DOCUMENT RESUME

ED 370 962 TM 021 502

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TITLE	Cognitive Diagnosis Using Latent Trait Models.
PUB DATE	Apr 94
NOTE	42p.; Paper presented at the Annual Meeting of the National Council on Measurement in Education (New Orleans, LA, April 4-8, 1994).
PUB TYPE	Reports - Evaluative/Feasibility (142) -- Speeches/Conference Papers (150)
EDRS PRICE	MF01/PC02 Plus Postage.
DESCRIPTORS	*Cognitive Processes; Cognitive Tests; *Computer Software Development; Educational Assessment; *Educational Diagnosis; Elementary Secondary Education; Higher Education; High Schools; High School Students; *Item Response Theory; Knowledge Level; *Problem Solving; Trigonometry
IDENTIFIERS	Cognitive Modeling; *Competency Space Approach; Computer Assisted Data Analysis; Japan

ABSTRACT

This paper discusses the competency space approach to diagnosing misconceptions, skill, and knowledge acquisition. In some approaches that combine misconceptions, skill, and knowledge acquisition, the latent ability theta is used more or less as an insignificant element, but in the competency space approach, a multidimensional latent space is used. The competency space approach is explained and an example is given of its use. It is necessary to expand the acceleration model to the multidimensional space if it is to be used effectively in the competency space approach. A slow and patient approach that attempts to identify the subspace of theta that deals with attribute mastery (domain knowledge) (theta b) of a relatively small dimensionality first may be productive. Once a unidimensional theta b has been identified, the acceleration model can be tested. The example given uses data from a trigonometry test for high school students in Japan. The competency approach can be effectively used in research in which an appropriate software is developed and used for intensive observation of an individual's problem-solving behavior. Two tables and seven figures illustrate the analysis. An appendix contains the test answers. (Contains 24 references.) (SLD)

COGNITIVE DIAGNOSIS USING LATENT TRAIT MODELS

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April 5, 1994

Approaches to Cognitive Modeling

1994 NCME Annual Meeting

New Orleans, LA

TABLE OF CONTENTS

I. Introduction

With a rapid progress of computer technologies, now we can *program* a well-controlled cognitive experiment in computer *software*, accomodate the software in a number of microcomputers, and have each trained experimenter carry one of the microcomputers and collect data, conducting the experiment on the individual basis. In this way, we can easily collect data for several hundred individual subjects without using too much time. With such a set of data not only we can observe each individual's behavior intensively but also observe his/her behavior *in perspective*, clarify individual differences, etc.

Tatsuoka (1985, 1990) has developed a method for diagnosing misconceptions, skill and knowledge acqusition combining deterministic concepts of *attribute* and *ideal response patterns* with a latent trait θ .

DiBello, Stout and Roussos (1993) have proposed a model for cognitive assessment:

$$P_g(\alpha,\theta) = (1-p)[d_g \; s_{\alpha g} \; P_g(\theta + \Delta c_g) + (1-d_g)P_g(\theta)] \quad ,$$

where g denotes *item*, α is an *attribute pattern*, which is expressed as a sequence of k 0's (= nonpossession) and 1's (= possession), θ is the *latent residual ability*, p is the slip probability, d_g is the probability to select the strategy represented by the vector q_g , for answering item g , which is expressed as a sequence of k 0's (= not needed) and 1's (= needed), $s_{\alpha g}$ is the conditional probability of applying q_g correctly, given that q_g is selected, Δ is a constant usually set equal to 2.0, c_g is a measure of completeness of q_g , $P_g(\theta)$ is the probability of correctly applying all required attributes other than those indicated in the q_g vector which follows the one-parameter logistic model (Rasch model), and $P_g(\alpha,\theta)$ is the conditional probability of correct answer to item g , given α and θ .

In these approaches, the latent ability θ is used more or less as an insignificant element. Samejima (1993a, 1994a) has proposed the competency space approach for cognitive assessment, in which a *multidimensional latent space* is used. Unlike other approaches, the competency space approach uses latent ability spaces agressively, making attribute mastery diagnosis easy and more precise, assessing ability to use mastered or partially mastered attributes, etc.

II. Developmental Changes

When our interest in assessing children's skills of addition, multiplication, fractions, negative numbers, etc., mastery or non-mastery of these skills may cover all important aspects. Developmental psychology tells, however, that there are more attributes to acquire and use as one grows and becomes capable of handling more complex problems, and also each attribute becomes more complex. Thus attribute mastery itself becomes more complicated, and also acquisition of attributes and *ability to identify necessary attributes, which have already mastered or partially mastered, structure them for solving a new problem, and use them innovatively*, etc., will gradually be differentiated. As one becomes more mature and challenges intellectual tasks, such *dynamics* become more and more important.

To give an example, we come across graduate students who are excellent in course work, but cannot design dissertation research, which is essential for a Ph. D. candidate. Such individuals lack ability to identify the necessary attributes in, and retrieve them from, our long term memory, structure them well, etc. Unlike domain knowledge, it is more difficult to assess such dynamics and screen applicants for graduate programs in terms of this kind of ability before decision has to be made as to whether an applicant should be accepted or rejected. Thus cognitive diagnosis should deal with both of these two differentiated aspects, whereas this may not be very important when we deal with children.

III. Competency Space Approach

Competency space approach assumes that mastery of an attribute is a continuous process, although observable indices may be just *mastery* and *non-mastery.* Thus a continuous latent variable is hypothesized behind each attribute mastery. The approach is more or less focused on situations in which:

1. Our target populations of individuals are on relatively high levels of intellectual maturity, and *ability to use mastered attributes in innovative ways* has become more important.

2. Thus two subspaces are considered in the competency space, although a clear-cut separation may not be feasible. Let Θ denote the competency space, which is multidimensional. This competency space is decomposed into two subspaces, such that

$$\Theta' = [\Theta_a', \ \Theta_b'] \ ,$$

where the subspace Θ_a deals with attribute masteries, or *domain knowledge*, and the subspace Θ_b concerns with ability to use the mastered attributes dynamically.

It is conceived that, as one grows and becomes intellectually more mature, the roles of the subspace Θ_b will become more important.

An advantage of the competency space approach is the possibility of *reducing* the dimensionality of Θ, as factor analytic ideas indicate. Operationally, reduction of dimensionality can be done by transforming the manifest variables to the set of *principal components*, and discarding those components the variances of which are negligibly small; this can also be done by using the factor analysis model, discarding the unique factors. Note, however, that these procedures are justifiable only when the unique factor variances are all negligibly small. If some of them are large enough, then these unique factors should be included in determining the dimensionality of Θ_a.

It may be advisable to use both factor analysis and the principal component analysis. Factor analysis will tell how many unique factor variances are substantially large, and with this number added by the number of common factors becomes the estimated dimensionality of Θ. This number should be checked against the results of the principal component analysis, to find out if it agrees with the number of principal components whose variances are not negligibly small. If it does, then those components with negligibly small variances can be discarded; if it does not, include more components to make it agree. This is important for making assessment in Θ_a accurate.

Once this has been done, the remaining principal components must be rotated to make each dimension meaningful. An appropriate method will be oblique pattern matrix rotation (e.g., Lawley & Maxwell, 1971), devising the pattern matrix to decompose Θ into the subspaces Θ_a and Θ_b .

It should be noted that the estimated dimensionality of Θ partly depends on the homogeneity of the target population. If individuals of the target population are educated fairly similarly, for example, the structure of the latent variables behind attribute masteries will be relatively simple. If the target population consists of people with variously different cultures, for example, the competency space will require a larger dimensionality. Thus cross-validation of the dimensionality is necessary, in order to make the usefulness of the operationally defined competency space broader.

ASSESSMENT IN Θ_a

Higher mental processes include not only more *complicated* attributes, but also a *larger* number of attributes. This will require more elaborate methodologies in assessing mastery or partial mastery of attributes in the subspace Θ_a , and also with a powerful methodology cognitive assessment will become more precise and informative.

Grades of Attainment

Assessment in the subspace Θ_a deals not only with the dichotomous categories of *mastery* and *non-mastery* but also *partial* mastery, with the introduction of the concept of the *grade of attainment*, in an effort to make the diagnosis more precise and *informative.*

Take *trigonometry*, for example. There are many problem solving tasks which require some levels of attainment toward mastery of trigonometry, but not the complete *mastery* of trigonometry. For some problems, understanding *sine*, *cosine* and *tangent* on the *triangle* level without dealing with negative values (grade 1) may be sufficient. For some others, understanding those concepts on the triangle level *with* possibly negative values (grade 2) may be required, or those on the *unit circle* level using *radians* (grade 3) may be mandatory.

Thus diagnosis will be made more precisely, and the number of attribute patterns, and hence that of *non-fuzzy* response patterns will increase. Table 1 presents sixteen conceivable states of attribute mastery when there are four attributes, A, B, C and D, on the left hand side. Since at least one attribute is needed to make an item, in this example there are fifteen conceivable types of items requiring one or more attributes. The *non-fuzzy* response patterns of these fifteen items corresponding to the sixteen states of attribute masteries are given on the right hand side of Table 1.

Insert Table 1 About Here

If the grade of attainment is accomodated and the states of attribute mastery are described by 0 , 1 (grade 1 attainment) and 2 (mastery), then the number of conceivable states of mastery will be increased to $3^4 = 81$, and there will be eighty conceivable types of items. If a test includes all these types, then there will be the same number of *non-fuzzy* response patterns. Also the number of possible *fuzzy* response patterns becomes enormous.

A strength of the competency space approach is that all these response patterns can be hadled without difficulty, by virtue of the *graded response model* (Samejima, 1969, 1972, 1994c). Thus there is no need to attempt to *reduce* the number of response patterns, nor to classify them into a smaller number of categories.

The relationship between each grade of attainment toward mastery of an attribute and Θ_a is probabilistic, rather than deterministic, with the reduction of the dimensionality of the latent space. This is illustrated by the curves in Figure 1, when Θ_a turns out to be unidimensional.

Insert Figure 1 About Here

It is important to select a suitable model out of the family of graded response models (Samejima, 1972, 1994c), and criteria for this selection will be discussed later. Also as an example of suitable models for cognitive assessment, the *acceleration model* (Samejima, 1994c) will be introduced and discussed later.

Attribute Diagnosis

Attribute diagnosis in the sense of Tatsuoka (1985, 1990) and of DiBello, Stout and Roussos (1993) can be done practically in the same way in the competency space approach.

Let g denote an attribute, and X_g be a grade of attainment toward mastery of each attribute, and $x_g \ (= 0, 1, ..., m_g)$ denote its realization. The *operating characteristic*, $P_{x_g}(\Theta_a)$, of the grade x_g means the conditional probability, given Θ_a , with which the individual gets x_g , that is,

$$P_{x_g}(\Theta_a) \equiv prob.[X_g = x_g \mid \Theta_a] \ .$$

For a set of $n \ (\geq 1)$ attributes, a response pattern, denoted by V , indicates a sequence of X_g for $g = 1, 2, ..., n$, and its realization, v , can be written as

$$v = \{ x_g \}' \ .$$

It is assumed that *local independence* (Lord & Novick, 1968) holds, so that within any group of individuals all characterized by the same value of θ the distributions of x_g 's are all independent of each other. (The acceptability of this assumption is determined when we decide on the dimensionality of Θ operationally.) Thus the operating characteristic, of the response pattern v is defined as

$$P_v(\Theta_a) \equiv prob.[V = v \mid \Theta_a] = \prod_{x_g \in v} P_{x_g}(\Theta_a) \quad ,$$

which is also the likelihood function, $L(v \mid \Theta_a)$, for $V = v$.

Using this likelihood function, the maximum likelihood estimate of the position of each non-fuzzy response pattern, as well as of each *fuzzy* response pattern, can be obtained. Thus an examinee with a given response pattern will be assigned to one of the *non-fuzzy* response patterns in terms of his/her state of attribute mastery, selecting one with the shortest distance in Θ_a or based on the likelihood ratio, etc.

Because of the introduction of the grade of attainment, this diagnosis can be done more *precisely*, that is, instead of deciding on either *mastery* or *non-mastery* of each attribute, the *grade of attainment* in the attribute in question is diagnosed for non-mastery individuals.

ASSESSMENT IN Θ_b

The graded response model (Samejima, 1969, 1972, 1994c) takes essential roles in the assessment in the subspece Θ_b . If the number of observable steps in, say, problem solving, then the amount of information will be increased and the estimation of the subject's position in Θ_b will become more accurate. This is important, for screening graduate program applicants in terms of this aspect of suitability for a future Ph. D., for example, will be done with little risk.

Suppose, for example, that a cognitive process, like *problem solving*, contains a finite or enumerable number of steps. For convenience, let g be a problem, and x_g be a step. The graded item score x_g should be assigned to the individuals who have successfully completed up to the step x_g but *failed* to complete the step $(x_g + 1)$.

Let $M_{x_g}(\Theta_b)$ be the *processing function* (Samejima, 1994c) of the graded item score x_g , which is the probability with which the individual completes the step x_g successfully, under the *joint* conditions that:

1. the individual's position is Θ_b , and
2. the steps up to $(x_g - 1)$ have already been followed and completed successfully.

It is assumed that $M_{x_g}(\Theta_b)$ is either strictly increasing in any dimension of Θ_b or constant, for $x_g = 1, 2, ..., m_g$. This assumption is reasonable considering that each item has some direct and positive significance to the ability measured.

Let $(m_g + 1)$ be the hypothesized graded item score adjacent to and above m_g . Since everyone can at least obtain the item score 0 , and no one is able to obtain the item score

$(m_g + 1)$, it is reasonable to set

$$M_{x_g}(\Theta_b) \begin{cases} = 1 & \text{for } x_g = 0 \\ = 0 & \text{for } x_g = m_g + 1 \end{cases} ,$$

for all Θ_b . Thus the operating characteristic, $P_{x_g}(\Theta_b)$, of the graded item score x_g is given by

$$P_{x_g}(\Theta_b) = \prod_{u \le x_g} M_u(\Theta_b) \, [1 - M_{(x_g+1)}(\Theta_b)] \ .$$

This provides the fundamental framework for the general graded response model.

Let $P^*_{x_g}(\Theta_b)$ denote the conditional probability with which the individual of ability Θ_b follows and completes the cognitive process successfully up to the step x_g , *or further*. Thus

$$P^*_{x_g}(\Theta_b) = \prod_{u \le x_g} M_u(\Theta_b) \ .$$

This function is called the *cumulative operating characteristic* (Samejima, 1994c), although cumulation is in the opposite direction. Thus the operating characteristic, $P_{x_g}(\Theta_b)$, can also be expressed by

$$P_{x_g}(\Theta_b) = P^*_{x_g}(\Theta_b) - P^*_{(x_g+1)}(\Theta_b) \ .$$

It is obvious that $P^*_{x_g}(\Theta_b)$ is also either strictly increasing in Θ_b or constant for all Θ_b , and assumes unity for $x_g = 0$ and zero for $x_g = m_g + 1$ for the entire range of Θ_b . From this $P_{x_g}(\Theta_b)$, the operating characteristic, $P_v(\Theta_b$ is obtained by

$$P_v(\Theta_b) \equiv prob.[V = v \mid \Theta_b] = \prod_{x_g \in v} P_{x_g}(\Theta_b) \ ,$$

which is also the likelihood function, $L(v \mid \Theta_b)$, for $V = v$. Thus the maximum likelihood estimate of the individual's position in Θ_b is obtained using this $L(v \mid \Theta_b)$.

Since experiments consist of continuous *discoveries*, it will be wise to use a nonparametric approach for estimating the operating characteristics (e.g., Leveine, 1984; Samejima, 1983, 1993a, 1994b) before fitting any parametric model. Figure 2 illustrates a nonparametrically estimated curve using simulated data, by the differential weight procedure of the conditional p.d.f. approach (Samejima, 1994b). Unlike the logistic curve, which was fitted to the theoretical curve and plotted in the same figure, the curve obtained by the differential weight procedure can disclose non-monotonicity of the theoretical curve. Thus for any identified pattern of behavior shared by a subgroup of individuals it will be better that the operating characteristic be estimated nonparametrically.

Insert Figure 2 About Here

If no strong relationship with any dimension of the subspace Θ_b is found out, that is, each and every curve turns out to be flat, the pattern of behavior is called *diffused*. Some possible interpretations are that the pattern of behavior under observation may have multiple meanings, it may be an indicator of the dimension Θ_b does not include and addition of a new dimension in Θ_b may be necessary, etc. In any case, search for the true reason is mandatory.

If a strong relationship between the pattern of behavior and any dimension of the competency space is discovered, the pattern of behavior is said to be *concentrated*. The next step is to tentatively parameterize the nonparametrically estimated operating characteristics to make mathematical handling easy, using a semi-parametric models such as Ramsay & Wang's (1993). After this, some appropriate graded response model, such as *acceleration model*, can be fitted (cf. Samejima, 1994c). A similar approach can be used for any identified *buggy* pattern of behavior, which can also be useful in cognitive diagnosis.

If we only have data of a limited number of subjects, we could replace the estimated operating characteristic by the sum of the estimated conditional distributions of the latent trait, given the maximum likelihood estimates of the trait, in order to obtain some amount of information. These conditional distributions can be obtained in the processes of nonparametric estimation of the operating characteristics, such as the differential weight procedure and the simple sum procedure of the conditional p.d.f. approach (Samejima, 1994b). Figure 3 illustrates *concentrated* and *diffused* patterns of behavior.

Insert Figure 3 About Here

INSTRUMENTS

It may be feasible to categorize cognitive processes into the following three categories with respect to their assessabilities:

1. Assessable by paper-and-pencil tests.
2. Difficult to be assessed by paper-and-pencil tests, but assessable by computerized tests, using figural responses, etc.
3. Not assessable by either of them, and intensive observations in experimental situations are needed.

With advancement of computer technologies, category 2 will take more and more from category 3. Category 1 has advantage over the other two categories in the economy of time and cost, however. Since we cannot aford to conduct research which belongs to category 2 without a sizable research money, it may be wise to start with paper-and-pencil tests, to find out how far we can go with them.

In practice, we may develop relatively simple items requiring a small numbers of attributes, which are basically used for diagnosing the grade of attainment toward mastery of the attributes, and also more complex multi-attribute items that require dynamics of innovative use of the mastered or partially mastered attributes, etc. The resulting test will consist of such different types of questions.

IV. Example

We are in the process of collecting data, using a paper-and-pencil test on *trigonometry* for the first year high school students in Japan. At the moment, there are only 41 protocols. From these protocols, the strategy matrix (DiBello, Stout & Roussos, 1993) has been made.

TEST ITEMS

There are 12 items in the test, and they are as shown below. They were taken from the original test of 17 items. Thus five items, (1)2., (2)2., (3)1., (4)1. and (4)4., are left blank.

(1) Obtain the values of the following.

1. $\sin 90° + \cos 180° + \tan 45°$
2.
3. $\sin\left[\frac{\pi}{3}\right] \cos\left[\frac{5}{6}\pi\right] - \cos\left[\frac{\pi}{3}\right] \sin\left[\frac{5}{6}\pi\right]$

(2) Simplify the following.

1. $(\sin\theta + \cos\theta)^2 + \left[\frac{(1-\tan\theta)^2}{1+\tan^2\theta}\right]$
2.

(3) Obtain the following when θ is an angle in the third quadrant and $\sin\theta\cos\theta = \frac{1}{2}$.

1.
2. $\sin\theta + \cos\theta$

(4) Solve the following trigonometric equation and inequality when x is as given in the brackets.

1.
2. $2\cos x + \sqrt{3} \le 0 \quad (0° \le x \le 360°)$
3. $2\sin^2 x + \cos x - 2 = 0 \quad (0 \le x \le 2\pi)$
4.

(5) Suppose that $AB = 5$, $BC = 8$ and $B = 60°$ in the triangle ABC.

1. Obtain the length of AC.

2. Obtain $\sin A$.
3. Obtain the radius of the circumscribed circle.
4. Obtain the area of the triangle ABC.

(6) In the triangle ABC, we set $BC = a$ and $CA = b$. When $a^2 \cos A \sin B = b^2 \sin A \cos B$, what shape does the triangle ABC have? Explain it in a concrete way.

(7) In the triangle ABC, suppose that $\sin A : \sin B : \sin C = 7 : 8 : 13$. Obtain the largest angle.

Since this test is a part of the term examination, and the main objective of the examination was to find out the students' achievement in trigonometry after a quarter of learning, most questions require orthodox approaches based on the mastery of attributes. A few items, including (6), may require some dynamics, however. The expected correct answers to these twelve items and the item score matrix are presented in Appendices 1 and 2, respectively.

ATTRIBUTES AND STRATEGY MATRIX

The attributes required in solving these trigonometry problems are *stratified*, in the sense that mastery of an attribute in each phase requires mastery of some or all attributes in the previous phases.

1. Phase 1: <u>prerequisites</u>

   In this phase, easy tasks, such as addition and multiplication involving negative numbers, fractions, etc., are excluded, for all examinees seem to have mastered them.

   (a) inequalities

   (b) factorization

   (c) distribution

   (d) Pythagoras theorem

2. Phase 2: <u>understanding of basic trigonometry</u>

   grade 1: sine, cosine, tangent on the *triangle* level, all *positive* values.

   grade 2: the above, but inclusive of *negative* values.

   grade 3: sine, cosine, tangent on the *unit circle* level, using *radians*.

3. Phase 3: <u>laws and theorems</u>

   (a) area of a triangle using trigonometry

(b) $\sin^2\theta + \cos^2\theta = 1$

(c) $\tan\ \theta = \left[\frac{\sin\ \theta}{\cos\ \theta}\right]$

(d) In a triangle, a side facing to a larger angle is longer than a side facing to a smaller angle.

(e) law of cosines: $c^2 = a^2 + b^2 - 2ab\ \cos C$

(f) law of sines:

grade 1: $\left[\frac{a}{\sin A}\right] = \left[\frac{b}{\sin B}\right] = \left[\frac{c}{\sin C}\right]$

grade 2: $\left[\frac{a}{\sin A}\right] = \left[\frac{b}{\sin B}\right] = \left[\frac{c}{\sin C}\right] = 2R$

NOTE: a, b, c: sides; A, B, C: angles; R: radius of the circumscribed circle; S: area

The strategy matrix of these items is provided by Table 2. In addition to the attributes, a column is assigned for ability to restructure attributes and use them innovatively, etc., under the heading, *dynamics*. When the grade of attainment is considered for an attribute, the grade required for solving the item is written in the column; when only mastery and non-mastery are used, * is given, meaning that mastery of the attribute is required for solving the item. This table will become the most important information source when the pattern matrix is determined in factor rotation for defining the competency space Θ , differentiating Θ_b from Θ_a .

Insert Table 2 About Here

V. Mathematical Models

Here we only discuss mathematical models in the unidimensional latent space, although many models suitable for cognitive assessment can be easily expanded to the multidimensional latent space. Let θ be the latent trait, which assumes any real number. Hereafter, all those functions observed in the multidimensional latent space earlier in this paper will be transfered to the corresponding functions of θ .

Samejima (1972) has proposed a general theoretical framework of the graded response model, in which the *homogeneous case* is distinguished from the *heterogeneous case*. The *general graded response model* represents a family of mathematical models which deal with *ordered polychotomous categories* in general. These ordered categories include: A, B, C, D and F in the evaluation of students' performance, *strongly disagree*, *disagree*, *agree* and *strongly agree* in a social attitude survey, *partial credit* given in accordance with the individual's degree of attainment toward the solution of a problem, to give some examples.

CRITERIA FOR EVALUATING MODELS

The general graded response model includes many specific mathematical models. In an effort to select a right model, or models, for a specific psychological reality, the following features will be considered as desirable.

1. The principle behind the model and the set of accompanied assumptions agree with the *psychological reality* in question.

   This is by far the most important criterion.

2. The model provides *additivity* in the operating characteristics of the item scores x_g 's (Samejima, 1994c).

   Additivity holds if the operating characteristics belong to the same mathematical model under *finer* recategorizations and *combinings* of two or more categories together.

   This implies that the unique maximum condition is satisfied by the resulting operating characteristics, if it is satisfied by those of the original x_g 's .

   Graded item scores, or partial credits, are more or less *incidental.* For example, it is a general practice to reevaluate the grades, A, B, C, D, and F, in a required course to *pass* and *fail.*

   Also, with the advancement of computer technologies, it is quite possible to obtain more abundant information from the individual's performance in computerized experiments as we proceed in research, and thus we need finer recategorizations of the whole cognitive process.

3. The model can be *naturally generalized to a continuous response model.* This criterion is a natural extension of *additivity.*

4. The model satisfies the *unique maximum condition* (Samejima, 1969, 1972).

   Satisfaction of this condition assures that the likelihood function of *any* response pattern consisting of such response categories has a unique local or terminal maximum.

5. The model provides the *ordered modal points* of the operating characteristics in accordance with the item scores.

The acceleration model introduced here is a model constructed with these considerations in mind. It is obvious that the first two criteria are most important for cognitive assessment.

ACCELERATION MODEL

The *acceleration model* is a family of models that belongs to the heterogeneous case, and is specifically appropriate in *cognitive assessment.*

Consider a as problem solving that requires a number of subprocesses before attaining the solution. It is assumed that there are more than one *step* in the whole process which are *observable.* Graded item scores, or partial credits, 1 through m_g , are assigned to the successful completions of these separate observable steps. The *processing function* for each x_g $(= 1, 2, ..., m_g)$ is given by

$$M_{x_g}(\theta) \ = \ [\Psi_{x_g}(\theta)]^{\xi_{x_g}} \ ,$$

where ξ_{x_g} (> 0) is the step *acceleration* parameter.

It is assumed that the whole process leading to the solution consists of a finite number of *clusters*, each containing one or more steps, and within each cluster the parameters in $\Psi_{x_g}(\theta)$ are common. Thus, if two or more adjacent x_g 's belong to the same cluster, then the parameters in $\Psi_{x_g}(\theta)$ are the same for these x_g 's , and, otherwise, at least one of the parameters is different.

Let w denote a subprocess, which is the smallest unit in the cognitive process. Thus each step contains one or more w 's . Let ξ_w (> 0) be the subprocess *acceleration* parameter, and then the step acceleration parameter, ξ_{x_g} , for each of $x_g = 1, 2, ..., m_g$ is given as the sum of ξ_w 's over all $w \in x_g$. Thus we can rewrite the processing function in the form:

$$M_{x_g}(\theta) \ = \ [\Psi_{x_g}(\theta)]^{\sum_{w \in x_g} \xi_w} \ .$$

The name, acceleration parameter, comes from the fact that, within each step, separate subprocesses contribute to *accelerate* the value of θ at which the discrimination power is maximal to its ultimate position (cf. Samejima, 1994c).

The acceleration model represents a family of models in which $\Psi_{x_g}(\theta)$ is specified by a strictly increasing, five times differentiable function of θ with zero and unity as its two asymptotes, and

$$\frac{\Psi_{x_g}(\theta) \ \frac{\partial^2}{\partial \theta^2} \ \Psi_{x_g}(\theta)}{[\frac{\partial}{\partial \theta} \ \Psi_{x_g}(\theta)]^2}$$

decreases with θ . Thus the cumulative operating characteristic, $P^*_{x_g}(\theta)$, is given by

$$P^*_{x_g}(\theta) \ = \ \prod_{u=0}^{x_g} \ [\Psi_u(\theta)]^{\xi_u} \ ,$$

and we obtain for the operating characteristic

$$P_{x_g}(\theta) \ = \ \prod_{u=0}^{x_g} \ [\Psi_u(\theta)]^{\xi_u} \ [1 - [\Psi_{(x_g+1)}(\theta)]^{\xi_{x_g+1}}] \ .$$

If our experimental setting is improved and allows to observe the individual's performance in more finely graded steps, then m_g will become larger. It is obvious (Samejima, 1994c) that the resulting operating characteristics still belong to the acceleration model: a partial satisfaction of the *additivity* criterion. It is also obvious that the model *can be generalized to a continuous* *response model* as the limiting situation in which there are infinitely many subprocesses in each step.

Here a *specific* model in this family will be introduced, in which $\Psi_{x_g}(\theta)$ is given by

$$\Psi_{x_g}(\theta) = \frac{1}{1 + \exp\left[-D\ \alpha_{x_g}(\theta - \beta_{x_g})\right]} \ , \tag{1}$$

where $D = 1.7$, and α_{x_g} (> 0) and β_{x_g} are the discrimination and location parameters, respectively. Figure 4 illustrates the six operating characteristics with $m_g = 5$ and the parameters $\alpha_{x_g} = 1.36517$, 1.03244 , 0.87524 , 1.09083 , 0.58824 , $\beta_{x_g} = -0.94260$, -0.76985 , 0.03941 , 1.35406 , 0.80000 , and $\xi_{x_g} = 0.41972$, 0.51741 , 0.54196 , 0.60004 , 1.00000 , for $x_g = 1, 2, 3, 4, 5$.

Insert Figure 4 About Here

Since we have

$$\theta_{dmax} = \Psi_{x_g}^{-1}\left[\frac{\xi_{x_g}}{1 + \xi_{x_g}}\right] \ ,$$

where θ_{dmax} indicates the value of θ at which the processing function $M_{x_g}(\theta)$ is steepest, or most discriminating, it is obvious that θ_{dmax} is a strictly increasing function of ξ_{x_g} . Applying this for the subprocess acceleration parameter ξ_w , we can say that within each step separate subprocesses contribute to *accelerate* θ_{dmax} to its ultimate position.

It has been shown (Samejima, 1994c) that the unique maximum condition is satisfied in this model. It has also been shown (Samejima, 1994c) that the orderliness of the modal points of the operating characteristics *practically* holds in this model, except for very unusual cases.

PARAMETER ESTIMATION

In cognitive assessment, it is suggested to follow the procedure described below in parameter estimation.

1. Use a nonparametric estimation method like Levine's (1984) or Samejima's (1983, 1993b, 1994b), and estimate the operating characteristics, $P_{x_g}(\theta)$'s .

2. Tentatively parameterize the results using a very general semiparametric method, such as Ramsay and Wong's (1993), in which not only the fit of $P_{x_g}(\theta)$ but also that of its first partical derivative with respect to θ are considered.

3. From these results obtain the estimated processing function $\hat{M}_{x_g}(\theta)$ and its partial derivative with respect to θ .

4. Select three arbitrary probabilities, p_1 , p_2 and p_3 , which are in an ascending order, and find out θ_1 , θ_2 and θ_3 , at which $\hat{M}_{x_g}(\theta)$ equals p_1 , p_2 and p_3 , respectively.

5. The estimated acceleration parameter $\hat{\xi}_{x_g}$ is obtained as the solution of

$$\frac{\theta_3 - \theta_2}{\theta_2 - \theta_1} = \frac{\log\ [(\ p_2\ )^{-1/\xi_{x_g}} - 1] - \log\ [(\ p_3\ )^{-1/\xi_{x_g}}\ - 1]}{\log\ [(\ p_1\ )^{-1/\xi_{x_g}} - 1] - \log\ [(\ p_2\ )^{-1/\xi_{x_g}} - 1]} \quad . \tag{2}$$

6. Obtain the estimate, $\hat{\beta}_{x_g}$, as the solution of

$$\hat{M}_{x_g}(\beta_{x_g}) \ = \ \left[\frac{1}{2}\right]^{\hat{\xi}_{x_g}} \quad . \tag{3}$$

7. From these results obtain the estimate of α_{x_g} by

$$\hat{\alpha}_{x_g} \ = \ \frac{2^{\hat{\xi}_{x_g}+1}}{D\ \hat{\xi}_{x_g}} \frac{\partial}{\partial\theta} \hat{M}_{x_g}(\theta) \qquad at \ \ \theta = \hat{\beta}_{x_g} \quad . \tag{4}$$

Note that this method can be applied for *any* curve as long as $\frac{\partial}{\partial\theta}\ \hat{M}_{x_g}(\theta)$ is available. Suppose, for example, a model based on the individual choice behavior has been used, and then the researcher decides to *switch* to this specific acceleration model for the rest of research. The method just explained can be used directly. Figure 5 presents the operating characteristics in Masters' (1982) partial credit model, using $\alpha_{x_g} = 1,2,3,4,5,6$ and $\beta_{x_g} = 1.0, 2.0, 3.0, 3.5, 1.8, 1.0$ in Bock's (1972) multinomial model represented by

$$P_{k_g}(\theta) \ = \ \frac{\exp[\alpha_{k_g}\ \theta + \beta_{k_g}]}{\sum_{u \in K_g}\ \exp[\alpha_u\ \theta + \beta_u]} \quad ,$$

with k_g replaced by x_g for $x_g = 0,1,2,3,4,5$, respectively.

Insert Figure 5 About Here

In fact, the parameters in the acceleration model used in Figure 4 were obtained by the parameter estimation method just described, from the $M_{x_g}(\theta)$'s in Masters' model with the above set of parameters, setting $p_1 = 0.1$, $p_2 = 0.5$ and $p_3 = 0.9$ in (2). Compare Figure 5 with Figure 4. These two sets of curves are practically indistinguishable! The same procedure was repeated by setting the values of p_1 , p_2 and p_3 to $(0.2, 0.5, 0.8)$, $(0.25, 0.50, 0.75)$, $(0.3, 0.5, 0.7)$ and $(0.3, 0.6, 0.7)$, respectively, and the results turned out to be very similar.

The similarity of the two sets of curves exemplifies the fact that the acceleration model provides varieties of different curves. This is a strength, for researchers who want to switch to the acceleration model can do that without reanalyzing the protocols following the acceleration model, if he/she uses the method just described.

The *item response information function* (Samejima, 1973, 1993b) of the graded item response x_g is defined by

$$\begin{aligned} I_{x_g}(\theta) \ &\equiv \ -\frac{\partial^2}{\partial\theta^2} \log\ P_{x_g}(\theta) \\ &= \sum_{u \le x_g} -\frac{\partial^2}{\partial\theta^2}\ \log\ M_u(\theta) - \frac{\partial^2}{\partial\theta^2}\ \log\ [1 - M_{(x_g+1)}(\theta)] \quad . \end{aligned}$$

Once this function is evaluated, The *item information function*, is obtained as its conditional expectation, given θ , that is,

$$I_g(\theta) \equiv E[I_{x_g}(\theta) \mid \theta] = \sum_{x_g} I_{x_g}(\theta) P_{x_g}(\theta) \ .$$

The item information functions obtained in these two models proved to be fairly close to each other (cf. Samejima, 1994c), as was expected from the similarity between the two sets of operating characteristics presented in Figures 4 and 5.

ROBUSTNESS OF THE MODEL

The assumption that a single set of α_{x_g} and β_{x_g} exists within each step may be violated, especially when m_g is small. Robustness of the acceleration model will handle this situation, however. Suppose that we did *not* know there were *two* clusters involved, and treated them as a single step, estimating the step parameters following this specific acceleration model, and, later, with the improvement of the experimental setting, they were disclosed as two separate steps which belonged to two different clusters. The results obtained by treating them as a single step still provides *good approximations*, showing the robustness of the model.

Suppose, on the contrary, we need to combine two steps which do not belong to the same cluster. Note that the resulting combined step will *not* belong to the acceleration model. Using the method described earlier, we can approximate the operating characteristic of the combined category following the acceleration model, and the result usually provides a good approximation.

Figure 6 illustrates six step processing functions, the first three of which belong to a cluster with $\alpha_{x_g} = 1.0$ and $\beta_{x_g} = -1.0$, and the second three to another with $\alpha_{x_g} = 1.0$ and $\beta_{x_g} = 1.0$, respectively, and the third parameters are $\xi_{x_g} = 0.5, 1.0, 1.5,$ for the three steps in each cluster. It is obvious that the operating characteristic of the combined category of any two adjacent steps still belongs to this specific acceleration model, except for the combination of $x_g = 2$ and $x_g = 3$. The two more curves in the same figure are the product of $M_{x_g}(\theta)$ ' for $x_g = 2$ and $x_g = 3$ (solid line) and its approximation following the acceleration model (a dash and two dots repeated), respectively.

Insert Figure 6 About Here

Figure 7 presents the operating characteristics of the seven steps, 0 through 6 , plus the *sum* of two operating characteristics for $x_g = 2$ and $x_g = 3$ (solid line), and the approximated operating characteristic (a dash and two dots repeated) obtained by fitting a single $M_{x_g}(\theta)$ to the product of the original $M_3(\theta)$ and $M_4(\theta)$ following this specific acceleration model. The p_1 , p_2 , p_3 used in this approximation were $0.21109, 0.48884, 0.79446$, and the corresponding $\theta_1, \theta_2, \theta_3$ were $-0.3, 0.5, 1.4$. The resulting estimated parameters obtained through (2), (3) and (4) turned out to be: $\hat{\xi}_{x_g} = 1.11338$, $\hat{\beta}_{x_g} = 0.43006$, and $\hat{\alpha}_{x_g} = 0.86888$.

Insert Figure 7 About Here

The two curves for the combined category in Figure 7 overlap almost completely, showing the robustness of the model. Thus *Additivity* of the operating characteristics *practically* holds for this model.

The reasons for this *robustness* of this model comes from the fact that the two parameters, α_{x_g} and ξ_{x_g} , work *compensatorily* to determine the *steepness* of $M_{x_g}(\theta)$, while ξ_{x_g} *alone* accounts for the shape of the curve. Thus a set of a large α_{x_g} and a small ξ_{x_g} will provide the steepness of the curve similar to the one resulted from a set of a small α_{x_g} and a large ξ_{x_g} . The shape of the curve is largely determined by ξ_{x_g} , however, as we can see in the earlier observation that θ_{dmax} changes as a function of ξ_{x_g} , thus together providing various shapes and steepnesses.

VI. Discussion

It is necessary to expand the acceleration model to the multidimensional latent space, in order to use it effectively in the competency space approach. It can be done in the same way the nomal ogive model on the continuous response level was expanded to the multidimensional latent space (Samejima, 1974). Several other directions of expansion are conceivable, including both compensatory and noncompensatory directions.

A slower, patient approach may be more productive, however, attempting to identify Θ_b of a relatively small dimensionality first. It is expected in the example given here that Θ_b will be unidimensional, for the population of individuals is relatively homogeneous and the problem solving is limited within trigonometry of the high school level in Japan. Once a unidimensional Θ_b has been identified, then the acceleration model can be tested to find out how well, or badly, it works.

The competency approach can be more effectively used in research that belong to category 2, in which an appropriate software for intensive observation of the individual's problem solving behavior is developed and used. The only reason that such research has not been done yet is that it takes a sizable amount of research money. It is the author's hope that both the competency space approach and the acceleration model will be used by other researchers in varieties of different situations and feedback information will be given in the near future, so that the approach and the model can further be developed in useful directions.

NCME941.TEX
March 16, 1994

References

[1] Birnbaum, A. (1968). Some latent trait models and their use in inferring an examinee's ability, Contributed chapters in Lord, F. M. and Novick, M. R., *Statistical theories of mental test scores*, Chapters 17-20, Reading, MA: Addison Wesley.

[2] Bock, R. D. (1972). Estimating item parameters and latent ability when responses are scored in two or more nominal categories, *Psychometrika*, Vol. 37, 29-51.

[3] DiBello, L. V., Roussos, L. A. & Stout, W. F. (1993). *Unified cognitive/psychometric diagnosis foundations and application*, Papaer presented at the 1993 American Educational Research Association Meeting, Atlanta, GA.

[4] Lawley, D. N. & Maxwell, A. E. (1971). *Factor analysis as a statistical method*, New York, NY: American Elsevier.

[5] Levine, M. (1984). An introduction to multilinear formula scoring theory, *Office of Naval Research Report*, 84-4.

[6] Lord, F. M. & Novick, M. R. (1968). *Statistical theories of mental test scores*, Reading, MA: Addison Wesley.

[7] Masters, G. N. (1982). A Rasch model for partial credit scoring, *Psychometrika*, 47, 1982, 149-174.

[8] Ramsay, J. O. & Wang, X. (1993). *Hybrid IRT models.* Paper presented at the 1993 Psychometric Society meeting, Berkeley, California.

[9] Samejima, F. (1969). Estimation of ability using a response pattern of graded scores, *Psychometrika Monograph, No. 17.*

[10] Samejima, F. (1972). A general model for free-response data, *Psychometrika Monograph*, No. 18.

[11] Samejima, F. (1973). A comment on Birnbaum's three-parameter logistic model in the latent trait theory, *Psychometrika*, *38*, 221-233.

[12] Samejima, F. (1974). Normal ogive model on the continuous response level in the multidimmensional latent space, *Psychometrika*, *39*, 111-121.

[13] Samejima, F. (1981). Efficient Methods of Estimating the Operating Characteristics of Item Response Categories and Challenge to a New Model for the Multiple-Choice Item, *Office of Naval Research Report*, Final Report of N00014-77-C-0360. Knoxville, TN: Department of Psychology, University of Tennessee.

[14] Samejima, F. (1983). Some Methods and Approaches of Estimating the Operating Characteristics of Discrete Item Responses, *Principals of Modern Psychological Measurement: a Festschrift for Frederic M. Lord*, (ed. H. Wainer and S. Messick), 159-182, Lawrence Erlbaum, New Jersey.

[15] Samejima, F. (1988). Advancement of Latent Trait Theory, *Office of Naval Research Report*, Final Report of N00014-81-C-0569. Knoxville, TN: Department of Psychology, University of Tennessee.

[16] Samejima, F. (1990). Differential Weight Procedure of the Conditional P.D.F. Approach for estimating the operating characteristics of discrete item responses. *Office of Naval Research Report*, 90-4, N00014-87-K0320. Knoxville, TN: Department of Psychology, University of Tennessee.

[17] Samejima, F. (1992). A design of cognitive diagnosis, Paper presented at the 1992 ONR Conference on Cognitive Diagnosis and Model-Based Measurement, Champaign, IL.

[18] Samejima, F. (1993a). Human psychological behavior: approached by latent trait models, *Proceedings of the IEEE International Workshop on Neuro-Fuzzy Control, Muroran, Japan*, 26-33.

[19] Samejima, F. (1993b). Roles of Fisher type information in latent trait models, *in Bozdogan, H. (Ed.), Proceedings of the First US/Japan Conference on the Frontiers of Statistical Modeling: an Informational Approach, May, 1992, Knoxville, Tennessee*, Kluwer Academic Publishers, Netherlands.

[20] Samejima, F. (1994a). A cognitive diagnosis method using latent trait models: competency space approach and its relationship with DiBello and Stout's unified cognitive/psychometric diagnosis model, in Proceedings of ACT/ONR Conference on Alternative Diagnostic Assessment May, 1993, Iowa City, Iowa (in press).

[21] Samejima, F. (1994b). Rationale and actual procedures of efficient nonparametric approaches for estimating the operating characteristics of discrete item responses, *Psychometrika*, (under review).

[22] Samejima, F. (1994c). Acceleration model in the heterogeneous case of the general graded response model, (in preparation).

[23] Tatsuoka, K. K. (1985). A probabilistic model for diagnosing misconceptions in the pattern classification approach. *Journal of Educational Statistics*, Vol. 12, 55-73.

[24] Tatsuoka, K. K. (1990). Toward an integration of item-response theory and cognitive error diagnosis. In N. Frederickson, R. L. Claser, A. M. Lesgold & M. G. Shafto (Eds.), *Diagnostic monitoring of skill and knowledge acquisition*. Hillsdale, NJ: Erlbaum.

APPENDIX 1

Correct Answers to the Twelve Test Items

(1) Obtain the values of the following.

1. $\sin 90° + \cos 180° + \tan 45°$

$= 1 + (-1) + 1 = 1$

3. $\sin\left[\frac{\pi}{3}\right] \cos\left[\frac{5}{6}\pi\right] - \cos\left[\frac{\pi}{3}\right] \sin\left[\frac{5}{6}\pi\right]$

$= \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\frac{1}{2} = -\frac{3}{4} - \frac{1}{4} = -1$

(2) Simplify the following.

1. $(\sin\theta + \cos\theta)^2 + \left[\frac{(1-\tan\theta)^2}{1+\tan^2\theta}\right]$

$= 1 + 2\sin\theta\ \cos\theta + \left[\frac{1-2\tan\theta+\tan^2\theta}{\frac{1}{\cos^2\theta}}\right]$

$= 1 + 2\sin\theta\ \cos\theta + \cos^2\theta - 2\sin\theta\ \cos\theta + \sin^2\theta$

$= 1 + 1 = 2$

(3) Obtain the following when θ is an angle in the third quadrant and $\sin\theta\ \cos\theta = \frac{1}{2}$.

2. $\sin\theta + \cos\theta$

$(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\ \cos\theta = 1 + 1 = 2$

$\sin\theta + \cos\theta = -\sqrt{2}$

(4) Solve the following trigonometric equation and inequality when x is as given in the brackets.

2. $2\cos x + \sqrt{3} \leq 0 \qquad (0° \leq x \leq 360°)$

$\cos x \leq -\frac{\sqrt{3}}{2}$

$150° \leq x \leq 210°$

3. $2\sin^2 x + \cos x - 2 = 0 \qquad (0 \leq x \leq 2\pi)$

$2(1 - \cos^2 x) + \cos x - 2 = 0$

$2\cos^2 x - \cos x = 0$

$\cos x\ (2\cos x - 1) = 0$

$x = \frac{\pi}{2}\ ,\ \frac{3}{2}\pi\ ,\ \frac{\pi}{3}\ ,\ \frac{5}{3}\pi$

(5) Suppose that $AB = 5$, $BC = 8$ and $B = 60°$ in the triangle ABC.

1. Obtain the length of AC.

$$b^2 = a^2 + c^2 - 2ac\ \cos B = 8^2 + 5^2 - 2 \cdot 8 \cdot 5 \cdot \tfrac{1}{2}$$
$$= 64 + 25 - 40 = 49$$
$$b = 7$$

2. Obtain $\sin A$.

$$\left[\frac{a}{\sin A}\right] = \left[\frac{b}{\sin B}\right]$$

$$\sin A = \left[\frac{a}{b}\right] \sin B = \frac{8}{7}\frac{\sqrt{3}}{2} = \frac{4}{7}\sqrt{3}$$

3. Obtain the radius of the circumscribed circle.

$$\left[\frac{b}{\sin B}\right] = 2R$$

$$R = \left[\frac{b}{2\sin B}\right] = \frac{7}{2 \cdot \frac{\sqrt{3}}{2}} = \frac{7}{\sqrt{3}} = \frac{7\sqrt{3}}{3}$$

4. Obtain the area of the triangle ABC.

$$S = \frac{1}{2}\ [a \cdot c \sin B] = \frac{1}{2} \cdot 8 \cdot 5 \cdot \frac{\sqrt{3}}{2} = 10\sqrt{3}$$

(6) In the triangle ABC, we set $BC = a$ and $CA = b$. When $a^2 \cos A\ \sin B = b^2 \sin A\ \cos B$, what shape does the triangle ABC have? Explain it concretely.

$$\cos A = \left[\frac{b^2+c^2-a^2}{2bc}\right]$$

$$\cos B = \left[\frac{a^2+c^2-b^2}{2ac}\right]$$

$$\sin A = \left[\frac{a}{2R}\right]$$

$$\sin B = \left[\frac{b}{2R}\right]$$

$$a^2 \cos A \sin B = a^2 \left[\frac{b^2+c^2-a^2}{2bc}\right] \frac{b}{2R}$$

$$b^2 \sin A \cos B = b^2\ \frac{a}{2R} \left[\frac{a^2+c^2-b^2}{2ac}\right]$$

$$a^2(b^2 + c^2 - a^2) = b^2(a^2 + c^2 - b^2)$$
$$a^2c^2 - a^4 = b^2c^2 - b^4$$
$$c^2(a^2 - b^2) - (a^2 + b^2)(a^2 - b^2) = 0$$
$$(a^2 - b^2)(c^2 - a^2 - b^2) = 0$$

$a^2 - b^2 = 0 \longrightarrow a = b$ isosceles triangle

or

$c^2 - a^2 - b^2 = 0 \longrightarrow c^2 = a^2 + b^2$ right triangle

(7) In the triangle ABC, suppose that $\sin A : \sin B : \sin C = 7 : 8 : 13$. Obtain the largest angle.

$$\left[\frac{a}{\sin A}\right] = \left[\frac{b}{\sin B}\right] = \left[\frac{c}{\sin C}\right]$$

$a = 7k$, $b = 8k$, $c = 13k$

C is the largest angle.

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\cos C = \left[\frac{a^2+b^2-c^2}{2ab}\right] = \left[\frac{47k^2+64k^2-169k^2}{2 \cdot 7k \cdot 8k}\right] = \left[\frac{-56}{112}\right] = -\left[\frac{1}{2}\right]$$

$C = 120°$

TABLE 1

Sixteen conceivable states of attribute mastery for four attributes, A, B, C and D, and the non-fuzzy response pattern corresponding to each state, based on the fifteen exhaustive types of items requiring mastery of one or more attributes.

	attribute				non-fazzy response pattern														
	A	B	C	D	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
4	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
5	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	1	1	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0
7	0	1	0	1	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0
8	0	1	1	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0
9	1	0	0	1	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0
10	1	0	1	0	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0
11	1	1	0	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0
12	0	1	1	1	0	1	1	1	0	0	0	1	1	1	0	0	0	1	0
13	1	0	1	1	1	0	1	1	0	1	1	0	0	1	0	0	1	0	0
14	1	1	0	1	1	1	0	1	1	0	1	0	1	0	0	1	0	0	0
15	1	1	1	0	1	1	1	0	1	1	0	1	0	0	1	0	0	0	0
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

TABLE 2

Strategy matrix of the twelve trigonometry items including ability to use the attributes dynamically.

item \ attr.	1(a)	1(b)	1(c)	1(d)	2	3(a)	3(b)	3(c)	3(d)	3(e)	3(f)	dynamics
(1)1.					2							
(1)3.					3							
(2)1.			*		1		*	*				
(3)2.			*		3		*					*
(4)2.	*				3							
(4)3.		*	*		3							*
(5)1.					1					*		
(5)2.					1						1	
(5)3.					1						2	
(5)4.					1	*						
(6)		*	*	*	1					*	2	*
(7)					1				*	*	1	

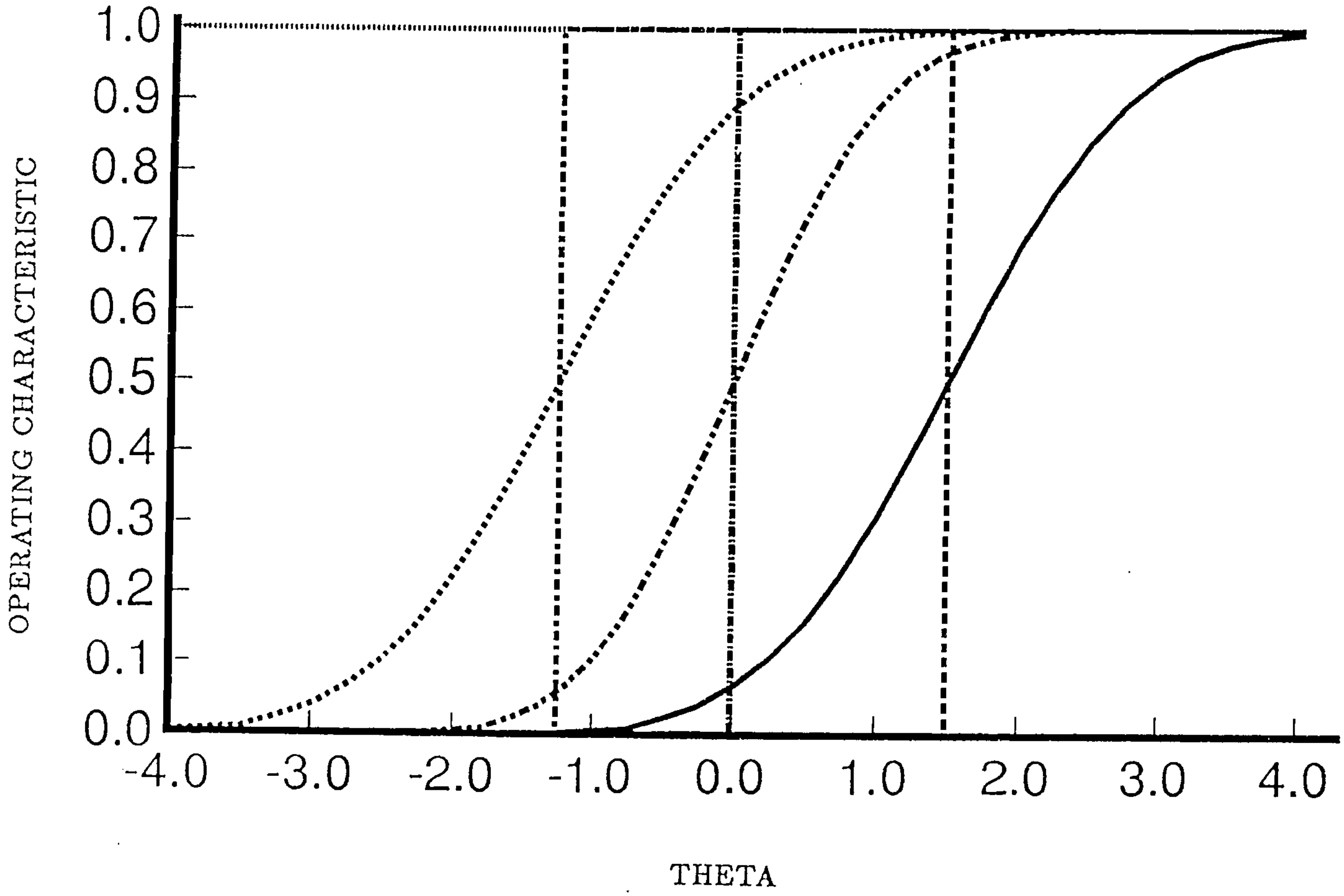

FIGURE 1

Probabilistic vs. deterministic relationships between each grade of attainment toward attribute mastery and Θ_a when it is unidimensional.

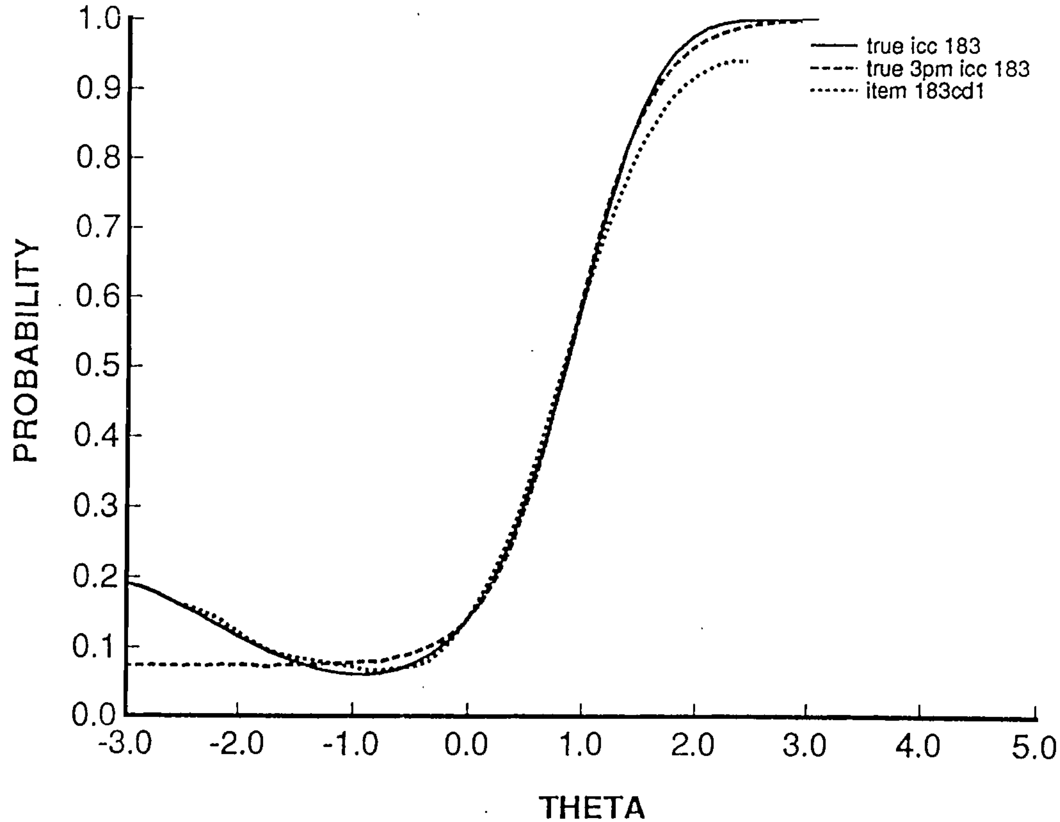

FIGURE 2

A nonparametrically estimated operating characteristic (dotted line) obtained by the differential weight procedure of the conditional p.d.f. approach, with the true operating characteristic (solid line) and the best fitted three-parameter logistic curve (dashed line).

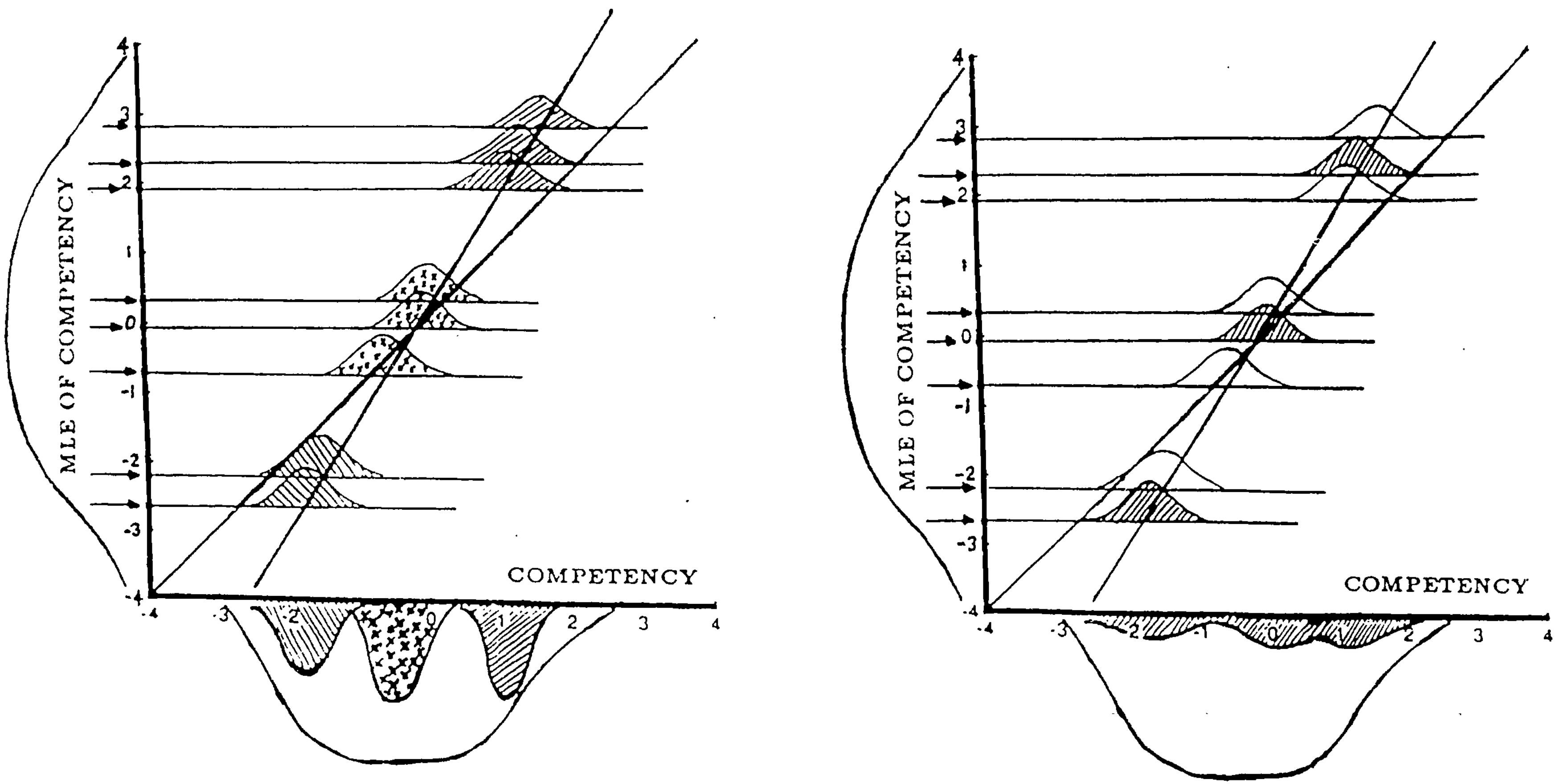

FIGURE 3

Graphical representation of concentrated and diffused patterns of behavior with respect to a dimension of the subspace Θ_b .

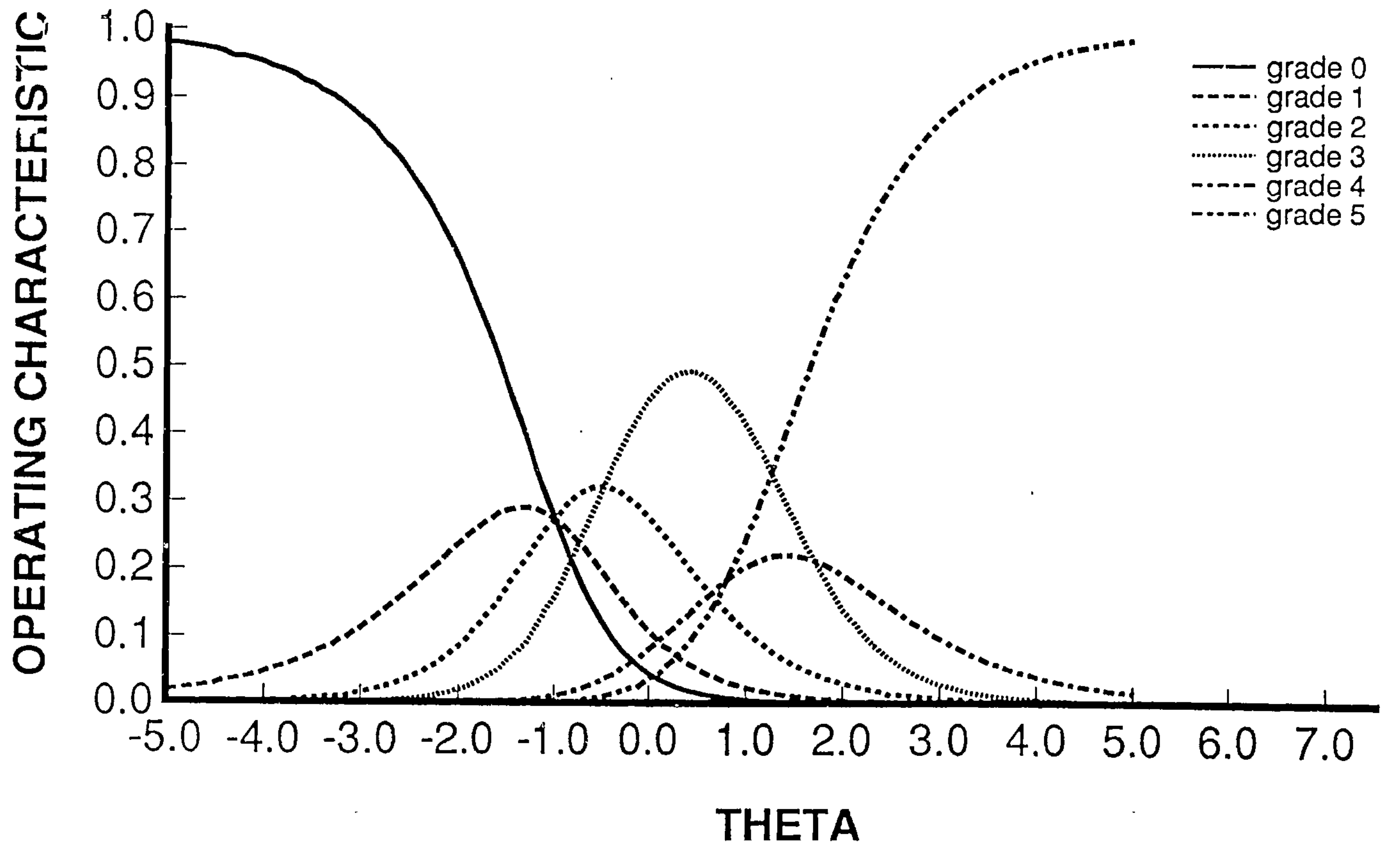

FIGURE 4

Example of a set of operating characteristics of six steps in the acceleration model.

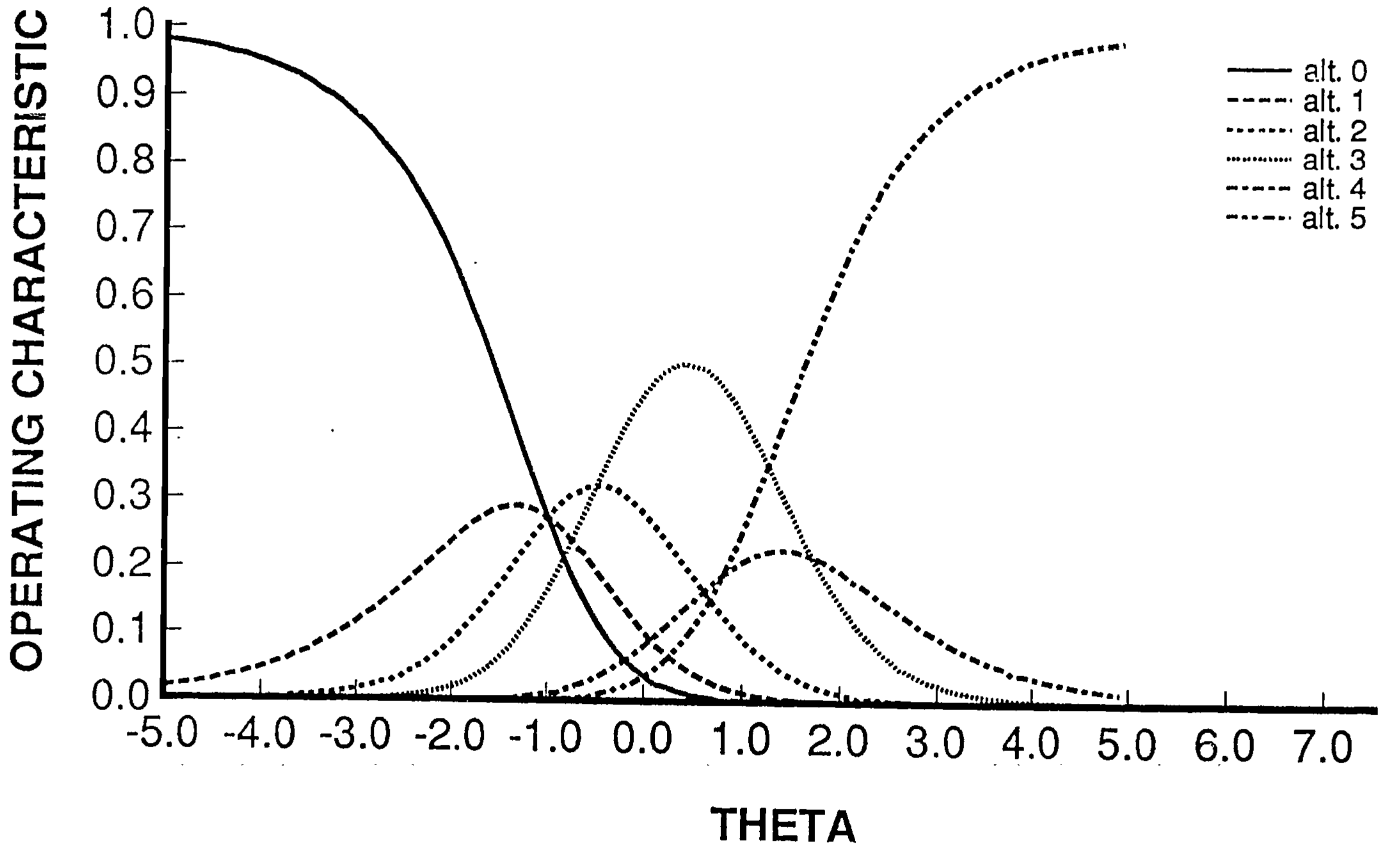

FIGURE 5

Example of a set of operating characteristics of six item scores in Masters' partial credit model.

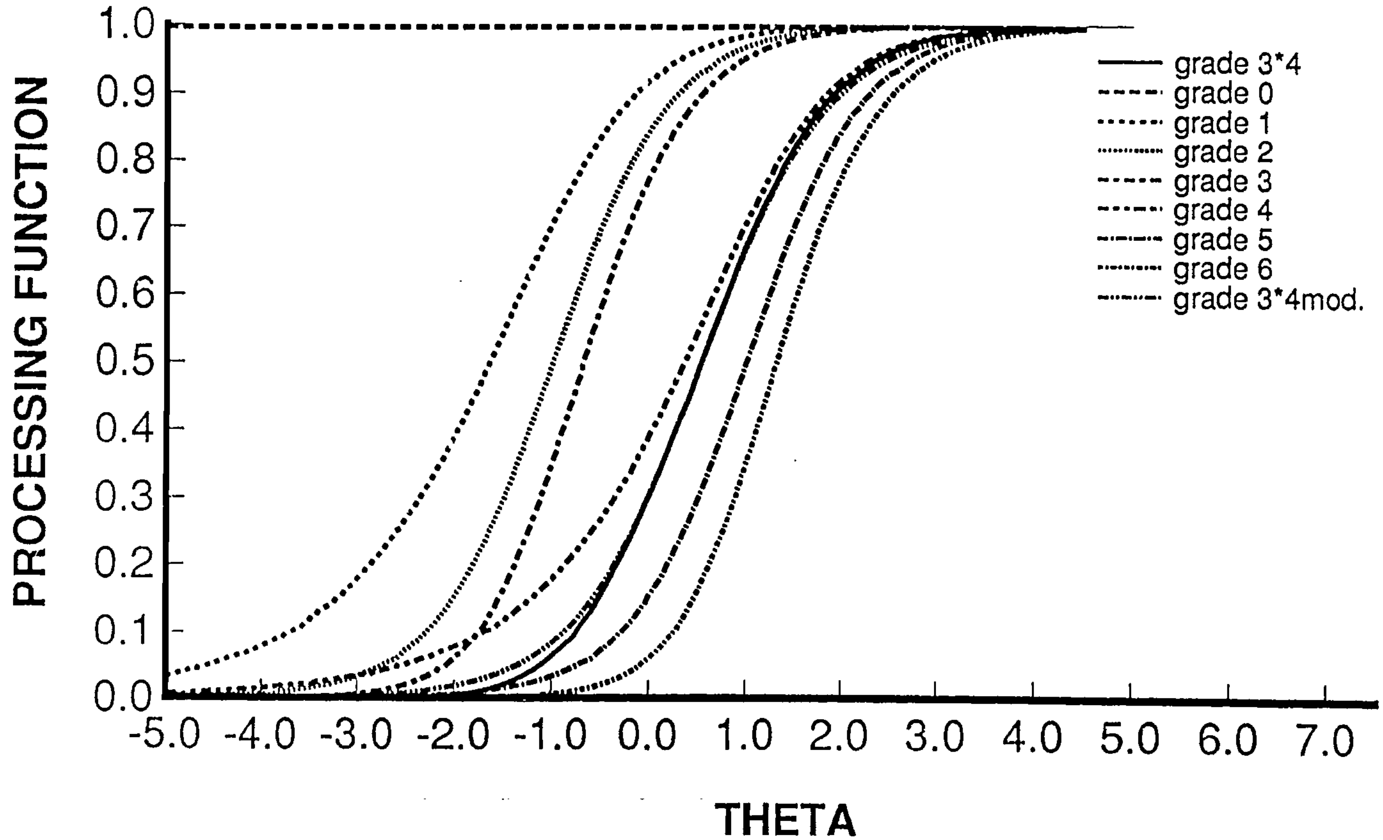

FIGURE 6

Six step processing functions, three of which belong to one cluster and the other three to another cluster.

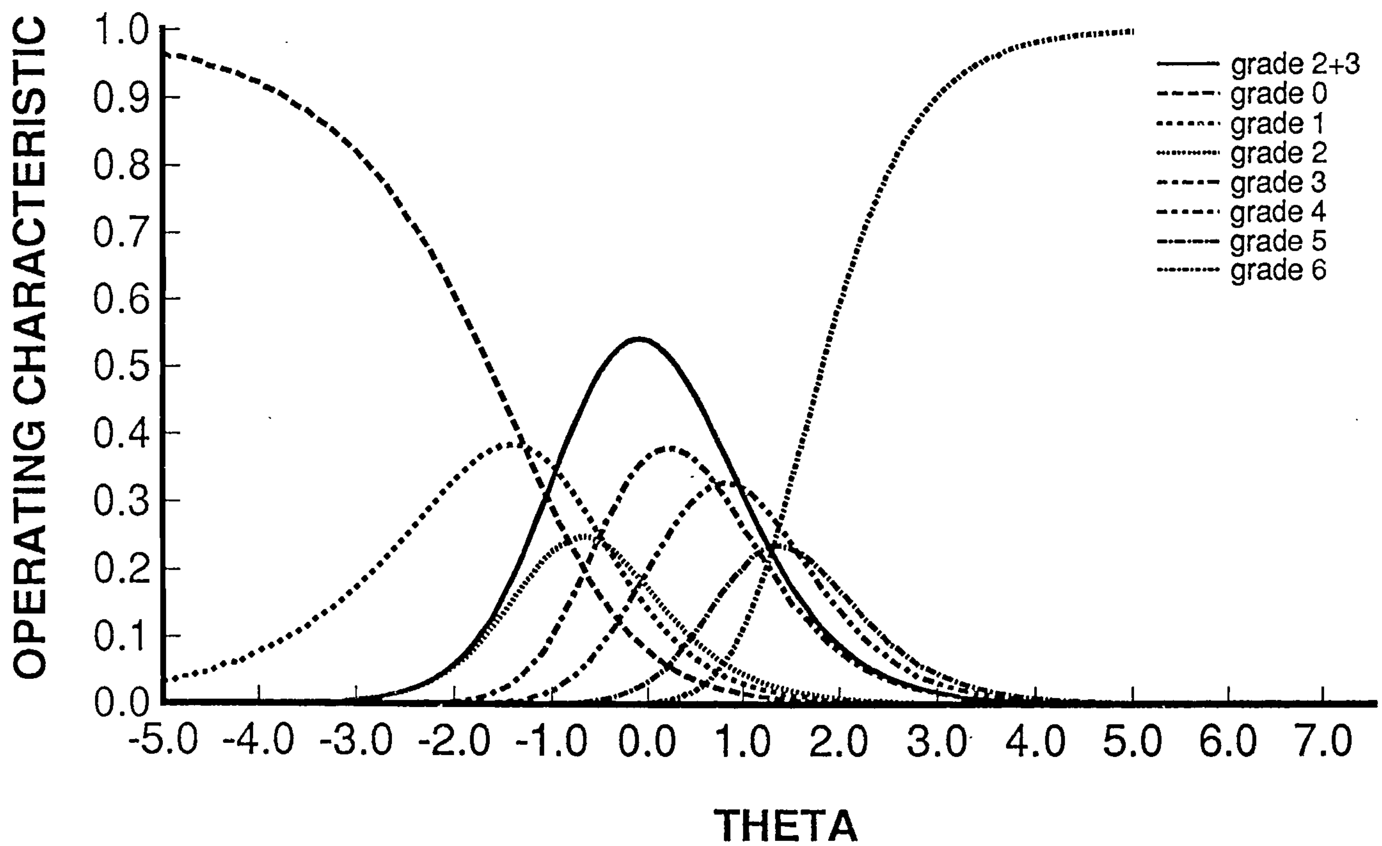

FIGURE 7

Operating characteristics of seven steps plus the sum of two operating characteristics for $x_g = 2$ and $x_g = 3$ (solid line) and the approximated operating characteristic for this combined category in the acceleration model (a dash and two dot. repeated).