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ABSTRACT

The International Mathematics Tournament of the Towns is a mathematics competition for junior and senior high school students all over the world. The tournament began in 1980 in the former Soviet Union. Participants write contest papers locally, with emphasis on solving within a very generous time allowance a small number of interesting problems. The tournament does not offer prizes, in order to de-emphasize its competitive aspect, though diplomas, certificates, and books are distributed to deserving participants. Supplementary activities include a year-round correspondence school and a summer school in Russia. A description of a participant at the summer school, which was hosted by the Beloretsk Computer School, is appended. Three of the problems proposed at the summer school are included. A selection of 50 problems from the tournaments for junior high students is also appended. (JDD)

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ED 371 556

The International Mathematics Tournament of the Towns

Andy Liu
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The Tournament is a very unusual mathematics competition for junior and senior highschool students. The participants from all over the world write the contest papers locally. The emphasis is on solving within a very generous time allowance, a small number of interesting problems, rather than on solving a large number of essentially routine problems at break-neck speed. The participants will learn the important skill of constructing written presentations of arguments.

The Tournament was born in 1980 in the former Soviet Union. It was organized by a group of dedicated mathematicians, most of whom were and still are based in Moscow, under the direction of Prof. Nikolay Konstantinov. Due to the efforts of Prof. Jordan Tabov of Bulgaria and Prof. Peter Taylor of Australia, the Tournament first spread to the former Eastern Block and then to the whole world.

There is a Junior Tournament for students in Grades 7 to 10, and a Senior Tournament for students in Grades 11 and 12. Students in lower grades within each Tournament write the same papers, but have their raw scores multiplied by a factor greater than one.

Each Tournament consists of a Fall Round and a Spring Round. Each Round consists of an Ordinary-Level Paper and an Advanced-Level Paper. A participant can write all four papers in a Tournament, with the final score being the best of the four.

There are four or five problems in an O-Level Paper. They are easier but worth less points. The participants are allowed four hours. There are six or seven problems in an A-Level Paper. They are harder and worth more points. The participants are allowed five hours. In each Paper, only the best three problems count.

A selection of fifty beautiful problems from the first ten Junior Tournaments are appended. They may serve to heighten or awaken the interest in mathematical problem-solving of your students or children. Ask them to have a go at it, and try them yourself too. While some problems may sound technical, others are certainly very down-to-earth.

If your youngsters have had a crack at the problems and would like someone to check their solutions, they may send them to me at the following address:

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Edmonton, Alberta, T6G2G1.

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Alternatively, you may purchase the following publications:
(1) Tournament of the Towns, Questions and Solutions, 1980-1984.
(2) Tournament of the Towns, Questions and Solutions, 1984-1989.
The first sells, in Australian currency, for $18, and the second $20. There is also a $1 handling charge per book. The third book may soon be ready. Payment by Visa or Mastercard is preferred. The books may be ordered from:

Australian Mathematics Trust,
The University of Canberra,
P.O.B. 1, Belconnen,
A.C.T. 2616, Australia.

I hope that there may be sufficient interest in your school or town to take part in the Tournament. If you are in the Greater Edmonton area, we have participated in three Tournamets already, and you are most welcome to join. Our students usually write the contest papers at the University of Alberta on Sunday afternoons.

The Tournament does not offer prizes, to de-emphasize its competitive aspect. Nevertheless, deserving participants will receive much valued diplomas from Moscow, in Russian. We augment them with book prizes, as well as certificates for all local participants.

If you are in another urban centre, I am more than happy to help you set up a local committee. There is an entry fee, in U.S.A. currency, of $50 plus $3 per 100,000 in local population, to be paid to Moscow. The students' papers are first graded locally, and the best ones are forwarded to Moscow. I will supply solutions and suggest grading schemes.

If you are in a rural area and have difficulty setting up a local committee, I am sure I can persuade Moscow to allow your youngsters to participate under the banners of an appropriate urban centre.

In recent years, two supplementary activities have also been implemented. The first is a year-round correspondence school. Training problems are supplied by Moscow. Students work on them and send their solutions to me for checking and feedbacks.

The second is a summer school in Russia. In 1993, it was held in Beloretsk. I accompanied two Greater Edmonton area highschool students on this mathematical journey. A write-up about it is also appended.

International Mathematics

Tournament of the Towns

Selected Junior Problems.

Problem 1.
Find all permutations $(a_1, a_2, \ldots, a_{101})$ of the numbers 2, 3, ..., 102 in which a_k is divisible by k for all k.

Problem 2.
ABCD is a convex quadrilateral inscribed in a circle with centre O. AC is perpendicular to BD. Prove that the broken line AOC divides the quadrilateral into two parts of equal area.

Problem 3.
Each of 64 friends simultaneously learns one different item of news. They begin to phone one another to tell them their news. Each conversation lasts exactly one hour, during which time it is possible for two friends to tell each other all of their news. What is the minimum number of hours needed in order for all the friends to know all of the news?

Problem 4.
A game is played on an infinite plane. There are fifty-one pieces, one "wolf" and fifty "sheep". There are two players. The first commences by moving the wolf. Then the second player moves one of the sheep, the first player moves the wolf, the second player moves a sheep, and so on. The wolf and the sheep can move in any direction through a distance of up to 1 metre per move. Is it true that for any starting position, the wolf will be able to capture at least one sheep?

Problem 5.
In a certain country, there are more than 101 towns. The capital of this country is connected by direct air routes with 100 towns, and each town, except for the capital, is connected by direct air routes with 10 towns. It is known that from any town, it is possible to travel by air to any other town, changing planes as many times as is necessary. Prove that it is possible to close down half of the air routes connected with the capital, and preserve the capability of traveling from any town to any other town within the country.

Problem 6.
(a) A circle is divided into 10 equal arcs by 10 points. These points are joined in pairs by 5 chords. Is it necessarily true that 2 of these chords are of equal length?
(b) A circle is divided into 20 equal arcs by 20 points. These points are joined in pairs by 10 chords. Is it necessarily true that 2 of these chords are of equal length?

Problem 7.
A pedestrian walked 3.5 hours. In every period of one hour's duration, he walked 5 kilometres. Is it true that his average speed was 5 kilometres per hour?

Problem 8.
The positive integer K is obtained from another positive integer M by scrambling its digits.
(a) Prove that the sum of the digits of 2M is equal to that of 2K, and the sum of the digits of M/2 is equal to that of K/2.
(b) Prove that the sum of the digits of 5M is equal to that of 5K.

Problem 9.
A version of billiard is played on a right triangular table, with a pocket in each of the three corners. A ball is played from just in front of the pocket at the 30° angle, towards the midpoint of the opposite side. Prove that if the ball is played hard enough, it will land in the pocket at the 60° angle after 8 bounces.

Problem 10.
Prove that in any set of 17 distinct positive integers, either there are five each dividing the next or there are five none of which divides any of the other four.

Problem 11.
In a ballroom dance class, 15 boys and 15 girls are lined up in parallel rows so that 15 couples are formed. It so happens that the difference in height between the boy and the girl in each couple is not more than 10 centimetres. The boys and girls are rearranged in their respective rows in descending order of height, and 15 new couples are formed, matching the tallest boy with the tallest girl. Prove that in each of the new couples, the difference in height is still not more than 10 centimetres.

Problem 12.
Show how to cut an isosceles right triangle into a finite number of isosceles right triangles every two of which are of different sizes.

Problem 13.
In each of the pairs (8,9) and (288,289), each number contains each of its prime factors to a power no less than 2. Prove that there are infinitely many such pairs of consecutive positive integers.

Problem 14.
A village consists of 9 blocks in a 3 by 3 formation, each block a square of side length 1. Each block has a paved road along each side. Starting from a corner of the village, what is the minimum distance we must travel along paved roads, if each section of paved road must be passed along at least once, and we are to finish at the same corner?

Problem 15.
A quadrilateral has a vertex on each side of a given rectangle. Prove that the perimeter of the quadrilateral is not smaller than double the length of a diagonal of the rectangle.

Problem 16.
M is a set of points in the plane, no three on a line. Some points are joined to others by line segments, with each point connected to no more than one line segment. If we have a pair of intersecting line segments AC and BD, we may replace them with AB and CD. In the resulting system of segments, if there are still pairs of intersecting segments, we may make a similar replacement. Is it possible for such replacements to continue indefinitely?

Problem 17.
Six musicians gathered at a chamber music festival. At each scheduled concert, some of these musicians played while the others listened as members of the audience. What is the minimum number of such concerts in order to enable each musician to listen, as a member of the audience, to all the other musicians?

Problem 18.
On the Island of Camelot live 13 grey, 15 brown and 17 crimson chameleons. If two chameleons of different colours meet, they both simultaneously change to the third colour. Is it possible that they will eventually all be the same colour?

Problem 19.
There are 68 coins, each having a different weight from that of one another. Show how to find the heaviest coin and the lightest coin in 100 weighings on a balance.

Problem 20.
Find all solutions of the system of equations $(x+y)^3=z$, $(y+z)^3=x$ and $(z+x)^3=y$.

Problem 21.
Three grasshoppers are on a straight line. Every second, one of the grasshoppers jumps across one, but not both, of the other two grasshoppers. Prove that after 1985 seconds, the grasshoppers cannot all be in their initial positions.

Problem 22.
The first number of a sequence is 1. Each subsequent number is the sum of the preceding number x and the sum of the digits of x. Can the number 123456 belong to this sequence?

Problem 23.
A square is divided into 5 rectangles in such a way that its 4 vertices belong to 4 of the rectangles, whose areas are equal, and the fifth rectangle has no points in common with the sides of the square. Prove that the fifth rectangle is a square.

Problem 24.
The digits 0, 1, 2, ..., 9 are written in a 10 by 10 table, each number appearing 10 times.

(a) Is it possible to write them in such a way that in any row or column, there would be no more than 4 different digits?
(b) Prove that there must be a row or column containing more than 3 different digits.

Problem 25.
In a tournament, each of eight football teams plays every other team once. There are no ties. Prove that at the end of the tournament, it is possible to find four teams A, B, C and D such that A beats B, C and D, B beats C and D and C beats D.

Problem 26.
Two people toss coins. One tosses his 10 times, the other tosses his 11 times. What is the probability that the second person has more "heads" than the first?

Problem 27.
Through vertices A and B of triangle ABC are drawn two lines which divide the triangle into three triangles and a quadrilateral. Three of these four regions have equal area. Prove that one of them is the quadrilateral.

Problem 28.
There are 20 football teams in a tournament. On the first day, all the teams play one game. On the second day, again all the teams play one game. Prove that after the second day, it is possible to select 10 teams no two of which have played each other.

Problem 29.
We are given two two-digit numbers x and y. It is known that x is twice as big as y. One of the digits of y is the sum, while the other digit of y is the difference, of the digits of x. Find all values of x and y.

Problem 30.
In a game with two players, there is a rectangular chocolate bar with 60 pieces arranged in a 6 by 10 formation. It can be broken only along the lines dividing the pieces. The first player breaks the bar along one line, discarding one section. The second player then breaks the remaining section, discarding one section. The first player repeats this process with the remaining section, and so on. The game is won by the player who leaves a single piece. In a perfectly played game, which player wins?

Problem 31.
Consider subsets of the set $\{1,2,...,N\}$. For each such subset, we compute the product of the reciprocals of all members. Find the sum of all such products.

Problem 32.
Prove that $3(1+a^2+a^4) \geq (1+a+a^2)^2$ for all real number a.

Problem 33.
We are given tiles in the form of right triangles having perpendicular sides of lengths 1 centimetre and 2 centimetres. Is it possible to form a square from 20 such tiles?

Problem 34.
A machine gives out five pennies for each nickel and five nickels for each penny. Can Peter, who starts out with one penny, use the machine several times to end up with an equal number of nickels and pennies?

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Problem 35.
Nine pawns form a 3 by 3 square at the lower left corner of an 8 by 8 chessboard. Any pawn may jump over another one standing next to it onto an empty square directly beyond. Jumps may be horizontal, vertical or diagonal. It is desired to reform the 3 by 3 square at another corner of the chessboard by means of such jumps. Can the pawns be so rearranged at the
(a) upper left hand corner;
(b) upper right hand corner?

Problem 36.
Prove that the second last digit of each power of three is even.

Problem 37.
In a game, two players alternately choose larger positive integers. At each turn, the difference between the new and the old number must be greater than zero but smaller than the old number. The original number is 2. The player who chooses the number 1987 wins the game. In a perfectly played game, which player wins?

Problem 38.
We are given a figure bound by arc AC of a circle and a broken line ABC, with the arc and the broken line on opposite sides of the chord AC. Construct a line passing through the midpoint of arc AC which divides the figure into two regions of equal area.

Problem 39.
There are 2000 apples, contained in several baskets. One can remove baskets as well as apples from the baskets. Prove that it is possible to leave behind an equal number of apples of each of the remaining baskets, with the total number of apples not being less than 100.

Problem 40.
ABCD is a convex quadrilateral. The midpoints of BC and DA are M and N respectively. The diagonal AC divides MN in half. Prove that the areas of triangles ABC and ACD are equal.

Problem 41.
(a) The vertices of a regular decagon are painted alternately black and white. Two players take turns drawing a diagonal connceting two vertices of the same colour. These diagonals must not intersect. The winner is the player who is able to make the last move. In a perfectly played game, which player wins?
(b) Answer the same question for the regular dodecagon.

Problem 42.
Let a, b and c be positive integers such that $a=b+c$. Prove that $a^4+b^4+c^4$ is double the square of a positive integer.

Problem 43.
Is it possible to cover a plane with circles in such a way that exactly 1988 circles pass through each point?

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Problem 44.
It is known that the proportion of people with fair hair among people with blue eyes is more than the proportion of people with fair hair among all people. Which is greater, the proportion of people with blue eyes among people with fair hair or the proportion of people with blue eyes among all people?

Problem 45.
In a triangle, two altitudes are not smaller than the sides on to which they are dropped. Find the angles of this triangle.

Problem 46.
To each vertex of a cube is assigned randomly the number 1 or the number -1. To each face of the cube is assigned the product of the four numbers at the vertices of the face. Is it possible that the sum of these 14 numbers is 0?

Problem 47.
Prove that $a^2+3b^2+5c^2\leq 1$ where a, b and c are positive numbers satisfying $a\geq b\geq c$ and $a+b+c\leq 1$.

Problem 48.
What digit must replace "?" in the number 888...88?999...99, where there are fifty 8's and fifty 9's, in order that the resulting number is divisible by 7?

Problem 49.
Two players alternately moves a pawn on a chessboard from one square to another, subject to the rule that, at each move, the distance moved is strictly greater than that of the previous move. Distance is measured from the centre of the starting square to the centre of the destination square. A player loses when unable to make a legal move on his turn. Who wins if both use the best strategy?

Problem 50.
(a) Prove that if 3n stars are placed on the squares of a 2n by 2n board, then it is possible to remove n rows and n columns in such a way that all stars will be removed.
(b) Prove that it is possible to place 3n+1 stars on the squares of a 2n by 2n board in such a way that after removing any n rows and n columns, at least one star remains.

A MATHEMATICAL JOURNEY(Andy Liu)

In the summer of 1993, Matthew Wong of Old Scona Academic High School, Edmonton, and Daniel van Vliet of Salisbury Composite High School, Sherwood Park, were invited to attend an International Mathematics Tournament of the Towns Conference in Beloretsk, Russia, along with me. It was chaired by Prof. Nikolay Konstantinov, President of the Tournament and recent winner of the Paul Erdos Award from the World Federation of National Mathematics Competitions. Prof. Nikolay Vasiliev, who chairs the Problem Committee of the Tournament, was also present.

There were 60 participants in all. The 15 non-Slavs consisted of 1 Englishman, 2 Austrians, 3 Canadians, 4 Germans and 5 Columbians. Apart from Prof. Gottfried Perz of Graz, Austria, and me, the others are highschool or university students. Among the Slavs were some Bulgarians, Armenians and Estonians.

Beloretsk is in the Bashkirian Republic of Russia. It is just west of the Ural Mountains and north east of the Caspian Sea. The train ride from Moscow takes 36 hours each way. The time difference from Edmonton is 12 hours. So we had come literally to the other side of the world.

It is quite hot in Beloretsk on an August day, but comfortably cool in the morning and the evening. The population is about 100,000, spread over quite a large rural area. It is not uncommon to be followed by chicken and sheep while walking on the streets. There is just enough industry to give the economy a big boost, but the air and water are refreshingly clean.

The Beloretsk Computer School which hosted the Conference is at the edge of the town. It consists of the original school building and a new five-floor dormitory. The three of us shared a spacious, comfortable and well-furnished room. We were next to the Austrians, with whom we shared a sink, a toilet, a shower and a refridgerator. The landscape around the school is very picturesque. A nearby river was a favourite spot for swimming, and the site of a traditional Russian tea-party by the bonfire one evening.

Our daily routine was roughly as follows. Breakfast was at 9 in the morning. From 10 to 12, there was usually a Mathematical Education forum. From 12 to 2 was a problem session for the students. Lunch was at 2. From 3 to 5 ran another problem session. Supper was at 7, and occasionally another Mathematics Education forum ran from 8 to 10. The food was good.

The Conference began officially on August 1, even though our appetite had already be whetted by a problem set distributed on the train. During the first two days, four problems were presented to the students. The proposers provided some relevant background information. This was done in Russian, with adequate translation into English, which all 15 non-Slavs understand.

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The students could work on the problems in their own rooms, in classrooms, on the meadows, or wherever they chose. They could work in teams, an option favoured mostly by the non-Slavs. The lone English student joined the Canadian team, but the lone Austrian student worked on his own.

The students had only until 10 o'clock in the evening of August 3 to solve the problems. Starting from August 4, solutions to those parts of the problems which had been solved were presented, along with a more challenging fifth problem.

The final deadline was at 10 o'clock in the evening of August 7. All solutions, as far as they are known to the proposers, were presented on August 8. During that afternoon, the participants were presented with diplomas, with very detailed descriptions of what they had accomplished, and whether their efforts were solo ones or in collaboration. I was most impressed with the meticulous care the jury had graded the students' work.

The Anglo-Canadian team did not win any prizes. We probably spent too much time socializing with the Russian students. However, I felt that this was just as important an aspect of our trip as working on the problems. The three of them did get some work done, and the jury commended them for formulating a generalization of one of the problems and making partial progress towards its solution. Matthew and Daniel continue to work on the problems. after their return.

On August 9, the last day of the Conference, solutions to the more difficult problems in this year's Tournament of the Towns were presented. Then we bid farewell to Prof. V. G. Khazankin, principal of the Beloretsk Computer School, and other friends. They included Mother Khazankin, Ilia, the eight-year-old son of one of the teachers, and Alexei. He is eight-three, a most interesting man who has collected lots of minerals and folklore from the Ural region.

We spent one night in Moscow on the way into, and two more on the way out of Russia. We stayed with Moscow mathematicians, who moved their families out temporarily so that we could have their apartments to ourselves. They are a very dedicated group. Besides running the Tournament, they organize the Independent University of Moscow, which keeps alive the famed tradition of the Moscow Mathematics Circles, without official recognition or financial support.

It was a wonderful experience, living in actual Russian homes. In the little time we had, we managed to get quite a bit of sightseeing done. We had an acute sense of the changing social fabrics at a very exciting time in the history of a nation which not many have had the privilege to observe first hand. It is a trip that leaves a lasting impression on each of us, mathematically and otherwise.

BELORETSK PROBLEMS(Andy Liu)

The following are three of the four problems proposed in the International Mathematics Tournament of the Towns Conference in Beloretsk, Russia, in August 1993. They are reconstructed from my notes and are not the exact formulations as were presented. Problem 3, which is very elaborate, is omitted.

I invite the readers to send me nice solutions to these problems. I will forward them to the proposers too. Bear in mind that they do not have all the solutions. If there is sufficient interest, I will discuss some of them in a follow-up article, which may include the missing Problem 3. I can reached at:

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Edmonton, Alberta, T6G2G1.

Problem 1.
Proposer: Prof. A. A. Egorov.
Prize Winners:
V. Zamjatin, highschool student from Kirov, Russia.
A. Barkhudarian and V. Poladian, highschool students from Yerevan, Armenia.
A. Bufetov, highschool student from Moscow, Russia.
I. Buchkina and D. Schwarz, highschool students from Moscow, Russia.

Consider the following diophantine equation in x and y:

(*) $x^2+(x+1)^2+\cdots+(x+n-1)^2=y^2$,

where n is a given positive integer.

If (*) has infinitely many solutions, we say that n is infinitely good.

(a) Prove that 2, 11, 24 and 26 are infinitely good.
(b) Prove that there are infinitely many infinitely good positive integers.

If (*) has at least one solution with $x>0$, we say that n is very good.

(c) Prove that an infinitely good positive integer is very good.
(d) Prove that a positive integer which is very good but not infinitely good cannot be even.
(e) Prove that 49 is very good but not infinitely good.
(f) Prove that there are infinitely many positive integers which are very good but not infinitely good.

If (*) has at least one solution, we say that n is good.

(g) Prove that 25 is good but not very good.
(h) Prove that there are no other positive integers which are good but not very good.

If (*) has no solutions, we say that n is bad.

(i) Prove that 3, 4, 5, 6, 7, 8, 9 and 10 are bad.
(j) Prove that there are infinitely many bad positive integers.
(k) Devise an efficient algorithm which classifies a given positive integer as infinitely good, very good but not infinitely good, good but not very good, or bad.

Problem 2.
Proposer: Prof. N. Vasiliev.
Prize Winner: Yu. Belous, university student, Ekaterinburg, Russia.

A partition of a convex polygon into at least two triangles is called an anti-triangulation if whenever two of the triangles share a common segment, this segment is not a complete side of at least one of the two triangles.

(a) Determine all integers $k>1$ such that there exists an anti-triangulation of a triangle into k triangles.
(b) Prove that no anti-triangulations exist for a convex polygon which is not a triangle.
(c) Generalize the result to partitions into convex n-gons not sharing common sides for $n>3$.
(d) Generalize the result to partitions into tetrahedra not sharing common faces or not sharing common sides.

Problem 4.
Proposer: Prof. K. A. Knop.
Prize Winners:
Oleg Popov, highschool student, Moscow, Russia.
E. Tsyganov and V. Kartak, university students, Beloretsk, Russia.

In Russia, there are 1, 2, 3, 5, 10, 15, 20 and 50 kopeck coins. To make up an integral amount, we take at every stage the largest coin not exceeding the remaining part of the amount. This method is called the Greedy Algorithm. For example, to make up 29 kopecks, the Algorithm yields 29=20+5+3+1.

A general coinage system consists of m coins of respective integral values $1=a_1<a_2<\cdots<a_m$. It is said to be suitable if for any integral amount, the number of coins used in the Greedy Algorithm is minimum.

(a) Prove that the Russian system is suitable.
(b) A new k kopeck coin is to be introduced into the Russian system. Determine all values of k for which the new system remains suitable.
(c) Prove that a general coinage system is suitable if a_{k+1} is divisible by a_k for $1\le k\le m-1$.
(d) Prove that a general coinage system is suitable if $a_{k+1}-a_k$ is constant for $1\le k\le m-1$.
(e) Devise an efficient algorithm for testing whether a given coinage system is suitable.

In a general coinage system S which is not necessarily suitable, denote by $f(S,k)$ the smallest number of coins required to make up the integral amount k. Denote by $g(S,n)$ the largest integral value such that $f(S,k)\le n$ whenever $k\le g(S,n)$, and by $g(m,n)$ the minimum value of $g(S,n)$ taken over all systems with m coins.

(f) Prove that $g(S,3)=28$ if S is the Russian coinage system.
(g) Determine $g(m,n)$ for specific values of m and n, or obtain upper and lower bounds.

Suppose we are only interested in making up integral amounts up to and including 100, but we wish to do so in as efficient a way as possible.

(h) Determine the minimum values of $m\cdot\max\{f(S,k):1\le k\le 100\}$ taken over all coinage systems S, where m is the number of coins in S.
(i) Determine the minimum value of $m(f(S,1)+f(S,2)+\cdots+f(S,100))$ taken over all coinage systems S, where m is the number of coins in S.