DOCUMENT RESUME

ED 370 766 SE 054 062

AUTHOR	English, Lyn D.
TITLE	Reasoning by Analogy in Constructing Mathematical Ideas.
PUB DATE	[93]
NOTE	57p.
PUB TYPE	Viewpoints (Opinion/Position Papers, Essays, etc.) (120)
EDRS PRICE	MF01/PC03 Plus Postage.
DESCRIPTORS	Abstract Reasoning; Basic Skills; Cognitive Structures; *Constructivism (Learning); Elementary Secondary Education; Foreign Countries; Learning Processes; Manipulative Materials; *Mathematical Models; *Mathematics Education; Mathematics Skills; Schemata (Cognition); *Thinking Skills
IDENTIFIERS	*Analogies; *Analogue Models; Analogy

ABSTRACT

A powerful way of understanding something new is by analogy with something already known. An analogy is defined as a mapping from one structure, which is already known (the base or source), to another structure that is to be inferred or discovered (the target). The research community has given considerable attention to analogical reasoning in the learning of science and in general problem solving, particularly as it enhances transfer of knowledge structures. Little work, however, has been directed towards its role in children's mathematical learning. This paper examines analogy as a general model of reasoning and discusses its role in several studies of children's mathematical learning. A number of principles for learning by analogy are proposed, including clarity of the source structure, clarity of mappings, conceptual coherence, and applicability to a range of instances. These form the basis for a critical analysis of some commonly used concrete analogs (colored counters, the abacus, money, the number line, and base-ten blocks). The final section of the paper addresses more abstract analogs, namely, established mental models or cognitive representations that serve as the source for the construction of new mathematical ideas. A reference list contains 78 citations. (MKR)

Reasoning by Analogy in Constructing Mathematical Ideas

Lyn D. English
Centre for Mathematics and Science Education
Queensland University of Technology
Australia

Running head: REASONING BY ANALOGY

Abstract

Analogy appears to be one of the most important mechanisms underlying human thought, at least from the age of about one year. A powerful way of understanding something new is by analogy with something which is known. The research community has given considerable attention to analogical reasoning in the learning of science and in general problem solving, particularly as it enhances transfer of knowledge structures. Little work, however, has been directed towards its role in children's learning of basic mathematical ideas. This paper examines analogy as a general model of reasoning and highlights its role in children's mathematical learning. A number of principles for learning by analogy are proposed. These form the basis for a critical analysis of some commonly used concrete analogs. The final section of the paper addresses more abstract analogs, namely, established mental models or cognitive representations which serve as the source for the construction of new mathematical ideas.

REASONING BY ANALOGY IN CONSTRUCTING MATHEMATICAL IDEAS

It has been argued that much of human inference is basically analogical and is performed by using schemas from everyday life as analogs (Gentner, 1989; Halford, 1992). Given that analogy is a very natural and ubiquitous aspect of human cognition, analogical reasoning would seem to lie at the very core of our cognitive processes. It is even used by very young children under appropriate conditions (Brown, Kane, & Echols, 1986; Crisafi & Brown, 1986; Gholson, Dattel, Morgan, & Eymard, 1989; Goswami, 1991). Such reasoning is also responsible for much of the power, flexibility, and creativity of our thought (Halford & Wilson, 1993; Holyoak & Thagard, 1993).

In 1954, Polya devoted an entire volume to the use of analogy and induction in mathematics. While he demonstrated how analogies can provide a fertile source of new problems and can enhance problem-solving performance, his ideas were not widely adopted, largely because they were descriptive rather than prescriptive (Schoenfeld, 1992). More recent studies however, have given greater attention to analogical reasoning in general problem solving, particularly as it enhances transfer of knowledge structures (e.g., Bassok & Holyoak, 1989; Gentner, 1989; Holyoak & Koh, 1987; Novick, 1990, 1992). While research has also addressed the role of analogy in science learning (e.g., Clement, 1988; Duit, 1991; Gentner, 1982; Gentner & Gentner, 1983), little work has been directed towards its role in children's learning of basic mathematical concepts and procedures. Studies which have focussed on mathematics have looked at how high school or college students apply newly learned formulas to related problems (e.g., Bassok & Holyoak, 1989; Novick, 1988).

The purpose of this paper is to examine analogy as a general model of reasoning and to highlight its role in children's learning of mathematics. Since the research community has focused largely on the role of analogical reasoning in general problem solving, the first section of this paper reviews some of the major findings in this area. From this review, a number of principles for learning by analogy are proposed. These form the basis for a critical analysis of some commonly used concrete analogs. The final section of the paper addresses more abstract analogs, namely, established mental models or cognitive representations which serve as the source for the construction of new mathematical ideas. Examples are drawn from the areas of numeration and algebra.

The Nature of Analogical Reasoning

Polya (1954) defined analogous systems as those that "agree in clearly definable relations of their respective parts" (p. 13). The definition commonly used today, and adopted in this paper, is that of Gentner's (1983; 1989), namely, an analogy is a mapping from one structure, the base or source, to another structure, the target. The system of relations that holds among the base elements also holds among the target elements. Normally the source is the part that is already known, whereas the target is the part that has to be inferred or discovered. A simple everyday example is shown in Figure 1. The source comprises two elements, dog and pup, and the relation, "parent of," between them. The target comprises the elements, cow and calf, with the same relation between them. There is a mapping from source to target such that dog is mapped into cow, and pup into calf, and the relation between dog and pup then corresponds to the relation between cow and calf. The important component here is the relation, "parent of." The attributes of the elements are not mapped, that is, the attribute "barks" is not mapped from dog to cow.

INSERT FIGURE 1 ABOUT HERE

An analogy utilizes information stored in memory (Halford, 1992). For example, the base in Figure 1 includes knowledge that a pup is an offspring of a dog. In this way, a model of analogical reasoning shares common features with knowledge-based models of reasoning (e.g., Carey, 1985; Chi & Ceci, 1987). However as Halford (1992) notes, analogies go beyond the information retrieved because the interaction of the base and the target produces a new structure that extends beyond previous experience. Furthermore, employing an analogy can open up new perspectives for both perceiving and restructuring the analog (Duit, 1991). The acquisition of this new structure is in accord with the constructivist views of children's learning; that is, learning is an active construction process that is only possible on the basis of previously acquired knowledge (Baroody & Ginsburg, 1990; Davis, Maher, & Noddings, 1990; Duit, 1991; von Glasersfeld, 1990). In other words, learning is fundamentally concerned with constructing similarities between new and existing ideas.

A typical case of analogical reasoning in elementary mathematics is the use of concrete aids in developing numeration understanding. The concrete representation is the source and the concept to be acquired is the target. The value of these analogs is that they can mirror the structure of the concept and thus enable the child to use the structure of the analog representation to construct a mental model of the concept. This is illustrated in Figure 2 where base-ten blocks are used to convey the meaning of a two-digit numeral, 27. Here, the 2 ten-blocks represent the digit "2" in the tens' place of the numeral, that is, there is a mapping from the source, the two ten-blocks, to the target, the digit 2. The 7 single blocks represent the 7 ones in the ones' place of the numeral (i.e., there is a mapping from the source, the set of 7 blocks, to the target, the digit 7). The MAB material is an effective analog since it clearly mirrors the target concept. However not all of the analogs commonly used in classrooms display this feature, as indicated later.

INSERT FIGURE 2 ABOUT HERE

Analogical Reasoning in General Problem Solving

Analogical reasoning plays a significant role in problem solving. The ability to access a known problem (i.e., a base or source problem) that has an identical goal structure to the new problem to be solved (target problem) can enhance problem-solving performance (Holyoak & Koh, 1987; Novick, 1988, 1992; Novick & Holyoak, 1991). This analogical transfer involves constructing a mapping between elements in the base and target problems, and adapting the solution model from the base problem to fit the requirements of the target problem (Novick, 1992). To illustrate this process, we consider some studies of children's skills in solving analogous problems.

In a study by Holyoak, Junn, & Billman (1984), children as young as 4 years were able to solve problems using a solution to an analogous problem. Two bowls were placed on a table, one which contained gumballs within the child's reach, and one out of reach. The child was provided with a walking cane, a large rectangular sheet of heavy paper, and a variety of other objects. The goal was to transfer the gumballs from the near bowl to the far bowl without leaving the seat. One solution is to use the cane to pull the far bowl within reach. Another is to roll the paper into a tube, and roll the gumballs down it into the other bowl. Children were told stories that required them to solve analogous problems, such as a genie who transferred jewels from one bottle to another by rolling his magic carpet into a tube, or by using his magic staff to move the distant bottle nearer. Four-year-olds were able to solve the problem even when the similarities between source stories and the target problem were relatively low. For example, in one experiment, the source story involved Miss Piggy rolling up a carpet to transfer jewels to a safe. Given that there are not many similarities between a magic carpet and a square of cardboard, or between a bowl and a safe, it appears that the children were using the relational mappings to some degree to help them solve the problems. However the young children's solution processes were fragile and easily disturbed by things such as adding extra characters to the stories or altering goals.

Evidence that children can use analogical reasoning in solving more complex problems has been provided in a series of studies by Gholson and his colleagues (Gholson, Eymard, Long, Morgan, & Leeming, 1988; Gholson et al., 1989). They used the well-known farmer's dilemma, the missionaries and cannibals, and the three-disk tower-of-Hanoi puzzles, together with a number of isomorphs, with children aged 4-10 years. There was a sequence of moves that was common to each type of problem, as can be seen in the farmer's dilemma. A farmer has to move a fox, a goose, and some corn in a wagon which will only transport one thing at a time. The problem is to move all three things without ever leaving the fox with the goose, or the goose with the corn, because in either case the former would eat the latter. The solution is to take the goose first, then go back, take the fox, then take the goose back, then take the corn, then go back, then take the goose again. The structure of this task is similar to the tower-of-Hanoi puzzle, in that both involve a sequence of forward and backward moves. Excellent isomorphic transfer was shown, even by the youngest children.

Gholson et al. (1989) suggest this might have been because extensive experience with the source tasks gave the children plenty of opportunity to acquire a high quality representation of the source.

In a recent study by English (reported in English & Halford, forthcoming), 9 to 12 year-olds from low, average, and high achievement levels in school mathematics were individually administered sets of novel combinatorial and deductive reasoning problems presented in concrete and isomorphic written formats. The order of presentation of these formats was counterbalanced for each problem type. The concrete combinatorial problems involved dressing toy bears in all possible combinations of colored T-shirts, pants, and tennis rackets. The number of combinations ranged from 9 to 12. The isomorphic written examples required the child to form all possible combinations of: a) colored buckets and spades, b) colored shirts, skirts, and shoes, and c) greeting cards featuring different colors, lettering, and messages. The hands-on deductive problems entailed working through a series of clues to determine how to: a) arrange a set of playing cards, b) stack a set of colored bricks, and c) match names to a set of toy animals. In the isomorphic written examples the child used given clues to determine: a) the locations of families in a street of houses, b) the location of a particular book in a stack of books, and c) the identification of personnel who played particular sports. Upon completion of each of the sets of combinatorial and deductive problems, children were asked whether solving one set (either hands-on or written) assisted them in solving the other set. Children were also asked if they could see ways in which the problem sets were similar.

Results to date indicate that, on the whole, the older children were better able to identify the structural similarities between the problems than the younger children. There were however, several cases in which the younger children performed better than their older counterparts in recognizing these similarities. This was also the case for children in the lower achievement levels who often performed just as well, if not better, than the high achievers. For example, 9 year-old Hayley, a low achiever, stated that the sets of combinatorial problems were similar because "you have to use combinations ... you have to do them in a method so you don't get get two exactly the same." On the other hand, Nicholas, a high-achieving 9 year-old, commented that the problems were "about dressing ... about matching colors." The older children frequently made mention of the similarity in the number of sets that had to be matched. For example, 12 year-old Natalie commented that the last two written problems (of the form, X x Y x Z) were like the last two hands-on examples because they had "three things to match up."

For the deductive reasoning problems, most children recognized that the problems involved an arrangement of items or a matching of names with items. As Kerry, a low-achieving 9 year-old stated, "In the books' problem, you had to stack them and in the cards' problem you had to arrange them across." Most children were also able to recognize the similarity in item arrangements, for example, "The houses problem is like the cards problem because you have to work out which ones go next to each other. And the tower (of blocks) is like this one (stack of books) because you have to stack them up in the right order" (Hayley, 9 year-old low achiever).

Few children however, commented on the nature of the clues per se, such as the extent of information they provided, or the need to look for related clues. James, a high-achieving 12 year-old commented on the fact that there was one clue which provided a starting point: "The five houses along the street is like the cards problem because you knew where one was and then you had to figure out where the others would go.... there's sort of a trick to it. You got one of them (referring this time, to the stacking problems) and you had to figure out which went on top and which went below." It is worth mentioning the response of 12 year-old Natalie when asked if solving one set of deductive reasoning problems helped her solve the other. She claimed, "I did each (set of problems) separately. I didn't relate them." When questioned on the similarities between the problem sets, she commented, "You've got to match stuff up with other stuff but otherwise I don't relate problems as I don't really look at that sort of thing."

Many studies have shown that novices tend not to focus on the structural features of isomorphic problems especially when they have different surface features or when the surface details provide misleading cues (Chi, Feltovich, & Glaser, 1981; Gentner, 1989; Gentner & Toupin, 1986; Holyoak et al., 1984; Novick, 1988, 1992; Reed, 1987; Silver, 1981; Smith, 1989). This means that the surface features in a novice's model of a target problem will likely serve as retrieval cues for a related problem in memory. On the other hand, studies have shown that similarity among surface details, or superficial similarity, promotes "reminding," that is, assists novices to notice a correspondence between their mental model of a base problem and the new target problem (Gentner & Landers, 1985; Reed, 1987; Ross, 1984, 1987). However while surface similarity can facilitate children's retrieval of the base problem. its usefulness for analogical transfer is once again governed by their ability to detect the structural correspondences between the base and target problems (Gentner & Landers, 1985).

Component Processes in Analogical Problem Solving

It is worth reviewing the component processes entailed in solving problems by analogy (Gholson, Morgan, Dattel, & Pierce, 1990), since these processes apply equally to the use of analogy in learning mathematical ideas. Firstly, the solution to the source or base problem (e.g., the gumballs problem cited earlier) must be learned. Secondly, the base problem must be represented in terms of the structural features of a generalizable mental model, rather than in terms of particular surface details such as the specific attributes of the items (Gentner, 1983; Holland, Holyoak, Nisbett, & Thagard, 1986). Thirdly, the child must notice the correspondence between the target problem (e.g., the genie problem cited earlier) and the base problem and retrieve the base in terms of its generalizable structure rather than in terms of specific surface details such as bottles or jewels (Gholson et al., 1990). Finally, the child must map, one-to-one, the structural features of the source and target and then carry out the required problem-solving activities (Gentner, 1983; Holyoak, 1985). As a result of successfully transferring the base solution to the target problem, students at all proficiency levels are likely to induce a more abstract knowledge structure encompassing the base and target problems (Novick, 1992). The ability to abstract the structural components of a problem domain facilitates solution of subsequent analogous problems and is particularly important in children's mathematical development.

We can view the solving of these analogous problems in terms of mapping the states, goals, and operators (or techniques) of the novel problem into the familiar one. These processes can be represented by a conventional structure-mapping diagram, as shown in Figure 3. These diagrams indicate how the elements of one structure map onto the elements of another such that any relations, functions, or transformations between elements of the first structure correspond to relations, functions, or transformations in the second structure (Halford, 1993).

INSERT FIGURE 3 ABOUT HERE

As indicated in Figure 3, the source is the problem-solving procedure used previously on a now-familiar problem. The components of the structure-mapping are states and goals, and the relations are the operators that transform the initial state into one or more subgoals and then into the final goal state. The target is the novel problem. As shown in the diagram, the states, goals, and operators of the novel problem are mapped into the familiar one. In the case of Holyoak et al.'s (1984) hollow-tube problem, the initial state is that the gumballs are in one bowl, the goal is to have them in the other bowl, and the operator is to move them down the tube. The subgoal is to construct the tube and the operator is to roll a sheet of cardboard to achieve this. The source is the similar "genie" problem, with the genie's jewels in one bottle being the initial state. The goal was to have them in another bottle, with the operator being to roll them down a tube made from the magic carpet (assuming the tube was a subgoal achieved by commanding the carpet to roll itself into a tube). More complex examples, such as the missionaries and cannibals problem, would obviously involve a greater number of subgoals and operators.

More complex processes are involved in solving nonisomorphic problems where one of the problems comprises concepts or relations that cannot be mapped into the concepts or relations in the other (Reed, 1987). Gentner (1989) uses the term, transparency, to define the ease with which it can be decided which attributes and relations in the base domain should be applied in the target domain. Transparency would obviously be highest for equivalent problems where both the story context and relational structures correspond, and lowest for unrelated problems in which neither of these corresponds. In the case of nonisomorphic problems where only some of the concepts and relations correspond, procedural adaptation (Novick, 1988, 1992) must be carried out. This involves correctly representing both the base and target problems in terms of their structural features, noticing the differences, and then modifying the procedures in the base to enable a one-to-one mapping between the modified base and the target (Gholson et al., 1990).

To illustrate this procedure, we consider two problems from Novick's (1992) work:

Base problem.

A small hose can fill a swimming pool in 10 hours and a large hose can fill the pool in 6 hours. How long will it take to fill the pool if both hoses are used at the same time?

Target problem.

It takes Alex 56 minutes to mow the lawn and it takes his older brother Dan 40 minutes to mow the lawn. Dan mowed half the lawn on Saturday. On Sunday the two boys work together to mow the other half of the lawn, but Dan starts 4 minutes after Alex. How long will each boy work on Sunday?

(Novick, 1992, p. 175).

The equation given to students for the base problem was $(1/10)h + (1/6)h = 1$. The equation which had to be generated for the target problem was $(1/56)m + (1/40)(m - 4) = 1/2$. Solving the target problem by analogy with the base problem requires students to realize firstly, that the right-hand-side of the base equation refers to the quantity of task completed together by the two hoses ("workers"), which is not necessarily the entire task. This generalization is reflected in the base/target correspondence $1 \equiv 1/2$. Secondly, students must realize that the workers need not work the same amount of time. If Dan corresponds to the large hose, the generalization can be seen in the correspondence $(1/6)h \equiv (1/40)(m - 4)$. The remaining components of the equation for the target problem (i.e., 1/40, 1/56, and (1/56)m) can be generated through substitution (Novick, 1992, p. 175).

Principles of Learning by Analogy

To this point, we have highlighted a number of key features of analogies and the processes involved in reasoning by analogy in problem solving. Since these have significant implications for mathematics learning, we review them in terms of a number of learning principles. In proposing these principles, we draw upon some of Gentner's (1982) criteria for effective analogs.

Recall that reasoning by analogy involves mapping from one structure which is already known (base or source) to another structure which is to be inferred or discovered (target).

Clarity of Source Structure

The structure of the source should be clearly displayed and explicitly understood by the child.

For an analogy to be effective, children need to know and understand the objects and relations in the base. It is particularly important that the child abstracts the structural properties of the base, not its superficial surface details. It will not be possible to map the base into the target, then use the base to generate inferences about the target, unless this understanding has been acquired and is readily available.

Clarity of Mappings

There should be an absence of ambiguity in the mappings from base to target.

The child should be able to clearly recognize this correspondence between base and target. When a base has to be recalled from memory, it should be retrieved in terms of its generalizable structure rather than in terms of particular surface details (Gholson et al., 1990). This is particularly important in the development of abstractions. These are formed from mappings in which the source, itself, is an abstract relational structure, with few or no attributes. Hence if children are to form meaningful abstractions, they must learn the structure of the examples they experience. Good analogs can assist here because mapping between an analog and a target example encourages children to focus on the corresponding relations in the two structures (Halford, 1993).

Conceptual Coherence

The relations that are mapped from source to target should form a cohesive conceptual structure, that is, a higher order structure.

According to Gentner's (1983) systematicity principle, relations are mapped selectively, that is, only those that are mapped enter into a higher order structure. For example, in using various concrete analogs to illustrate grouping by ten, attention must be focussed on the corresponding relations between the groups of items, not between the materials themselves (e.g., the physical size relation between a bundling stick and an MAB mini is not mapped).

Scope

An analogy should be applicable to a range of instances.

Analogies with high scope can help children form meaningful connections between mathematical situations. For example, the "sharing" analogy in teaching the division concept can be applied readily to both whole numbers and fractions. Likewise, the area model can effectively demonstrate a range of fraction concepts and procedures.

These principles prove to be particularly useful in assessing the effectiveness of the analogs used in children's mathematical learning. While we consider initially a selection of concrete analogs, they are by no means the only analogs available. There are more abstract analogs such as a mental model of arithmetic relations which can serve as an effective source for algebraic learning; we address these in the final section. Considerable concern has been expressed over teachers' selection (or lack thereof) of concrete learning aids and the fact that teachers are offered little assistance in making appropriate choices (Ball, 1992; Baroody, 1990; Hiebert & Wearne, 1992; Kaput, 1987). It is understandable then, why some children see as many different concepts as there are analogs, even though only one concept is being conveyed, and why teachers often fail to consider the representations they are using when trying to help children overcome these difficulties (Dufour-Janvier, Bednarz, & Belanger, 1987). In the next section, we take a critical look at some of these analogs and, using the principles we have established, offer an assessment of their appropriateness for conveying intended concepts.

The Appropriateness of Concrete Analogs

Concrete analogs are generally considered to enhance learning by helping children understand the meaning of mathematical ideas and their applications. Analogs can model problem situations effectively, can facilitate retrieval of information from memory, can verify the truth of what is learned, can increase flexibility of thinking, and can generate new ideas and unknown facts (Dienes, 1960; Fuson, Fraivillig, & Burghardt, 1992; Grover, Hojnacki, Paulson, & Matern, in press; Halford & Boulton-Lewis, 1992; Kennedy, 1986; McCoy, 1990; NCTM, 1989; Sowder, 1989). However analogs, in and of themselves, cannot impart meaning; mathematical ideas do not actually reside within wood and plastic models (Ball, 1992; Wearne & Hiebert, 1985). Furthermore, while analogs display many relevant features, they frequently contain many irrelevant, potentially confusing features (Hiebert, 1992). We cannot automatically assume that children will make the appropriate mappings from the analog to the abstract construct, especially when some of the analogs themselves, are complex.

Despite their significance in the mathematics curriculum, these analogs have received little critical analysis, especially from a psychological perspective (Ball, 1992). Furthermore, as Thompson (1992) points out, the research findings on their effectiveness have been equivocal. Some studies (e.g., Labinowicz, 1985; Resnick & Omanson, 1987) found little impact of the base-ten blocks on children's facility with algorithms. Other studies (e.g., Wearne & Hiebert, 1988; Fuson & Briars, 1990) reported a positive effect of these materials on children's understanding of, and skill with, decimal numeration and multi-digit addition and subtraction. Still other studies (e.g. Gilbert & Bush, 1988) have indicated that concrete analogs are not widely used, with their overall use decreasing as grade level and length of teaching experience increase.

As we indicate in our analysis of these analogs, some materials may be structurally simple, yet prove to be complex learning aids when applied to target concepts which comprise inherently complex relations. This places an additional processing load on children as they attempt to interpret the arbitrary structure that has been imposed on the concrete analog to mirror the structure of the target concept. This can result in a failure to acquire the concept. We have chosen to analyze some of the well known analogs, including colored counters or chips, the abacus, money, the number line, and the base-ten blocks. By considering the processes involved in interpreting these analogs, we attempt to illustrate how they can enhance learning when their structure clearly mirrors the target but how they can become complex aids when assigned an arbitrary, implicit structure.

Colored Counters or Chips

Discrete items such as counters and other simple environmental items are typically used in the study of elementary number and computation. These analogs do not possess inherent structure as such, that is, they do not display in-built numerical relationships. However they can effectively demonstrate the cardinality of the single-digit numbers. In this instance there is just one mapping from the base (the set of counters) to the target (the number name). When applied to the learning of basic number concepts, colored counters score highly on clarity of source structure and mappings. When used with the appropriate language and manipulative procedures, these analogs can promote a cohesive understanding of single-digit numbers and of the elementary number operations.

The complexity of this analog increases significantly however, when it is applied to the development of place-value ideas. In this instance the analog takes on an arbitrary structure in order to mirror the structure of the target and, as such, the mappings between the source and target become more complicated. This implied structure is of a grouping nature where groups of counters or chips of one color are traded for a chip of a different color to represent a new group. This single chip represents a number of objects rather than a single object (LeBlanc, 1976). The analog thus becomes an abstract representation because the value of a chip is determined only by its color, which is arbitrary, and not by its size. For example, if a red chip is worth one hundred, a blue chip worth one ten, and a green chip worth one unit, then the numeral 364 would be represented by three red chips, six blue chips, and four green chips. Because there is no obvious indication of each chip's value, there is not a clear mapping from the base material to its corresponding target numeral. In fact, there is a two-stage mapping process involved, namely, from chip to color, then from color to value. That is, the child must firstly identify the color of the chip and then remember the value that has been assigned to that color (the same situation exists with the Cuisenaire rods). This naturally places an additional processing load on the child, especially if she does not readily recall this value. Given the lack of clarity in its source structure and the multiple mappings required, this material does not seem an appropriate analog for introducing grouping and place-value ideas. It appears more suitable for enrichment work.

The counters analog also increases in complexity when it is used as a source for the part/whole notion of a fraction. For example, to interpret the fraction of red counters in a set comprising 3 red and 5 blue counters, the child must initially conceive of the set as a whole entity to determine the name of the fraction being considered. An added difficulty here is that the items do not have to be the same size or shape (in contrast to an area model comprising, say, a rectangle partitioned into 8 equal parts). Hence the child must see the items of the set as equal parts of a whole, irrespective of whether the items themselves, are unequal. While keeping the whole set in mind, the child must identify all the red counters and conceive of them as a fraction of this whole set. Since it is difficult to ascertain the whole and the parts, which more or less requires simultaneous mapping processes, it is not uncommon for children to treat the red and blue counters as discrete entities and interpret the fraction as a ratio (i.e., "3 parts to 5 parts;" Behr, Wachsmuth, & Post, 1988; Novillis, 1976). It is for this reason that the analog comprising sets of counters is inappropriate for introducing the part/whole construct (Hope & Owens, 1987).

In sum then, colored counters do have considerable scope and can be an effective analog for early number and computation activities where there is clarity of source structure and unambiguous mappings between source and target. When the target concept increases in complexity however, the analog also becomes more complex and does not mirror the target as readily as before. The analog adopts an implied structure which makes it difficult to form clear and unambiguous mappings between source and target. In the case of the fraction example, the analog's structure encourages children to focus on the inappropriate relation, namely, the relation between the two colored sets instead of the relation between one colored set and the whole set. This means the analog does not establish the conceptual coherence required. However when used in conjunction with other fraction analogs (such as area models) and when accompanied by the appropriate language and manipulative procedures, this particular analog can enrich children's conceptual understanding of the fraction concept.

The Abacus

The traditional classroom abacus consists of nine beads on each of several vertical wires which designate the places in our number system. There are no more than nine beads in any one column since "ten" is represented by one bead in the column immediately to the left (reflecting the Egyptian system).

Since nine (not ten) beads on one wire are swapped for a single bead on the next wire, it is more difficult for the child to see the intended correspondence between the source and the target place-value ideas. While all the beads are identical, except perhaps in color, they adopt different numerical values depending on the position of the wire. The new single bead has a value ten times greater than a single bead to its right, however this relation is not explicit.

In interpreting a number on the abacus, the child must undergo a three-stage mapping process, namely, from the number of beads on a particular wire to the wire's position, then to the value of this position, and finally, to the target numeral. This poses quite a high processing load for the child. Given these complexities, the abacus is not an appropriate analog for introducing grouping and place-value concepts. In fact, the child must apply a prior understanding of these concepts when representing numbers on this device. Hence the abacus is more appropriately used when the child has acquired this knowledge.

Money

At first glance, money seems an appropriate and appealing analog. It certainly has the desirable features of being real world and "hands-on" for the child. However, this does not automatically qualify it as a suitable analog for teaching number concepts and operations. Money is not unlike the colored chip material in that the relationships between the denominations are not immediately discernible. Furthermore, in some currencies, there is a conflict of size and value. For example, in the USA, the dime is smaller than the penny; in Australia, the two-dollar coin is smaller than the one-dollar coin. There is also the problem of some coins not fitting nicely within the "ten-for-one" trades of our decimal system, for example, the US nickel and quarter (Fuson, 1990).

Because the base-ten feature of decimal currencies is not explicit in the material, the use of money to illustrate grouping and place-value concepts presents complex mapping processes for the child. This is particularly the case when money is used to illustrate decimal fractions. Children have difficulty in seeing a particular coin as being a fraction of another, particularly since the relative sizes of the coins do not suggest a fractional relationship. Furthermore, through their everyday transactions with money, children (and adults) come to see a particular denomination as an entity in its own right, not as a fraction of some other denomination. Hence for children to see 45 cents as 45 hundredths of a dollar, they must firstly identify the four ten-cent coins as equivalent to forty cents and the one five-cent coin as equivalent to five cents. Secondly, the child must identify the one-dollar coin or note as one whole unit comprising 100 cents. There is no visual indication, of course, that this is the case. Finally, children must apply their understanding of the part/whole fraction concept to the recognition that 45 cents is 45 hundredths of a dollar. Again, there are no visual cues for this (that is, the child cannot place the 45 cents on top of the one dollar to see that it "covers" only 45 hundredths of the dollar). The use of money for this purpose thus entails several mappings and places a considerable cognitive load on the child. As such, money is not a suitable analog for establishing decimal fraction concepts and serves better as a source of application activities. It is doubtful whether money would ever be used as an analog if it were not so pervasive in our society.

<u>The Number Line</u>

The number line is also an abstract analog which has enjoyed popularity in the study of single-digit numbers and computations. However because the number line is a continuous, rather than discrete, analog it is not appropriate for children's early number experiences. Furthermore, the analog does not display clarity of structure, nor clarity of mapping, because the number of gradations on the number line does not correspond to the numerals represented. For example, even though a child might be instructed to make four "jumps" to reach the numeral "4," as shown in Figure 4, there are, in fact, five gradations to this point. That is, the number of gradations is one more than the corresponding numeral.

INSERT FIGURE 4 ABOUT HERE

Dufour-Janvier, Bednarz, and Belanger (1987) cite other problems associated with this analog. Included here is the tendency for children to see the number line as a series of "stepping stones." Each step is conceived of as a rock with a hole between each two successive rocks. This may explain why so many secondary students say that there are no numbers, or at the most, one, between two whole numbers.

A further difficulty with this analog is that it does not effectively promote conceptual coherence. For example, since it is difficult to represent the multiplication concept in ways other than repeated addition, there is the danger of children seeing this operation simply in terms of repeated "jumps." The number line is also limited in promoting understanding of the other operations, such as subtraction where there is not a clear mapping from the analog to the basic "take-away" notion to which children are initially introduced. Given the difficulties associated with this abstract analog, it would seem to be more appropriate for application activities where children can demonstrate their previously acquired numerical understandings.

An even more difficult application of the number line (and other comparable continuous models) is i~ the representation of fractions (Bright, Behr, Post, & Wachsmuth, 1988; Hiebert, Wearne, & Taber, 1991; Larson, 1980). As noted by Bright et al. (1988), there is firstly the problem of length representing the unit. Since the number line acts like a ruler, there is not only iteration of the unit but also simultaneous subdivisions of all iterated units. Secondly, the model is totally continuous, that is, there is no visual separation between consecutive units. This is in contrast to the visual discreteness of the set and area analogs. As a consequence, children may count the iterations rather than the intervals when attempting to identify a given fraction. For example, the fraction marked on the number line of Figure 5 could be interpreted as "four fifths" instead of "three fourths." The third important difference between the number line and the other fraction analogs is that it requires the use of symbols to convey the fraction notion. As Bright et al. comment, the number line is made all the more complex by its integration of two forms of information, namely, visual and symbolic. The symbols can distract the student from any visual embodiment of the fraction concept.

INSERT FIGURE 5 ABOUT HERE

In analyzing these complexities, it is easy to see why the number line is a difficult analog for children. As indicated in Figure 5, several mappings are entailed in interpreting the target concept. Firstly, the spaces, not the iterations, must be interpreted as fractional components. This involves a mapping from the spaces (distance between iterations) to the notion of equal parts of a whole. Secondly, the number of spaces comprising a whole unit must be identified and mapped onto the fraction name, that is, four equal spaces -> fourths. Thirdly, the point marked by the cross must be interpreted as encompassing all of the spaces from the zero point. The number of spaces to the cross must then be mapped onto the number of fourths being considered (three spaces -> three fourths). Given the complexity of this interpretation process, it appears that this analog would serve better as a source of application activities, not as a means of introducing the fraction concept.

<u>Base-ten Blocks</u>

The base-ten blocks (Dienes, 1960) are probably the most commonly used analogs in the teaching of numeration and computation. Because the size relations between the bocks clearly reflect the magnitude relations between the quantities being represented, the blocks display clarity of source structure and clear mappings to the target concept. The analog also demonstrates high scope since it can be applied to a range of instances. For example, when used in conjunction with a place-value chart, the base-ten blocks can assist children in their understanding of the numeration of multidigit numbers. The blocks can also demonstrate the regrouping and renaming of whole numbers, and hence, can foster conceptual coherence of our numeration system.

While the blocks represent a highly appropriate analog, their effectiveness will be limited if children do not form the correct mappings between the analog representations and the target concepts and between their manipulations with the analog and the target procedures. This can happen if the blocks are not arranged in accordance with the positional scheme of our number system or if sets of blocks are combined in any order, beginning with any size block and moving back and forth to trade for another size when necessary (Hiebert, 1992). Children's failure to form connections between the analog representation and the target ideas has been reported in several studies (Baroody, 1990; Davis, 1984; Resnick & Omanson, 1987). Findings from other studies have shown that the nature of the teacher's explanations during the learning sequence is a crucial component in this process, with appropriate and frequent verbal explanations seen to enhance learning (Fuson, 1992; Leinhardt, 1987; Stigler & Baranes, 1988). The importance of children's verbal explanations, with an emphasis on the quantities they are manipulating, has also been highlighted (Resnick & Omanson, 1987).

While the base-ten blocks serve as an effective analog for whole numbers, they take on an added complexity when representing decimal fractions. Changing the values of the blocks to accommodate decimal fractions poses a higher processing load for the child. For whole numbers, the values assigned to the blocks normally remain fixed and children associate a given block with its whole number value. When the blocks take on new values, children are faced with additional mapping processes. For example, if the flat block is assigned the value, one unit (or one one), the long block is equal to one tenth and the mini, one hundredth. This means that, to interpret the representation shown in Figure 6, children must firstly identify the flat block as representing one whole unit. They must then recall that the flat block is equivalent to ten long blocks as well as one hundred mini blocks. Next, children have to perceive the long block as equivalent to one tenth and the mini, one hundredth, of the flat block. This process, itself, involves an application of the fraction concept. Once the respective values of the blocks have been established, children must interpret the decimal fraction being represented. If children do not make all of the mappings required, there is the danger that they will interpret the decimal fraction as a whole number, record it as such, and simply insert a decimal point.

INSERT FIGURE 6 ABOUT HERE

The complexity of the mapping processes involved here means that the base-ten blocks can lose clarity of both source structure and mappings when used as an analog for the initial representation of decimal fractions. Since children have to apply an understanding of fractions in interpreting this analog, it would seem more appropriate to employ less complex analogs, such as partitioned region models, in introducing decimal fractions and reserve the base-ten blocks for application activities.

In this section we have focussed on concrete analogs. We now turn to a consideration of more abstract analogs, namely, established mental models, which serve as the source for the learning of a new target concept or procedure.

Mental Models as Analogs

The term, mental models, has been used extensively in the psychological literature (e.g., Johnson-Laird, 1983; Halford, 1993; Rouse & Morris, 1986). Johnson-Laird considers mental models to be structural analogs of the world. Halford (1993) adopts a broader perspective and views mental models as representations wnich are active while solving a particular problem and provide the workspace for inference and mental operations. According to Halford, cognitive representations are the workspace of thinking and understanding and must have a high degree of correspondence to the environment that they represent.

Mental models can be retrieved from memory where a particular representation has been associated with that situation in the past. They can also be transferred from another situation and used by analogy (Collins & Gentner, 1987), or they can be constructed from components obtained from both of these sources (Halford, 1993). Because mental models comprise representations, and since analogies are mappings from one representation to another, mental models can serve as analogies. We provide some instances of these in the remainder of this section.

Use of Analogy in Multidigit Numeration

In using mental models as analogs, children need to explicitly recognize the correspondence between their model of a particular mathematical construct (i.e., the source) and the targeted construct. Consider firstly, children's learning of the relationships inherent in our place-value system. Children's introduction to multidigit numbers presents a new relational construct for the child, namely, the periods within our number system. The important feature of these is that the same set of relationships exists in each period, that is, the ones' period comprises hundreds, tens, and ones of ones, the thousands' period comprises hundreds, tens, and ones of thousands, and the millions' period, hundreds, tens, and ones of millions. This is readily demonstrated on a place-value chart. A meaningful mental model of the "hundreds, tens, and ones" relations within the ones' period can serve as an effective analog for the learning of larger numbers. As an analog, it displays clarity of mappings since it is readily mapped onto each period of a multidigit number, as indicated in the next paragraph. The analog also promotes conceptual coherence of our number system because it highlights the important place-value relations.

Using this analog to interpret a multidigit number involves a process of mapping the "hundreds, tens, ones" model onto each new period within the number and assigning the appropriate period name. For example, to interpret the numeral, 435 537, we firstly recognize that there are two periods, the left-hand being the thousands' period and the right-hand, the ones' period. By mapping the "hundreds, tens, ones" model onto each period, the value of the number can be determined, that is, four hundred and thirty-five thousands and five hundred and thirty-seven ones. The value of each digit can also be readily discerned, for example, the left-hand "3" has a value of 3 ten-thousands because it is in the tens' place of the thousands' period. Likewise, the right-hand "3" is worth 3 tens because it is in the tens' place of the ones' period. Applying a mental model of these periods is a less complex process for the child than the common procedure for reading numerals identified by Fuson et al. (1992). They argue that while number words are written down, left to right, in the order in which they are said, the reading of a multidigit number involves a reverse right-to-left process. That is, to read a numeral such as 4 289, the child must look along the digits from right to left in order to decide the name of the farthest left place (i.e., "ones, tens, hundreds, thousands"). The child can then proceed to read the number name from left to right.

Use of Analogy in Interpreting Metric Measurements

A mental model of the positional relationships within the decimal number system can serve as a useful analog in the understanding of metric measurements. For example, to interpret a metric measurement such as 5.2 meters, we firstly map our knowledge of length relationships (i.e., 1 meter is equivalent to 100 centimeters and 1000 millimeters) onto our knowledge of decimal number positional relationships (i.e., 1 unit is equivalent to 100 hundredths and 1000 thousandths). This means the metric unit, meter, is mapped onto the ones' or units' place, centimeter is mapped onto the hundredths' place, and millimeter is mapped onto the thousandths' place, as shown in Figure 7. By recording the metric measurement on the place-value chart, it can be seen that 5.2 meters represents 5 meters and 20 centimeters as well as 5 meters and 200 millimeters.

INSERT FIGURE 7 ABOUT HERE

Use of Analogy in Algebraic Learning

The final example we will consider is the use of analogy in algebraic learning. It is posited that understanding algebra, as with other domains, depends on constructing appropriate mental models of the essential concepts. While arithmetic is primarily concerned with relations between constants, algebra focuses on relations between variables. To understand algebra then, means to have a mental model of these relations, and be able to use this mental model to guide the development of appropriate operations and strategies.

Variables can only be understood in terms of their relations to other numbers, which may be either variables (e.g., $x = 5y$) or constants (e.g., $\log N^2$). Where these relations are defined by convention, and are part of the number system, they will be relatively fixed. For example, the value of $\log N^2$ will always have the same relation to the value of N, because of the way $\log^2$ is defined. However when the relation is defined by a specific expression, such as with $x = 5y$, it is not fixed, and is not necessarily part of any predefined system. In such cases, interpretation of a variable symbol cannot be based on past experience with that particular symbol, but must be based on conventions used to interpret such expressions. Children who fail to understand this may try to relate the meaning of a variable to past experience, and believe that, for example, y must represent an item like yachts (Booth, 1988) or must be equal to 25 because y is the 25th letter of the alphabet. In the latter case, students are using their familiar arithmetic frame of reference (Chalouh & Herscovics, 1988). Given that this numerical frame of reference is the only one available to beginning algebra students, it seems appropriate to capitalize on it in the early stages.

The meaning of $\underline{x}$ in the expression $\underline{x} = 5\underline{y}$, for example, can be demonstrated by analogy with arithmetic. Just as relations between sets of objects can be used as analogs of relations between constants, relations between constants can be used as analogs of relations between variables. Thus a mental model of $\underline{x} = 5\underline{y}$ can be formed through one or more arithmetic examples such as, 15 = 5 x 3, 30 = 5 x 6, 25 = 5 x 5, and so on. Recognition of the correspondence between each of these examples and the expression $\underline{x} = 5\underline{y}$ is a form of analogical reasoning. So too, is the use of specific numerical examples to illustrate the general assertion, $\log N^a = a \log N$ for any for any $N > 0$ (NCTM, 1989).

Since algebraic relations can be understood by analogy with arithmetic relations, we can analyze the processes involved in terms of analogy theory. One point which emerges immediately is that the arithmetic relation is the source and the algebraic relation is the target. For example, to develop a mental model of the relation, $a(b + c) = d$, we can take as the source, an arithmetic example such as $3(2 + 1) = 9$. For the analogy to be effective however, the arithmetic relation, such as distributivity, must be well learned. This was emphasized in our first principle of learning by analogy, namely the need for clarity of source structure. The important point which emerges from analogy theory is that learning algebra will depend crucially on how well arithmetic relations are learned, because arithmetic relations are the source for the initial understanding of algebraic relations (Booth, 1989; Chaiklin & Lesgold, 1984). Understanding arithmetic relations gives children both a rationale for the arithmetic procedures which they learn in their elementary years, and provides a basis for the more abstract understanding that is required in algebra (English & Halford, forthcoming).

Understanding algebra does not end with arithmetic analogs, of course. One reason is that there are some inconsistencies between arithmetic and algebra, as noted extensively in the literature (e.g., Booth, 1988; Chalouh & Herscovics, 1988; Herscovics, 1989; Kieran, 1990, 1992; Matz, 1979, 1982; Vergnaud, 1984). One of the most important differences is that in arithmetic the answer to a problem is a specific constant, whereas in algebra the answer is itself a relation. The solution to an equation is usually a relation between one variable, on the left-hand side, and an expression containing one or more other variables and constants on the right-hand side. However while differences between algebra and arithmetic are clearly important, it does not follow that arithmetic cannot be a useful analog for algebra. There are also important differences between sets of discrete items and numbers, yet sets are a useful analog for understanding elementary number relations and operations. The value of analogies is partly that they transcend domains which may be very different apart from the relations they have in common. However the most important feature about analogies in the present context is that they are an excellent way of learning about relations, and they are a means by which relations that are learned in arithmetic can be transferred to algebra. The better this is done the more readily children can progress to a more abstract understanding of algebra.

Another reason why understanding algebra does not remain tied to arithmetic analogs, is that elementary algebraic relations can serve as mental models for more advanced relations. In this case an elementary relation serves as a source and the more advanced relation as the target. In other instances, an elementary relation in a conventional form can serve as the source for a more sophisticated form of the same relation. For example, consider the following two expressions: $\underline{f}\,\underline{g} = \underline{c}$ and $\underline{j}\,(\underline{N} + \underline{x}) = \underline{s}$. If we substitute $\underline{g}$ for $(\underline{N} + \underline{x})$, the expressions are essentially the same. The fact that different symbols are used is irrelevant. This correspondence is due to common relations in the two expressions; specifically, one variable is expressed as the product of two others. Therefore recognition of the correspondence between the expressions is essentially the same kind of process as is involved in analogical reasoning.

It follows that recognition of common relations in different expressions is also a form of analogical reasoning. This process is commonly used when deciding how to solve a problem such as $x\sqrt{y} = 1 + 2x\sqrt{1+y}$. Mathematicians would recognize this as a case of the equation, $\underline{ax} = \underline{b} + \underline{cx}$, with a solution of the form, $x = b/(a - c)$, if $a \neq c$ (Wenger, 1987). Recognizing the given equation as a case of $\underline{ax} = \underline{b} + \underline{cx}$ is a form of analogy, in which the latter is the source and former is the target. Once recognized, it should be clear what procedure to adopt.

Concluding Points

Analogy appears to be one of the most important mechanisms underlying human thought, at least from the age of about one year. A powerful way of understanding something new is by analogy with something which is known. This paper has examined analogy as a general model of reasoning and has demonstrated its role in novel problem solving and in children's basic mathematical learning. An analogy was defined as a mapping from one structure, which is usually already known (the base or source), to another structure that is to be inferred or discovered (the target). Mathematical analogs range from elementary concrete models such as counters, to abstract mental models such as arithmetic and algebraic relations. The value of these analogs is that they mirror the structure of the targeted mathematical idea and thus enable children to use the structure of the analog representation to construct a mental model of the new idea.

The important feature of analogies is that the structural correspondence between the source and target is mapped, not the superficial attributes of these elements. Relations are mapped selectively, that is, only those relations that enter into a coherent structure are mapped. One of the values of analogies is that they transcend domains which may be vei, different, apart from the relations they have in common. Since analogies focus on common relational structures, reasoning by analogy is an important process in children's mathematical learning. Effective analogs can help children form the desired connections between mathematical ideas. As noted in the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989), children need to see mathematics as an integrated whole and should be able to use a mathematical idea to further their understanding of other mathematical ideas (p. 84).

While analogs may have the potential to enhance children's mathematical learning, they often possess inherent or arbitrary structures which can detract from their effectiveness. Effective analogs are those in which the child clearly recognizes and understands the structure of the source, can clearly recognize the correspondence between source and target, and can make the required mappings from source to target. The analog should facilitate the mapping of appropriate relations, that is, those that form a cohesive conceptual structure. Analogs which are applicable to a range of instances can help children form meaningful connections between mathematical ideas. On the other hand, a given analog can prove to be complex and confusing for children when its structure does not fully correspond with that of the target. In this instance, the analog is given an arbitrary, implicit structure so that it will mirror the relations inherent in the target. This modification can often result in loss of clarity of source structure and hence of clarity of mappings from source to target.

As mathematics educators, we need to critically analyze the analogs we use with our students and ensure that they do in fact, reflect the intended mathematical ideas. We also need to make greater use of children's existing mental models as analogs for new understandings. Provided these models are well established, they can serve as a powerful source for the learning of more complex relations. The use of analogy to construct these abstractions is likely to result in more meaningful and productive learning.

References

Ball, D. (1992). Manipulatives and the reform of math education. American Educator, Summer, 14-18, 46-47.

Baroody, A. J. (1990). How and when should place-value concepts and skills be taught? Journal for Research in Mathematics Education, 21, (4), 281-286.

Bassok, M., & Holyoak, K. (1989). Interdomain transfer between isomorphic topics in algebra and physics. Journal of Experimental Psychology: Learning, Memory, and Cognition, 15, 153-166.

Behr, M. J., Wachsmuth, I., & Post, T. (1988). Rational number learning aids: Transfer from continuous models to discrete models. Focus on Learning Problems in Mathematics, 10, (4), 1-18.

Booth, L. R. (1988). Children's difficulties in beginning algebra. In A. F. Coxford & A. P. Shulte (Eds.), The ideas of algebra, K-12 (pp. 20-32). Reston, Virginia: National Council of Teachers of Mathematics.

Booth, L. R. (1989). A question of structure. In S. Wagner & C. Kieran (Eds.), Research issues in the learning and teaching of algebra (pp.57-59). Reston, Viginia: National Council of Teachers of Mathematics; Hillsdale, NJ: Lawrence Erlbaum Associates.

Bright, G. W., Behr, M., Post, T. R., & Wachmuth, I. (1988). Identifying fractions on the number line. Journal for Research in Mathematics Education, 19 (3), 215-233.

Brown, A. L., Kane, M. J., & Echols, C. H. (1986). Young children's mental models determine analogical transfer across problems with a common goal structure. Cognitive Development, 1, 103-121.

Carey, S. (1985). Conceptual change in childhood. Cambridge, MA: MIT press.

Chaiklin, S., & Lesgold, S. (1984, April). Prealgebra students' knowledge of algebraic tasks with arithmetic expressions. Paper presented at the annual meeting of the American Educational Research Association, New Orleons, LA.

Chalouh, L., & Herscovics, N. (1988). Teaching algebraic expressions in a meaningful way. In F. Coxford & A. P. Shulte (Eds.), The ideas of algebra, K-12 (pp. 33-42). Reston, Virginia: National Council of Teachers of Mathematics.

Chi, M. T. H., & Ceci, S. J. (1987). Content knowledge: Its role, representation and restructuring in memory development. Advances in Child Development and Behavior, 20, 91-142.

Chi, M. T. H., Feltovich, P. J., & Glaser, R. (1981). Categorization and representation of physics problems by experts and novices. Cognitive Science, 5, 121-152.

Clement, J. (1988). Observed methods for generating analogies in scientific problem solving. Cognitive Science, 12, 563-586.

Collins, A., & Gentner, D. (1987). How people construct mental models. In N. Quinn & D. Holland (Eds.), Cultural models in language and thought (pp. 243-265). Cambridge, UK: Cambridge University Press.

Crisafi, M. A., & Brown, A. L. (1986). Analogical transfer in very young children: Combining two separately learned solutions to reach a goal. Child Development, 57 (4), 953-968.

Davis, R. B. (1984). Learning mathematics: The cognitive science approach to mathematics education. London: Croom Helm.

Davis, R. B., C. A. Maher, & N. Noddings (1990). (Eds.). Constructivist views on the teaching and learning of mathematics. Reston, VA: National Council of Teachers of Mathematics.

Dienes, Z. P. (1960). Building up mathematics. London: Hutchinson Education.

Duit, R. (1991). On the role of analogies and metaphors in learning science. Science Education, 75 (6), 649-672.

English, L. D., & Halford, G. S. (forthcoming). Mathematics education: Models and processes. Hillsdale, NJ: Lawrence Erlbaum Associates.

Fuson, K. C. (1992). Research on learning and teaching addition and subtraction of whole numbers. In G. Leinhardt, R. T. Putnam, & R. Hattrup (Eds.), Analysis of arithmetic for mathematics teaching. Hillsdale, NJ: Lawrence Erlbaum.

Fuson, K. C., & Briars, D. J. (1990). Using a base-ten blocks learning/teaching approach for first- and second-grade place-value and multidigit addition and subtraction. Journal for Research in Mathematics Education, 21, 180-206.

Fuson, K. C., Fraivillig, J. L., & Burghardt, B. H. (1992). Relationships children construct among English number words, multiunit base-ten blocks, and written multidigit addition. In J. I. D. Campbell (Ed.), The nature and origins of mathematical skills. Amsterdam: Elsevier Science Publishers.

Gelman, R., & Meck, E. (1986). The notion of principle: The case of counting. In J. Hiebert (Ed.), The relationship between procedural and conceptual competence (pp. 29-57). Hillsdale, NJ: Lawrence Erlbaum Associates.

Gentner, D. (1982). Are scientific analogies metaphors? In D. S. Miall (Ed.), Metaphor: Problems and perspectives (pp. 106-132). Atlantic Highlands, NJ: Humanities Press.

Gentner, D. (1983). Structure mapping: A theoretical framework for analogy. Cognitive Science, 7, 155-170.

Gentner, D. (1989). The mechanisms of analogical learning. In S. Vosniadou & Ortony A. (Eds.), Similarity and analogical reasoning (pp. 199-241). Cambridge: Cambridge University Press.

Gentner, D., & Gentner, D. R. (1983). Flowing waters or teaming crowd: Mental models of electricity. In D. Gentner & A. L. Stevens (Eds.), Mental models (pp. 99-129). Hillsdale, NJ: Lawrence Erlbaum Associates.

Gentner, D., & Landers, R. (1985, November). Analogical reminding: A good match is hard to find. Proceedings of the International Conference on Systems, Tucson, AZ.

Gentner, D., & Toupin, C. (1986). Systematicity and surface similarity in the development of analogy. Cognitive Science, 10, 277-300.

Gholson, B., Dattel, A. R., Morgan, D., & Eymard, L. A. (1989). Problem solving, recall, and mapping relations in isomorphic transfer and non-isomorphic transfer among preschoolers and elementary school children. Child Development, 60 (5), 1172-1187.

Gholson, B., Eymard, L. A., Long, D., Morgan, D., & Leeming, F. C. (1988). Problem solving, recall, isomorphic transfer, and nonisomorphic transfer among third-grade and fourth-grade children. Child Development, 3, 37-53.

Gholson, B., Morgan, D., Dattel, A. R., & Pierce, K. A. (1990). The development of analogical problem solving: Strategic processes in schema acquisition and transfer. In D. F. Bjorklund (Ed.), Children's strategies: Contemporary views of cognitive development (pp. 269-308). Hillsdale, NJ: Lawrence Erlbaum.

Gilbert, R. K., & Bush, W. S. (1988). Familiarity, availability, and use of manipulative devices in mathematics at the primary level. School Science and Mathematics, 88 (6), 459-469.

Goswami, U. (1991). Analogical reasoning: What develops? Child Development, 62 (1), 1-22.

Grover, B., Hojnacki, S. K., Paulson, D., & Matern, C. (in press). Legs + heads + 1,000,000,000 floor tiles = thinking mathematics with learning disabled students. In C. A. Thornton (Ed.), Windows of opportunity: Mathematics for students with special needs. Reston, VA: National Council of Teachers of Mathematics.

Halford, G. S. (1992). Analogical reasoning and conceptual complexity in cognitive development. Human Development, 35 (4), 193-217.

Halford, G. S. (1993). Children's understanding: The development of mental models. Hillsdale, NJ: Lawrence Erlbaum Associates.

Halford, G. S., & Boulton-Lewis, G. M. (1992). Value and limitations of analogs in mathematics teaching. In A. Demetriou, A. Efkliades, & M. Shayer (Eds.), The modern theories of cognitive development go to school (pp. 183-209). London: Routledge.

Halford, G. S., & Wilson, W. H. (1993). Creativity and capacity for representation: Why are humans so creative? Newsletter of the Society for the Study of Artificial Intelligence and Simulation of Behaviour, Special Theme: AI and Creativity, 85, 32-41.

Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. In S. Wagner & C. Kieran (Eds.), Research issues in the learning and teaching of algebra (pp. 60-86). Reston, VA: National Council of Teachers of Mathematics; Hi'lsdale, NJ: Lawrence Erlbaum Associates.

Hiebert, J. (1992). Mathematical, cognitive, and instructional analyses of decimal fractions. In G. Leinhardt, R. T. Putnam, & R. Hattrup (Eds.), Analysis of arithmetic for mathematics teaching (pp. 283-322). Hillsdale, NJ: Lawrence Erlbaum.

Hiebert, J., & Wearne, D. (1992). Links between teaching and learning place value with understanding in first grade. Journal for Research in Mathematics Education, 23, (2), 98-122.

Hiebert, J., Wearne, D., & Taber, S. (1991). Fourth graders' gradual construction of decimal fractions during instruction using different physical representations. The Elementary School Journal, 91, (4), 321-341.

Holland, J. H., Holyoak, K. J., Nisbett, R. E., & Thagard, P. R. (1986). Induction: Processes of inference, learning and discovery. Cambridge, MA: Bradford Books/MIT Press.

Holyoak, K. J. (1985). The pragmatics of analogical transfer. In G. H. Bower (Ed.), The psychology of learning and motivation (vol. 19, pp. 59-87). San Diego, CA: Academic Press.

Holyoak, K. J., & Koh, K. (1987). Surface and structural similarity in analogical transfer. Memory & Cognition, 15, 332-340.

Holyoak, K. J., & Thagard, P. (1993). Mental Leaps. Cambridge, MA: MIT Press.

Holyoak, K. J., Junn, E. N., & Billman, D. O. (1984). Development of analogical problem-solving skill. Child Development, 55, 2042-2055.

Hope, J. A., & Owens, D.T. (1987). An analysis of the difficulty of learning fractions. Focus on Learning Problems in Mathematics, 9 (4), 25-40.

Johnson-Laird, P. N. (1983). Mental models. Cambridge, UK: Cambrige University Press.

Kaput, J. J. (1987). Representation systems and mathematics. In C. Janvier (Ed.), Problems of representation in the teaching and learning of mathematics (pp. 19-26). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kennedy, L. M. (1986). A rationale. Arithmetic Teacher, 33 (6), 6-7, 32.

Kieran, C. (1990). Cognitive processes involved in learning school algebra. Mathematics and cognition: A resc ·ch synthesis by the International Group for the Psychology of Mathematics Education (pp. 96-112). Cambridge: Cambridge University Press.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws, (Ed.), Handbook of research on mathematics teaching and learning (pp. 390-419). New York: Macmillan Publishing Co.

Labinowicz, E. (1985). Learning from children: New beginnings for teaching numerical thinking. Menlo Park, CA: Addison-Wesley.

Larson, C. N. (1980). Locating proper fractions on number lines: Effect of length and equivalence. School Science and Mathematics, 80, 423-428.

LeBlanc, J. F. (1976). (Ed.). Numeration. Massachusetts, CA: Addison-Wesley Publishing Company.

Leinhardt, G. (1987). The development of an expert explanation: An analysis of a sequence of subtraction lessons. Cognition and Instruction, 4, 225-282.

Matz, M. (1979). Towards a process model for high school algebra errors. (Working Paper 181). Cambridge: Massachusetts Institute of Technology, Artificial Intelligence Laboratory.

Matz, M. (1982). Towards a process model for high school algebra errors. In D. Sleeman & J. S. Brown (Eds.), Intelligent tutoring systems (pp. 25-50). New York: Academic Press.

McCoy, L. P. (1990). Correlates of mathematics anxiety. Paper presented at the Annual Meeting of the American Educational Research Association, April, Boston.

National Council of Teachers of Mathematics (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: National Council of Teachers of Mathematics.

Novick, L. R. (1988). Analogical transfer, problem similarity, and expertise. Journal of Experimental Psychology: Learning, Memory, and Cognition, 14 (3), 510-520.

Novick, L. R. (1990). Representational transfer in problem solving. Psychological Science, 1, (2), 128-1332.

Novick, L. R. (1992). The role of expertise in solving arithmetic and algebra word problems by analogy (pp. 155-188). In J. I. D. Campbell (Ed.), The nature and origins of mathematical skills. Amsterdam: Elsevier Science Publishers.

Novick, L. R., & Holyoak, K. J. (1991). Mathematical problem solving by analogy. Journal of Experimental Psychology: Learning, Memory, and Cognition, 17, 398-415.

Novillis, C. G. (1976). An analysis of the fraction concept into a hierarchy of selected subconcepts and the testing of the hierarchical dependencies. Journal for Research in Mathematics Education, 7, 131-144.

Polya, G. (1954). Mathematics and plausible reasoning, vol.1: Induction and Analogy in mathematics. Princeton, NJ: Princeton University Press.

Reed, S. (1987). A structure-mapping model for word problems. Journal of Experimental Psychology: Learning, Memory, and Cognition, 13, 124-139.

Resnick, L. B., & Omanson, S. F. (1987). Learning to understand arithmetic. In R. Glaser (Ed.), Advances in instructional psychology (Vol.3). Hillsdale, NJ: Lawrence Erlbaum Associates.

Ross, B. H. (1984). Remindings and their effects in learning a cognitive skill. Cognitive Psychology, 16, 371-416.

Ross, B. H. (1987). This is like that: The use of earlier problems and the separation of similarity effects. Journal of Experimental Psychology: Learning, Memory, and Cognition, 13, 629-639.

Rouse, W. B., & Morris, N. M. (1986). On looking into the black box: Prospects and limits in the search for mental models. Psychological Bulletin, 100 (3), 349-363.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws, (Ed.), Handbook of research on mathematics teaching and learning (pp. 334-370). New York: Macmillan Publishing Co.

Silver, E. A. (1981). Recall of mathematical problem formulation: Solving related problems. Journal for Research in Mathematics Education, 12 (1), 54-64.

Smith, L. B. (1989). From global similarities to kinds of similarities: The construction of dimensions in development. In S. Vosniadou & Ortony A. (Eds.), Similarity and analogical reasoning (pp. 146-178). Cambridge: Cambridge University Press.

Sowder, J. T. (1989). Setting a research agenda, vol. 5. Hillsdale, NJ: Lawrence Erlbaum Associates; Reston, VA: National Council of Teachers of Mathematics.

Stigler, J. W. & Baranes, R. (1988). Culture and mathematics learning. In E. Z. Rothkopf (Ed.). Review of research in education (pp. 253-306). Washington, D.C.: American Educational Research Association.

Thompson, P. W. (1992). Notations, conventions, and constraints: Contributions to effective uses of concrete materials in elementary mathematics. Journal for Research in Mathematics Education, 23 (2), 123-147.

Vergnaud, G. (1984). Understanding mathematics at the secondary-school level. In A. Bell, B. Low, & J. Kilpatrick (Eds.), Theory, research and practice in mathematical education (Report of ICME 5 Working Group on Research in Mathematics Education, pp. 27-35). Nottingham, UK: Shell Centre for Mathematical Education.

von Glasersfeld, E. (1990). Environment and communication. In L. P. Steffe & T. Wood (Eds.), Transforming children's mathematics education: International perspectives (pp. 30-38). Hillsdale, NJ: Lawrence Erlbaum Associates.

Wearne, D., & Hiebert, J. (1985). Teaching for thinking in mathematics. Curriculum Review, 25, (1), 65-68.

Wearne, D., & Hiebert, J. (1988). A cognitive approach to meaningful mathematics instruction: Testing a local theory using decimal numbers. Journal for Research in Mathematics Education, 19 (5), 371-384.

Author Notes

Dr Lyn English

Associate Professor of Mathematics Education

Centre for Mathematics and Science Education

Queensland University of Technology

Locked Bag #2

Red Hill

Brisbane

Australia, 4059.

Figure Captions

Figure 1. An example of a simple analogy

Figure 2. An example of a mathematical analog: the base-ten blocks

Figure 3. A structure-mapping diagram of analogical problem solving

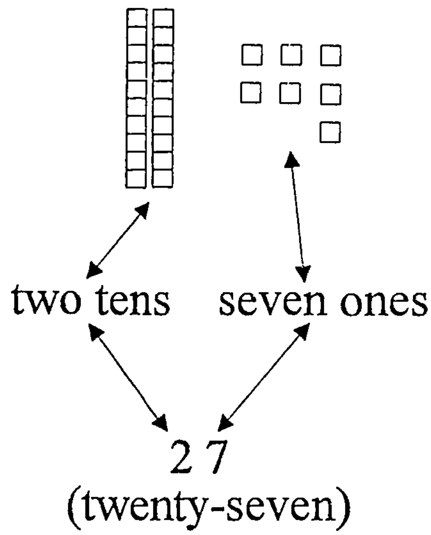
Figure 4. Lack of clarity of structure and of mapping in the number line analog

Figure 5. Complexity of the number line analog in representing fractions

Figure 6. The base-ten blocks as an analog for decimal fraction ideas

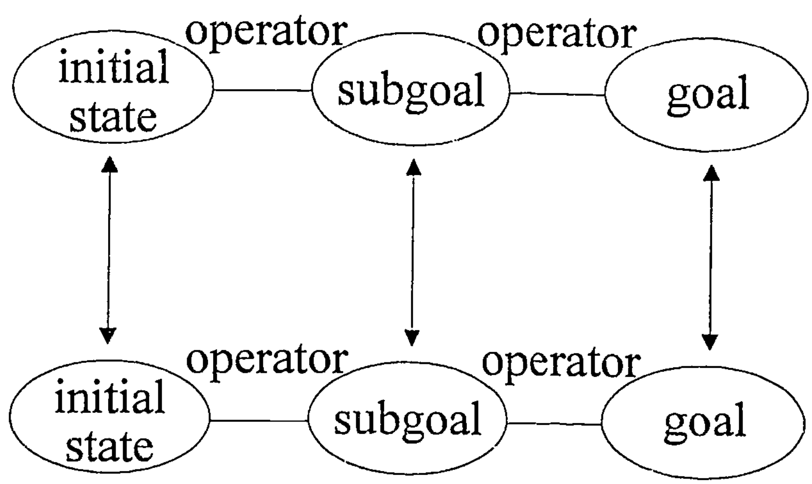
Figure 7. Mapping a metric measurement onto a model of decimal number positional relationships

Source	dog	parent of	pup
Target	cow	parent of	calf

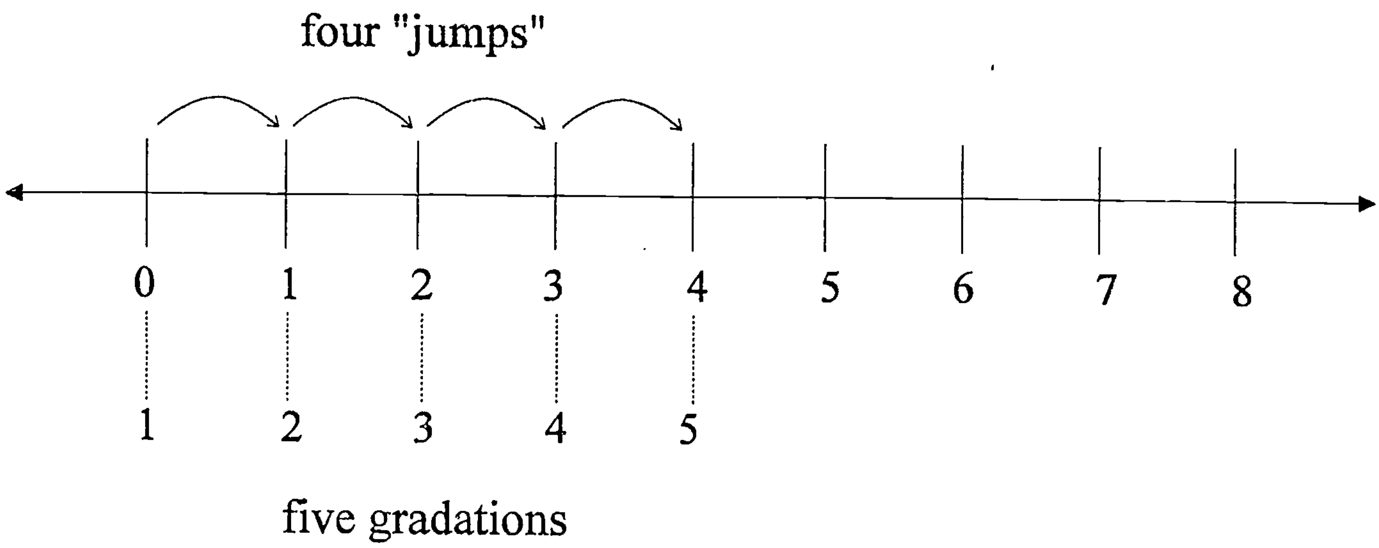

two tens
seven ones
2 7
(twenty-seven)

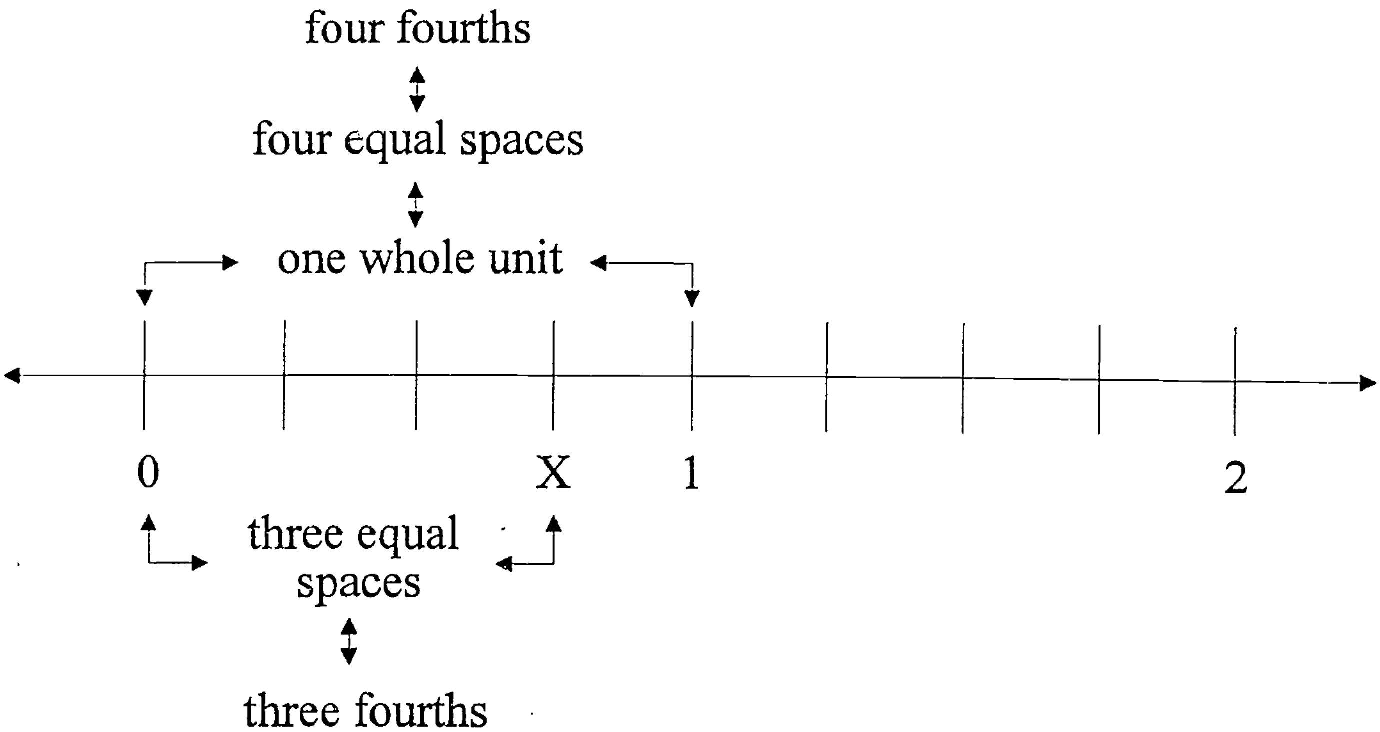
Familiar problem (source)

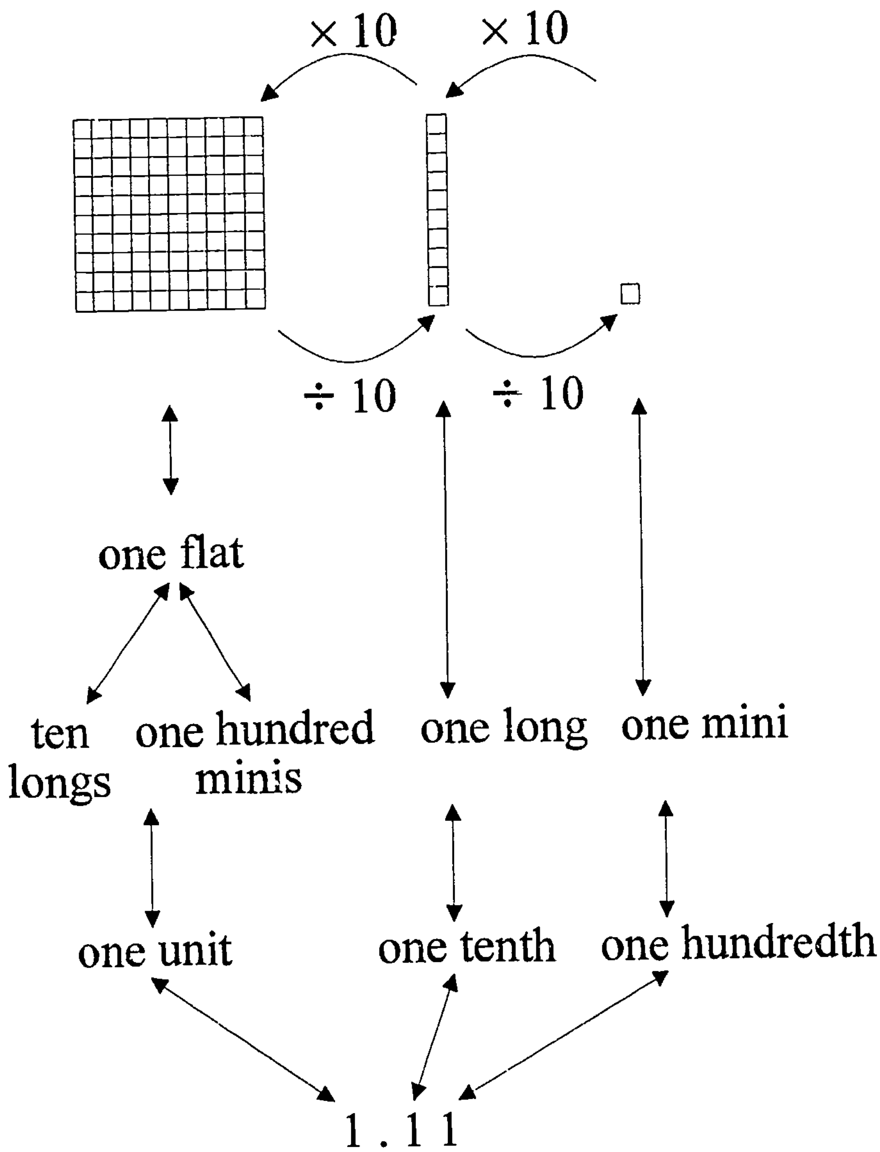
Unfamiliar problem (target)



(Halford, 1993, p.213)


four "jumps"
0 1 2 3 4 5 6 7 8
1 2 3 4 5
five gradations


four fourths
four equal spaces
one whole unit
0
X
1
2
three equal spaces
three fourths


× 10
× 10
÷ 10
÷ 10
one flat
ten longs
one hundred minis
one long
one mini
one unit
one tenth
one hundredth
1 . 1 1

Tens	Ones or Units	tenths	hundredths	thousandths
	↕ meters		↕ centimeters	↕ millimeters

5 . 2

five meters and twenty centimeters
five meters and two hundred millimeters