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AUTHOR	Ebert, Christine L.
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ABSTRACT

The study discussed in this paper examined the pedagogical content knowledge of prospective secondary mathematics teachers with respect to the content area of functions and graphs. The pedagogical content knowledge was examined through an analysis of the transformation of knowledge and beliefs about the content and the learner into instructional practices. It is proposed that the transformation takes place through developing explanations, planning lessons, simulating teaching, and reflecting on teaching. The study documented the subject matter knowledge, the pedagogical content knowledge, and the beliefs about learners and mathematics as this potential transformation was taking place. An assessment of 11 prospective secondary mathematics teachers based on 5 tasks was designed to present a composite view of their pedagogical content knowledge in the area of functions and graphs. The task discussed in this paper was an audiotaped interview of prospective teachers' responses to vignettes of students solving problems in this content area. This task provided the opportunity for the prospective teachers to construct explanations in response to students' misconceptions. This task is considered important because it serves as the link between those tasks which examine baseline data sources and those tasks which examine the transformation process. Vignettes of teaching functions and graphs are included. (Contains 30 references.) (LL)

AN ASSESSMENT OF PROSPECTIVE SECONDARY TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE ABOUT FUNCTIONS AND GRAPHS

Christine L. Ebert

College of Education

University of Delaware

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ABSTRACT

This study deals with the pedagogical content knowledge of prospective secondary teachers with respect to the content area of functions and graphs. Shulman's description of pedagogical content knowledge as the knowledge " which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (Shulman, 1986, p.9) served as the working definition for this study. Thus, pedagogical content knowledge was examined through an analysis of the sources of pedagogical content knowledge and the transformation of the knowledge and beliefs about the content and the learner into instructional practices. The model which served to illuminate the transformation of subject matter knowledge into pedagogical content knowledge suggests that this transformation takes place through developing explanations, planning lessons, simulating teaching, and reflecting on teaching. The primary objective of this study was to document the subject-matter knowledge, the pedagogical content knowledge, and the beliefs about learners and mathematics as this potential transformation took place.

An assessment of 11 prospective secondary mathematics teachers on a variety of tasks designed to present a composite view of their pedagogical content knowledge in the context of functions and graphs was conducted. Tasks were designed to assess different aspects of pedagogical content knowledge within the discipline of mathematics and the content area of functions and graphs. The methods class of these prospective teachers was chosen as a research site because it presented the opportunity to examine evidence of these transformations on a variety of assessment tasks. These tasks include 1) a written assessment of subject-matter knowledge of functions and graphs; 2) audiotaped interviews of the prospective teachers' responses to vignettes of students solving problems in this content area; 3) a unit plan dealing with key topics within the content domain; 4) a video-taped peer-lesson on quadratic functions; and 5) two stimulated recall interviews of the quadratic functions videotape both before and after reading <u>Standard 6</u> on functions; and several other post task interviews.

For the purposes of this paper, one aspect of this study will be presented - the relationship between subject-matter knowledge and pedagogical content knowledge suggested by the results of the vignettes dealing with student misconceptions of functions and graphs. This particular task, which provided the opportunity to construct explanations in response to students' misconceptions, is important because it serves as the link between those tasks which examine baseline data sources of pedagogical content knowledge and those tasks which examine the transformation process. Thus, it serves two purposes: 1) to reveal evidence of subject-matter knowledge, beliefs about learners and learning mathematics, beliefs about mathematics, and pedagogical knowledge; and 2) to provide information about the initial step in the transformation process - the development of explanations.

STATEMENT OF THE PROBLEM

The Call for Reform

The current reform movement in mathematics education is in direct response to the widespread dissatisfaction with the way in which school mathematics is being taught. During the past two decades, school mathematics was driven by a preoccupation with behavioral objectives and thus, placed too much emphasis on the memorization of procedures and isolated facts. Though test scores on basic skills showed some improvement during this era, test scores on reasoning, problem-solving, and higher order thinking skills declined. This decline has been attributed to instruction that did not examine the central ideas of mathematics in ways that promote understanding. Fortunately and coincidentally with the revolution in how mathematics is practiced, there has also been a revolution in understanding how children learn mathematics. Research on learning shows that children are not passive learners who simply absorb knowledge but learners with information structured in long term memory who actively structure incoming information and attempt to fit it into their established cognitive framework. Thus, the emphasis in mathematics learning as a result of reform movement, characterized by the Curriculum and Evaluation Standards, is placed on examining various representations of a concept and developing connection between those representations. Solving multi-step problems and utilizing appropriate representations in the solution process replaces the memorization of isolated facts and displays of algorithmic dexterity. As a result, classroom teachers (expert, novice, and prospective) who are encouraged to convey to their students the processes in which mathematics is discovered and communicated experience new demands on their teaching.

The Arena of Change

Much of the failure in school mathematics is due to a tradition of teaching that is inappropriate to the way most students learn. Though research on learning shows that most students cannot learn mathematics effectively by listening and imitating, most teachers teach mathematics in just this way. If the changes in teaching and learning advocated by the National Council of Teachers of Mathematics are to be accomplished, they are dependent on teachers (both current and prospective) restructuring their knowledge and thus, their instructional practices.

Research on teaching indicates that it is an extremely complex endeavor. Shulman (Shulman & Grossman, 1988; see also Wilson, Shulman, Richert, 1987) developed a theoretical model of domains of teachers' professional knowledge. They hypothesized that teachers draw from seven domains of knowledge - or sets of cognitive schemata - as they plan and implement instruction: knowledge of subject-matter, pedagogical content knowledge, knowledge of other content, knowledge of the curriculum, knowledge of learners, knowledge of educational aims, and general pedagogical content knowledge. In their research program, they focused primarily on the subject matter knowledge and pedagogical content knowledge - providing extremely detailed definitions and findings about their relationships to classroom practice. Thus, we know that teachers' knowledge is an important and integral component of how they teach even as we struggle to understand all of the factors involved.

It has been suggested (Ball, 1988) that the reform view of what it means to know and do mathematics is very different from the mathematics instruction of both current and prospective teachers. Researchers (Stein, Baxter, & Leinhardt, 1990) suggest that the subject-matter knowledge necessary to support the instruction that will foster this new view of mathematical competence remains underspecified. Thus, the current reform movement in mathematics education advocates both an increased attention to the mathematical preparation of prospective teachers and to how that preparation impacts advocated instructional practices. They strongly suggest (Stein et al., 1990) that the realization of this new view of mathematical competence will not take place without systematic attention to subject-matter knowledge and a careful investigation of the links between the current and desired levels of teacher knowledge and their instructional practices. Thus, it is clear that subject-matter knowledge is considered a key source of pedagogical content knowledge.

This reform view of mathematics classrooms and mathematical competence also suggests that teachers need to acquire and implement very different pedagogical content knowledge. Brown and Borko (1992) suggest that because pedagogical content knowledge is unique to the profession of teaching, we expect it to be relatively undeveloped in novice teachers, and thus a primary focus of their educational experiences. They note that in several studies, "novice teachers showed evidence of growth in pedagogical content knowledge as a result of teaching and preparing to teach" (p.221). However they go on to suggest that the growth processes of student teachers exhibited discontinuities. Struggle, and often failure to construct powerful ways of representing the subject matter to students, as well as inefficiency in presenting the subject matter are characteristic of student teachers' experiences. In the studies of in-service programs for experienced teachers, the results indicate that they too struggle to expand their pedagogical content knowledge and in fact, " growth in content knowledge appears easier than growth in pedagogical content knowledge" (Brown & Borko, p. 221). These findings support the recommendation that "acquisition of pedagogical content knowledge should be a central priority in pre-service teacher education programs as well as a continuing objective of in-service programs" (Brown & Borko, p. 221).

These research findings provide a first step in understanding the pedagogical content knowledge of pre-service and in-service teachers. To understand more fully the sources of mathematics teachers' instructional practices, it would be useful to assemble more complete accounts of the relevant knowledge base and their beliefs about the learner, learning mathematics, and mathematics from which they work. As we examine both subject-matter knowledge, pedagogical content knowledge and the links between them, it is important to study the prospective teacher's pedagogical content knowledge as evidenced in planning and teaching. By examining these links we may also access the beliefs of these prospective teachers about learners, learning mathematics, and mathematics itself. Through opportunities to assess the sources of pedagogical content knowledge as baseline data and to observe how these sources change and are integrated through the process of designing, implementing, and reflecting on instruction, we may begin to illuminate the steps in the transformation of subject-matter knowledge into pedagogical content knowledge.

The Focus of the Study

The purpose of this study is to describe the knowledge base of prospective mathematics teachers. I have chosen to examine secondary mathematics teachers and to consider their relevant knowledge of a central topic in the secondary curriculum - functions and graphs. The knowledge which will be examined is that which is most directly relevant to their future instructional practices. Many researchers describe this knowledge that enables a teacher to make a subject comprehensible to a student as pedagogical content knowledge.

The present study was designed to investigate the pedagogical content knowledge of secondary teachers with respect to the content domain of functions and graphs. Pedagogical content knowledge was examined through an assessment of subject-matter knowledge in the content area, instructional practices and strategies revealed through unit plans, teaching simulations, and reflection on teaching, and the beliefs about the learner and mathematics revealed through both baseline assessment tasks and instructional design tasks. An important goal of this study was to obtain descriptions of *prospective teachers as they transformed subject-*

matter knowledge into pedagogical content knowledge through the process of designing instruction. The study proposes a framework of analysis drawn from the research on teaching (Brophy, in press; Shulman, 1986) and learning in the content area (Leinhardt, Zaslavsky & Stein, 1990) designed to illuminate the steps in this transformation. Whereas previous work considered teacher change as experienced by prospective elementary teachers and did not concentrate on a particular content domain, this study focuses on describing and documenting: 1) the sources of pedagogical content knowledge -- specifically, subject-matter knowledge, pedagogical knowledge, beliefs about learners and learning mathematics, and beliefs about mathematics; 2a) the transformation of their knowledge and beliefs about the content and the learners into pedagogical content knowledge revealed through developing explanations, planning, teaching, and reflecting on teaching; and 2b) the features of pedagogical content knowledge displayed through their explanations, analogies, examples, representations, and demonstrations.

BACKGROUND AND RATIONALE

Sources of Pedagogical Content Knowledge

Shulman describes pedagogical content knowledge as the knowledge " which goes beyond knowledge of subject matter per se to the dimension of subject-matter knowledge for teaching" (Shulman, 1986, p.9). He suggests that for a particular subject area such as mathematics, pedagogical content knowledge includes " the most useful representation of those ideas, the most powerful analogies, illustrations, examples, explanations, demonstrations -in a word the ways of representing and formatting the subject that make it comprehensible to others" (Shulman, 1986, p.10). The following diagram provides a model of the sources of pedagogical content knowledge which serve as the theoretical framework for this investigation. This model served as a vehicle to examine pedagogical content knowledge by describing and documenting some of the sources of pedagogical content knowledge and by illuminating the steps in the transformation of subject-matter knowledge into pedagogical content knowledge.

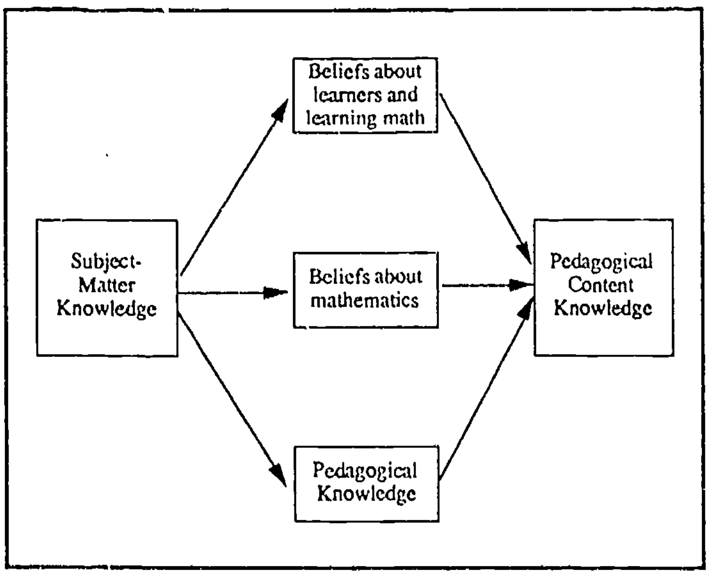


DIAGRAM I: SOURCES OF PEDAGOGICAL CONTENT KNOWLEDGE

Pedagogical content knowledge is viewed as the transformation of subject-matter knowledge through the lenses of beliefs about learners and learning mathematics, beliefs about mathematics, and pedagogical knowledge. Though subject matter knowledge is clearly an essential component of pedagogical content knowledge, it is not sufficient to define it. Beliefs about the learner and the content as well as knowledge about teaching also serve as sources of pedagogical content knowledge. It is in the process of designing and presenting instruction (developing explanations, planning lessons, teaching lessons, and reflecting on teaching) that the teachers' subject-matter knowledge is transformed. It is subject-matter knowledge that empowers the teachers with confidence and enables them to make connections, build analogies, create examples, and take intellectual excursions that lead to future mathematical discoveries. However, this transformation is also based on the knowledge and beliefs of the teachers about the learners and about mathematics. This study proposes to investigate pedagogical content knowledge because it is the knowledge that most fundamentally provides the links between subject-matter knowledge and instructional practice.

Subject-Matter Knowledge

An investigation of prospective secondary teachers' pedagogical content knowledge may be operationalized in the context of a specific content domain. Subject-matter knowledge of mathematics at the secondary level may be examined through a specific topic within the content domain. The mathematical topic functions, graphs, and graphing has received considerable attention of late (Leinhardt, Zaslavsky, & Stein, 1990). The significance of this study is best understood if it is viewed within the conceptual frameworks dealing with research on teaching (Brophy, in press; Shulman, 1986) and research on learning in the content area (Leinhardt et al., 1990). Within the content area of functions and graphs, a significant body of research exists which describes students' misconceptions and difficulty learning the concept of function. While prospective secondary teachers generally exhibit fewer flaws in their conceptual understanding than prospective elementary teachers, these misconceptions do exist and are somewhat resistant to change (Leinhardt et al., 1990; Ebert, 1991). Thus, research which examines prospective teachers' baseline knowledge as they make the journey from student of mathematics to novice teacher may utilize previous research to analyze this knowledge. In this context, it is important to compare the prospective teachers' concept image of function (Vinner, 1982) with those of students. The question of how closely a students' concept image matches the mathematical definition of function have been investigated (Vinner, 1983; Even, 1989; Lovell, 1971; Markovits, Eylon, & Bruckheimer, 1983; 1986; Marnyanski, 1975; Thomas, 1975). Students tend to hold an image of functions as linear and expect graphs of functions to be "reasonable" and represented by equations. Constant functions, piece-wise functions, and functions obtained by composition are usually not included in their concept image of function. The hypotheses concerning prospective teachers' concept image of function do not predict the same level of understanding as that of students. However, if prospective teachers hold a limited and somewhat impoverished concept image of functions and graphs, they may have few tools with which to construct the analogies, examples, explanations, representations, and demonstrations that Shulman (1986) describes in his definition of pedagogical content knowledge. The current study provides this kind of information by investigating subject-matter knowledge of prospective teachers through responses to vignettes describing students' misconceptions.

The significance of a study on functions and graphs can best be described in terms of the importance of the functions and graphs in the secondary curriculum (NCTM Standards, 1989) and the view by Leinhardt, Zaslavsky, & Stein, (1990) that functions and graphs " represent one of the earliest points in mathematics in which a student uses one symbol system to expand and understand another (e.g. algebraic functions & graph, data patterns and their graphs, etc)" (Leinhardt et al.,1990, p.2). They suggest that algebraic and graphical representations are " two very different symbol systems that articulate in such a way as to jointly construct and define the mathematical concept of function" (Leinhardt et al., 1990, p.3). The current reform movement in mathematics education also emphasizes the importance of forming the connections between these two representation of functions. "Students who are able to apply and translate among different representations of the same problem situation or of the same mathematical concept will have at once a powerful, flexible set of tools for solving problems and a deeper appreciation of the consistency and beauty of mathematics" (NCTM Standards 1989, p.146). Thus, the content domain of functions and graphs is important both from the perspective of being a central topic in the secondary curriculum and also from the perspective of serving as a site in which conceptual understanding is dependent on connecting different representational forms. The study of pedagogical content knowledge of prospective secondary teachers within the content domain of functions and graphs provides the opportunity to examine the transformation of subject-matter knowledge into pedagogical content knowledge in an important and central topic in secondary mathematics.

Beliefs about Learners and Learning Mathematics

The close association of cognitive psychology with research in the teaching and learning of mathematics has contributed extensively to the view of the learner in mathematics. Students are no longer viewed as "blank slates" but as active constructivists of their own knowledge. The basic tenet that learners are active in structuring and inventing knowledge has important implications for teaching mathematics. Instruction cannot be viewed as the simple presentation, however carefully done, of the knowledge and skills to be acquired. Instruction must focus on the means to facilitate construction of mathematical knowledge (built on existing knowledge) through providing classroom settings in which students (learners) explore relationships, use those relationships as tools to solve problems, and communicate those findings with each other and the teacher. Thus a major component of pedagogical content knowledge is knowing how students think within specific content domains.

In addition to viewing pedagogical content knowledge as the ability to represent and formulate subject-matter knowledge so that it is comprehensible to others, some researchers (Even & Markovits,in press) also view pedagogical content knowledge as an understanding of what makes the learning of specific topics easy or difficult and the knowledge of conceptions and preconceptions that students of different ages and backgrounds bring with them to learning a specific topic. In particular, within the formal preparation of teachers, there has been little emphasis on the understanding of students' ways of thinking related to specific topics and the development of appropriate ways of responding to students' questions, remarks and ideas. To a large extent, prospective teachers construct their pedagogical content knowledge through reflection on their own schooling and watching their own teachers teach. Knowledge of students' conceptions and preconceptions is limited to at most a rational assessment of their own difficulties within a particular content domain. While this knowledge may be limited, the design of experiences to help them construct pedagogical content knowledge, must be based on accurate assessment of teachers' initial knowledge (beliefs) of students' conceptions in a specific content domain. These experiences can then enable the teachers to construct their pedagogical content knowledge in much the same way that students' understanding (conception) of a particular topic enables the teachers to build from these limited conceptions towards more sophisticated conceptions.

The studies of Carpenter, Fennema, Peterson, Chiang, & Loef (1989) use knowledge of children's thinking to study teachers' knowledge and suggest that teachers' knowledge in this area might have an important influence on classroom learning. The potential to conduct a study of teachers' decisions based on knowledge about children's thinking in a specific content domain was realized due to the extensive body of research on children's thinking in this content domain--specifically children's learning of addition and subtraction (Carpenter & Moser, 1983). The central question investigated through a series of studies funded as part of a National Science Foundation-sponsored project called Cognitively Guided Instruction, was to determine if the knowledge about addition and subtraction gathered through research would make a difference in the instructional decisions of teachers. These findings revealed that knowledge from research on learners' thinking can be used by teachers in ways that have impact on educational outcomes. The CGI studies provide evidence that teachers can attend to individual students when they have appropriate and well-organized knowledge. The knowledge which the CGI teachers had access to was both specific and well-organized. It appears that this knowledge empowers teachers to understand children's thinking in a ways they had been unable to do before.

These results emphasize the importance of teachers' knowledge about the learner in general (as a constructor of knowledge) and specifically (in a particular content domain) and how this knowledge has the potential to impact instruction and subsequently learning. However, as Hiebert and Wearne (1988b) point out, there does not exist a robust integrated set of knowledge available about most content areas within the mathematics curriculum. Thus, the efforts to empower and enhance teacher knowledge, especially teacher knowledge of students' understanding, must be based on research findings about students' understanding within a particular content domain.

Within the content domain of functions and graphing there have emerged a number of studies which utilize the research on students' thinking in this domain to investigate teachers' knowledge and beliefs about the learner (Even & Markovits, in press; Cooney, Wilson, & Shealy, 1992; Monk, 1991; Evan, 1993). Though the research on learning about functions and graphs continues, an extensive and integrated summary of research in this content area exists (Leinhardt et al., 1990). These studies which utilize research on students' thinking about functions and graphs to investigate teachers' knowledge and beliefs serve as a link to the type of research utilized in Cognitively Guided Instruction. This study also provides this kind of information by investigating pedagogical content knowledge of prospective teachers through vignettes of students' comments to problems dealing with the key topics within a unit on functions and graphs and the misconceptions many students exhibit in these topics. Studies which analyze teachers' responses to potential student misconceptions provide this linking information to designing studies that utilize this information to guide teacher development of this aspect of pedagogical content knowledge.

Beliefs About Mathematics

The beliefs or conceptions of mathematics held by teachers may impact the ways in which mathematics is characterized in classroom teaching. The model of the transformation of subject-matter knowledge into pedagogical content knowledge suggests that these beliefs about mathematics, the learner, and learning mathematics act as lenses through which subject-matter knowledge is transformed. Beliefs about mathematics are particularly important because they may affect the form in which the concepts and skills are conveyed. Cooney (1985) has argued that substantive changes in the teaching of mathematics such as those proposed by the current reform documents (NCTM Standards, 1989) will be slow in coming and difficult to achieve because of the basic beliefs teachers hold about mathematics. He notes that the most prevalent verb used by preservice teachers to describe their teaching is *present*. This suggests the existence of a fixed body of knowledge to be transmitted to the learners. The extension of this conception of how mathematics relates to education and its practice is an important one. The teachers' view of how teaching should take place in the classroom is strongly influenced by the teachers' understanding of the nature of mathematics rather than on what they believe to be the best way to teach (Hersh, 1986).

The use of vignettes of students' comments to problems within the content domain provide a means to assess the prospective teachers' beliefs about mathematics. In the study of Even and Markovits (in press), they describe several dimensions of analysis which relate to teachers' beliefs about the nature of mathematics. In their analysis of experienced teachers responses to vignettes of students' misconceptions, they noted that the majority of the responses were teacher-centered indicating a transfer of knowledge by direct telling, and few were rich in conceptual meaning with the majority comprising essentially procedural explanations of the problem. These kinds of results provide evidence of teachers' beliefs about the nature of mathematics and suggest the strong influence of beliefs about mathematics on pedagogical content knowledge.

Pedagogical Knowledge

The existence of schools of education and institutes of pedagogy serve as indicators of the importance of the acquisition of pedagogical knowledge. Our *methods classes* are devoted to the acquisition of pedagogical knowledge in general and also in the content. Thus it is important to define what we mean by pedagogical knowledge (as opposed to pedagogical content knowledge) and to be able to find evidence of its existence in the instructional practices of teachers. Pedagogical knowledge as defined by the model and in this study is the knowledge of the various schemas of classrooms which include different means of instruction (lecture, cooperative-group, guided discovery, for example) and different means of assessment. While pedagogical knowledge may exist without subject-matter knowledge, in the model suggested by this study, pedagogical knowledge is enriched and strongly influenced by subject-matter knowledge. However, it also is a source of pedagogical content knowledge in that subject-matter knowledge is not sufficient to describe the kinds of knowledge necessary for the transformation of students of mathematics to novice teachers of mathematics.

The use of vignettes of students' responses to problems in the content domain provide a means of assessing pedagogical knowledge by examining the structure of the teachers' responses and the types of activities they prescribe to answer the students' questions and alleviate the students' confusion. The prospective teachers' responses were closely tied to particular aspects of the content but, they also reflect choices about activities that reflect pedagogical knowledge. These kinds of results provide evidence of the contribution of pedagogical knowledge to pedagogical content knowledge and support the importance of pedagogical knowledge in designing instructional strategies.

The Transformation into Pedagogical Content Knowledge

The importance of pedagogical content knowledge and its impact on quality instruction is evident within the research on teaching. However, though pedagogical content knowledge is considered important, there have not been enough direct efforts to help teachers construct this knowledge (Even & Markovits, in press). Most subject-matter courses that teachers take are pedagogy-free and most teacher-education courses are content-free. With the exception of the traditional " methods" course, there has been little attempt to integrate both content and pedagogy. However, pedagogical content knowledge is the product of integrating knowledge (Ball, 1988). "When teachers represent mathematics they are influenced by what they know across different domains of knowledge: mathematics, learning, learners, and context" (Even & Markovits, in press, p.25). Thus teachers should be provided with the opportunities to construct their pedagogical content knowledge and research should focus on those sites where the construction or transformation takes place. Given the current situation in teacher education, it is too easy to simply conclude that prospective teachers don't have a comprehensive and well-articulated pedagogical content knowledge when they finish their formal preparation. The following model represents an attempt to illuminate the steps in the transformation of subject-matter knowledge into pedagogical content knowledge.

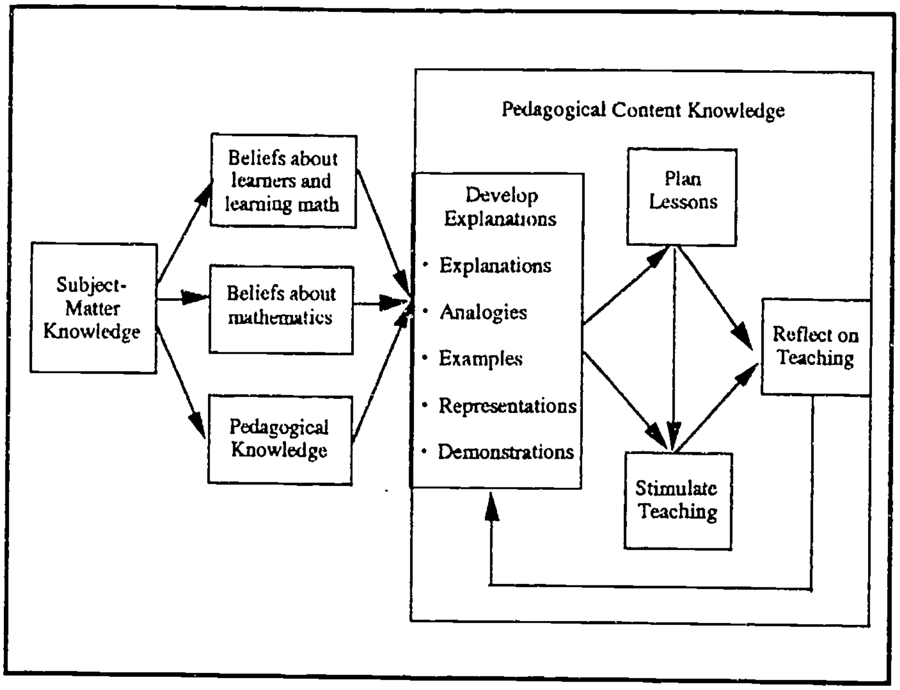


DIAGRAM II: THE TRANSFORMATION OF SUBJECT MATTER KNOWLEDGE INTO PEDAGOGICAL CONTENT KNOWLEDGE

In this model the sources of knowledge and beliefs which contribute to pedagogical content knowledge are indicated as well as the activities which facilitate the transformation of the knowledge and beliefs into pedagogical content knowledge. These activities include the development or construction of explanations which include not only explanations but also analogies, representations, examples, and demonstrations, the planning of lessons, the teaching or simulation of teaching, and reflection on teaching. The model also reflects, to some extent, the order in which these activities usually occur. The development of explanations is viewed as the initial means by which teachers transform their subject-matter knowledge, influenced by their beliefs about learners and learning mathematics, their beliefs about mathematics, and their pedagogical knowledge into pedagogical content knowledge. The planning, teaching and reflecting on teaching are all dependent of the strength of these explanations.

The important contributions of this kind of research lie in describing, within a particular content domain, the pedagogical content knowledge revealed through the development of explanations, the planning of lessons, the teaching of lessons, and the reflecting on that teaching. The current study provides this kind of information by investigating the pedagogical content knowledge of prospective teachers through vignettes of prospective teachers' responses to students' comments and questions in the content domain. The vignettes provided an opportunity to develop explanations prior to the task of writing unit plans. Thus, they enable the researcher to investigate the initial steps in the transformation process directly as well as providing evidence of subject-matter knowledge, beliefs about learners and mathematics, and pedagogical knowledge indirectly.

The following research questions which served to define the entire study of pedagogical content knowledge in the content domain of functions and graphs also provide the important framework for this study. While a series of tasks were used to investigate pedagogical content knowledge, the task of responding to vignettes of students' comments and questions was the only task that directly addressed both questions. Thus, the questions for the entire study also serve as questions for the study presented in this paper.

RESEARCH QUESTIONS

The primary objective of this study--to investigate the pedagogical content knowledge of 11 prospective secondary teachers with respect to the content domain of functions and graphs through an analysis of the sources of pedagogical content knowledge and the transformation of the knowledge and beliefs about the content and the learner into instructional practices--will be addressed by investigating the following questions.

1. What components of pedagogical content knowledge do these prospective teachers possess--specifically what evidence about their subject-matter knowledge, pedagogical knowledge, beliefs about learners and learning, and beliefs about mathematics does the baseline data reveal?
2. How will these prospective secondary teachers transform their knowledge and beliefs about the content and the learners into instructional practices?
    a) What evidence of their pedagogical content knowledge is revealed as they:
        1) develop explanations
        2) plan lessons
        3) simulate teaching
        4) reflect on teaching
    b) As they utilize these sources of knowledge and beliefs to develop explanations which are used in their instructional practices, what features of pedagogical knowledge will they display through their:

1) <u>explanations</u> - means of describing the concepts and procedures in the content domain of functions and graphs.

2) <u>analogies</u> - means of making the connections between the concepts and procedures a meaningful instantiation from the students' experience.

3) <u>examples</u> - those contextual situations or numerical problems that the teacher chooses to illustrate the concepts and procedures.

4) <u>representations</u> - those specific symbolic vehicles which the teacher chooses to illustrate a particular concept. In the context of functions and graphs the mode of representation may be a graph, equation, table of values, or a specific type of narrative situation that is characteristic of a particular class of functions.

5) <u>demonstrations</u> - those instructional strategies which are usually used to make the concepts and procedures more concrete. These may include demonstrations of some physical phenomena or a demonstration of graphing functions using a graphing calculator.

METHODS

<u>Sample</u>

The sample for this study consists of 11 prospective secondary mathematics teachers enrolled in a secondary methods class at a major state university in the mid-atlantic states. The methods course occurs in the Fall semester prior to student-teaching in the Spring of their senior year. Eight of the students are scheduled to do their student-teaching during the upcoming semester at the area high schools. There are 8 female students and 3 male students participating in the study.

Tasks

This study will provide an assessment of prospective secondary mathematics teachers on a variety of tasks designed to present a composite view of their pedagogical content knowledge about functions and graphs. These tasks include 1) a written assessment of subject-matter knowledge of functions and graphs; 2) written responses and audiotaped post-task interviews of the prospective teachers' responses to vignettes of students solving problems in this content area; 3) a unit plan dealing with key topics within the content domain of functions and graphs; 4) a video-taped peer lesson on quadratic functions; 5) two stimulated recall interviews of the quadratic functions videotape both before and after reading Standard 6 on functions; and follow-up data including other post-task interviews. The five tasks will provide a composite view of the prospective teacher's pedagogical content knowledge. The following schedule indicates how and when the tasks and post-task interviews were scheduled throughout the semester.

Week 2	Beliefs Scale	Background information and Beliefs Scale
Week 3	Task I	Written assessment of subject-matter knowledge.

Week 4	Interview I	Interview on written assessment
Week 5	Task II	Student vignettes
Week 7	Interview II	Audio-taped Interview on vignettes
Week 9	Task III	Lesson Plans due
Week 11	Task IV	Video-taped peer lesson on quadratic functions
Week 12	Interview III	Stimulated recall interview on the video-taped quadratic function lesson.
Week 13	Interview IV	Interview on the video-taped quadratic function lesson after reading Standard 6.

Task II and Interview II

The second task was an assessment of prospective teachers understanding of student conceptions and misconceptions in the content domain of functions and graphs (Research questions 1 & 2). In this task the subjects were presented with five scenarios or vignettes describing student comments. The subjects were asked to explain the students' ideas or comments and then were asked to describe how they would respond to the student/s in the situation. The content of the vignettes represents the four major topics on functions and graphs: 1) definition; 2) notation and evaluation; 3) composition of functions; and 4) inverse functions as well as one of the major student misconceptions in the literature on functions and graphs - 5) picture-as-graph misconception.

Insert Table 1 about here

The first vignette presents a function in which the graph does provide a correct image of the motion of the projectile. The subject is asked to respond to the student's comment and (hopefully) offer some further explanation. Research on learning functions suggest that students can deal with functions in a point-wise manner but experience difficulty examining the behavior of a function over an interval, in a global way, or as an entity (Bell & Janvier, 1981; Even, 1989; Janvier, 1978; Marnyanskii, 1975; Monk, 1988). These studies also point to the picture-as-graph misconception and examine its persistence in students thinking. Thus, including this vignette provides the opportunity to assess prospective teachers' subject-matter knowledge of this potentially difficult concept as well as examine their knowledge of students' conceptions.

The second vignette presents a piece-wise defined function. The subject is asked to respond to the student's comment that suggests the misconception that functions must be linear. Thus this vignette addresses the misconception evident from the literature and the basic concept of definition of function. Its inclusion provides another opportunity to examine the prospective teachers' knowledge of the definition of function and assess their knowledge of students' conception of function.

The third vignette presents an evaluation problem in which the student must possess a clear understanding of how to construct the function of x+1 when given the rule for f(x). The subject is asked to evaluate the incorrect solution in terms of the source of the errors and suggest a response to these solutions. This vignette is closely related to the research on student's concept image of function as well as being a major topic within this content domain.

The fourth vignette presents a function obtained by composition and asks the subject to respond to student comments about the nature of the two functions f(x) and g(x) where h(x) = f[g(x)]. This vignette is also closely related to the nature of the student's concept image and the way they use the symbols to express operations with functions. The subject is asked to evaluate the comments and suggest responses to clear up the confusion. This vignette also provides the opportunity to examine subject-matter knowledge and assess the prospective teachers' responses to a genuine student dilemma.

The fifth vignette presents the problem in which students have been asked to determine the inverse function of a given function. The students' comments basically reflect confusion with the order of operations but also are related to how they use symbols to express operations with functions. The subject is asked to respond to the comments and suggest responses to clear up the confusion.

After the subjects had responded to these vignettes in writing, they were be scheduled for a post-task interview. These interviews were audio-recorded and explored their responses in detail. For each of the vignettes they were asked to explain the students' thinking and comments and why they responded as they indicated. These interviews provided the opportunity to further explore their evolving pedagogical content knowledge and continue to assess their subject-matter knowledge.

Data Analysis

The data analysis of the responses to the five vignettes for each of the 11 prospective teachers consisted of recording the kinds responses in each of the following categories: subject-matter knowledge; beliefs about learners and learning mathematics; beliefs about mathematics; pedagogical knowledge; and explanations. These were recorded for each vignette and each of the eleven teachers. From this initial analysis, clear strengths and weaknesses for each of the categories emerged. These relative strengths and weaknesses seemed to fit well within the framework proposed by Thompson (1991) of the development of teachers' conceptions of mathematics teaching. This framework consists of three levels, each of which is characterized by conceptions of:

1. What mathematics is.
2. What it means to learn mathematics.
3. What one teachers when teaching mathematics.
4. What the role of the teacher and student should be.
5. What constitutes evidence of student knowledge and criteria for judging correctness, accuracy, or acceptability of mathematical results and conclusions.

While these conceptions don't match the categories which describe the sources of pedagogical content knowledge and the initial step in the transformation of subject-matter knowledge into pedagogical content knowledge exactly, the fit is sufficiently close to examine the results in terms of the levels proposed by Thompson.

Level 0

At level 0 the view of mathematics is based of the utility of basic skills in everyday life. The implication for instructional practices is that the major emphasis is on developing students' arithmetic skills through the acquisition of facts, rules, formulas, and procedures which are not connected to the underlying concepts. The role of the teacher is one of demonstrating well-established procedures to the students who will in turn practice them until they become habituated.

Level 1

At level 1 the conception of mathematical knowledge is broadened "from rote procedural proficiency to include an emerging appreciation for understanding the concepts and principles behind the rules" (Thompson, 1991, p.10). At level 1 there is also an emerging awareness of the use of instructional representations to help students develop meaning and understanding. However, the utilization of these representations is limited to explaining isolated concepts, procedures, algorithms, and formulas. The ability to generalize these instructional strategies to other topics is usually not present. Thus, the conceptions of teaching at this level are characterized by a limited view of the " possible uses of representations for achieving cognitive objectives of instruction" (Thompson, 1991. p.11).

Level 2

At this level the view of how mathematics should be taught is characterized by the belief that students must engage in mathematical discussions and inquiry. " The development of students' mathematical reasoning in the context of investigating and constructing mathematical ideas is as important a goal of instruction as their understanding of the ideas themselves" (Thompson, 1991, p.12). Thus, the view of teaching for understanding which develops at Level 1 is replaced at Level 2 by the view that mathematical understanding emerges from the process of doing mathematics - conjecturing, refuting and validating conjectures, and generalizing. At level 2, representations now serve as vehicles for making connections between concepts and procedures. The role of the teacher is one of facilitator, guiding students' thinking in mathematically productive ways while questions serve to stimulate mathematical inquiry rather than to merely elicit answers. There is also the increasing awareness of the " subtleties inherent in mathematical ideas that pose cognitive obstacles for students and lead to common misconceptions" (p. 13). Thus, the explicit utilization of cognitively based principles for instruction, is the chief characteristic of this level.

RESULTS

The results reported here are related to the categories which describe the sources of pedagogical content knowledge and the initial step in the transformation of subject-matter knowledge into pedagogical content knowledge in the content area of functions and graphs. Five vignettes of students' conceptions and misconceptions were presented to the prospective teachers. For each of the sources of pedagogical content knowledge, the subjects' responses as characterized by Thompson's framework are described and a comprehensive analysis is provided. Examples of the kinds of explanations which form the initial step in the transformation process are also described and analyzed. Interpretations of the prospective teachers' sources of pedagogical content knowledge and the transformation process are discussed in the discussion section.

Subject-Matter Knowledge

In the prospective teachers' responses to the vignettes, subject-matter knowledge is revealed indirectly through their understanding of the students' misconceptions in the four major topics on functions and graphs: 1) definition; 2) notation and evaluation; 3) composition of functions; 4) inverse functions; and 5) the picture-as-graph misconception (Table 1). The hypotheses concerning prospective teachers' understanding of these topics and their concept image of function do not predict the same level of understanding as that of students. However, several of the teachers (3 out of 10) did exhibit inadequate (Level 0) subject-matter knowledge. This level was characterized by an inability to express the definition of function correctly, to use function notation sensibly, to diagnose errors, and by the presence of misconceptions concerning the path of a projectile. In an attempt to address the student misconception that functions must be linear, one prospective teacher wrote, " Some functions are linear, but not all of them are lines".

The majority of the teachers (4 out of 10) exhibited good subject-matter knowledge (level 1). These prospective teachers expressed the definition of function correctly and interpreted the graphical representation of a function to obtain information and suggest possible real-world situations described by the graph. Teachers at this level did experience some difficulty in diagnosing student errors in the function-evaluation vignette and did not distinguish between the graphical representation and the actual path of the projectile in picture-as-graph misconception vignette. These teachers also had some difficulty expressing the distinctions between constant versus variable functions. They focused on the surface features to address the student misconception that a possible decomposition of $h(x) = 2(x-5)^2$ could be $g(x) = (x-5)$ and $f(x) = 2$ where $h(x) = f[g(x)]$. Rather than describing $f(x)$ as a constant function, they provided the rationale that the variable, x, must be present.

According to Thompson's framework, teachers at level 2 facilitate students' understandings of the concepts and procedures used in one problem-setting to extend these mathematical ideas to other, seemingly different, problem settings. Based on these types of observations, several of the prospective teachers (3 out of 10) possessed strong (level 2) subject-matter knowledge. All of these teachers were able to use the definition of function correctly, to diagnose all of the evaluation errors, and to express the distinctions between the graphical representation and the path of the projectile. They were also characterized by the ability to extend students' conceptions in a partic.lar topic to future mathematics topics. One prospective teacher, Helen, was able to evaluate the potential decompositions suggested in the 4th vignette and suggest, " I might emphasize the third students' decomposition in a precalculus class since it lays a better foundation for the 'inner' and 'outer' functions used to apply the chain rule for derivatives".

Subject-matter knowledge, which serves as an essential component of pedagogical content knowledge, is also reflected through the beliefs of the teachers about learners and learning mathematics, about mathematics, and also through their pedagogical knowledge. Thus, it is difficult to separate evidence of subject-matter knowledge from evidence of the prospective teachers' beliefs about learners and learning mathematics. As we examine these teachers' beliefs about learners and learning mathematics through the instructional decisions they suggest in response to the vignettes, evidence of their subject-matter knowledge continues to emerge.

Beliefs about Learners and Learning Mathematics

Prospective teachers' beliefs about learners and learning mathematics may include conceptions of the role of students and teachers, beliefs about students' conceptions and misconceptions, and an understanding of what makes the learning of a particular topic easy or difficult. These beliefs are exhibited through the instructional decisions made by the prospective teachers and can also be characterized in terms of three levels of beliefs.

Three of the teacher (level 0) view responding to student misconceptions as an opportunity to set the student straight by telling them the rule or procedure. In some cases, as in the picture-as-graph vignette, they believe the student is actually correct. In all other opportunities to respond to students, these teachers chose to tell them the rule (vertical line test for determining whether the graph of a relation is a function) and show them their "algebraic" errors. These observations are certainly consistent with Thompson's framework in which the role of teachers at this level is to demonstrate well-established procedures while the students are expected to practice these until they are mastered.

The majority of the teachers (4 out of 10 at level 1) exhibit beliefs about learners and learning mathematics consistent with Thompson's framework. The importance of inquiry in seeking justifications for procedures on the part of the students is emerging. These teachers indicated an appreciation for discussion to resolve the conflicts posed in the fourth and fifth vignettes. They indicated that discussion in general is useful both for resolving conflicts and for validating conjectures. With respect to the issue of remediating the errors they observed in the third vignette, all of them agreed that re-learning was necessary. They suggested using numerical examples to uncover the conceptual errors (linearity - $(a+b)^2 = a^2+b^2$ and a replacement error) and following that investigation with practice problems. As with the discussion of subject-matter knowledge, teachers at this level believe that mathematics should be demonstrated and that learners should be told. They chose to resolve the issue of the graphical representation being equivalent to the path of the projectile by simply telling the students that there was a distinction and they employed the vertical line test as a means of demonstrating other classes of functions that were non-linear.

In contrast to the teachers in levels 0 and 1, those prospective teachers in level 2 (3 out of 10) exhibited beliefs about learners and learning mathematics that were significantly different and quite consistent with Thompson's framework. They indicated beliefs that students could and should examine various representations of functions to test their conjectures, and that making connections between representations is an important part of doing mathematics. With respect to the issue of whether functions are solely characterized by straight lines, Barbara suggested that the students could generate and examine characteristics of functions to test this hypothesis. Another prospective teacher, Mark, would build on students understanding of linear functions to examine cases of functions that are non-linear. All of the teachers at this level suggested examining the definition of function, citing the univalence criterion, rather than the vertical line test. With respect to the issue of remediating the errors evident in the third vignette, these prospective teachers indicated the importance of examining these misconceptions and expressed an awareness of the difficulties inherent in repairing these two misconceptions simultaneously. Thus, they exhibited an awareness of the " subtleties inherent in mathematical ideas that pose cognitive obstacles for students and lead to common misconceptions" and for these three in particular, they gave "careful consideration to shaping instruction so that it helps students make these subtleties explicit to themselves" (Thompson, 1991, p.13).

Beliefs About Mathematics

Beliefs about mathematics are particularly important because they may affect the form in which the concepts and skills are conveyed. Cooney (19857) notes that the most prevalent verb used by preservice teachers to describe their teaching is *present*. This suggests the existence of a fixed body of knowledge to be transmitted to the learners. The extension of this conception of how mathematics relates to education and its practice is an important one. The teachers' view of how teaching should take place in the classroom is strongly influenced by the teachers' understanding of the nature of mathematics rather than on what they believe to be the best way to teach (Hersh, 1986).

The prospective teachers in this study probably fall into two groups with respect to their beliefs about mathematics. Using Thompson's framework, six of the teachers are at level 1 or level 0 with few distinctions between them. These teachers hold the view that mathematics is largely rule-based and that it is the responsibility of the teacher to make students aware of these rules and procedures. They view misconceptions as simple " algebraic" errors due to carelessness and though they do suggest examining these errors, the "cure" is largely one of practicing procedures until they are habituated. If they believe that mathematics should make sense to them and to their students, they were unable to convey that belief.

The remaining teachers, (4 out of 10) who could be classified at level 2, are all characterized by their strong belief that mathematics should make sense and that students should engage in making sense of mathematics. The belief that mathematics should make sense comes through in all of their instructional practices. With respect to resolving the distinctions between the graphical representation and the path of the projectile, Mark suggests the importance of making the distinction between the visualization of the path which is distance vs. distance with the graphical representation which is distance vs. time. He suggest using all possible representations - table, graph, equation to make the connections and the distinctions explicit. With respect to the issue of remediating the misconceptions in evaluation evident in the third vignette, Helen suggests graphing all of the functions for f(x) and f(x+1) and examining which graph of f(x+1) represents a horizontal shift of one unit to the left. She would utilize technology, in the form of the TI-81 graphing calculator, to examine these graphical representations and validate the conclusions algebraically. It is evident from her suggestion that she believes that the graphical representation is clear and filled with meaning for the students. It is also evident that she believes that confronting these graphical discrepancies will be beneficial in addressing the algebraic misconceptions. With respect to the issue of whether functions are solely represented by straight lines, Barbara would use the analogy of "all squares are rectangle but not all rectangles are squares" to examine whether all functions must be straight lines. She would also encourage students confused about possible decompositions of $h(x) = (x-5)^2$ to graph both $f(x) = 2$ and $f(x) = 2x$ to resolve their confusion. Only after the students have explored these functions and their graphs and discussed the issues would she remind them of the distinctions between constant functions and variable functions. All of these teachers have consistently expressed beliefs consistent with Thompson's framework - that mathematics should make sense and that understanding emerges from the processes of engaging in authentic mathematical activities -making conjectures, examining those conjectures, refuting and validating those conjectures and generalizing.

Pedagogical Knowledge

Pedagogical knowledge as defined by the model and in this study is the knowledge of the various schemas of classrooms which include different means of instruction (lecture, cooperative-group, guided discovery) and different means of assessment. The use of vignettes of students' responses to problems in the content domain provide a means of assessing pedagogical knowledge by examining the structure of the teachers' responses and the types of activities they prescribe to answer the students' questions and alleviate the students' confusion.

The prospective teachers in this study also fall into two groups with respect to their pedagogical knowledge. Six of the teachers exhibit pedagogical knowledge that could be classified as level 1 or level 0, according to Thompson's framework, with few distinctions between them. These teachers value students' observations and believe that it is important to affirm their contributions. They also value making connections but rather than allowing students to make these connections, these teachers believe that they should make the connections explicit for the students. The importance of introducing procedures after concepts is also characteristic of these teachers. This is a valuable pedagogical tool, however the impact may be lost because of their belief that they are the sole source of authority. They all suggest having students demonstrate their solutions at the chalk board, as a means of resolving conflict. They view their role as one of advising, admonishing, and apprising the students. Thus their pedagogical tools are limited by focusing most of the flow of information on the path between teacher and student.

The other four teachers (level 1 or level 2) also value students input and offer praise for their inciteful comments. However these teachers also suggest a significant amount of student-to-student interactions and claim that group work is useful for sharing strategies and exploring and resolving conflicts. They also suggest the use of technology so that the source of authority shifts from the teacher to the students and the technological tools. These teachers are also characterized by the value they place on building on student understanding and extending that understanding with questions that elicit further mathematical knowledge. They stress the need for conceptual examinations of errors rather than re-teaching. Thus these teachers, who also possess strong subject-matter knowledge, also possess the pedagogical tools to construct the analogies, examples, explanations, representations, and demonstrations that Shulman (1986) suggests in his definition of pedagogical content knowledge. They also possess the beliefs about learners and learning mathematics, as well as beliefs about mathematics, that allow them to facilitate students construction of mathematical knowledge through authentic mathematical inquiry.

The Initial Step in the Transformation - Constructing Explanations

In this model the sources of knowledge and beliefs which contribute to pedagogical content knowledge are indicated as well as the activities which facilitate the transformation of the knowledge and beliefs into pedagogical content knowledge. These activities include the development or construction of explanations which include not only explanations but also analogies, representations, examples, and demonstrations, the planning of lessons, the teaching or simulation of teaching, and reflection on teaching. The model also reflects, to some extent, the order in which these activities occur. The development of explanations is viewed as the initial means by which teachers transform their subject-matter knowledge, influenced by their beliefs about learners and learning mathematics, their beliefs about mathematics, and their pedagogical knowledge into pedagogical content knowledge. The planning, teaching, and reflecting on teaching are all dependent on the strength of these explanations. The vignettes provided an opportunity to develop explanations prior to the task of writing unit plans. Thus they enabled the research to investigate the initial steps in the transformation process directly as well as providing evidence of subject-matter knowledge, beliefs about learners and mathematics, and pedagogical knowledge indirectly.

The kinds of explanations the prospective teachers provided for each of the vignettes reflect the evidence provided by the analysis of subject-matter knowledge, beliefs about learners and learning mathematics, beliefs about mathematics, and pedagogical knowledge. The explanations, while not surprising, do fall within the levels proposed by Thompson's framework. At this point it is useful to characterize these explanations in much the same way that subject-matter knowledge was characterized - <u>inadequate</u> (level 0), <u>good</u> (level 1), and <u>strong</u> (level 2). Explanations that could be characterized as <u>inadequate</u> consisted of just telling the students how to do the procedure, that their conjecture is false, or that a distinction exists. None of these explanations offered conceptual examinations of the topic explored in the vignette and most relied on purely procedural techniques. The use of representations was not present in these explanations nor were students encouraged to engage in mathematical inquiry. Students were viewed as passive recipients of information presented by the teacher. Three of the ten prospective teachers exhibited these types of explanations.

Explanations that could be characterized as <u>good</u> were generally based on concepts rather than procedures and provided an examination of why certain procedures were successful. Students were much more actively involved in these explanations. They were encouraged to explore numerical examples to resolve symbolic conflicts and when examples were presented by the teachers, group work involved with solving similar problems was suggested. Explanations were based on an examination of an error rather than the re-teaching of an incorrectly executed procedure. Four of the prospective teachers exhibited these types of explanations.

Explanations that could be characterized as strong were based on examining various representations of a concept and engaging in authentic mathematical activities to acquire understanding. Many times students served as demonstrators and the teacher as a facilitator. With respect to the first vignette dealing with the distinctions between the path of the projectile and the graphical representation, Mark's explanation utilized an enactive, iconic and symbolic trace of the path followed by the graph of the equation which characterizes the motion of the projectile as a function of time rather than distance. His explanation was a demonstration of the path with an examination of the connections and distinctions. With respect to the second vignette which deals with the issue of whether all functions are characterized by straight lines, Barbara suggests having students generate the characteristics of functions to determine whether other graphs (non-linear) may also fit the definition of functions. With respect to the third vignette which deals with the two evaluation errors, Helen's examination of the graphical discrepancies between f(x) and the various choices for f(x+1) utilizes technological tools, relies on students engaging in mathematical inquiry, and provides a model of the steps involved in examining and validating or refuting conjectures. Three of the prospective teachers out of the ten exhibited these types of explanations.

Discussion

What is evident from these explanations (the strong ones) is the lack of what we have traditionally thought of as explaining in mathematics - that is, telling or presenting. It is, of course, also the case, that these same three teachers also exhibited the strongest subject-matter knowledge. Thus, the connection between strong subject-matter knowledge and the willingness to abandon the traditional view of teaching mathematics as presenting information, is certainly present for these three teachers. Their beliefs about learners and learning mathematics also played an important role in determining the nature of these explanations. The belief that students can and do learn mathematics through their own mathematical inquiry in essential to assuming the role of facilitator rather than one of transmitting knowledge. It was also essential that these teachers believed that mathematics makes sense - that it is much more than a collection or rules and procedures. These beliefs about mathematics provided the teachers with the foundation to engage in mathematical inquiry and to value the importance of students making and testing conjectures. The pedagogical tools of utilizing student-to-student interactions and the use of technology as a source of authority were also important components of these explanations. Other means of engaging in mathematics were necessary to replace the traditional method of lecture/discussion. Thus, the strong explanations exhibited by these three teachers were dependent on all of the sources of pedagogical content knowledge - strong subject-matter knowledge, beliefs about learners and learning mathematics, beliefs about mathematics, and pedagogical knowledge. These explanations do not provide the entire profile of pedagogical content knowledge for these ten teachers. They do provide a glimpse of the first step in the transformation process and they do provide evidence of pedagogical content knowledge which may be revealed through the process of planning lessons, simulating teaching, and reflecting on their teaching.

CONCLUSION

The acquisition of pedagogical content knowledge is an extremely complex endeavor. While subject-matter knowledge is obviously an important source of pedagogical content knowledge, it is not sufficient to define pedagogical content knowledge nor does it serve as a proxy for research of pedagogical content knowledge. The model proposed in this study (diagrams I and II) suggest that pedagogical content knowledge may be investigated by an examination of its sources which include subject-matter knowledge, beliefs about learners and learning mathematics, beliefs about mathematics, and pedagogical knowledge, and by an examination of the transformation process which includes activities such as developing explanations, planning lessons, teaching, and reflecting on teaching. The task of responding to vignettes of students, conceptions and misconceptions served as the vehicle for investigation the sources of pedagogical content knowledge and the initial step in the transformation process - the development of explanations. Thus, both the scope of the investigation and the kind of task suggest that it is important to investigate pedagogical content knowledge through much more than just subject-matter knowledge alone.

The results of the prospective teachers' responses to the vignettes suggest the viability of investigating beliefs about the learners and the content and pedagogical knowledge. The kinds of responses, the choices about activities, and the subsequent nature of their explanations all provided evidence about their beliefs. The results also suggest that these initial explanations provide evidence of their emerging pedagogical content knowledge. When the teachers subsequently planned their units on functions and graphs, they had several sources of information including several textbooks to help them. Thus, their unit plans reflect other sources of what is important in the teaching of functions and graphs. However, the responses from the vignettes were prior to writing the units and undoubtedly, prior to examining many textbooks. Thus, their responses which reflect their beliefs were relatively uncontaminated by outside sources. These beliefs about learners, learning mathematics, and about mathematics itself will undoubtedly undergo some change as they begin their teaching careers. However, to attempt to provide the types of activities in the methods classes which will enable prospective teachers to examine their beliefs as they struggle to integrate them with their subject-matter knowledge and pedagogical knowledge, it is important to examine this kind of evidence.

The results concerning the explanations, which serve as the initial step in the transformation process, clearly indicate the strong connection between subject-matter knowledge and pedagogical content knowledge. Those teachers who were strong in subject-matter knowledge also provided the strongest explanations. It was interesting to discover that their explanations were very non-traditional. Clearly their beliefs and their pedagogical knowledge play an important role in their willingness to abandon the traditional methods of teaching mathematics.

These three teachers whose pedagogical content knowledge is in such sharp contrast to that of their peers provide hope for the future of mathematics instruction at the secondary level. The evidence provided from the study about the sources of pedagogical content knowledge for these teachers may serve to guide us as we develop programs of teacher enhancement and teacher change. It is clear that we must provide opportunities for teachers to view mathematics as sensible and learners as capable of engaging in authentic mathematical inquiry.

Table 1: Vignettes of Teaching Functions and Graphs

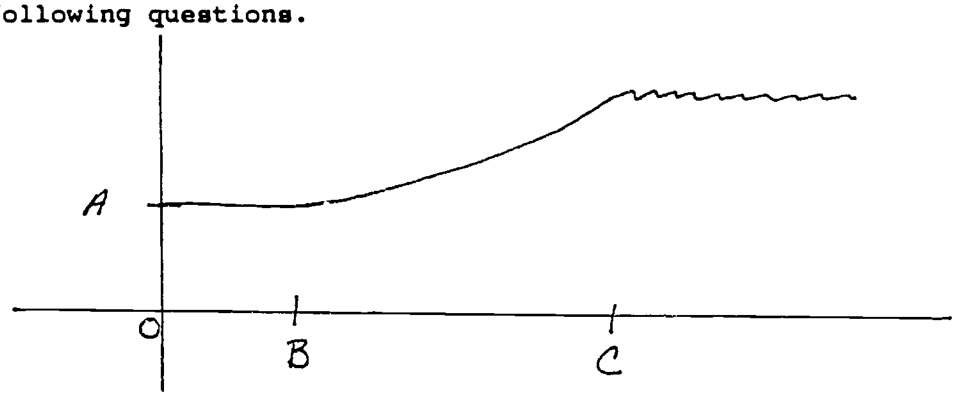
Task A

Suppose that in teaching a unit on functions and graphs you are explaining a typical projectile motion problem in which a ball is tossed at t=0 seconds and hits the ground again at t=7 seconds. [Let the height, $h(t) = -4t^2+28t$]. You sketch the graph of h(t) in order to explain the relationships between height and time. A student says, "Wow, the graph looks just like the ball going up and coming back down!".

How do you respond to this student's comment?

Task B

Suppose that you are discussing the definition of function in class. You illustrate the concept with a qualitative graph like the one below and pose the following questions.



1. How would you describe the temperature in the following time intervals? [0,B], [B,C], and [C, +∞].

2. What real-world situation could be described by this graph? Explain the temperature in the various time intervals in terms of your situation.

A student responds to this problem, "I thought that functions had to be straight lines". How would you respond to this student's comment? What kinds of responses would you hope for from these questions?

Task C

Suppose that several students chose the following solutions to this problem involving evaluation of functions.

<u>Problem</u>: Evaluate $f(x+1)$ if $f(x) = x^2+x+1$.

a) x^2+3x+2 b) x^2+x+2

c) x^2+x+3 d) x^2+3x+3

For each of the incorrect solutions:

1. What is the source of the mistake? (Show how they may have found this solution.)

2. How would you respond to these incorrect solutions.

Task D

You have been discussing the concept of composition of functions in class. You pose the following problem in class.

Let $h(x) = f[g(x)]$ and determine $f(x)$ and $g(x)$ if $h(x) = 2(x-5)^2$.

One student suggest that, "$g(x) = (x-5)^2$ and $f(x)=2$". Another student interrupts, " No, $f(x)$ must be equal to $2x$ if $g(x) = (x-5)^2$". A third student remarks, " Well I think $g(x)=x-5$ and $f(x) = 2x^2$. The class seems confused.

How would you respond to these comments and clear up the confusion?

Task E

You have been discussing the concept of inverse functions in class. You pose the following problem in class.

Determine the inverse (f^{-1}) of the function $f(x) = x/7 + 4$.

One student suggest that, " $f^{-1}(x) = 7x-4$". Another student says, " No I think it's $f^{-1}(x) = 7(x-4)$".

How would you respond to these comments?

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