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ABSTRACT

In this paper it is argued that the characterization of mathematical justifications as inductive or deductive is incomplete. Promoting in classrooms the development of and discourse about a third type of reasoning, transformational reasoning, which considers the results of operations on a set of objects, may contribute to mathematics learning in significant ways. Topics discussed include: contrasting the presence and absence of transformational reasoning, defining transformational reasoning, transformational reasoning as a way of thinking, and implications. Contains 19 references. (MKR)

Beyond Inductive and Deductive Reasoning: The Search for a Sense of Knowing

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Paper prepared for the symposium entitled: Mathematical Argumentation and Knowledge Development: Focusing on Mathematics Students

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Current mathematics reform efforts (NCTM, 1989) emphasize the importance of students engaging not only in the generation of mathematical ideas, but also in the validation and modification of those ideas. Classroom mathematics activity strives to embody key aspects of the practice of communities of mathematicians (Chazan, 1990; Hanna, 1990; Lampert, 1990), acculturating (Cobb, Yackel, and Wood, 1988) students into accepted (by the mathematics community) practices of proof and refutation (Lakatos, 1976). Thus, mathematics classroom communities, like communities of mathematicians engage in arguing and evaluating the validity of mathematical ideas for the purpose of increasing the mathematical knowledge of the community and its members.

The mathematics education research community has come to see proof as involving an interaction of social and cognitive processes (Simon & Blume, 1993; Hanna, 1990, Balacheff, 1987). Further the learning to create proofs has been characterized as proceeding from inductive (empirical) reasoning to deductive reasoning (Simon & Blume, in press-b; Balacheff, 1987, Van Dormolen, 1977; Bell, 1976).

In this paper, I argue that the characterization of justifications as inductive and deductive is incomplete. The quest of mathematics learners to understand mathematics and to determine mathematical validity leads not only to inductive and deductive reasoning, but also to a third type of reasoning. Promoting in classrooms the development of and discourse about this third type of reasoning (along with deductive and inductive) may contribute in significant ways to mathematics learning.

This paper is theoretical rather than empirical, although my positing an additional category of reasoning is the result of reflection on data from two research projects, the Construction of Mathematics (CEM) Project (Simon & Blume, in press-a) and the Educational Leaders in Mathematics (ELM) Project (Simon & Schifter, 1991) as well as informal data from other mathematics teaching situations. The contents of this paper was not my intended area of study. Rather, I have developed this hypothesis to account for numerous observations which I could neither overlook nor account for with available conceptualizations.

Consider the following scenario from the CEM project (Simon & Blume in press-b). A class of prospective teachers were asked for ways to determine the area of an ameba-shaped figure that was drawn on the chalk board. The students worked on this problem in small groups. Following the group work, the teacher requested that students report all of the strategies that had been generated. No evaluation of strategies was to take place until all of the strategies were described. Lilly shared a strategy that her group had generated. ". . . if you took some kind of rope or something and measured the whole outside of the area and then pulled [the rope] out into a shape like a rectangle or a square, you'd get the area." Ignoring the teachers intention to postpone evaluation of strategies, the students immediately began to locate string, belts, etc. to explore the proposed strategy. There was a high level of energy and activity in the group as the students worked together to determine the validity of the "string strategy."

What were these students doing? Observation of their process revealed that they were <u>not</u> collecting data on multiple examples of the string strategy. Thus, their process was not an inductive one. Neither were they trying to generate a deductive proof for the strategy. Rather, the students' messing around seemed to be an attempt to develop a feel for what happens to the area when the string is reshaped from a "blob" to a rectangle. Three points are worth noting regarding this scenario. First, as mentioned, the students were engaged in an attempt at validation that was neither inductive nor deductive. Second, the students had a sense that by examining this strategy in action, they could know whether it preserved the area of the blob. Third, the students had a spontaneous desire to pursue this way of knowing. These three characteristics can be seen in other examples that I will refer to later in this paper.

These and other data have led to a hypothesis that although inductive and/or deductive reasoning may lead to students persuading themselves of the truth of an idea, that often what they are seeking is not inherently inductive or deductive. Rather they are seeking a sense of how the mathematical system in question works. Such knowledge is often the result of "running" the system, not to accumulate outputs as in an inductive approach, but rather to develop a feel for the system. I call this *transformational reasoning* [1].

I will present some examples which illustrate the nature of the knowledge that results from transformational reasoning and contrast it with examples where such knowledge is lacking. I will then define and describe transformational reasoning directly.

Contrasting the Presence and Absence of Transformational Reasoning

An example from the ELM Project:

Mary, a tenth-grade geometry student in Ms. Goodhue's geometry class, was participating in an exploratory lesson on isosceles triangles. Her class had not yet worked with the theorem that states that the base angles of an isosceles triangle are congruent. The students were asked to explore isosceles triangles using the Geometric Supposer software (Schwartz & Yerushalmy, 1985) and record the side lengths and angle measurements in the table provided.

Mary was creating isosceles triangles by specifying the dimensions of the three sides.

Ms. Goodhue: Mary, could you make an isosceles triangle by specifying two angles and the included side?

Mary pauses and then punches in equal angles.

Ms. Goodhue: Can you tell me what you did?

Mary: Well, I know that if two people walked from the ends of this side at equal angles towards each other, when they meet, they would have walked the same distance.

Author: What would happen if the person on the left walked at a smaller angle to the side?

Mary: (without hesitation) Then that person would walk further [than the person on the right] before they meet (Simon, 1989, p. 373)[2].

In this example, Ms. Goodhue anticipated that her students would generate several different isosceles triangles and notice a pattern (inductive reasoning), that the base angles are equal. This would lead to a conjecture about all isosceles triangles, which would create a need for deductive proof. However, Mary showed evidence of a different way of knowing the relationship between the legs of an isosceles triangle and its base angles. Mary was able to see an isosceles triangle, not as a static figure of particular dimensions, but rather as a dynamic process that generates triangles from the two ends of a line segment. Her dynamic mental model allowed her to know, not only the relationship between the base angles of an isosceles triangle, but also to reason about the relative lengths of the legs of a triangle given unequal base angles. Due to the nature of her reasoning, these two ideas were connected; the isosceles triangle was a particular case of a more general understanding about angles and sides of triangles.

Mary's reasoning about triangles can be contrasted with Bill's. Bill was also confronted with the issue of the relationship of the lengths of the sides and the opposite angles. Bill was an undergraduate student who was attempting to draw a diagram to represent triangle side lengths of x, 4x, and 4x+1. He had no information about the angles. He drew the following diagram see Figure 1):

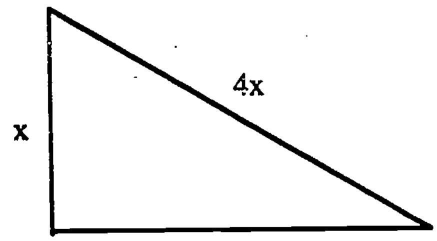

Fig. 1

Reflecting on what he drew, he said,

> Uh oh! I drew a right triangle, and the way I labeled it, this side [pointing to the side labeled "4x"] would be the longest side. But the hypotenuse should be the longest side...(pause) I know! I'll make this angle [the one appearing to be 90 degrees] 91 degrees.

Bill then labeled that angle 91 degrees and proceeded with the problem, satisfied that he had resolved his dilemma (Simon, 1989).

Bill's reasoning about triangles was very different from Mary's. While Mary engaged in transformational reasoning, Bill relied on bits of recalled knowledge. He did not see a right triangle as a static example of a dynamic system in which side lengths and angle measures vary in a related fashion. He did not see the right triangle as a special case, one instance of the phenomenon that the longest side is opposite the largest angle. Bill *remembered* that the hypotenuse is the side opposite the right angle and that the hypotenuse is the longest side.

Let's consider a second example of transformational reasoning. Consider the problem:

> Triangle ABC has a right angle at vertex B. I is a point on side AB between the endpoints. How does the sum of the measures of line segments AB + BC compare to the sum of the measures of AI + IC (see Figure 2)?[3]

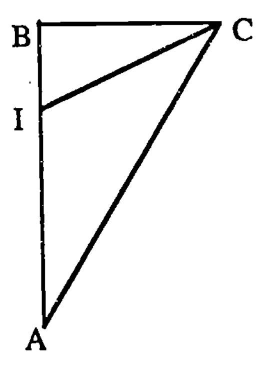

Fig. 2

A deductive proof can be generated showing that the sum of the measures of line segments AB and BC is greater than the sum of the measures of AI and IC. However, a transformational reasoning approach can make use of the learner's experience and provide a different kind of knowing related to this problem. Consider Sam's approach:

> Point A is my house and Point C is my school. Inside triangle ABC is the wooded area. Normally, I walk from my house (A) up the block to the corner (B), turn right and walk to the school (C). Sometimes, as I am walking up the block, I cut through the woods (from point I). I know that when I cut through the woods, the walk is shorter. I also know that the sooner [closer to A] that I cut through the woods, the shorter I have to walk.

Note that the final sentence of Sam's transformational reasoning allows him to posit a more complex and more encompassing theorem than was stated in the original problem.

Defining Transformational Reasoning

What is transformational reasoning? **Transformational reasoning is the mental or physical enactment of an operation or set of operations on an object or set of objects which allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated.** If we use the example of Mary's reasoning about isosceles triangles, the set of *objects* being operated on are line segments which are potentially the bases of isosceles triangles. The *operation* which transforms these objects is the generation of line segments (paths) from the endpoints of the original segment. The *results* are the triangles generated.

In the definition, I *mental* enactment refers to operations carried out in mental images. Piaget and Inhelder (Gruber & Vonèche, 1977)[4] made two relevant distinctions with respect to mental images. They categorized mental images as either *reproductive* or *anticipatory*, indicating that a mental image may recreate a previous perception or may anticipate transformations which have not previously been perceived. Piaget and Inhelder went on to elaborate reproductive images as including static, kinetic, and transformational. Transformational reasoning is supported by transformational reproductive images or by anticipatory images. In either case, the problem solver is able to visualize the transformation resulting from an operation. However, transformational reasoning is not restricted to mental imaging of transformations. Physical representations may be used to examine the results of a transformation. For example, a student who is exploring the validity of the statement, "If you know the perimeter of a rectangle, you know its area," might work with a loop of string observing what happens to the area as she makes the rectangle longer and thinner. However, a *mental anticipation* was required in order for the student to think to model this problem using the loop of string; that is, she knew before handling the string how she would model the rectangles and how she would use the string to observe the results of the operation.

Transformational Reasoning as a Way of Thinking

As with other types of reasoning (e.g., inductive, deductive), instances of transformational reasoning may range from relatively trivial to extremely powerful. Transformational reasoning tends to be more powerful when it considers a range of potential situations rather than just one. For example, according to the definition above, we could label as transformational reasoning a child envisioning a set of five blocks and a set of three blocks becoming two rows of four blocks, which she recognizes as a total of eight . However, the same child might envision two odd numbers, each as a set of paired blocks plus a single, coming together resulting in the pairing of the single blocks. Such an image allows the student to "see" the result of adding (any) two odd numbers. In the latter example, transformational reasoning provides a way of thinking about a broader set of mathematical objects and therefore an opportunity for generalization.

Although a well-developed ability to generate transformational mental images seems to be an important asset in transformational reasoning, it is not sufficient to insure that transformational reasoning will be used regularly and productively. Thinking back on the examples of transformational reasoning described earlier in this paper, one might reach the following conclusions. Many students of mathematics (as well as readers of this paper) would not generate Mary's transformational reasoning with respect to isosceles triangles nor Sam's reasoning for comparing the lengths of paths ABC and AIC nor any other transformational reasoning approach to those situations. However, many of those who would not spontaneously generate the transformational reasoning would have no trouble producing the images once they heard them described. Thus, in order to understand transformational reasoning as a way of thinking, we must look beyond the ability to generate transformational mental images and ask how a particular incidence of transformational reasoning is generated. Unfortunately, I have little more than the question to contribute at this time. However, it seems important to recognize that transformational reasoning requires both the inquisitiveness with respect to the workings of a mathematical system and the developed ability to translate the system into a mental or physical representation which can be "run."

Transformational reasoning may not only produce a different way of thinking about mathematical situations, it may also involve a different set of questions. The intent of these questions is both validity and understanding. This can be illustrated by the following scenario from the ELM project. In Ms. Barnett's class, the students were considering whether equilateral pentagons (all sides of equal length) are equiangular (all angles congruent) and vice versa. After determining that it was not, they were asked to extend the problem to other polygons. Of those that they investigated, equilateral only implied equiangular for the triangle. Ms Barnett and her students were satisfied with the results of the students' work when Mr. Allen, an observer shook things up by asking, "What is it about a triangle that makes it unique in this way?" Note that this question inquires into *how it works*. Understanding and validation come together in response to such a question. Insight into the workings of the system leads to knowledge of what results to expect and why. Validity is inherent in such understanding, not because it is deductively established, but because the learner has "seen" the relationship between the initial state and the result.

Consider the following transformational reasoning in response to Mr. Allen's question. (The reader might picture a line segment as a narrow strip of wood and an angle as two strips of wood joined at one end by a rivet.)

> Picture two line segments, AB and BC each having a specified length. Imagine, the two segments, joined at B are free to move so as to change angle ABC. If we now, attach line segment AC, having a given length, we can no longer move line segments AB and BC; angle ABC is fixed. We have a rigid figure. (Note that by not specifying that the sides of triangle ABC are of equal length, this provides a transformational reasoning approach to the side-side-side determination of unique, rigid, or congruent triangles.) Now if instead of joining points A and C with a rigid line segment, we join it by a set of segments with movable joints (angles). In this case the polygon formed is not rigid. Changes in angle ABC can be compensated by changes in the other angles (up to a point).

The transformational reasoning described here provides a sense of understanding the uniqueness of a triangle as a rigid figure among convex polygons.

To conclude this section, we can reflect on the relationship of transformational reasoning with inductive and deductive reasoning. Although this paper, focuses on transformational reasoning, effective mathematical problem solving involves a coordinated use of all three. Furthermore, transformational reasoning in many cases overlaps with both inductive and deductive reasoning. This overlap is represented spatially in Figure 3. Let me explicate this overlap. Some inductive strategies may resemble operating a "black box" in order to collect results (region A in the figure), while some inductive strategies lead to an understanding of the workings of the transformations involved (region B). Similarly, while some deductive arguments determine validity without offering insight into the nature of the phenomenon (region E), others break down the mechanism of the relevant transformations (region D). Region C refers to transformational reasoning that is neither inductive nor deductive.

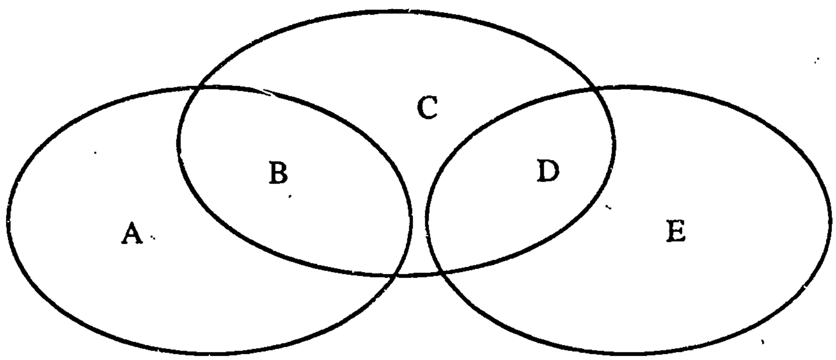

Fig. 3

Non Geometric Examples of Transformational Reasoning

In this discussion, the examples of transformational reasoning have been in relation to geometric problem situations. Does transformational reasoning extend to other areas of mathematics? I believe it does. Consider the following problem:

> In jar A, a solution of 50% alcohol and a solution of 75% alcohol are mixed. In jar B, a solution of 65% alcohol and a solution of 95% alcohol are mixed. What can we say about the relative concentrations of alcohol in jars A and B?

My experience is that this is a difficult problem for many students. Many assume that since 65% is stronger than 50% and 95% is stronger than 75%, that jar B will have a stronger concentration than jar A. The following is an example of what transformational reasoning might contribute:

> Picture two spigots filling jar A. One is connected to a vat of 50% solution and the other to a vat of 75% solution. (Vats are used to imply amounts greater than what can be held by a jar.) The flow from each spigot can be adjusted. We can "see" that if the 50% spigot is opened fully and the 75% spigot is barely open, the solution in jar A will be only a bit above 50%.
>
> The 50% spigot could be increasingly closed down and the 75% spigot could be increasingly opened, causing the concentration in the jar to be stronger. If the 75% spigot is opened fully and the 50% spigot is barely open, the solution in jar A will be only a bit below 75%. Thus, the solution in jar A will be between 50% and 75%.
>
> Similar thinking about jar B suggests that jar B would have a concentration between 65% and 95%. Therefore, either jar could have the stronger concentration, or they could be equally strong, depending on the relative amounts of each component concentration that was used.

A second example involves what Fischbein has called a "primitive model" (Fischbein, Deri, Nello, & Marino, 1985). Students tend to believe that "division makes smaller." One way that textbooks try to address this well-documented conception is to demonstrate inductively what happens as the divisor gets smaller. For example:

$$4 \div 2 = 2 \qquad 4 \div 1 = 4 \qquad 4 \div \frac{1}{2} = 8 \qquad 4 \div \frac{1}{4} = 16$$

As happens when students perceive a pattern, many are persuaded that the quotients do get larger. However, finding the pattern alone does not develop an understanding of why the quotients get larger as the divisors get smaller, nor why divisors less than 1 yield quotients greater than the dividend.

Transformational reasoning might involve picturing 4 objects (e.g., candy bars) and seeing what happens to the number of groups as the size of the group is diminished. The experience of seeing the putting together of objects when the group size is bigger than 1, the counting of the objects when the group size is 1, and the subdividing the objects when the group size is less than 1 might contribute to an understanding of division which would transcend the notion that division makes smaller.

What is common to the geometric examples described earlier and the quantitative examples provided in this section is the experience of mentally or physically "running" the relevant operation in order to examine the transformations and elicit the results of those transformations.

Summary and Implications

I have postulated that transformational reasoning is a natural inclination of the human learner who seeks to understand and to validate mathematical ideas The inclination, like many other inclinations (the desire to draw what one sees, to find patterns in one's world of experience) must be nurtured and developed. The assertion that it is a natural inclination is based on numerous examples in which the learners reasoning was not the result of instruction. Because mathematics educators have not recognized its importance, school mathematics has failed to encourage or develop transformational reasoning, causing the inclination to reason transformationally to be expressed less universally. Reversing this situation may significantly increase the sophistication of classroom discourse and the learning and reasoning of individual learners.

Transformational reasoning involves envisioning the transformation of a mathematical situation and the results of that transformation. The affective consequence of transformational reasoning is often a *sense of understanding how it works*. Based on examples that I have considered, it seems that transformational reasoning can serve several functions including, theorem generation, making of connections among mathematical ideas, and validation of mathematical ideas.

The conceptual work described in this paper, invites empirical study of transformational reasoning. Perhaps most puzzling is the question of how one comes to generate a particular transformational reasoning approach. For example, how can we explain Mary's reasoning about isosceles triangles? A second issue is the study of how transformational reasoning affects students' beliefs of what is true. Martin and Harel (1989) have shown that students are not necessarily persuaded by deductive proofs. Schoenfeld (1986) demonstrated that students who had successfully completed a proof of a theorem, failed to make use of the theorem in doing a construction on the same sheet of paper. Might greater emphasis on transformational reasoning have a significant role in students beliefs about the mathematics that they validate and thus their usable knowledge?

Investigation is also needed into the role of transformational reasoning in classroom discourse aimed at validation of mathematical ideas. I have advanced the hypothesis that transformational reasoning contributes to satisfying students' innate desire to understand and to know what is valid. Thus, attempts at establishing classroom norms for determining validity, that do not include the role of transformational reasoning, may be unsuccessful. While naturalistic observations of classroom discourse may provide some relevant data, teaching experiments, in which transformational reasoning is intentionally developed and encouraged will probably be required in order to explore effectively the relationship between transformational reasoning and classroom discussions of validity.

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Footnotes

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1 This term resulted from discussions with Guershon Harel and is a modification of his term "transformational observation."

2 In Simon (1989), I used the term *intuitive knowledge* to describe reasoning of this type. Transformational reasoning seems to contribute to what is commonly referred to as intuitive knowledge (Fischbein, 1987). However, I now avoid describing this phenomenon as intuitive knowledge because of the imprecision of the term and the wide range of connotations connected with thenotion of intuition.

3 This example is adapted from one described in Simon (1989).

4 I am indebted to Guershon Harel for bringing to my attention the connection between the ideas developed in this paper and the work of Piaget and Inhelder on mental images.