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ABSTRACT

The focus of this report is children's construction and analogical transfer of mathematical knowledge during novel problem solving, as reflected in their strategies for dealing with isomorphic combinatorial problems presented in "hands-on" and written form. Case studies of 9-year-olds, one low and one high achieving in school mathematics, serve to illustrate a general progression through three identified stages of strategy construction (non-planning stage, transitional stage, and odometer stage). The important role of domain-general strategies in this development is highlighted. It was found that achievement level in school mathematics does not predict children's attainment of the third stage, as evidenced by the low-achieving student's construction of sophisticated combinatorial knowledge and the high-achieving student's failure to do so. Children's ability to recognize structural correspondence between two isomorphic problem sets and the extent to which this facilitates problem solution are also reported. The study concludes that: (1) Children can construct important mathematical ideas through solving novel problems; (2) Level of achievement in school mathematics is not a reliable predictor of ability to solve novel problems; (3) Bright students' ability to generate ideas for themselves can be inhibited by formal mathematical rules; and (4) Assessment of students' mathematical competence must include a range of novel problems. (Contains 72 references.) (MDH)

CHILDREN'S CONSTRUCTION OF MATHEMATICAL KNOWLEDGE IN SOLVING NOVEL ISOMORPHIC PROBLEMS IN CONCRETE AND WRITTEN FORM

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RUNNING HEAD: Children's Construction of Mathematical Knowledge

Abstract

The focus of this report is children's construction and analogical transfer of mathematical knowledge during novel problem solving, as reflected in their strategies for dealing with isomorphic combinatorial problems presented in "hands-on" and written form. Case studies of low and high achieving 9 year-olds in school mathematics serve to illustrate a general progression through three identified stages of strategy construction. The important role of domain-general strategies in this development is highlighted. Included in the study's findings is the fact that achievement level in school mathematics does not predict children's attainment of the third stage, as is evidenced by the low-achieving student's construction of sophisticated combinatorial knowledge and the high-achieving student's failure to do so. Children's ability to recognize structural correspondence between the two isomorphic problem sets and the extent to which this facilitates problem solution are also reported.

CHILDREN'S CONSTRUCTION OF MATHEMATICAL KNOWLEDGE IN SOLVING NOVEL ISOMORPHIC PROBLEMS IN CONCRETE AND WRITTEN FORM

While the mathematics education community has espoused the importance of developing children's problem solving and reasoning skills (e.g., National Council of Teachers of Mathematics, 1989), studies on children's competence in solving novel mathematical problems have not been prolific. The bulk of the research in this field has examined children's skills in solving routine arithmetic problems (e.g. Bisanz & Lefevre, 1990; Carpenter, Moser, & Bebout, 1988; Hamann & Ashcraft, 1985; Siegler & Jenkins, 1989). These studies have shed considerable light on children's strategy discovery and generalization in numerical operations where accuracy, speed, and retrieval have been of prime concern (e.g. Siegler & Jenkins, 1989). Considerably less attention has been devoted to children's construction of strategies in solving novel problems that do not involve routine computations. Problems where children can create, test, and modify their own solution strategies, while at the same time acquire important mathematical principles, play a significant role in the development of children's mathematical power. Such problems are particularly worthy of investigation and should form a significant component of the mathematics curriculum.

The focus of this report is children's construction and analogical transfer of mathematical knowledge during novel problem solving, as reflected in their strategies for dealing with isomorphic combinatorial problems presented in "hands-on" and written form. The present study extends previous work on children's learning in the combinatorial domain (e.g., English, 1991a,b; 1992) and is part of a larger project on children's combinatorial and deductive reasoning (English, 1993a). Prior to addressing the present study, we review briefly the different opinions on what constitutes a strategy and then examine some ideas on children's construction of mathematical knowledge as reflected in their strategy development. The role of domain-general strategies in children's knowledge construction is highlighted here. We conclude this introductory section with a discussion on children's analogical transfer in solving isomorphic problems.

The Nature of Strategies

While there are diverse opinions on what constitutes a strategy (Bisanz & LeFevre, 1990), there is nevertheless some agreement on its key features. It is usually accepted that strategies are "goal-directed operations employed to facilitate task performance" (Bjorklund & Harnishfeger, 1990, p.1). They are frequently seen as domain specific (Pressley, Borkowski, & Schneider, 1987) and designed to facilitate both knowledge acquisition and utilization (Prawatt, 1989). Some view strategies as necessarily involving a choice of procedures (Siegler & Jenkins, 1989), with the procedure being invoked in a "flexible, goal-directed manner ... that influences the selection and implementation of subsequent procedures" (Bisanz & LeFevre, 1990, p.236). Procedures which create new procedures or alter old ones in flexible ways are also considered strategic (Bisanz & LeFevre, 1990).

There are others who emphasize the "potentially conscious and controllable" nature of strategies (Bjorklund, Muir-Broaddus, & Schneider, 1990; Pressley et al., 1987), as well as the "dynamic interaction" of strategies, one's knowledge of the strategies, and one's monitoring of their implementation (Pressley, Forrest-Pressley, Elliott-Faust, & Miller, 1985). Within the framework of the present study, strategies are viewed as goal-directed procedures which facilitate both problem solution and acquisition of domain-specific knowledge. They are also seen as potentially conscious and controllable.

Children's Construction of Mathematical Knowledge During Problem Solving

Children's construction of mathematical ideas during the course of problem solving is a fundamental goal of mathematics education (Davis, 1984; English, 1992; Schoenfeld, 1992; Maher & Martino, 1992). This knowledge construction is reflected in children's strategy development as they attempt to master a challenging problem situation (English, 1992; Ericsson & Oliver, 1988). Research in the last decade has presented convincing evidence that children do behave strategically, that they are able to direct their own learning, and can acquire a knowledge of the domain in which they are working (e.g., Burton, 1992; DeLoache, Sugarman, & Brown, 1985; English, 1991a; Gelman & Brown, 1986; Gelman & Greeno, 1989; Karmiloff-Smith, 1979, 1984; Martino & Maher, 1991).

When challenged with problem situations, children are thought to cycle through various steps as they build representations of those situations (Davis, 1984). We review these steps in the light of our current work on children's development of mathematical models (English & Halford, forthcoming). Children must firstly examine the problem for cues or clues that might guide the retrieval from memory of a relevant mental model of a related problem or situation. We define mental models as "representations that are active while solving a particular problem and that provide the workspace for inference and mental operations" (Halford, 1993, p.23). After retrieving a model that might be useful in solving or in trying to solve the problem, children attempt to map the model onto the problem data. If the mapping appears adequate, that is, if there is a correspondence between the elements of the mental model and the data of the problem, the model can be used to commence the solution process. However retrieving an appropriate mental model may not be automatic or easy for children, especially when the problem presents a novel situation. Children's attempts at making a suitable mapping may involve rejecting, modifying, or extending the retrieved model or perhaps replacing it with another model. This necessitates frequent checking of the correspondence between the model and the problem data. When a suitable mental model of the problem situation has been constructed, other techniques (e.g., setting subgoals) may be brought into play to assist in the solution process. As children progress on the problem, they may recycle through the previous steps in an effort to construct a more powerful model of the problem situation and its solution process. This construction process is considered responsible for children's development of new mathematical ideas.

Children's progression through such a cycle was evident in the longitudinal research of Maher, Martino, and Alston (1993) and Maher, Martino, and Davis (in press). They identified three major stages through which children proceed in solving a novel problem. New ideas are seen to emerge in each stage and form the basis and motivation for the development of other ideas. One such problem required children to build as many towers as possible of height four cubes, from plastic cubes in two colors. The children were to convince their peers that they had solved the problem, namely, that there were no duplicates and that none had been omitted. In the first stage of solving this problem, children displayed "random creativity" where they made whatever towers they could think of that had not already been constructed. They then began to detect relationships between two towers, such as opposite colors in corresponding positions. They monitored for any duplicated arrangements by comparing a newly formed tower with those already constructed. The children's proof of completeness included statements of the type, "I can't think of any more!" In the second stage, children progressed from these random methods to using local systems of organization where where they noticed patterns among sets of towers and discovered new ways of grouping them, such as inverting a tower and its opposite to form another set of towers. Some children began to realize though, that their local organization schemes were inadequate to account for all the possibilities. They then began to search for an overall scheme for organizing and exhausting all possible combinations, for example, forming towers with no red cubes, then towers with one red cube, two red cubes, and so on.

Children's development of mathematical ideas may also be viewed in terms of principled knowledge of the problem domain (Gelman & Meck, 1986; Gelman & Greeno, 1989). This theory posits that children's initial understanding of the domain is principled, albeit in a "limited and implicit way." (Gelman and Greeno, 1989, p.126). As children gain experience with this domain, they acquire a knowledge of its principles and also demonstrate a more explicit or stateable understanding of these. Early principled knowledge directs attention towards domain-relevant inputs and guides the learning of new principles. This view is consistent with the well established notion that prior knowledge in a domain determines what and how other information is encoded and learned (Resnick, 1986; Siegler & Jenkins, 1989).

A model of principled learning also provides a means of determining whether children have an implicit theory about a domain. The more children's knowledge can be characterized in terms of the principles of the particular domain, the more it can be said they have a "theory" (Gelman & Greeno, 1989, p. 130). Children's competence in the domain is viewed in terms of their ability to generate competent plans of action that meet the constraints of the knowledge principles in that domain. Because this planning component must determine whether a chosen strategy meets the requisites dictated by the principles, it can serve as a potential source of feedback to children solving a novel problem. If the requisite conditions are not met, the plan or its execution can be rejected or terminated. This means that the child can start again, without being explicitly told to do so.

An important component in children's knowledge construction during problem solving is their application of domain-general strategies (Alexander & Judy, 1988; English, 1992; English & Halford, forthcoming; Kuhn, Amsel, & O'Loughlin, 1988). These general strategies include those that perform a self-regulatory or metacognitive function and include skills such as planning, predicting, monitoring, checking, and revising (e.g., Brown & Campione, 1981; Lawson, 1984; Pressley & Ghatala, 1990; Schoenfeld, 1992; Sternberg, 1985). Davis (1984) refers to similar skills in his discussion on children's "meta-analyses" of their construction activities where they assess their progress or lack of progress during problem solution (p. 307). The process of knowledge construction and the application of domain-general strategies are seen to occur simultaneously during the problem-solving episode (Davis, 1984), with the nature of their interaction determining the extent of goal attainment (English, 1992).

The fact that, in the absence of instruction, children do modify their ideas and actions in their efforts to solve a novel problem highlights the important contribution of these general strategies (Kuhn &

Phelps, 1982; Kuhn et al., 1988; Pressley et al., 1987). Children's capacity to monitor their actions, noting the relationship between their outputs and the problem goal, can foster the construction of mathematical ideas. Through problem experience, children acquire not only knowledge about the particular problem domain, but also knowledge about their own strategies as they apply to the problem. That is, they come to realize how a particular strategy works, why it works, and why it is the most appropriate for the problem. They also become aware of less efficient strategies, why these do not work or why they are inappropriate for the problem, and the errors that can result from their use (Kuhn et al., 1988). It has been claimed that these self-regulatory strategies contribute not only to children's knowledge construction during a problem-solving episode but also to continued growth of the cognitive system (Scardamalia & Bereiter, 1985).

Children's Analogical Transfer in Solving Isomorphic Problems

A significant factor in children's construction of ideas during problem solving is their ability to access a known problem (base or source problem) that has an identical goal structure to the new problem to be solved (target problem). Many studies have shown that exploiting the structural correspondences between a base problem and an analogous target problem can enhance problem-solving performance (Holyoak & Koh, 1987; Novick, 1988, 1992; Novick & Holyoak, 1991). This analogical transfer involves constructing a mapping between elements in the base and target problems, and adapting the solution model from the base problem to meet the requirements of the target problem (Novick, 1992).

Before children can make use of analogical transfer however, they must notice the correspondence between the target problem and the base problem and retrieve the base in terms of its generalizable structure (Gholson, Morgan, Dattel, & Pierce, 1990). Studies have shown though, that novice problem solvers often have difficulty in detecting structural similarities between problems that have different surface features (Novick, 1988). This is largely because novices tend to focus on salient surface features such as the specific objects and terms mentioned, rather than the structural features, such as how the entities in the problem are causally interrelated (Chi, Feltovich, & Glaser, 1981; Gholson et al., 1990; Novick, 1988, 1992; Silver, 1981). This means that the surface features in a novice's model of a target problem will likely serve as retrieval cues for a related problem in memory. On the other hand, studies have shown that similarity among surface details, or superficial similarity, promotes "reminding," that is, assists novices to notice a correspondence between their mental model of a base problem and the new target problem (Gentner & Landers, 1985; Reed, 1987; Ross, 1984, 1987). While surface similarity can facilitate children's retrieval of the base problem, its usefulness for analogical transfer is governed by their ability to detect the structural correspondences between the base and target problems (Gentner & Landers, 1985).

Although children may be particularly dependent on surface cues for the retrieval of a base problem, there is some evidence that even preschoolers can overcome similarity in appearance in categorizing objects if they are given relevant information, including age-appropriate materials and procedures (Carey, 1985; Gelman & Markman, 1986). Gentner (1989) reports on a study (Gentner & Toupin, 1986) in which 4-6 year-olds and 8-10 year-olds were asked to transfer a story plot from one group of characters to another. Two factors were varied, namely, systematicity of the base domain, that is, the relational structure of the original story, and the transparency of the mapping, that is, the degree to which the target objects resembled the corresponding base objects. The systematicity of the original story was manipulated by adding beginning and ending sentences that expressed a causal or moral summary. Transparency was varied by changing the similarity of the corresponding characters. In a high-transparency mapping, the new characters (e.g., squirrel, elk, toad) resembled the original characters (e.g., chipmunk, moose, frog). In the medium-transparency mapping, three unrelated animals were used, while in the low-transparency mapping, the new characters were similar to the original characters but occupied noncorresponding roles.

Gentner and Toupin (1986) found both transparency and systematicity to be important in determining transfer, with the two age groups displaying different patterns. For both age groups, transfer accuracy was almost perfect in the high-transparency condition, but lower in the medium-transparency and lower still, in the low-transparency condition. Systematicity also had strong effects for the older age group. Even in the most difficult mapping conditions, the 9 year-olds performed almost perfectly when they had a systematic relational structure to hold onto. Informal observations showed that these children monitored any object-similarity-based errors they made and corrected them by utilizing the systematic causal structure of the story. The 5 year-olds, in contrast, showed no significant effects of systematic base structure. All that mattered to these younger children was the transparency of the object correspondences. The results of this study and others (e.g., Holyoak, Junn, & Billman, 1984; Smith, 1989) indicate a developmental shift from a reliance on surface similarity, and especially on transparency of object correspondences, to the use of relational structure in analogical mapping (Gentner, 1989). This has significant implications for the development of children's mathematical problem solving, in particular, their ability to abstract important structural principles of a problem domain.

In our discussion to date we have argued firstly, that children construct important mathematical ideas as they develop strategies for solving a novel problem. Secondly, we have claimed that they cycle through a sequence of steps in building and refining these ideas. Thirdly, we have suggested that these ideas may reflect an underlying principled knowledge of the problem domain. We have also highlighted the interactive role of domain-general strategies in this construction process. Finally, we have emphasized the importance of children's ability to access a known or base problem that has an identical goal structure to the target problem to be solved. As children reflect on earlier problem-solving experiences and re-analyze them in the light of their newly constructed knowledge, they are frequently able to generalize particular solution strategies (Brown, 1989; Maher, Martino, & Alston, 1993).

We address these issues in the present report. More specifically, we examine two case studies in an effort to shed light on the following questions:

1. How do children construct strategies for solving two sets of combinatorial problems, one presented in hands-on format and the other, as isomorphic written problems? What role does their use of domain-general strategies play in this construction?

2. How does this strategy development change between the problem sets?

3. What does children's strategy development suggest about their construction of combinatorial knowledge?

4. Do children recognize the structural correspondence between the problem sets and use this to facilitate problem solution?

The tasks for this study were deliberately chosen to challenge children's thinking and encourage them to develop their own ideas about a novel domain. The combinatorial domain was considered eminently suited to this purpose.

The Combinatorial Domain

Combinatorics, involving the selection and arrangement of objects in a finite set, lends itself to the design of problems that are challenging while at the same time, meaningful, to children. Furthermore, combinatorial tasks can be solved at different levels of sophistication and can be readily modified to accommodate individual needs.

The domain is of significance from a mathematical perspective. It comprises a rich structure of important mathematical principles which underlie several areas of the curriculum, including counting, computation, and probability. In simple mathematical terms, combinatorics may be viewed as the operation of <u>cross product</u>. The cross product of two sets, X and Y, is the set of combinations obtained by systematically pairing each member of X in turn with each member of Y, as shown in Figure 1 (v). In more complex examples involving combinations of three elements, each member of set X must be systematically matched with each member of set Y <u>and</u> set Z, as shown in Figure 2 (v). This tree diagram represents the most efficient way of forming X x Y x Z combinations. However there are several other, less efficient, methods of generating these combinations, as can be seen in the remaining tree diagrams. We revisit these in a later section.

INSERT FIGURES 1 AND 2 ABOUT HERE

In addition to its mathematical importance, the combinatorial domain is of developmental significance. It is a major component of Piaget's theory where it plays a significant role in cognitive development (Piaget, 1957; Flavell, 1963). The combinatorial system is evident in a subject's ability to "link a set of base associations or correspondences with each other in all possible ways so as to draw from them the relationships of implication, disjunction, exclusion etc." (Inhelder & Piaget, 1958, p.107). The key cognitive strategies here are isolation or control of variables, and systematic combination. The appearance of a systematic method of generating combinations is said to occur at the onset of the formal operations stage (Piaget & Inhelder, 1975).

In sum, the problems chosen for this study draw upon a clearly defined body of mathematical knowledge which is within the grasp of elementary school children. The tasks allow for different levels of solution which enable children to apply their existing, informal knowledge to initial problem solution and to subsequently build on this knowledge as they construct new ways of tackling the problem. The problem tasks are described in the next section.

Method

Subjects

The current study is part of a larger project involving 288 children from grade levels 4 to 7 (8 years 11 months to 12 years 7 months). The children who feature in this report are both 9 year-olds in their fourth year of school. James attends a small state school in a middle class suburb of Brisbane, Australia, while Kerry attends a large catholic school in a predominantly low socio-economic suburb of the same city. James is considered by his teacher to be a high achiever in school mathematics while Kerry is regarded as a low achiever.

Instruments

The present study involved two sets of isomorphic combinatorial problems. One set comprised three hands-on tasks and the other, three written tasks. The hands-on problems required children to dress toy bears in all possible combinations of colored tops and pants (first problem in the set) or colored tops, pants, and tennis rackets (remaining two problems). The first hands-on problem was a two-dimensional task ($\underline{X} \times \underline{Y}$) comprising 3 sets of colored tops and 3 sets of colored pants (9 combinations altogether). The remaining two problems were of a three-dimensional structure ($\underline{X} \times \underline{Y} \times \underline{Z}$), with one problem involving 2 sets of tops, 2 sets of pants, and 2 sets of tennis rackets (8 combinations), and the other, 2 sets of tops, 3 sets of pants, and 2 sets of tennis rackets (12 combinations). The bears were made of thin wood and were placed on a stand. This enabled the children to see clearly the outfits they had formed. The clothing items were made of colored card and were backed with adhesive material to facilitate the dressing process.

The set of written problems corresponded in structure to the hands-on problems, that is, the first problem was of the form, $\underline{X} \times \underline{Y}$, with a total of 9 combinations. The remaining two problems were of the form, $\underline{X} \times \underline{Y} \times \underline{Z}$, with 8 and 12 possible combinations respectively (these appear as an appendix). The three written problems also shared some of the surface features of the hands-on tasks, in particular, that of color. Problems 1 and 3 were set in different contexts to that of the hands-on problems, while the second written problem featured a similar clothing context to the hands-on problems. In contrast to the concrete problems, these written examples required children to construct an abstract representation of the problem. However children were able to use recorded notation (e.g., listing, diagrams) as an aid in formulating their ideas.

Designing the problems from simple to more complex in each set facilitated observation of children's strategy development as they tried to accommodate the more difficult three-dimensional examples. It was hypothesized that this would encourage children to construct a more sophisticated knowledge of the combinatorial domain, a knowledge that comprised at least an implicit understanding of domain-specific principles. To assist in the subsequent analysis of children's knowledge construction, we review briefly the structure of these problems with specific reference to the concrete examples.

Problem Structure

As children attempt to solve these problems, they must meet certain constraints imposed by the problem goal (Glaser & Pellegrino, 1982). The minimum set of constraints that children must meet in solving these problems is as follows:

1. A constraint on the types of items to be combined. That is, items of the same type cannot be combined, such as two tops or two pants. A combination must comprise one top and one pair of pants (and a tennis racket).

2. A constraint on similarity across combinations. That is, given the ordered pairs of items (a, b) and (c, d) where a and c represent any tops and b and d any pants (in the case of the two-dimensional problems), different combinations will result if any of the following is adhered to:

i. a is different in color from c, and b is different in color from d;

ii. a is the same color as c, and b is different in color from d;

iii. a is different in color from c, and b is the same color as d.

A particular case of these constraints warrants citing:

iv. a is the same color as b, and c is the same color as d, but a is not the same color as c.

This fourth constraint allows items within a combination to be the same color (e.g. red top/red pants) while items across combinations must be different (e.g. red top/red pants and blue top/blue pants).

An awareness of the above constraints would be sufficient for a trial-and-error approach to problem solution where items would be generated in a random fashion, as indicated in Figure 1 (i), then selected and combined according to the above rules. Since there would be no evidence of forward planning in such behavior (Rogoff, Gauvain, & Gardner, 1987, the children's self-monitoring processes would be particularly important.

On the other hand, the most efficient strategy for solving the problems would reflect a clear plan of action with a focus on the overall goal of generating all possible combinations (Rogoff et al., 1988). In contrast to the novice strategies, where an item is not selected more than once in succession, the expert strategy involves the repeated selection of an item (referred to here as, "holding an item constant") and systematically matching it with each of the other, "varying" items. These latter items are varied in a cyclic fashion, as shown in Figure 1 (v) (Y_1, Y_2, Y_3, Y_1, Y_2, Y_3....). Because this method resembles the working of an odometer in a car, it has been labelled the "odometer" strategy (English, 1988; Scardamalia, 1977). In the case of the two-dimensional problems (X x Y) there is only one item to be held constant at any one time, as indicated by items X_{1-3} in Figure 1 (v). For the three-dimensional problems however, there are two items (X_{1-2} and Y_{1-3}) which are held constant at any one time. The item which is changed least often (X_{1-2}), that is, the slowest moving dimension, is referred to here as the major constant item; the item that is changed more frequently (Y_{1-3}), the faster moving dimension, is termed the minor constant item (refer Figure 2 v).

It is worthwhile noting that, for young children in particular, the repeated selection of an item seemingly goes against the problem goal of different combinations. An earlier study (English, 1988) had shown that some children are initially reluctant to select an item more than once in succession, perhaps because they interpret "different" to mean "different in all ways" and thus see the goal of "all different outfits" as an indication to make each new outfit completely different from the previous outfit(s). It thus seems that children avoid repeating the selection of an item because they see it as going against the problem goal. Such behavior reflects the difference-reduction method of problem solving (Anderson, 1985, p. 206) where problem solvers attempt to make the current state as similar as possible to the desired goal state. However a correct solution frequently involves going against the grain of similarity (Anderson, 1985). In the case of the present problems, selecting the same item in succession is a key feature of the most efficient combinatorial strategies.

Procedure

The children were administered the problems on an individual basis by a research assistant who is a qualified teacher. Each child's responses were videotaped for subsequent analyses. The presentation of the two problem sets was counterbalanced, that is, half the children in the original sample received the hands-on problems prior to the written problems, while the other half received the reverse of this.

For the hands-on problems, children were initially given one familiarization task to ensure they understood the idea of forming different outfits. The goal here was to simply "dress the bears." The task was designed to test children's color recognition, as well as to establish an understanding of the terms, "outfit,"and "same/different outfits." The latter term was crucial in the interpretation of the problem goal, especially when a common item was present. For example, the outfits, red top/blue pants, and red top/yellow pants are different from each other even though they have a common item. In working the familiarization task, children were not given any information that could bias their performance on the two problem sets.

In each of the hands-on tasks, children were provided with more materials (both bears and items) than were needed. This was to ensure the children did not use item depletion as a signal that they had solved the problem. The children were expected to complete each problem without assistance and were asked to explain their procedure at the end of each problem.

The written problems were administered to each child on a worksheet with space provided for the child's working. The research assistant read through each problem with the child and ensured that the child understood the problem goal. For example, in the first problem, it was emphasized that a set must comprise one bucket and one spade and that each set must be different from all other sets. It was also explained that a particular bucket (or spade) could be used more than once. For the few children who asked, it was stated that a blue bucket could be matched with a blue spade. As the children worked through the written problems, they were encouraged to "talk aloud" about what they were doing. It was explained that they could use the space on the sheet to help them work out the problem, however no mention was made of any particular strategy, such as drawing a diagram.

Analysis of children's responses

Each child's videotaped response on each problem was analyzed in terms of accuracy (i.e., whether the correct solution was produced), the solution strategy employed (i.e., the way in which items were selected and combined), and, in addition for the written problems, the type of written procedure used (e.g., use of systematic listing, tree diagram). The solution strategies children employ in solving two- and three-dimensional combinatorial problems had been identified previously (e.g., English, 1991a,b; 1993b) and are reviewed in the next section. These strategies were used here to classify children's responses on each problem. To facilitate the classification process, each child's response on each problem was converted to a tree diagram. These diagrams provide an effective visual representation of the generation of combinations (DeGuire, 1991; Graham, 1991) and enabled the child's solution strategy to be identified readily.

Children's Combinatorial Strategies

For each of the two-dimensional and three-dimensional problems, children's strategies reflect three main stages of development (cf. Maher, Martino, & Alston, 1993). The first stage entails random, trial-and-error procedures that are devoid of any global planning components. In the second stage, children display transitional strategies where they adopt an identifiable pattern in their selection of items to combine, however the pattern is not the most efficient for task solution. In the final stage, children construct odometer (or "almost odometer") strategies. These are the most efficient for problem solution because of their generative nature, that is, they provide an organizational structure for generating all possible combinations. These stages are described below for each of the two- and three-dimensional problems. Reference is made to the tree diagrams shown in Figures 1 and 2.

Two-dimensional Strategies

Non-planning Stage

In this stage, children adopt a trial-and-error approach to problem solution, selecting items in a random manner. This strategy is represented by the first tree diagram of Figure 1. Children's checking actions play an important role here, with the effectiveness of these actions largely determining goal attainment (English, 1992).

Transitional Stage

During this stage, children construct a pattern for selecting their items. This pattern is of an alternating or cyclic nature and is usually confined to one item type, as indicated in Figure 1 (ii) and (iii) (X_1, X_2, X_3, X_1, X_2, X_3.....). At the beginning of this transitional stage however, children do not continue their pattern throughout problem execution and revert to a trail-and error approach (as indicated by the final three combinations of Figure 1 [ii], namely, X_2/Y_3, X_3/Y_1, X_1/Y_2).

Odometer Stage

This is the most sophisticated stage where children construct a procedure which provides a global framework for solving the problem. This procedure is characterized by an odometer pattern in item selection where an item of one type is held constant (e.g., item X_1 in Figure 1 [v]) while items of the other type are varied systematically (e.g., Y_1, Y_2, Y_3 of Figure 1 [v]). Notice how the cyclic pattern of the transitional stage is retained here and coordinated with the use of a constant item. Once this constant item has been exhausted (i.e., all possible combinations with that item have been formed), a new constant item is chosen and the process repeated (i.e., X_2 is matched, in turn, with Y_1, Y_2, and Y_3 of Figure 1[v]). The use of a constant item reduces the number of new selections of item X, hence rendering this stage the most efficient.

During children's construction of this odometer strategy, they frequently display one or more weaknesses. These include a failure to exhaust a constant item (frequently the omitted combination is formed at the end of task execution, as indicated in Figure 1[iv]), an "over-exhaustion" of a constant item (the child normally detects the duplicated combination and corrects this without requesting assistance), or a failure to recognize problem completion upon exhaustion of all constant items (in this instance, the child attempts to create further combinations but soon realizes this cannot be done).

Three-dimensional strategies

The key feature of the three-dimensional strategies is children's ability to deal simultaneously with the major and minor constant items. The ability to hold these items constant until they are exhausted does not emerge until the third stage.

Non-planning Stage

This is the least efficient of the three-dimensional stages, since children do not follow any identifiable pattern and make the greatest number of new selections of items X and Y, as indicated in Figure 2 (i).

They do not exhaust any major constant items and exhaust less than half of the minor constant items.

Transitional Stage

During this stage, children begin to exhaust more of the minor constant items, thus reducing their number of new item selections (refer Figures 2 [ii] and [iii]). In the early part of this stage, children may not exhaust all of the minor constant items, as shown in Figure 2 (ii), but do so later in the stage (Figure 2 [iii]). They do not exhaust either of the major constant items. This occurs in the odometer stage.

Odometer Stage

In this final stage, children are able to coordinate the repeated selection and subsequent exhaustion of both major and minor constant items. Initially, children may only exhaust one of the major items, as indicated in Figure 2 (iv) (i.e., X_1 is matched, in turn, with each of Y_{1-3}, which is also systematically matched with each of Z_{1-2}). They finally master the exhaustion of both constant items, as shown in Figure 2 (v). This strategy involves the least possible number of new X and Y item selections.

Results

We now illustrate children's construction of these strategies by considering the two case studies. We address our original research questions by firstly focussing on this strategy construction and on the role of domain-general strategies in this process. This analysis will then enable us to compare children's strategy development across the problem sets (both within and between subjects) and will provide the bases for inferring their construction of combinatorial knowledge. Finally, we will examine children's ability to detect the structural correspondence between the problem sets and consider whether this facilitated problem solution. We begin our discussion with James' responses.

James

Recall that James was aged 9 years 4 months and was in his fourth year of elementary school. He attended a state school situated in a middle class suburb and was considered a high achieving student in school mathematics. James was administered the written problems prior to the hands-on problems.

Like several of the other high achievers in the larger sample, James did not use any notation to represent his ideas in solving the written problems, preferring instead to determine the combinations mentally. Given the number of combinations to be formed, it is of no surprise that he failed to generate the correct combinations for each of the written problems. His response to the first written problem was as follows:

> O.K., let's see. Two.... no, that is not exactly right because you can make much more. I like to do these in my head. I'm counting up each one. I'm going 2, 4, like that. Green and orange, red and purple, green and purple, that's 6.

The research assistant asked him how many sets he had made. James repeated the sets he had already made and then continued:

> Blue and orange, red and blue, five sets, and green and blue. O.K. Red and purple, oh, I said that. Green and orange, red and purple. That's 8 sets. That's all I can think of.

While James applied some general monitoring procedures in an effort to keep track of his combinations, he nevertheless had difficulty in generating all of the possible sets. Had he been able to follow some pattern, he might have been more successful here.

His performance on the remaining two written problems was not significantly better, although he did progress towards the transitional stage in the second problem where he exhausted a few of the minor constant items (but not all), as indicated in the following transcript:

O.K. She can do green, yellow, and orange.
Oh! She can do red skirt, white T-shirt, and blue sandals. Two.
And she can have red, yellow, and orange.
She can have green, yellow, and orange.
She can also have green, white, and blue.
I think that's 4, no 5.
And she can have red, yellow, and orange.
She can have white, yellow, and orange. Oh no.
She can have green, yellow, and orange.
She can wear red, yellow, and orange.
She can have red, white, and orange. Nine.
She can have white, green, and orange. That's ten.
She can have white, green, and blue. Eleven.
She can have yellow, green, and blue. That is twelve.
She can also have yellow, green, and orange, and yellow, red, and blue.
That's 14 altogether.

In the above problem, James demonstrated quite an amazing mental capacity in his effort to generate and monitor combinations without the aid of notation. Because he duplicated combinations, James produced nearly double the number required. Had he been able to exhaust more of his items, James would have been better able to keep track of his combinations.

James adopted an unusual approach to the third written problem. He read the problem silently to himself after the research assistant had read it aloud with him. After thinking for a few minutes, he stated:

> Forty-nine. I squared it. I counted up each type. I counted up green, yellow, Christmas, birthday, Easter, gold lettering and silver lettering, and I squared it. My answer is forty-nine.

When asked why he squared the seven, James gave a somewhat confused reply:

> Well, I was thinking about this problem (referring back to the previous problem), yellow and white, that's two. Three, four, five, six (counting the other items in the second problem). Now true, there might be more but I thought maybe if I squared it, I might get the answer because ..... it's kind of hard to explain. I'm trying .... what I'm getting at is, if you squared it, you could find out that you could ... make all different combinations like that, but you couldn't go over what you could square. You couldn't go over the square root once you'd squared it.

It is interesting here how James used the previous problem to help him recall a rule that might expedite the solution of this more complex example. Unfortunately, James incorrectly mapped his mental model of this rule onto the data of the third problem. His mental representation of this rule seems to have included some understanding of the relationship between squaring a number and finding the square root; this, of course, was not relevant to the present problem. The fact that James linked the squaring of a number with the generation of all possible combinations could mean that he had met the formal combinatorial rule some time in his informal experiences with number (he had not been taught combinatorics in school). However his mental model of such a rule was clearly erroneous.

James' lack of significant improvement on the second set of problems (dressing the bears) suggests that he simply mapped some of his procedures for solving the written examples onto these hands-on problems, without building on these strategies. His strategy for solving the first hands-on problem was basically one of trial-and-error, although there was an emerging awareness of the efficiency of repeating the selection of an item, as can be seen in the explanation of his strategy given shortly. James solved this first problem by initially forming 8 combinations, as follows:

orange pants and blue top
blue pants and orange top
green top and blue pants
blue top and pink pants
blue top and blue pants
green top and orange pants
orange pants and orange top
green top and pink pants

At this point, the child spent considerable time checking for further possible combinations by making trial outfits with the remaining items in front of him. He then stated, "I'm afraid there aren't any more, I don't think .... oh, there might be ...." He checked again and then completed the final combination, namely, pink pants and orange top. On completion, he stated, "I'm afraid that's all I can do."

When asked to explain his strategy for solving this first hands-on problem, James simply recalled his selection of items for the first four combinations. When he reached the fifth outfit (blue top and blue pants), he explained:

> ... when I got to this one, I thought, O.K., maybe I should just put the whole lot on like a tracksuit. I thought when I got to the fifth one, I thought I might as well not waste my patterns here. I thought I might as well put in a matching thing (pointing to the blue top of the fourth bear) and the same with this one (pointing to the orange pants of the seventh bear and indicating it was also used to dress the sixth bear). On this bear (the seventh, orange pants and orange top) I was about to put blue and orange but I thought, "No." And then for this one (the eighth, green top and pink pants) I thought of green and thought, "yes," I can do green and pink and with the last one I did orange and pink, and I thought, well, I can't do any more.
>
> Every time I finished one, I'd check up on all the others. If I did find a pattern to be the same, then I'd take that one off and think hard about the next one. You may have noticed when I was stuck a bit, I went like that (putting his hand to his head). I thought of the pattern.

James' reference to a "pattern" in the above explanation involved repeating the selection of an item to generate a new combination. He seemed aware of the need to develop some generative procedure, suggesting that he was beginning to construct some knowledge of the combinatorial domain. However he did not build on these ideas as he worked the remaining problems. The increase in problem complexity could have been a contributing factor here, although our next case study, Kerry, moved into the final odometer stage on a complex, 2 x 3 x 2 example.

Notice how James monitored his actions in this first hands-on problem by checking each combination after he had formed it (a discussion on the different types of scanning actions children display in checking their combination: ppears in English, 1991a). He obviously considered it important to use such thorough checking; in fact, without it, he would not have solved the problem. This highlights the important role of domain-general strategies in the absence of a strong body of domain knowledge (English, 1992).

In solving the second hands-on problem, James again used a trial-and-error approach in forming the first four outfits. He then repeated the selection of two items in forming the fifth and sixth combinations. After the seventh outfit, he stated that he had solved the problem. When encouraged to look for further combinations, James explained:

> I could do more, but the thing is, if I did do more, he (the bear) would still have the same colors but he'd be different. I'll show you what I mean. Instead of having his blue racket in his right hand, he'd have it in his left hand.

On being reminded of the meaning of a "different outfit," James proceeded to form two additional combinations, one of which was a duplicate of a previous outfit. James' explanation of his strategy was similar to the previous problem and indicated that he was aware of the value of repeating an item selection. However he did not appreciate the need to exhaust an item. When he had difficulty in constructing a new combination, he just looked back at an outfit that he had formed earlier and used it again with modification (e.g., changed the tennis racket). He was clearly still at the beginning of the transitional stage in his construction of combinatorial ideas.

The final hands-on problem was considerably more difficult for James; in fact, he reverted to the non-planning stage in solving this. He duplicated combinations in constructing the first six outfits but soon corrected these. In forming the remaining combinations, James spent considerable time in making trial combinations and checking them against the previous outfits. He commented at this point, "It's getting hard." However he managed to form all but one of the remaining outfits. His explanation of his approach reflected a reliance on trial-and-error and efficient checking. He also appeared to be more interested in matching colors of items within a bear (e.g., giving a bear an orange racket and orange pants) than in repeating the selection of items, as is evident in his explanation:

> .... The next one (yellow top, pink pants, and orange tennis racket) was all different and so was the next one. The next one (yellow top, orange pants, and orange tennis racket) had the tennis racket matching the pants and the next two didn't have anything the same .... then the ninth one, I gave him an orange racket to match his orange pants and a blue shirt which I hadn't used .... and the last one I gave an orange racket, a blue shirt, and pink pants because I hadn't used these. And this poor bloke (pointing to a remaining undressed bear), well, he didn't get dressed.

On completing all of the problems, James was asked whether solving the first written set helped him with the second hands-on set. He replied:

> No, with some and yes, with others.

> The first one (the first written problem) gave me an idea about making different sets and everything. I found it a bit harder though because I didn't actually have anything like I did with the bears, that you could put on and look at. The second (written) problem was good because that helped me match up things like with the tennis ones (the hands-on problems). The third (written) problem didn't really help me because I just took a guess at that and used my own theory of squaring. I used my own theory about it and I just squared it.

James was also asked if he could see any ways in which the hands-on problems were similar to the written examples. He responded:

> Yes, the second one (written problem) is the one that has the most in common with these (the hands-on problems) because to match up a T-shirt with a skirt and sandals is kind of like how you have to match up a tennis racket, pants, and a T-shirt, and that's mainly the one I'd say helped most. This one (the first written problem), you just had to match up two things, but they did help me a bit. They prepared me in matching them all up, gave me an idea. That's right.

It is interesting that James commented on the usefulness of the hands-on materials in the second problem set, especially since he did not use any form of notation to represent his ideas in the written problems. It may be that he had not been encouraged to do so in school or more likely, he felt that he had no need to use notation because of his confidence in his mental computation. In reflecting on problem similarity, James could see the parallel between the clothing problem in the written set and those of the hands-on set. This is not surprising, given that the resemblance between the problem items would create a high-transparency mapping condition for the child (Gentner, 1989). This similarity in surface features appeared to alert James to their structural correspondence which he saw in terms of matching items. He did not comment though, on the idea of all possible different matchings. Nevertheless, his responses here support our earlier discussion on how superficial similarity can help novices notice a correspondence between their mental model of a base problem and a new target problem. It is clear that the third written problem, in which James "guessed" and constructed his "own theory of squaring," did not assist him in solving the hands-on problems. The child's discussion gave no indication of whether he recognized any structural similarity between this problem and the other examples.

In reviewing James' performance, there appear a couple of interesting features worth mentioning. Firstly, while James could effectively describe and justify his strategies, he did not make much progress in his construction of combinatorial ideas. This is despite the fact that he seemed very aware of his own thinking. However he was skilled in his use of domain-general strategies, as could be seen in the way he monitored his actions throughout problem solution. Had James not been competent here, he would have had great difficulty in solving the problems. These general strategies clearly compensated for his lack of sophisticated combinatorial knowledge.

The second factor pertains to James' mental capacity for generating combinations without the use of notation. Given his precocity here, it is tempting to speculate that James was of the opinion that "people who are good at math do it in their heads." The application of his "squaring theory" also suggest that he had been exposed to advanced formal mathematics for which he was not ready. Our second case study, Kerry, a low achiever in school mathematics, provides an interesting contrast to James.

Kerry

Recall that Kerry was aged 9 years 5 months and was a grade four student at a large catholic school in a predominantly low socio-economic area. She was considered a low achieving student in mathematics. Kerry was administered the hands-on problems prior to the written examples.

In contrast to James, Kerry was not as vocal in her discussion of her actions, yet made greater progress in her strategy construction. She commenced the hands-on problems at the non-planning stage where she relied on a trial-and-error procedure. She correctly formed the first five combinations, then paused for some time to generate a further three outfits. She realized that she had made the eighth outfit the same as the first and so removed the items. When asked if she could make further combinations, Kerry continued to create an additional two, making 10 in total. She realized that she had duplicated outfits and, although she made some modifications, was unable to correct all of her errors. When asked to describe the method she used, Kerry stated, "I just kept on checking back to the last one so that I knew I didn't do the same." This checking procedure however, was not sufficiently thorough for her to detect all of the duplicated combinations.

Kerry also repeated outfits in the second and third hands-on problems and again, was unable to make all of the required modifications. However, she demonstrated a distinct improvement in her solution strategy. She was now working in the transitional stage where she reduced her number of new item selections by repeating the selection of several minor items. However she was unable to exhaust all of these items. She demonstrated an efficient dressing procedure which facilitated her procedure. Instead of completing the dressing of a bear prior to moving onto the next one, she dressed the bears in pairs (or occasionally in threes). She accomplished this by placing two identical items on both bears (e.g., two green tops), followed by another two identical items (e.g., two orange pants), and finally, changed the third item in an alternating fashion (e.g., orange tennis racket, blue tennis racket, orange tennis racket). This can be seen in the transcript of her actions as she solved the third hands-on problem involving 12 combinations. Kerry formed her first 8 combinations as follows:

green top, pink pants, and orange tennis racket
green top, pink pants, and blue tennis racket

orange pants, blue tops, and orange tennis racket
orange pants, blue tops, and blue tennis racket

pink pants, yellow top, and orange tennis racket
pink pants, yellow top, and blue tennis racket

pink pants, green top, and orange tennis racket
orange pants, yellow top, and blue tennis racket

She then constructed the ninth, tenth, and eleventh bears simultaneously, as follows:

yellow top, orange pants, and blue tennis racket
yellow top, pink pants, and blue tennis racket
yellow top, orange pants, and orange tennis racket

On realizing that the eighth and ninth bears were identical, Kerry removed the top and tennis racket of the ninth bear and subsequently formed three new outfits:

orange pants, blue top, and orange tennis racket orange pants, blue top, and blue racket

pink pants, green top, and orange tennis racket

She again attempted to correct the duplicated combinations but was unable to make all the necessary modifications. Kerry was apparently unaware that there were still corrections to be made for she claimed that all of the outfits were different and that she had solved the problem. Her explanation of how she constructed her combinations was simply, "I just kept on changing the clothes every time I went and kept on looking back." It would seem that she was aware, at least implicitly, of her decision to dress two or three bears in the same two items and vary the third.

Kerry's response to the first written problem was interesting. After reading the problem, she stated immediately that there were three combinations. When asked whether there were any more possible combinations, Kerry began to use the clothing items from the hands-on tasks to represent the problem data. She tried to match the colors of the clothing items to the colors of the buckets and spades. She used a green top for the green bucket, a blue top for the blue bucket, a pink top for the red bucket, and orange and blue pants for the orange and blue spades respectively. Kerry then placed an orange spade with a blue bucket and stated, "Orange bucket, no, orange spade with a blue bucket, and that's one. The red bucket and a blue spade, that's two." However she became very confused when she attempted to use the tennis rackets for spades because she did not have the correct colors. When asked if there was another way in which she could solve the problem, Kerry replied, "I could do it in my head but I can't work it out that way." She then decided to use notation to represent the problem data.

On the worksheet in front of her, Kerry recorded the colors of the buckets under the heading "B" for buckets and repeated this for the spades. She then drew lines to connect pairs of items, generating the nine combinations, as shown in Figure 3. Her use of a uniform cyclic pattern in selecting the spades (i.e., orange spade, purple spade, blue spade, orange spade, purple spade....) facilitated the construction of these combinations.

INSERT FIGURE 3 ABOUT HERE

Kerry remained in this transitional stage as she solved the second written problem. While she repeated the selection of each T-shirt, she did not exhaust these items and was unable to generate all eight items. Kerry applied a similar procedure for representing the problem data, recording the letters of the T-shirt colors under the heading "T", the colors of the skirts under "S," and the colors of the sandals under "SA." She then rewrote each set of colors and drew lines between letters to form her combinations. After making three outfits, Kerry recorded the digit, "3". After constructing two more outfits, she crossed out this digit and replaced it with "5". She claimed she was unable to form any more than five outfits. This may have been due partly to the untidy nature of her diagram.

It was on the third written problem that Kerry demonstrated the greatest progress in her construction of combinatorial knowledge. Kerry again represented the problem data by listing the first letter of each item, that is, she listed, one under the other, the letters, G, Y, C, B, E, g, and S for the colors, greetings, and letterings respectively. As before, she drew lines to link the different items and verbalized her actions as follows:

A green Christmas with gold.

A green birthday card with gold writing, that's two.

A green card with Easter and gold writing, three.

At this point, the child recorded the digit, "3."

A yellow card with Christmas and gold lettering.

A yellow card with birthday and gold lettering.

A yellow card, Easter, and gold writing. Six.

The digit, "6" was recorded.

A green card with Christmas and silver writing.

A green card, birthday, and silver writing.

A green Easter and silver writing. That's nine.

The digit, "9" was recorded.

A yellow card with Christmas and silver writing.

A yellow card with birthday and silver writing.

A yellow card with Easter and silver writing. That's twelve and that's all that you can make.

The digit, "12," was recorded.

Kerry had now progressed to the third stage where she was employing a complete odometer strategy to solve the problem. That is, she chose the gold and silver lettering as her major constant items and used them repeatedly until they were exhausted. She did the same with the minor constant items (the colors, namely, green and yellow) while systematically cycling through the remaining items (the greetings, namely, Christmas, birthday, and Easter).

Kerry's progress from her initial non-planning stage through to this final stage reflects considerable growth in her construction of combinatorial ideas. Her commencing mental model of problem solution was simply one of joining items and ensuring that each new combination was different from all preceding combinations. While her general monitoring strategies were not as effective as James', she nevertheless continued to modify and extend her model as she worked through the problems.

There were two major developments in Kerry's strategies for solving the hands-on tasks. Firstly, she reduced the number of new item selections by holding items constant and secondly, she incorporated a cyclic pattern in her item selection. These were evident when she chose to dress two or three bears at a time using pairs of identical items and systematically varying the third. The weakness of her model however, was that she did not exhaust her major and minor items before selecting new ones. This did not occur until she tackled the final written problem.

On the first written problem, Kerry only used the cyclic component of her model. Unlike James who was confident that he could solve the problem mentally, Kerry attempted a concrete representation. Presumedly, her experience in solving the hands-on problems taught her the value of representing a problem in concrete terms. When this representation did not work for Kerry, she turned to a diagrammatic format involving literal symbols connected by lines to denote the possible combinations. This enabled her to generate all of the required combinations. On the second and third written problems however, Kerry incorporated her strategy of holding items constant and finally succeeded in exhausting each constant item.

We will revisit Kerry's construction of combinatorial knowledge in the discussion section and will turn now to her comments on the similarities between the two problem sets. When asked whether solving the hands-on problems assisted her in solving the written examples, Kerry replied:

> Yes, because you knew what you were supposed to do. Well, you could look at them and see what colors they were and then you could just join them up like I did.

When questioned on ways in which the two problem sets were similar, Kerry responded, "All of them are. They have all got colors." She was unable to suggest any other ways in which the problems were similar. Given Kerry's limited explanations throughout the problem session, it is not surprising that she was unable to comment further. However it seems that the transparency feature of the problems, that is, the similarity in the nature of the problem items, had a greater impact on Kerry than the systematicity component, that is, the similarity in relational structure. This contrasts with James' ability to detect the structural correspondences as well as similarities and differences in surface features.

Discussion

It is interesting to speculate on why Kerry was better able to construct combinatorial ideas than James, given that neither child had met these problems in school. The fact that Kerry worked the hands-on problems prior to the written examples may have contributed to her progress. On the other hand, unlike James, Kerry did not seem to perceive the structural correspondence between the two problem sets. She was also not as efficient in her use of monitoring procedures.

What Kerry did demonstrate though, was an active search for more efficient methods of problem solution and, in so doing, was able to modify and extend her initial mental model of the problem. Being aware of the limitations of generating a solution mentally, she appreciated the value of concrete and diagrammatic representations in formulating her ideas and made effective use of these. Kerry's efforts at streamlining the solution process were particularly evident in the second and third hands-on problems ($\underline{X} \times \underline{Y} \times \underline{Z}$) where she formed two or three outfits simultaneously by holding two item pairs constant and varying the third pair. Her preoccupation with this procedure however, meant a decline in her monitoring processes with the result that she could not correct all of her errors. Her acquisition of the expert odometer strategy alleviated this difficulty. However Kerry could not rely solely on such a procedure if she were presented with a similar problem which comprised a hidden constraint on goal attainment. One such task involves the use of clothing items with different numbers of buttons instead of colors, with the goal being to form all possible different button totals. Identical totals could result from combinations formed from different items (e.g., a shirt with 2 buttons combined with a skirt with 5 buttons and a shirt with 3 buttons matched with a skirt with 4 buttons). Children who rely on their efficient procedures for solving such a problem and do not monitor their actions thoroughly, or at best, monitor only the items and not the button totals, do not detect the duplicated combinations (English, 1992).

James presents a particularly interesting case. He was rated as a high achieving student in mathematics, had an effective command of language, was skilled in monitoring his actions, and demonstrated insight into his own thought processes. In contrast to Kerry, he had confidence in his mental capacity and relied on this to solve the written problems. While he did an admirable job of keeping track of his combinations, he nevertheless was unable to solve the problem. His reluctance to use some form of written notation was a major stumbling block. In his search for an efficient method of solving the most difficult written problem, James reflected on the data of the previous problem. This prompted him to retrieve a formal mathematical "theory" which he inappropriately mapped onto the data of this final written problem. It is tempting to speculate that James viewed problem solving in terms of retrieving an (apparently) appropriate rule and applying it to the problem at hand. Nevertheless, James did see the value of concrete representation after solving the hands-on problems. It is difficult to say whether James would have made better progress if he had been given these hands-on problems initially. While he did recognize the structural similarity between the problems, he still did not construct any sophisticated combinatorial ideas in solving these latter problems. This suggests that he might not have been as successful as Kerry.

At this point, it is worth reviewing the children's knowledge construction in terms of domain-specific principles (English, 1990; Gelman & Greeno, 1989; Gelman & Meck, 1986). Recall that our analysis of the problem structure highlighted the constraints imposed on the child by the problem goal. The children's detection and correction of duplicated combinations indicates that they were attempting to meet the constraints on similarity across combinations. This suggests that they had at least an implicit knowledge of the principle of difference. This principle asserts that two or more combinations of items will be different from each other if they differ in at least one item (English, 1990). An understanding of this principle is the minimum requirement for operating in the first stage of strategy construction.

As the children moved into the transitional stage they began to construct strategies that involved selecting items in a systematic cyclic pattern, as well as holding one or more items constant. These actions imply at least an implicit knowledge of two major combinatorial principles, namely, the principles of systematic variation and constancy (English, 1990). We define these principles as follows:

Principle of systematic variation.

Within the one cycle of selection from discrete sets of items (i.e., the selection of an item from each set), different combinations will result if at least one type of item is varied systematically. For example, given the sets of items X_{1-2}, Y_{1-3}, and Z_{1-2}, different combinations will result if the Y items are systematically varied (e.g., $X_1/Y_1/Z_1$; $X_2/Y_2/Z_2$; $X_1/Y_3/Z_2$).

Principle of constancy.

Within the one cycle of selection from discrete sets of items, different combinations will result if at least one type of item is held constant while items of at least one other type are varied systematically (e.g., $X_1/Y_1/Z_1$; $X_1/Y_2/Z_2$; $X_1/Y_3/Z_1$).

Kerry's transition into the final odometer stage reflected the construction of two new combinatorial ideas, namely, the exhaustion of constant items and recognition of problem completion (English, 1990). We refer to these as the principles of exhaustion and constancy:

Principle of exhaustion.

A given constant item is exhausted when it can no longer generate combinations which are different from existing combinations.

Principle of completion.

All possible combinations will have been generated when all constant items have been exhausted.

An implicit understanding of the four principles of constancy, systematic variation, exhaustion, and completion may be regarded as essential to children's progression to the final stage of strategy construction. In contrast to Kerry, James did not demonstrate a knowledge of constancy and hence did not move past the transitional stage. Previous studies (e.g., English, 1988) have shown that once children have constructed the odometer strategy, they can clearly explain their procedure and can justify why it is the most efficient for problem solution. Their explanations reflect an understanding of each of these principles.

The children's responses to the present problems raise a number of important issues for mathematics education. Firstly, the study illustrates how children can construct important mathematical ideas during the course of solving novel problems set within meaningful contexts. When children lack formal domain knowledge, they rely on their existing informal models of the problem situation, together with their domain-general strategies, to generate a solution.

Secondly, children's level of achievement in school mathematics is not a reliable predictor of their ability to solve novel problems. This was evident in analyses of the entire sample of study participants where low achievers were observed to progress to the odometer stage on the present problems and also outperformed their high-achieving counterparts in solving deductive reasoning problems (English, 1993a). These low achievers were able to quickly detect relationships and connections between items of information and used these to streamline the solution process.

The third issue pertains to the mathematics presented to seemingly "bright" students in mathematics. While not denying that many high-achieving children thrive on abstract rules and procedures, there is the danger of advancing these students in an inappropriate direction. Challenging students with sophisticated mathematical concepts, without affording them the opportunities to generate the ideas themselves, can contribute to children's construction of inadequate (and erroneous) mathematical models. While James' may not have acquired his "theory" in this manner, his preoccupation with a formal mathematical rule did inhibit his approach to problem solution and limited his construction of more advanced mathematical concepts. His reluctance to use concrete representations in modeling and solving the written problems was also a major contributing factor.

The final issue addresses assessment practices in mathematics classrooms. Assessment of students' mathematical competence must include a range of novel problems that do not require children to draw upon previously learnt mathematical rules. Such problems should permit a variety of solution strategies and enable those children who experience difficulty with formal computational procedures to construct their own methods of problem solution. Children of this type, who are often classified as low achievers, are capable of building sophisticated mathematical concepts and should be given frequent opportunities to demonstrate this.

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Appendix

Set of Written Problems

1. Tim has a green bucket, a red bucket, and a blue bucket. He also has an orange spade, a purple spade, and a blue spade. How many different spade and bucket sets can he make (a set has one bucket and one spade)?

2. Marianne has a yellow T-shirt and a white T-shirt. She also has a green skirt and a red skirt. With these, she can wear orange sandals or blue sandals. How many different outfits can she make with these items? (an outfit must contain a top, a skirt, and a pair of sandals)

3. The Select-A-Card company plans to make new boxes of greeting cards. In each box there will be greeting cards that are:

either GREEN or YELLOW, and have

either CHRISTMAS greetings or BIRTHDAY greetings or EASTER greetings, and have either GOLD LETTERING or SILVER LETTERING.

How many different greeting cards will there be in each box?

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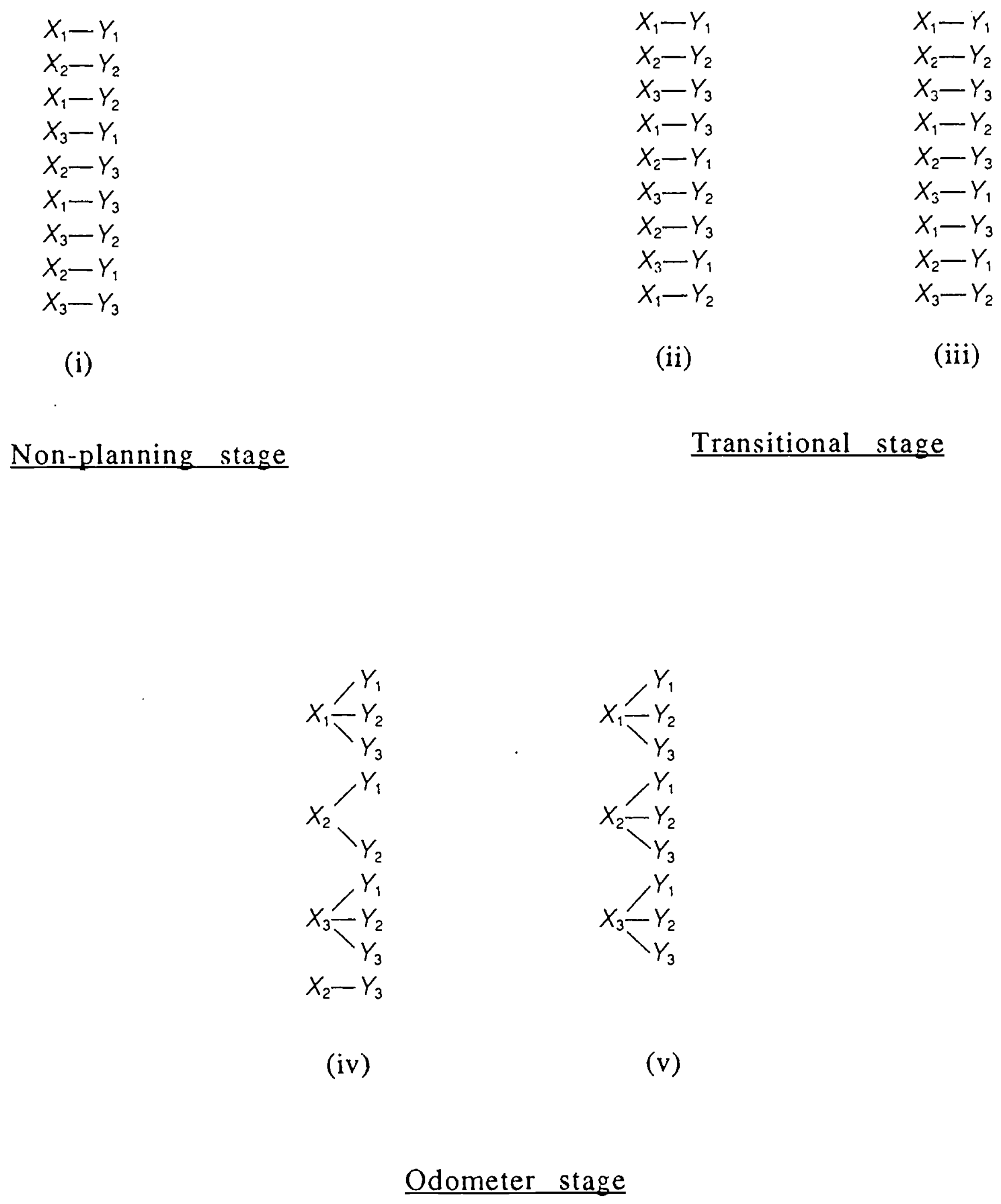

Figure 1. Three stages in the construction of two-dimensional combinatorial strategies.

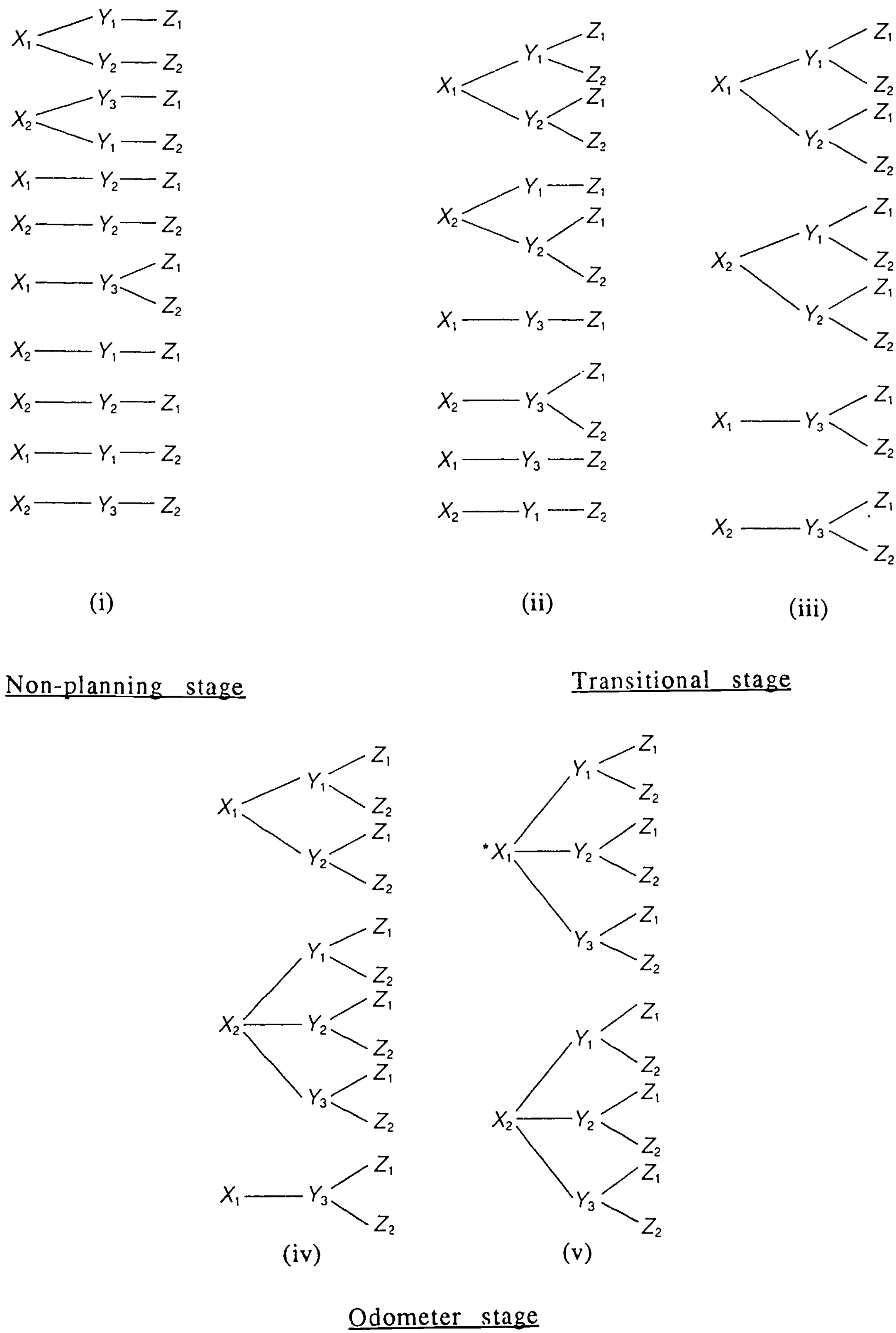

Figure 2. Three stages in the construction three-dimensional combinatorial strategies.

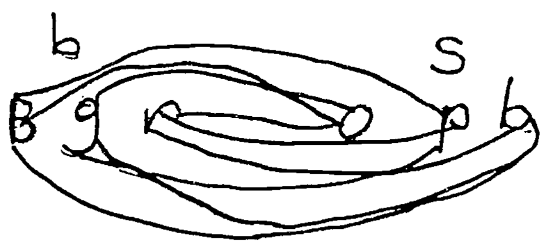

Figure 3. Kerry's notation for the first written problem.