DOCUMENT RESUME

ED 364 411 SE 053 841

AUTHOR Wilson, Melvin R.

TITLE One Preservice Secondary Mathematics Teacher's Evolving Understanding of Mathematical Functions.

SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 93
CONTRACT NSF-TPE-9050016
NOTE 39p.; Paper presented at the Annual Meeting of the American Educational Research Association (Atlanta, GA, April 12-16, 1993).
PUB TYPE Speeches/Conference Papers (150) -- Reports - Descriptive (141)

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS Beliefs; Case Studies; Cognitive Development; *Functions (Mathematics); High Schools; Mathematics Instruction; Mathematics Teachers; *Preservice Teacher Education; Qualitative Research; Schemata (Cognition); Secondary Education; Secondary School Mathematics; *Teacher Education Programs
IDENTIFIERS Pedagogical Content Knowledge; Preservice Teachers; Subject Content Knowledge

ABSTRACT

This case study report provides information that can be useful in implementing rational changes in mathematics teacher education. Considering teachers' thinking about a specific mathematical topic, such as functions, allows one to better understand the broader domain of teachers' mathematical thinking and its influence on teaching and learning. This study used the categories of concept definition and concept image as described by Vinner and Dreyfus to examine the evolving knowledge and beliefs of a preservice secondary mathematics teacher as she participated in a mathematics education course that emphasized mathematical and pedagogical connections and applications of the function concept. Her conceptions were revealed over a 10-week period through interviews, observations, and written work. The teacher's initial understanding of functions as computational activities (e.g., function machines, point plotting, vertical line test) was consistent with her larger view of mathematics as a collection of concrete procedures. Although her understanding of function grew substantially during the study, her anticipated approach to teaching, which was dominated by her narrow view of mathematics, was less significantly affected by course activities. Contains 26 references. (Author/MPN)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED 364 411

One Preservice Secondary Mathematics Teacher's Evolving Understanding of Mathematical Functions

Melvin R. (Skip) Wilson

University of Michigan

Paper presented at the annual meeting of the American Educational Research Association, Atlanta, Georgia, April 12-16, 1993. Session 11.37, *Subject Matter Knowledge and Classroom Teaching in Math and Science,* April 13, 1993.

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY

Melvin R. Wilson

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

This document has been reproduced as received from the person or organization originating it.

Minor changes have been made to improve reproduction quality

Points of view or opinions stated in this document do not necessarily represent official OERI position or policy

Inquiries should be directed to
Melvin R. (Skip) Wilson
610 E. University, 1228 SEB
Ann Arbor, MI 48109
(313) 747-2907
Internet:
skip.wilson@um.cc.umich.edu

SE 053 841

The research reported in this paper was supported by a grant from the National Science Foundation (Grant No. TPE-9050016). The conclusions and recommendations expressed here are those of the author and do not represent official positions of NSF.

Abstract

This study examines the evolving knowledge and beliefs of a preservice secondary mathematics teacher as she participated in a mathematics education course that emphasized mathematical and pedagogical connections and applications of the function concept. Her conceptions were revealed during a ten-week period through interviews, observations, and written work. The teacher's initial understanding of functions as computational activities (e.g., function machines, point plotting, vertical line test) was consistent with her larger view of mathematics as a collection of "concrete" procedures. Although her understanding of function grew substantially during the study, her anticipated approach to teaching , which was dominated by her narrow view of mathematics, was less significantly affected by course activities.

One Preservice Secondary Mathematics Teacher's Evolving Understanding of Mathematical Functions

Secondary teacher education is one of the underrepresented research themes in mathematics education (Sowder, 1989). Indeed, many popular calls for reform in teacher education (e.g., Carnegie Forum on Education and the Economy, 1986; Holmes Group, 1986) are not based on research evidence concerning the potential effects of recommended changes; rather, "teacher education is moved by forces better characterized as whimsical than as rational" (Cooney, 1980, p. 460). This article describes how one prospective secondary mathematics teacher understands an important part of the mathematics she will be expected to teach. It also traces her experience in an undergraduate mathematics education course and documents the influence of course activities on her mathematical and pedagogical understanding. Its intent is to provide information that can be useful in implementing rational changes to mathematics teacher education.

Considering teachers' thinking about a specific mathematical topic, such as functions, allows one to better understand the broader domain of teachers' mathematical thinking and its influence on teaching and learning (Cooney & Wilson, 1993). In other words, study of a specific case can enrich one's conceptualization of the general. The case reported in this article about Molly, a preservice secondary mathematics teacher, is part of a larger study that examined several preservice teachers' mathematical conceptions (Wilson, 1992). The article explores the meanings and understandings Molly communicated about mathematics in general and mathematical functions in particular.

Previous studies investigating teachers' knowledge of function contributed significantly to the conceptualization of the current study. For example, this study made use of the categories of concept definition and concept image described by

ERIC

Vinner and Dreyfus (1989). It also considered ways in which teachers represent functions in general and functions of the high school curriculum (Even, 1990). Also influencing the conceptualization of this study was the literature on students' understanding of function (e.g., Leinhardt, Zaslavsky, & Stein, 1990) and more general documents describing the recommended mathematical knowledge of high school teachers and students (e.g., Mathematical Association of America, 1990; National Council of Teachers of Mathematics [NCTM], 1989). Within this literature there is general consensus concerning which aspects of the function concept are most crucial for deep understanding. Areas identified include: (1) interpreting functions represented by graphs, situation descriptions, formulas, and tables, (2) modeling real-world situations using functions, (3) translating among multiple representations of functions, (4) analyzing the effects of parameter changes on the graphs of functions, (5) examining operations on and properties of classes of functions, and (6) applying technology to represent functions.

Research on mathematics teachers beliefs (e.g., Cooney, 1985; Thompson, 1984, 1992) also provided empirical, theoretical, and methodological impetus for the design of this study. This study builds on previous research in two major ways: (1) it provides greater depth of information concerning preservice teachers' knowledge of function than has previously been available, and (2) it goes beyond carefully describing teachers' beliefs about mathematics and mathematics teaching. It identifies important aspects of a teacher's knowledge of a specific mathematical topic (function) *and* explores relationships between this knowledge and her more general beliefs, understandings, and dispositions.

Participant

Molly is a 20-year-old university junior from a small southern town. Before the study, she had no formal teaching experience, although she had provided private mathematics tutoring for several of her friends and family members.

ERIC

Molly is a cheerful, confident, and energetic individual who was very willing to share her experiences with me. She is a serious student who considers herself to be "intelligent, but not extremely so." Molly refers to herself as a perfectionist; she insists that things be done in particular ways. Her success in high school mathematics was the biggest factor motivating Molly's decision to become a mathematics teacher.

Molly's high school mathematics coursework included *Algebra I, Algebra II, Geometry*, and *Pre-calculus*. At the time of the study Molly had completed the following university mathematics courses: a three-quarter calculus sequence, a linear algebra course, two geometry courses for prospective teachers, and a course titled *Introduction to Higher Mathematics*, which focuses on formal proof in advanced mathematics. Molly had also completed two courses emphasizing uses of computers in mathematics teaching and one on the history of mathematics. Her overall university grade point average was 2.9 out of a possible 4.

Method

Research Site

The purpose of the course in which Molly was enrolled during this study is to help preservice teachers determine content for the secondary school mathematics curriculum and demonstrate competency in this content. The course generally precedes a field-based methods course and student teaching. The specific section in which this study took place met for two hours, two (and sometimes three) times per week, for ten weeks. The course dealt with two prominent secondary mathematics topics: Euclidean geometry during the first half of the course and functions during the second half. The curriculum materials used during the second half of course, hereafter referred to as the *functions unit*, were created as part of a National Science Foundation (NSF#TPE-9050016) project, *Integrating*

Mathematics Pedagogy and Content in Teaching. The functions unit is described in more detail in a later section of this article.

Functions were studied during the second half of the course, so Molly's understanding of function was assessed initially in a context independent of instruction about functions, at least in this course. Information regarding Molly's initial (pre-functions unit) conceptions of function was obtained during the first four weeks of the study, using a written instrument and three interviews. Observations in classroom settings throughout the term, Molly's written work, and later interviews, provided opportunities to gain additional insights concerning her mathematical and pedagogical conceptions and how her interaction with course materials and activities influenced those conceptions.

Instruments

Written Instrument. During the first week of the term Molly completed a written instrument requesting her to solve specific mathematical problems. For example, she was asked to interpret functional situations, define function, and classify relations as functions or non-functions. While responding to the problems in writing, she verbally described her thinking on audio-tape.

Interviews. The formal interviews in which Molly participated varied in structure and format, and each lasted about one hour. All seven interviews were audio recorded and transcribed. The first interview provided information about how Molly interpreted and identified functions, as well as how she believed functions should be defined. During this interview, Molly elaborated on her responses to the written instrument and also completed a sorting activity involving definitions of function as obtained from various textbooks published between 1909 and 1990. The second interview engaged Molly in a sorting activity wherein she demonstrated how she organized her knowledge about several function classes common to the secondary curriculum: linear, quadratic, polynomial, exponential, logarithmic, and periodic. These activities were based on Kelly's (1955) repertoire grid technique. Molly was given 24 cards, each describing a different function. Functions were represented graphically, algebraically, in tables (as sets of ordered pairs), and as real-world situations. Molly was asked to sort the cards, or subsets of the cards, in different ways, and describe and justify her thinking while sorting.

The third interview explored the nature of Molly's knowledge and beliefs about calculators and computers, particularly her views about graphing calculators. This interview had two foci: (1) how Molly used a graphing calculator to represent and apply functions and (2) how she anticipated using technology in teaching and problem solving in general. One task in this interview involved her generating potential problems that could be solved using a graphing calculator, then explaining how she would solve the problems.

Although many insights into Molly's general views of mathematics and mathematics teaching were gained through analyzing her responses to the written instrument and first three interviews, four additional interviews were conducted to elicit more direct information concerning her beliefs about mathematics and mathematics teaching. Initial questions for the first beliefs interview were pre-determined, while follow-up questions were based on Molly's responses. The second beliefs interview explored issues raised during the first, together with other pre-determined issues. The questions for these two interviews consisted largely of *episodes* (Brown, Brown, Cooney, & Smith, 1982)--hypothetical situations designed to stimulate in-depth discussions about mathematics and mathematics teaching.

The final two interviews were more open-ended. Brief outlines were prepared before the interviews and referred to during conversations with Molly. In one of these interviews, Molly was engaged in a clustering activity (Cooney, 1985) in which she was asked to organize and comment on statements she had made

ERIC

during previous interviews. The final interview, conducted at the end of the course, was used to gain additional data regarding Molly's perceptions of the influence of the course, to probe in greater depth prominent themes identified during previous interviews and observations, and to bring closure to the research process by encouraging Molly to comment on issues she believed to be interesting and important.

Teacher/Student Interview. Near the end of the term (during the ninth week) Molly planned and conducted a short interview with an undergraduate student (a preservice middle school mathematics teacher), to assess the student's understanding of function. I met with Molly as she planned the interview, observed the actual interview, and conducted a debriefing session immediately following the interview. Each of these meetings was audio recorded and transcribed.

Class Observation and Artifacts. I observed all sessions of the mathematics education course in which Molly was enrolled during the study. I took careful fieldnotes and occasionally conducted brief, informal interviews with Molly. Additional data sources included copies of Molly's written work (e.g., assignments and tests).

Table 1 summarizes when in the term each of the data collection activities occurred.

Analysis

Data were analyzed using a modified constant comparison approach (Glaser & Strauss, 1967; Strauss, 1987). Analysis progressed through several stages, including one in which Molly participated in the analysis of her own data. Initial data analysis, or *open coding,* involved reading interview transcripts and fieldnotes holistically and writing memos describing general impressions and overall tone. For interviews, such coding occurred soon after the interviews had been transcribed (before subsequent interviews). Open coding of classroom observation fieldnotes was conducted immediately after each class meeting.

The sixth interview consisted of a clustering activity. Preparation for this interview involved a more detailed analysis of each of the first five interview transcripts. During this line-by-line analysis, 15 categories or themes were identified, and 40 of Molly's previous statements were selected. Her statements represented the major themes identified in the line-by-line analysis, as well as discrepant (negative or contradictory) and neutral cases (Erickson, 1986).

During the clustering interview Molly was asked to elaborate on her 40 statements, and to sort them into categories using whatever criteria she wanted. After sorting the statements, she was asked to provide a one or two word heading for each category and a brief sentence to describe what the statements expressed.

Subsequent analysis relied extensively on Molly's clustering. *Selective coding* involved analyzing ways in which Molly commented on and clustered her statements during the sixth interview. During this selective coding I also reviewed in detail the transcripts of the sixth and seventh interviews, the teacher/student interview, fieldnotes of class observations, and written classwork and tests. I merged categories within and across data sources and defined emergent categories by referring to quotations from the interviews, and examples from fieldnotes and written work.

Results

Following are descriptions of Molly's general views of mathematics and mathematics teaching, of her conceptions of function both before and after receiving instruction about functions, and of the functions unit in which Molly participated. The description of her general views forms a backdrop or context that allows the reader to understand how Molly's specific conceptions of function are related to her more general views. By considering relationships between Molly's specific and general conceptions, both are understood more completely. A description of the activities in which Molly was engaged as she participated in the functions unit will also provide a sense of context; it will help the reader interpret Molly's conceptions of function and how specific course activities influenced her to build new understandings.

Molly's Beliefs about Mathematics and Mathematics Teaching

The aspects of mathematics that appeal most to Molly are its consistency, order, and, "correctness." Although she acknowledged that some aspects of mathematics are not "concrete," the mathematics that Molly likes best is "solid and down to earth." She explained,

> I like solving equations. I like coming up with something that's concrete, something that I can see, something I can put my hands on. . . . But I'm not very happy unless I can get a concrete answer, [unless] I can get something simple that I can look at and understand, otherwise I'm not very happy, I'm not very pleased with mathematics.

Molly dislikes "theoretical" branches of mathematics, such as abstract algebra and hyperbolic geometry, that do not emphasize procedures and concrete answers. She commented,

> Hyperbolic geometry [is] really interesting, but it's really difficult. I mean we'll never see it. I mean in real life it doesn't exist. That bothers me to have math that's totally theoretical. I mean I understand that we need to have it and that it does exist but only theoretically. I like real, hard, concrete stuff that you can see. That's what I like about mathematics.

While discussing what aspects of mathematics she believes teachers should emphasize, Molly pointed out that "[in] math [there are] some things you have to follow a certain way, you have certain procedures you need to follow." An assumption communicated by Molly throughout our discussions was that the major source for important mathematical rules and procedures is school textbooks. Molly believes that teachers do not necessarily have to follow exactly what the textbook says, but, "[Mathematics] is in the book, that's why they write books." In several of our discussions I asked Molly how she felt about non-traditional teaching methods, such as independent student-directed projects, or small group cooperative learning. She never expressed opposition to such activities, and at the end of the study she even claimed that such activities were very important. However, Molly cautioned that teachers should always be careful to "relate what's in the books to what the children are doing so they'll get some *math* background." During one such discussion about the teaching of geometry, she was more specific, stating that regardless of what else happens in the classroom, it is important for the teacher to spend time teaching "straight geometry--like angle bisectors and midpoints and things like that."

Molly is mixed in her view of what constitutes "thorough understanding" of mathematics. Although she feels that it is important for mathematics teachers to understand both how and why mathematical procedures work, she believes that it is sufficient for students to know only how to correctly apply procedures. The following dialogue, which occurred during our final interview, amplifies this view.

M: My students don't really care how it works, as long as it works and they get it right on the test, you know, then that's fine. And very few students are going to say, "Why does that work? . . . Why does it do that?" But I think we [teachers] do need to, in cases where it comes up, explain and be prepared to explain why.

I: But you don't think it's essential that students . . . ?

M: No, I think you should be ready to answer it, just in case it comes up, but not every day say, "well this works because blank." But if the students are wondering, "why?, that totally looks different than what we did yesterday, why can you do that today? I don't understand." I mean if that comes up.

I: But if they don't have any problems with it?

M: Yah, I think you've got to do so much else. If they're comfortable, if they're understanding, [if] they're not questioning it, then [there is] no need to rock the boat. But if they are questioning it, [if] they don't understand, then you can use that to help them understand. But if I look around and I don't see anybody with wrinkled faces then I'm going to go on. If they have problems with it individually then they can come and ask me.

Molly believes that it is the teacher's responsibility to teach correct rules and procedures in an organized fashion, explaining exactly which procedures students are expected to use, so there is no confusion. Her interview responses, such as the one cited in the preceding dialogue, consistently emphasized how teachers should appropriately explain things. During an early interview she described a personal experience in which one of her university professors used an unstructured teaching approach that she found to be frustrating, because the professor "just gave us one sheet and let us go at it." She explained, "We did it one way, and this other group did it another way and on certain given information, you could do it both ways. . . . I just think he should have said 'do it this way, these are the guidelines that I want you to follow.'"

Molly's Understanding of Function before Completing the Functions Unit

Definition and Image. On the written instrument (during the first week of the study) Molly suggested the following "definitions" of function:

(1) Graphically, it has to pass the vertical line test, which means that you can draw a vertical line through the graph and the vertical line will touch no more than one point on the graph.

(2) A function is written in the form *f* of *x* is equal to something that contains *x*'s, such as *x* squared, because it's a function of *x*. You're using *x* to tell you what you're doing.

Her second definition suggests that Molly thought of functions as computational activities or operations as defined by equations or algebraic expressions. Molly confirmed this idea during the first interview, when she elaborated on her definitions:

> What I mean . . . is with the *f*, the *f(x)*, you have to put, it's like a grinder. You put a number in right here and this *f* is going to change this *x* in some way. You know it maybe is going to square it, it's going to cube it, it's going to take one-half of it. It's going to *do* something to it. If it's just *x* then it's going to leave it the same. But it's going to give you something else, it's going to give you a number that has had some, I don't want to use the word function, but it has had some kind of, something performed on it.

Molly's understanding of functions as numerical operations was consistent with the way many beginning students think about functions (Sfard, 1987, 1989).

Molly's concept image conflicted somewhat with her concept definition. For example, she believed that the graphs of functions should be continuous, or "flowing." On several occasions she classified as non-functions relations that "look[ed] funny," or were "disjoint or not continuous," even though they passed her vertical line test. In referring to one discontinuous graph, Molly commented, "That may be wrong but I'm thinking that a function can be graphed so that it's continuous."

Knowledge Organization. During the function sorting interview (second interview), Molly preferred to organize a set of 24 functions by representation. She consistently wanted to put all the graphs together, all the equations together, all the tables together and all the real-world descriptions together. When encouraged to sort the entire set and smaller subsets of the 24 functions in other ways, she had difficulty but was somewhat successful in seeing beyond the surface features of the functions.

Molly's first attempt to organize the 24 functions in a way other than by representation is shown in Figure 1. As she organized the functions, she first tried to identify which functions were indeed functions. This process evolved into a less well-defined process involving several criteria, including whether or not she "felt comfortable" with them and whether or not she expected that most high school students would understand the functions. For example, in deciding to put the quadratic word problem into her "harder to understand" category, she explained, "I have to think about this, area and diameter, formulas and stuff like that I've got to recall." For the exponential equation she explained, "The only thing I'm worried about is with the exponential--I know it's a function but I don't know if your ordinary tenth or eleventh grader would know it."

Functions I am comfortable with

x	-2	-1	0	1	2
y	6	3	0	-3	-6

x	0	1	2	3	4	5	6	7	8
y	0	1	2	4	0	1	2	4	0

x	-1	0	1	3	4
y	-8	-1	0	8	27

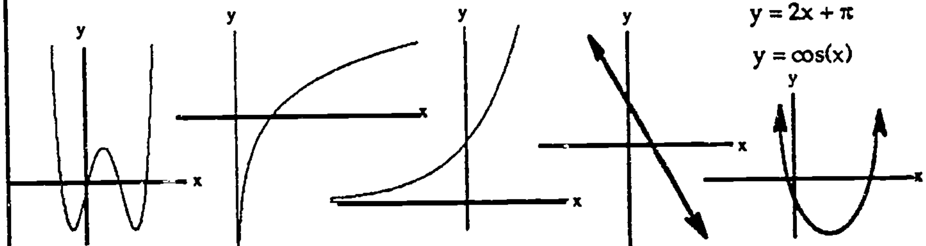
x	-2	-1	0	1	2
y	.25	.5	1	2	4

x	.01	.1	1	$\sqrt{10}$	10	100
y	-2	-1	0	.5	1	2

x	-2	-1	0	1	2
y	1	0	1	4	9

$y = 2x + \pi$

$y = \cos(x)$



The 1990 census shows that Central City has a population of 40,000 people. Social scientists predict Central City will experience a growth rate of 2% per year over the next 20 years. How can one predict Central City's population for each of the next 15 years? *(exponential)*

Mr Washington has noticed an increase of 3 cents per gallon in the price of regular unleaded gasoline over the past four weeks. If the current price is $1.309, he wonders what the price will be in the coming weeks if this same price increase continues each week. *(linear)*

To locate the blockage of a sewage line, workers start at the midpoint to determine which half of the line contains the blockage thus decreasing the length of the line where the problem occurs by one-half. They then divide the remaining part of the line into two equal parts and repeat the process of determining which half of the line (in this section) contains the blockage. This partitioning process continues until the distance between the partition and the blockage is less than 8 feet. Mr. Davis wants to know the number of partitions necessary to guarantee finding the blockage within 8 feet. *(logarithmic)*

Denise is filling with water a cubical container measuring one foot on each edge. She notices that it takes a lot more water as each dimension of the cube is increased. She wonders how much the volume of the cube increases when each dimension is increased x units. *(polynomial)*

The Art Museum has a bulletin at its entrance showing its weekly hours:
Monday: Closed; Tuesday-Friday: 9am - 6pm; Saturday: 10 am - 6 pm; Sunday: Noon - 5 pm
Ms. Carter decides to have her students graph the number of hours the museum is open for each day in March. *(periodic)*

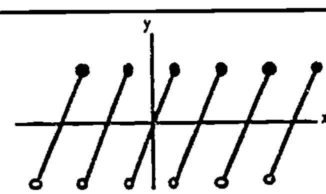
Harder to Understand

Fred is deciding which size pizza is the best buy. He wonders how the area of the pizza is related to its diameter. *(quadratic)*

$y = \frac{1}{3} 4^x$ $\quad y = \frac{1}{3} x^2 - x + 1$ $\quad y = 3x^5 - x^4 + 4x^3 - 10x^2 + x - 6$

Not Comfortable with

$y = \log_4 (x + 4)$



Note: Word problems in the actual activity were not identified by type

Figure 1. Molly's first organization of 24 functions.

When asked to sort the cards depicting linear, quadratic, and exponential functions (12 cards), Molly again suggested an organization by representation. An organization other than by representation did not occur immediately after the request for a different organization; Molly thought carefully and extensively before suggesting the organization shown in Figure 2.

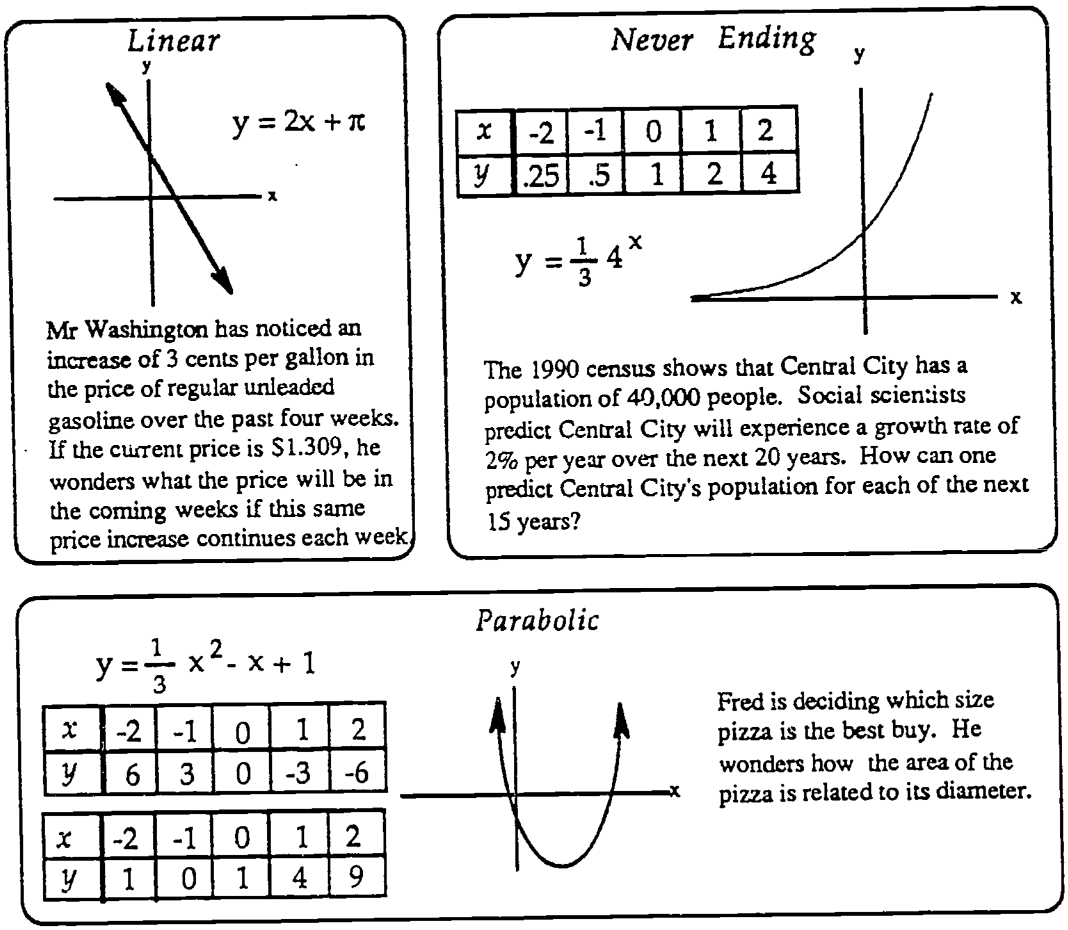
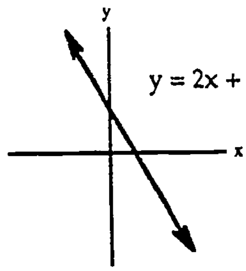


Figure 2. Molly's organization of a set of linear, quadratic, and exponential functions.

After other intermediate exercises in which she sorted various subsets of the 24 functions, Molly was again asked to sort the entire set of 24 functions. Figure 3 shows her final organization. Her three categories, "linear," "parabolic properties," and "never ending," were the same as those she identified for the subset of linear, quadratic, and exponential functions.

Linear



$y = 2x + \pi$

Mr Washington has noticed an increase of 3 cents per gallon in the price of regular unleaded gasoline over the past four weeks. If the current price is $1.309, he wonders what the price will be in the coming weeks if this same price increase continues each week.

Parabolic Properties

x	-2	-1	0	1	2
y	6	3	0	-3	-6

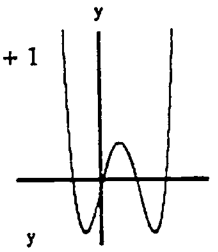
x	0	1	2	3	4	5	6	7	8
y	0	1	2	4	0	1	2	4	0

x	-1	0	1	3	4
y	-8	-1	0	8	27

$y = 3x^5 - x^4 + 4x^3 - 10x^2 + x - 6$

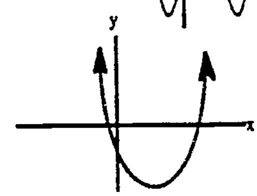
x	-2	-1	0	1	2
y	1	0	1	4	9

$y = \frac{1}{3}x^2 - x + 1$



Fred is deciding which size pizza is the best buy. He wonders how the area of the pizza is related to its diameter.

The Art Museum has a bulletin at its entrance showing its weekly hours: Monday: Closed; Tuesday-Friday: 9am - 6pm; Saturday: 10 am - 6 pm; Sunday: Noon - 5 pm
Ms. Carter decides to have her students graph the number of hours the museum is open for each day in March.

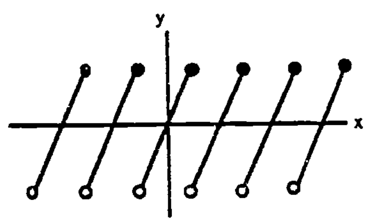


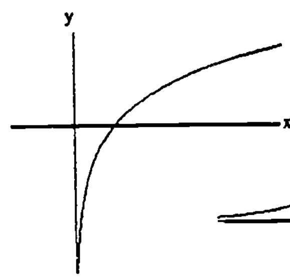
Never Ending

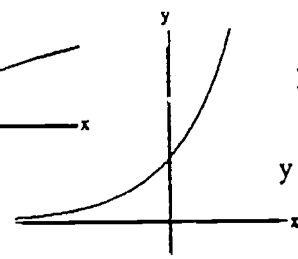
x	.01	.1	1	$\sqrt{10}$	10	100
y	-2	-1	0	.5	1	2

x	-2	-1	0	1	2
y	.25	.5	1	2	4

$y = \log_4(x + 4)$







$y = \frac{1}{3}4^x$

$y = \cos(x)$

Mr Washington has noticed an increase of 3 cents per gallon in the price of regular unleaded gasoline over the past four weeks. If the current price is $1.309, he wonders what the price will be in the coming weeks if this same price increase continues each week.

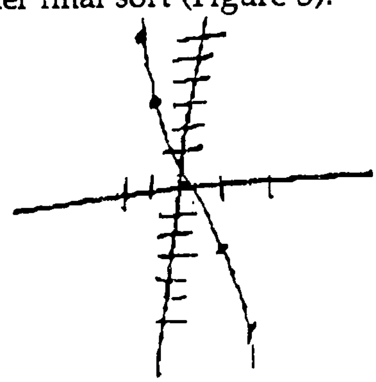
To locate the blockage of a sewage line, workers start at the midpoint to determine which half of the line contains the blockage thus decreasing the length of the line where the problem occurs by one-half. They then divide the remaining part of the line into two equal parts and repeat the process of determining which half of the line (in this section) contains the blockage. This partitioning process continues until the distance between the partition and the blockage is less than 8 feet. Mr. Davis wants to know the number of partitions necessary to guarantee finding the blockage within 8 feet.

Denise is filling with water a cubical container measuring one foot on each edge. She notices that it takes a lot more water as each dimension of the cube is increased. She wonders how much the volume of the cube increases when each dimension is increased x units.

Figure 3. Molly's final organizations of 24 functions.

Perhaps as interesting as the ways in which Molly organized the functions were the criteria she used in organizing them and the processes she used to understand or familiarize herself with the functions. In "coming to terms with" the equations, she sketched Cartesian graphs. Her traditional equation graphing procedure of plotting points whose coordinates she obtained by substituting values of x into the equations and computing corresponding y-values, was consistent with her understanding of functions primarily as computational operations (i.e., "function machines"). Even her statement about the linear equation, that it would *"end up* being linear," suggests that she did not see this equation as an object to be described by its graph, but as a signal to perform the graphing operation of plotting points.

Molly similarly graphed the functions represented by tables--she plotted points one by one and connected them with a smooth curve. Molly placed the linear table (i.e., a table generated using a linear equation) in the "parabolic" category (see Figures 2 and 4) because her sketch (see Figure 4) was somewhat "curvy," as opposed to straight or asymptotic (i.e., "never ending"). Similar thinking accounted for her decision to put the polynomial graph in the "parabolic properties" category during her final sort (Figure 3).



Molly's Graph of

x	-2	-1	0	1	2
y	6	3	0	-3	-6

Figure 4. Molly's graph of a linear table.

Molly explained that she was "comfortable with" the graphs because they passed the vertical line test and therefore represented functions. The lone exception to this was the periodic graph which she disliked "because it's got holes in it and I think a function should be continuous." She did not mention that she could or needed to generate equations corresponding to these graphs. Feeling comfortable with a word problem required that she understand the situation described in the problem but not necessarily know how to solve it. She did not use an equation, table, or graph to describe any of the word problems.

Molly's rapid "comfort" with the graphs, together with her insistence on graphing the equations and tables, indicate that the graphical representation was central to her thinking, at least for the purpose of categorizing the equations, graphs and tables. She never, for example, spontaneously thought about generating an equation corresponding to a graph or word problem. Neither did she attempt to identify numerical patterns for the tables, let alone equations to generate the tables. In referring to the functions represented by tables, she explained, "these are just ordered pairs, you really don't know anything about them. . . . You don't know if x is squared, if y is half of x, . . . these could just be arbitrary numbers." Even after graphing the points listed in the tables she demonstrated no inclination to note a pattern or equation corresponding to the tables. In contrast to Molly's almost exclusive reliance on graphs to organize functions, she was reluctant to consider graphical representations in other situations in which the graph might help communicate useful information about the situation. This idea is illustrated more completely in the section, *Using Functions to Solve Problems.*

Importance of Functions. Before this study, Molly could not recall having seriously considered why functions are important. During one of our early interviews she told me that she thought functions were important, although she admitted that she could not remember why. She explained,

> I don't think I know enough to really intelligently answer [why]. . . . I don't know what part [functions] play in anything else. . . . I don't remember what you do after you learn about functions. You can say that you know that certain equations are functions but that's about all

> I can remember from high school algebra. . . . You've got to know about domain and range but I don't, I just can't remember what it has to do [with], how it relates to anything else. That's just a memory blank. I can look in an Algebra II textbook . . . [and] maybe get it back. . . . You learn about polynomials, and binomials and things like that and then maybe having to determine if they're functions or not. I'm not really sure. . . . I know they have [a use] but I don't know what it is.

Using Functions to Solve Problems

Throughout the study Molly was asked to interpret or solve problems related to situations commonly thought to involve functional relationships. Some of these *functional situations* involved real-world phenomena, while others were limited to purely mathematical applications of functions (e.g., using graphs to solve equations or systems of equations). Molly used a numerical approach to solve real-world problems involving functional situations. Never did she spontaneously generate a graph, equation, or table to describe a given situation; rather, she conducted a series of numerical computations to describe them. This idea is illustrated by considering, for example, Molly's responses to the following problem:

> The 1990 census shows that Central City has a population of 40,000 people. Social scientists predict Central City will experience a growth rate of 2% per year over the next 20 years. How can one predict Central City's population for each of the next 15 years?

The following dialogue illustrates Molly's approach to this problem.

> M: The first thing I'd do is write down all my information that I'd need, write down you know, 40,000 people and that it's expected growth rate of 2 percent per year over the next 20 years, then what's going to be the population for each of the next 15. I'd write

down everything that I had been given in my problem, and then I would take 2 percent of 40,000 and that would be my growth rate. If in 1990 it was 40,000 then in 1991 it's going to be 40,000 plus 2 percent of 40,000. And then for '92 it's going to be the 40,000 pus 2 percent of 40,000 plus 2 percent of that whole thing. And then you would just keep doing that.

I: Keep going, so you'd have to do 15 calculations?

M: Well, you could probably figure out a formula. Well, like n plus one factorial, I don't know.

Molly used a similar approach to solve a problem that asked her to describe a hypothetical relationship between the diameters and areas of circular pizzas. She explained that she would figure out the areas of several pizzas of different diameters, then compare these areas. She could not compute any such areas, because she could not remember a formula for the area of a circle. After being reminded of the formula
$A = \pi r^2$, she generated the formula $A = \pi (d/2)^2$, but did not believe the formula was a solution to the problem. To actually solve the problem, she insisted on considering several pizzas of varying diameters, calculating their areas, and comparing.

Molly's responses to situations involving potential uses of a graphing calculator further illustrate her tendency to solve problems related to functional situations in non-functional (particularly non-graphical) ways. After receiving brief instructions about how to generate graphs of simple equations using a graphing calculator, she graphed several equations, including the equations $y = 3x + 2$ and $y = x^3 + 5x^2 - 4$. When asked how she would solve the system defined by these equations, Molly pointed out that her preferred method of solution was algebraic--she would use "substitution." She explained that the points of intersection of the equation's graphs would correspond to the solution, but she was not sure how many solutions there would be (all intersection points were not displayed on the screen). She also could not explain why the intersection points of the graphs represented a solution to the system of equations.

Molly showed similar tendencies while attempting to solve the inequality $x^3 + 5x^2 - 4 < 0$ (while viewing the graph of $y = x^3 + 5x^2 - 4$). She said she thought the solution would correspond to "the part of the graph that lies below the x-axis." Although she identified a strategy that could lead to a solution, Molly was unable to articulate *how* the graph would relate to the solution. Her suggested solution involved shading a region on the graph rather than identifying values of x that would satisfy the inequality.

Summary. Molly's conceptions of function before the functions unit were consistent with her larger view of mathematics--as a collection of "concrete" procedures to be applied in isolated contexts to obtain correct answers to well-defined problems. To Molly, the study of functions was primarily an exercise in applying remotely related rules such as the vertical line test and constructing graphs by point plotting. Throughout her work in solving problems, her comments about the definition of function, and her decisions during the function sorting activities, Molly demonstrated weak understanding of relationships among various representations and procedures. She showed little appreciation for the usefulness of functions and was extremely limited in her ability to operate on and use functions in meaningful ways. Her confusion throughout the early interviews illustrates that the limited knowledge she had in this area was extremely fragmented.

The Functions Unit

The functions unit consisted of activities designed to assist preservice teachers in developing an appreciation for the power of functions and an ability to interpret functional situations and solve problems related to functions. Teachers were given opportunities to reflect on their own images of function, mathematics, and mathematics teaching, and were encouraged to develop strategies and activities that they could use in their future teaching. Although substantial emphasis was given to small group activity (teachers were required to complete several course assignments in groups), individual reflection and whole-class discussions were also important components of class activities.

An introductory activity engaged teachers in a dialogue about the teaching of functions. Teachers reacted to and discussed a vignette depicting a high school teacher and her students discussing the use of polynomial functions to solve a maximum value problem. Following this discussion the preservice teachers reflected on their own understanding of function and their views about the importance of teaching functions. They also considered and generated hypothetical situations involving functional relationships, including examples of such relationships in the media.

Following these introductory activities, teachers met in groups of four and discussed ways of organizing a collection of 32 functions (see Figure 1 to get an idea of the types of functions involved in this activity). Emphasis in subsequent class discussions was given to the relative levels of sophistication, as well as the advantages of, various ways of organizing the functions. For example, the class concluded that if one focuses on the graphical or algebraic properties of the functions, the traditional "families" of functions emerge (e.g., linear, polynomial, exponential). If one chooses instead to focus on the representations themselves, then four categories emerge: graphs, tables, formulas, and word descriptions.

A set of experiments allowed teachers to collect data (in small groups) and generate functions related to that data using tables, graphs, and equations. For example, teachers constructed simple pendulums and considered relationships between pendulum length and period, as well as between pendulum weight and period. Building upon these and other hands-on experiences, the preservice teachers considered various problem situations related to functions, including word problems, algebraic inequalities, maximum-minimum value problems, and the effects on graphs of changing algebraic parameters of functions defined by formulas. Many of these topics were explored using graphing calculators and graphing computer software. Throughout the activities, topics were considered from both mathematical and pedagogical perspectives. For example, teachers were assigned to write papers about the history of functions, considering both mathematical and pedagogical implications of that history. Table 1 illustrates when various course activities occurred during the term.

	Data Collection Activities	Course Activities
Week 1	Written Task (with audio-taped comments), Selection of participants	Euclidean Geometry Activities--constructions, properties of polygons
Week 2	Written Task Follow-up Interview (audio-taped)	
Week 3	Function Sorting Interview (video-taped)	
Week 4	Graphing Calculator Interview (audio-taped)	
Week 5	First Beliefs Interview (audio-taped)	Introduction to Functions Unit
Week 6	Second Beliefs Interview (audio-taped)	Reflections on personal understanding of functions, Importance of teaching functions
Week 7	No interviews while preparing data for Clustering Interviews	Function sort, Experiments
Week 8	Clustering Interview (audio-taped)	History of functions, Definition of function, Functions in school mathematics
Week 9	Teacher/Student Interview (audio-taped)	Graphing calculators--effects of changing parameters, using functions to solve problems, using graphs to solve equations and inequalities
Week 10	Exit Interview (audio-taped)	Test on Functions

Table 1: Timeline of data collection and course activities

ERIC

Changes in Molly's Understanding of Function

Definition and Image. Before participating in the activities of the functions unit, when Molly attempted to define function, she referred to the symbolism of functions (i.e., $f(x)$ notation) and properties of the graphs (i.e., vertical line test). She also demonstrated some confusion about various definitions of function, particularly formal, set-theoretical definitions. This confusion was consistent with her understanding of functions primarily as computational operations. Through participation in the functions unit, Molly gained a deeper understanding of both formal and informal definitions of function and she became flexible in relating formal definitions to informal ones. This flexibility was based on her new understanding of functions as "relationships."

During the clustering interview, which occurred about midway through the functions unit, Molly commented on several of her early statements about functions. Reacting to her statement, "A function is something written f of x equals something that contains x's," Molly commented, "well that's true, but it means a lot more to me now then it did when I said it." She discussed how class activities such as the experiments and an activity in which she had to find examples of functions in the newspaper had helped her "understand some more things about relationships." Her understanding grew to include the idea of functions describing relationships between variables that did not necessarily have to be defined by equations or form smooth, continuous curves when graphed. In a paper she wrote about the history of functions, she elaborated on the Dirichlet function, or what she called the "salt and pepper" function (a function that is not continuous and not defined by a single equation).

The function sorting activity also contributed significantly to Molly's growth in this area. Because the activity occurred in groups of four teachers, Molly was not only able to generate ideas based on her own understanding, but was also able to share in the understandings of the other group members. For example, at the beginning of the activity Molly suggested that two of the functions (which were represented by non-continuous graphs) were not even functions, and should therefore be placed in a separate category. One group member seemed to concur completely with Molly, while the other two members agreed that although one could think of non-continuous functions as a separate category, the graphs in question were indeed functions. After some discussion, this interpretation was accepted by all of the group members (and subsequently by the whole class). In a later interview Molly commented that she had learned during the function sorting activities about "step functions" and "piece-wise continuous functions" and that considering these examples had "convinced" her that all functions are not continuous or defined by a single equation.

Molly's performance on a test at the end of the functions unit provided evidence that, after participating in the functions unit, she could relate her image of function to accepted definitions. She explained the difference between the following two definitions:

> *Definition A*: If with each element of a set A there is associated in some way exactly one element of a set B, then this association is called a *function* from A to B.
>
> *Definition B*: If two variables, such as x and y, are so related that to each value of x there corresponds a definite value or set of values of y, y is called a *function* of x.

To illustrate how these definitions differed, Molly discussed the binary relation defined by the equation $x = 2$, with domain {2} and range {2, 3, 4}. She wrote,

> This relation is not a function under Definition A because x is not paired with exactly one element of set $\underline{y}$. The line $x = 2$ does satisfy Definition B because x corresponds to a set of values of y. This is the difference between

Definition A and Definition B: [Definition] A says unique correspondence, [Definition] B does not.

Thus Molly's understanding of the definition of function was deepened to include an understanding of both formal and informal definitions, as well as relationships between them.

Knowledge Organization. A related area of growth in Molly's understanding of function was her ability to operate on and organize her thinking about functions common to the secondary curriculum. She demonstrated an increased ability to translate among various representations and to organize sets of functions in multiple ways. This growth was illustrative of how course activities helped her "connect" her understanding of functions. She began to see relationships among important mathematical ideas that she did not see before. Additionally, she no longer thought of functions simply as isolated computational operations, but had begun to consider them as dynamic objects that were related and could be operated upon themselves.

Molly's experiences conducting experiments were particularly beneficial in helping her learn to translate among various common functional representations. The process of collecting experimental data, tabulating the data, and constructing graphs and equations to represent the data, helped her attach meaning to each of these representations. Reflecting on relationships among graphical, tabular, and symbolic representations, as these representations pertained to concrete experiences, and discussing these representations with fellow teachers, helped her build important connections among them. Later course activities allowed her to extend her understanding in this area to include more abstract and general examples--examples not necessarily tied to concrete experiences. Her competency in translating between tabular and algebraic representations is exemplified by her suggestion of equations that could be used to generate the tables illustrated in

Figure 5. After plotting graphs, she determined that the equations $y = 2^{-x}$ and $y = x^2$ could be used to generate the tables.

x	-1	0	1	2	3	4
y	2	1	1/2	1/4	1/8	1/16

x	-1	0	1	2	3	4
y	1	0	1	4	9	16

Figure 5. Functions represented by tables.

Molly's improved skill in operating on functions was paralleled by an enhanced understanding of important relationships among functions. The function sorting activity was particularly instrumental in helping Molly build this relational understanding. Group interactions, such as those described in the previous section, as well as individual reflection and whole-class discussions, helped Molly develop flexibility in organizing her thinking about the 32 functions considered. Some of the classification criteria she used to sort functions during and after this activity included representation (e.g., equations, graphs, tables, word problems), graphical properties (e.g., shape, continuity, asymptotic behavior), algebraic properties (e.g., functions that contain $1/x$ or e^x), and "family" (e.g., linear, exponential, logarithmic).

Importance of Functions. At the end of the functions unit Molly was very articulate in describing why she thought functions are important. This ability contrasts with her original position of "I don't know why functions are important." Speaking about the benefits of the experiments and the assignment to identify functional situations in the media, Molly commented,

> [The activities] helped me realize that functions just aren't in school, they're everywhere. Everybody uses them. They may use them in a different context. . . . IBM may be talking about its decreasing sales because of its increasing prices on its personal computer, but it's still a

function. . . . So I could see how many different areas in a business or psychology or just regular fictional writing use functions.

On a test Molly further explained why she believed functions are important. She wrote, "functions prepare students for higher mathematics." To support this claim she pointed out that logarithmic and exponential functions are important for calculus, and that to succeed in higher mathematics, students "need to be able to translate from one representation to another." She also wrote that "functions can help [students] understand real-world situations." She listed several applications of functions, including some related to business (e.g., marketing trends) and others she had considered in class (e.g., predicting gas prices and finding areas of pizzas). Molly also mentioned that to understand functions, one has to "know algebra, arithmetic, and a little geometry." She wrote, "Functions tell students that all those years of math classes did have relevance." These statements supported her claim that functions are important because they "tie in everything the student has learned previously." That is, understanding functions helps students understand relationships among important mathematical ideas.

Using Functions to Solve Problems. Consistent with Molly's claim that functions are useful in modeling real-world situations, she demonstrated growth in her ability to use functions to interpret functional situations and solve problems. Her experiences throughout the functions unit, including the experiments, her consideration of various other real-world problem situations, and her work using graphing calculators to construct graphs related to functional situations, contributed to this increased ability. Her solution process to the following problem illustrates how Molly was able to generate an equation and table to model a functional situation and solve a related problem.

A helicopter shuttle service operates between an airport and the center of a city. It charges a fare of $10 and carries 300 people a day. The manager estimates that he will lose 15 passengers for each increase of $1 in the fare. Find the most profitable fare he can charge.

Molly generated the equation $y = (300 - 15x)(10 + x)$ to represent this situation, where x represented the number of dollars of fare increase and y represented the profit. She also constructed a table, part of which is shown in Figure 6 (x-values in her table ranged from 0 to 11). She used her equation to generate values for the table. After concluding that the maximum profit occurred when x is 5, she wrote, "When $x = 5$, the number of people using the helicopter service is 225. The fare charged is $15. The manager will make $3375 a day."

x	y
3	3315
4	3360
5	3375
6	3360

Figure 6. Partial table Molly used to represent the function $y = (300 - 15x)(10 + x)$

Although Molly demonstrated increased proficiency in solving problems related to "real-world" functional situations, she was still unsure about how to use graphs of functions to model and help interpret more abstract functional situations, such as solving equations and inequalities. This confusion persisted despite considerable instructional attention given to using graphical approaches to solve traditionally algebraic problems. Two class periods were devoted largely to discussing relationships among and relative merits of various solution approaches to inequalities. For example, the class discussed how to solve the inequality $|x - 1| < |x - 2|$. Many of the teachers in the class, including Molly, had been unable to solve this problem on a previous assignment. The class, guided by the instructor, discussed two solution methods, one algebraic and the other graphical.

The algebraic method involved considering various cases of equality to find critcal values (e.g., when does $x - 1 = -(x - 2)$?). The graphical approach (suggested by another teacher in Molly's group) involved analyzing the graph of the function $y = |x - 1| - |x - 2|$. Following this discussion, Molly liked the algebraic/case method better, even though the class consensu was that for most problems of this type, the graphical method was more efficient (particularly if one has access to a graphing calculator).

On a test after the functions unit, Molly attempted to solve the inequality $x^3 + 4x^2 < -4x$ using both methods. Her first approach was algebraic. She wrote:

(1) $x^3 + 4x^2 < -4x$	(5) $x < 0, x < -2, x < -2$
(2) $x^3 + 4x^2 + 4x < 0$	(6) $x = -3$ $(-)(-)(-) = -$
(3) $x(x^2 + 4x + 4) < 0$	(7) $x = -1$ $(-)(+)(+) = -$
(4) $x(x + 2)(x + 2) < 0$	(8) $x = 1$ $(+)(+)(+) =$

Molly also sketched the graph in Figure 7. Molly's final answer to the problem was, "$x^3 + 4x^2 + 4x < 0$ when $x < 0$ and $x < -2$" (the correct answer is $x^3 + 4x^2 + 4x < 0$ when $x < 0$ and $x \neq -2$). Although Molly recognized that this problem could be solved graphically as well as algebraically, she did not demonstrate an understanding of relationships between the two methods. Neither did she seem clear about why these methods worked.

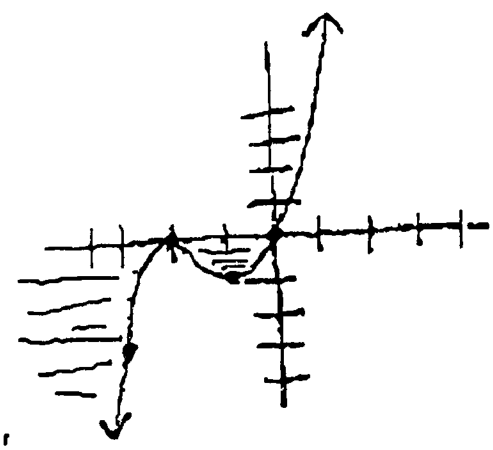


Figure 7. Molly's graph used to solve $x^3 + 4x^2 < -4x$.

Summary. Although Molly demonstrated some confusion with several functional concepts after completing the functions unit, her understanding had grown immensely as a result of her participation in the activities suggested by the unit. Before the unit she understood functions primarily as computational operations. She demonstrated little understanding of any mathematical or practical applications and a limited ability to use functions to solve problems. However, after the unit, because she understood an important aspect of functions to be that they describe mathematical and real-world relationships, she was able to operate flexibly with classes of functions given in different representations, use functions to solve problems, and identify other important connections among function concepts.

Changes in Molly's Conceptions of Teaching

Molly indicated that she liked the course in which this study took place because it "opened [her] eyes to a lot of different ways to . . . teach things like functions." She discussed how she no longer felt she had to teach functions "straight out of the book" because she could involve her students in activities like the function sort and the experiments. She explained how course activities helped her better understand functions *and* provided her with teaching ideas:

> Those are activities I would not have thought of, . . . activities I would like to use when I teach functions, because they help me understand some more things about relationships. . . . If it helps me then maybe it'll help my students understand a little more, when it's not going to be so cut and dried. It'll be something with a little more flare to it. . . . These activities are a hands-on experience for students to learn about relationships. The students can actually *see* the relationship [between] the independent and dependent variables."

Molly insisted that the principles described above would carry over into her teaching of other topics. To break the monotony of lectures and homework assignments, she said she would have her future students participate in games, computer simulations, and other "hands-on" activities. She expressed confidence that such activities would help keep her students interested in mathematics.

Molly believed that her view of teaching had significantly changed as a result of her experience in the course. The above quote indicates that perhaps course activities will have a substantial impact on her teaching practice. However, there was also evidence that her new conceptions of teaching did not really represent a radical shift. At the end of the term Molly insisted that it will not be important for her future students to understand the meaning of mathematical procedures. This claim seems to conflict with her statements about how the meaningful learning she experienced in the course would carry over to her own teaching. However, a careful analysis of Molly's plan for using innovative instructional activities reveals that she will use such activities primarily because they will make mathematics class more interesting (e.g., break the monotony of lectures), not necessarily because such activities will communicate a more vibrant and meaningful image of mathematics. Thus, although her participation in the functions unit helped her *begin* to see alternative ways of teaching mathematics, her views of mathematics and mathematics teaching were still relatively narrow at the end of the study.

Discussion

When Molly becomes a teacher, she will face many obstacles. Some of these obstacles, like "covering the material," and "lazy students," she admitted may discourage or prevent her from implementing the innovative ideas she encountered and enthusiastically embraced in the course in which this study took place. But perhaps Molly's greatest obstacle, one of which she is not consciously aware, will be her own dualistic (Perry, 1970) conceptions about the nature of mathematics and mathematics teaching.

It will be difficult for Molly to communicate to her students that mathematics is not "cut-and-dried" when she understands it to be. It will be equally difficult for her to encourage students to be creative and explore mathematical relationships when deep down she believes that the most important part of mathematics involves obtaining correct answers by applying standard procedures, many of which she herself does not recognize as being related. While the course activities provided Molly with a glimpse of a different perspective about mathematics and mathematics teaching, it is unlikely that her experience in this course, *by itself*, will be sufficient to profoundly change her conceptions of teaching.

One of the most glaring weaknesses in Molly's understanding of function before the functions unit was her relative ignorance of the power and usefulness of functions, both within mathematics and in the real world. To Molly, the value of functions was related mostly to determining whether or not relations are functions. She could neither comment on the unifying mathematical power of functions nor on the prevalence of functional relationships in the real world. Furthermore, although she used graphs to help her organize collections of functions, she could not use functions to help her solve problems. This study provided no evidence that Molly's previous experience (including several university mathematics courses for which function was a central concept) emphasized either mathematical or practical connections of the function concept. A study by Even (1989, 1990) suggests that many other preservice teachers have deficiencies in their understanding of function similar to Molly's.

There were "gaps" in Molly's understanding of function even after she completed the functions unit. However, the approach taken in the functions unit was effective in helping Molly develop an awareness of important connections for the function concept, as well as an increased ability to operate with and use functions. The materials suggested numerous activities that encouraged Molly to make connections between functions and the real world, as well as between functions and other areas of mathematics. Molly's participation in these activities helped her become familiar with and more competent in aspects of mathematics currently considered to be very important (e.g., NCTM, 1989, 1991; National Research Council, 1989).

Most people take as an axiom that preservice teachers should understand the mathematics they will be teaching at and beyond the level expected of their future students (Committee on the Mathematical Preparation of Teachers, 1990). Some reform efforts have aimed to improve the mathematical competence of prospective teachers (e.g., Holmes Group, 1986). Most of these reform efforts suggest that prospective teachers should be required to complete more advanced mathematics.

Such a solution is appealing because it is relatively simple. Molly's case provides evidence that the solution to this problem of teachers' mathematical preparation involves more than simply helping them become familiar with advanced mathematics, as important as this is. Molly's experience in university mathematics courses, while familiarizing her with topics in advanced mathematics, did not broaden her view of the subject. In fact, her participation in such courses reinforced her narrow views of mathematics. In this respect, she was not unlike other preservice mathematics teachers (Helms, 1989; Owens, 1987).

While it is important for preservice teachers to consider advanced mathematical topics, it may be easier to effect change in teachers' views of the nature of mathematics and mathematics teaching by giving them opportunities to reflect on their own conceptions while learning (or re-learning) mathematics that they will have to teach themselves. Such experiences will allow them to make accommodations in their belief systems to first *acknowledge* alternative

ERIC

perspectives of mathematics and mathematics teaching, and then (perhaps) *embrace* them. The course within which this study was conducted attempted to do this by integrating mathematics content and pedagogy, allowing teachers to reflect on their mathematical and pedagogical understanding. Although the course activities did not "cause" Molly to modify her beliefs to be consistent with current recommendations for teaching (this would be impossible in such a short time), she had *begun* to consider alternative ways of thinking about mathematics and teaching.

Models for mathematics teacher education should seriously consider the idea of integrating mathematics content and pedagogy, with a significant component of that integration consisting of activities that encourage teachers to reflect on their own views of mathematics and mathematics teaching while actively exploring important mathematical concepts and processes that they will be required to teach. Such an approach will allow teachers to make important connections in their own mathematical understanding and improve the chances that such an integrated approach will be reflected in their future teaching.

References

Brown, C., Brown, S., Cooney, T., & Smith, D. (1982). Episodes. Unpublished manuscript, University of Georgia, Athens, Georgia

Carnegie Forum on Education and the Economy. (1986). *A nation prepared: Teachers for the 21st century.* The report of the Task Force on Teaching as a Profession. Washington DC: Author.

Committee on the Mathematical Education of Teachers (1990). *A call for change: Recommendations for the mathematical preparation of teachers of mathematics.* Washington DC: The Mathematical Association of America.

Cooney, T. (1985). A beginning teacher's view of problem solving. *Journal for Research in Mathematics Education, 16*, 324-336.

Cooney, T. J. (1980). Research on teaching and teacher education. In R. J. Shumway (Ed.), *Research in mathematics education* (pp. 433-442). Reston, VA: National Council of Teachers of Mathematics.

Cooney, T. J., & Wilson, M. R. (1993, in press). Teachers' thinking about functions: Historical and research perspectives. In T. Romberg, E. Fennema, & T. Carpenter (Eds.), *Integrating research on the graphical representation of function* (pp. 131-158). Hillsdale NJ: Lawrence Erlbaum Associates.

Denzin, N. (1970). *The research act.* Chicago: Aldine.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Whitrock (Ed.), *Handbook of research on teaching* (3rd ed.) (pp. 119-161). New York: Macmillan.

Even, R. D. (1989). *Prospective secondary mathematics teachers' knowledge and understanding about mathematical functions.* (Doctoral dissertation, Michigan State University, 1989). Dissertation Abstracts International, 50, 642A.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *International Journal of Mathematics Education in Science and Technology, 14*, 293-305.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory.* New York: Aldine.

Helms, J. M. (1989). Preservice secondary mathematics teachers' beliefs about mathematics and the teaching of mathematics: Two case studies. (Doctoral dissertation, University of Georgia, 1989). *Dissertation Abstracts International, 51*, 440A.

ERIC

Holmes Group. (1986). *Tomorrow's teachers: A report of the Holmes Group.* East Lansing, MI: Holmes Group.

Kelly, G. A. (1955). *The psychology of personal constructs* (Vol. 1). New York: Norton.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Research Council. (1989). *Everybody counts.* Washington, DC: National Academy Press.

Owens, J. E. (1987). A study of four preservice secondary mathematics teachers' constructs of mathematics and mathematics teaching. (Doctoral dissertation, University of Georgia, 1987). *Dissertation Abstracts International, 48,* 588A.

Perry, W. G. (1970). *Forms of intellectual and ethical development in the college years.* New York: Holt, Rinehart, and Winston.

Sfard, A. (1987). Two conceptions of mathematical notions: Operational and structural. In J. C. Bergeron, N. Herscovics, & C. Kieren (Eds.) *Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education* (Vol. 3, pp. 162-169). Montreal, Canada: Universite' de Montreal.

Sfard, A. (1989). Transition from operational to structural conception: The notion of function revisited. In *Proceedings of the Thirteenth Annual International Conference for the Psychology of Mathematics Education,* Paris, France.

Sowder, J. T. (Ed.). (1989). *Setting a research agenda.* Reston, VA: National Council of Teachers of Mathematics.

Strauss, A. L. (1987). *Qualitative analysis for social scientists.* Cambridge: Cambridge University Press

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan Publishing Company.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20* (4), 356-366.

Wilson, M. R. (1991). A study of three preservice secondary teachers' knowledge and beliefs about mathematical functions. Unpublished doctoral dissertation, University of Georgia.