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ABSTRACT

Through timely questioning, teachers can estimate a child's understanding of mathematical ideas. Hence, the type of question asked by the teacher must be connected to the student's present thinking about a solution. This paper illustrates and analyzes how timely questioning on the part of teachers helped third, fourth, and fifth grade students (n=150) from three school districts in New Jersey: (1) build powerful justifications to their solutions, (2) connect two problems which were isomorphic in structure, and (3) come to understand the strategies of others. Four episodes are presented from selected videotape data of teacher/student interactions to demonstrate the use of timely questioning. Contains 41 references. (MKR)

Teacher Questioning to Stimulate Justification and Generalization in Mathematics[1]

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Background

The National Council of Teachers of Mathematics Standards (1991) reflect current educational research findings that learning occurs as students actively assimilate new information and experiences and construct their own meanings. The call for reform in the teaching of mathematics asks teachers to make a major shift in their teaching from dispensers of information to facilitators of children's construction of mathematical ideas. To support students in their building of ideas, teachers are urged to provide mathematical tasks that require thoughtfulness; encourage student discourse in small-group settings and within a classroom community; make available tools to enhance students' mathematical experiences; and monitor and facilitate the development of students' mathematical ideas (NCTM, 1991).

This approach to mathematics teaching places heavy emphasis on teachers' ability to pay close attention to children's thinking and reasoning. In order for teachers to be successful in their attempts to listen carefully to students and build upon their students' current knowledge, they must possess a rich understanding of mathematics on which they can build a vision for their students mathematical experiences (Maher, 1988). In addition, they must be alert to the range and depth of their students' mathematical thinking as they solve problems (Fennema, Carpenter, Franke & Carey, 1993; Maher & Martino, 1992a).

This direction towards reform that has been most recently associated with "constructivist perspectives" has its origin in earlier work on students learning (See for example, Cobb & Steffe, 1983; Davis, 1984; Kamii, 1985; Davenport & Narode, 1991; Kieran & Pirie, 1992; Maher, Davis & Alston, 1992; Maher & Martino, 1992a; Schoenfeld, Smith & Arcavi, 1993; Cobb, Wood, Yackel & McNeal, 1993.) and the findings of recent mathematics teacher development projects. (See for example, Carpenter, Fennema & Peterson, 1986; Schifter & Simon, 1991; Davenport & Narode, 1992; Hart & Najee-ullah, 1992; Maher, Davis & Alston, 1992.)

Our work at Rutgers began in 1984 with an intensive teacher development project in a blue-collar school. A model for teacher development which originates from that work is described by Maher (1988). [2] In describing the Rutgers approach to teacher development, Maher emphasizes the importance of the teacher's growing ability to pay attention to children's mathematical thinking. Thus, teachers who share this vision for teaching mathematics present problem tasks, listen carefully to the thinking of their students and challenge students to explain and justify that thinking at appropriate moments.

This model of teaching requires that the teacher develop the skill of questioning student thinking. As teacher knowledge of student thinking continues to grow and evolve with time, it provides the teacher with a framework for posing timely questions to facilitate the cognitive growth of his/her students (Maher, 1988; NCTM, 1991; Maher, Martino & Alston, 1993). Skillful questioning of student thinking can provide the teacher with essential knowledge about students' developing mathematical ideas, knowledge which might be otherwise inaccessible. The art of questioning may take years to develop for it requires an in-depth knowledge of both mathematics and children's learning of mathematics. Once acquired, the teacher has available a powerful tool to support students in their building of mathematical ideas.

International and cross-cultural studies which have been conducted in elementary mathematics classrooms studying the effects of teacher questions on the growth of students' conceptual knowledge support our contention that the teacher plays crucial role in the advancement of students mathematical thinking (Klinzing et al.,1985; Sullivan and Clarke, 1992; and Perry et al., 1993). All of these studies conclude that asking more open-ended questions aimed at conceptual knowledge and problem-solving strategies can contribute to the construction of more sophisticated mathematical knowledge by students. Our work specifically examines modes of teacher questioning which promote justification and generalization on the part of students, and then traces how these questions appear to stimulate students' growth of conceptual knowledge.

Theoretical Framework

In the classrooms where we conduct research, children are challenged by their teacher to build solutions to problems and develop an understanding of the problem task. In this process students may build models either with physical objects or with a notation that they invent (Maher & Martino, 1992b; Maher, Martino & Alston, 1993). Strategies are applied that include guess and check, looking for patterns, or organizing information using a chart or table. Once students have had an opportunity to build a personal representation of the problem, they share their ideas with each other and then the teacher (Maher & Martino, 1991; Martino, 1992, Maher & Martino, 1992b). At this time, individuals are challenged to consider their solutions through questions asked by both their classmates and teacher. [3]

The environment which we attempt to provide for students is consistent with Burns' (1985) description of a mathematics classroom which emphasizes the importance of teacher questioning, "Questioning is an important part of the teacher's ability to establish a classroom atmosphere conducive to the development of mathematical thinking", and Burns further points out that:

> children's classroom experiences need to lead them to make predictions, formulate generalizations, justify their thinking, consider how ideas can be expanded or shifted, look for alternative approaches and search for those insights that, rather than converging toward an answer, open up new areas to investigate (p. 17).

Classroom setting: Students working alone, in pairs and in small groups

Students, usually begin by building their own personal representation of the problem. Once an individual solution has been constructed the child becomes interested in the ideas of others and often looks to other students for feedback ready to engage in conversation about the idea (Maher & Martino, 1992b). Hence, after a child has developed a solution to a problem, he/she is encouraged to compare his/her solution with those of other students to determine if the solutions agree or disagree. When there is disagreement between students' work, individuals are challenged to convince their classmates of the validity of their work. This type of discussion may result in validation or refinement of an individual solution. Sometimes it triggers student interest in returning to the original problem and rebuilding a solution.

Many times this discussion among students occurs naturally as students begin to look around the classroom to share their ideas. In some instances, when children do not seek discussion with other students, it is not unusual for the teacher to suggest that certain children discuss their findings with each other. This teacher intervention is strategic in the sense that children with conflicting ideas may be brought together to discuss their findings in anticipation that student disagreement may precipitate the development of justifications and possibly modification of an idea.

As students discuss their ideas in pairs or a small-group setting, the teacher's role shifts to one of moderator and observer. This situation provides the opportunity for the teacher to listen to the thinking of the students involved in the discussion. We have found that students must have sufficient time to build their solution *without any teacher intervention.* This time is crucial. The resulting student-student interaction provides an opportunity to test, consolidate, or modify an idea. In some cases, an original solution is rejected when another is judged to be more reasonable.

Classroom setting: When teacher questioning comes upon the scene

We have found in general that students, either alone or in consultation with a partner or in a small group, do not *naturally* seek to build a justification or proof of the validity of a solution (Maher & Martino, 1991; Maher & Martino, 1992b). Furthermore, many children believe that proposing of a solution, *per se*, is sufficient evidence for justification. We have also observed that children working in pairs or small groups do not *naturally* question each other about the details of their arguments. [4] This may be due to the relatively low level of questions which students tend to ask each other (Wilkinson & Martino, 1993). [5] It is for these reasons that the teacher's role of questioning students becomes critical *after* students working alone or together have taken their ideas as far as they can. After students have built a solution, consulted with each other and proposed a solution that they believe is valid they are ready for the challenge to justify and/or generalize their solutions. It is at this time, that the teacher's role of interaction with students becomes critical. This paper examines teacher questioning under these circumstances.

Teacher questioning which enhances students' building of arguments

Through timely questioning the teacher can estimate a child's understanding of a mathematical idea. Hence, the type of question asked by the teacher must be connected to the student's present thinking about a solution. After listening to student discussion, this use of questioning enables a teacher to gain understanding of his/her students' current thinking. A form of encouraging the child to explain his/her solution to the problem task might be approached through timely questions such as: "Can you explain your solution to me?" or "I don't understand, can you show me how this works?".

Once a child has posed a solution and explains his/her thinking, the teacher may see that some component of the student's solution is vague or incomplete. The teacher may then ask questions which direct the child's attention on that component of his/her argument. For example, the teacher may respond, "I'm not convinced about this particular aspect of your argument, you might consider how you are going to convince your classmates of this". This type of response assists the student in focusing on that aspect of his/her argument which may ultimately resolve the difficulty.

Sometimes children pose a solution to the problem (either complete or incomplete), and are quick to abandon further thought about the task. In these situations, the teachers' questions might be designed to sustain student thought about the mathematical idea. The teacher may ask: "Have you considered 'this' possibility?" or "What if we changed the problem to consider 'this'?". The result of such questioning can be the development of a more complete argument or an extension of current thinking about an idea.

Students who have developed some form of justification for their solution may be asked by the teacher to consider how they will convince other students. Our previous work suggests that questions posed by the teacher which are aimed at *justification* of an asserted solution can stimulate further thought about the problem situation, and even lead to a re-organization of the student's solution (Maher, Martino & Alston, 1993). This process of re-organization frequently results in the creation of a more sophisticated form of justification. Questions which encourage mathematical justification include "How did you reach that conclusion?", "Could you explain to me what you did?" and "Can you convince the rest of us that your method works?".

To extend present individual knowledge, it is sometimes useful to ask a student to consider a justification produced by another student so that similarities and differences between the two approaches can be considered. Questions which assist students in focusing on the ideas of others include: "Did anyone have the same answer, but a different approach?", "Is there anything about your solution that's the same as your classmate's?" and "Can you explain what your classmate has done?". The goal is to extend student thinking.

To broaden the knowledge of a student who has already developed a powerful form of justification, the teacher may ask him/her to think about his/her previous work in light of newly developed methods of organization. This type of questioning may trigger generalizations of a solution to similar problem tasks. A question which facilitates students in making generalizations and mathematical connections might be "Have you ever worked on a problem like this one before?". Questions such as these which are aimed at *generalization* can enable the student to form mathematical connections between classes of problems.

It is important to emphasize that in these situations teachers must allocate adequate classroom time for students to consider those questions which have been posed.[6] Earlier work by Rowe (1974) which emphasizes the importance of wait time is relevant to this study because we have found that students require sufficient time to formulate a solution once a question has been asked by the teacher. Rather than impose closure on students' engagement with ideas, teachers may decide to return to a problem task at a later time for further exploration and discussion.

Background of the Longitudinal Study

Subjects - Approximately 150 3rd, 4th and 5th grade students are members of classrooms which are involved in a longitudinal study of the development of mathematical ideas in children .[7] The teacher/facilitators in this long-term study have had opportunity to "do mathematics" and reconsider what is important for students to know (Davis & Maher, 1990; Maher, Davis & Alston, 1992). They have also had the opportunity to develop their skills of paying attention to students' mathematical thinking.

Setting - Participants in the longitudinal study come from 3rd, 4th and 5th grade classrooms from three school districts in New Jersey, sites of a long-term teacher development program in mathematics (Maher, Davis & Alston, 1992). The sites include: an urban community, a blue collar community and a suburban community.

Methods and Procedures

Data Source - The data for this study, in the form of videotape transcripts, the accompanying written work of the students, and researcher notes that document the mathematical activity of the student and their interactions with both teacher and other students during the time in which the development of a mathematical idea is being traced comprise a video portfolio.

Procedure for Collection of Videotaped Data - All mathematics lessons are videotaped with three or four cameras to capture both students working on problem tasks with a partner and the teacher circulating throughout the classroom. Sometimes students are interviewed by the teacher following the problem task. These interview sessions are videotaped with two cameras: one focusing on the student(s) and the teacher, and the other recording the work produced by the student(s).

Procedure for Data Analysis - The videotapes from both the classroom sessions and interview sessions are transcribed and checked for accuracy. All teacher/facilitator questions are identified within these transcripts and the interactions which follow these questions are traced for the development of the child's mathematical ideas. In addition, questions which stimulate children's building of solutions are followed within and across problem tasks in order to determine whether any mathematical connections are made between different problem activities. Four episodes are presented from selected videotape data of teacher/student interactions to demonstrate the use of timely questioning.

The Present Study

Two isomorphic problem tasks were administered to 151 third, fourth and fifth grade children in three school districts in New Jersey over the course of the 1992-1993 school year. This paper illustrates and analyzes how timely questioning on the part of one teacher helped third and fourth grade students in one school district: (1) build powerful justifications to their solutions, (2) connect two problems which were isomorphic in structure, and (3) come to understand the strategies of others.

The Problem Activities Used in this Research - Videotape transcripts of two problem activities and follow-up interviews are analyzed for this study. The two problems contain isomorphic mathematical structures whose solution can be represented using different methods of proof, and be expressed in a generalized way. [8]

The Tower Problem - The task required students to build as many towers as possible of a certain height (for example, all possible towers four cubes tall) when plastic cubes (Unifix cubes) in two colors were available, and then to convince other students that there were no duplicates and that none had been omitted.

The Pizza Problem - Pizza Hut has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms and pepperoni.

1. How many different choices for pizza does a customer have?
2. List all the possible choices.
3. Find a way to convince each other that you have accounted for all possible choices.

Previous Findings Relevant to the Present Study

An observed sequence of movement of student solutions from use of "trial and error" to use of more "global and all-inclusive" methods of organization

The methods used and the data collected have made it possible to follow many children from several school districts. We have observed that their problem-solving for the Tower and Pizza Problems generally moves through certain sequences. [9] In each sequence, the appearance of new ideas provides a foundation and motivation for the next (Davis & Maher, 1993, Maher, Martino & Alston, 1993; Maher & Martino, under review; Maher, Martino & Davis, under review).

Children generally begin by searching for new tower/pizza combinations in a random manner and check for duplicates by comparing a newly generated combination to the ones already built. They sometimes invent names for particular combinations. For example, a tower with alternating colors is sometimes referred to as having a "checkerboard" pattern or a pizza with all four toppings as the "pizza with everything". For the Tower Problem, children may form pairs of towers which they sometimes call "opposites" or "upside-down" pairs (see Figure 1).

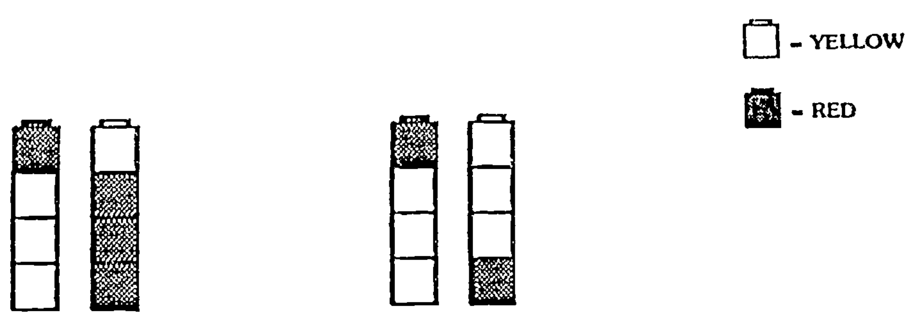

Figure 1. Towers called "opposites" (left) and "upside-down" pairs (right).

For many children there is a natural progression from use of random methods to the use of systematic "local" organizing aids that serve to simplify the task of finding new combinations and keeping track of their collections.

While developing these local groupings for combinations, some children begin to experience disequilibrium as they come to realize that their local organization schemes are inadequate to account for all possible towers.

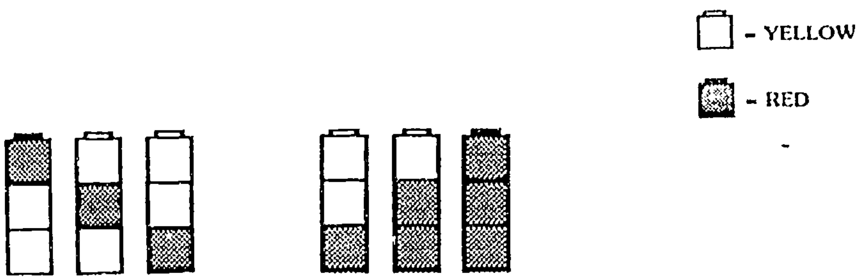

Figure 2. Two sets of towers which overlap and produce duplicate arrangements.

This state of disequilibrium often leads to children recognizing conflicts between their different local organizations (see Figure 2), and recognizing the need for an overall global scheme to account for all possible towers. This realization frequently precipitates students developing an all-inclusive organization which provides a framework for the student to develop a method of proof. Figure 3 shows a proof by cases developed by a third grade child for all possible towers four cubes tall when selecting from two colors.

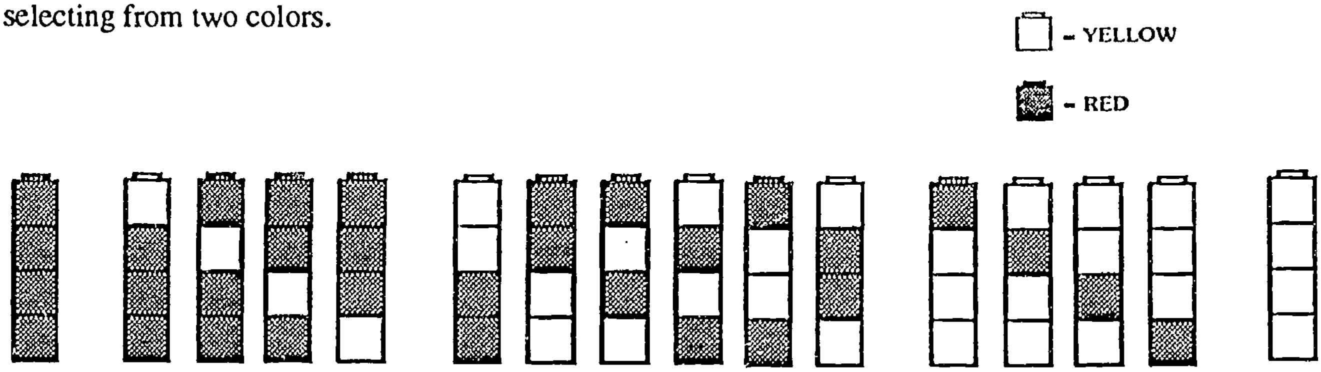

Figure 3. Meredith's organization of towers as a proof by cases.

Once students have developed their own form of justification, many are receptive to listening to other children's arguments, considering alternate methods, and expressing a preference for those methods that are more elegant (Maher and Martino, 1992b).

The purpose of the present study is to analyze how timely questioning on the part of the teacher which builds upon a child's present construction of knowledge can enhance this sequence of development towards proof-like mathematical arguments.

Results

A series of four teacher interventions with two third grade students (Meredith and Sarah) and one fourth grade student (Brandon) illustrate how we use teacher questioning to stimulate solution justification, solution re-organization and in some cases generalizations of a problem solution.

Episode 1: Teacher questions which facilitate justification

Grade 3 - The Tower Problem - How teacher questioning stimulated Meredith's re-organization of a problem solution (December 10, 1992).

This first episode illustrates how timely teacher questioning can trigger student re-organization of a solution into a more global justification. Meredith, a third grade student in her fourth month of school, worked with partner Jackie to build all possible towers four cubes tall when selecting from red and yellow plastic cubes. She and her partner built their towers by trial and error methods then quickly organized their towers into pairs of "opposites". An arrangement of towers into pairs of "opposites" is a early form of organization used by many of the children who have worked on this activity (see Figure 1).

After Meredith and Jackie completed their search for towers, the teacher listened carefully to Meredith's argument, and observed that Meredith described individual towers by the color distribution of cubes within that tower (e.g." a tower with three yellow cubes"). The teacher realized that an organization by color distribution was a means of providing a more global method of organization than Meredith and Jackie's current organization by pairs of "opposites". Keeping this information in mind, the teacher asked Meredith and Jackie to explain how their organization of towers by "pairs of opposites" would help them to justify that they had built every possible arrangement without any duplication of combinations.

Teacher	*How many towers did you find that are different?*
Meredith:	Sixteen.
Teacher	*How do you know you don't have any (towers) in here that are the same, the same tower?*
Meredith:	First we took the first one and... [She takes a tower and compares it to all the other towers in the row.] We went like that and we found out that none of them were the same.
Jackie:	And then, to get... to get another one, we did the "opposite" of it.
Meredith:	Went like that, we kept on doing it. [Meredith demonstrates how she checked each tower.]

Thus, Meredith and Jackie were able to demonstrate that their method for building towers eliminat l the possibility of producing any duplicate arrangements. The teacher had used this questioning to monitor whether the children had developed a method to account for the creation of duplicate combinations. The children were then asked by their teacher to consider whether any combinations might be missing from their solution. The teacher's goal was to ascertain whether Jackie and Meredith had devised a method for being exhaustive in their search for towers. When challenged to consider the strength of her argument, Meredith began a re-organization of towers of height four into five groups: no yellow cubes, one yellow cube, two yellow cubes, three yellow cubes and four yellow cubes (see Figure 3).

Teacher:	*How are you going to convince people, though, that maybe there aren't seventeen towers, or eighteen towers? How do you know you have them all? Do you feel sure you have them all?*
Meredith:	Yup, I am, cause I looked at all of them and I said that all of them had doubles [an "opposite"]. I know that, and if uhm, like there was another one, I like went down the row. I didn't put them in order, but, say like there was three yellows [towers of height four with three yellow cubes] and I would go, then I would start off with, uhm, where is it? this one [a tower with three yellow cubes in the top three tower positions], Then I'd say that I'd make it into this one [a tower with three yellow cubes in the bottom three tower positions], and I'd make it into this one. [a tower with three yellow cubes configured differently from the other two selected] I just traded.

Teacher:	*Ah, so those [towers] have three yellows [three yellow cubes]. Are there any others [towers of height four] with three yellows?*

Thus, the teacher began to build upon Meredith's spontaneous organization of tov.ers by their color distribution, in this case, all the towers with three yellow cubes. Meredith was then encouraged to pursue this new organization by the number of yellow cubes in each tower while the teacher listened to her reasoning. Without any additional teacher intervention, Meredith located four towers with three yellow cubes, six towers with two yellow cubes, four towers with one yellow cube and one tower with four yellow cubes and grouped each set by their color distribution (see Figure 3).

Once Jackie and Meredith had arranged the five groups of towers, the teacher returned and asked them to consider the completeness within each of these new ca'egories for towers. Meredith re-arranged her group of four towers with one yellow cube into a staircase pattern (see Figure 2). This arrangement for all towers with exactly one yellow cube was clearly exhaustive showing that she had considered every possible position for placement of the yellow cube. The teacher's intent was to encourage Meredith and Jackie to refine their justification by considering each piece of their argument for completeness.

Teacher:	*This is interesting, these groups. Now within these groups are you sure you have all [towers] with say, one yellow [cube]?*
Meredith:	Yeah.
Teacher:	*How do you know that?*
Meredith:	Well, I would really put it like this [she arranged the four towers of height four with one yellow cube in a staircase pattern placing the one with the yellow cube in the bottom position, next to the one with the yellow cube in the next to the bottom position etc], instead of putting it like that. I would go like that and go up like that, like, sort of like a step [referred to the staircase pattern].
Teacher:	*Oh. How do you know there aren't any more?*
Meredith:	and you can't go any higher...
Teacher:	Why not?
Meredith:	... the step, because we can only do four [four cubes tall] and there's four steps.

Meredith made a similar arrangement of towers for the group she called "three yellows".

Teacher:	*Is there a way that might help me better be sure that we have all the ones [towers] with three... three yellows?*
Meredith:	Wait, this should go... [she begins to rearrange her set of towers with exactly three yellow cubes into a staircase pattern where the red cube moves down one position at a time beginning in the top position]
Teacher:	I have a real hard time, because I have...
Meredith:	There we go. That's better.

Likewise, Meredith and Jackie were able to convince the teacher that there was only one tower with exactly four yellow cubes and one tower with ex ıctly no yellow cubes (or four red cubes). The group containing six towers with two yellow cubes was more difficult to explain. Both Meredith and Jackie attempted to re-organize this set and satisfied themselves that all towers in this subset had been accounted for. Although the teacher was not convinced, she did not show the students an alternate organization. She simply indicated that she wasn't persuaded that the organization within this fifth group of towers (the ones with exactly two yellow cubes) accounted for all possibilities thus suggesting that the girls to continue to think more about it leaving open that this was not settled. The followin, teacher intervention sustained student thought about one aspect of the problem over an extended period of time.

Teacher:	I'm pretty convinced about some of these groups [the group of towers with exactly one yellow cube and the group of towers with exactly three yellow cubes]. This one really [the group of towers of height four with exactly two yellow cubes] I have a very hard time seeing it. Maybe because I wasn't here when you actually built them, but I'm wondering how other people are going to react to that. So think about it as you're recording these. *Think about how you might convince people about this group, okay? Alright I'm going to let you work and record.*

Meredith did continue to think about these five cases of towers and the organization of towers within each case. In particular she thought about the group of towers with exactly two yellow cubes. The teacher's questioning left open for consideration her ideas and helped her to focus on the incomplete pieces of her argument further refining her justification which now took the form of a proof by cases. In an interview which followed the problem-solving activity three days later, Meredith built a global all-inclusive organization for her towers and presented it as a proof in five cases. She quickly rebuilt her five groups for towers, and argued that there were no other possible groups since the towers were only four cubes tall. Thus, the range of possibilities for tower groupings went from "none of a color" to "four of a color" which made five groups.

At this interview, Meredith did indeed focus on the troublesome group of towers with "exactly two yellow cubes" and was able to justify that there were six and only six towers in this group. Meredith organized her six towers of height four into three pairs (see Figure 4): a tower with exactly two yellow cubes separated by no red cubes and it's "opposite", a tower with exactly two yellow cubes separated by one red cube and it's "opposite" and a tower with exactly two yellow cubes separated by two red cubes and its "opposite". She then explained that there could not be a tower of height four with exactly two yellow cubes separated by three red cubes unless the tower violated the initial condition that it be four cubes tall.

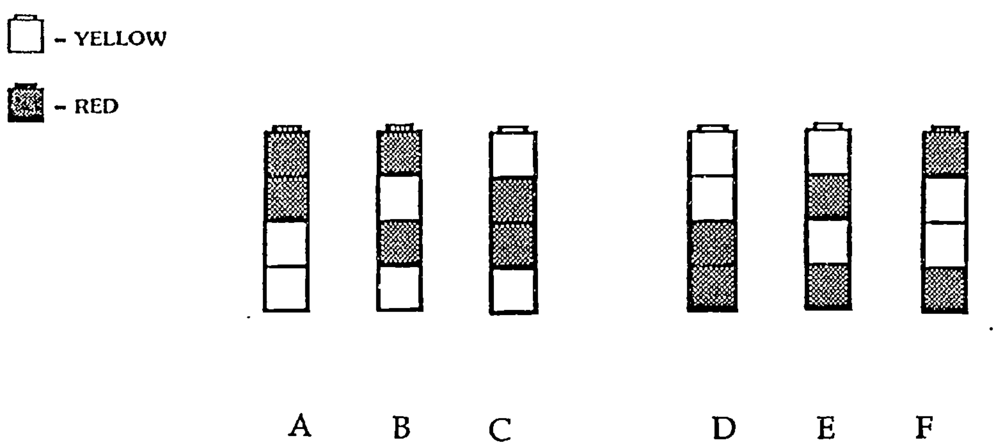

Figure 4. Meredith's revised organization for all towers with exactly two yellow cubes.

Teacher: *Okay. What about this group in the middle, [towers with] exactly two yellows [yellow cubes]? How could you convince me that you've made all the towers that are four [cubes] high with exactly two yellows [yellow cubes]?*

Meredith:	Well, first I started with this, at the bottom, two [yellow cubes] at the bottom. That's the closest you can get to the bottom. [Figure 4, Tower a]
Teacher:	Okay
Meredith:	And then I just did the "opposite" of it [Figure 4, Tower d]. Then I knew I'd have to separate them, so I separated them like that [Figure 4, Tower b]. That's the closest you can get to the bottom with that.
Teacher:	*What do you mean by that?*
Meredith:	I said I just turned this one [yellow cube] into a red one and moved it [the yellow cube] up [one position].
Teacher:	*Oh, so you separated the two yellows [yellow cubes] with one red [cube]?*
Meredith:	Yes.
Teacher:	Oh, I'm following you. Go ahead.
Meredith:	And then I just changed it around, it turned into that [Figure 4, Tower e].
Teacher:	Ah hmm.
Meredith:	And then I separated the yellows with two [red] blocks [Figure 4, Tower c] and then it, uhm, then I turned it around like that and... [Figure 4, Tower f]
Teacher:	*This is very interesting. Could you have separated the two yellows [yellow cubes] by three red blocks?*
Meredith:	No, because that would be [a tower with] five blocks.

The teacher's inquiry about the group of towers with exactly two yellow cubes helped to sustain Meredith's interest in the problem. Her sustained interest enabled Meredith to modify a piece of her argument which was previously unclear to the teacher and her classmates.

Episode 2: Questioning which facilitates generalization

Grade 3 - Interview with Meredith- How teacher questioning encouraged Meredith to consider towers of different heights (December 14, 1992).

During the individual interview which followed the Tower Problem, Meredith was challenged by her teacher to consider towers of different heights. Having built all possible towers of height four, Meredith was asked to predict the total number of towers three cubes tall which could be built selecting from two colors of cubes. She decided that there would again be sixteen towers, the same as there had been for towers of height four. She hypothesized that removing one block from the top of each tower four cubes tall would in no way alter the total number of towers. She then removed the top block from each of her sixteen towers which were four cubes tall and discovered that eight of the sixteen towers of height three were duplicates (see Figure 5).

Teacher: *Now we made towers of [height] four. What if we made... we had two colors, red and yellow, and we made towers that were three [cubes] high instead of four [cubes] high. How many do you think there'd be?*

Meredith: Well, maybe there would, there would still be sixteen.

Teacher: Think so?

Meredith: Cause if you just took one [cube] off each one [tower four cubes tall]...

Teacher: Now that's an interesting thought. If you took one off each of these. Want to try it, see what happens?

Meredith: Okay. [She took the top block off each tower and detected eight duplicate towers.]

Meredith was then asked to consider why the number of towers decreased from sixteen to eight.

Teacher: We had sixteen towers that were four [cubes] high, so we had eight duplicates over here then... right? Once we took the top block off of them [towers four cubes tall]? *Can you explain why you think that happened?*

Meredith: Because this [tower] used to be four yellows... [She takes a tower with three yellow cubes and picks up the discarded duplicate tower and points to the missing cubes on top.] ... and for this duplicate this used to be one red on top of it and when you take the red off and you take the yellow off, it would be the same.

Teacher: They become the same. Does that happen for all of those?

Meredith: Yup.

Teacher: That's interesting.

Meredith: Cause they each had a duplicate and we have to take eight [towers] away and eight [towers] would still be there.

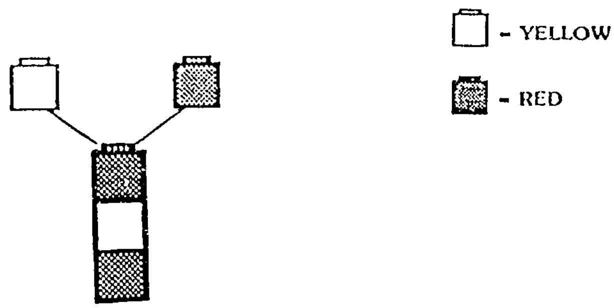

Figure 5. Meredith's model demonstrates how sixteen towers four cubes tall become eight towers three cubes tall.

The teacher's question had encouraged Meredith to formulate a hypothesis and then to test that hypothesis. Thus, Meredith now indicated an understanding that decreasing the height of the tower would decrease the number of different tower arrangements.

Episode 3: Questions which facilitate mathematical connections

Grade 4 - The Pizza Problem - Brandon's invention of a global organization for the Pizza Problem (March 11, 1993).[10]

In the process of searching for all possible pizza choices when selecting from four pizza toppings, Brandon, a fourth grade student, invented a code to represent the presence of specific toppings on a pizza. He constructed a chart with four columns, one column for each of the four toppings, and represented the toppings as follows: "P" for pepper, "M" for mushroom, "S" for sausage and "peponi" for pepperoni. Brandon decided to represent a pizza by placing either a 1 or a 0 in each of the topping columns to indicate whether or not a topping was present on the pizza under consideration.

Thus, when he wrote "0" in the peppers column it was meant to represent that the topping peppers was not included on the pizza under consideration. He wrote "1" in the sausage column to indicate that the topping sausage was included on the pizza. Using this system of coding, Brandon was able to organize his data to account for all possibilities. His organization was by five cases: (1) zero toppings [a plain pizza with cheese and tomato sauce], (2) one topping, (3) two toppings, (4) three toppings, and (5) four toppings (see Figure 6). Thus, Brandon had developed a *proof by cases* for all possible pizzas selecting from four toppings.

Figure 6. Brandon's zero/one code for representing pizza choices.

Grade 4 - Interview with Brandon - Teacher questioning which supported Brandon in connecting the Pizza Problem to the Tower Problem and applying his organization by cases initially developed for pizzas to towers (April 5, 1993).

Using his system of coding and organization by the number of toppings, Brandon was able to provide an all-inclusive organization in five cases for his toppings and accounted for all possible pizzas. His cases were pizzas with: no toppings, one topping, two toppings, three toppings and four toppings. In an attempt to ascertain whether Brandon could apply his new problem organization to a problem he had worked on prior to the Pizza Problem, the teacher asked Brandon if the Pizza Problem reminded him of any others that he had worked on in the past, and Brandon recalled his work on the Towers Problem.

Teacher: *In any way does it remind you of any of the problems we've done?*

Brandon: It kind of a little reminds me of the blocks [towers] cause my partner and I... whoever it was... I think it was Colin... did them in order like... yellow, red, yellow, red and then switch around and do the "opposite"... Red, yellow, red, yellow. It's kind of like what you do here [He refers to the case groupings for pizzas.] Just do it in groups... that's what we did with the blocks...

	how many ways can you make towers... how many ways can you make pizzas... the same problem, sort of.
Teacher:	The same kind of thing? Do you remember how many towers there were?
Brandon:	I think I can remember...

The teacher's question stimulated Brandon's reconsideration of his solution to the Pizza Problem which ultimately led to his re-organization of his solution for towers. Brandon took red and yellow cubes and quickly reproduced towers of height four as pairs of "opposites". This was the same strategy he used when he solved the problem four months earlier. [11] When asked by the teacher to defend how the organization of towers by "opposites" was an effective method to account for all possible towers, Brandon paused for about one minute. Brandon then organized the eight towers shown in Figure 7 into two staircases, each with four towers, and verbalized his recognition of a connection between the Tower and Pizza Problems. He enthusiastically shared with the teacher his "discovery".

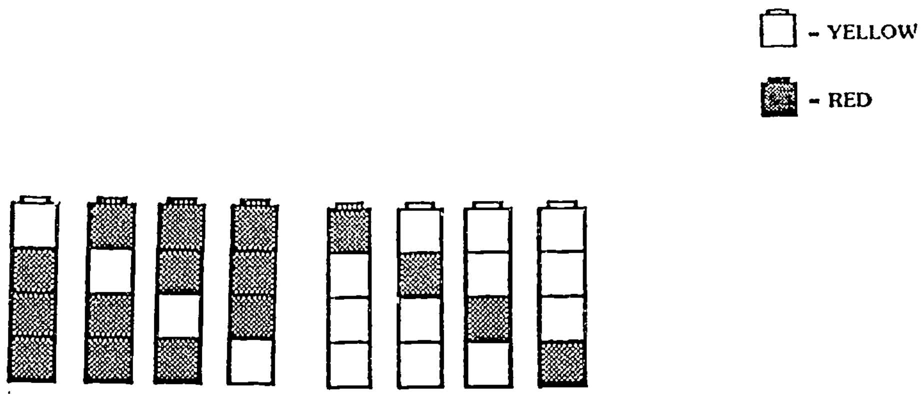

Figure 7. Brandon's eight towers four cubes tall with a 1-3 color distribution.

Brandon:	It's kind of like the Pizza Problem... like this would be the one's group. [referring to the eight towers of height four with three cubes of one color and one cube of the opposite color]
Teacher:	Let's see...
Brandon:	Oh, yeah... I see! this now... this is like the one's group [referring to the eight towers in Figure 7]. You only have one of the opposite color in these [towers with exactly one red or one yellow cube]... This isn't how I did it, but I just noticed this...
Teacher:	This is very interesting.

The teacher had asked Brandon about the similarity of the Pizza Problem to other problems in an attempt to give Brandon an opportunity to apply his new global method of organization to prior activities. This is precisely what Brandon began to do. For Brandon's new organization of towers, he moved from eight sets of paired "opposites" to three cases (towers with all one color, one of a color or two of a color) and quickly moved from three cases to five cases: (1) towers with no yellow cubes, (2) towers with one yellow cube, (3) towers with two yellow cubes, (4) towers with three yellow cubes and (5) towers with four yellow cubes (see Figure 3).

The teacher then asked Brandon to explain this new organization in an attempt to have him synthesize his new construction. It was at this point in the interview that Brandon, enthusiastically, expressed that the group of four towers with exactly one yellow cube were like the four pizzas with one topping in his chart, and placed each tower made of plastic cubes on top of the associated zero/one pizza code on his chart establishing a one-to-one correspondence between each pizza and its analogous tower of height four (see Figure 8). He explained how the red cubes in each tower corresponded to the "zero's" on his pizza chart and how the yellow cubes in each tower were analogous to the "one's" on his chart. He then confidently proceeded to match each of the sixteen towers to each of the sixteen pizzas represented on his chart.

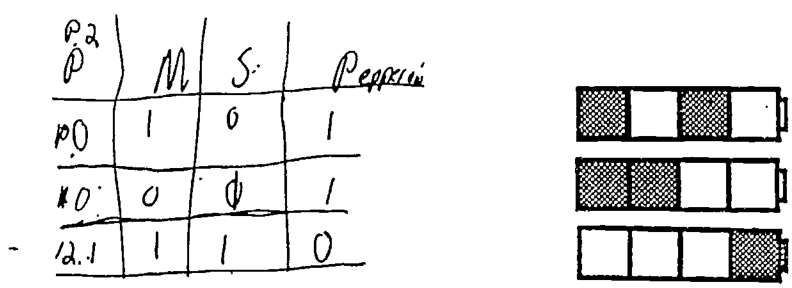

Figure 8. Brandon's one-to-one correspondence between towers and pizzas.

Teacher: *Tell me again how this is like the pizzas?".*

Brandon: You have the one pepperoni... since we're looking at yellows [cubes] the yellow cube would be 1 and the red would be 0. [He compared the red cubes in the towers to the digit "0" used in his pizza chart which represent the absence of a topping.] You could have one pepper... then it's like stairs if I draw a line down here

> like this... it's sort of like here you'd have one pepperoni, one mushroom, one sausage and one pepper.

As he spoke, Brandon pointed to the one yellow cube in each of the four towers. He noted the similarity between the four towers of height four with exactly one yellow cube and entries two through five in his pizza chart (see Figure 6), and then traced a diagonal line on the chart to show the similarity.

Brandon:	Right here... you would have the yellow stand for one. So it would be yellow for one, red zero, yellow one, red zero.
Teacher:	I see.
Brandon:	That'd be another one.
Teacher:	So this would come next.
Brandon:	... yellow, red, red, yellow. [He pointed to the 1,0,0,0 code on his chart as he spoke.] Yellow, red, red, yellow. [He pointed to the red and yellow cubes in the corresponding tower as he spoke.]
Teacher:	*I see. So what would this pizza look like? This one? [referring to a tower with yellow cubes in the top and bottom tower positions]*
Brandon:	That would be pepperoni and... pepper and pepperoni.

The teacher asked Brandon to explain which pizza the four cubes represented. Thus, the teacher was able to follow Brandon's thinking about the two different representations, the zero/one code and the towers built with plastic cubes.

Thus, through questioning which stimulated thought about prior problem activities, Brandon was able to establish the equivalent structural relationship between the Tower Problem and the Pizza Problem. This empowered him to apply a global argument by cases originally developed for the Pizza Problem to the Tower Problem, and thereby give a convincing justification for its solution.

Episode 4: Questions which facilitate focusing on solutions presented by other students

Grade 3 - The Pizza Problem - Sarah and Meredith's initial work with pizzas (March 15, 1993).

Meredith decided to tackle the Pizza Problem by using the heuristic of constructing a chart to record the choices of pizzas. Like Brandon, her chart had four columns, one to represent each topping: peppers, sausage, mushrooms and pepperoni. She then constructed five cases: pizzas with one topping, pizzas with two toppings, pizzas with three toppings, pizzas with four toppings and pizzas with zero toppings, which she commonly referred to as the plain pizza (see Figure 9). [12]

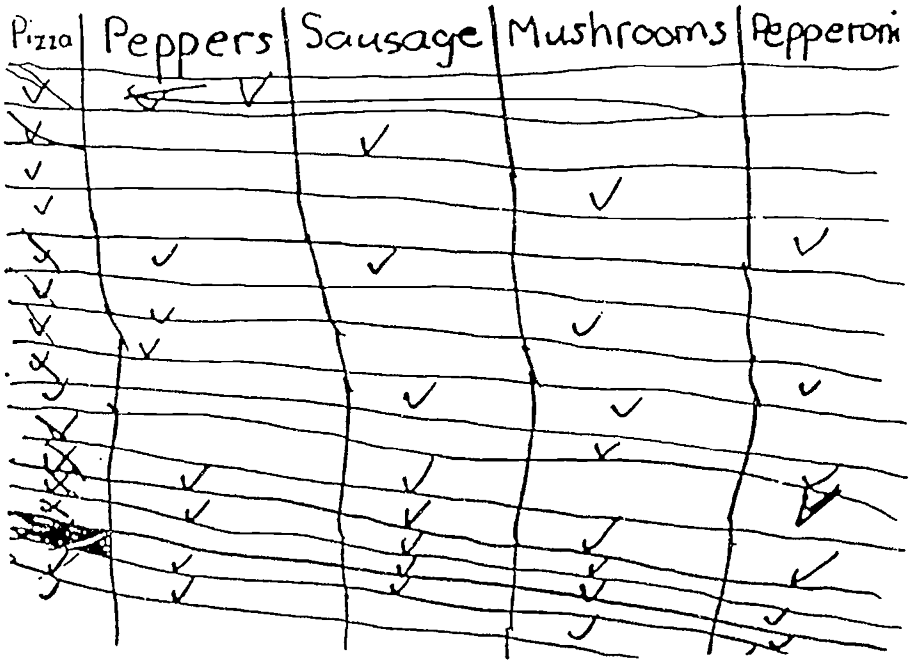

Figure 9. Meredith's chart organization by cases.

Meredith developed a global organization for considering and exhausting all possible pizza choices. Her chart helped her to keep track of all possible pizzas within individual cases by providing a systematic format for the formation of combinations. She demonstrated this by first combining peppers with sausage; then by skipping the sausage column and combining peppers and mushrooms; then, by skipping the sausage and mushroom columns, and combining peppers with pepperoni.

Meredith's partner Sarah made a list to record pizza combinations. Instead of organizing her pizzas by the number of toppings, Sarah made all combinations that she could arrive at with peppers, then sausage, and so on.

Grade 3 - The Pizza Problem - How teacher questioning encouraged Sarah and Meredith to connect the Pizza Problem to other problems (March 15, 1993).

The teacher decided that it might be useful to get Sarah and Meredith to consider previous problem activities to determine whether they could connect this activity to others within the same class of problems. When the teacher asked Meredith and Sarah whether the Pizza Problem reminded them of any other problem they had worked on in the past, both girls commented that it reminded them of several combinatorial problems given throughout the year and among those mentioned was the Tower Problem. [13]

Teacher:	*I want to ask you both to think back over the things we've done this year... does this problem remind you of any of the others we've worked on and if so why?*
Meredith:	Maybe the towers...
Sarah:	Yeah, the towers.
Teacher:	*Why?*
Meredith:	Cause it's pattern-like, and also the uniforms [She referred to another combinatorial problem where they were asked to make as many combinations as possible with three shirts and two pairs of jeans.] The uniforms and the towers... because the towers you couldn't have duplicates and you had to find as many as possible... you had to explain your answer and find as many as possible.

The teacher continued to pursue the girls comments about the sameness of the Tower and Pizza Activities in an attempt to assess the degree to which they recognized this "sameness".

Teacher:	That's interesting... that you thought of the towers because it looks very different from the way you did the Pizza Problem.
Meredith:	It looks different, but it's not really.
Teacher:	*Why not?*
Meredith:	Cause you find different ideas like in the towers... for example... you would find one [tower] with a yellow here two reds inside and one yellow... well it's the same actually... cause if you have

four toppings here... it would be like just standing them up... the pepper on top of the sausage, a mushroom and a pepperoni.

Teacher: *Could you show me if I gave you some blocks?*

Meredith: Yeah. Okay, let's pretend that this is peppers [yellow cube], this is sausage [blue cube] this is mushrooms [brown] and this is pepperoni [white].

Meredith used four colors of Unifix cubes to represent the four pizza toppings. She represented a two-topping pizza with pepperoni and sausage using two cubes, one blue and one yellow. Although this was a valid model for representing her pizza combinations with cubes, use of this model did not uncover the structural sameness between the Tower and Pizza Problems as Brandon's solution had done. Through teacher probing about her model, Meredith recognized a connection with the Tower Problem. However, her new representation for pizzas with four colors of plastic cubes had not become a vehicle to clarify her understanding of the structural sameness of the two problems.

Grade 3 - Interview with Sarah and Meredith - Teacher questioning which facilitated Sarah and Meredith's exploration of Brandon's solution (May 3, 1993).

The teacher did not impose closure by establishing the isomorphism between the Tower and Pizza Problem for the girls, but instead decided to ask Sarah and Meredith to examine Brandon's solution. The teacher resolved that studying someone else's solution might stimulate renewed interest in pursuing the similarities between towers four cubes tall selecting from two colors and pizzas selecting from four topping choices. This discussion occurred in a paired interview which occurred several weeks after Meredith and Sarah's initial exposure to the Pizza Problem. In this case, the timeliness of focusing on another student's solution helped both Meredith and Sarah to build the knowledge necessary to establish the structural isomorphism between the two tasks.

Sarah and Meredith were presented with Brandon's written solution which used a binary code to represent each pizza (see Figure 6). The girls spent several minutes reading Brandon's solution and Meredith attempted to interpret Brandon's code reading off each of Brandon's pizzas and referring to them by the number to toppings present on each combination. Sarah explained that Brandon had placed a numeral "0" in a topping column to represent the absence of that topping on the pizza and a numeral "1" in a topping column signified the presence of that t· ·pping on the pizza. Both girls could translate Brandon's code, Me ·edith did this with written language like "pizza with peppers" and Sarah reconstructed Brandon's chart.

After Meredith and Sarah had studied Brandon's solution, the teacher made the decision to ask the girls to revisit the Tower Problem. The teacher's goal was to determine whether having seen Brandon's organization the girls would detect the structural sameness of the two problem activities.

Teacher:	*The reason that I'm showing you Brandon's work is that Brandon also said that this problem kind of reminded him of the Towers Problem... which he had also worked on... remember the towers four tall when we used the two colors. And he said that by using his system here for pizzas...*
Sarah:	It [the zero and one code] would be like red, yellow.

When presented with another student's solution by her teacher, Sarah had made the initial correspondence between the zero/one code and the red and yellow cubes in the towers. The teacher pursued Sarah's thinking about Brandon's strategy by asking her to justify her statement.

Teacher:	*What do you mean by that?*
Sarah:	Like the 1's [in Brandon's code] could be red [cubes] and the 0's could be yellow.

The teacher now encouraged the girls to use materials (red and yellow cubes) to demonstrate their thinking.

Teacher:	I have some blocks here...
Meredith:	The 0's...
Sarah:	The 0's could be yellow (cubes) and like...
Teacher:	*Do you think it matters?...*
Girls:	Well it doesn't really matter...
Sarah:	...but I just picked the 1 for red and it could be the 0 for yellow.
Teacher:	*What would a tower look like?*

Meredith:	You'd actually need four colors really to do it.
Teacher:	*Okay, why?*
Meredith:	Because the different toppings you'd need four... because there's four toppings... four different colors.

The girls then proceeded to assign a color to represent each topping which was precisely what they did on March 15 in their earlier attempt to connect the Tower and Pizza Problems. The teacher did not reject Meredith's representation. In an attempt to re-focus their attention on Brandon's binary system, the teacher asked Sarah and Meredith if it was possible to represent all pizza combinations with only two colors of plastic cubes.

Teacher:	*Okay, but that isn't what I heard you saying at first to me... you said something about the colors here being like the 0's and 1's? What if we only had two colors of cubes, but we wanted to show all these pizzas. Could we do something like that?*
Sarah:	Yes.
Teacher:	*Let's decide. What would pizza number one look like? [This was Brandon's "pizza with no topping" represented as "0,0,0,0".]*
Sarah:	Number one would look like you have nothing.
Meredith:	Nothing. It would just look like cheese.
Sarah:	The second one would be a red and three yellows [referred to the numerical code of 1,0,0,0] [Meredith used the cubes to make the tower that Sarah described with one red cube in the top position and yellow cubes in the three bottom positions.]
Teacher:	*Okay so show me how that works. So that's a pizza you're telling me... can you show me what it has?*
Sarah:	It would have pepper and it had nothing else on it.

The girls then proceeded to take Brandon's zero/one code and build all the pizza combinations using red and yellow plastic cubes. Once the girls had built each pizza combination in Brandon's chart with plastic cubes (see Figure 3), the teacher in an attempt to have the girls make a comparison, asked them to recall their work on the Tower Problem.

Teacher:	*Did you have groups like this when you made towers?*
Sarah:	Yes.
Teacher:	*Do you remember what your groups for towers were?*
Meredith:	I had them grouped like this.
Teacher:	*And what were those groups called?*
Meredith:	Threes, twos, ones, fours and this is the zeros... these were the threes cause they had three yellows.

Sarah and Meredith were asked by the teacher to consider Brandon's solution. Interpreting the solution of another student provided a vehicle for recognizing the structural isomorphism between the Tower Problem and the Pizza Problem where Sarah and Meredith mapped each tower to its corresponding pizza.

Conclusions

Analyses of the videotaped data indicate strong relationships between a teacher's monitoring the process of a student's constructions of a problem solution and the teacher's posing a timely question which challenges the student to advance to a higher level of mathematical understanding.

Initially, children become engaged with a problem task; they then tend to move through a sequence of re-organization of their original representation. For example, children may begin by using random methods to build a solution to a problem task, such as trial and error and guess and check. Often, in an effort to explain their solution, a transition is made from the use of random methods to the use of systematic local organizations. If justification does not follow, different local organizations are attempted. Often these attempts lead to a further re-organization of the problem solution which is global in nature and perhaps the invention of a method of proof.

Originally, students indicated satisfaction with their original solutions obtained by trial and error methods with limited or no teacher intervention. When they were then approached by a teacher and asked to clarify or justify their results, they re-examined their solution and reorganized their initial representation to build a partial justification based on a new local organization of their data. This was demonstrated by the teacher's timely inquiry into Meredith' s current tower organization by pairs of "opposites" which built upon Meredith's present thinking, and eventually led to Meredith's construction of a global organization of proof by cases. The teacher, acting as a catalyst, provided the stimulus for further student thinking. Again, it was individual judgment based upon careful observation which led the teacher to ask Brandon and Sarah to reflect on their prior problem-solving experiences to determine if they could apply newly developed strategies to prior problem activities. In this instance, the children came to recognize the structural isomorphism between the Tower and Pizza Problems, and used a sophisticated organization originally developed by Brandon for pizzas to organize their towers. Thus, teacher questioning triggered student reflection on earlier problems and stimulated recognition of connections between them. Teacher questioning provided the incentive for student re-examination and reorganization of the problem data, and in some cases the development of methods of proof and mathematical generalizations.

Educational Significance

Movement from a classroom that is teacher-centered to one that is more student-centered suggests a critical and central role for the classroom teacher. Results from this research suggest that teacher questioning that is directed to probe for student justification of solutions has the effect of stimulating students to re-examine their original solution in an attempt to offer a more adequate explanation, justification and or generalization. This implies that teacher questioning is crucial in stimulating students' construction of higher level justifications, and underscores the importance of teachers' attention to the thinking and reasoning of individual students.

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Endnotes

1. This work was supported in part by grant number MDR 9053597 from the National Science Foundation. The opinions expressed are not necessarily those of the sponsoring agency, and no endorsement should be inferred.
2. For more detail on Maher's model for teacher development in mathematics which characterizes the teacher as a designer, implementer and evaluator in the mathematics classroom see Maher 1988.
3. For research on student to student questioning, see Wilkinson & Martino, 1993.
4. One of our goals is that students become more skillful at different modes of inquiry. Eventually, it is expected that they will become empowered to ask questions of themselves and each other. Preliminary data suggest that this phenomenon has begun to happen at several of our project sites where we have worked in classrooms for several years (Maher & Martino, in press; Maher & Martino, submitted for review). When children work together over a several month period of time, their level of questioning becomes more sophisticated.
5. This may be explained by the relatively low level of questioning that the teacher is modeling. McCullough and Findley (1983) and Proudfit (1992) conclude that over time teachers who demonstrate good questioning techniques in the elementary mathematics classroom provide students with a model for asking questions of themselves and other students.
6. Time for student reflection is also important. We have found that asking students to write about the way they are thinking about a problem during class and at home facilitates a prolonged interest in thinking about the questions posed by the teacher.
7. The children in this study come from classes that have been involved in the longitudinal study: A Three and One Half Year Study of Children's Development of Mathematical Knowledge (NSF grant # MDR-9053597) which is currently ending its third year. Leder (1993) has argued for the importance of intense, long-term studies to obtain detailed accounts rather than snapshots of children's mathematical experiences. In this way, we are able to study the building of mathematical knowledge over time. This report focuses on one aspect of this research, that is, the effect of teacher questioning upon the growth of this knowledge.
8. Because we were interested in determining whether or not students' ideas develop in a parallel way when confronted with a problem with the same mathematical structure, we selected the Pizza Problem which has a structure which is isomorphic to the Tower Problem. The Pizza Problem asks the children to determine the number of pizzas that can be made when there is the option of selecting from among n pizza toppings (for this study, $n = 4$), where n is an arbitrary non-negative integer. The student can make two choices for each of the n pizza toppings, either to include it on the pizza or not. None of the individual topping choices are restricted by the

choice made with regard to the choice of the other toppings. Since there are two possible choices for each of the n toppings and these choices are independent, there are 2 x 2 x ... x 2 or 2^n possible pizzas that can be made from n possible toppings.

Both problems lend themselves to building a justification for the solution by either *proof by mathematical induction* or *proof by cases.* The number of possible towers n blocks high that can be made from two colors can be generalized as 2^n and, similarly, the number of pizzas that can be made from n toppings as 2^n (Alston and Maher, 1993; Davis, 1992; Maher and Martino, 1993).

For the towers, focusing on one of the two colors in certain positions, one could partition the set of towers into $(n + 1)$ cases in which there are exactly none, one, two, ..., n of that color within a set, to develop a *proof by cases.* Similarly, one could have organized the set of pizzas according to the number of toppings. (See Maher and Martino, under review; Maher, Martino and Alston, 1993.)

9. For a more complete discussion of this sequence of development, see Maher and Martino, under review.

10. For a more complete analysis of Brandon's connection between isomorphic problems see Maher, Martino & Alston, 1993; Maher & Martino, under review).

11. <u>Brandon's use of local organization for the isomorphic tower problem which came prior to the Pizza Problem - November 17, 1992 - Grade 4</u> When Brandon and his partner, Justin, were first challenged to solve the Tower Problem for towers four cubes tall, they generated new towers by trial and error, then built a "partner" tower for each new tower. Sometimes they made an "opposite" partner and other times an "upside-down" partner (see Figure 1). Brandon and Justin eventually recognized that using two different "partner" groupings sometimes resulted in producing duplicate towers. They adjusted for this difficulty by discarding the "upside-down" pairing strategy and grouped solely by "opposite" pairs. This procedure resulted in eight pairs of "opposite" towers.

12. For more detail on Meredith's method of solution see Maher, Martino & Alston, 1993.

13. Recall that Meredith had developed a proof by cases to justify that she had accounted for all possible towers. Sarah, who had initially built towers with two partners, Beth and Lori organized her towers into two sets of four towers with a "staircase" pattern and the other towers were arranged into pairs which they called "opposites". For Sarah, "opposites" referred to the two types of relationships shown in Figure 1.