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ABSTRACT

This study contributes to a growing body of research on the development of elementary teacher's content knowledge of mathematics. Individual clinical interviews were conducted with preservice elementary teachers (N=21) enrolled in a professional development course called "Foundations of Mathematics for Teachers." An instrument that allowed for the flexibility to probe and clarify participant understandings of elementary number theoretical concepts was employed. Questions were designed to clarify participants' understandings of procedures and concepts relating to divisibility and to investigate their abilities to make connections and inferences from them. A constructivist-oriented phenomenological analysis of reflective abstraction was adapted as a framework for interpreting data acquired in the interviews. Data analysis supports the general claim that teacher's content knowledge is "weak" and teacher's conceptual understanding is "insufficient" at times to teach arithmetic even in the elementary grades. Results provide a preliminary basis of a descriptive theory for the development of mental constructions involving elementary numbers concepts, their properties and relationships. An illustration of the model used to guide analysis and excerpts from the interviews are included. (Contains 16 references.) (LL)

DIVISIBILITY AND MULTIPLICATIVE STRUCTURE OF NATURAL NUMBERS: PRESERVICE TEACHERS' UNDERSTANDING *

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Abstract

This study contributes to a growing body of research on teacher's content knowledge of mathematics. The focus of this study is in the domain of elementary Number Theory. We have adapted a constructivist-oriented phenomenological analysis of reflective abstraction as a framework for analyzing and interpreting data acquired in clinical interviews with pre-service teachers. The findings of this study support the general claim that teacher's content knowledge is "weak" and teacher's conceptual understanding is "insufficient" at times to teach arithmetic even in the elementary grades. The results of this study provide a preliminary basis of a descriptive theory for the development of mental constructions involving elementary number concepts, their properties and relationships.

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Divisibility and Multiplicative Structure of Natural Numbers: Preservice Teachers' Understanding

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Introduction

This study is a contribution to the growing body of research on teachers' subject content knowledge in elementary mathematics and its development. The specific concepts involved in this study pertain to introductory Number Theory. The main emphasis in this report is given to concepts involving divisibility and the multiplicative structure of non-negative integers.

Previous discussion of multiplicative structures (e.g. Vergnaud, 1988; Graeber, et. al., 1989; Greer, 1992; Ball, 1990; Schwartz, 1988) focused mainly upon contextual situations. In this research we are interested in the development of cognitive structures of number concepts and operations *per se*. A previous study using elementary number theory concepts has been conducted by Martin and Harel (1989), however their focus was on the concept of proof and the concepts of Number Theory they used were but a means to that end.

We use the term "multiplicative structure" having in mind not the multiplicative structure of a problem situation, that is, a story(word) problem that can be solved applying either multiplication or division. In this study "multiplicative structure" is the structure of number independently of situation. Thus, here we use the term "multiplicative structure" having in mind decomposition of natural numbers as a product of prime factors, which is, by virtue of the fundamental theorem of arithmetic, part of the formal and conceptual foundation of mathematics.

Our objectives in this study are three-fold:

(1) to explore pre-service teacher's understanding of elementary concepts in the theory of numbers with emphasis given to concepts involving divisibility and the multiplicative structure of non-negative integers;

(2) to analyze and describe cognitive strategies of solving unfamiliar problems involving and combining those concepts;

(3) to adapt and extend a constructivist oriented theoretical framework for the analysis and interpretation of those strategies and the cognitive structures supporting them.

The results of this study provide a preliminary basis of a descriptive theory for the development of cognitive constructs involving elementary number theory concepts, their properties and relationships. The purpose and utility of such a theory is to eventually design and implement pedagogical methods that meet contemporary professional development requirements for conceptual understanding of mathematics.

Participants

Twenty one preservice elementary school teachers participated in the study. These people were volunteers from the group of students involved in a professional development course called "Foundations of Mathematics for Teachers". The concepts/topics of factorization, least common multiple, greatest common divisor, prime and composite numbers, prime decomposition and the fundamental theorem of arithmetic, divisibility and alternative 'divisibility rules' for the numbers 2, 3, 5, and 9 were part of their curriculum. The mathematical background and experience of the participants varied considerably, even though the interviews were conducted after the number theory related topics were 'covered' in their course.

Theoretical Framework

The particular interpretation of constructivism used in this study is based upon Dubinsky's action-process-object developmental framework (Dubinsky, 1991). Dubinsky developed this framework as an adaptation of some of Piaget's central ideas to the studies of *advanced* mathematical thinking. Previously, this framework has been used in studies of undergraduate mathematics topics such as calculus and abstract algebra (Ayeres, *et. al.*, 1988;

Breidenbach, *et. al.*, 1992; Dubinsky, *et. al.*, in press). Rather than limiting the application of this framework to domains of advanced mathematical understanding, we hope this study will contribute to determining the extent to which this theoretical perspective can be useful for investigating the development of mathematical understanding in general.

A central tenet of Piaget's theory is that an individual, dis-equilibrated by a perceived problem situation in a particular context, will attempt to re-equilibrate by assimilating the situation to existing schemas or, if necessary, reconstruct particular schemas enabling the individual to accommodate the situation. Dubinsky holds that the constructions which may involved are mainly of three kinds—actions, processes, objects. An *action* is any repeatable physical or mental manipulation that transforms objects in some way. When the total action can take place entirely in the mind of an individual, or just be imagined as taking place, without necessarily running through all of the specific steps, the action has been *interiorized* to become a *process*. New processes can also be constructed by *inverting* or *coordinating* existing processes. When it becomes possible for a process to be transformed by some action, then we say that it has been *encapsulated* to become an *object*. We express the construction of connections which relate disparate actions, processes or objects to a particular object as the *thematization* of the *schema* associated with that object. In this way we take each object to be a kernel of a schema. These notions will be illustrated below as we explore the extent to which this theoretical framework contributes to an understanding of number theoretical knowledge construction and development.

The cognitive events of interiorizing an activity into a process, encapsulating a process into an object or thematizing a schema are accounted for in this framework in terms of *equilibration*: For instance, some form of dis-equilibration within a particular context, such as a need to perform an action on an existing process, serves as a precondition or motivation to encapsulation - with encapsulation subsequently restoring equilibration. Equilibration then, as an operational postulate, provides an explanatory function in this otherwise descriptive framework.

Methodology

Individual clinical interviews with preservice elementary teachers were conducted using an instrument that allowed for the flexibility to probe and clarify participant understanding of elementary number theoretical concepts. The instrument was designed to reveal our participants' ability to address problems by recall or construction of connections within their existing content knowledge. Our objective in this study was not to determine statistical occurrences of particular understandings in any detail but rather to probe for and determine distinctive qualitative features of conceptual structure commonly exhibited in this domain. The questions covered a spectrum ranging from elementary number concepts (e.g. What does it mean to you that a number is an even number?) to more subtle and sophisticated problems requiring deeper conceptual insight into elementary number theoretical properties and relationships (e.g. What is the smallest positive integer divisible by every integer, 1 through 10?). The specific subset of questions analyzed in this study is described in the next section.

During the interviews, each of which lasted for about one hour, the participants were probed, when appropriate, for understanding that may not have been apparent from their initial response. A consequence of this methodological strategy was that not all, nor necessarily the same, questions were addressed by our participants. In circumstances where participants experienced difficulties with a particular question they were encouraged to reflect upon and articulate, or otherwise express, the nature of those difficulties. In cases where such activity proved inadequate in leading the participant to a realization of a solution or, alternatively, a resolution of their difficulties, the interviewer would progressively allude to, or provide, additional information. This method proved conducive to uncovering and identifying the possible source and depth of conceptual and/or procedural difficulties. In addition, this method incorporates an important diagnostic dimension, described by Simon as a 'continuum of connectability', demarcated by clear distinctions regarding the *connectability* of an individual's knowledge as exhibited within the context of a clinical interview. Simon characterizes connectibility as a continuum with the following four levels as different points within this continuum: (1) "The information was well connected. The subject had the relationship immediately available"; (2) "A relevant task and the demand for verbalization of thought processes stimulated the subject to make the connection during the interview"; (3) "The subject did not make the connection until a critical experience was initiated by the interviewer"; (4) "The subject's current knowledge was not sufficient to allow the particular connection to be made" (Simon, 1993). Although these distinctions are descriptively clear it is important to note that the interplay of diverse cognitive functions involved in making, inhibiting the making or even in the manifestation of such connections within the course of the interview often remains difficult or impossible to determine from the available data.

Asses. ment instrument

The questions were designed to clarify our participants' understandings of procedures and concepts relating to divisibility and to investigate their abilities to make connections and inferences from them. All questions presented specific examples of numbers in order to minimize any complexities added by algebraic abstraction. The interview question sets analyzed for this report and the rationale for their formulation are as follows:

<u>Questions Set 1</u>

Consider the number $M=3^3 x 5^2 x 7$.
Is M divisible by 7? Explain.
Is M divisible by 5, 2, 9, 63, 11, 15? Explain.

These questions were designed to investigate our participants understanding of the connection between the concepts of divisibility and prime decomposition. We were interested to determine whether our participants would take advantage of the fact that M is given as a product of its prime factors or if they would actually calculate the value of, and then divide, the number M in order to infer divisibility. In our choice of numbers as divisors we included prime factors of M, prime non-factors of M and composite factors of M to investigate the extent to which the nature of the divisor would educe differences in our participants' approach to inferring divisibility.

<u>Questions Set 2</u>

(a) Is 391 divisible by 23?
(b) Is 391 divisible by 46?
(c) What is the next number divisible by 23?
(d) How many positive number smaller than 391 are divisible by 23?

Parts (a) and (b) of this question were designed to investigate our participants' understanding of the connection between the operation of division and the concept of divisibility by providing cases where division is essential (part a) and where division is subsequently unnecessary (part b). The numbers were carefully chosen to eliminate any applications of the divisibility properties. Parts (c) and (d) were designed to investigate our participants' understanding of the modular distribution of numbers sharing the same divisibility property—that is to say, that starting from 1, every n-th number is divisible by n.

<u>Questions set 3</u>

Consider the numbers 12358 and 12368.
Is there a number between theses two numbers that is divisible by 7? by 12?

This question set conceptually expands upon part (c) from the question set 2. These questions served to assess our participants' ability to minimize or forego calculation and argue for the existence, or non-existence, of a number with a particular property. We were particularly interested in determining specific procedural and conceptual strategies our participants would use in addressing these questions and the connections they would make, if any, to the previous question set.

Questions set 4

(a) The number, 15, has exactly 4 divisors. Can you list them all. Can you think of several other numbers that have exactly 4 divisors?

(b) The number 45 has exactly 6 divisors. Can you list them all. Can you think of several other numbers that have exactly 6 divisors?

The participants were presented with either part (a) or part (b) of this question set depending upon their acumen with previous questions. This question set was designed to determine the extent to which the participants would systematically construct numbers with the desired properties.

Analysis

The action-process-object framework was used to guide the analysis of the interviews for the manners in which the participants appeared to think about the specific topics and problems presented to them. We illustrate an application of this model with an abbreviated phenomenological analysis or 'genetic decomposition' of the concept of divisibility:

Divisibility—an abbreviated illustration of genetic decomposition

A construction of the concept of divisibility as a conceptual object starts with specific examples of divisors. These divisors are usually small numbers such as 2, 3, 4 and 5. Initially, divisibility by 3, for example, is an *action*: a learner has to actively perform division and obtain a quotient of a whole number (no remainder) in order to conclude *a posteriori* that a number is indeed divisible, or not divisible, by 3. Later, the activity of division may be *interiorized* as a conceptual *process*, such that the action is intended but not actually performed. In this case, the student has conceptualized the notion that it is the division procedure itself that determines whether or not a whole number satisfies the 'rule' or criterion for divisibility. In this way, the action/process distinction can be used to distinguish between *procedural activity* and *procedural understanding*.

Processes of divisibility with particular numbers may be *coordinated* to create new processes of divisibility, that is, processes for new numbers. For example, coordination may be demonstrated when divisibility by 2 and 3 is used to infer divisibility by 6. Furthermore, some processes can be *inverted*: For instance, knowing that the sum of a whole number's digits is divisible by 3 implies the number itself is also divisible by 3 can be inverted and used to construct numbers divisible by 3. *Encapsulation* of divisibility as an *object* begins with an awareness of the concept of divisibility as an essential property of whole numbers independently of the procedural aspects of division. The concept of divisibility at this stage is conceived as a bivalent, yes or no, property of whole numbers. That is to say, where a and d are whole numbers, a is *a priori* either divisible by d or not divisible by d. More generally, when related to other cognitive structures such as those involving factorization and prime decomposition, divisibility comes to be *thematized* to form a higher order object or *schema*. Such a general schema could be in evidence as an object when used in actions or processes involving greatest common divisors or least common multiples.

Paths of analysis

The interviews were transcribed and categorized in terms of different questions, their difficulty, and identifiable cognitive patterns of various degrees of sophistication exhibited by the participants. Responses to the interview questions were coded in accordance to their contribution to the chosen paths of analysis. Several paths were predetermined in terms of the purposes for which the interview questions were originally designed. From the following paths we explored: (1) the general development of divisibility concepts in terms of the action-process-object theoretical framework, (2) relationships between divisibility and division, and (3) the role of verification and refutation with respect to divisibility. Several other paths were encountered during and after the interviews that emerged as recurring issues relevant to knowledge of divisibility and its construction. Of these paths we will consider: (4) the (ab)use of divisibility rules, and (5) additive vs. multiplicative structures. Other paths covered areas involving .

mathematical vocabulary, a variety of 'hit and miss' strategies and notions regarding mathematical certainty. As large segments from these paths were uncharted in this study, we will restrict our consideration of them to the more general discussion that follows. The matrix below illustrates the relation between the various question sets and paths of analysis covered in this section.

Path / Question Set \	(1)	(2)	(3)	(4)	(5)
Q 1	x		x	x	
Q 2		(a)	(b)	(a)	x
Q 3	x			x	
Q 4	x				

Results and Interpretations

In the following sections we provide results and interpretations of findings from paths (1) through (5) respectively. We begin with a spectrum of developmental levels of our participants' understanding of divisibility in terms of the action-process-object framework. We then proceed with aspects of divisibility related to division, multiplication, prime decomposition, divisibility rules and conclude with some observations regarding numerical structure.

(1) Development of divisibility concepts

Actions

The majority of participants in this study group were not able to discuss divisibility as a relation or property of numbers without performing division. This tendency is an indication that their construction of divisibility has not developed beyond action or process. This reliance upon procedural forms of understanding justifies and supports adherence to and dependence upon specific examples that, in turn, reinforce a strictly empirical attitude towards mathematics - as this excerpt from the interview (from Question set 1) with Nicole exemplifies:

Rina: Suppose I have an even number which is divisible by 7. Say I've now divided it by 7. Would I still end up with an even number?

Nicole: You'd have to try. You'd have to try to see if it works.

The claim "you'd have to try to see if it works" or "you cannot be sure that the result is a whole number if you don't know what the result is" seemed to be typical in this group of preservice teachers. Even participants who provided reasonable explanations in terms of multiples and divisors often made statements expressing their tendency "to work it out to make sure". This level of acumen requires the carrying out an action not only to obtain, but to assure one's confidence in, the answer. Thinking of divisibility as an action is further exemplified in the following excerpt (from Question set 3):

Rina: Do you think there is a number between 12358 and 12368 that is divisible by 7?

Nicole: I'll have to try them all, to divide them all, to make sure. Can I use my calculator?

Rina: Yes, you may, but in a minute. Before you do the divisions, what is your guess, what is your bet?

Nicole: I really don't know. If it were 3 or 9 I could sum up the digits. But for 7 we didn't have anything like that. So I will have to divide them all.

[Indeed, Nicole performed several divisions to find the number that gives a whole quotient when divided by 7 and only then answered the original question positively]

...

Nicole:	Yes, there is one. 12362 divided by 7 is 1766 exactly. No decimal part. So this is the number.
Rina:	Do you think there is another number in this interval that is divisible by 7?
Nicole:	I'll just keep checking, 'cause I can't see a pattern happening, I don't know an easier way that you do it to find -- in a glance.

For Nicole the approach to decide whether there is a number between 12358 and 12368 which is divisible by 7, is to divide by 7 all the numbers in the given interval. "keep checking" is the main strategy and maybe the only strategy Nicole is aware of, since she claims she doesn't "know an easier way that you do it". Another action is exemplified in the following excerpt:

Rina:	I'm asking you to look at the number which is $3^3 \times 5^2 \times 7$, do you think this number is divisible by 7?
Armin:	Do I work it out. or do I just do my first thought?
Rina:	Whatever....
(pause)	
Rina:	Would you please tell me what you are doing.
Armin:	Okay, first I'm just multiplying 27 x 25 x 7 and I get 4,725 and now I need to divide them all by 7.
Rina:	Okay.
Armin:	So get 625, so you have it divisible.
Rina:	So this number is divisible by 7. Could you know this without using the calculator and without finding out the product of all the numbers.
Armin:	Could I know it? Um, well, I know we discussed something in class about if A, if one number is divisible by 7, then another number is divisible, or what was it, A, this number is divided by 7 and this number, and this number is divided by 7, then the sum of those numbers should divide by 7. ...
Rina:	If I ask you whether this number was divisible by 5, what would you do?
Armin:	I'd do the same thing.

Armin, responding here in question set 1, preferred to calculate the number M and decide about its divisibility by 7 by performing division. Unlike Nicole, Armin seems to be aware that there are ways other than dividing to conclude divisibility. She recalls a theorem related to divisibility, which is not readily applicable in this case. When subsequently asked about divisibility by 5, and having the number calculated, Armin still claims - "I'd do the same thing"- which is, division.

Interiorization: from action to process

Betty, similarly to Armin, prefers to find out the value of M and then to divide by 7. But her response seems to be a bit more sophisticated than Armin's:

> Betty: Well, I would just multiply them all together, and then, and then after I have the number, then divide it by 7, unless I could just tell by looking at the number. . .

Betty's words "unless I could just tell by looking at the number" reflects an understanding that division may not be the only means to conclude divisibility.

Andy is in the "transition stage". She is making an attempt to interiorize divisibility .

Rina:	Would you please look at the number that is $3^3 \times 5^2 \times 7$. I would call this number M. ...
Andy:	Okay.
Rina:	Is M divisible by 7?
(pause)	

Andy:	Um, okay, I know that this is 27 and this is 25, ??, and you're asking divisible by 7?
Rina:	Um hm.
Andy:	Oh um, I'd say no.
Rina:	And why do you think so?
Andy:	Um, I guessed no because 25 isn't divisible by 7 and 27, oh no, maybe not, I wouldn't be able to guess, I'd have to multiply it out.
Rina:	Do you think there is another way?
(pause)	
Andy:	Or could I do, oh no, I could do 3x5 is 15 and then add those two, (pause) Could I do that . . ..
Rina:	So you have written $15^6 \times 7$.
Andy:	Um hm, and then I would say that it would be divisible by 7 because you're multiplying it by 7, because 7 is a, a factor.
Rina:	Factor. Please help me understand something. You looked at this expression: 27 x 25 x 7 and you couldn't draw your conclusion from here, and then you looked at this expression, which is 15^6 x7 and this helped you to draw your conclusion. Why to look at this [$15^6 \times 7$] was easier for you than to look at this [27 x 25 x 7]?
Andy:	Um, (pause) because this [15^6] is one number.

In the beginning of confronting Question set 1, Andy attempted to carry out the action, "to multiply it out" , in order to solve the problem. Following interviewer's suggestion to think of "another way", Andy 'chunked' $3^3 \times 5^2$ to 15^6 to conclude divisibility. Her computational error was ignored in an attempt to detect in what way the new expression could help her to draw a conclusion. It becomes evident from her answer that she fails to associate the product 27x25 as one numerical entity. On the other hand, when 7 has one multiplier, Andy claims it "would be divisible by 7 because you're multiplying it by 7". She was able to recognize 7 as a factor when it was one of the two factors, but couldn't see it as a factor in the list of more than two factors. Minutes later in the interview she claimed divisibility by 5 and denied divisibility by 2:

Andy:	Um, oh guess, yeah I could. You probably could say that 5 would be a factor, but that 2 wouldn't be a factor.
Rina:	And why ...?
Andy:	Because, because 5 is, 5 is a factor of this number.
Rina:	How do you know?
Andy:	Because you've multiplied it to get the answer, to get the sum or total, whatever.

We find Andy in the transition stage from an action to a process, and perhaps the interview stimulated this transition. Unfortunately, in the course of the interview, it was difficult to ascertain if Andy actually constructed new knowledge here or belatedly recalled it. Also unresolved is Andy's understanding of the relationship between factors and divisors. Be this as it may, she began with a reference to explicit action, but later the action becomes intended. Andy seems to be thinking of (possibly prime) factors of M as something you "multiply by" and this is a step towards procedural understanding. Lena (also in an excerpt from Question set 1) expresses a similar idea more explicitly, noting a seemingly trivial and yet profoundly important relationship between divisibility and multiplication:

Lena:	Yeah, well I was thinking that um (pause), I don't know what I was thinking I guess, well no, I was thinking that if the number was multiplied by 7 then is it divisible by 7, but I don't know if that really means anything.

Thinking of divisibility as a process is exemplified in the following excerpt from Question set 3 in the interview with Jane. It illustrates a procedural understanding, rather than just the procedural activity, of division.

> Rina: Do you think there is a number between 12358 and 12368 that is divisible by 7?
> Jane: Let's see. [performs long division] 12359 divided by 7 gives remainder 4. So., 60, 61, 62... 12362 will be divisible by 7.
> Rina: It's interesting. How did you know? I haven't seen you doing division.
> Jane: If this one [12359] gave remainder 4, the next one will give remainder 5, and the next one--6, and the next one 7, which means zero or no remainder. So if you divide 12362 by 7 there will be no remainder, it will be divisible.

Jane demonstrates procedural understanding of the fact that an increment of one in the dividend will result in an increment of 1 in the remainder, and that the latter is taken modulo the divisor 7. Having figured out that 12358 leaves remainder of 4 when divided by 7, she counts up three numbers to reach the number divisible by 7. Still, Jane did not claim the existence of such a number before she actually found one.

It is interesting to note Jane's and Andy's use of the future tense in the discussion of divisibility. Their statements "it would be divisible" or "if you divide...it will be divisible" may be interpreted as a transitional dependence on the procedural activity of division. Indeed, a definition of divisibility can be implicitly procedural: e.g., a is said to be divisible by b if and only if the quotient a/b is a whole number. But how can one be certain that the quotient is a whole number without knowing what this number is? A more conceptual definition that reveals the significance of Lena's seemingly trivial observation would be: a divides b if and only if there is an integer d such that $ad = b$. However, here as well, in many cases it would seem quite natural to calculate $d = a/b$ in order to determine if d is an integer or not.

New processes: inverting and coordinating

One of the tenets of Dubinsky's theoretical perspective is that new processes may be obtained from existing processes by coordinating existing processes or by *inverting* existing processes. The tasks presented to participants in the interviews made it possible to observe their constructions and their struggles with coordination as well as with inverting. The process of divisibility by 15 is a coordination of divisibility by 5 and by 3. It was found that tasks involving coordination were problematic for many of our participants. For example, 13 out of 21 participants were able to infer divisibility of the number M in the first question by 7, 3 and 5 on the basis of being factors in M's prime decomposition. But only 6 of these 17 were able to coordinate these processes of divisibility to infer divisibility by 15 or 63.

Another major difficulty appeared when participants were asked to do 'reversed tasks'. The ability to check whether or not an object has a certain property appears to be easier than to construct an object that has such a property. In a pilot study, carried out in the previous year with similar participants and in similar circumstances, sixteen participants who were able to check successfully whether a number was divisible by 15 using simple divisibility rules for 3 and 5, had significant difficulties providing examples of 6 digit numbers divisible by 15 that are quite readily constructable using those very same rules. Most examples provided were based on extension of known structures. These included numbers like 150,000, 300,000, 151,515 or 153,045 (concatenating 15-30-45). When asked to determine the largest 6-digit number divisible by 15, or to give an example of a number without repeating digits - most of the participants resorted to trial and error That is to say, they would guess a particular 6-digit number and then divide it to see if it was divisible by 15.

Another example where we have found evidence of "inverting the process" was via question set 4: When asked to give example of a number that had exactly 4 (or 6) divisors, all but 3 participants preferred to choose a number, and then "check it" by listing and counting its divisors. If the "guess" was successful, participants were dis-equilibrated with a request to generate 10 more examples. We had anticipated that somehow our participants would be prompted to interiorize or encapsulate any particular procedure that they may have been using. Armin had "guessed" that the number 6 had 4 divisors and after noting that "it's any two prime numbers" that give the desired solution, she could easily generate additional examples like 21, 35 and 55. Tara, after identifying the example of $45 = 3^2 \times 5$ as a number with 6 divisors, claimed that $3^2 \times 7 = 63$ and $3^2 \times 2 = 18$ had exactly 6 divisors. Unfortunately, it was not clear from the interview whether Tara would be able to generalize '3' to mean 'any prime' when asked to provide more examples. Melinda, for instance, was very close to Tara in recognizing a pattern of this kind. Upon considering 45 and 18 she concluded "it must be 3^2 x something", but her attempt to try and list the divisors of $3^2 \times 4$ put her theory in question.

Encapsulation: from process to object

Encapsulation of divisibility as an object starts when a learner begins to distinguish the concept of divisibility from the procedure of division. The following is an excerpt from the interview with Bob, who explains divisibility of M by 7 and 5 in terms of factors in the prime decomposition of M:

Sen: Bob, I'm going to ask you to write down a number please. And that number is $3^3 \times 5^2 \times 7$, and we're going to call this number M. Now, my first question is, is M divisible by 7?
Bob: Yes, it is.
Sen: And would you explain why?
Bob: Well if 7 (pause), let's see (laugh), M is, or let's see, so 7 is a factor of M, therefore, it's divisible by M, pardon me, by 7.
Sen: And how about 5?
Bob: 5 is also a factor of M.
Sen: Okay, and would M be divisible by 2?
Bob: No, it would not, since 2 is um, (pause) since 2 is not seen here, it's not a factor of M.
Sen: Hmm, okay, and why do you feel that that's the case?
Bob: Um, explain this clearly (pause), since 2 is not one of the numbers that's being multiplied, the product therefore, cannot be divided by 2.
[...]
Bob: Okay. And that since obviously 2 is a prime number, the prime number of 2 is not in this solution, therefore, whatever the product M turns out to be, 2 cannot divide into that.

This excerpt illustrates that Bob has made some important connections regarding relationships between factors of multiplication with divisors, and that he may have encapsulated divisibility as an object. Encapsulation may be indicated in determining divisibility by 7 and in refuting divisibility by 2. But even this is not always so. Apparently, the ability to infer divisibility on the basis of a connection with prime decomposition or multiplicative factors is not necessarily achieved simultaneously with the ability to infer non-divisibility. Patty for example noted that both 7 and 5 as factors, were divisors of M, but then regressed to procedure when asked about divisibility by 2 and 11.

Rina: Okay. And will it be divisible by 2?
Patty: I would multiply each one and find out what the total number is. So 3 x 3 is 9 x 3 is 27, and this 25 is x 7, (pause) it's not, 2 doesn't go into it evenly.
Rina: So you computed the number and you got 4,725, and now you are sure that it is not divisible by 2.
Patty: Right.

Rina:	But you were able to conclude about divisibility by 7 with, before you knew what was the number. . .
Patty:	Um hm.
Rina:	So how is it?
Patty:	Because 7 is a factor of it, so it's, what is it, the commutative law or associate law - 7 is a factor of it. . .
Rina:	And what about divisibility of M by 11?
Patty:	I would divide 4725 by 11 to find out.

Despite some confusing rationale as to what constitutes a factor, it appears that Patty may not have made the connection that all factors of M are divisors of M. In other words, she may have been thinking that 2 and 11 could possibly be divisors of M even if they are not actually factors of M.

Unlike Patty, Bob had discussed both divisibility and indivisibility in terms of M's divisors and non-divisors. It is indeed tempting to conclude from those responses that Bob had encapsulated divisibility. The next excerpt illustrates that this is not quite the case:

Sen:	Would you think that 81 would divide M?
Bob:	I'd want to find out what M would be. I guess that's the. the best thing, that's what I'd prefer.
Sen:	Um hm.
Bob:	I guess knowing what M would equal, and then from there working backwards, finding which numbers can go into that.
Sen:	Um hm.
Bob:	Um, right now I can't see whether or not 81 can go in there.
Sen:	Okay. Uh, how about 63?
Bob:	(pause) Once again, um, we have 7 now, 7 can go into 63, well ?? 3 can as well (pause), once again I'd have to solve for M, in order to find out whether 63 can divide M.
Sen:	Okay, so when you say solve for M, you mean like multiply it out and then divide by 63?
Bob:	Yeah, exactly, exactly.
Sen:	Okay. How about if you wish to divide M by 15?
Bob:	(pause) Um, well since there's 5, 5^2 in this problem, we know that the, that the units digit will be 5, now 15 obviously has a 5 in it as well, therefore quite possibly 15 will go into M, and once again I'd have to solve for that.

Bob provided mathematically literate arguments when discussed divisibility of M by 7 and 5 and non-divisibility of M by 2. When asked about 81, 63 and 15 Bob describes his strategy as "to solve for M", that is, to find out the value of the number and then to perform division. From the previous excerpt it seems fairly evident that Bob recognizes that a factor of M will also be a divisor of M. It is likely that his subsequent difficulties reflect deficiencies in fully understanding what constitutes a factor. This in would explain his compromised ability to fully connect prime decomposition with divisibility. Has Bob not fully encapsulated divisibility or has he yet to fully realize the relations between divisibility, prime decomposition and multiplicative structure? It is possible that his construction of divisibility as an object is not solid enough to accommodate composite numbers. This is a difficult call. It is possible that Bob has encapsulated divisibility as an object in that it is fairly evident that he recognizes *a priori* that any given number will either divide M or not. With the possibility that Bob has yet to make certain connections with other objects, such as factor, may reflect that he has yet to fully thematize divisibility as a schema. Be this as it may, Patty and Bob's interviews provide evidence that encapsulating divisibility as an object involves coordination of many specific examples of divisibility by specific numbers and thematizing divisibility as a schema involves constructing many specific relations with other objects.

While Bob encapsulated divisibility by small prime factors, Dana shared Bob's initial difficulties with composites, but was able to overcome them. After justifying divisibility by 7 and 5, Dana was asked about divisibility by 63. She explained why for her the case of 63 was different:

Sen:	Okay, and 63?
Dana:	(Pause) Oh, I don't know, I'm confused. (Laugh) 3 x 7 x 3, but, I guess so.
Sen:	Okay. And when you say you guess so...
Dana:	Yes, ugh, well just because, yeah it would, okay, it does (laugh), just because the numbers aren't actual prime numbers, like 63, there's something you think twice about.

Thematization: from object to schema

Apparently, some of our interviewees, like Bob, preferred to resort to procedural computations when confronted with a more complex example, rather than "think twice" about it. The following excerpts illustrate that Anita and Karen were able to more smoothly coordinate divisibility by primes by drawing conclusions with composite cases without the need to "think twice":

Sen:	Okay. And 9?
Anita:	Yes, because um, because you have 3 to the power of 3 and 9 is 3^2, you can make 9 from 3^2.
Sen:	Okay. And how about 63?
Anita:	Yes, because 63 is 9 x 7, and you have the 7 and you can make 9 by 3^2.

Sen:	Okay. How about 63?
Karen:	Okay, well I'll, I'll actually try it then, because I can't remember if 63 is divisible by anything. (pause) 9, 9 x 7 , yeah, okay, so 63 is divisible by 9 and 7, so yeah, I think it's divisible by 63.
Sen:	Okay.
Karen:	Because we could get 3 x 3 which is 9, and 7 from M, M's factors.

Anita and Karen demonstrate their awareness of M's divisibility not only by its prime factors, but also by composite products of its prime factors. Making this connection between factors and prime decomposition with divisibility, if not an essential prerequisite, seems quite helpful to the encapsulation of divisibility as an object and contributes to thematizing divisibility as a schema.

Our concluding example in this section indicates in one sense how deeply thematization of a schema can reach and in another how deeply entrenched the obstacles to encapsulation can be. The following excerpt shows that Pam perceived divisibility by 7 not only as an *a priori* property but also could explain how often this property is found within a contiguous set of whole numbers.

Rina:	Do you think there is a number between 12358 and 12368 that is divisible by 7?
Pam:	I think there is.
Rina:	Do you know which number it is?
Pam:	Not yet, but I can find it if you want me to.
Rina:	No, you don't have to find it. But if you don't know what it is, how do you know it is there?
Pam:	Here we have 9 numbers. And I know that if I take any 7 numbers there will be one divisible by 7. And here I have 9, which is more than 7.
Rina:	Are you saying that if I pick any 7 numbers I wish there will be one divisible by 7?
Pam:	I didn't mean that, what I mean is if you take these numbers one after another there will be one of them divisible by 7.

The understanding of this modular distribution of numbers that share a certain divisibility property is an indication of the depth of Pam's schema for divisibility in that she has made a strong connection between it and multiplication. Pam's idea that "every seventh number is divisible by 7" was part of the repertoire of only seven participants this group of twenty one preservice teachers. For example, Nicole, after finding the number divisible by 7 in the given interval, was asked whether she could have predicted the existence of such a number without calculating it. Her answer was negative, followed with the explanation: "the further you go, the more they grow apart ... (and) by the time you get up into numbers that are this high, the difference between the two numbers is only 10, there would be a larger difference between the two numbers (divisible by 7)". This description was accompanied with a hand-waving that indicated progressively increasing intervals. It is possible that Nicole was confusing multiplication with exponentiation here. Nevertheless, this and other instances led us to suspect that, for many of our participants, the process of repeated addition had not been properly encapsulated in terms of multiplication as an object. This deficiency in relating multiplication to divisibility appears partially responsible for her reliance upon procedure and thus would serve as an obstacle to encapsulation. We return to this issue below in the section on "additive vs. multiplicative structures".

(2) Divisibility and division

Is the number 41,418 divisible by 177? Unless one's answer comes from divine inspiration or one has memorized the multiplication table for 177, there is but one obvious way to find out - divide. It was mentioned above that encapsulation of divisibility as an object must begin by discerning between divisibility as an outcome and division as a procedure. But even with a clear understanding of divisibility, there is no evident alternative strategy for answering the above question without performing division. Indeed, a majority of our participants applied this strategy when addressing the first part of question set 2: "Is 391 divisible by 23?". These numbers were carefully chosen to make it difficult to 'guess' or to determine an answer using divisibility rules, yet easy enough to perform division, even without the help of a calculator. Surprisingly, 6 out of the 21 participants applied strategies other than carrying out division. On one hand their choices may be explained as a search for sophistication by avoiding the obvious. On the other, it may be the case that relationships between division and divisibility have yet to be constructed.

Lena and Joan concluded divisibility by performing multiplication. Here, Lena looked for a number that gives 391 when multiplied by 23.

Sen: I'm going to ask you, is 391 divisible by 23?
Lena: Hmm, (pause) I'm not sure if it's divisible evenly or not.
Sen: Um hm. How would you go about answering that question for yourself?
Lena: Okay, this is what I would do. 23, um, if I was given this question, I would honestly just, you know, plug in a few numbers and multiply them by 23 to see how close I get to 391. So, I'm going to try that, ...
Sen: It was bingo!
Lena: Right, okay, well the only reason I chose 17 is because I know that 7 x 3 is 21, and I know that it's not 7, because 7 x 23 is too small of a number, so I put a 1 in front of it. . .

Lena's 'lucky' guess was creatively based on estimation and considering the number patterns for the last digit. Joan used a rough estimation to establish a starting point, and then converged towards the product via a tedious set of incremental multiplications.

Sen: Is 391 divisible by 23?
Joan: (pause) Do you want me to figure it out, or just. . .
Sen: Sure, go ahead, take your time.

Joan:	(pause) Okay, yeah, it's 17 times.
Sen:	Okay. Okay, could you explain to me how you've approached this problem, Joan?
Joan:	Um, well I just took as many, I kept sort of going up, well from 23 x 10 it would be 230, so I went up from there, I multiplied by 13, it might be a bit more than that, and then ultimately I kept going up and up until I multiplied it by 17 and found that 23 x 17 is 391, so the 391 divided by 23 would equal 17.

Joan's approach consisted of multiplying (with paper and pencil) 23 by 13, 14, 15, 16, and 17. To the question about her choice of strategy, Joan indicated:

Joan:	I always feel that it's easier just to keep multiplying, because I have to multiply anyways to figure out what this is and what this is.
...	
Joan:	Um, I suppose I could have (done) long division, but it, um, whenever I get one like that, I always multiply to divide, because it's just, I find it easier ...

Joan demonstrates clear understanding of the relationship between multiplication and division, and a clear preference towards the former. To emphasize, she preferred to perform five long multiplications instead of one long division. It may be the case that both Lena and Joan carry with them from elementary school some discomfort with 'long division'. But during the rest of the interview we observed Joan and Lena performing 'long division' several times, with ease and without seeking for the help of calculator that was placed in front of them on the desk. Therefore we suggest that, for these participants, that the concept of divisibility was related to multiplication by the definition: b is divisible by a, or a divides b, if there exists a natural number d such that $ad=b$. Joan's activity demonstrated a search for such a d.

Karen, Anabelle and Tara claimed that 391 was not divisible by 23 since it was prime. The 'primeness' of 391 was inferred in different ways. Karen claimed that 391 was a prime number since the sum of its digits was a prime number

Sen:	Let's take the number 391. Is 391 divisible by 23?
Karen:	Ugh, (pause) um, I don't think so, because when, when I add 3 + 9 is 12 and 1 is 13, and 13 is not really divisible, like the sum of the digits in 391 aren't really divisible. 13 is not really divisible by anything, it's sort of a prime, like the prime number. Um, so basically I don't think 391 is divisible by anything, because, because the sum of the digits is 13, I don't think it's divisible by anything, except for um 1 and itself.

Anabelle and Tara reached the conclusion about the primeness of 391 after they were not able to find a small prime number by which it was divisible.

Sen:	Let's take the number 391. Would 391 be divisible by 23?
Anabelle:	23, (pause). I don't know. I don't think so.
Sen:	Hmm, and why do you think not?
Anabelle:	391, I think, is a prime.
Sen:	And why do you think that 391 is a prime?
Anabelle:	Because I don't think it had been divided by 2 or 3, or 5, or 7 (laugh)...

In the next step Anabelle divided 391 by 11, and the fractional result on the calculator confirmed her conclusion about the primeness of 391. Tara didn't stop at 11, but proceeded a little further:

Sen:	Okay, alright, um, so if you were to determine for yourself one way or another whether or not 391 was divisible by 23, how would you go about it?
Tara:	Um, I would find the prime factorization of 391. . .
...	
Tara:	I have a feeling this is probably a prime number. . .
Sen:	And can you tell me how you're going about this?
Tara:	Oh, yeah, um, I'm trying to find a prime that divides 391 evenly, and I tried 11 for some reason, but I thought maybe it would work, it didn't, I tried 13, it didn't work, so, so I have to come to the conclusion that maybe it's a prime number, uh, (pause) . .

After about 10 minutes of prompting in search of divisors of 391, attempts to establish its primeness, and facing the evident with the help of calculator, Tara was asked why she didn't simply divide 391 by 23 from the beginning. Her answer was:

Tara:	I don't know. I guess, like I, um, like I was saying with, I know there's a way to do it, prime factorization, and I know that 23 is a prime number, but I guess, um, I was assuming, for some reason, that as long as 391 was not a prime number, it would have a factor smaller than 23, a prime factor smaller than 23.

It our research we found additional evidence supporting that some students believe that 'prime decomposition' means 'decomposition into *small* primes' and this belief co-exists with their awareness of existence of 'very big' primes. A detailed discussion of this issue can be found in Zazkis and Campbell (1994, PME-NA, in preparation).

Stanley, similarly to Karen, tried to generalize a rule for divisibility by considering the sum of the digits. In his opinion 391 was not divisible by 23 since the sum of the digits was not divisible by 23.

Sen:	Okay, is 391 divisible by 23?
Stanley:	By 23? Um, let me think. (pause) I don't think so. . . .
Sen:	Okay, can you tell me uh a bit about what you've done here in terms of how you've thought about it, and how that corresponds to what you've written?
Stanley:	Okay. Um, basically I'm just going on assumption, that I just learned in my last class there, in that uh if I add the digits of 391, it'll go 13, and if 13 is divisible by 23, then the number itself should be divisible by 23, but uh seeing as it's not, then the number isn't divisible by 23. . .
Sen:	Okay, and before you used that uh, or learned that rule, um, how would you have gone about answering the question?
Stanley:	Uh, I probably would have sat and counted up 23 enough times until it gets close enough to 391, it'll either be under it or over it, or dead on, um, and depending what the result was, I would decide...

The "rule" that Stanley "learned in his last class" was most likely a misgeneralization of divisibility by 3. But even when specifically asked not to apply this rule, Stanley does not opt to use division, nor does he choose multiplication. He would have "sat and counted up" by 23. Apparently, Stanley would be more at ease with addition in determining divisibility than with either division or multiplication.

As a matter of fact, 4 out of these 6 students gave reasonable and mostly correct answers on question 1, arguing divisibility of the number M in terms of its divisors and non-divisors. It may be the case that they tried to avoid division in a search for a more powerful strategies accompanied by a belief that such strategies do exist for most, if not for all, cases. But it also may be the case that some of our participants had yet to progress to a stage in which they had properly constructed relationships between division and divisibility.

(3) Verification and refutation with respect to divisibility

In the first question of the interview participants were asked to consider the number $M=3^3 x 5^2 x 7$ and decide whether it was divisible by each of the numbers 7, 5, 3, 2, 15, 11, 9 and 63. Asking participants to consider the number given in its prime decomposition, we hoped to divert their attention from procedures when determining divisibility and motivate a focus towards the multiplicative structure of M. Even so, there were several individuals who calculated the value of M and then performed division by 7. As we have seen, there were some individuals who easily concluded divisibility by 7, 5, and 3, since "those were among the factors", but had to calculate the value of M and divide in order to check for divisibility by 15 or by 63. Determining the decomposition of 63 as $3^2 x 7$ and then comparing the prime exponents of 63 to the prime exponents of M and noting they were factors was a strategy used by only five of our participants. There was no indication by any of the participants in this study of explicit factoring such as $M=3^2 x 7(3x5^2)$.

Another point of interest was that the 'proof' or verification of divisibility was in most cases more readily achieved than the refutation. For some, it may seem obvious to claim that M is not divisible by either 2 or 11 since these are not factors in the prime decomposition of M. However, more than half of our participants, like Patty in the example in section 1 above, inferred divisibility by 7 or 5 by considering the prime factors of M, but could not infer indivisibility by 2 or 11 using the same approach. For some participants, the question about 2 seemed easier than the question about 11, when they noted that "M is an odd number (as a product of odd numbers), so "2 can't go into it". For them, the mystery of divisibility by 11 remained unsolved unless the actual division was performed. We suggest that these students may not 'believe', or at least not 'believe in practice', in the fundamental theorem of arithmetic that assures the uniqueness of prime decomposition. Most students could quote the fundamental theorem of arithmetic and exemplify how the factors in the prime decomposition of a given number will be the same (but for the change of order) and will not depend on which factor was found first. But when asked about divisibility of M by 11, it was apparent that many of our participants did not over-rule the possibility of a "different decomposition". This issue is discussed in detail in (Zazkis and Campbell, PME-NA 1994, in preparation). Another possible explanation for this phenomenon may be that divisibility has been encapsulated to an object whereas 'indivisibility' has not.

In the second part of Question set 2 - 'Is 391 divisible by 46?' - there was no unanimous conclusion among the 17 participants who were asked this question. It is important to note that this part of the second question was presented only to participants who 'succeeded' with part one of the question, that is, who concluded divisibility of 391 by 23, by whatever means. Four participants immediately used a calculator and based their conclusion on the calculator's result. After concluding that 391 was not divisible by 46 with the help of her calculator, Armin was invited to think of another strategy:

Rina: My question is: if you didn't have your calculator with you, how would you think about this? 391, is it divisible by 46? What would you do?

Armin: (Pause) Um, I guess I'd just have to guess out of the blue. I would say, no, but, I mean, I would never trust my own opinion, I always have to work it out just to see (laugh).

It seems that a certainty achieved by carrying out the action is so important for Armin that she wouldn't dare to try another route of thought. For her an alternative for action is not more advanced mathematical construction, but just "guess(ing) out of the blue".

Attempts to use more advanced mathematical reasoning do not necessarily lead to correct conclusions. On the same question, 5 out of the 17 participants claimed that 391 was indeed divisible by 46, since "46 is just 23 doubled". This was a pitfall on the way to process construction. Anita started to fall into this pit, but eventually pulled herself out with a specific numerical example:

Sen:	Okay, right. Um, would 391 be divisible by 46?
Anita:	Yes.
Sen:	And why so?
Anita:	Oh, maybe not.
Sen:	I'm, I'm interested in both of those things that just happened to you. I'm interested in the 'yes' and the 'maybe not'.
Anita:	Well, first I said yes because I thought 46 is, well 23 is a factor, is a factor of 46, it's 23 x 2, um, but then again, I thought the 5 is a factor, like, for example, 5 is a factor of 25 but 10 isn't, and so just because it's doubled doesn't mean it's a factor of, so I'm not too sure ??. I think I'd have to say no.

Bob, Patty and Anabelle demonstrated in their answers various levels or degrees of sophistication in the interiorization of action to process:

Sen:	Okay. How about 46, would 391 be divisible by 46?
Bob:	(pause) No, it wouldn't because uh in 46 the unit digit is 6, and the units digit of 391 is 1, and 6, knowing the multiples of 6, I know that there will not be a units digit of 1 after being multiplied by 6. For example, 6 x 6 is 36, units digit and that is obviously 6.

Bob's conclusion is based on considering the last digit and multiples of 6. Aware of the multiples of 6, his conclusion that no multiple of 6 will end with 1 is based on intended, rather than explicit action. Here Bob had constructed a novel procedural understanding of indivisibility of 391 by 46. Patty makes a further step when she, similarly to Bob, considers the last digit, but also considers the evenness and oddness of the numbers in question.

Patty:	Because 46 ends with an even number and 391 is an odd number. .
	Rina: Um hm.
	Patty: And 6 is even, it won't fit into an odd number.

That Anabelle's explanation is yet further refined is evidenced in her consideration of 46 itself as an even number, and not just considering its even last digit.

Sen:	Okay. Alright. Now I'd like to ask you if 391 would be divisible by 46?
Anabelle:	No, because it (46) was an even number, and this one (391) is an odd number.

Here we have evidence that interiorization of division as a process may involve an accretion of activities involving division. At an entirely different level, Dana illustrates the power of thematization of divisibility as a schema by applying the connection with prime decomposition to refute divisibility of 391 by 46:

Sen:	Okay, um, would you say 391 is divisible by 46?
Dana:	(Pause) No, because 23 and 17 are both prime numbers, there is no 2 involved in there, it's just 23 times 17.

Apparently, concepts of divisibility and non-divisibility are not developed simultaneously. If this happens, it is usually the case that divisibility is more advanced than non-divisibility and that some degree of thematization may be involved.

(4) (Ab)use of Divisibility Rules

Awareness of and application of divisibility rules is a step towards encapsulation of divisibility as a conceptual object. Divisibility rules give the learner legitimation to conclude divisibility without performing division. Such 'permission' may help the learner to separate between performing division and considering divisibility as an intrinsic property of a number. On the other hand there is a danger that divisibility as a property of a number may be procedurally reduced to seeking patterns of digits. For example, when Andy was asked "Do you think there is a number between 12358 and 12368 that is divisible by 7?", she replied that 12358 was probably such a number since the sum of its digits is 14, which is divisible by 7. A more common (repeated three times in this group of 21 students) answer was: "I would guess that 12363 is divisible by 7 since 63 is divisible by 7". These responses do well to summarize the misapplication of the divisibility rules learned or reviewed by this group of students. As mentioned above, the participants were familiar with the divisibility rules for 3 and 9 (the sum of the digits rule) and with the divisibility rules for 2,4,5 and 10 (the last digit(s) rule). Eight out of 21 participants managed to incorrectly generalize at least one of the rules in response to at least one of the questions. In section (2) we observed Karen's conclusion that 391 was prime since the sum of its digits, *viz*. 13, was prime, and Stanley's conclusion that 391 was not divisible by 23 since the sum of its digits was not divisible by 23. Here is another example:

> Jennifer: [Question set 1] I went 225 x 7 and got 1.575 and figured 1.575 isn't divisible by 7.
> Sen: Okay, and the, the basis of your conclusion there?
> Jennifer: Because I looked at the last digit.....
> [...]
> Sen: Okay. And 9?
> Jennifer: Um, 9 as well is not divisible, or 1,575 isn't divisible by 9 because I'm looking at the last two numbers, 75 and know that 75 isn't divisible by 9.

Jennifer denied divisibility by 7 and by 9 by considering the last digit and the last two digits of number 1.575. When Anabelle is asked the third question - is there a number between 12358 and 12368 divisible by 7, she claims: " I don't know how to figure out how a number is divisible by 7"; And this, just after she had tested divisibility by 23 by long division. Anabelle's answer may indicate more than an unawareness of a divisibility rule for 7 - she may be assuming the existence of such rules for most, if not for all, numbers.

Over generalization of divisibility rules have taken place for our participants, when explicit rules were not available. According to Matz (1982) these errors may be explained as students' reasonable, all be they unsuccessful, attempts to adapt previously acquired knowledge to a new situation. We also note, from these pseudo divisibility rules, not only students' propensities to grasp for procedures in the absence of conceptual understanding, but also a sense of dis-equilibration in the absence of a rule to follow and a subsequent sense of re-equilibration from the creation of such pseudo rules.

(5) Additive vs. Multiplicative Structures

In question set 2, all the students who established divisibility of 391 by 23 by whatever means had no problem in identifying the next number divisible by 23. Some (3 out of 21) used "multiplicative" reasoning by claiming that the next number on the number line divisible by 23 should be 18x23. The majority used "additive" reasoning and found the number in question by adding 23 to 391. Even though both strategies are equally efficient, students' choices may indicate their preference towards and their ease with additive structures. The question of "How many positive numbers divisible by 23 and smaller than 391 are there?" was more challenging for our interviewees. Even though eventually most of them arrived at the correct answer, there was an evident struggle in making a connection between multiplication and addition. Using variations upon counting up or down by 23's, as typified by Anita, Bob and Jennifer below, was a more popular strategy than taking advantage of the multiplicative structure of 391 as 23x17, as we will see later in Armin's interview:

Sen: How many positive numbers smaller than 391 would be divisible by 23?
Anita: Um, (pause) 16.
Sen: And why so?
Anita: Because there is, because 23 x 17 is 391, and I think it's 16 because, because basically what that means is you're going 23 + 23 + 23 + 23 + 23, 17 times, and then less than 391, they still have to be divisible by 23, so it would be 23 x 16, so you have 23 + 23 + 23 + 23, 16 times, and if you stop after each of those times, you'll have either 16 or 15, I'm not sure of the exact number, different numbers.

Sen: ... how many numbers smaller than 391 would be divisible by 23?
Bob: (pause) I'd say 16.
Sen: Okay, and can you tell me how you see that?
Bob: (pause) Well I know that 23 x 17 is 391, therefore, whenever you subtract 23 from 391, no, when you subtract 23 from 391 you're gong to get uh 16 groups of 23, you subtract another 23 from that number, and it will be going, 23 will go into that 15 and so on...

Sen: Okay, and how many positive numbers smaller than 391 would be divisible by 23?
Jennifer: Um, well I would just take 391 and divide it by 23, (pause) um, I don't know if that would work. . . (pause) I would probably do this the long way and I would just keep subtracting 23, 23, 23, and making myself a whole list and then just counting them.
Sen: Um hm, okay. Um, any, if you had to guess at how many there would be on your list?
Jennifer: Well I would say I know there is more than 10, (pause) am I getting this straight?
Sen: I think so.
Jennifer: Okay. Um, I know there's more than 10, because there's, 10 would be 230, another 5 of those would be 115, (pause) well I know for sure there's 15 of them, and then I would just have to work that out, so there's 17 of them. Because I knew there were, I knew there were more than 10 because that would make it 230, I knew there was another 5 because that would be 115, and I got 345 out of that, and I subtracted 391, or 345 from 391 and got 46, so I know that that's another 220, two groups of 23, so I figured out that there's 17.

We note that while for Bob and Anita the connection between additive and multiplicative structures was made fairly smoothly, Jennifer avoided this connection and actually found a way to count all the multiples of 23 without listing them all. While Jennifer's image of numbers divisible by 23 is numbers apart from one another by 23 units, Armin renders 'numbers divisible by 23' to 'multiples of 23'. Such a translation is evident in her list of numbers.

Sen: Okay. Now, um, I'm going to ask uh this question here, how many positive numbers smaller than 391 would be divisible by 23?
Armin: 16.
Rina: 16, and what makes you think of that?
Armin: Okay, well if 23 x 17 is um 391, and there's also 23 x 16, 23 x 15, 23 x 14, all the way down to times 1.

It may be the case that this 'translation' done by Armin and some others was not apparent for all the participants. It may also be the case that additive structures are much stronger than multiplicative structures in the mathematical understanding of our participants, so whenever additive structure leads to a solution it seems to be the more desired choice. It may also be the case that the bridge to connect between additive and multiplicative structures has yet to be constructed. Such a bridge is mathematically formalized in the distributive law. Subsequently, in confirmation of this conjecture, additional interviews were conducted with this same study group in order to investigate their conceptualization of distributivity in greater detail, the results of which are reported elsewhere (Campbell & Zazki PME-NA 1994, in prep).

In the beginning of this section we discussed our participants' preference for additive structures over multiplicative structures, when both were appropriate for solving the given problem. In what follows we show two examples of additive thinking that lead to mathematically incorrect conclusions. Karen, in order to find the prime decomposition of 391 considers the prime decomposition of 91. Her conclusion that 91 is prime confirms her previous idea that 391 may be prime:

> Karen: Well, I, I think what I'd do is, I'd take 391 and try to think about um breaking it down into the prime factorization, if I could.
>
> Sen: Um hm.
>
> Karen: But see, um, I sort of realized that if this were 390 it would automatically be divisible by 10 and 39. I think, . . . 39. But because it's 391, um, and also like if you look at the, even the 91 here, um, we know that 91, we don't have anything in the times tables that actually equals 91 too, we know that that's prime.

A point to note from this excerpt, aside from our main discussion here, is the claim that "we don't have anything in the times tables that actually equals 91". Indeed, in times tables for one digit numbers memorized in grade 3, nothing equals 91. Nevertheless, it is a fact that 91= 13x7. Is it just a computational omission or is there a fundamental dependence on "times tables"?

Tara, in the excerpt below, repeats Karen's assumption that "391 would be prime if 91 was prime", and later tests her conjecture that the prime decomposition of 391 can be found as a sum, whatever the 'sum' could mean here, of prime decomposition of 300, 80 and 11.

> Tara: Yeah, I'm trying to prime. find the prime factorization of 391, so that I can determine whether or not 23 divides into 391. Umm, (pause) hmm, and 391 is not prime? Can you tell me that?
>
> Sen: No comment.
>
> Tara: (Laugh) I have to determine that, right? Well, I guess, I mean 391 would be prime if 91 was prime. . .
>
> Sen: And how so?
>
> Tara: Hmm, I don't know. I'm not sure of that statement either, but, hmm, for some reason like it seems to make sense to me and I don't know why (laugh). Hmm, (pause) actually, I have a question for you and I know you're not going to answer. I was going to say, now if I write it in this form and I prime factorize this, wouldn't that be the same thing as to find the prime factorization of this number?
>
> Sen: Oh, I see, um, you've written 391 as 300 plus 80 plus 11, and you'd like to know whether or not it would be equivalent to do the prime factorization of the three numbers in that sum as an equivalent procedure to finding the prime factorization of 391. . .
>
> Tara: That's right.
>
> Sen: Okay. Um, well, I would suggest that you experiment with that,
>
> [Tara experiments]
>
> Tara: Well, that wouldn't be the same. I couldn't do the prime factorization of 300 plus 80 plus 11 if I wanted to get the prime factorization of 391. (Pause) Hmm, well, uh. . .

Tara's conjecture, even though rejected by herself after 'experimenting', demonstrates, again, a predilection towards additive structures. Her conjecture may be an additional example of what is described by Matz (1982) as a 'misapplication of linearity' or an 'over generalization of distributivity'. The erroneous claim that sin(a+b) = sin(a) + sin(b) is one of the "classical" examples of such over generalization. In Tara's case, a similar over generalization led to the conjecture that: PRIME_DECOMPOSITION(a+b) = PRIME_DECOMPOSITION(a) + PRIME_DECOMPOSITION(b).

Discussion

Basic concepts, vocabulary and definitions

Elementary number theory related concepts introduce learners to a new mathematical vocabulary—a vocabulary of words that may sound familiar but carry with them new meanings and interpretations. It was found that a significant percentage of our participants experienced difficulties grasping conceptual aspects of mathematical definitions, seemingly invoking various forms of linguistic inference that provided a rationale for preserving meaning in terms of more familiar procedural activities and understandings. A frequent claim was, for instance, that 3 is a multiple of 18, since "you multiply 3 by 6 to get 18". In this case the meaning of the word 'multiple' was conflated with the concept of factor, apparently due to a linguistic association with the activity of multiplication: "A multiple is something you multiply with". Another common claim was that 5 is divisible by 2, since "you can divide any number by whatever you want" or "5 is divisible by 2, but the result isn't a whole number". In this case the concept of divisibility was conflated with the process of division.

Increasingly evident from the analysis and interpretation of the data was the central role of defining divisibility in terms of both multiplication and division. The conceptual subtleties of these two formally equivalent definitions can hardly be appreciated without a clear understanding of the basic vocabulary involved. For this study group it seems evident that the two definitions were a source of conceptual conflicts and confusion. If so, this would be quite ironic in the sense that these differences are, in actuality, conceptually complementary and could be used pedagogically to highlight the important inverse relationships between multiplication and division upon which multiplicative structure is based. In order to address the common gaps in understanding, it is evident that some remedial attention to basic *concepts* such as multiplication, division, distributivity, exponentiation, factorization, etc., may be warranted. These gaps in understanding appear to serve as obstacles to encapsulation and thematization of higher order objects and schemas.

Mathematical Certainty

Some of the interview questions presented disequilibration between the desire to apply recently acquired mathematical knowledge on one hand and the desire to feel certain with the answer on the other hand. Calculating the number M in question set 1 or dividing by 7 all the numbers in the given interval in question set 3 left the participants certain in their conclusion, although many of them made statements indicating their dissatisfaction with the chosen approach, like "there should be a better way", "I did it the long way, I couldn't think of any shortcut". But can you be certain in your claim of existence of number divisible by 7 between 12358 and 12368 without actually finding one? Can you be certain that the result of division would turn to be a whole number without knowing what this number would be? Achieving such certainty is a step towards mathematical maturity as well as a step towards encapsulation of divisibility as a mathematical object. Many of our participants who demonstrated process or even object constructions of divisibility, claimed they would have preferred to carry out the action in order to feel sure in their conclusion.

However, without some basis in conceptualization, students tend towards an uncertainty that engenders a paranoia that can only be quelled through brute calculation. At best, such procedural dependencies without some form of conceptual guidance are immensely frustrating and time consuming and at worse can result in pathetic 'hit and miss' strategies and total disenchantment with the subject. So long as procedures lead to some form of conceptual understanding and in many cases procedural emancipation, 'hit and miss' strategies can be quite fruitful. We've witnessed several times in our discussion our participants' taking an experimental attitude towards mathematics which resulted in working out explicit division in order to conclude divisibility (Question set 1) and in pointing out the number with desired property in order to claim its existence (Question set 3). For some students the strategy of 'hit and miss' or a more sophisticated strategy of 'intelligent guesses' is essential as a midway between experimentation and logical argumentation. In some cases just a handful of successful intelligent guesses is all that is needed to achieve certainty and uncover the logic underneath such guesses.

Complexity of structure:

Understanding 'divisibility by n' as a generalized object is likely to be proceeded by the encapsulation of separate processes of divisibility for specific numbers. These encapsulations need not occur simultaneously for all numbers. For example, in Joan's interview we suggest that she may be thinking of 'evenness', or divisibility by 2, as an object; of divisibility by 3 as a process; and of divisibility by 7 as an action. Also, student's knowledge isn't static and meaningful construction may occur during the interview. Nicole was not the only one who demonstrated action construction in the beginning of the interview and interiorization of the process of divisibility as the interview proceeded. At some point, one may speculate, a critical mass is obtained and a generalization of the process of encapsulating divisibility for specific numbers may serve as a catalyst for the encapsulation of 'divisibility by n'. These issues and others will be the subject of further refinements of our theoretical framework and empirical investigations.

Summary and Conclusion

The action-process-object framework has proved useful to describe the construction of mathematical knowledge and provides a reasonable vocabulary to describe students' difficulties in these constructions. When analyzed and interpreted in terms of this framework, responses to questions and tasks such as considered herein have worked particularly well in revealing the pervasiveness of procedural attachments even when some degree of conceptual understanding is in evidence. Given the level of theoretical sophistication we have used to study this area we are most impressed with the need for further refinement to assimilate and accommodate the subtleties and complexities encountered in this domain.

In order to obtain better resolution of this cognitive terrain we are preparing further reports in the areas of prime decomposition and the relations between multiplicative and additive structure. Further studies are anticipated to investigate the concept of variable and even more basic concepts such as equality and reflexivity. We believe that our studies are providing evidence that the development of conceptual understanding in algebra requires a firm grounding in the conceptual understanding of elementary number theory. One fifth grade teacher described the number theory section in a grade five textbook, that included prime factorization, as "nothing new to learn, just doing multiplication tables backwards". We believe such trivialization of elementary number theory concepts has been convincingly refuted in this study and their importance underscored.

We believe with Steffe (1990) and many others that improvement of mathematics education starts with improvement of mathematical knowledge of teachers. Improvement of mathematical knowledge of teachers starts with a deeper understanding of their existing knowledge and its construction. This study provides details, or, using Shoenfeld's terminology (Schoenfeld, Smith, & Arcavi, 1992), serves to provide a finer granularity of knowledge in the domain of divisibility and factorization. Developing a conceptual understanding of divisibility and factorization is essential in the development of conceptual understanding of the multiplicative structure of numbers in general.

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