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ABSTRACT

In this paper four programs are described in which children learn multidigit number concepts and operations with understanding: (1) the Supporting Ten-Structured Thinking projects, (2) the Conceptually Based Instruction project, (3) Cognitively Guided Instruction projects, and (4) the Problem Centered Mathematics Project. The diversity in these programs indicates that learning with understanding is possible under a variety of conditions; however, a critical feature shared by the four programs is that they engage students in building connections. In each of the programs, treating the development of arithmetic procedures as a problem-solving activity and asking students to share and explain their answers encourages students to reflect on procedures and on the properties of the whole number system and to learn these topics with understanding. Contains 16 references. (MKR)

Teaching Mathematics for Learning with Understanding in the Primary Grades

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Four programs are described in which children learn multidigit number concepts and operations with understanding. The diversity in these programs indicates that learning with understanding is possible under a variety of conditions; however, a critical feature shared by the four programs is that they engage students in building connections. In each of the programs, treating the development of arithmetic procedures as a problem-solving activity and asking students to share and explain their procedures encourages students to reflect on procedures and on the properties of the whole number system and to learn these topics with understanding.

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Teaching Mathematics for Learning with Understanding in the Primary Grades

Research is beginning to provide some perspective on what it means for students to learn mathematics with understanding (Hiebert & Carpenter, 1992), but there is little definitive evidence about how instruction should be designed to facilitate learning with understanding. Classical experimental paradigms in which variables are systematically manipulated have not proved particularly successful in resolving the issue. A thesis of this paper is that learning with understanding is possible under a variety of conditions. It is not the use of particular instructional activities or materials that leads to learning with understanding; rather it is critical is that instruction provides an environment that fosters understanding. Exactly what that means is the subject of the remainder of this paper.

We attempt to shed some light on how learning with understanding occurs in classrooms by describing critical features of four instructional programs designed to foster understanding of mathematical concepts and skills. To provide a focus for our analysis, we consider how understanding develops within a specified content domain, multidigit numbers and operations. This selection was based on the belief that learning of what has traditionally been considered computational skills offers an ideal site for developing understanding of fundamental number concepts. However, our conceptions of fundamental number concepts and skills and how those concepts and skills are acquired are very different from what has been the norm in traditional classrooms. Furthermore, the projects described in this paper share the assumption that the learning of what has traditionally been considered computational skills can support the development of an attitude toward learning mathematics in which understanding is thought to be critical and reasoning and communication are viewed as an essential component of learning mathematics.

We start with a brief discussion of understanding. Next we describe how multidigit number concepts and operations are learned in each of the four programs. Finally, we discuss some factors that are common to all four programs as well as some salient differences among them.

Understanding Understanding

Our working definition of understanding follows the analysis of Hiebert and Carpenter (1992) and others that characterizes understanding in terms of how knowledge is connected. Broadly speaking, knowledge that is understood is rich in connections. But this does not imply that all connections among constructs signify understanding. Particular connections among constructs may in fact represent misconceptions, whereas true understanding of a particular construct may require that knowledge is connected in very specific ways. Thus, it is critical to have some knowledge about the nature of the connections that are constructed.

Starting with the assumption that understanding can be described in terms of how knowledge is connected, differences in goals of instruction of different programs may be characterized in terms of the kinds of connections that are considered to be most central. Any program that is concerned with learning with understanding must develop connections between conceptual knowledge and the procedures children use to solve problems, but how these connections are established and which connections are vital may differ among programs.

Much work in mathematics is based on representing quantitative situations and then operating on these representations. Connections can be built among the different forms of representation (physical, pictorial, symbolic, spoken and written words) and within a particular representation form.

Connections between different forms of representation include relating written numerals with spoken words, relating physical and pictorial collections grouped by tens with numerals and spoken number words, and relating manipulations of physical materials with procedures with numerals. Symbols and symbolic procedures also can be given meaning by relating them to problem situations that children understand. For example, operations of addition and subtraction can be defined by relating them to joining, separating, comparing, and part-whole problem situations that most young children intuitively understand well before they encounter the formal operations of arithmetic (Carpenter, 1985; Fuson, 1992).

By connections within a representation form we mean noticing patterns and regularities within a system. This would include becoming aware of patterns within the standard notation system and using these regularities to develop additional concepts and procedures and to solve problems. For example, children who understand regrouping for two and three digit numbers can extrapolate that knowledge to larger numbers. Connections within a representational form also are represented by children justifying or extending a procedure by appealing to the properties of the number system itself rather than to some external referent like base ten blocks. For example, a child may draw on implicit knowledge of commutativity and associativity to add $58 + 36$ by first adding 50 and 30 and then adding on 8 and then 6 more. This does not mean that the child could state the principles or apply them in more general cases, but the child does appear to have some implicit knowledge of properties of numbers and operations to which new concepts and procedures can be connected.

Another critical kind of connection is between a mathematical construct or procedure and its purposes. An example is the procedure of adding from right to left in most conventional algorithms for adding multidigit numbers. The purpose of adding this way instead of from left to right is to eliminate the need to write intermediate partial answers when a column adds up to more than ten. The procedure is efficient, but there is little evidence that many children understand the purpose for adding numbers in this way. Without this understanding the procedure can appear arbitrary.

Specific examples of how these three dimensions of understanding are played out in children's learning of multidigit number concepts and operations are discussed in some detail in the sections that follow.

Teaching for Understanding: Four Examples

Four examples of programs designed to help children learn number concepts and operations with understanding are: (a) the Supporting Ten-Structured Thinking projects (STST), directed by Karen Fuson at Northwestern University, (b) the Conceptually Based Instruction project (CBI), directed by James Hiebert and Diana Wearne at the University of Delaware, (c) Cognitively Guided Instruction (CGI), directed by Thomas Carpenter, Elizabeth Fennema, and Megan Franke at the University of Wisconsin, and (d) the Problem Centered Mathematics Project (PCMP), directed by Piet Human, Hanlie Murray, and Alwyn Olivier at the University of Stellenbosch in South Africa.

All four programs take a problem-centered approach to teaching multidigit number concepts and operations. What this means is that the learning of multidigit concepts and procedures is perceived as a problem-solving activity rather than as the transmission of established rules and procedures. Teachers do not demonstrate procedures or expect all children to use a particular algorithm. Children spend a great deal of time working out their own procedures for solving place value and addition and subtraction problems and sharing and discussing alternate strategies with their classmates. The intent is to convey to students the importance of working out a strategy for solving the problem and then sharing and reflecting on alternative strategies.

In all four projects, the teacher plays an active role in the classroom by posing the problems, coordinating the discussion of strategies, joining the students in asking questions about strategies, and occasionally sharing an alternative strategy. The intent is to create an environment in which the construction of strategies is problematic and in which teachers support students' efforts to deal with such problems. In traditional instruction, teachers often do more than serve as a resource or guide. By intervening too much or too deeply, teachers can easily remove the problematic nature of learning multidigit operations. This does not mean that teachers in the four project classes do not provide input to classroom discussion, but they must be sensitive to convey to the students that they can figure out strategies for dealing with multidigit numbers and do not need to appeal to the authority of the teacher to ascertain whether a procedure is correct or acceptable.

Supporting Ten-Structured Thinking (STST)

The Supporting Ten-Structured Thinking (STST) project has engaged in three related series of studies. The first studies focused on small groups of second grade children using base-ten blocks to invent procedures to add and subtract four-digit numbers(Fuson & Fraivillig, 1993; Fuson, Fraivillig, & Burghardt, 1992). The second set of studies were case studies of two children using a computer base-ten microworld to invent multidigit procedures (Fraivillig, Fuson, & Thompson, 1993). The current studies focus on supporting urban, low SES Latino children's mathematical thinking (Fuson & Perry, 1993; Fuson & Smith, 1994). The recent studies go beyond the earlier studies in that they provide specific supports for linking single-digit and multidigit numbers and for relating written numerals and spoken number words. The recent studies also present problems in real world situations familiar to children.

The STST project provides the most explicit guidance and support for making connections between representations of the four projects. As with the other three projects, multidigit procedures are constructed by the children themselves, either individually or collectively, so that the learning of multidigit procedures is viewed as problem-solving rather than as the acquisition of established procedures. The recent STST studies also rely primarily on word problems set in a context rather than horizontally or vertically presented calculations presented on worksheets. What distinguishes the STST project from the other three projects is (1) connections are much more explicitly drawn between representational forms in order to give meaning to multiunit numbers and operations on them and (2) children are provided specific conceptual supports to provide quantitative meaning to procedures and word problem situations.

When children or the teacher discuss multiunit numbers and operations on them, they consistently specify relations between number words, numerals, and quantities; and conceptual supports are provided that make the connections more apparent. Initially children are expected to demonstrate explicit connections between symbolic procedures for adding and subtracting and operations on base ten blocks or other materials that show ones, tens, and hundreds. Thus, when children describe a procedure that they have invented, they are expected to be able to justify each step in the procedure by showing how it corresponds to a legitimate manipulation of the physical multiunit quantities. Children describe and justify their solution in words as they relate their written solutions to the corresponding physical representation. Having to describe how their operations on written symbols correspond to manipulations of the multiunit quantities makes it less likely that children simply imitate the solutions of other children and more likely that they understand the solutions that they generate.

STST instruction also provides conceptual supports to make connections between number words and multiunit groupings more explicit. The English words used to denote numbers less than one hundred do not support connections as well as the words for larger numbers do. With numbers larger than one hundred, the spoken number words specifically designate the multiunits and the number of each multiunit. We say "eight thousand three hundred" to designate 8 groups of a thousand and 3 groups of a hundred. With numbers less than one hundred the designation of units in European languages is less explicit and is irregular in several ways. The number names do not clearly emphasize the groupings of tens; in English we say "forty" rather than "four ten." The problems are even more acute for numbers in the teens. Numbers in the teens are designated by a single word, and the first syllable of the word denotes the units rather than the tens. For example, in the word seventeen the number of ones is said before the number of tens, which often results in children writing nineteen as 71.

To make the connection between spoken number words and multiunit groupings more explicit, STST students learn a spoken number word system in which multiunits are explicitly named. Children say "four tens and five ones" rather than "forty-five" and "one ten and seven ones" rather than "seventeen." When children are first learning multidigit concepts, they use this system as well as than the standard number words when they talk about numerical situations.

STST also structures instruction to take advantage of the properties of larger numbers that support the development of multiunit concepts. Because the English spoken number names for larger numbers support connections between number names and collected multiunits, instruction moves quickly to incorporate three- and four-digit numbers.

STST instruction also supports specific solutions of single-digit addition and subtraction that involve grouping by ten. Sums and differences are chunked to make a ten and some ones (e.g. $8 + 5 = 8 + 2 + 3 = 10 + 3$). There is a direct connection between these solutions and the Name Ten number system described above ($10 + 3$ translates directly to one ten three), and they are easily integrated into children's solutions of multidigit addition and subtraction problems.

Conceptually Based Instruction (CBI)

The Conceptually Based instruction project (CBI; Hiebert and Wearne,1992; 1993) starts with the hypothesis that connections do not have to be formed by developing step-by-step one-one correspondences between written notation and physical representations. Rather the focus is on having children reflect on similarities and differences highlighted by the different representation forms. Hiebert and Wearne also assume that these comparisons can be drawn most productively when the materials and symbols are both used as aids or tools for solving problems rather than studied outside of a problem context or used only for demonstration.

Several principles have guided the development of instruction to support students' efforts to make connections. First, external representations (physical, pictorial, verbal, symbolic) are used as tools for demonstrating and recording quantities, acting on quantities, and communicating about quantities. Second, once a particular representation is introduced (e.g., base-ten blocks), it is used consistently to allow students to practice using it as a tool and to become familiar with the uses and connections it affords. Third, the representations are used to solve problems as well as being analyzed as interesting artifacts in their own right. Fourth, class discussions focus on how the representations can be used and on how they are similar and different. The general aim of these principles is to help students become comfortable with different forms of representation and to build relationships between them.

Several additional guidelines were followed in designing instruction. Physical materials and verbal stories are used as the initial representations for quantities and actions on quantities. Pictures of the physical materials that have been manipulated by the students are then used for convenience and for focusing class discussions. Finally, written symbols are introduced as efficient ways of recording the quantities and actions that have been explored and discussed using the other representations. Once a particular form of representation is introduced, it is used continually and interchangeably with previous forms.

Classroom lessons are organized around the solving of several problems, usually drawn from a common theme or scenario. The problems are constructed so students can solve them with strategies already in their repertoire or with new, more efficient strategies. This feature is intended as a support for connecting new knowledge with prior knowledge.

During the first few days, only blocks are used to solve problems. The reasons for this are (1) to encourage all the students to become familiar with the features of the blocks, (2) to develop class discussions in a context in which all students can participate (The referents for the discussion are relatively unambiguous and all students are equally familiar with them.), and (3) to provide a referential base that students can later use to support their inventions and/or explanations (The blocks can serve as a ready referent for students when they explain or defend their strategies.). After students have some experience with the blocks, written symbols are used, and students develop more sophisticated strategies with blocks and written symbols simultaneously. Each child has a full set of blocks, and they are always available. By the end of first-grade, almost all students are familiar with the tool-like power of the blocks. At the least, they can be used as a default option and/or can be used to check the outcome of a newly invented mental or written strategy for adding or subtracting two-digit numbers. By the end of second grade, most students use written symbols to solve two- and three-digit problems, but they continue to use blocks to check their answers or to help them invent strategies for new kinds of problems.

During a usual class period, the scenario for the day is presented along with a first problem.

After the problem is solved, either individually or in small (spontaneously constructed) groups, students share their strategies. Because base-ten blocks are always available, the strategies rely either on written symbols, blocks, or a combination of the two. Occasionally, students solve the problem without blocks or written symbols but this is not common, probably because they are asked to recall later how they solved the problem and because mental calculation is not explicitly encouraged. Strategies are then discussed by encouraging students to ask questions if they do not understand, to comment on the strategy, and to compare it to others they have used or shared. Explicit attention is paid to the similarities between strategies with the blocks and strategies with written symbols. After the discussion, a second problem is posed, and the lesson continues in this way.

It should be noted that each form of representation affords and constrains strategies in particular ways. The base-ten blocks, for example, support regrouping strategies, especially in subtraction contexts that students perceive to require a take-away action. Discussions of connections between written and physical strategies in this context help to illuminate how regrouping is used in subtraction, because one must trade a ten for 10 ones or put a ten with the ones in order to take away more ones than are initially present.

Cognitively Guided Instruction

Cognitively Guided Instruction (Carpenter & Fennema, 1992; Carpenter, Fennema, & Franke, 1992) is not a program of instruction in a traditional sense. There is no curriculum or recommended activities. CGI focuses on teachers' knowledge. The goal is to help teachers better understand children's thinking so that they can build on the knowledge that children already have. It is up to the teachers to decide how to use that knowledge. However, in spite of the fact that there is no specified program of instruction, there are certain common features in many CGI classrooms. Although there are wide variations among classes, the following description captures some of the features that characterize instruction of multidigit number concepts based on in depth case studies of selected classes (Carpenter, Franke, Fennema, Weisbeck, & Ansell, in preparation).

In these Cognitively Guided Instruction classes, procedures for operating on multidigit numbers develop as natural extension of the procedures that children use to solve problems involving single units. When children enter school, most of them are able to solve a variety of basic word problems by modeling the action or relationships described in the problems. Initially they model the problems by using some kind of counters to represent the quantities, action, and relationships in the problems. Over time these physical modeling strategies are abstracted and abbreviated as children begin to use counting strategies and derived facts. Essentially the same pattern occurs for children's solutions of problems with multidigit numbers. Children's symbolic procedures evolve out of direct modeling strategies with tens materials.

Word problems provide the basis for almost all instruction. In the early grades, teachers begin by giving children a variety of word problems that can be solved by modeling and counting using single counters. Teachers do not demonstrate the solution to problems, but a great deal of time is spent discussing alternative strategies for solving each problem. The discussions serve as models for other children, and they provide an opportunity for children to reflect on their own solutions. Initially children solve problems involving multidigit numbers by modeling the problems with single unit counters. These solutions do not require any real conceptions of place value beyond the ability to count.

In the classes studied there was very little specific instruction devoted to place value instruction per se. Children engaged in a few activities in which they grouped collections of objects by ten or counted objects grouped by ten, but these activities did not seem to play a major role in children acquiring place value concepts. Essentially children appear to learn place value concepts as they explore the use of ten blocks and other base-ten materials to solve word problems and listen to other children explain their solutions with the blocks.

Typically base-ten materials, connected ten blocks or stacking cubes stored in rods of ten cubes, may be made available as early as the first or second week of school in the first or second grade. Children initially use these materials as single units to solve problems, counting each of the individual units in the ten blocks. The ten blocks simply serve as convenient collections of unit counters that do not get mixed up. With teacher encouragement, some children come to recognize that they do not have to count all the individual units in the tens block each time they construct a set. At first they may count on by one from ten. Soon they construct two-digit quantities by making collections of tens and ones.

At first most children are relatively inflexible in constructing and counting sets using tens. They may solve an addition problem by making each of the addends by counting collections of ten but then find the sum by counting the total by ones. Place value concepts emerge over time. There are a variety of direct modeling strategies that children use to solve different word problems. Some of the strategies are more sophisticated than others in that they are more efficient and involve more flexible use of place value concepts. The transition to using more sophisticated strategies may be facilitated by teacher probes ("Is there an easier way to count those? Is there something you can do so that you can take away 8?"), by the selection of problem types and numbers in problems, and by listening to other children demonstrate how they solved a given problem.

Teachers also use a mix of addition, subtraction, multiplication, and division problems involving different number combinations in a given lesson. For some problems children can use tens relatively easily, but for others it is more difficult. As a consequence, even children who can use tens blocks to solve some problems frequently fall back to using single unit counters for others. For example, children can combine tens and ones in adding 23 and 48 without actually exchanging ones for tens. Children can simply count on 11 more after combining the tens to get 60. On the other hand, partitioning a collection of 42 objects into 3 equivalent sets does require a tens-ones exchange. Thus, children have to evaluate whether they can use tens to solve a particular problem. There is not a single way of solving problems or using ten blocks that is practiced repeatedly.

Over time children become increasingly flexible and efficient in the use of base-ten materials. As their use of the materials becomes more automatic, they come to depend less on the manipulations of the physical materials themselves. Over time they are able to abstract their solutions with physical materials so that they can add and subtract multidigit numbers without them.

Throughout the year different children in a CGI class operate at many different levels with respect to place-value knowledge. One important consequence is that there is no prevalent strategy that all children use at a particular point in time. Children have the latitude to use a strategy that makes sense to them at the time. A consequence of the variety of strategies in use at any given time is that children have the opportunity to learn more advanced strategies by interacting with other students who are using them. Thus, although children are not asked to relate specific components of different representations to one another, they continuously shift among representations both in their own solutions of different problems and in their discussions of different strategies for the same problems with classmates. It is hypothesized that the continuing discussion of multiple representations and moving back and forth among representational types is what helps children to see the connections among different representations even though specific mappings are not specified.

Problem Centered Mathematics Project (PCMP)

The Problem Centered Mathematics Project (PCMP; Murray, Olivier, & Human, 1992; Olivier, Murray, & Human, 1990) is based on an analysis which portrays the development of children's number concepts and computational strategies as proceeding through three basic levels. Level 1 is characterized by the ability to count a number of objects and a knowledge of the number names and their associated numerals, without assigning meaning to the individual digits of a multidigit number. Typically children at this level solve addition and subtraction problems by direct modeling with single counters. Level 2 understanding of number is characterized by the ability to conceptualize number in the abstract, independent of immediate physical models. This ability comes into play in children's use of counting on and counting back strategies. It is not until Level 3 that children begin to use knowledge of place value to solve problems. At Level 3 they are able to use the property of additive composition of numbers to replace a given number with two or more numbers that are more convenient for computation. For example, they can interpret 34 as 30 + 4. But children do not only group numbers into tens and ones. For some problems, particularly those involving division, other groupings are more appropriate. For example, to share 51 candies among three children, it would not be useful to decompose 51 into 50 and 1; a partition of 51 into 30 and 21 or 30, 12, and 9 would make more sense.

Thus, the levels can be described in terms of children constructing increasingly abstract units of number. At Level 1 a number like 27 means 27 ones; at Level 2 it means 27 ones and also one 27; at Level 3 it means 27 ones and one 27, but also 20 and one 7 or 25 and one 2, etc. The basic approach in PCMP classes is to help children to construct these increasingly sophisticated concepts of different units, including ten, and to build these concepts on children's counting-based meanings by encouraging increasingly abstract counting strategies and child-generated computational algorithms.

Instruction in PCMP classrooms systematically facilitates the transition through the three levels of development described above by encouraging children to reflect upon their strategies for solving problems and discuss them with other children. Number concept development goes hand in hand with children's construction of computational algorithms, and little distinction is made between the two. Each of the levels affords certain kinds of strategies, and children within each level use methods for solving problems that are appropriate for that level. For example, children at Level 1 might add 28 + 15 by modeling the problem with counters and counting by ones. Children at Level 2 may count on 15 from 28, and children at Level 3 may decompose the numbers so that tens and ones can be added separately (20 plus 10 is 30, and 8 more is 38, and 2 more is 40 and then the 3 that is left from the 5 makes 43).

Children do not use structured manipulatives like base-ten blocks that embody base ten groupings. Instead, they use loose counters, collect them into groups of ten, and count 10, 20, 30, 31, 32, 33, 34. Children have two sets of numeral cards: multiples of ten and ones. To represent the numeral 34, they take the 30 card and the 4 card and place the 4 over the zero of 30. Once children reach Level 3, the representation of two-digit numerals is therefore handled as the juxtaposition of two numbers, a tens number and a ones number.

Computational procedures build directly on children's number concepts and their knowledge of properties of number operations rather than on connections to operations with manipulative materials. Any computational procedure involves transforming the given task to one or more easier tasks that the child already knows how to do. The process of changing the task to equivalent but easier sub tasks involves three distinguishable sets of sub tasks; (1) transformation of the number, (2) transformation of the given computational task, and (3) carrying out the computation. For example, the addition of 24 + 38 involves the transformation of the numbers (24 = 20 + 4 and 38 = 30 + 8). The computational task (20 + 4) + (30 + 8) is transformed to the equivalent task [(20 + 30) + 8] + 4. Transformations of computational tasks depends on at least implicit awareness of certain properties of operations (theorems in action), in this case the commutative and associative properties. The resulting computations involve tasks that are familiar and may be based on recall of known number combinations together with knowledge of number concepts. 20 + 30 = 50; 50 + 8 = 58; 58 + 4 = 62.

The standard vertical addition algorithm depends on these very transformations, but the transformations are hidden. In using a standard vertical algorithm, children often loose sight of the fact that they are actually adding 20 and 30; they think of the addition in terms of columns of numbers. 5 + 8 and 2 + 3. In PCMP classes the procedures are carried out at the conceptual level; children actually think of the addition as 20 + 30, not 2 + 3. Children never are expected to use standard computational algorithms.

The instructional approach emphasizes the role of negotiation, interaction, and communication between teacher and students and among students in the evolution of their cognitive processes. Problems are set to students in small groups. Students are expected to demonstrate and explain their methods, both verbally and in writing, with the teacher providing needed support with respect to notation and terminology. Children are also encouraged to discuss, compare, and reflect on different strategies, trying to make sense of other students' explanations, thereby learning from each other. Teachers spend a great deal of time listening to pupils, accepting their explanations and justifications in a nonevaluative manner, with the purpose of understanding and interpreting children's available cognitive structures. This enables the teacher to provide appropriate further learning experiences that will facilitate the child's development.

Commonalities and Differences Among the Four Programs

In all four of the programs, the vast majority of time is spent engaged in activities in which connections between or within representational forms are made explicit. Children discuss multiple strategies involving multiple representations; they use symbolic procedures in which connections to basic number concepts and properties of operations are explicitly drawn upon. They are not presented with procedures to follow; they are not expected to engage in the syntactic manipulation of symbols. All four programs at least defer the use of traditional vertical algorithms for addition and subtraction until children have demonstrated some basic understanding of multidigit numbers and procedures. In the Problem Centered Mathematics Project standard vertical algorithms are never introduced. In contrast, traditional programs of instruction focus on one desired procedure and attempt to move quickly to the practice of symbolic skills. In these four programs, the time is spent insuring that children's knowledge is connected rather than practicing skills.

A key feature of all four programs that insures that children must connect the concepts and procedures that they are learning to their existing knowledge base is that all learning, including in particular the learning of multidigit procedures, is taken as a problem-solving activity. Children are not provided with algorithms to learn; they must construct them themselves. Multidigit problems can be solved with understanding at a number of levels of sophistication, ranging from direct modeling with counters or base ten blocks up to very abstract invented algorithms. Because children construct their own procedures there is no reason to imitate a procedure that they do not understand. As a consequence, children should recognize the reason for each step in a procedure, because they are the ones who decide what steps to follow. In other words, because children construct and explain their own procedures, they should be able to connect the steps in the procedures to their purposes. Teachers clearly communicate that specific solution strategies are not expected, and the classroom environment is structured to encourage children to construct alternative strategies. One of the critical factors in establishing this problem-solving environment is that children are asked to describe and explain the strategies they used to solve any given problem. Children talk about how they solved a problem to the teacher, to other children or small groups of children, and to the whole class. This discussion of alternative strategies serves four important functions: (1) It communicates to children that alternative strategies are valued. (2) It forces children to use procedures that they understand, because they need to understand whatever procedure they use well enough to explain it. (3) Explaining procedures encourages children to reflect upon them. Many researchers have pointed out the centr-' role that reflective abstraction plays in the construction of abstract number concepts. Earlier in this paper we discussed how operations with manipulative materials and abstract symbols can be linked through explanation and reflection. (4) Children can learn from one another. The explanations of other children provide models of alternative strategies that children can use for themselves. This social construction of knowledge is very different from situations in which the teacher presents a strategy for all children to imitate. Children are not expected to adopt specific strategies that other children present. Although interactions with other children influence the strategies that any child adopts, they are not in a position that they have to adopt any strategy that they do not understand.

Fundamental Differences

Although the four programs all provide extensive opportunity for students to connect emerging number concepts and procedures to previously established concepts and procedures, there are fundamental differences in the nature of the connections and how they are formed. One of the fundamental differences is in the roles played by connections between and within representational forms. These differences are manifested in the children's use of manipulative materials. The Problem Centered Mathematics Project does not employ structured base-ten manipulative materials like base-ten blocks. In the other three programs structured ten materials play a prominent role.

Within the three programs in which structured ten materials are used, there are critical differences in the ways in which materials are used. In the Supporting Ten-Structured Thinking project, specific attention is drawn to the connections between operations on base-ten blocks and operations on symbols. Each step in the symbolic procedure is linked to the corresponding operation on the base-ten blocks. In Conceptually Based Instruction step-by-step mapping enters the class discussion as one way of justifying a particular procedure. However, step-by-step mapping is not required; for some students procedures with blocks and written symbols do not develop simultaneously. Procedures with blocks are developed first, and procedures with written symbols are then developed by reflecting on the blocks procedures. In Cognitively Guided Instruction, manipulations with blocks generally are not linked step-by-step to manipulations with symbols. Symbolic procedures emerge as more efficient variants of procedures with blocks. Connections between blocks and symbols are constructed as children abstract the operations on the blocks in creating their own invented symbol procedures. Having children explain how they solved problems with the blocks may play a significant role in extending the physical modeling strategies with base-ten blocks to more abstract symbolic procedures. When children talk about combining tens, trading tens for ones, and the like, their verbal descriptions of operations with physical materials come to sound very much like the invented symbolic procedures that replace them.

There has been a great deal of research on the question of the use of manipulative materials. Initially the questions revolved around whether instruction with manipulative materials was more effective than instruction in which children did not use manipulatives. Over time we have come to recognize that the question is not so simple as whether manipulatives are used or not; rather the ways in which manipulatives are used and how they support understanding of fundamental constructs is critical. We also see from these four projects that there may not be a single best material or single best way to use materials to support the learning of specific concepts. Instead, genuine understanding may occur under diverse conditions and along different paths.

This diversity also is reflected in the sequences in which concepts may be learned. There has been an extended debate in the literature whether children should develop a relatively solid understanding of basic multiunit concepts before they are asked to apply those concepts to add and subtract multidigit numbers or whether the multiunit concepts are more effectively learned in the process of using them to add and subtract (Baroody, 1990; Fuson, 1990). Our evidence suggests that children can learn with understanding under both conditions. In the Conceptually Based Instruction project children spend a substantial amount of time on grouping activities designed to develop multiunit concepts before they are given problems involving addition and subtraction of multidigit numbers. In the other three projects the learning of multiunit concepts is more integrated with multidigit addition and subtraction from the start.

Conclusion

We began the paper by defining the development of understanding as the process of building connections. It is not surprising that the critical features shared by the four programs engage students in building connections. Treating the development of procedures as a problem-solving activity and asking students to share and explain their procedures encourages students to reflect on procedures and on the properties of the whole number system. Reflection of this kind involves drawing connections between forms of representation or drawing connections within a particular form of representation, or drawing connections between a procedure and its purpose. We cannot yet say exactly which connections are critical, and it appears that understanding may be generated through a variety of different connections. Consequently, some differences between the programs that may be quite salient, such as whether physical materials are used and how they are used, may not be essential for learning with understanding. What appears essential is that students are provided with many opportunities to create connections through developing and reflecting on procedures.

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