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ABSTRACT

The importance accorded mathematical connections in the professional literature is not properly reflected in the relatively small number of empirical investigations of students' mathematical connections. The purpose of this study was to investigate the mathematical connections that students form and use in solving nonroutine problems. Two main types of mathematical connections are: internal (across mathematical topics) and external (between mathematics and its applications in other fields or in the real world). Tenth-grade students (n=17) were interviewed while solving one of two nonroutine problems, one involving algebra and internal connections, the other involving geometry and external connections. The two problems elicited a wide variety of responses and seemed to require students to operate metacognitively, to recognize the purposes for the mathematical tools they study, and to identify subgoals along the road to a solution, and therefore seem to be a worthwhile means of exploring students' abilities with mathematical connections. (MKR)

Research Symposium:

Mathematical Connections: Instances from Research

Mathematical Connections:

Two Cases from an Evaluation of Students' Mathematical Problem Solving

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Mathematical Connections:

Two Cases from an Evaluation of Students' Mathematical Problem Solving

"Mathematical Connections" is one of four main themes running through the National Council of Teachers of Mathematics (NCTM) *Curriculum Standards* (NCTM, 1989), yet a cursory review of mathematics education research does not reveal a great deal of research that seems to be specifically focussed on students' abilities to form mathematical connections or on teachers' instructional activities designed to promote mathematical connections. Part of the reason for the apparent lack of research may be that what disciplined inquiry there is in this area is found in reports of evaluation or assessment projects, rather than in work identified as research. If so, students' abilities with connections may be only one of a large number of valued outcomes, and hence not prominent in the report of findings. In any case, the importance given to mathematical connections in the professional literature does not seem to be matched by reports of empirical investigations of mathematical connections. The purpose of this paper is to present and discuss some findings of an evaluation project focussing on qualities of students' mathematical problem solving (Schroeder, 1992b) that bear on the question, "What abilities to form and to use mathematical connections do students demonstrate when solving non-routine mathematical problems?"

The notion of mathematical connections in the *Curriculum and Evaluation Standards for School Mathematics* is introduced as follows:

> The fourth curriculum standard at each level is titled Mathematical Connections. This label emphasizes our belief that although it is often necessary to teach specific concepts and procedures, mathematics must be approached as a whole. Concepts, procedures and intellectual processes are interrelated. In an important sense, "the whole is greater than the sum of its parts." Thus the curriculum should include deliberate attempts, through specific instructional activities, to connect ideas and procedures both *among different mathematical topics* and *with other content areas* [emphasis added]. (NCTM, 1989, p. 11)

Later in the document the notion is elaborated with examples of questions and activities for students in each of three grade ranges: K-4, 5-8, 9-12. Curiously, there is no mention in the evaluation standards of assessment and evaluation of mathematical connections, other than a brief reference under "mathematical disposition" to seeking information about students' "valuing of the application of mathematics to situations arising in other disciplines and everyday experiences" (NCTM, 1989, p. 233). One way of conceiving the two main types of mathematical connections mentioned in the quotation above would be to call them either *internal* connections (i.e. within mathematics, among mathematical topics) or *external* connections (i.e. connecting mathematics with its uses and applications in other fields or in the "real world"). In this paper, some results are reported of an evaluation of Grade 10 students' solving of two non-routine mathematics problems, one involving internal connections, the other involving external connections.

The purpose of the original study was to provide a qualitative assessment of students' problem solving focussing on the nature of students' thinking, their problem-solving strategies and heuristics, the mathematical approaches that they selected, and the ways they monitored their progress. In this paper certain findings of the project are reconsidered in terms of their implications regarding students' abilities to identify and use mathematical connections of both types. The data of the study include interviewers' field notes and students' written responses to non-routine problems presented in task-based interviews with individual students or pairs of students. A wide range of apparatus, tools, and supplies were provided for students to use in solving the problems. Since the particular items at hand might influence students' perception of the nature of the problem, and since some tools might facilitate or enhance the students' performance, care was taken to provide items that could support the students' efforts (e.g. calculators, geometry sets, squared paper, etc.). There was no assumption that students should use any particular ones of the tools available, but evidence from the interviews suggests that some students took the presence of some items as hints about how to solve the problems.

Before conducting the interviews, the project team produced an evaluator's guide describing typical responses that students might be expected to give in solving each problem. This was done to provide background for the interviewers and to prepare them by sensitizing them to the range of approaches that students might take in solving the problems. Particular behaviors that might be observed, as well as variations of the main approaches that could be anticipated, were discussed. Interviewers were cautioned not to assume that these would be the only possibilities and to remember that students' responses might include features of several of the basic approaches discussed.

The evaluator's guide also suggested a number of possible interventions on the part of the interviewer. Hints and follow-up questions that the interviewer might wish to use, their purposes, and the circumstances under which it might be appropriate to use them were discussed. The overall purposes of the follow-up questions and hints were to help students to demonstrate all that they are capable of doing, to enable interviewers to understand more fully the nature of the students' thinking, and, in some cases, to judge the students' abilities to make use of information provided by the interviewer. In particular, some follow-up questions were developed that would assess whether students could make use of mathematical connections for which the question provides a hint.

For each problem a data collection form, designed to be a convenient and standardized way of summarizing and reporting the students' work on the problem, was developed. It was to be completed following the interview on the basis of notes taken by the interviewer and the students' written work. The record sheet was intended to guide the interviewer in reflecting on the interview, drawing conclusions from it, and reporting observations in an informative and systematic way.

Case 1: A Problem Involving *Internal* Connections

The first problem, called the page number problem, was presented in written form:

> A textbook is opened at random. The product of the numbers of the facing pages is 3192. To what pages is the book opened?

Students were provided a sheet on which the problem was printed, a 5-function (+, −, ×, ÷, √) or scientific calculator, and additional plain paper for figuring.

Anticipated Solutions

Four fundamentally different methods or approaches for solving this problem were anticipated: a *guess-and-test* or *successive-approximations* approach, a *factorization* approach, an *algebraic* method, or a method using the *square root* operation. The guess-and-test or successive-approximations method involves estimating approximately how large the two page numbers must be and then multiplying them to see whether the estimate is too high or too low. Using this approach, the problem-solver may try to establish a range within which the answer lies. For example, noting that 10 x 11 = 110 and that 98 x 99 = 9702 shows that the page numbers must be two-digit numbers. Noting that 50 x 50 = 2500 and that 60 x 60 = 3600 shows that the page numbers must be in the fifties. Further analysis of the second approximation suggests that the two desired numbers are in the middle fifties.

The factorization approach is based on the idea that the two factors of 3192 that are to be found may be determined by breaking 3192 down (perhaps all the way to its prime factors) and reassembling the pieces into two factors that are consecutive whole numbers. If the problem solver adopting this approach happens to factor 3192 as 3192 = 2 x 2 x 2 x 7 x 57, it is fairly easy to see that the two desired factors are 56 and 57. The factorization 3192 = 2 x 2 x 2 x 3 x 133 or the factorization to primes, 3192 = 2 x 2 x 2 x 3 x 7 x 19, are not so helpful, but if they are combined with guessing and testing, there may only be a few possibilities to test.

The algebraic approach entails assigning a variable to one of the two unknown page numbers, writing the other page number in terms of the variable, expressing the condition stated in the problem as an equation, solving the equation, and interpreting the algebraic solution in terms of the problem. For example, if x is one page number then (x + 1) could be the other. When their product x (x + 1) is set equal to 3192, the solution process proceeds as follows.

$$x(x+1) = 3192, \quad \text{so} \quad x^2 + x = 3192, \quad \text{and} \quad x^2 + x - 3192 = 0$$

This quadratic equation could be solved by factoring if the problem-solver knew two integers whose difference is 1 and whose product is 3192, but that is just another way of restating the problem, and the two numbers that satisfy both these conditions are not obvious. However, reaching this point may suggest to the problem solver the factorization method, which can be used directly as described above, or as a way of solving the quadratic by factorizing it to (x + 57) (x – 56). If the problem solver decides that factorization is not feasible, the quadratic formula may be applied with a = 1, b = 1, and c = -3192. This gives

$$x = \frac{-1 \pm \sqrt{1 - (4)\bullet(-3192)}}{2} = \frac{-1 \pm \sqrt{12769}}{2} = \frac{-1 \pm 113}{2} = 56, -57$$

Since page numbers in books are not negative numbers, the required solution is x = 56 and (x + 1) = 57.

The square root method is based on a recognition of the fact that the square root of 3192 will give the value of two equal numbers whose product is 3192. Thus, the two consecutive whole numbers whose product is 3192 will be the whole numbers that lie on either side of $\sqrt{3192}$ = 56.4977… , i. e. 56 and 57. This method seems to be the most direct and efficient one, but to think of applying the square root concept in this way requires insight or inspiration. That is, the problem solver must see a mathematical connection, and exploit it.

Hints and Follow-up Questions

A number of hints and follow-up questions were prepared for different situations that might arise in the course of the interviews. Some were designed to help students having trouble getting started with the problem, or to check that they understood that the term "product" implies multiplication. Other questions were designed to suggest an approach to students or even to steer them in the direction of a particular approach such as guess-and-test or estimation, or an algebraic strategy.

Questions specifically related to the square root approach, most likely to be used after the student had found the solution by another approach, were also developed, some before the interviews were conducted, others as a result of that experience. The intention of these questions is to explore whether the student is able to understand the use of the square root operation to solve the problem directly, even if he or she does not think of using it on his or her own. These questions included, "If you had known the value of the square root of 3192, would that have helped you to solve the problem? How?" and "Do you expect the square root of 3192 to be a whole number?" If the student did not seem to see the connection being hinted at, the interviewer might suggest that he or she find $\sqrt{3192}$ using the calculator, and then ask, "Can you explain why the square root of 3192 falls right between the two numbers that answer the problem?"

Results

Seventeen volunteers (9 females and 8 males) took part in interviews based on this problem, fifteen were interviewed individually, the other two (both males) were interviewed as they worked together. In the discussion which follows, the unit of analysis is the interview, of which there were 16. The overall results showed that in eight of the interviews (50%) the students solved the problem on their own, and that in eight interviews (50%) the students solved the problem with help from the interviewer; in none of the interviews did the students fail to solve the problem.

The total length of time spent in each interview ranged from 12 to 45 minutes with a median of 27 and a mean of 26, and the time taken to reach a solution ranged from 2 to 32 minutes with a median and a mean of 13. Interviews in which the students solved the problem on their own were noticeably shorter in time to solution than were the interviews in which students received help from the interviewer. The median time to solution for students who solved the problem on their own was 7 minutes as opposed to 15 minutes for students who received help; the means were 6 minutes and 19 minutes respectively. In fact, the two groups' distributions of time to solution did not overlap. However, the average overall lengths of the interviews was about the same for both groups. All these time measures include about three minutes spent exchanging introductions, recording facts such as names and birthdates, summarizing interview procedures, and obtaining students' consent to participate. An additional two or three minutes was taken showing students the photographs and presenting the problem orally. Although these gross measures give a sense of the extent of the interviews and an idea of how well the students performed, they were not the focus of the analysis; the qualities of students' work was the main concern.

The initial approach adopted by most of the students was algebraic; in eleven interviews (69%) students began by writing expressions or equations to represent the situation. In a few cases, the students used two variables for the two page numbers, but all of them soon switched over to single-variable expressions. In six of the interviews (38%) the initial algebraic expression modelled the sum, rather than the product, of the page numbers, and in most of these cases the error was not corrected until after the student obtained a decimal value for one or both of the page numbers. In seven of the interviews (44%) students made errors transforming equations involving "$x(x + 1)$" or "$x \times x + 1$.

In twelve of the interviews (75%) the students sooner or later (and sometimes with help from the interviewer) produced a correct quadratic equation to represent the situation, i. e. $x^2 + x = 3192$ or $x^2 + x - 3192 = 0$. However, none of the students found their first solution to the problem by solving the quadratic. Several students commented that they had studied "equations like these" (i.e. quadratic equations) quite a bit earlier in the school year, but only two (13%) were able to recall how to solve the quadratic later in their interview.

Having tried to use algebra to solve the problem and having gotten stuck, two students tried a guess and test approach, as did three other students for whom guess and test was their initial approach. One student, asked if he could think of another way to solve the problem, suggested factoring. His approach involved identifying successive small numbers that are factors of 3192 and also listing the corresponding other factors. Before abandoning this approach the student had constructed the following set of factors of 3192: {1, 2, 3, 4, 6, 7, ..., 456, 532, 768 [incorrect, should be 798], 1064, 1596, 3192}. He gave up on this approach with the comment that it might take the whole day!

For the majority of students (67%) it was finding the square root that led to the solution. In one case, the student applied this function to 3191 rather than 3192, having incorrectly moved from "$x (x + 1) = 3192$" to "$x^2 + 1 = 3192$" to "$x^2 = 3191$" to "$x = \sqrt{3191}$." In only three of these cases did the interviewer suggest using the square root; in the rest of the interviews the students came up with the idea on their own. But in most of these cases the students checked out their "guesses" of the page numbers (after taking the square root) by multiplying them. Having "rounded" the square root 56.4977 to 56, several of the students were unsure whether the other page number would be 55 or 57 and just under half of them tried 55 x 56 before trying 56 x 57. These facts suggest that the students who used the square root did so on a hunch, rather than understanding the connection between "two nearly equal numbers whose product is 3196" and "two identical numbers whose product is 3196."

Discussion

Although this problem may be seen as involving external connections to the extent that students to represent the "real world" page numbers as mathematical expressions such as x and (x + 1) and so on, the most interesting aspects of this problem have to do with students' abilities to exploit or at least to understand the internal mathematical connections between one or more of the approaches identified in advance. In the following paragraphs we consider the evidence from the interviews that bears on the question of how and how well students developed, understood, or used mathematical connections.

It was anticipated that the connection between multiplication and factorization (as inverse processes) might be used by the students, but in fact it was evident in the work of only two interviews (13%). In one, the student came up with the idea of factoring on his own, and systematically carried out his plan of writing all the factors of 3192 in order. He persisted in this plan for some time, but gave up after listing factors from 1 up to 7 and their counterparts from 3192 down to 456. The decision to abandon this method was based on an assessment that it would take too long. Another student, having found the solution initially by guess and test, and having been asked if she could do it another way, was also asked whether she could "find out what numbers make 2500," since she had just used 2500 and 3600 to establish the interval in which to search. The student replied confidently that she could make a factor tree, and proceeded to do so. Then the interviewer asked, "could you do the same with 3192?" This hint was greeted with the exclamation, "Oh yes, my God!" (an expression of "Aha!" in the local dialect). Having completed the factorization of 3192 to primes, the student commented, "I should be able to extract 56 x

57." But only a short while later this confidence seemed to vanish as the student reported, "I don't know why I'm doing this." These two instances both suggest that the students at least understood, and perhaps could also develop for themselves, helpful mathematical connections. In the latter case, at least, the strength and stability of the student's insight seemed to be undermined by the need to wade through the details.

Possible connections between the quadratic equation in the form $x^2 + x = 3192$ or $x^2 + x - 3192 = 0$ and other mathematical ideas can also be seen. Although most of the students had not had sufficient experience working with quadratic equations to be fluent and confident using them, three students (19%) did take some halting steps in that direction. In two interviews, the students wrote the form "$x^2 + x - 3192 = 0$" and underneath it "$(x - \_)(x + \_) = 0$," and in both cases the solution had already been found by another method. In one of these cases the student recognized that the previously found solution $x = 56$ meant that $(x - 56)$ was a factor of the trinomial (thinking backward?), but the other student did not see the connection and commented, "I've forgotten how to do this."

The connections from "two consecutive numbers whose product is 3196" to "two nearly equal numbers whose product is 3196" to "two identical numbers whose product is 3196" to "the square root of 3196" are ones that are quite useful in this problem. Of course, the connection need not be elaborated in this way or to this extent; a person who "sees" this connection may experience it as an intuition or a flash of insight, and that appears to be what happened for more than half the students interviewed. An important aspect of this connection has to do with the fact that because the square root of 3192 is between 56 and 57, these two numbers must be the factors. Quite a few students who found the square root of 3192 or of 3191 followed up with a "guess" of the two page numbers, one being the root "rounded off" to the nearest whole number, and the other being the next number up or down. In actuality, this may not be so much a case of intuition as of elimination, since the problem contains only one obvious bit or numerical data, the 3192, and the calculator has only one readily available unary (i.e. single input) operation, the square root function.

Case 2: A Problem Involving *External* Connections

A second problem, called the dock problem, had content related to the Pythagorean theorem. This task was presented to students orally by an interviewer who explained the problem situation with the aid of photographs. Students were told that the waterfront restaurant shown in Figure 1 has a dock, one part of which rises and falls with the tide. Access from the fixed part of the dock to the floating part is by means of a ramp which is relatively steep at low tide (Figure 1a), but less steep at high tide (Figure 1b). Currently, the lower end of the ramp rests on the floating dock (Figure 2a). As the floating dock rises and falls, the end of the ramp scrapes across it causing scratches in the surface. In order to prevent this damage, the owners of the dock plan to mount wheels on the lower end of the ramp and tracks on the floating dock for the wheels to run in – an arrangement similar to the one shown in Figure 2b, which shows another dock nearby. The problem is to determine how long the track needs to be. The data provided are that (1) the ramp is 18 m long, (2) when the tide is at its highest, the floating dock is 1 m below the fixed dock, and (3) when the tide is at its lowest, the floating dock is 6 m below the fixed dock.

Figure 1 and Figure 2 here.

Anticipated Solutions

It was anticipated that students would approach the problem by drawing a diagram and by recognizing the importance of two right triangles, both having the 18 m long ramp as the hypotenuse, one 1 m on its vertical side, the other 6 m tall. The difference between the lengths of the horizontal sides of these two triangles is the required length of the track. Thus the simplest method of solving the problem is to use the Pythagorean theorem to determine the two unknown sides and then subtract. It was anticipated that students might use other means of solving the problem such as constructing a scale drawing or applying trigonometric ratios to calculate the unknown lengths.

Results

Seventeen volunteers (11 females and 6 males) took part in interviews based on this problem; thirteen were interviewed individually, the remaining four in two same-sex pairs. The students were provided with the photographs, a summary of the data, a scientific calculator, tables of square roots and trigonometric functions, squared paper and plain paper, and a geometry set (ruler, protractor, set square, and compass). In the discussion which follows, the unit of analysis is the interview, of which there were 15. The overall results showed that in five interviews (33%) the students solved the problem on their own, and that in ten interviews (67%) the students solved the problem with help from the interviewer; in none of the interviews did the students fail to solve the problem.

The total length of time spent in each interview varied from 22 to 50 minutes with a median of 38 and a mean of 39, and the time taken to reach a solution ranged from 9 to 50 minutes with a median of 18 and a mean of 22. Interviews in which the students solved the problem on their own tended to be somewhat shorter overall and noticeably shorter in time to solution. The median time to solution for students who solved the problem on their own was 10 minutes as opposed to 24 minutes for students who received help; the means were 13 minutes and 27 minutes respectively. All these time measures include about three minutes spent exchanging introductions, recording facts such as names and birthdates, summarizing interview procedures, and obtaining students' consent to participate. An additional two or three minutes was taken showing students the photographs and presenting the problem orally. Although these gross measures give a sense of the extent of the interviews and an idea of how well the students performed, they were not the focus of the analysis; the qualities of students' work was the main concern.

It was anticipated that students would draw diagrams both as a means of understanding the problem and to facilitate their work on it. Although all students eventually solved the problem using diagrams that included two right triangles, there were wide variations in their initial drawings, some of which are shown in Figure 3. In these early diagrams different features of the problem are prominent, and in some of them critical features of the problem are misrepresented. For example, in two interviews students first represented the situation as in Figure 3a with three parallel lines. Two students initially drew diagrams similar to the one in Figure 3b with the track in the plane of the ramp rather than the plane of the floating dock. The student who drew the diagram in Figure 3c focussed on the rotation of the ramp about its upper end. As she examined the photos she rotated her pencil holding the upper end stationary just above the paper and allowing the point to trace out an arc on the paper.

Figure 3 here.

The five students who solved the problem on their own quickly produced appropriate diagrams in which the two needed right triangles were prominent. In four cases the triangles were drawn in two separate figures; in one case they were overlapping as in Figure 3f. In five of the ten interviews where the problem was solved with help from the interviewer, students produced diagrams on their own which they used to make progress toward a solution; the help they received was unrelated to representing the problem in a diagram.

In the remaining five interviews students received help that was related to representing the situation in diagrams and identifying the relevant parts of diagrams they had drawn. In one case, after the student had spent some time studying the photographs and appeared to be stuck, the interviewer suggested that it might help to draw a diagram. In two cases the students had drawn appropriate overlapping diagrams (similar to Figure 3f), but after several minutes had not made progress using them. In one of these cases the interviewer asked, "Are there any triangles in the diagram that you could use?" and in the other he said, "Would it help to draw two separate diagrams?" to which the student immediately replied, "You mean one for high tide and one for low tide?" Each of these hints led immediately to progress toward a solution. The difficulties experienced in the remaining two interviews seemed to be related to misconceptions regarding the names and relative positions of the fixed dock, the ramp, the floating dock, and the linkages between them. These students produced the initial diagrams shown in Figures 3g and 3h. Their difficulties were resolved in question and answer exchanges between the interviewer and the students which focussed the terms, the photos, the data, and the students' diagrams.

It was anticipated that students would use the Pythagorean theorem to find the horizontal dimensions of the two right triangles formed by the floating dock, the vertical, and the ramp, and in all 15 interviews students obtained a solution in this way. In 11 of the interviews (73%) students used the Pythagorean theorem without being given a hint that they should do so, and without receiving any help in applying it to the figures they had drawn. The helps and hints given in the remaining four interviews (27%) ranged from the fairly oblique, "Is there any way you could relate the side you want to the sides you know?" in one case, to the quite direct, "Would Pythagoras's theorem help?" in another. A third student wondered aloud whether the angle was a 90° angle, and was asked by the interviewer, "What if it was 90° and what if it wasn't?" to which she replied, "If it was, I could use Pythagoras." In the fourth interview, the two students had spent more than 40 minutes trying various approaches without success, when one of them asked, "What's the square root table for?" The students decided on their own that it could be a hint to use Pythagoras, and before long they reached a solution by this method.

All of the students seemed to be quite familiar with the Pythagorean theorem, although one student referred to his method as "using a theory," and another referred to it as "Mr. So-and-So's method," presumably because that teacher had taught or reviewed it. In cases where the students received hints related to the Pythagorean theorem, the hints were mostly vague questions rather than direct hints, and they concerned whether to use the Pythagorean theorem, not how to use it. There were no instances in which students made errors using the Pythagorean theorem that they did not detect and correct by themselves (e.g. failing to square or take the square root, adding the squares rather than subtracting them, making computational errors, etc.).

Application of the Pythagorean theorem was not, however, the first approach adopted in all the interviews; in seven interviews (47%) students began by using or proposing to use trigonometry. One student produced his first solution using trigonometry, but most of the students abandoned this approach either because they ran into difficulties with it or because they noticed that applying the Pythagorean theorem would be simpler. In all cases where there was time available, students who had found the solution were asked if they could solve the problem in another way. In six interviews (40%) students solved the problem using trigonometry. In two interviews the students produced a trigonometric solution without help from the interviewer, but assistance of various types was given in the other four. Two of the students commented that they were just starting to learn trigonometry in their mathematics class; they thought trigonometry could be used, but they weren't sure they could do so successfully. The fact that the students had only recently begun studying trigonometry probably accounts for the large number of students who thought of using it and for the difficulties they encountered in doing so.

Before the interviews were conducted it was anticipated that some students might use scale drawing as a means of solving the problem, and for that reason a geometry set was provided. None of the students proposed solving the problem with a scale drawing, and two students thought it would not be possible when the interviewer suggested it.

Discussion

One of the most remarkable findings of this study was the amount of time that the students spent working on the problem. By comparison with multiple-choice test items, which students are expected to answer at the rate of about one per minute, or constructed-response items, which take on the order of five minutes, this task was quite time consuming, and there is a question whether the time required is justified by the information obtained. The amount of time that the students spent is a measure of their perseverance with the task and their willingness to reflect on and extend their work. One student, when asked whether she could solve the problem in a different way, commented that solving problems in more than one way was not something that was ever done in her mathematics class.

Students solved the dock problem without help from the interviewer in only 33% of the interviews. One way of interpreting this result is to say that the problem was relatively difficult for them, but an analysis of the nature of the hints and help provided by the interviewer suggests that what they needed was not direction about what to do, but encouragement and help thinking about their plans for proceeding. The hints in the form of questions which were described earlier which resulted in progress are typical of the internal dialogue that many researchers have identified as crucial for success in problem solving. The interviews suggest that students' cognitive monitoring needs to be developed, and this problem may provide an appropriate context in which this development can take place.

Although in five of the interviews (33%) students readily drew the two right triangles that are key to solving the problem, in the majority of interviews (67%) they did not. Representing the problem situation in a useful diagram was a major source of difficulty. One reason for this may be that only one side of the two needed triangles, the sloping ramp, is concrete; the other two sides must be constructed or imagined. One of them is a vertical line extending downward from the point where the upper end of the ramp meets the fixed dock to the plane of the surface of the floating dock. In the photographs one cannot "see" this line, since it does not correspond to any structural element of the dock system; it is an imaginary line through empty space. Similarly, the horizontal side is a line in the plane of the floating dock that starts on the surface of the dock where the scratches are and that extends over the surface of the water (a fraction of a metre above it) until it intersects with the previously described vertical. The students who made initial diagrams like the one shown in Figure 3a have drawn horizontal lines representing the planes of the fixed dock and the floating dock at high tide and at low tide, but the numeric values they have written on this diagram are not measurements along these lines. Before these numeric values can be put to use, vertical lines must be added to the diagram and the measurements must be appropriately related to them. A key understanding required in order to construct an adequate representation of the problem is that the given distances below the fixed dock are measured along vertical lines, and that these verticals may be placed wherever it is convenient or necessary, even through empty space.

The student who drew the diagram shown in Figure 3c saw the sloping ramp not as the hypotenuse of a right triangle but as the radius of an arc, the path travelled by the lower end of the ramp where the wheel is to be placed. Her diagram also contains two vertical lines which could be marked to correspond to the given distances below the fixed dock, but there are no horizontal lines corresponding to the fixed dock, or the track, or the horizontal components of the ramp at high and low tides. Without such horizontal lines appropriately identified with elements of the problem situation, this diagram is not particularly helpful. If the student had added the dashed horizontal line shown in Figure 4 and had extended the two original vertical lines to meet it, she would have had a complete and useful diagram in which the lengths of the track and the horizontal components of the ramp can be found in the plane of the fixed dock rather than in the plane of the floating dock as discussed previously. However, it is probably unlikely to expect this, since the two triangles would then be "upside down," and the horizontals would not be found in the plane of the floating dock, but as projections onto the plane of the fixed dock. In one interview a diagram with two separate triangles "upside down" was drawn; overlapping diagrams like the one shown in Figure 3f were made in three of the fifteen interviews (20%).

Figure 4 here.

Conclusions and Implications

The two problems discussed in this paper with their two types of mathematical connections are quite different in the logical, conceptual, and skill demands that they make. Problems involving internal connections may be thought of as requiring understanding of mathematical concepts and procedures while those involving external connections seem to put more importance on understanding of "real world" situations and conventional ways of representing situations with diagrams and quantitative and spatial language (e.g., "the floating dock is 6 metres below the fixed dock."

Both internal and external connections seem to require students to operate metacognively, to recognize the purposes for the mathematical tools they are studying, and to identify subgoals along the road to a solution.

Although substantial variability was seen across students in the extent of their mathematical connections and the rapidity with which they were formed, it seems safe to say that many mathematical connections are not obvious to most students. Substantial amounts of time are required for students to ponder about them. Even when fairly broad hints are given, students may not catch on quickly.

The two problems used in this study elicited a wide variety of responses. Therefore they seem to be worthwhile means of exploring students' abilities with mathematical connections. For students who do not solve these problems readily, the experience of attempting them, and in particular of considering the hints and follow-up questions that have been discussed, may be powerful in stimulating the formation and the valuing of mathematical connections. Further work to develop additional appropriate tasks for assessment and for instruction would be warranted.

References

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(a) (b)

Figure 1: A waterfront restaurant and its dock at low tide and at high tide.

(a) (b)

Figure 2: The lower end of the ramp at present and as proposed.

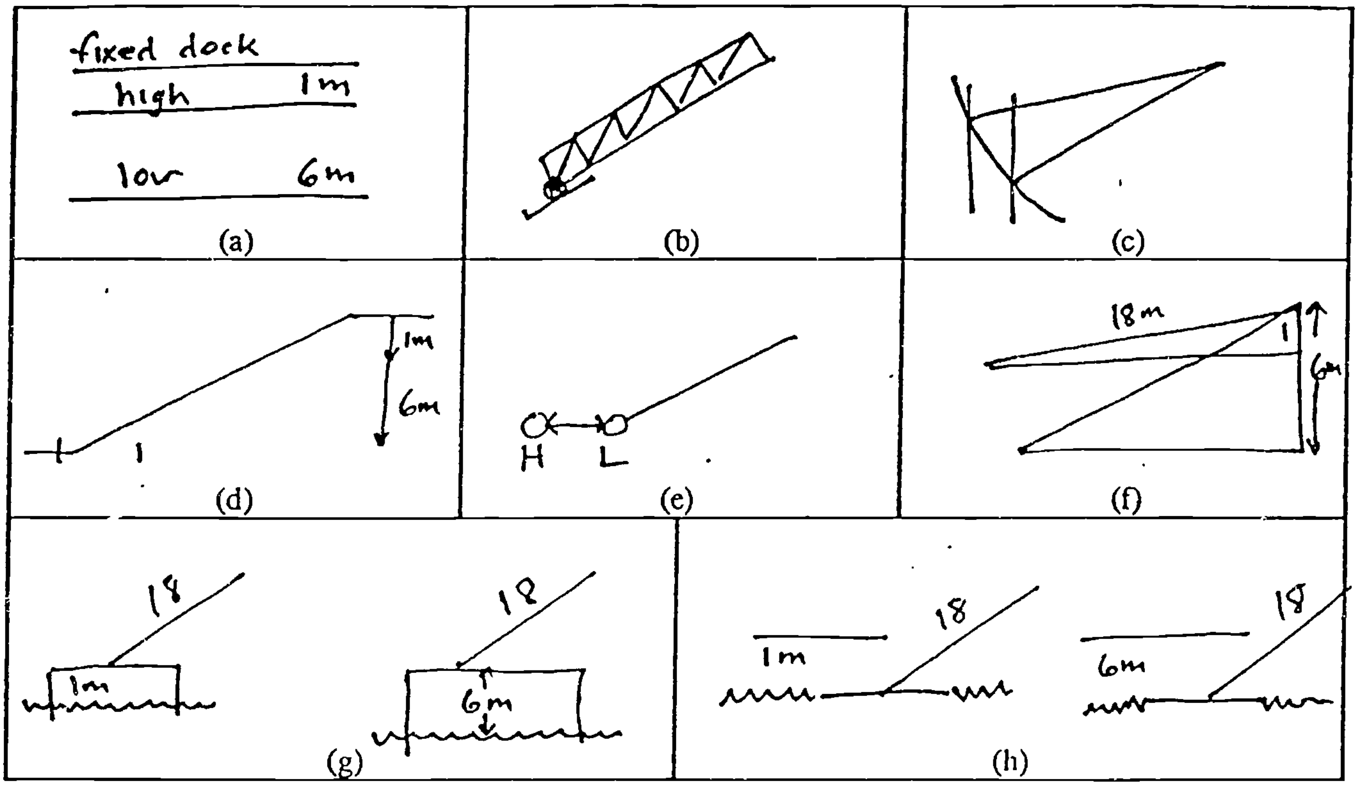

Figure 3: Initial diagrams drawn by students.

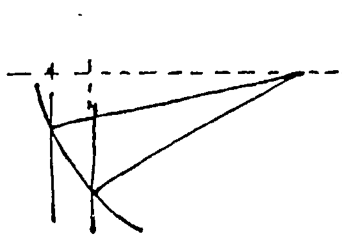

Figure 4: Modified version of diagram in Figure 3 (c)