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ABSTRACT

Defining better implicit models of children's actions in a series of situations is of paramount importance to understanding how knowledge is constructed. The objective of this study was to analyze the implicit mental models used by children in complex change problems to understand the stability of the models and their evolution with the child's development. The study was structured in two phases. First, a written test was given to 198 fourth-, fifth-, and sixth-graders in order to identify different stable resolution patterns used by children in a set of problems involving the reconstruction of a change. The second phase consisted of individual interviews with 15 children of each level representing all procedures. Three models were identified from the study: sequential model, state comparison model, and change comparison model. Two conceptual leaps in the transition from the different models were identified: the first related to the representation of the problem structure and the other deals with the concept of number. Contains 14 references. (MKR)

Title: Conceptual and developmental analysis of mental models: an example with complex change problems.

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The learning of the four arithmetic operations constitutes an important element of the mathematics curriculum in primary school. One of the goals for this learning is to bring the child to understand each operation and to recognize which one to use in a problem solving context. However, this is not easily achieved.

In a previous research (Bednarz, Schmidt and Janvier, 1989), observations led us to study additive change problems wherein an initial state is changed into a new state of similar nature in which it is included. Some problems involving an arithmetic change provide the initial state as well as the final and the question centers around the change (what happened). The child must therefore reconstruct the change when the problem does not explicitly provide it. (This is what we call the reconstruction of an arithmetic change).

Example 1: Mary has 5 marbles. Her father gives her some more. She now has 8 marbles. How many did her father give her?

In this problem, the initial state (5 marbles) is changed into a final state (8 marbles) and the question centers on the change.

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Several studies (Carpenter and Moser, 1982; Vergnaud, 1982; Riley, Greeno and Heller, 1983; DeCorte and Verschaffel, 1985...) have shown the difficulties of such problems for grade school pupils. In these cases, children recognize only the states, combining them in different ways, but cannot perceive the intermediate change which occurred. Other studies have also indicated that such difficulties are not limited to young children but are still present with older children, even in high school (Conne, 1979; Marthe, 1979; Bednarz et al, 1989...) when they are confronted with more complex problems involving a sequence of changes. Similar erroneous procedures are employed by students of different levels.

Example 2: John plays with marbles. In the first game, he lost 7 marbles. He plays a second game; we are not telling you what happened. If, after the two games, he has won 5 marbles altogether, has he won or lost during the second game and how many?

On one hand, studies have demonstrated the difficulties encountered at the early stage of elementary school, to solve additive problems. On the other hand, other studies have highlighted similar difficulties with older children when confronted to more complex problems of similar structure. Such results are puzzling. Why do we find the same errors at different levels (primary and even high school)? How can we explain the stability of such errors and the difficulty for children to represent, in a given situation, the change that happened?

Theoritical Framework

Many studies held in mathematics and science education try to get a better grasp at the underlying errors and to understand more precisely the child's mathematical thinking. These concentrate on the error and its role, thereby looking in a global fashion at the mechanisms of knowledge construction as well as at the teaching and learning phenomena.

This constructivist view concentrates on the learning activity whereby the error plays a central role. With this approach, errors originate from a knowledge of limited applicability rather than the result of a random process. Errors are systematic and stable; they are not unpredictable. They play a central role in the knowledge construction process: knowledge is built from and/or in opposition to previous experiences and knowledges. Hence, constructivists perceive the learning process as the child's adaptation in reaction to his environment. Our reseach is along that line. By looking at the procedures used to solve different problems in which the child has to mentally reconstruct a change, we want to discover on which internal representations of the relationship between the data are these procedures based.

We postulate that, behind these stable errors and processes, lie certain invariant organizations of methods which lead to the resolution. These organizations are studied along with the notion of "implicit mental model", a notion central to the process of problem-solving (Greeno, 1991). Greeno is hypothesizing that reasoning is partly based upon the construction and manipulation of mental models. He sees a mental model as a particular type of mental representation. Therefore, learning to reason with models having constraints germane to the conceptual field is crucial to the learning of this field.

Brousseau (1972) defines "mental models" as follows:

> "When a child in a series of comparable situations (same structure) shows a series of comparable behaviours (same reaction), one can conclude that this child has perceived a certain number of elements and relations of this structure. He, therefore, has a certain mental model of this situation." (1972, p. 58)

Rouse and Morris (1985) have produced a synthesis of various definitions of mental models and they have shown that they share a series of functions and goals:

> "The common themes are describing, explaning and predicting, regardless of whether the human is performing internal experiments, scanning displays or executing control actions" (1985, p. 11)

A mental model has therefore a heuristic function. It represents a structured entity and its structure must relate to the reality it represents. Fischbein (1990) is defining a model as follows:

> "Given two systems A and B, B may be considered a model of A, if, on the basis of a certain isomorphism between A and B, a description or a solution produced in terms of A may be reflected, consistently, in terms of B and vice versa." (1989, p.9)

This definition emphasizes certain aspects of a model. First of all, it mentions that the model must be able to replace the original. In addition, the relation between the original and the model must be based on some type of structural correspondence. Finally the model must be autonomous from the original. Fischbein, Tirosh, Staby and Oster (1990) have studied this last feature:

> "Being structurally unitary and autonomous, the model often imposes its constraints on the original and not vice versa! Consequently, a model is not simply a substitute, an auxiliary device (more simple, more familiar, more accessible)." (1990, p. 24)

Fischbein states that this autonomy of models is a condition to their heuristic efficiency. Even though a mental model must be a substitute to the original, it cannot just be a mere reflection of the original but rather a structure governed by its very own rules and parameters. In conjunction with its autonomy, Fischbein mentions that the model must also be stable:

> "The autonomy and stability of mental models seem to suggest that they are not mere products, mere reflections of the originals. They belong to the mental structure of the individual, well integrated into this structure, reflecting its requirements, its particularities, its schemata, its laws" (1990, p. 29)

This ties in very well with Brousseau's definition presented above. The autonomy of the model with respect to the original and its stability mean that the model originates from the mental structure of the subject. As the mental model guides the child's action when solving problems, it will bring about stable procedures, sometimes erroneous, which will be a reflection of his own mental structure.

The importance of better defining the implicit models leading to the children's action in a series of situations is of paramount importance to understand how knowledge is constructed.

Objective

The objective of this study is to analyze the implicit mental models used by children in complex change problems, to understand their stability and their evolution with the child's development.

Method.

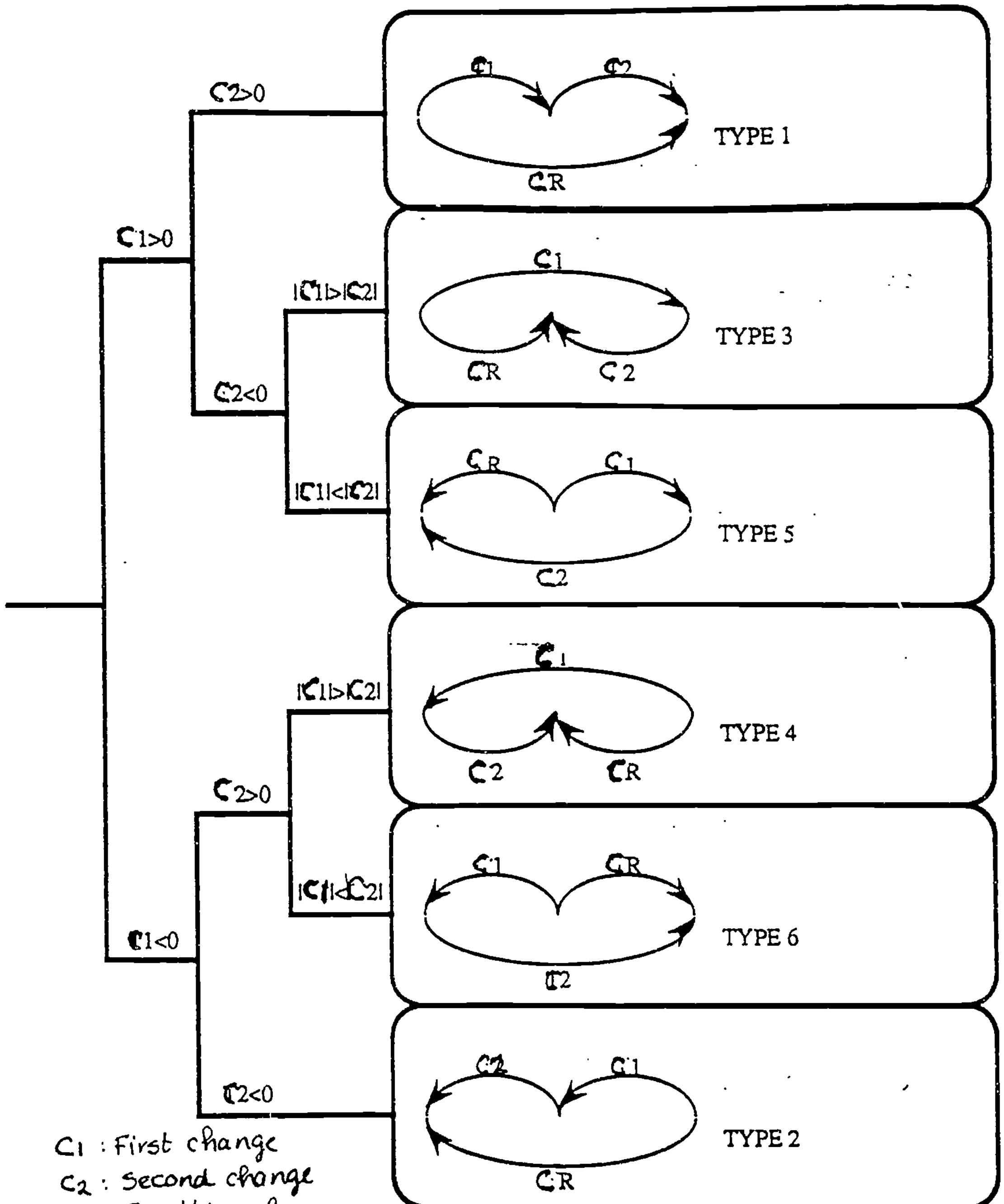
The study is structured in two phases. First, we used a written test given to three groups of each level (4th, 5th and 6th grade, ages from 9 to 12) for a total of 198 students, to identify different stable resolution patterns used by children in a set of problems involving the reconstruction of a change (see table 1). This type of problem was presented in two distinct contexts (discrete collection and continuous length) in order to verify the stability of the procedures against the context.

Discrete collection:

> John plays with marbles. In the first game, he lost 7 marbles. He plays a second game; we are not telling you what happened. If, after the two games, he has won 5 marbles altogether, has he won or lost during the second game and how many?

Continuous length:

> Window cleaners are installed on a platform. First, the platform is raised (lowered) by 20 meters. It is moved once more. If we know that after these two dispacements, the platform is 30 meters higher (lower), was the platform moved the second time upwards or downwards and by how much?



C_1 : First change
C_2 : Second change
CR: Resulting change

Table 1
Different types of complex change problems
(Type 1 to 6 are of increasing complexity)

Problems such as above have two specific features relevant to this study. First, their underlying structure demands a reconstruction [(a + ? = b) where a and b are known]. Second, "a" and "b" being associated with changes instead of states will require an interpretation by the child which goes beyond the usual natural numbers.

An a priori analysis of the problems have highlighted different types, type 1 to 6, corresponding to their complexity (see Vergnaud's analysis). By studying the main errors made by children in solving problems of the C_1XC_R type, a series of procedures were identified:

Procedures

- C_R	pupil answers in terms of the resulting change
- C_1+C_R	pupil adds the two data of the problem
- $C_1 \xrightarrow{C_R} X$	pupil applies the resulting change on the first one to get the final state
- Requires an initial state	
- $S_1 \overset{?}{\frown} S_F$	pupil compares the two changes (as states) and find their differences
-Success	pupil uses a procedure leading to the correct answer

Can any of these procedures stem from the same reasoning? How do they differ from each other? When confronted to the two distinct contexts, will the children use the same procedures? Will the procedures differ for different levels and for different schools?

In an attempt to answer the above, we have retained the problems most pertinent to a written test, namely type 2, 3, 4, 5 and 6 problems for both contexts; the test also included problems of C_1C_2X type (direct sequence).

Example: John plays with marbles. In the first game, he loses 8 marbles. He plays again and wins 6 marbles. After the two games, has he won or lost and how much?

By starting the problem with a loss, what will a child requiring an initial state do? Children who add the two values or use the $C_1 \xrightarrow{C_R} X$ procedure should use the same procedure as in solving C_1XC_R problems. However, children that compare the two values should use an alternate procedure.

The absence of an initial state appear to play a central role in explaining the difficulty with the C_1XC_R problems. In order to confirm this need for an initial state and to verify if children are using different procedures when an initial state is provided, we have included also a ISC_1XC_R type problem in the written test. These are put at the end to ensure that they will not influence the resolution of the problems where the initial state is not provided.

Example: Sabine has 17 marbles. She plays a first game and loses 9 marbles. She plays once more. If we know that after these two games she has won 5 marbles, has she won or lost in the second game and how much?

The following table summarizes the content of the written test:

5 types of C_1XC_R problems of increasing complexity					C_1C_2X	IS C_1XC_R
type 2	type 3	type 4	type 5	type 6		
two different contexts for each type of problems						

Each and every child was given a booklet with all the problems in random order except for the two problems giving an initial state which were placed at the end of each booklet; a total of 198 students (4th, 5th and 6th graders) passed the written test.

The written test was reviewed in terms of procedures used by each student for each problem instead of looking for right or wrong answers. Each procedure was then coded as follows:

Procedure	Code
C_R	2
C_1+C_R	3
$C_1 \xrightarrow{C_R} X$	4
Initial state needed	5
$S_1 \xrightarrow{X} S_F$	6
Success	7
No answer	1
Other	9

Results were then collated in a table where each line was showing the characteristic of a given child (his assigned number, school level and school A, B or C) as well as the procedures used at each and every problems of the test.

Student	Level	School	C_1XC_R					C_1C_2X	IS C_1XC_R
			Type 2	Type 3	Type 4	Type 5	Type 6		
			M W	M W	M W	M W	M W	M W	M W

The results of the written test were processed using a cluster analysis (analyse de correspondance). The cluster analysis is a statistical process that can be used to identify representative groupings of procedures used by the children. This statistical analysis has determined that some class of procedures consistently used over a series of situations regardless of the context and the school, with each class including procedures related to the same basic reasoning. For example, in grade 4, a class regroups $C_1 \xrightarrow{C_R} X$, C_1+C_R, whereas in grade 5, it regroups $C_1 \xrightarrow{C_R} X$, C_1+C_R et C_R, and in grade 6, Need of an initial state, C_1+C_R et C_R are grouped together. We have given this class the name "Sequential". Two other groupings of procedures were identified (state comparison and change comparison).

The second phase consisted in individual interviews with fifteen children of each level representing all procedures. These interviews were used to substantiate and further understand the qualitative meaning of each grouping of procedures. This analysis has shown that specific erroneous procedures identified in the written test relate to the same reasoning or mental model. The following three general models have been identified (they will be described using the problem "John" presented as example 2, page 2).

1- Sequential Model:

The child considers the first change as an initial state on which the resulting change operates thus producing a final state. Generally, the subject treats "lost 7 marbles" as an initial state "John had 7 marbles"; he then operates the resulting change "has won 5 marbles" thus obtaining a final state "John has now 12 marbles"

$$7 \xrightarrow{+5} 12$$

The child's answer will vary depending if he answers in terms of the final state "He now has 12 marbles" or in terms of the resulting change which has become the change (the crux of the question) "He has won 5 marbles". Some show signs of requiring an initial state to solve the problem and, when given one, they are using a sequential model.

Kingsley (4th grade):

"John is playing with marbles and he had 7 marbles. He plays again and he wins 5 marbles. And the question is if he has won or lost and how many. He has won. He has now 12 marbles."

Julien (5th grade):

"He had 7 marbles. He wins 5 other. He has 12 marbles alltogether. So, during the second game, he has won 5 marbles."

(He then draws:)

o o o o o o o ⟶ before

o o o o o ⟶ second game

Sonia (5th grade):

> "I can't do it".
> Interviewer: "Why not?"
> Sonia: "Because I don't know how many marbles he had at the beginning of the game."
> Interviewer:" If I tell you that he had 15 marbles, would that help you?"
> Sonia: "Sure. So, he had 15 marbles. He then lost 7 marbles. He had 8 marbles left. Then, he won 5 marbles. He has 13 marbles alltogether."

With this model, the child does not understand the underlying structure a+?=b but rather treats the problem as a+b=?; he handles the data sequentially and as states instead of changes. He does not perceive that he has to reconstruct a change. This model will always lead to an erroneous answer.

2. State Comparison Model

The child is still treating the changes as states as in the first model, but he understands that there is a reconstruction involved (he sees the structure a+?=b). He thus compares the two states given to get their differences. He simplifies the problem by treating it as a reconstruction of states.

Anne-Marie (5th grade):

> "During the first game, he had 7 marbles and they are not telling you what happened during the second game. After that, he has won 5 marbles. He lost 2 marbles. He lost 2 marbles and he has 5 marbles left."
> (she then draws) John o o
> o o
> o o
> o o
> o o
> ———
> o lost 2 marbles
> o 5 marbles left.

Mélanie (6th grade):

He had 7 marbles and then at the end he only had 5 marbles left. He had to loose 2 marbles during the second game."

This model will provide the right answer with the simpler types of problems.

Example: Amélie plays with marbles. In the first game, she has won 7 marbles. She plays a second game; we are not telling you what happened. If, after the two games, she has won 5 marbles altogether, has she won or lost during the second game and how many?

Anne-Marie (5th grade):

"At first she had 7 marbles then she played another game. At the end, she had 5 marbles left. She must have lost 2 marbles during the second game."

Julien (6th grade) is the only one who used this comparison of states procedure that led him to wright answers. He sets a ficticious inital state first and then follows with intermediate states:

Julien : (6th grade)

"Lets say that he started with 15 marbles. Now then, he has lost 7 marbles. So 15 minus 7, he now has 8 marbles. If we say that he has won 5 marbles, you take this starting number 15 plus 5, that makest it 20 marbles. He must now make sure that he will end with 20 marbles. So you do 20 marbles minus 8 marbles, he therefore has won 12 marbles during the second game."

$$\begin{array}{r} 15 \\ \underline{-\ 7} \\ 8 \end{array} \qquad \begin{array}{r} 15 \\ \underline{+\ 5} \\ 20 \end{array} \qquad \begin{array}{r} 20 \\ \underline{-\ 8} \\ 12 \end{array}$$

3- <u>Change Comparison</u>

As in the previous model, the child sees the reconstruction but treats the data as changes, therefore fully understanding the problem and getting the right answer all the time.

<u>Frédéric</u> (5th grade):

> "In the first game, he has lost 7 and at the end he had won 5 marbles. So, from minus 7 to get to 5. He has won 12 marbles during the second game."

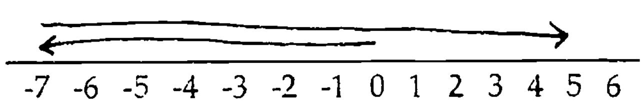
While solving the problem, Frédéric writes:

> First game: 7 marbles lost
> Second game: ?
> Alltogether: 5 marbles won.

Some of them have used a number line or a thermometer to find the difference between -7 and +5:

<u>Mélanie</u> (6th grade):

> "There, he lost 7. These are negative values. So, he goes to -7. He must have won since he went up ot zero plus 5 marbles. He has won 12 marbles.
> She draws:



This model is however used by very few children.

Interpretation

The cluster analysis shows a slight evolution in the model used: from a sequential model used predominantly by the fourth graders towards comparison models used by a majority of sixth graders. By associating these three mental models to the three distincts groupings, the statistical analysis provided the following results.

Models	Grade 4	Grade 5	Grade 6
Sequential	58%	44%	37%
State comparison	25%	35%	41%
Change comparison	2%	6%	11%
Other / no answer	15%	15%	11%

One can also identify from these results two conceptual leaps in the transition from the different models.

a) The first, linked to the passage from sequential to state comparison models, relates to the representation of the problem structure. To make the transition, the child must acquire a global anticipation of the problem: he must construct a "portrait" c the relations between all data before solving the problem.

b) The second conceptual leap, appearing from our results to be the most difficult, deals with the very concept of number. The child must reject the well established idea of natural number, (a measure of a collection or length) and replace it by the broader and more generalized concept of relative number. Being capable of making this transition from considering numbers as states to considering them as changes constitutes a considerable conceptual evolution. Vergnaud associates this transition as an epistemological obstacle.

> "Un obstacle épistémologique évident est la réduction par les élèves du concept de nombre à celui de mesure des grandeurs et quantités".
> Vergnaud, 1989, p. 83
> ("An obvious epistemological obstacle is the limitation of numbers to mere measures of length and quantity")

How can this absence of re-construction and this concentration on states can be explained? As the first two models center around numbers as states, are we witnessing the consequence of the current teaching methods or are we in front of a more fundamental phenomenon?

Piaget and his collaborators (1980, 1983, 1990) have studied these cognitive processes enabling the mind to acquire the concept of changes. Piaget identifies two categories of processes: correspondances involving comparison and changes (or transformations). Piaget feels that studying these cognitive processes is of prime importance:

> "...such questions are key to any constructivist epistemology which must distinguish if not oppose the two main function of reasoning: to compare and to transform."
> (Piaget, Heneriques and Ascher, 1990, p.15)

Transformation or change acts, modifies, constructs new things and leads to reversibility whereas a correspondance (comparison) is dependent on states under comparison. Piaget has identified a general development process characterized by the transition from concentration on states to relations that compare states to changes therefore leading to reversible operations. The three models in our study fit very well with Piaget's generalized process of development.

Conclusion .

This study has identified two fundamental conceptual leaps requiring special attention at school. However, our analysis of school textbooks (used in the province of Québec) showed that the present curriculum does not provide the sup to help children in their evolution towards the acquisition of those concepts.

We are presently involved in a study of the models and underlying obstacles to the resolution of these problems. Our first objective is to study if and how these models are altered in high school when relative numbers are introduced. Our second objective is to study the importance of the formulation of problem on the stability of these models (work being done in cooperation with Michel Fayol, Dijon, France). Results from this study will be used as the foundation of a new pedagogical approach.

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