, and analogical reasoning. Unlike situated reasoning which only relies on the resources in the current environment, metaphor reasoning requires employment of a prior situation. Although example-based reasoning also uses a prior situation, it searches for an identical problem to replicate the prior solution. As a result, example-based reasoning is vulnerable to critical differences between the two problems. In contrast, metaphorical reasoning acknowledges the inherent differences and creates an imaginative interpretation of the current situation by aligning aspects of it with a prior situation. Symbolic mnemonics rely on a notation's serendipitous characteristics, whereas metaphors rely on a person's meaningful experiences. Finally, the analogical reasoning through abstraction view fails to explain entire classes of metaphors, requires significant prior understanding of the target, and provides reasoning tools that are neither necessary nor sufficient for analogical reasoning.

Research Questions Regarding Negative Number Metaphors

The empirical portion of this paper explores the nature of the subjects' metaphors and their uses while solving problems involving negative numbers. In particular, which metaphors, if any, do they invoke? How detailed are they? How do their metaphors differ from one another?

Also, how do subjects' metaphorical reasoning facilitate their problem solving? Do they metaphorically compute results? Do they evaluate their progress through metaphorical constraints? Do they metaphorically justify their solutions? Can they readily access these metaphors? How reliable are these metaphors?

Method

This study employed semi-structured interviews (Bernard, 1988) to examine the subjects' metaphorical reasoning through a triangulation of verbal, written and behavioral data (Jick, 1983). The analysis consisted of two major sections: identifying different metaphors (and their variations) and coding subjects' metaphorical reasoning. After incorporating subjects' gestures (Kendon, 1981, 1990; McNeil, 1992) from my field notes into the transcripts of the audio tapes, I employed open coding (Strauss, 1987) to collect comments and gestures that could be construed as parts of metaphors (Lakoff & Turner, 1991). Through axial coding (Strauss, 1987), I refined the metaphor categories by examining variations among the subjects' metaphors. Then the second half of the analysis coded the subject's metaphorical reasoning as: understanding the problem, computing results, evaluating their progress, and/or justifying their solutions. In addition, the analysis included coding the accessibility and the reliability of their metaphors. Data sources included audio-taped interviews of 12 middle school students and 5 masters graduates, their written work, and field notes. (Videotape is preferable, but the author lacked access to such resources.)

Procedure

Each subject worked alone on three consecutive tasks in the given order. The interviewer encouraged the subjects to describe their solution process by thinking aloud, occasionally prompting for greater detail. Although pen and paper were provided, the subjects were not required to use them. After each subject completed the set of tasks, the interviewer asked for clarification of unfamiliar actions.

Participants

The novices differed from the experts both in schooling and in academic ability. The twelve novices were private school students from a seventh grade algebra class. Six to eight weeks before their interviews (which spanned three weeks), they completed a unit on arithmetic computations with negative numbers. According to their math teacher, they were mostly of average math ability. (On the negative number exam, all scored above 80%, two above 90%).

The math teacher described her instruction on negative numbers as "pretty consistent with the textbook."[15] After she had introduced negative numbers with temperatures below zero and owing money, she presented the number line and the special rules for negative number arithmetic. The students practiced the computations, but were not given any word problems to solve.

In contrast, the five experts were all high academic achievers as indicated by their masters degrees. Furthermore, each of them completed at least two years of college mathematics, suggesting a high level of mathematics sophistication as well. Aside from number lines, these subjects could not recall much about their classroom instruction.

All subjects were paid for their participation.

Tasks

This study examined the subjects' metaphors and their uses through a triangulation of data from three tasks. I chose three tasks that were likely to induce metaphorical reasoning. The first problem, set in a stock market, required the subjects to perform several arithmetic computations with negative numbers. The second, an ordering problem, can be solved quickly by reasoning through a spatial metaphor (ARITHMETIC IS MOTION). Finally, the last task asked them directly for their images of negative numbers and ways of teaching the topic.

The first half of the stock market problem (See Appendix B) asked the subject to calculate the day's earnings or losses given a particular set of transactions. The interviewer provided a short written description of buying and selling metals (gold, silver, copper or platinum) to which subjects could refer at any time. The transactions listed on a computer printout included both the number of ounces of a metal either purchased or sold and the change in price per ounce. The solution entailed multiplying the ounces by the change in price per ounce for each transaction and totaling the results. On this particular day, the overall result was a loss. In the second half of the problem, each subject could break into the computer account and change a single number by 5 (+5 or -5) to maximize their profit. The second question asked for the number of different ways in which the person wins money overall.

The second problem asks the subject to order the following six numbers from least to greatest: 15, -7/8, -21, 4, 1/6, -3 (see Appendix C). One possible solution is to find the least number through pair-wise comparisons using mathematical rules (e.g. positive numbers are greater than negative ones), then find the least of the remaining numbers and so on until all the numbers are ordered.

In the last task, the interviewer asks the subject for images of various expressions such as -5 + 8, 7- -2, -2 x 3, and -2 x -3 (see Appendix D) and then to simulate teaching each expression to a sixth grade friend.

Coding and Sco

This sec· ·on describes the coding scheme and presents examples of each category. After describing evidence for a metaphor, I discuss the coding for particular types of metaphorical reasoning and for the metaphors' accessibility and perceived reliability.

Evidence for a subject employing a metaphor included a) introducing a different situation, b) coherence within the invoked situation, and c) using an inference pattern from the invoked situation in the problem situation. For example, Ana might say, "Minus forty minus ten is minus thirty [-40 - 10 = -30]. Wait, I'm moving ten to the left, so it'll be more negative." By introducing motion into the problem, she interprets "-10" as "moving ten to the left." The spatial inference of moving further into a region manifests itself in arithmetic as a "more negative" result. Subjects' gestures (bouncing her finger to her left ten times) and drawings (hatch marks on a line) provided additional evidence. If Ana said, "On the calculator, -40 goes down to -50" however, she would not be metaphorically solving the problem. Although she refers to a different situation, she recalls only the result, not an inference pattern. Her use of "down" suggests that she interprets the relation between the numbers through a MORE IS UP metaphor, but she does not invoke a spatial inference pattern to compute the result. Hence, she may understand the situation spatially, but does not solve the problem spatially. (Since students can alternate between arithmetic facts (5 + 3 = 8) and metaphorical operations at any time, each metaphorical operation between two numbers was counted as a separate instance of metaphorical reasoning.)

This study examines whether people metaphorically a) understand problems, b) compute results, c) evaluate via metaphorical constraints and d) justify answers. For example, a person may metaphorically understand the ordering problem (15, -7/8, -21, 4, 1/6, -3) as identifying relative location on a number line. To compute the answer metaphorically, Ana draws a line and places each number at its appropriate location. She may employ metaphorical constraints to evaluate the status of her solution, for instance, "negative numbers must be on the left side of zero." Finally, she may justify her answer on the basis of this spatial metaphor, "1/6 is greater than -21 because 1/6 is on the right side of -21." Another person, say Brett, may solve it differently. Brett understands the numbers as greater and lesser values, not locations. To solve the problem, he compares pairs of numbers using mathematical rules such as "for $x, y \in \Re^-$, if $|x| > |y|$, then $x < y$." His constraints are likewise mathematical, such as "the result $a < b$, $b < c$, and $c < a$ indicates an error." Finally, Brett justifies his results with mathematical axioms.

The author and the coder also evaluated the accessibility and reliability of each person's metaphor(s). To classify a person's accessibility to a metaphor, the subjects were asked for their images and metaphors directly in the third part of the interview. In addition, the coders checked for difficult situations in which the subjects could productively apply a metaphor they used earlier, but did not. In these situations, the coders categorized these subjects' metaphorical reasoning as context-dependent. Otherwise, their metaphorical reasoning was classified as readily available. To determine the reliability of a person's metaphor, each subject was asked if the metaphor(s) worked for all arithmetic problems. In addition, the coders examined if the subject accepted the conclusion of the metaphor when it conflicted with another result computed in a different manner.

The author and another coder scored the transcripts separately using the author's guidelines described above. Troublesome cases were discussed and consensually scored.[16]

Results

The subjects did not simply invoke arithmetic facts or rely on the problem situation to solve the problems. Nearly every subject (16/17, 94%) explained arithmetic expressions with negative numbers metaphorically. Fifteen people (88%) spontaneously reasoned metaphorically at least once while solving the first two problems. A large percentage of both novices (8/12, 67%) and experts (4/5, 80%) reasoned with a spatial metaphor in the ordering problem. However, only one expert (20%) reasoned metaphorically in the stock market problem whereas nine novices (75%) did so. Table 1 summarizes the percentage of subjects who reasoned metaphorically in each problem. After describing the variety of metaphors among the subjects, I compared experts' and novices' reasoning through these metaphors.

Insert Table 1 about here

A Variety of Metaphors

The subjects in this study did not reason through one common metaphor but through a wide range of them, with varying parts, actions, and forms. These metaphors fall into two categories: space and quantity. Experts articulated significantly more metaphors (m=2.80) than novices ($\underline{m}$=0.91, t_{15}= 10.4588, $\underline{p}$ < .01, two-tailed test). Whereas all experts reasoned with at least one metaphor from each category, novices used at most one metaphor. Nine of the twelve novices reasoned with spatial metaphors and two with quantity metaphors.

Spatial Metaphors. The most common spatial metaphor in this study imagines each number to be a location along a linear path, the number line described in many mathematics textbooks. OH_e, an expert, employed a minimal number line to solve the ordering problem (15, -7/8, -21, 4, 1/6, -3). (See Figure 6. Subjects are indicated by their initials (OH) and the subscript indicates a novice ($_n$) or an expert ($_e$). "I" refers to the interviewer.)

OH_e:	Fifteen,	[writes "15" near the center of the paper]
	negative seven-eighths,	[writes "-7/8" two inches to the left of 15]
	negative twenty-one,	[writes "-21" two inches to the left of -7/8]
	four,	[writes "4" between -7/8 and 15, half an inch to the right of -7/8]
	one-sixth,	[writes "1/6" between -7/8 and 4]
	negative three.	[writes "-3" between -7/8 and -21]

Insert Figure 6 about here

OH_e wrote down the numbers in the order given by the problem at particular places, suggesting that he identified the locations of each number relative to "15." OH_e both interpreted and solved this problem metaphorically. All of the subjects' spatial metaphors seemed to be variations of this one.

The remainder of this section describes how the operations, components and orientations of the subjects' spatial metaphors varied. Most subjects discussed addition or subtraction of a positive number as a unidirectional movement, forward or backwards, regardless of the location. In contrast, JO_n differentiated the consequences of his metaphorical actions according to the region, positive or negative. In following excerpt, he computed -120 - 40 during his solution of the stock market problem.

JO_n: I have a negative one twenty. I'd go one sixty, and minus forty, then I'd go forward instead of going backward. Instead of going, it would go forward and in this scale,[17] except it would be going backwards, you know. It'd be...

I: Because it's a negative?

JO_n: Yeah, because it's a negative...

I: You switch everything?

JO_n: Yeah, you switch them. Think of the opposite of what I'm saying. It would be going forward, but it would be going sort of backward, you know. It would be going backward except it would be going forward.

JO_n argued that the magnitude of the result, -160, increases while the result itself is mathematically less, adding that subtracting 40 in the negative region has the opposite effect that it has in the positive region.

I: So--

JO_n: I'm just thinking about how to say this. Instead of going this way [points to the right] and adding, I would be adding, but it would be going this way [points to the left].

I: If this were addition?

JO_n: Yes. This is for subtraction. It would be going forward. It's sort of like if you're adding something except it's going to be the opposite way.

JO_n's focus on going forward suggested a focus on increasing magnitude, rather than mathematically greater. Although JO_n did not draw a diagram for his metaphor, one can depict it as two distinct regions with complementary location-dependent operations (see Figure 7).[18]

Insert Figure 7 about here

In the negative region, subtraction had the effect of adding or "going forward." By switching them --"Think of the opposite of what I'm saying" --addition decreased a number's magnitude in the negative region and thus had the effect of subtraction. In addition to understanding this problem and computing this solution metaphorically, JO_n also integrated addition and subtraction through this metaphor.

Only one subject demonstrated a metaphorical action for subtracting a negative number (7 - -2). Other subjects explained that subtracting a negative was equivalent to adding a positive (7 + 2) by appealing to slogans ("minus minus is plus") or to authority ("that's what Ms. G_____ told us"). Three experts also employed symbolic mnemonics ("the two negatives join together and make a plus"). However, only YO_e reasoned about this situation metaphorically. Routinely subtracting smaller positive numbers from larger positive numbers (e.g. 130- 80), he suddenly had to compute 130 - -40. Noting that he had passed the "zero line," he demonstrated his solution by drawing a number line and a curve highlighting the distance between -40 and 130 (see Figure 8).

Insert Figure 8 about here

Other subjects' spatial metaphors lacked particular components. For example, NI_n's spatial metaphor omitted fractions. She solved the ordering problem through pair-wise comparisons to obtain the following solution: 1/6 -7/8 -3 -21 4 15. Next, she evaluated her initial solution metaphorically.

NI_n: Wait a minute, negative three is further to the right on the scale so it's not as worse off, so twenty-one is smaller then.[19]

Then, the interviewer asked her to explain parts of her solution, including her ordering of -21 and -7/8.

I: How about negative twenty-one and negative seven-eighths?

NI_n: Negative seven-eighths is smaller because fractions is smaller than a whole number.

I: And one-sixth is less than negative twenty-one?

NI_n: Um-hmm, because it's a fraction.

I: So all fractions are less than whole numbers?

NI_n: Yes! Especially since these don't even out into a whole number.

I: Umm, are fractions on the scale?

NI_n: Scale? Oh, no. That's just for whole numbers.

NI_n's spatial metaphor did not include fractions. Other novices, such as FA_n, omitted zero from their metaphors. While calculating -4 + 5, she drew a vertical line and labeled evenly spaced hash marks with integers from -5 to 4, but omits 0 (see Figure 9).

Insert Figure 9 about here

FA_n: We're at negative four, we want to go up from negative four,[20] so we add five to it and [her pen begins at "-4" and bounces upward five times, but skipping zero] we get positive two.

By omitting zero from her diagram, her metaphorical computation yielded an incorrect answer.

Like many novices, FA_n's vertical metaphor also differed in orientation from the textbook horizontal number line. Novices also employed mirror image and diagonal number lines. In EL_n's mirror image number line, he switched the negative and positive regions while computing 1 - 5 (see Figure 10).

Insert Figure 10 about here

However, EL_n used a traditional number line at other times and commented on his atypical construction, "our teacher usually does it the other way, so it's positive over here [points to the "-4"]." Finally, CY_n mapped a strikingly graphic image on to her integers.

I: Are there any pictures that you use to help you [to compute -5 + 8]?

CY_n: Well, there's one, kind of that my mom did when she was trying to help me with my homework one night, kind of like, um if there's sea level and there's a mountain coming up and then there's like a big ravine going below the water, then it's minus five plus eight, you walk up toward sea level and keep going and then you'd be like up to three on the mountain, so that'd be positive three, and that's what I usually think of, that's what I try to remember, cause then it makes it easier, then I don't get as confused as it's just one long line that just keeps going.

Later, CY_n drew a graphical representation of her metaphor (see Figure 11).

Insert Figure 11 about here

In short, the subjects' spatial metaphors generally resembled a number line. However, the actions, components and orientations of the subjects' metaphors varied. Several novices employed different corresponding actions in their metaphors without derailing their problem solving. In contrast, the components (or their absences) were central to the subjects' accurate arithmetic computations. Finally, they recognized that the metaphor's orientation could vary without significantly affecting its use.

Quantity metaphors. The subjects discussed two types of quantity metaphors. Both employed quantities of physical objects for positive numbers, and the metaphors differed primarily in their interpretation of negative numbers.

Subjects who reasoned through object opposition metaphors viewed positive and negative numbers as quantities of different objects. Moreover, the different objects neutralized each other on a one to one basis. For example, AM_n described his marbles and holes metaphor.

AM_n: [reads] What images if any come to mind when you think of -5 + 8?

[answers] plus three.

I: Do you have any pictures or images?

AM_n: I think of like holes and stuff. Like the negative five are like holes you know, [draws five circles] and the, um, eight, positive eight are like marbles [draws eight dots, five inside the five circles (see Figure 12)]. So the holes, um, eat up five of them, and so there's three left, so the answer's three, positive three.

Insert Figure 12 about here

Although AM_n solved this problem by factual recall, he understood it through a metaphor that treats number as the quantity of a group of objects.[20] Each hole matches up with a marble and they cancel one another, leaving behind a quantity of marbles (or holes).

Subjects who reasoned through attribute opposition metaphors imagined both positive and negative numbers to be quantities of the same object, but assigned attribute oppositions to them.

For example, YO_e said that he would teach "-5 + 3" through a metaphor based on owning and owing.

> YO_e: I would change it to 3 - 5. I would bring in the notion of borrowing, um, I have three toys let's say, and I have three toys and I have to give, i s say I was giving gifts and I had to give five toys as gifts and I only have three, so I have to borrow two more toys from someone, so I borrow it. I wind up with five I give the five toys away, but at the same time I owe someone two toys that I borrowed from. So that owing is minus two.

YO_e's metaphor relied on an understanding of social conventions, as well as physical objects. Unlike AM_n's metaphor, YO_e employed the same objects (toys) for both positive and negative numbers and used social attributes, possession and obligation, to distinguish them. The three toys in his possession were mapped on to 3 while the five toys to be given over to someone else were mapped on to "- 5." To fulfill the social commitment of giving gifts, he employed another social practice, borrowing. Consequently, his result was also social, an obligation to repay two toys.

The quantity metaphors' actions and components did not vary, but the subjects used many objects. Like most of their spatial metaphors, none of the subjects demonstrated a metaphorical action for subtraction of a negative number (7 - -2). Finally, the specific form of the objects need not be marbles or toys. SU_e said outright that any objects may be used: "it's money, or it's candies, or it's apples, or whatever you want."

In short, several subjects employed metaphors based on the quantity of groups of objects. In both types of metaphors, positive and negative numbers were understood as quantities of objects. Object opposition metaphors envisioned negative numbers as quantities of different objects that neutralized positive objects on a one to one basis. On the other hand, attribute opposition metaphors used the same objects for both positive and negative numbers, but assigned attribute oppositions to each. In addition, the source of a metaphor can be socio-cultural rather than physical. Like most of the spatial metaphors, the quantity metaphors also lacked a corresponding action for subtraction of a negative. Finally, each of these versions differed in form, as the objects and attributes varied.

Summary of Variety of Metaphors. Subjects described a wide range of metaphors that were classified into two categories: space and quantity. (These two types of metaphors may be part of a broader system of duality metaphors discussed in Lakoff (1992).) Then, I further divided the quantity metaphors into object oppositions and attribute oppositions. Within each category, some of the subject's metaphors differed with respect to components, actions and form. The differences in the spatial metaphors' components and actions led to different behaviors and different computations. However, the orientation of the spatial metaphors and the types of objects and properties in the quantity metaphors did not significantly vary the students' reasoning.

Comparison of Experts and Novices

Expert and novice metaphors. The metaphors of experts generally resembled those of novices. However, experts' metaphors corresponded more closely to standard arithmetic (see Table 2). Moreover, experts reasoned with significantly more types of metaphors ($m = 2.80$) than novices ($m = 0.91$, $t_{15} = 10.4588$, $p < .01$, two-tailed test.), and they used multiple metaphors to teach a single computation.

Insert Table 2 about here

Expert and novice metaphorical reasoning generally a) shared central correspondences, b) lacked a metaphorical action for subtraction of a negative number (e.g. 7 - -2), and c) had little detail. Both expert and novice spatial metaphors mapped locations on to numbers and motions on to operations. In both expert and novice quantity metaphors, the objects mapped on to the numbers. However, only one expert could metaphorically explain the subtraction of a negative number (e.g. 7 - -2). The other subjects computed these arithmetic expressions through symbolic mnemonics or through memorized facts. Finally, both groups expressed little detail in their metaphorical reasoning.

Expert metaphorical reasoning, however, aligned more closely with standard arithmetic than novice reasoning. All expert metaphors included corresponding components for zero and for fractions, which some novice metaphors lacked. As a result, some novices metaphorically computed incorrect results. In their spatial metaphors, all experts employed location-independent actions and used textbook, horizontal number lines. In contrast, some novices used location-dependent actions and drew vertical, diagonal, and mirror image number lines. Unlike the absence of zero and fractions though, these differences in action and in form did not derail novice problem solving. Finally, the two novice object opposition metaphors did not differ significantly from the experts'.

Experts also described significantly more types of metaphors than novices. Whereas each novice described at most one, experts had at least two. Moreover, each expert articulated at least one spatial metaphor and at least one quantity metaphor. Experts also used multiple metaphors to explain a single computation. For example, SU_e explained -5 + 8 in the following ways:

SU_e: -5 + 8 is you get a loss and you get a plus, it's my stock price, and or you could think of this, minus five is your investment, at first you have to put some money to get some profit.

I: Is that how you think of minus five plus eight?

SU_e: Minus five is your loss, minus is loss and plus is your gain. Or minus five is your debt and plus is your credit.

SU_e presented three subtly different economic metaphors: changes in a stock price, investment and profit, and credits and debts. Then, she presents an owning and owing metaphor.

> SU_e: He has eight candies and he told me he would give me five candies, how many candies would he have in his hand?

Finally, she added a spatial metaphor.

> SU_e: It's like when you go somewhere, and your parents give you directions that you could go to five blocks to right hand side and you go uh, two blocks, two blocks to the left side or whatever, but you draw the line here [draws horizontal line], then it's like uh, you start from the zero point [writes zero above the center of the line and marks the line underneath with a hatch mark] and you move back and whenever you find a positive, you move forward.

SU_e imposed many different situations on to the expression -5 + 8, demonstrating both flexibility and additional connections that novices seemed to lack.

Metaphorical Understanding. Novices typically employed metaphorical understanding for both the arithmetic and ordering problems, whereas experts generally interpreted only the ordering problem metaphorically (see Table 3). Since many novices experienced difficulty with negative number computations, they introduced an additional layer of metaphorical comprehension to interpret them significantly more often than experts ($t_{15} = 2.181$, $p < .05$, two-tailed test). As discussed earlier, the only instance of expert metaphorical understanding of arithmetic occurred when YO_e suddenly confronted a difficult problem. Otherwise, both novices and experts typically employed simpler and more efficient number facts. In the ordering problem however, many novices and experts interpreted it metaphorically to facilitate a simpler solution.

Insert Table 3 about here

Metaphorical Problem Solving. Experts avoided unnecessary metaphorical problem solving in the stock market problem, but exploited it's efficiency in the ordering problem(see Table 4). Novices metaphorically computed arithmetic problems involving negative numbers significantly more often ($t_{15} = 2.154$, $p < .05$, two-tailed test), but also required significantly more time ($m = 44.41$ minutes) than experts ($m = 12.58$ min.) to complete the first problem ($t_{15} = 3.997$, $p < .005$, one-tailed test).

Insert Table 4 about here

In contrast, experts employed significantly more metaphorical operations in the ordering problem than in the arithmetic computations (paired t-test. $m = 3.8$, $t_4 = 3.919$, $p < .02$, two-tailed). They also required less time than the novices to solve the ordering problem (expert m = 0.49 minutes, novice $m = 1.31$, $t_{15} = 1.991$, $p < .06$, one-tailed). Although experts reasoned through metaphorical operations more often than novices in the ordering problem, this difference was not statistically significant ($t_{15} = 1.530$, $p > .10$). Experts required less metaphorical interpretation, but recognized when to capitalize on it.

Evaluating Solutions through Metaphorical Constraints. A few novices, but no experts, evaluated their progress through metaphorical constraints (see Table 5). Most novices (67%) did not stop to evaluate their work at all, but simply moved from one problem to the next. The remaining novices applied mathematical rules and symbolic mnemonics as well as metaphorical constraints to evaluate their solutions. In contrast, the experts checked their work for mistakes, but relied on their knowledge of mathematical facts (20 + 30 = 50) and relationships (each positive number is greater than each negative number). In the words of OH_e, "I just look at it [-5 + 8], and I know it's 3, that's how I check it." Most novices did not examine their work at all while experts used arithmetic facts and mathematical rules to evaluate their solutions.

Insert Table 5 about here

Justifying Solutions Metaphorically. Novices justified their solutions by referring to a metaphorical explanation significantly more often than experts ($t_{15} = 2.256$, $p < .05$, two-tailed test, see Table 6). Novices not only reasoned metaphorically, but also relied on metaphorical explanations as the foundation for their work. In contrast, experts never invoked metaphorical explanations for their solutions. In the first problem, they often referred to the stock market situation as the basis of their solution. To justify their answers in the ordering problem, they pointed to mathematical rules, even though they used the number line to identify the location of each number. Whereas many novices based their solutions on metaphorical reasoning, experts appealed to either the problem situation or the mathematical rules.

Insert Table 6 about here

Access to Metaphorical Reasoning. Nearly all subjects readily accessed their metaphors as needed (see Table 7). Each subject described the different metaphors they had used when asked to teach addition and subtraction of negative numbers to a twelve-year old. CH_e claimed that he "instinctively imagine[d] a number line." However, some novices did not invoke their metaphors in applicable situations. Despite employing their metaphors earlier in similar situations, they did not invoke them when their metaphorical reasoning could have helped them overcome impasses.

Insert Table 7 about here

Perceived Reliability of Metaphorical Reasoning. Most subjects also believed that they could rely on their metaphorical reasoning to solve any arithmetic expression (see table 8).[21] However, four novices successfully employed metaphors to compute arithmetic expressions, but either doubted the validity of a metaphor-based solution or said that their metaphorical reasoning was not always reliable. In addition, five novices faced contradictory conclusions from an unexplained memorized rule and their metaphorical reasoning. Three chose the metaphor-based results, one did not choose, and one chose the rule, (suggesting a preference for metaphor-based conclusions, but not a unanimous one).

Insert Table 8 about here

Summary of Comparison. Expert and novice metaphorical reasoning differed both in quality and in usage. Experts readily accessed their reliable metaphors and recognized applicable problem situations. However, they generally used mathematical facts and rules for efficient computation. In contrast, novices lacked ready access to their metaphors and had more doubts about their reliability. Nevertheless, they reasoned metaphorically more often both to understand and to solve problems. Few novices and no experts evaluated their solutions through metaphorical constraints. When asked to justify their results, experts appealed to either mathematical rules or the problem situation. In contrast, many novices relied on metaphors to justify their results. While experts reasoned through metaphors proficiently and rarely used them, novices displayed weaker metaphorical reasoning and used it frequently.

Discussion

In this section, I discuss the subjects' metaphorical reasoning and speculate on the development of novices into experts.

Subjects' Metaphorical Reasoning

I begin by discussing the variety of the subjects' metaphors. Then, I examine where these metaphors fall along the six dimensions described earlier. Finally, I analyze how the subjects benefited by reasoning through these metaphors.

Variety of Subjects' Metaphors. There are at least three possible causes for the variation among the subjects' metaphors. Since both the textbook and the teacher introduce the number line, we would expect many students to invoke a spatial metaphor. However, several students also used metaphors that did not originate within the classroom. For example, CY_n explicitly cited a conversation with her mother (the mountain ravine metaphor). Others may have constructed their own metaphors. In addition, all experts extended their metaphors to include additional mathematical concepts such as fractions but some novices did not. Finally, students may adapt their metaphor to the particular situation as EL_n did when he employed his mirror image number line for computing "1-5." In short, metaphorical reasoning among individuals may vary due to different origins, added extensions, or adaptations to problem situations.

Dimensions of Subjects' Metaphors. Although all subjects demonstrated adequate source comprehension during metaphorical reasoning, experts differed from novices with respect to target comprehension, systematicity, operational level, detail, and automaticity. No subject committed errors in their source, indicating competent source comprehension. However, experts demonstrated greater target understanding as indicated by their multiple metaphors and their arithmetic facts and rules. Moreover, expert metaphorical reasoning displayed more systematicity in contrast to novice reasoning which occasionally omitted components (fractions, zero) from their spatial metaphors. Consequently, expert metaphors had a higher operational level than novice ones even though novices reasoned through metaphors more frequently. Since novices employed more detailed and less automatic metaphors, this study supports the following claim: As a person develops expertise, he reasons with less detailed metaphors and automates them.

Subjects' Benefits from Metaphor Reasoning. This study demonstrated how metaphorical reasoning facilitated both conceptual understanding and problem solving through a) intuitive justification, b) arithmetic integration, c) an enhanced computational environment, and d) improved recall. Since these metaphors were built on an intuitive foundation, novices confidently appealed to them to justify their mathematical operations. Subjects also used their intuitive source experiences metaphorically to integrate multiple arithmetic operations. In addition, they solved problems through metaphorical computations and novices evaluated their solutions with metaphorical constraints. Finally, the tight connection between familiar intuitions and arithmetic helped subjects reconstruct explanations for their arithmetic operations. In short, subjects reasoned metaphorically to justify mathematical actions, integrate arithmetic relationships, compute mathematical expressions, evaluate their solutions and recall explanations.

Development from Novices to Experts

This section discusses the pitfalls of an expert-novice paradigm and the role of metaphors in mathematical development.

Expert-novice intervention paradigm supporters argue that novices should imitate experts. Since the stock market task results suggest that experts, unlike novices, do not reason metaphorically, expert-novice paradigm advocates would discourage novices from reasoning metaphorically. Stripping novices of their metaphorical reasoning however, neglects the potential scaffolding role of metaphor as a transition to expertise. Moreover, the results of the ordering and the image/teaching tasks both show that experts can reason through many systematic metaphors.

Therefore, metaphors may play a greater role in mathematical development than is currently acknowledged. Novices may initially understand a novel problem metaphorically. As they familiarize themselves with more problems, they recognize the central operational parts of the metaphor and begin omitting unneeded details. Eventually, they automate their computations so that their reasoning is no longer metaphorical, using the target results without resorting to the source. Therefore, expert-novice paradigm advocates risk neglecting the scaffolding available through metaphorical reasoning.

Expert reasoning through multiple metaphors suggests greater problem solving power, emphasizes application conditions, and mitigates the limitations of individual metaphors. The multiplicity of expert metaphors resonates strongly with Moschkovich, Arcavi, & Schoenfeld (1992) multiple views of linear functions, Dowker's (1992) multiple arithmetic strategies and Smith's (1991) multiple strategies for manipulating fractions. Moreover, metaphorical reasoning is only one of many tools available to experts (others include symbolic mnemonics, examples, etc.). With this diversity of mathematical tools, experts can solve a wider range of problems and do so more efficiently. Rather than simply applying a single tool repeatedly to every problem, expertise requires choosing the appropriate tool from a list of possibilities. Finally, the diversity of an expert toolbox mitigates the limitations of a specific tool (such as those inherent in any metaphor) because the expert can employ other methods to cover otherwise vulnerable areas. An expert with a diverse mathematical toolbox must recognize more application conditions, but he can solve more problems, solve them more efficiently, and tolerate greater limitations on particular tools.

Conclusion

In this study, I have argued that abstract representations, complex situations, and metaphorical reasoning fit together theoretically and have shown empirically that both experts and novices reason metaphorically in mathematics. Initially, students build up their understanding of abstract representations metaphorically through meaningful complex situations, and then they trim down the details during repeated metaphorical uses of abstract forms. They eventually learn to apply the abstract forms without relying on the scaffolding of metaphorical reasoning. Empirically, these subjects generated a variety of spatial and quantity metaphors. Some of the differences between their metaphors arose from subjects extending their metaphors to include additional mathematical concepts. Others adapted their metaphors to particular situations. Although experts can reason more adroitly through more metaphors, they do so relatively infrequently, preferring more efficient mathematical facts and procedures. In contrast, novices reason less proficiently through fewer metaphors, but do so more frequently. Subjects reasoned metaphorically to justify mathematical actions, integrate arithmetic relationships, compute mathematical expressions, evaluate their solutions and recall explanations. As a result, this study suggests that metaphors may play a central role in mathematical development by bridging novices' intuitive understanding to expert mathematics.

Educators can use systematic metaphorical reasoning as a powerful, pedagogical tool to cover mathematical topics and build on students' intuitive knowledge. When choosing metaphors, curriculum designers must maximize both breadth and depth of coverage. As exemplified by ARITHMETIC IS MOTION, reasoning through a single metaphor can explain many related mathematical topics in rigorous detail. Since students must build mathematical understanding on the foundation of their prior knowledge, they are more likely to reason through metaphors that rely on intuitive sources (See Chiu & Gutwill (1993), diSessa (1993), Johnson (1987), Mandler (1992a, 1992b), Ogborn & Bliss (1990) and Talmy (1988) for discussions of intuitions). Virtually universal experiences such as eating and moving are likely candidates, but educators can also capitalize on common cultural experiences. Promising pedagogical metaphors exhibit both breadth and depth of coverage and build upon intuitive sources.

This study also shows that reasoning through a metaphor is not an atomic process, but occurs in varying degrees. As a result, students require teacher guidance to use their metaphorical reasoning appropriately. Firstly, the teacher must decide how to introduce a particular metaphor (e.g. posing a problem in an environment in which students are likely to generate the metaphor). Then, she must focus their attention on important aspects and encourage them to extend their metaphorical understanding. Therefore, the teacher plays an important metacognitive role by helping students negotiate their way through the benefits and the pitfalls of metaphorical reasoning.

Metaphors have a great deal of potential, but their successful implementation requires further evidence of their efficacy. In particular, how do students learn to reason through particular metaphors? How do student create composite metaphors? The answers to these and related questions may help Ana and other students challenge de Morgan's claim that some mathematical entities simply do not make sense.

References

Anderson, J. R. (1987). Skill acquisition: Compilation of weak method problem solutions. Psychological Review, 94, 192-210.

Bernard, H.R. (1988). Research methods in cultural anthropology. Beverly Hills, CA: Sage.

Black, M. (1979). More about metaphor. In A. Ortony's (Ed.) Metaphor and thought. Cambridge: Cambridge.

Boyer, C. B. (1991). A history of mathematics. New York: Wiley.

Brown, D.E. (1990). Facilitating Conceptual Change Using Mechanistic Conceptual Metaphors. Paper presented at the American Educational Research Association, Boston, MA, April.

Brown, D. E. & Clement, J. (1989). Overcoming misconceptions via analogical reasoning: abstract transfer versus explanatory model construction. Instructional Science, 18, 237-261.

Brown, J. S., & Burton, R.B. (1978). Diagnostic models for procedural bugs in basic mathematical skills. Cognitive Science, 2, 155-192.

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. Educational Researcher, 18, 32-42.

Brown, J. S., & van Lehn, K. (1980). Repair theory: A generative theory of bugs in procedural skills. Cognitive Science, 4, 370-426.

Carraher, T. N. (1986). From drawings to buildings: Working with mathematical scales. International Journal of Behavioral Development, 9, 527-44.

Chiu, M. M. (1993). Student intuitions of lines: Origins, uses and interactions. Proceedings of the Third International Seminar on Misconceptions in Math and Science Education. Ithaca, NY: Cornell.

Chiu, M. M., & Gutwill, J. (1993). Architecture of Intuition. Unpublished manuscript.

Clement, J. (1982). Students' preconceptions in introductory mechanics. American Journal of Physics, 50, 1, 66-71.

Clement, J., Brown, D. E., & Zietsman, A. (1989). Not all preconceptions are misconceptions. Paper presented at the American Educational Research Association, San Francisco, CA, March.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. Journal for Research in Mathematics Education, 23, 1, 2-33.

Crowley, M. L., & Dunn, K. A. (1985). On multiplying negative numbers. Mathematics Teacher, 78, 252-256.

Cummins, D. D., Kintsch, W., Reusser, K., & Weimer, R. (1988). The role of understanding in solving word problems. Cognitive Psychology, 20, 405-438.

Davis, P.J., & Hersh, R. (1981). The mathematical experience. Boston: Houghton Mifflin.

Davis, R. B. (1984). Learning mathematics: The cognitive approach to mathematics education. Norwood, NJ: Ablex.

Davis, R. B. & Maher, C. A. (1993). The reality of negative numbers. In R. B. Davis & C. A. Maher's (Eds.) School Mathematics and the World of Reality. Boston: Allyn & Brown.

Davis, R. B. & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconceptions stages. Journal of Mathematical Behavior, 5, 281-303.

De Morgan, Augustus. (1898). On the study and difficulties of mathematics. Chicago: Open Court.

diSessa, A. A. (1993). Toward an Epistemology of Physics. To appear in Cognition and Instruction.

Dolciani, M., Sorgenfrey, R.H., & Graham, J.A. (1985). Mathematics: Structure and function. Boston: Houghton Mifflin.

Donaldson, D. (1978). What metaphors mean. Critical Inquiry, 5, 1, 31-47.

Dowker, A. (1992). Computational Estimation Strategies of Professional Mathematicians. Journal for Research in Mathematics Education, 23, 45-55.

Gentner, D. (1989). The mechanisms of analogical learning. In S. Vosniadou & A. Ortony (Eds.) Similarity and analogical reasoning. New York: Cambridge.

Gentner D., & Gentner, D. R. (1982). Flow waters or teeming crowds: Metaphors of electricity. in D. Gentner & A. L. Stevens's (Eds.) Mental Models. Hillsdale, N.J.:Erlbaum.

Gibson, J.J. (1950). The perception of the visual world. Boston: Houghton Mifflin.

Gibson, J. J. (1966). The senses considered as perceptual systems. Boston: Houghton Mifflin.

Gibson, J.J. (1979). The ecological approach to visual perception. Boston: Houghton Mifflin.

Gick, M. L. & Holyoak, K. J. (1980). Analgoical problem solving. Cognitive Psychology, 12, 306-355.

Gick, M. L. & Holyoak, K. J. (1983). Schema induction and analogical transfer. Cognitive Psychology, 15, 1-38.

Graham, J.A., & Sorgenfrey, R.H. (1983). Trigonometry with applications. Boston: Houghton Mifflin.

Greeno, J. G. (1987). Instructional representations based on research about understanding. In A. H.Schoenfeld (Ed.) Cognitive science and mathematics education. Hillsdale, N.J.: Erlbaum.

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. Journal for Research in Mathematics Education, 23, 3, 170-218.

Hall, R., Kibler, D., Wenger, E., & Truxaw, C. (1989). Exploring the episodic structure of algebra story problem solving. Cognition and Instruction, 6, 223-283.

Holland, J. H., Holyoak, K. J., Nisbett, R. E. and Thagard, P. R. (1986). Induction: Processes of inference, learning and discovery. Cambridge, MA: MIT.

Holyoak, K.J. & Thagard, P.R. (1989). A computational model of analogical problem solving. In S. Vosniadou, S. & Ortony, A. (Eds.) Similarity and Analogical Reasoning. New York: Cambridge.

Hudson, T. (1983). Correspondences and numerical differences between disjoint sets. Child

Development, 54, 84-90.

Jick, T. D. (1983). Mixing qualitative and quantitative methods: Triangulation in action. In J. V. Maanen (Ed.), Qualitative Methodology (pp. 135-148). Beverly Hills.

Johnson, M. (1987). The body in the mind: The bodily basis of meaning, imagination, and reason. Chicago: Chicago.

Johnson-Laird, P. N. (1983). Mental models: Towards a cognitive science of language, inference, and consciousness. Cambridge, MA: Harvard.

Kendon, A. (1988). Sign languages of Aboriginal Australia : Cultural, semiotic, and communicative perspectives. New York : Cambridge.

Kendon, A. (1990). Conducting interaction: Patterns of behavior in focused encounters. New York: Cambridge University Press.

Lakoff, G. (1987). Women, fire, and dangerous things. Chicago: Chicago.

Lakoff, G. (1990). The invariance hypothesis: is abstract reasoning based on image schemas? Cognitive Linguistics 1, 39-74.

Lakoff, G. (1992). The contemporary theory of metaphor. Unpublished manuscript. Univeristy of California, Berkeley.

Lakoff, G., & Johnson, M. (1980). Metaphors we live by.Chicago: Chicago.

Lakoff, G., & Turner, M. (1989). More than cool reason. Chicago: Chicago.

Lave, J. (1988). Cognition in practice : Mind, mathematics, and culture in everyday life. New York: Cambridge University Press.

Lesh, R. Landau, M. & Hamilton, E. (1983). Conceptual models in applied mathematical problem solving. In R. Lesh & M. Landau (Eds.) Acquisition of mathematical concepts and processes (pp 263-343). New York: Academic.

Lobato, J. E. (1991). The development of a framework for studying computational estimation as a sense-making activity. Unpublished manuscript, University of California, Berkeley, CA.

Mandler, J. M. (1992a). The Foundations of Conceptual Thought in Infancy. Cognitive Development, 7, 273-285.

Mandler, J. (1992b). How to build a baby: II. Conceptual primitives. Psychological Review, 99(4), 587-604.

McNeil, D. (1992). Hand and mind: What gestures reveal about thought. Chicago: Chicago.

Musser, G. L. (1972). Additive inverses and negatives – two different concepts. Mathematics Teacher, 64, 301-306.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1992). What does it mean to understand a domain?: A study of multiple perspectives and representations of linear functions, and the connections among them. In T. Romberg, E. Fennema, & T. Carpenter (Eds.), Integrating research on the graphical representation of function. Hillsdale, NJ: Erlbaum.

Neves, P. M. (1981). Learning procedures from examples. Unpublished doctoral dissertation, Carnegie Mellon University.

Newell, A. & Simon, H. (1972). Human problem solving. Englewood Cliffs, NJ: Prentice-Hall.

Nolder, R. (1991). Mixing Metaphor and Mathematics in the Secondary Classroom. In K. Durkin & B. Shire's (Eds.) Language in mathematical education: Research and practice.

Nunes, T. (1993). Learning Mathematics: Perspectives from everyday life. In R. B. Davis & C. A. Maher's (Eds.) School Mathematics and the World of Reality. Boston: Allyn & Brown.

Ogborn, J. & Bliss, J. (1990). A psycho-logic of motion. European Journal of Psychology of Education, 5, 379-390.

Pimm, D. (1987). Speaking mathematically. New York: Routledge.

Resnick & Omanson (1987). Learning to understand arithmetic. In R. Glaser's (Ed.) Advances in instructional psychology, vol. 3, 41-95.

Reusser, K. (1985). From situation to equations: On formulating, understanding and solving situation problems. Technical Report No. 143, Institute of Cognitive Science, University of Colorado, Boulder.

Rissland, E. L. (1985). Artificial intelligence and the learning of mathematics: A tutorial sampling. In E. A. Silver (Ed.) Teaching and learning mathematical problem solving: Multiple research perspectives, 147-176. Hillsdale, NJ: Erlbaum.

Saxe, G. B. (1992). Studying children's learning in context: problems and prospects. The Journal of Learning Sciences, 2, 215-234.

Saxon, J. H. (1985). Geometry, trigonometry, algebra III: An incremental approach. New York: Grassdale.

Schoenfeld, A. H. (Ed.) (1987). Cognitive science and mathematics education. Hillsdale, N.J.: Erlbaum.

Schoenfeld, A. (1990). On mathematics as sense making: An informal attack on the unfortunate divorce of formal and informal mathematics. In J. F. Voss, D. N. Perkins, &J. W. Segal (Ed.) Informal reasoning and education. Hillsdale, NJ: Lawrence Erlbaum Associates.

Schoenfeld, A. H., Smith, J. P. I'', & Arcavi, A . (1990). Learning : The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.) Advances in instructional psychology, vol.4 Hillsdale NJ.: Erlbaum.

Scribner, S. (1986). Thinking in action: Some characteristics of practical thought. In R. J.Sternberg & R. K. Wagner (Eds.) Practical intelligence: Nature and origins of competence in the everyday world. New York: Cambridge Press.

Searle, J. (1979). Metaphor. In A. Ortony's (Ed.) Metaphor and thought. Cambridge: Cambridge.

Shalin,V.L. & Bee, N. V. (1985). Structural differences between two-step word problems. Paper presented at the Annual Meeting of the American Educational Research Association.

Chicago, IL.

Sobel, M.A., Maletsky, E. M., Golden, B. N., Lerner, N., & Cohen, L.S. (1986). Harper & Row geometry. New York: Scribner.

Stein, E.I. (1978). First course in fundamentals of mathematics. Boston: Allyn and Bacon.

Strauss, A. L. (1987). Qualitative analysis for social scientists. Cambridge: Cambridge.

Talmy, L. (1988). Force Dynamics in Language and Cognition. Cognitive Science, 12, 49-100.

Tatsouka, K. K. (1984). Changes in error types over learning stages. Journal of Educational Psychology, 76, 120-129.

Tatsouka, K. K., Birenbaum, M., & Arnold, J. (1989). On the stability of students' rules of operation for solving arithmetic problems. Journal of Educational Measurement, 26, 351-361.

van Dijk, T.A., & Kintsch, W. (1983). Strategies of discourse comprehension. New York: Academic.

van Engen, H. (1972). 'Tain't necessarily so, or minus minus A equals A. Mathematics Teacher, 64, 299-300.

Williams, C. K., & Kamii, C. (1986). How do children learn by handling objects? Young Children, 42, 23-46.

Yanker, L.E., Vannatta, G.D., & Crosswhite, F. J. (1981). Merrill advanced mathematical concepts. Toronto: Merrill.

Appendix A: Arithmetic is Motion Along a Linear Path metaphor

Number (target domain)		Motion (source domain)
0	<--	origin/starting point
number	<--	location relative to the origin
positive number N	<--	location N steps to the right of the origin
negative number -N	<--	location N steps to the left of the origin
quantity	<--	movement
absolute value	<--	distance from origin
additive inverse	<--	direction and steps to return to origin
N is greater than M	<--	location N is to the right of location M
N is less than M	<--	location N is to the left of location M

Operations:

All arithmetic operations begin at the origin

The default operation is to face right and move forward in that direction

- N indicates N steps backwards

add N <-- move N steps (if N=0, then hop in place)

subtract N <-- turn around and move N steps

Multiplication

M x N e.g. -7 x 4

M determines number of steps e.g. turn around, and move 7 times

N determines the new step size e.g. each step is 4 units long

First execute the "-" signs by turning around the appropriate number of times.

Repeat M times: Move step size N

Division

P / Q e.g. -20 / 2

P is the final destination

Q is the step size

How many steps of size Q should a person take to go to P?

(Do you need to turn around "-"?)

-20 / 2 :

Turn around and make 10 steps, so the answer is -10.

4 / 0:

How many steps of size 0 are needed to go to 4? impossible

0 / 0:

How many steps of size 0 are needed to go to 0? any ...0,1,2...

Appendix B: Stock Market Problem

The stock market is a place for gambling, like a casino. In stock market gambling, you can buy and sell things like gold and silver.

Suppose you BUY an ounce of gold for $100.

If the price increases, by $1, the next day, then you have $101 and you win.

If the price decreases (- $1), then you lose.

On the other hand, you SELL some of your silver for $50 an ounce.

If the price increases ($5) the next day, then you lose because you should have kept your silver which is now worth $55 instead of $50.

If the price decreases (- $3), you win by selling early and getting more money for it, $50 instead of $47.

Arrowsmith Stock Market Summary

	bought / sold(-) ounces	gain / loss(-) change per ounce	Total
Copper	10	- $4	
Gold	10	$8	
Platinum	- 40	- $1	
Silver	- 30	$5	

How much money did you win or lose?

It looks pretty bad, but fortunately, you're a computer expert. You can break into the computer account and change any one of the numbers by 5 (either +5 or -5).

How many different ways can you change one number and win money overall?

Appendix C: Ordering Problem

Put these in order from least to greatest:

15 $\frac{-7}{8}$ -21 4 $\frac{1}{6}$ -3

Appendix D: Images of Arithmetic Expressions

What images, if any, come to mind when you think of -5 + 8?

[The interviewer then asks how the subject would teach this and then follow with similar questions about 7 - -2, 3 - 5, -2 x 3, -2 x -3]

Footnotes

[1]Davis (1984) describes students building understanding of negative numbers through the activity of adding and removing pebbles from a bag.

[2]All metaphor names are in small capitals.

[3]Also, see Clement & Brown's (1987) bridging analogies and Davis's (1982, 1984) pre-assimilation paradigms.

[4]I prefer to view the situations as a source transparency placed over a target transparency, so I use the term "alignment" rather than "mapping" to emphasize the relationships within the source and within the target.

[5]Metaphorical reasoning can create new entities and relationships in poorly understood targets.

[6]The reader may wonder if building on familiar situations leads to infinite regress. Lakoff (1987) argues that the human conceptual system rests on two complementary foundations: image schemas and basic level concepts. Johnson (1987), Lakoff (1987), and Mandler (1992a, 1992b) have argued that young infants construct *image schemas* from frequent bodily experiences. For example, Johnson's (1987) source-path-goal image schema arises from an infant's experience with motion. Rosch (1976) argues that people form *basic-level concepts* through perceptual and functional interactions with the world. For example, people perceive an overall shape of a chair and interact with it, by sitting on it and leaning against its back. Therefore, people build metaphors upon situations interpreted through image schemas and basic level concepts. Unfortunately, a detailed explanation of the relationship between image schemas, basic-level concepts and metaphorical reasoning is beyond the scope of this paper.

[7]Lakoff's (1992) Invariance Hypothesis. In practice, a person must know something about the target to recognize/experience it, so the source situation is never entirely imposed on the target.

[8]Assume P_n is the largest prime. But adding one to the product of P_n and all smaller primes ($1 + P P_i$, $P_i = 2,3,5,...P_n$) yields a number that is either prime or has a prime factor larger than P_n.

[9]See MacGregor (1991) for four additional metaphors for equations.

[10]Reflexivity (a = a), symmetry (b = a, a = b), and transitivity (a = b, b = c, a = c).

[11]In contrast, Searle's (1979) claims that metaphorical phrases understood automatically must be dead metaphors, in which the source is lost and unrecoverable.

[12]I will refer to this metaphor as ARITHMETIC IS MOTION.

[13]A step size of zero is jumping up and landing in the same place.

[14]Lakoff & Johnson (1980) argue that metaphors need not be consistent, but must be

coherent. For example, the following sentence combines the ARGUMENTS ARE PATHS and ARGUMENTS ARE CONTAINERS : "If we keep going the way we're going, we'll eventually fit in all the facts." Metaphorical coherence requires using different parts of the metaphor to avoid direct contradiction (progress from PATHS and content from CONTAINERS).

[15]The teacher described her lecture to me four weeks after the students took their negative numbers exam. However, she did not recall the students' comments. (Future studies would benefit from classroom observations.) She used the Keedy & Bittinger (1987) Addison-Wesley textbook.

[16]The author found 207 instances of metaphorical reasoning, and another coder found five additional instances (212). There was 98% inter-coder reliability on categorizing the types of metaphors, 91% on metaphorical understanding, 96% on metaphorical computations, and 94% on both evaluations and justifications via metaphors. Judgments of both the subjects' accessibility to their metaphors and their perceived reliability was at 94%.

[17]Since several students referred to a scale (a bathroom scale as I learned upon further questioning), they may have discussed this during class or outside of class.

[18]See Davis & Maher (1993) for a discussion of two-attribute vs. one-attribute perspectives.

[19]This reference to "worse" occurs frequently, and may be an alignment of two metaphors: GOOD IS UP and MORE IS UP. As a result, more corresponds to good/better and less corresponds to bad/worse (Lakoff & Johnson, 1980).

[20]Indicated by his quick answer and his later comment, "I know it's three, but I think about it this way."

[21]Except for subtraction of negative numbers 7- (-2) which they knew they could not perform metaphorically.

TABLE 1.

% of Novices and Experts using at least one Metaphor in each Problem

Problem	Novices (n = 12)	Experts (n = 5)
Stock Market	75	20
Ordering	67	80
Images	92	100

TABLES 2.

% of Novices and Experts who Used Spatial or Quantity Metaphors Displaying Particular Properties

Metaphor	Properties	Novices (n = 12)	Experts (n = 5)
Spatial		75	100
	Standard Operations	58	100
	Included Zero	58	100
	Included Fractions	50	100
	Standard Horizontal Form	42	100
	a - (-b) included	0	20
Quantity		17	100
	Opposing Objects	17	100
	Opposing Properties	0	80
	Multiple Types of Objects	0	100
	a - (-b) included	0	0

Table 3.

Means of Novice and Expert Instances of Metaphorical Understanding while Solving each Problem

Problem	Novices ($n = 12$)	Experts ($n = 5$)
Arithmetic	3.92	0.20
Ordering	2.58	4.00

Table 4.

Means of Novice and Expert Instances of Metaphorical Operations while Solving each Problem

Problem	Novices ($n = 12$)	Experts ($n = 5$)
Arithmetic	3.17	0.20
Ordering	2.58	4.00

*$t_{15} = 2.154$, $p < 0.05$, two-tailed test.

Table 5.

Means of Novice and Expert Instances of Evaluations via Metaphorical Constraints in each Problem

Problem	Novices ($\underline{n}$ = 12)	Experts ($\underline{n}$ = 5)
Arithmetic	0.25	0.00
Ordering	0.41	0.00

Table 6.

Means of Novice and Expert Instances of Metaphorical Justifications in each Problem

Problem	Novices ($n = 12$)	Experts ($n = 5$)
Arithmetic	1.25	0.00
Ordering	0.67	0.00

Table 7.

% of Novices and Experts with each type of Accessibility to Their Metaphors

Problem	Novices ($\underline{n}$ = 12)[a]	Experts ($\underline{n}$ = 5)
Readily Accessible	75	100
Context dependent Access	17	0

[a]One novice did not reason metaphorically to solve any of the problems.

Table 8.
% of Novices and Experts who Perceived Their Metaphorical Reasoning as Reliable.

Problem	Novices ($\underline{n}$ = 12)[a]	Experts ($\underline{n}$ = 5)
Reliable	58	100
Uncertain	33	0

[a]One novice did not reason metaphorically to solve any of the problems.

Figure Captions

Figure 1. A translation view of mathematical problem solving: representing the key problem features, manipulating the representations, translating the result back into the problem situation, and checking the solution.

Figure 2. A typical calculation (500 < 83 x 6) with mathematical representations.

Figure 3. Shalin and Bee's box representation for arithmetic word problems (6 chairs at $83 each).

Figure 4. Ana metaphorically computes "4 - 6" with the ARITHMETIC IS MOTION metaphor by walking four steps to location "4," turning around, and taking 6 steps to "-2."

Figure 5. Students learn to use mathematical symbols by interpreting them metaphorically through their intuitive understanding of the parade situation. When faced with a banking problem, the students pick out important aspects such as the deposits and payments and decide that arithmetic operations are necessary. Not yet familiar with the intricacies of multi-digit subtraction, they recall their actions during the parade activity and reason metaphorically to perform the mathematical manipulations.

Figure 6. OH_e metaphorically solves the ordering problem by placing each number at a particular location.

Figure 7. One possible representation of JO_n's metaphor which employs different sets of metaphorical operations for the positive and negative regions.

Figure 8. YO_e's diagram for solving 130 - (-40), finding the distance between the locations "-40" and "130."

Figure 9. FA_n's vertical spatial metaphor without zero.

Figure 10. EL_n's mirror image number line for solving 1 - 5.

Figure 11. CY_n's mountain-ravine metaphor.

Figure 12. AM_n's holes and marbles metaphor for solving -5 + 8.

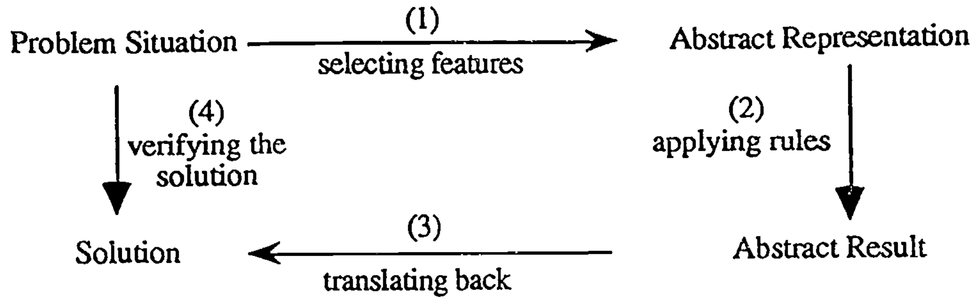
Problem Situation
(1)
selecting features
Abstract Representation
(2)
applying rules
Abstract Result
(3)
translating back
Solution
(4)
verifying the
solution

Figure 1.

$500 \geq 6 \times 83$? -->

$$\begin{array}{r} 83 \\ \underline{\times\ 6} \\ 18 \\ \underline{480} \\ 498 \end{array} \quad \text{or} \quad \begin{array}{r} {}^{4\ 1} \\ 83 \\ \underline{\times\ 6} \\ 498 \end{array}$$

Figure 2

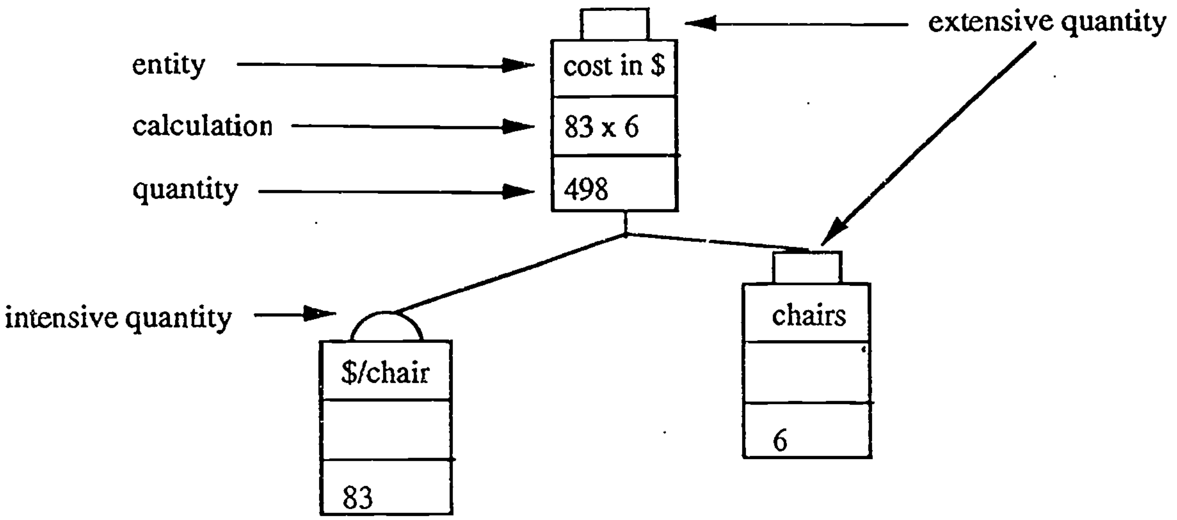
extensive quantity
entity
cost in $
calculation
83 x 6
quantity
498
intensive quantity
chairs
$/chair
6
83

Figure 3

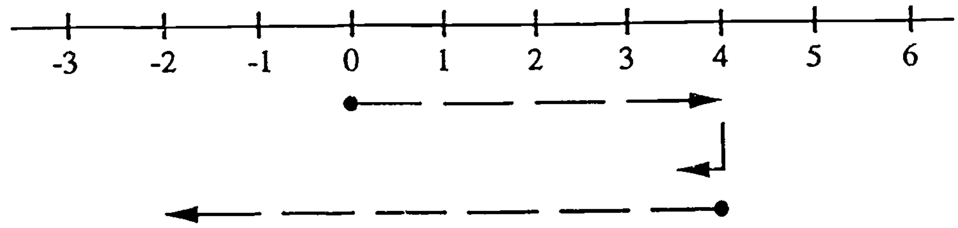

Figure 4

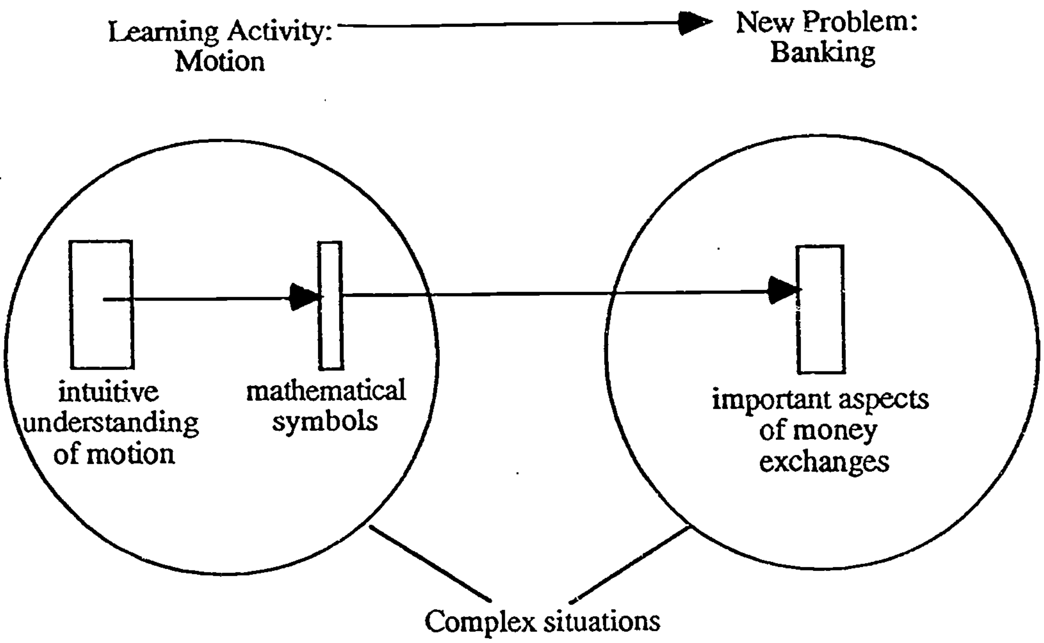
Learning Activity:
Motion
New Problem:
Banking
intuitive
understanding
of motion
mathematical
symbols
important aspects
of money
exchanges
Complex situations

Figure 5

-21 -3 $-\frac{7}{8}$ $\frac{1}{6}$ 4 15

Figure 6

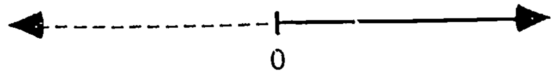

Figure 7

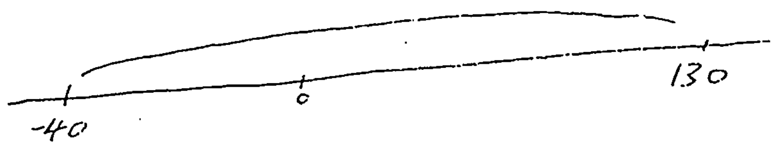

Figure 8

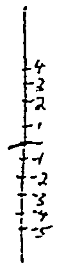

Figure 9

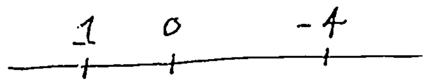

Figure 10

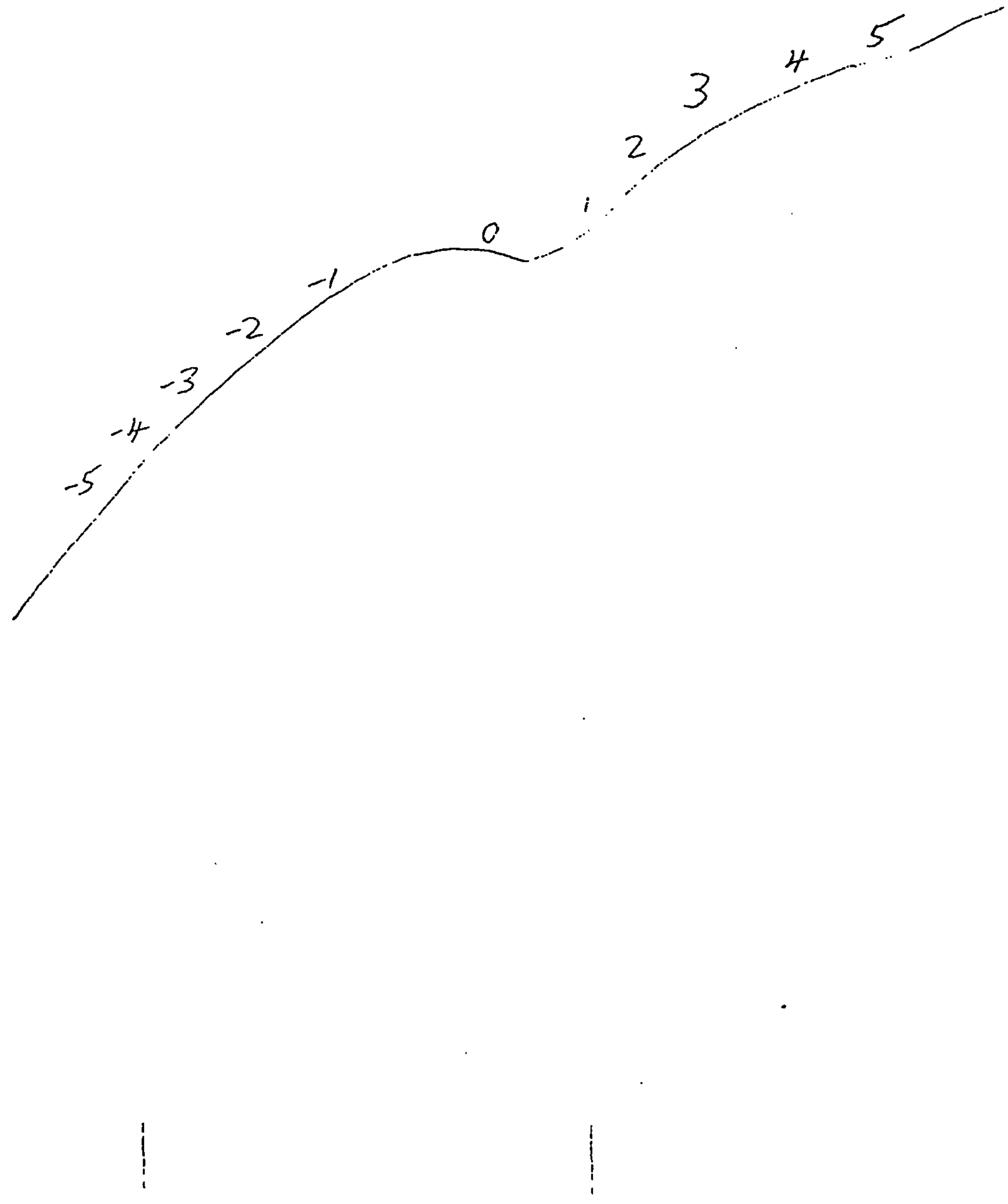

Figure 11

Figure 12