DOCUMENT RESUME

ED 364 410 SE 053 836

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TITLE	Beliefs, Autonomy, and Mathematical Knowledge.
PUB DATE	Apr 93
NOTE	60p.; Paper presented at the Annual Meeting of the American Educational Research Association (Atlanta, GA, April 1993).
PUB TYPE	Reports - Research/Technical (143) -- Speeches/Conference Papers (150)
EDRS PRICE	MF01/PC03 Plus Postage.
DESCRIPTORS	*Beliefs; Case Studies; Cognitive Structures; Functions (Mathematics); High Schools; High School Students; Interviews; *Knowledge Level; *Mathematical Concepts; Mathematics Education; *Personal Autonomy; Qualitative Research; *Schemata (Cognition); Self Efficacy; Self Motivation; Student Role; Teacher Role
IDENTIFIERS	Solution Methods (Mathematics)

ABSTRACT

This paper describes a study of the connections between beliefs about mathematics, autonomy, and knowledge structures in mathematics. Assumptions underlying the study were (1) that students' beliefs and knowledge are constructs of the individual and play a dynamic role in the learning and doing of mathematics; and (2) that autonomy theoretically affects persistence, confidence, and mathematical growth. The study was conducted in a high school in a university town in New Hampshire. Six volunteers distributed evenly across Algebra I and II courses were interviewed after participating with their class in a unit on functions. Eight 45-minute interviews were conducted with each student to gather information on their beliefs about mathematics as conceptual versus procedural, their beliefs about their role and the teacher's role in learning mathematics, their autonomy with mathematics, and their constructed knowledge from the unit on functions. The interviews were treated as case studies and were examined individually and then cross-compared. Autonomy and beliefs were found to be integral to the students' conception of mathematics and influenced how problems were approached and mathematics learned. Further study in the formation and modification of beliefs and in the interplay between beliefs, autonomy, and learning in actual classroom contexts is suggested. Contains 71 references. (PDD)

BELIEFS, AUTONOMY, AND MATHEMATICAL KNOWLEDGE

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A paper presented at the annual meeting of the American Educational Research Association, Atlanta, Georgia, April, 1993.

Beliefs, Autonomy, and Mathematical Knowledge

Introduction

A recurrent theme in mathematics education has been the impact of students' attitudes on their achievement in mathematics. Evidence for this interaction is suggested by the results of the recent National Assessments of Education Progress (NAEP). In the fourth NAEP, data was collected on both students' proficiency on basic arithmetic skills, problem solving, and conceptual understanding; and on students' attitudes towards mathematics (Brown, Carpenter, Kouba, Lindquist, Silver & Swafford, 1988b; Swafford & Brown, 1989). The mathematical results indicate a strong reliance on algorithms with little conceptual underpinning to facilitate applications or problem solving. Equally alarming are the attitudinal results. A majority of the seventh and eleventh grade students surveyed responded that they perceived mathematics as merely following rules and over half felt learning mathematics is mostly memorization. Approximately 20 percent of the students agreed with the statement that mathematics is made up of unrelated topics and about 35 percent agreed that new discoveries are seldom made in mathematics and that mathematicians work with symbols and not ideas.

For Silver (1987), these beliefs about mathematics reflect a 'hidden' component in the mathematics curriculum.

> These statements and others like them, reflect students' beliefs about or attitudes toward mathematics. The students' beliefs and attitudes have been shaped by their school mathematics experiences. Despite the fact that neither the authors of the mathematics curriculum nor the teachers who taught the courses had intentional curricular objectives related to students' attitudes toward and beliefs about mathematics, students emerged from their experience with the curriculum and the instruction with these attitudes and beliefs. Since the students' viewpoint represented by these statements is clearly inadequate, and potentially harmful to their future progress in mathematics, we need to focus our attention more clearly on those hidden products of the mathematics curriculum. (Silver, 1987, p. 57)

In his review of the literature on beliefs about mathematics, Underhill (1988) pleads that:

> As we know more of learners' beliefs, we are struck by the disparities between what we believe and what they believe, what we intend to be learned and what is learned. Further study can surely help us improve mathematics instruction by providing a new type of feedback. Too many learners have no sense of empowerment; they are looking only for correct answers; they are memorizing facts and procedures. Far too few feel mathematically empowered; far too few feel in charge of their own learning, feel in charge of the growth and development of their own mathematical knowledge. (p. 66)

While the results from the NAEP and the observations made by Silver and Underhill suggest the importance of beliefs, little is known about the actual nature of the interaction between attitudes/beliefs and achievement. A review of the attitude literature reveals numerous statistical studies showing a consistently low to medium correlation (.19 to .54) between the measures of attitudes towards mathematics and achievement (Aiken, 1970a,b, 1971, 1976; Kulm, 1980; Reyes, 1984). These statistical results have been interpreted as indicating that attitudes have a secondary rather than a causal effect on achievement.

Although numerous, the research results have proven inconclusive and fragmented. Reviewers of the literature have criticized the research on several key points: (a) no unifying definition of attitude, (b) the absence of a theoretical basis for interpreting statistical data or directing research questions, (c) little cross referencing or building on previous work, and (d) inadequate models to explain the interaction between attitude and achievement. Aiken (1976), in his review, indicates that attitude research has relied too heavily on correlation methods and indirect measures of attitudes such as questionnaires. He recommends that future research consider the distinction between the cognitive and emotional subcomponents of attitude when developing attitudinal instruments. Kulm (1980) further argues for theory development studies utilizing qualitative methods which are sensitive to nuances in beliefs, opinions, and behavior.

The recent qualitative research into students' problem-solving strategies has reopened the debate as to the effect of beliefs on students' cognitive processes. The studies of Buchanan (1984), Cobb (1985, 1986), Frank (1985), and Schoenfeld (1983, 1985) have pointed to students' beliefs about mathematics as a limiting factor in their problem-solving behavior. Beliefs by setting up expectations appear to constrain students' choice of heuristics and even restrict the type of problems students perceive as mathematics (Kouba & MacDonald, 1987, 1991). Collectively, these studies suggest that beliefs have an interactive role as students solve problems and learn mathematics.

In this investigation, a multiple case study design was utilized to examine six Algebra II students' beliefs about mathematics and their possible effect on learning mathematics. The research plan consisted of a pilot study and three data gathering phases: (a) classroom observations and assessment of the teacher's perception of her role in the learning process, (b) an assessment of student participants' beliefs about mathematics and autonomy with mathematics, and (c) an assessment of the students' newly formed mathematical constructs on functions.

Briefly, the rationale behind the research plan was to develop a detailed portrait of each individual's beliefs about mathematics, to observe these individuals within the social context of their mathematics classes, and finally, to carefully examine the mathematical constructs that the individuals formed from their classroom experiences. By synthesizing the information on individual students' beliefs with the classroom observations and expectations and comparing these results with the individual student's mathematical constructs, a description was developed for each student. These descriptions attempted to explain the type and depth of the mathematical constructs relative to the students' beliefs and to the classroom expectations. Finally, these individual descriptions were compared for any patterns and similarities among students' beliefs about mathematics, students' autonomy with mathematics, and students' understanding of functions. In exploring the relationships between students' beliefs about mathematics and autonomy, and students' knowledge of functions in the classroom environment, the following questions were addressed.

Students' Beliefs

1. To what extent do the students hold beliefs about mathematics as conceptual or procedural in nature?
2. To what extent do the students hold the belief that they are the source of authority for their knowledge?
3. What do students perceive as their role and the teacher's role in learning mathematics?
4. How are the students' beliefs about mathematics as conceptual or procedural related to their source of authority? For example, do students with a view of mathematics as conceptual have an internal source of authority?

Relationship between Beliefs and Knowledge of Functions

1. To what extent do the students reveal a conceptual or procedural understanding of functions which

includes the definition of function; the concepts of domain and range; function notation; composition of functions; and linear functions?

2. How do the students' beliefs about mathematics as conceptual or procedural fit with their knowledge of functions? For example, do students who hold a view of mathematics as procedural construct a procedural or conceptual understanding of functions?

Theoretical Framework and Research Assumptions

This research study is premised on several theories and perspectives. Foremost are the assumptions that beliefs and knowledge are constructs of the individual and that these constructs are formed in response to experience and self-reflection. Individuals utilize their beliefs and knowledge to create meaning from experience and subsequently to anticipate future events (Kelly, 1963; Green, 1971; Rokeach, 1968; Cobb and von Glaserfeld, 1983). Within the perspective of constructivism, the individual is seen as a scientist actively developing theories to explain observation and experience. Like scientists, individuals test their theories against experience.

One immediate consequence of this perspective is that all knowledge and beliefs are perceived as individualized. Yet individuals do not develop their understanding of the world in isolation from the culture and era in which they live. Toulmin (1972) suggests that the way an individual chooses to interpret experience is influenced by society. Kuhn (1962) further argues that knowledge can be viewed as socially justified beliefs. That is, those individual constructs or meanings which are shared and common to a society would be considered knowledge. Under this description, discipline knowledge would represent the common constructs of the practitioners of that discipline. The discipline also would share a common criteria for validating statements. Thus, even within this individualized perspective, the construction of knowledge is seen as a flow between the individual and society. Those individual constructs that were not shared or common to a discipline would be designated beliefs rather than knowledge.

The individual and the discipline also interact in other ways. While all mathematical statements are scrutinized within a well-defined deductive system, the practitioners of the discipline still engage in a personal affirmation process to reestablish the meaning and validity of these statements (MacLane, 1986;

Davis and Hersh, 1986). Mischler (1978) describes the mathematician's search for meaning and the processes used to create mathematics by saying:

> When a mathematician says he understands a mathematical theory, he possesses much more knowledge than that which concerns the deductive aspects of theorems and proofs. He knows about examples and heuristics and how they are related. He has a sense of what to use and when to use it and what is worth remembering. He has an intuitive feel for the subject, how it hangs together, how it relates to other theories. He knows how not to be swamped by details, but also to reference them when he needs them. (p. 361)

Thus mathematical understanding or knowledge entails more than the specific content knowledge. It also includes the processes by which the knowledge is developed and verified, and the perspective through which an object or idea is viewed within the discipline.

The processes and perspective associated with mathematics represent a metacognitive level in the discipline. It is one of the hypotheses of this study that students' beliefs about mathematics as a discipline and beliefs about learning and doing mathematics reveal their attempts to describe this secondary level in mathematics.

It is further hypothesized that these beliefs, rather than being extraneous, have a dynamic role in the learning and doing of mathematics. The rationale for this premise comes from multiple sources. Kilpatrick (1985), Schoenfeld (1983), and Shaughnessy and Haladyna (1984) suggest that beliefs act in a metacognitive fashion. Kilpatrick (1985) notes that: "metacognitive processes rather than being imposed on top of acquired knowledge, interact with knowledge as it is being acquired (p. 9)." Beliefs are metacognitive in the sense that they set-up expectations and anticipations which in turn delimit choices (Cobb, 1986; Schoenfeld, 1983, 1985). These expectations are theorized to affect both how knowledge is structured (Skemp, 1987) and how it is used (Buchanan, 1984; Cobb, 1986; Frank, 1985; Schoenfeld, 1985).

These mathematical constructs are theorized to form principally within the classroom content. Researchers have even proposed that dysfunctional beliefs inhibit students' ability to engage in mathematical activities, especially problem solving (Anderson, 1984; Baroody & Ginsburg, 1986; Borasi, 1990; Buerk, 1985; Frank, 1988).

What is being proposed, then, is that beliefs about mathematics and oneself as a doer of mathematics by setting up expectations impinge in the cognitive processes at vital decision making junctures. The following scenario suggests how and when this might occur in a problem solving situation. When confronted with a situation, a student must first decide whether or not that situation is indeed a mathematical problem. Even the acknowledgement of a situation as mathematical depends on the very elements--context and clues--that an individual perceives as relevant (Kouba & MacDonald, 1987, 1991). Here then is a first juncture where students' beliefs about the nature of mathematical problems can influence their response to a problem. For example, does the student expect a problem to be identical to ones seen in the classroom? Must it contain numbers? Does the student look for the mathematical structure inherent in the problem?

Once a problem is acknowledged as legitimately a mathematical situation, the student must then decide what strategies to employ to solve the problem. It is here that Cobb (1985, 1986) and Schoenfeld (1983, 1985) propose that beliefs again enter into the deliberations. Does the student approach the problem with the expectation that its solution resides in the quick execution of a known procedure? Does the student expect to use trial and error to explore a problem before a solution process is devised?

Also, within the solution process, beliefs are theorized to affect executive-monitoring (Confrey, 1982; Confrey & Lipton, 1985; Gelman & Meek, 1986; Schoenfeld, 1985). Does the student expect the solution to make internal sense? Are the solutions expected to be consistent with other knowledge? Are solutions validated solely on the basis of a careful execution of an algorithm? Are the solutions correct only if a teacher or answer key indicates them as such?

In each of these junctures, beliefs about mathematics were conjectured to impinge upon the cognitive processes in problem solving. In addition, beliefs are hypothesized also to affect knowledge formation by establishing expectations for what is valued and attended to in an experience and how it is expected to be utilized in the future.

Skemp (1987) explains the development of knowledge or understanding in terms of "assimilate [ion] into an appropriate schema (p. 29)." Skemp defines a schema as a conceptual structure that an individual constructs in response to (a) experiences in the world, (b) interactions with others' ideas, and (c) internal reflections. An individual tests the validity of that schema against (a) physical events, (b) others' ideas, and (c) personal knowledge or beliefs. The schema are then utilized by the individual to (a) integrate knowledge and make predictions about future events, (b) facilitate communication with others, and (c) aid future growth and reflection.

Skemp also believes that the assimilation process involves acceptance of the new information by the individual. This acceptance can arise in two ways. In the first way, "acceptance of an assertation depends on the acceptance of the teacher's authority, and acting on it partakes more of the nature of obedience than of comprehension (p. 87)." In contrast, in the second way, "the assimilation of meaningful material depends on its acceptability to the intelligence of the student. Acting on it results from, and consolidates, enlargement of the learner's schemas (p. 87)."

Thus schema or beliefs can be viewed as a structuring mechanism which individuals attach or assimilate new information into existing knowledge structure. The schema or belief would direct how the information is placed into the structure--whether it is tied to other ideas either directly or by a process of reflection or held apart, and whether the validity of the information is associated within or outside the individual.

Skemp (1987) proposes that students' mathematical knowledge structures or schema can be characterized as either relational (conceptual) or instrumental (procedural). By relational, Skemp means knowledge or understanding in mathematics that is integrated--"knowing what to do and why (p. 153)", while instrumental denotes knowledge or understanding based on the execution of rules without reference to their rationale. While Skemp designates these two categories as types of understanding, they are closely associated with students' beliefs about mathematical knowledge and with students' autonomy with mathematics. Cobb (1986), Frank (1985), and Buchanan (1984) identify these categorizations with differing belief systems.

Other writers and researchers also have delineated mathematical knowledge into broad categories based on the type of internal structure associated with the knowledge. Notably Hiebert and Lefevre (1986) categorize mathematical knowledge as conceptual or procedural. Like Skemp, Hiebert and Lefevre do not identify these categories directly with any belief systems but their language is suggestive of beliefs. Hiebert and LeFevre outline several benefits that arise when mathematics is understood relationally or in their terminology when conceptual and procedural knowledge are joined. These benefits provide a means for observing and distinguishing relational or joined knowledge from instrumental or procedural knowledge. First, symbols develop meaning. Second, procedures are now perceived as reasonable. Since the procedures are understood, they are more easily remembered and recalled. Third, problem solution is enhanced. This enhancement is achieved by (a) simplifying procedural demands, (b) monitoring procedure selection and execution, and (c) promoting transference. These advantages are realized since "the conceptualization of a task enables one to anticipate the consequences of possible actions. This information can be used to select and coordinate appropriate procedures (p. 12)." The transference is furthered since procedures are no longer tied to the surface context in which they were learned. As a result of this freedom from context, procedures are more readily generalized. Finally, procedural outcomes are monitored. Conceptual knowledge functions in this regard as a validating criteria, judging the reasonableness of the answer.

Skemp's theory of schemas links an individual's global view of the discipline with the manner in which new information is accepted or assimilated. This linkage is also visible in Perry's (1981) theory of intellectual development. In that theory the maturation process is tied to a change in authority for one's knowledge from external to internal. Confrey (1985) identifies an internal source of validation with autonomy in mathematics. For Confrey autonomy reflects a belief that the individual is responsible for the truthfulness or correctness of one's knowledge and answers and that mathematics is valid or acceptable when it makes sense to the individual. Confrey further proposes that without the acceptance of this responsibility that students will remain dependent on outside authority, teacher or text; develop knowledge that is formalized and isolated from the rest of their experience; and feel powerless with respect to their use and knowledge of mathematics.

Fennema and Peterson (1985) associate autonomy with the development of higher-level cognitive skills. To develop these skills individuals' need to participate in autonomous learning behaviors which they describe as "working independently on high-level tasks, persisting at such tasks, choosing to do, and achieving success in such tasks (p. 20)." Thus, for Fennema and Peterson, autonomy is linked to motivation and confidence.

In summary these various perspectives suggest that autonomy involves both an acceptance of oneself as having the primary responsibility for one's learning of mathematics and the acceptance of oneself as the source for validating one's knowledge and solutions. Autonomy is an independence theorized to affect persistence, confidence and intellectual growth.

Methodology

Setting

The setting for the study was a small high school in a university town in southern New Hampshire. Approximately 500 students were enrolled in this school which has a strong college preparatory program. Approximately 78% of the student body attends either a four-year or two-year college after high school. The student participants and classroom observations were from an Algebra II and an Algebra II/Trigonometry class both taught by Mrs. Thomas (pseudonym). The Algebra II course was one of three sections offered by the high school while the Algebra II/Trigonometry course was the sole section of its type. The six student participants were all volunteers from a pool of students from both algebra classes. The pool consisted of any student who scored 75% or higher on the Algebra I Placement Exam (College Board, 1972). Three student volunteers were selected from each class.

Data Collection

Phase I: Classroom Observation and Teacher Assessment

The first major phase in the data gathering was the video-taping of both classes during their respective units on functions. This phase (a) documented the teacher's presentation of the unit on functions, (b) observed the student-teacher interaction in the classroom, and (c) collected relevant classroom materials as examples of the teacher's expectations for the classes.

To further document the unit, copies of all quizzes, tests, handouts and homework assignments were collected. Right of privacy considerations and the university's human subjects restrictions prohibited the collection of the student participants' actual papers and assignments from the classes. In addition to the video-tapes and sample assignments, the researcher kept a daily journal of impressions from each day's observations.

The second component of the classroom assessment entailed a series of five audio-taped interviews with the classroom teacher. These interviews (a) clarified any issues that arose from the classroom observations, (b) obtained background information on the teacher's experience and professional activities, (c) ascertained the teacher's expectations concerning the functions unit for each class, (d) recorded the teacher's philosophy of teaching and classroom policies, and (e) collected data concerning the teacher's own beliefs about mathematics.

Phase II: Beliefs Assessment

The purpose of this second phase was to gather information on the students' background and beliefs about mathematics. Three primary belief categories were targeted: (a) students' beliefs about mathematics as conceptual or procedural, (b) students' beliefs about their own role and the teacher's role in learning mathematics, and (c) students' autonomy with mathematics.

This phase in the data collection involved a series of five interviews with each of the six participants. The audio-taped interviews occurred once a week during the students' free period and lasted approximately 45 minutes. The interviews were conducted in a small, enclosed study carrel in the library of the high school. A calculator, straight edge, pencil, and scratch paper were always available for student use.

The interview schedule including the questions and instruments is given in the appendix. Each interview began with a few minutes of informal conversation to relax the participants. The interviews were conducted in a neutral manner with regard to the validity of students' mathematical work and opinions so as to avoid the researcher assuming an authoritative or expert's role and to empower the students' own voice.

Several techniques were employed to collect information on the belief categories. (See Table 1.) One component of this assessment entailed observing and questioning students while they solved various mathematics problems. These problems ranged from simple arithmetic algorithms to problem-solving situations. Follow-up probes explored the students' rationale for their strategies and their dependency on rules and algorithms when solving problems particularly in problem-solving situations.

To further corroborate any beliefs that might be expressed by the students or inferred from their solution to mathematics problems, the students completed several additional activities. These activities included marking and discussing a mathematics topics ranking grid and vocabulary lists, grading a sample algebra test, and responding to various scenarios on student's/teacher's roles.

Table 1

Triangulation of Instruments

	Beliefs			
Instrument	Procedural/ Conceptual	Student/ Teacher	Autonomy	Interview Questions
Problems	X		X	19, 20, 21, 23 24, 25, 26, 27 28, 29, 30, 31 33, 34, 35
Sample test	X		X	32
Vocabulary list	X		X	22
Ranking grid	X			36
Student/teacher scenarios		X		37, 38, 39, 40 41, 42, 43
Background questions		X		1-18

Phase III: Functions Assessment

In this phase, the students participated in an additional three interviews. These were again audio-taped and lasted approximately 45 minutes each. The purpose of this series of interviews was to investigate the knowledge that the six participants had constructed from their classroom unit on functions. (See interviews six, seven, and eight in appendix for a list of the content questions used in this assessment.) The questions covered the concepts of slope, function (both the definition and notation), domain and range, graphing functions, intercepts, composition of functions, and word problems using function notation. With the exception of graphing straight lines, the functions unit represented new material for the students in both classes.

The content questions were developed prior to the study, but the final selection of questions was delayed until after the completion of the functions unit in each class. By delaying the selection, it was possible to include problems that were familiar to the students as well as extension and transfer problems.

Analysis of the Data

Beliefs Assessment

The assessment of the student participants' beliefs about mathematics formed the cornerstone of this research study. Since each of the six students was treated as a separate case study the analysis proceeded on a case by case basis. For the purpose of the initial analysis, the data from the instruments was divided into two major groups: (a) data from the vocabulary list, the ranking grids, and the student/teacher scenarios and (b) data from the mathematics problems and sample test. The data from the first group was analyzed using the qualitative technique of typological analysis (Goetz & LeCompte, 1984) while the data from the later group was coded and enumerated.

The analysis of the first group of instruments proceeded in several stages. The data from these instruments served as a potential confirmation (triangulation with) of the data from the students' problem solutions and, as such, needed to be analyzed separately. The data in this first group was further divided into those questions and instruments (e.g. vocabulary lists and ranking grid) that were related to general beliefs about the nature of mathematics and those instruments relevant to student/teacher roles. The analysis in both subsections proceeded in a similar manner. First, the transcriptions were reviewed for statements suggestive of a view of mathematics as procedural or conceptual, and statements indicative of the students' view of their own autonomy with mathematics. While these beliefs were the focus of the search through the transcriptions, any statement which expressed a clear belief about the nature of mathematics was flagged for inclusion in the report of the data. Once the data from the individual instruments was analyzed, the students' comments on each instrument were re-examined for cross instrument agreement. A similar process occurred when examining and then summarizing the individual students' beliefs about the student/teacher roles in learning mathematics.

A different process of analysis was used for the second group of data: the students' problem solutions and evaluation of the sample test. Unlike the first group, the analysis of this data entailed inferences from the students' work and subsequently necessitated corroboration for those inferences from independent sources. The transcriptions of the students' solution processes were coded according to the criteria given in the appendix. Actual passages within the transcriptions were annotated as evidence supporting the various coding categories.

The coding of the data occurred in two phases. The data was rated first for the conceptual/procedural criteria and, second, for the autonomous/nonautonomous criteria. The codings were based on a positive instance of a category rather than on its absence. For example, in order for an episode in the transcription to be designated as nonautonomous, the student would have had to made a comment or shown by her actions one the behaviors listed in the coding criteria. She would not receive a nonautonomous coding because she failed to show any autonomous behaviors. Also the categories were not assumed to be mutually exclusive, that is, an episode could potentially be coded both conceptual and procedural if the student exhibited actions or comments exemplary of both categories. If no positive instances of either category were observed, then the episode/problem was designated undecided. Finally the coding of the data was independent of whether the student correctly solved the problem.

Once the initial coding of the data for the six participants had been completed, the data was re-analyzed by two independent coders using the definitions and coding categories. The two coders were responsible for re-evaluating the data from two of the student participants: Ann and Tara. The coders were given clean copies of the transcriptions and the scratch work along with the audio-tapes and coding sheets. The coders worked through the data once to code it for the conceptual/procedural categories and a second time for the autonomous/nonautonomous categories.

The coders' evaluations and comments were then compared to the researcher's own coding of the data. This comparison yielded an 85% and 75% overall match in the selection of coding categories. Using both sets of codings and the comments, the researcher then wrote a summary of the evaluation of the students' problem solutions. The written summaries for Ann and Tara were then submitted to the coders for their corroboration as representative of the data they had examined and coded. The final conclusions reflect the coders' criticisms and comments.

The final step in the analysis of the students' beliefs entailed a re-examination of the data from both of the major groups of instruments. This examination searched for any consistencies in or discrepancies between the students' beliefs as inferred from the analysis of the problem solutions and the beliefs as reported from the other instruments.

Functions Assessment

The purpose of this assessment was to evaluate the student participants' understanding of the concept of function. This evaluation included questions on the definition of function, function notation, the concept of domain and range, graphs of functions, intercepts, composition of functions, and word problems using function notation. These topics represented the core of the content common to both classes. Like the beliefs analysis, the analysis of the data in this section progressed through several stages and involved the independent coders.

The analysis began with researcher coding the students' solutions and comments on each of the functions questions. Although the primary coding was either correct or incorrect, these codings were qualified by comments and by the demarkation of C+, C and C- which indicated the relative strength of the students' correct response. That is, how complete the response was and how readily it was forthcoming (without prompting or probing by the researcher).

The data from Ann's and Tara's functions questions was recoded by the same two coders who had evaluated their beliefs data. As before, the coders were given a clean copy of the transcriptions, scratch work, audio-tapes, and coding sheets. The coders were instructed to code Ann's and Tara's responses as mathematically correct or incorrect and to express their rationale for their coding selection. Since the interviews were interactive, the possibility existed that the researcher might have inadvertently led the student to the problem solution or curtailed the student's own response. In recognition of this possibility, the coders also were asked to comment on the interaction especially if in their perception the dialogue may have unduly influenced the student's responses.

The match on the categories selection between the cr ' ;rs' and the researcher's codings was fairly high: 90% and 80%. The few discrepancies that occurred were all instances where the students' comments reflected a very weak or partial understanding of the concepts implied in the question. For example, one coder rated the student's response C--, while the researcher evaluated it as W (incorrect). Since the percentages were based exclusively on category match, this example would have been tallied as a mismatch, yet both codings and the accompanying comments conveyed similar messages about the incompleteness or vagueness of the student's understanding.

Using both sets of codings and comments, the researcher then summarized the students' responses to functions questions. The summary was organized around three topics: (a) definition of function, (b) function notation, and (c) related concepts. Next the data from the beliefs assessment and the functions assessment were re-examined for any apparent consistencies or discrepancies. For example, would a student whose comments and problem solutions suggest a view of mathematics as conceptual develop connections or ties in his knowledge of functions or would his knowledge appear fragmented and rule based?

Secondary Analysis

For the primary level of analysis, each of the six students was treated and reported as a separate case study. For the secondary level of analysis, the data and conclusions for each student were compared. The purpose was to examine the case studies for apparent trends or contradictions. For example, were the students described as conceptual and autonomous in orientation also those who developed a well-connected understanding of functions? Was there any relationship between the designation as conceptual/procedural in orientation and autonomous/nonautonomous in orientation? Were there any students whose data appeared incongruent with the others?

Discussion and Conclusions

The theoretical framework suggests connections among beliefs about mathematics, autonomy, and knowledge structures. In particular the theory suggests that a conceptual view of mathematics is associated with autonomy and with a relational knowledge structure. Analogously, a procedural view of mathematics is associated with an external source of validity and with an instrumental knowledge structure. The results summarized in the Student Ranking Table (see appendix) support the plausibility of these connections. The table reveals that generally those participants with a higher percentage in the conceptual coding category also were those coded high on autonomy and who subsequently were coded correct on a higher percentage of the function questions.

These results, however, cannot establish causality between the factors. In a similar way the specific comments made in the belief interviews cannot be causally linked to responses in the function interviews. Yet, the interviews suggest plausible inferences between the two data sets. The following discussion summarizes and compares each participants' beliefs about mathematics and autonomy with their knowledge of functions.

Discussion of Individual Cases

Keith

Keith's beliefs assessment conveyed a view of mathematics as procedural. Keith described mathematics as "a brick wall" since it allowed no room for error. On the vocabulary list Keith selected the terms *rigid*, *controlled*, and *absolute* to reflect the right-wrong aspect of mathematics. "You use a specific formula to get specific answers" and "you have to get this answer or else you're wrong." Keith saw mathematics as useful but not beautiful or exciting. However, he did feel that the solution of word problems and proofs required original thinking. He also felt that one needed to be clever to succeed at mathematics. Keith saw real-world applications as important for motivation. He indicated that he expected tests to be like homework or class problems, although he did not expect the teacher to prepare him for each problem type.

The problem protocols in Keith's beliefs assessment also pointed towards a procedural view of mathematics. He was coded conceptual on several problems because he was able to justify intuitively arithmetic and algebraic procedures, and because he showed flexibility in his solution techniques. The procedural coding resulted from his use of unmonitored trial and error and dependence on the execution of formulas in the problem-solving situations. Keith also graded the sample test on the basis of familiarity of form rather than process. Keith's autonomy during the problem protocols was limited. He showed some monitoring of his processes and an expectation that his solutions should be justifiable.

Keith's functions interviews stand in contrast to his beliefs assessment. The conceptual and autonomous codings from the belief's assessment showed that 62% of the problem episodes received a conceptual rating while 40% received an autonomous rating. These relatively low percentages along with Keith's description of mathematics as prescribed rules suggested that Keith viewed mathematics as primarily procedural in nature. Based on this conclusion, Keith's functions assessment was anticipated to show a reliance on procedures. Instead his protocols implied an integrated understanding of and ease with many of the topics. Keith was able to use function notation in evaluating expressions like $f(x)$ and $f(g(x))$ and in solving word problems. Keith also associated $f(x)$ with y and quickly graphed $f(x) = 2x + 3$ both by a table of values and by using the slope-intercept form. He articulately described the components of $y = mx + b$ including the roles of x and y in the equation. Keith did, however, have difficulty applying the vertical line test to graphs of function. He tacitly assumed that continuity was implied in the definition. He also limited his definition of domain and range to the explicit x and y coordinates he computed. Finally Keith insisted that the graph of $y = (x - 1)(x - 2)(x - 3)(x - 4)$ would represent a series of 4 lines each with the same y-intercept of 24.

The researcher hypothesized that this incongruity between the beliefs assessment and the function assessment was partially attributable to Keith's reserved nature. As noted in the analysis of his interviews, Keith rarely volunteered information. This silence made the autonomous coding especially difficult to determine since the coding was premised on voluntary actions or comments. That is, a student only received an autonomous or nonautonomous coding for a problem episode if he or she showed a voluntary action or comment that indicated either coding category criteria. Hence the actual number of problem episodes that could be evaluated for Keith was fairly small. For instance, Ann's comments permitted coding on 12 problem episodes, while only 5 problem episodes could be coded for Keith. Thus Keith's evaluation was based on fewer problem episodes. In a similar fashion, Keith's quiet demeanor may have also unduly influenced his conceptual/procedural coding. He may have been less willing to volunteer his thoughts or conjectures. The researcher observed that Keith tended to become quieter when he was confused or uncertain.

Ann

Ann's interviews revealed a view of mathematics as conceptual. Ann repeatedly asserted that it was important to understand the rationale in mathematics. This rationale for procedures not only made the ideas reasonable but it also helped to extend the procedures to new situations. Ann felt that it was unnecessary to memorize mathematics if one understood it. She enjoyed the challenge of applying mathematics to new situations and to real-life applications. In fact, she saw mathematics everywhere in the world around her. Ann also explained that in mathematics, especially in word problems, it was often necessary to use trial and error to gain insight into the problem. To her the creation of mathematics required creativity and cleverness. While Ann saw mathematics problems as having only one right answer, she felt that there could be multiple-solutions techniques. Finally she saw mathematical knowledge as integrated concepts and cumulative in nature.

Ann's verbal reports were consistent with her actions and comments during the problem phase of the beliefs assessment. Ann demonstrated that she could use multiple approaches, justify procedures, summarize her solutions, and use number sense to check an answer. In addition to these actions, Ann evaluated the sample test on the basis of process rather than answers alone. Throughout the problem protocols, Ann revealed her autonomy with mathematics. She constantly monitored her progress checking not only that the procedures were executed correctly, but that the solutions made sense to her. She even challenged the researcher's questions and suggestions. These actions showed Ann's pervasive expectation that mathematics should make sense. That the rationales behind the procedures were knowable and vital.

Ann's autonomy and her view of mathematics as conceptual appeared consistent with the type of knowledge she constructed from the functions unit. Ann illustrated her definition of function and one-to-one with examples and graphs. She also was able to extend her use of the definition of function to the abstract situations in questions #61 and #63. (See appendix for a st...ment of the problem.) While confused by function notation, Ann still was able to describe the domain and range of several functions. Ann also demonstrated that she had integrated graphs with equations and the coordinates of points. She moved easily between these three representations and used them in conjunction to validate her work. Ann utilized this facility when she attempted to locate the intercepts in problem #53. Like the other participants, she had originally expected this problem to be linear. However, she persisted until she was satisfied that the graph accurately reflected the multiple intercepts. This tenacity and need to find closure was also evident in her solution to the word problem in question #59. Here Ann utilized number sense to compensate for her confusion with function notation. She expected her answer to make sense and actively sought alternate strategies to verify her solutions.

Overall Ann demonstrated a clear and integrated knowledge of functions and linear equations. While uncertain of function notation, she showed a meaningful understanding of the symbols in the equation $y = mx + b$. Ann was also able to transfer her understanding of functions and intercepts to several non-routine problems. Of the six participants Ann alone cited a specific application, the parking garage fee scale, in her rationale for studying functions. Ann's interviews suggested that she had developed a conceptual (relational) knowledge structure.

Ann's autonomy and conceptual view of mathematics appeared consistent with her knowledge structure of functions. She had emphasized the importance of understanding of the rationale and the expectation that this rationale behind procedures would facilitate her thinking in unfamiliar situations. Ann's functions assessment demonstrated that had she developed this rationale and indeed could apply it. Ann also had indicated her interest in the applications of mathematics. Again, her interviews showed that she had remembered one of the few examples of real life applications given in class. As in her beliefs assessment Ann repeatedly modeled her expectation that her solutions should make sense and be consistent with her knowledge. Thus, Ann's autonomy and conceptual view of mathematics were matched with her conceptual understanding of functions.

Tara

Tara's comments during the beliefs interviews presented a view of mathematics as procedural. Tara's selection of vocabulary terms emphasized mathematics as prescribed rules that were evaluated right or wrong. She saw mathematics as controlled since "There's a way it's to be done. . . .There's only one kind of answer you can get. It goes along with being right or wrong." She also explained that: "Everything that I ever learned in math seems to be like memorization." Tara also believed strongly that teachers were responsible for presenting step-by-step instructions for all problems that their students were expected to solve on tests.

These beliefs coincided with Tara's actions as she discussed and solved the problems in this second phase of the beliefs' analysis. During her work on the problems, especially the problem-solving questions, it was apparent that Tara expected to apply known algorithms or equations to solve the problems. She rarely utilized exploration or trial and error as a motivational tool and she readily expressed her expectation that the researcher should supply hints or answers to these nonstandard problems. Also when grading the sample test, Tara based her assessment on the familiarity of the form rather than on the validity of the process. Thus in the beliefs' assessment, Tara revealed a procedural view of mathematics and a dependence on an outside authority to validate her solutions and her knowledge. This dependency also extended to the expectation that all problem-solution techniques should be presented by the teacher.

Her explicit beliefs and her actions while solving problems appeared consistent with the type of knowledge she constructed from the functions unit. Tara's definition of function stressed the procedural aspect of evaluating functions: "A formula into [which] a number is put to get an answer." Tara could readily evaluate function notation, but failed to associate it with the y variable. She insisted, when asked about domain and range of an equation in function notation, that the equation had no range. While she recognized the graphs of functions by the "function test" (vertical line test), her usage of the test seemed mechanical rather than based on the definition of function. This hypothesis was confirmed by her failure to extend the idea of function to the more abstract situations of problems #61 and #63. (See appendix)

Tara could not extend the concept of intercepts to the nonlinear equation in problem #53. Her discussion showed that she could only generate the y-intercept by recognizing an equation in the form $y = mx + b$. She also did not appear to realize that each ordered pair from the graph must satisfy the equation. Overall Tara's knowledge of functions seemed to be based on procedures using symbols. Her knowledge of the various procedures appeared segregated. When confused or faced with contradictory evidence, Tara again turned to the researcher with the expectation that the dilemma would be explained and the correct procedure illustrated.

Tara's knowledge of functions appeared consistent with the assessment of her beliefs about mathematics. In her discussion, she had emphasized mathematics as memorized rules and that problems should conform to class procedures and examples. Her function assessment demonstrated that she had constructed a collection of procedures that could be applied to routine problems. These procedures, however, were not flexible, did not readily extend to new situations, and appeared disjointed. She also seemed dependent on the recognition of a familiar form to trigger an appropriate response. Thus, Tara's procedural view of mathematics and her lack of autonomy seemed congruent with her procedural knowledge of functions.

Tom

In his interviews, Tom conveyed a view of mathematics as utilizing rules, common sense, and logical thinking. While rules were a significant part, he felt that mathematics was "being able to do the thinking to get to those steps." Tom found that many problems just solved themselves, but when a solution was not immediate he suggested looking for similar examples, trying alternate approaches or simplifying expressions. Like Ann, Tom felt that memorization was not necessary if one understood the concept or procedure. Not only was mathematics useful, but Tom found it challenging and interesting, as well. In fact, mathematics was the only school subject that actively engaged his interest and efforts.

Tom's actions and comments as he solved the problems in the beliefs assessment were consistent with his views of mathematics. Tom demonstrated that he could justify procedures, summarize solutions, and apply number sense to check the reasonableness of answers. Tom also evaluated the sample test on the basis of process rather than familiarity of form. Tom revealed his autonomy with mathematics by voluntarily checking his answers and by monitoring his solution processes.

These expectations and actions from the beliefs assessment appeared consistent with Tom's responses during the function interviews. Tom demonstrated his understanding of functions through his recall of the definition, recognition of graphs of functions, and his application of the function concept to the abstract situations in problems #61 and #63. (See appendix) Tom was equally adept at determining the domain and range of a function, graphing linear equations, and computing the slope of a line. While Tom's responses showed a clear and an integrated understanding of functions and graphing, he was uncertain and confused by function notation. This confusion included a failure to recognize f(x) as y, evaluate expressions like f(2), and use composition notation. With the exception of function notation, Tom's responses generally showed a conceptual understanding of functions and related topics.

As the above discussion indicates Tom valued the rationale behind procedures. His function interviews revealed that he had put that belief into action by constructing an integrated knowledge of functions and graphing. He easily moved between graphs and equations and could cite specific examples to explain his ideas. However, Tom's codings in the Student Ranking Table (see appendix) would suggest a higher percentage of correct responses in the functions category. This discrepancy is tenable. Throughout the interviews, Tom indicated that he was often inattentive to his schoolwork. Although Tom enjoyed mathematics, he often found the homework boring and tedious and so would skip it. While not blaming his teacher, Tom also confessed that he had not read or completed the homework on the composition section even though he had been absent during the class discussion. Thus, Tom's uncertainty with the composition notation is understandable given his confession and attitude toward schoolwork.

<u>Sue</u>

In her beliefs assessment Sue indicated that mathematics required analysis and logic but did not utilize imagination or creativity. While Sue felt that every problem had a correct answer, most problems had multiple-solution techniques. In addition word problems often necessitated exploration before an appropriate equation could be written. Sue felt that mathematical knowledge was initially learned by memorization, but through repeated usage that knowledge became integrated into one's thinking. Her remarks suggested that she saw mathematics as automatized rules and formulas. Sue also expressed her frustration at the irrelevance of much of the mathematics she studied in school. She felt her schoolwork in mathematics would help her be well-rounded academically, but that it had no tie to her real life.

Sue's comments presented a view of mathematics as both conceptual and procedural. This mixture of views was also manifest in her discussion of and solutions to the problems in the beliefs assessment. Sue demonstrated in these interviews that she could justify solutions by using number sense, apply multiple approaches to a problem, and summarize results. At the same time, Sue also graded the sample test primarily on the basis of familiarity of form rather than process. In addition, Sue began the problem-solving protocols with the expectation that an equation or formula could be applied or that one could be written. While Sue held this expectation, she also consciously monitored her progress and abandoned that view when it failed to produce any tangible or immediate results. This monitoring along with her expectation that her results should make sense appeared to override her expectation for an equation or formula and hence facilitated her problem solving.

Sue's belief's assessment revealed that she had beliefs and actions indicative of both a procedural and conceptual view of mathematics. Along with this mixed perspective, Sue also demonstrated a consistent autonomy with mathematics by monitoring her actions and by expecting her answers and solution techniques to make sense. This same mixture was evident in her interviews on functions. Sue's interviews demonstrated that she could give the definition for function, apply it to determine whether a graph was a function, and extend the concept to the abstract situations in questions #61 and #63. (See appendix.) She also was able to graph linear equations and determine the slope from points. In contrast, her discussion of the intercept problem suggested that her use of the slope-intercept equation was more procedural than conceptual. She seemed only to be able to find the y-intercept by locating the b position of the equation. She was also uncertain how to check an ordered pair in an equation and even what the role of x and y were in the equation $y = mx + b$. While these difficulties suggested a more procedural understanding of the unit, she constantly checked her answers against her graphs. By doing so, she gave the impression that she was looking for and expecting the various procedures to be consistent and intuitively correct. Sue's autonomy again aided her solution process even when her factual and conceptual knowledge was inadequate.

Sue's beliefs assessment presented a view of mathematics as a mixture of both memorized procedures and logic. Her problem solutions in this assessment also revealed a pervasive sense of autonomy with mathematics. Sue's functions assessment mirrored this same mixture of dependence on known procedures with the expectation that the solutions should be consistent with her knowledge. It was, in fact, Sue's autonomy that moderated her procedural view of mathematics and facilitated her problem solving.

<u>Steve</u>

Like Sue, Steve presented a mixed view of mathematics. He perceived mathematics as a language and as a collection of prescribed rules or techniques. For Steve, the rules of mathematics represented established results, so he felt it was unnecessary to re-verify them. Foremost though, Steve saw mathematics as a tool of science. Scientist used mathematical equations to describe theoretical ideas or complex phenomenon. While Steve saw the rules as fixed, he also believed they could be adapted to fit new situations or applications. Steve felt that to apply mathematics often necessitated trial and error and insight. It was the applications of mathematics that Steve found interesting and creative. Steve also believed that mathematics was not instinctual but learned and that this learning required memorization, experience, and understanding. Using a computer analogy, Steve described his own understanding of mathematics as: "I learn how it to do it and I store [it] up [in] my brain and later I just pull it out and use it whenever I need to." Overall, Steve's discussion of mathematics stressed its procedures which required strict adherence to the rules and its applications which required insight to adapt or apply the rules.

Steve's problem protocols in the beliefs assessment also demonstrated a mixed view of mathematics. Steve received his conceptual codings because he justified algorithms, generalized results, utilized number sense, and evaluated the sample test on the basis of process. In contrast, Steve received his procedural coding because on the problem-solving questions he relied almost exclusively on equations and formulas to solve the problems. It was noted in the discussion of Steve's protocols that even in the problems coded conceptual, Steve based his arguments and solutions on symbol manipulation, application of rules, and solving equations. Steve's autonomous and nonautonomous codings followed a similar pattern. Steve demonstrated his autonomy by monitoring his results, by voluntarily checking his answers, and by summarizing his results. Steve's also received nonautonomous codings because of his reliance on authority to justify algorithms and because of his relegation of the validity of results to the execution of an equation. Thus, in both Steve's comments and problem solutions a common theme emerged. This theme stressed the accurate execution of rules and equations as a tool primarily for solving and verifying results.

Steve's function interviews showed a clear understanding of function notation. In particular, Steve grasped the relationship between f(x) and y and conveyed an understanding of the roles of the independent and dependent variables. He also easily discussed the components of $y = mx + b$, graphed a linear equation, and computed a slope. While Steve was comfortable with these topics, he was confused on the definition of function. Steve associated function with any equation that defined a relationship between two variables. Steve also expressed some frustration with the concept of function because it could not be represented by numbers or an equation: "That's the problem with math. Cause math is very quantitative and it's tough trying to stick qualitative things to it. Trying to describe it in something other than numbers." Two strategies were prominent in Steve's protocol. Whenever possible, Steve looked for some algebraic manipulation to solve the problem and whenever he graphed points, he would attempt to write an equation to describe that set of points. While at times these strategies were productive, at other times his search for an equation overshadowed the original question or concept. He gave the impression that he needed an equation in order to provide concrete evidence for his thoughts. Steve's solution process was aided by his monitoring of his execution of procedures and his expectation that his solutions should be consistent with his equations and knowledge.

Steve's beliefs assessment and function assessment seemed to coincide. In the beliefs assessment Steve expressed in many ways that mathematics entailed the execution and application of procedures. Steve's comments and actions suggested a reliance on finding and solving equations. He described mathematics as truthful because "equations don't lie." Overall, he saw "math as just a tool rather than an end." He also believed that any mathematical statement could be proved valid or invalid because "If you had the right knowledge you could prove it one way or the other. It may take forever to work out the equation. But it's still possible." These remarks support the contention that to Steve, equations and their solutions represented a very vital element in mathematics. The problem protocols in Steve's beliefs assessment also affirmed this supposition. This emphasis on equations and manipulation appeared again in his function interviews. Steve demonstrated mastery of those aspects that involved the use or description of equations. In contrast though, Steve had difficulty with the definition of function since it was a "qualitative thing." In both the beliefs assessment and in the function interviews, Steve's autonomy facilitated his use of these procedures.

Cross-case Discussion

As the preceding summaries illustrate, the six participants differed in their views of mathematics, in their autonomy, in their approach to problems, and in their knowledge of functions. The following discussion will highlight those areas which most clearly distinguish among the participants.

Beliefs about Mathematics

The students' beliefs about mathematics can be distinguished by their views on the nature of mathematics, the intellectual characteristics needed for mathematics, the utility of mathematics, the way to learn mathematics, and the teacher's role.

Nature of Mathematics. Ann's and Tara's views of mathematics provide the most striking contrast. Ann perceived mathematics as a way of thinking and stressed the importance of knowing the rationales underlying procedures. Tara stressed memorizing and executing procedures and rules. Tara also saw mathematics as rigid and inaccessible to individual choices. While acknowledging that mathematics problems have one right answer, Ann felt that the solution techniques could vary depending on the individual's insight and preference.

Keith, Tom, Sue, and Steve held views of mathematics that were a mix of these two perspectives.

Like Ann, Tom valued the reasoning behind problem solutions and Sue shared Ann's belief that problems often had multiple solutions. Keith, however, was like Tara in that he saw mathematics as rigid and as a collection of prescribed rules. While Steve also stressed this perspective, he added that mathematics was a language and a tool of science.

Intellectual Characteristics. Ann was unique in her perspective on the intellectual characteristics needed for mathematics. She felt that to do mathematics one must be creative, clever, insightful, and logical. Tara again provided a stark contrast to this view. She emphasized that she did mathematics by copying and repeating strategies demonstrated by the teacher. She, unlike Ann, did not hold the expectation that one should be able to modify rules or procedures to apply them new situations. On word problems, or in problem-solving situations if Tara was unable to apply a known formula or write an equation, she tried trial and error. However, for Tara this process entailed more guessing than deduction and conjectures.

Again, the other participants take positions between these two extremes. Tom, Sue, Steve, and Keith all shared Ann's view that word problems and proofs required insight, logic, originality, and trial and error. In addition Steve felt strongly that to apply mathematics required cleverness. Although generally concurring with Ann's view, the others did not agree that one needed to be creative in mathematics. For example, Sue saw creativity as a trait unrelated to mathematics, since for her it implied freedom of thought and self-expression. For Sue, mathematics was too rigid and rule-oriented to permit the individual a voice or choice.

Utility of Mathematics. Among the participants, Ann and Steve saw mathematics as relevant or useful in their lives. Ann perceived mathematics to be everywhere in life. She even saw mathematics in the deductions and inferences that one made while reading a newspaper. For Steve, mathematics was useful for its applications in the sciences and in architecture. Apart from consumer applications Tara, Sue, Keith, and Tom saw no real need or use for their mathematics. They also shared the belief that their mathematics coursework should to be useful in advanced courses like calculus and useful in learning reasoning, but this view was expressed vaguely and skeptically.

Learning Mathematics. The participants were divided on the issue of whether or not mathematics is learned primarily through memorization. While acknowledging that some basic facts needed to be memorized or made routine, Ann and Tom felt they did not need to memorize. They both stated that once they understood a concept that it naturally became part of their thinking. Tom also believed that mathematical ability was partially innate. Sue and Keith felt that they learned mathematics through repetition and some conscious memorization. In contrast, Tara openly admitted that all of her mathematical learning was achieved through memorization.

Teacher's Role. Among the participants, Tara alone held the expectation that the teacher should present step-by-step explanations of each type of homework problem. While the others did not share Tara's view, Ann went further to say that she wished the teacher would not demonstrate everything. She enjoyed challenges and used the occasions when the teacher did not explain everything to test her own understanding of the concepts. There was a general consensus among the participants that tests should mirror homework exercises and class examples. Tara was adamant in this view, saying she would quit any mathematics class that did not conform. While Ann agreed to this view of tests, she did so because she felt non-routine problems would be unfair to the less mathematically adept students. She felt that more difficult problems could be given as optional bonus problems. There was also universal agreement among the participants that the mathematics instruction should contain examples of real-life applications. They felt such applications were essential for motivation and for maintaining interest.

Autonomy

Each of the participants exhibited some autonomy either by voluntarily monitoring their work, checking their answers, or summarizing their results. Each held the expectation that mathematical processes and solutions should make sense, although they differed on the degree to which they were able to capitalize on that expectation. For example, Sue and Tara reached a point in several solutions where they realized their answers were inappropriate. Because they lacked some domain-specific knowledge, or held inflexible procedures, they were not able to modify their responses to be more appropriate. Of all the participants, Ann exhibited the most consistent and ongoing use of self-monitoring and self-reflection. She rarely put a problem aside until she was satisfied with her response. Like Ann, Tom, Steve, Sue, and

Keith frequently used number sense and alternate techniques to verify solutions. For Steve and Sue, their autonomy also seemed to mediate their procedural expectations in the problem-solving protocols and in the functions assessment.

Approach to Problems

The problem-solving situations and the evaluation of the sample test revealed distinctions among the participants. When confronted with a problem-solving situation, Tara expected to apply a formula or write an equation. Failing this, she used unmonitored trial and error. She was easily frustrated and insisted that the researcher should supply hints, clues, or the necessary formula or equation. In contrast, Ann would begin by exploring the situation, either by thinking of a simpler case or using trial values. She used these strategies effectively to refine her approach and to suggest reasonable conjectures. Like Ann, Tom also demonstrated an ease with problem-solving heuristics. He stated that he tried to look for logical connections between the problem and its solution. Like Tara; Keith, Sue, and Steve approached the problem-solving situation with the expectations that they should be solved by applying a formula or writing an equation. Unlike Tara, they were willing to abandon this strategy when their self-monitoring showed it to be ineffective.

The students' evaluation of the sample test also revealed distinctions. Ann, Tom, and Steve graded the test on the basis of the validity of the processes used while Keith, Sue, and Tara evaluated the problem solutions by their match to standard or familiar procedures.

Knowledge of Functions

In addition to the differences in beliefs about mathematics, autonomy, and approach to problems, the students demonstrated differing understandings of the function concepts. One problem in particular, question #53, highlighted those differences especially well. (See appendix.) In this question the students were asked to determine the x- and y-intercepts of the nonlinear equation $y = (x-1)(x-2)(x-3)(x-4)(x-5)$. The students differed not only in how they approached the problem, but on the flexibility of their content knowledge. Tara and Sue had only one strategy for solving for the y-intercept. They tried to put the equation in $y = mx + b$ form and read the b term. While Sue, Keith, and Tom each were able to recognize the form of the intercepts as (x, 0) and (0, y), they could not utilize this information readily. Sue found it confusing to substitute for y instead of x. All three were baffled when the researcher suggested that they substitute simultaneously both coordinates of a point into the equation. Not only was this action new to them, they saw no reason for it

Only Ann had integrated into her understanding the relationship between the ordered pairs, the original equation, and the graph. She moved quickly among these three representations or perspectives and ̧w motivation as well as confirmation from them. With the exception of Ann, the other participants were not especially willing to consider the possibility that the graph was nonlinear. In fact, Keith forced the graph to be linear by dividing the original equation into five separate linear equations: $y = x - 1$; $y = x - 2$; $y = x - 3$; $y = x - 4$; and $y = x - 5$. Sue admitted that she did not know how to determine when an equation might be linear. While initially confused, Tom and Steve were able to eventually ascertain the intercepts and suggest that the graph should be nonlinear. These observations, however, were not made without considerable prompting and questioning by the researcher. Here again Ann's autonomy and need for the concepts to make sense seemed to compel her to explore the question until she understood what type of graph could have multiple intercepts.

Additional Comments

Ann's interviews revealed not only a conceptual view of mathematics, but a facility with content as well. Ann was able to give spontaneous examples of graphs and equations to illustrate her definitions and explanations. These explanations were clear and succinct. Her conversations were not broken by false starts and ramblings. Thus, Ann's interviews demonstrated a qualitative difference beyond the substance of the discussion.

Just as Ann's interviews were characterized by their clarity, Tara's were often confused and vague. During the interviews Tara repeatedly asked for assistance and confirmation for her answers. When none was forthcoming, she would attempt to read the researcher's facial expressions for clues. While mathematical discussions were not easy for Tara, she was very articulate and confident in discussions of English literature and personal matters.

The discussion of the other participants also revealed individual characteristics. Like Ann, Tom also was able to give examples, although his discussion was marked by a terseness. Keith's discussions were characterized by their brevity, since he rarely volunteered information. Sue's discussions were punctuated by a jovial forthrightness. She would quickly tell you when she did not understand a question or concept and then laugh at her own confusion. Steve, like Keith, was somewhat reserved. His remarks usually were tied to references to the sciences or to computers. He often described his mind and thought processes in terms of a computer model.

These differences are not always reflected in the preceding discussions of the students' beliefs, autonomy, and knowledge of functions. They do, however, provide additional insight into the distinctions among the participants. For example, Ann, Keith, and Tom were close in the number of function questions they answered correctly but this proximity does not illustrate the distinctions in clarity, succinctness, and spontaneity exhibited in their remarks.

Summary of Results

The results suggested that generally the students' actions and knowledge in the function protocols were consistent with their beliefs about mathematics and their autonomy. Within this consistency though, each student conveyed a unique set of beliefs about mathematics and demonstrated different degrees of autonomy. The Student Ranking Table in Appendix summarized the differences among the students.

The differences observed among the students were in keeping with the results found by Buchanan (1984) in her investigation of beliefs and problem solving. Buchanan noted that students whose primary beliefs about mathematics were relational or instrumental showed both autonomous and non-autonomous actions in their problem-solving approach. They also had differing sources for their motivation. Collectively these results suggested a continuum of beliefs from conceptual to procedural rather than the dichotomous views of mathematics proposed by Skemp (1987). In this study, Ann and Tara represented views of mathematics at the extremes of the continuum with the remaining students located between them.

Skemp (1987) also conjectured that students' autonomy was related to their beliefs about mathematics. Briefly, he proposed that relational (conceptual) views of mathematics would coincide with autonomy, while instrumental (procedural) views would be associated with a lack of autonomy. This simple dichotomous view was not reflective of the data reported here. While Ann and Tara seemed to fit Skemp's dualistic model, the other participants did not. In fact, the protocols of Steve and Sue demonstrated that autonomy can enhance problem solving and hence mediate an otherwise procedural expectation that all problems should be solved by applying a formula or solving an equation.

Another hypothesis suggested by Skemp (1987) was that different beliefs and autonomy generate divergent knowledge structures. The students' beliefs and autonomy did appear consistent with their knowledge of functions, although the results did not confirm the two divergent structures proposed by Skemp. Just as the students' beliefs were a mixture of conceptual and procedural views, so were their understandings of functions a mixture of memorized procedures and integrated concepts.

While not validating the dichotomous views that were originally hypothesized, the results did confirm interrelationships among the factors investigated: beliefs about mathematics, autonomy, and knowledge structure. What appeared was a complex and subtle interdependency. For example, Steve's and Sue's autonomy seemed to mediate their procedural expectation in the problem-solving protocols and in the function assessment. For Tara, her attempts to validate her answers or explore her ideas often were frustrated by her apparent lack of domain-specific knowledge or her inflexibility with procedures.

In addition to the results reported above, several other observation arose from the data. Like Frank (1985), the researcher noted that the participants often prefaced their comments with phases such as: "I don't remember," or "we were never told this," or "I'm not sure this is right." Frank designated this as 'bailing out' and attributed it to "an attempt to gracefully get out of an uncomfortable or unprofitable situation (p. 95)." The researcher noted a similar implication in the use of these phrases in the participants' problem protocols. The results also seemed to confirm the conclusions drawn by Cobb (1985), Frank (1985), and Schoenfeld (1985) that students' problems-solving heuristics were in keeping with their global beliefs about mathematics.

One final issue needs to be discussed concerning these results. Doyle (1983) noted that what students attend to in class often reflected their perception of the class' requirements. In both the Algebra

II and the Algebra II/Trigonometry classes, the teacher's expectation on the function unit seemed to reflect the procedural aspects of the topic. Mrs. Thomas also held the belief that for the Algebra II class it was important to present samples of all homework problems and that tests should be fairly consistent with these problems. The tests and quizzes given in both classes conformed to this expectation and also tended to include procedural and recall type questions. During the lecture portion of the class period, Mrs Thomas reinforced this expectation by her continual use of procedural questions. However, the class format provided some opportunity for autonomy with the students presenting their homework solutions at the board.

The participants' understanding of functions needs to be considered against the background of this classroom environment. With the exception of Ann, the other participants strongly believed that mathematics test questions should match homework assignments and class examples and hence agreed with their instructor. Primarily the classroom analysis showed that the students' procedural beliefs about mathematics in general and their procedural expectations for the content were not challenged by the teacher's actions. Thus, Tara's view of mathematics as a disjoint collection of memorized rules seemed to be further reinforced in her class. Ann's strong belief that mathematics involved ideas and creativity stood in stark contrast to the teacher's own beliefs that mathematics was useful, but not interesting. Also the teacher's belief that the applications of the mathematics were not important, was in contrast with the students' desire for real-life examples to provide motivation for studying functions or mathematics in general.

Implications for Future Research and for Teaching

This research study investigated the relationship between students' beliefs, autonomy, and knowledge of functions against the background of the classroom environment. While the environment was observed and analyzed, it was not an integral component of the investigation. Thus the effects of that classroom on beliefs and knowledge were not explicitly studied. This aspect, then, is one possible area that should be studied further.An additional area of research suggested by the results is students' autonomy. The apparent power of these expectations to influence students' actions, especially in problem-solving situations, had not been anticipated by the researcher. Their potential warrants further study.

The participants' discussion of the their past experiences, the teacher's role in learning mathematics, and their own learning strategies suggests that beliefs about mathematics are connected with classroom experiences and classroom expectations. This further suggests that the classroom environment, which includes the teacher's own beliefs about mathematics and the teacher's presentation of and expectations for the mathematical content, may convey unspoken messages to the students about the nature and processes of mathematics. If subsequent research affirms the influence of beliefs and autonomy on learning, then the classroom teacher needs to be cognizant of these unspoken messages and perhaps modify classroom activities to foster a more conceptual view of mathematics. This view might be encouraged through student-centered activities, especially problem-solving situations, and the establishment of classroom expectations which include explanations, explorations, and autonomy.

Conclusions

The results from this research investigation suggest three hypotheses concerning students' beliefs about mathematics, autonomy, and mathematical knowledge. First, students' beliefs about mathematics rather than being dichotomous form a continuum from strongly conceptual in outlook to strongly procedural. Second, students' autonomy augments their beliefs about mathematics and often mediates them. Third, students' beliefs and autonomy appear to concur with their problem-solving strategies and with their knowledge of mathematics. Collectively these hypotheses suggest that students' beliefs and autonomy are an integral component of students' conception of mathematics and influence both how problems are approached and how mathematics is learned. Further study needs to be done on how and when these beliefs are formed and under what conditions these beliefs are modified or changed. Finally, the interplay among beliefs, autonomy, and learning need to be investigated in the actual classroom context.

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Student Interview Protocols

Interview #1 (Background Questions)

1) How old are you?
2) Are you a junior?
3) What courses are you taking now?
4) Do you belong to any clubs or groups?
5) Do you belong to the mathematics team?
6) Have you ever used a computer?
7) Do you own a calculator?
8) Do you ever read any game or puzzle books?
9) Do you have a job?

   Do you use mathematics in your job?
10) What other mathematics classes have you taken?

    Could you tell me about those classes?
11) What do you plan to do after high school?
12) Do you anticipate taking mathematics in college?
13) In New Hampshire you are required to take only 2 years of mathematics, so why are you taking a third year?
14) How do you use mathematics in your everyday life?
15) Now use your imagination. Name something that is the most unlike mathematics that you can think of. What makes it unlike math? Name something that is the mostlike mathematics. What makes it like math?
16) How would you describe your own ability to do mathematics?
17) Would you describe someone in your class that is good at mathematics? What makes them good?
18) How does someone get to be good at mathematics?

Problems. General instructions on all mathematics problems was: Read the problem out loud and tell me what you're thinking as you do the problem.

19) 1/4 + 2/3

Follow up:

Why do you need common denominators?

20) 6 divided by 3/8

Follow Up:

Why do you invert the divisor?

Does your answer make sense? Why? When you do divide, your answer gets smaller. Does your answer make sense?

Interview #2 (Problem Protocols and Belief Questions)

21) 1.50 x .25

Follow Up:

How do you know where to place the decimal point?

Does your answer make sense? Why could it not have been 37.5 or 3.75?

22) Vocabulary list. (see following list)

Instructions: Circle the words that you associate with mathematics (English, history or science). Read them out loud as you go. Are there any words that you think go together? Why?

VOCABULARY LIST

absolute	abstract	analyze	ancient
anxiety	arbitrary	beauty	boring
capricious	cause & effect	changing	chronological
classical	clever	common sense	concentration
controlled	controversial	creative	cultural
current	deductive	depth	detailed
diagrams	discovery	dogmatic	dull
easy	elegant	exciting	experiential
expressive	factual	fixed	flexible
formulas	fragmented	free	fun
general	goals	hard	humanistic
ideas	individualistic	insight	instinctive
integrating	intensive	interpretative	known
language	logical	mechanical	memorize
multi-dimensional	multi-perspective	new	objective
old	open ended	opinionated	ordered
organized	practical	precise	rational
right/wrong	rigid	routine	rules
sequenced	short cuts	simple	solemn
specific	structured	symbolic	truthful
theoretical	thorough	thought provoking	themes
trial and error	universal	uncertain	useful
valid	varying	verbal	visual
well-defined	writing		

Interview #3 (Problem Protocols and Belief Questions)

23) Which of the following fractions is more than 3/4? 35/71, 13/20, 71/101, 19/24, 15/20.

Follow Up:

Could you have done it without a calculator?

24) Which is the least of the following numbers?

1/5, $\sqrt{5}$, 1/$\sqrt{5}$, $\sqrt{5}$/5, 1/5 $\sqrt{5}$.

Follow Up:

How else could you do the problem?

25) Jimmy was trying a number trick on Sandy. He told her to pick a number, add 5 to it, multiply the sum by 3 then subtract 10 and double the result. Sandy's final answer was 28. What number did she start with?

Follow Up:

Do you believe your answer?

26) Find 20% of 85.

Follow Up:

Do you believe that number? Why does it seem reasonable?

27) What is the smallest positive number which when it is divided by 3, 4 or 5 will leave a remainder of 2? Note: Two is the smallest integer that satisfies this relationship. However, all the student participants tacitly assumed that divided by meant a factor greater than zero (e. g. 3k + 2, k > 0).

28) Which is larger the value in column A or in column B?

A	B
$\frac{543 \times 29}{32}$	$\frac{30 \times 543}{28}$

Follow-up:

Could you have answered the question without multiplying it out (or using the calculator)?

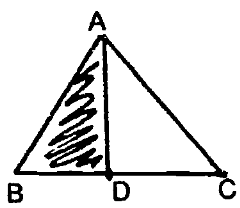
29) Which is larger the value in column A or in column B?

$P<O$

A	B
P^5	O

Follow-up:

If students gave an incorrect response, the researcher asks the students to try various numbers, including a counter-example to the students statement.



30) In the diagram above, if BD = DC and the area of the shaded region is 8, the area of the triangle ABC is.

Follow-up:

If stuck, the researcher asked the student to tell what they felt they needed to know in order to solve the problem. Since this usually involved the length of BD or BC, the researchers suggested the student make up a number and try it out.

Interview #4 (Problem Protocols and Belief Questions)

31) Smith gave a hotel clerk $15 for his cleaning bill. The clerk discovered he had overcharged and sent a bellboy to Smith's room with five $1.00 bills. The dishonest bellboy gave three to Smith, keeping two for himself. Smith has now paid $12.00. The bellboy has acquired $2.00. This accounts for $14.00. Where is the missing dollar?

32) Sample test. (see following Interview #4)

Instructions:

Pretend now that you are the teacher. I want you to grade this test. The first five problems are worth six points each, and the last four are worth ten points each. Grade it, and tell me why you are taking off the points that you are.

Follow Up:

After grading the test, the researcher queries the student about any misconceptions they may have allowed to stand. Also on question II (a) the researcher usually attempted to clarify if the students believed that the problem was done incorrectly, inefficiently or the student marked it based on not matching the standard procedure.

33) Which is larger, the value in column A or in column B?

A	B
P + 2	2 - P

Follow-up:

Again, counterexamples were suggested if students offered incorrect solutions.

34) Which is larger, the value in column A or in column B?

R > O

A	B
R^2	R

Follow-up:

Counterexamples were suggested if students offered incorrect solutions.

35) The radius of the earth is approximately 4,000 miles. What length of rope would be

needed to "fit" around the equator? Now suppose we add about 6 feet to the length of the rope (i.e. 2 feet), how far above the ground would the rope be? Would a piece of paper fit between the ground and the rope? Could a mouse crawl through? Could a person walk under it?

Follow-up

Since the purpose of the question included the students' reaction to the feasibility of the answer, the researcher interacted with the students, assisting if necessary, the students' understanding of the question and monitoring the appropriate usage of units. How certain do you feel about your answer? On a scale of 1 to 10 how confident are you?

ALGEBRA REVIEW TEST

** Show all work **
** Write neatly **
**Circle your answers **

Chris
name

I. Simplify the ~~following~~ (6 points each):

(a) $\left(x^{13}\right)^2 = x^{15}$

(b) $\sqrt{16 + 9} = \sqrt{16} + \sqrt{9}$
$= 4 + 3$
$= 7$

(c) $\frac{\cancel{X} - 1}{\cancel{X} - 1} = x + 1$

(d) $\frac{\cancel{5}}{6} + \frac{2}{\cancel{12}} = \frac{1}{6} + \frac{2}{2}$
$= \frac{1}{6} + 1$
$= \frac{7}{6}$

(e) $\frac{6}{0} = 0$

II. Solve for X. Write answers in set notation. (10 points each)

(a) 3X - 7 = 4

$3x - 7 + 4 = 4 + 4$
$3x - 3 = 8$
$3x - 3 + 3 = 8 + 3$
$3x = 11$
$\frac{3x}{3} = \frac{11}{3}$
$x = 11/3$
$\{11/3\}$

(b) 3(X - 2) + 12 = X + 2(X + 2) + 2

$3x - 6 + 12 = x + 2x + 4 + 2$
$3x + 6 = 3x + 6$
$6 = 6$
$\{x = 6\}$

(c) 1/2 (4X - 6) = 5X - 3(X - 4)

$2x - 3 = 5x - 3x + 12$

$2x - 3 = 2x + 12$

$-3 = 12$

$\{-3, 12\}$

(d) $|X + 3| < 2$

$-2 < x + 3 < 2$

$-2 - 3 < x + 3 - 3 < 2 - 3$

$-5 < x < -1$

No solution, absolute value must be a positive number.

$\emptyset$

Interview #5 (Problem Protocols, Belief Questions, and Student/ Teacher Roles Questions

36) Ranking chart. (see following chart)

Instructions:

What I would like you to do is rank order the topics across the top from one to eleven. For example, if you find decimals the most interesting, you would give it a one, the most boring topic an eleven. Let me know what you're thinking as you fill out the chart.

37) I would like you to imagine there is a new kid in school. This student is an English speaking foreign student who is unfamiliar with American school. The guidance office calls you down and asks that you show him the ropes in your mathematics class. What kind of advice would you give him?

Follow-up questions if necessary:

What would you tell him about homework? Tests? Lectures? Your teacher? What should he do if he gets stuck on his homework?

38) In a few years I will be teaching teachers how to teach mathematics. Do you have any advice to pass on to them?

39) If you could change anything about mathematics, or the way it is taught, what would you change?

40) How would you fill in the blank "a good math teacher is someone who"?

41) I have a friend who likes to make mathematics tests what he calls a learning experience. He puts problems on the test that the students have never seen before but are related to the ideas they have had in class. Do you agree or disagree with my friend's philosophy?

Follow-up, if necessary:

How would you convince him that this is not right?

42) I have had students say to me "You didn't go over that in class, but you gave us homework on it anyway. I don't think that's fair." Do you agree or disagree with those students?

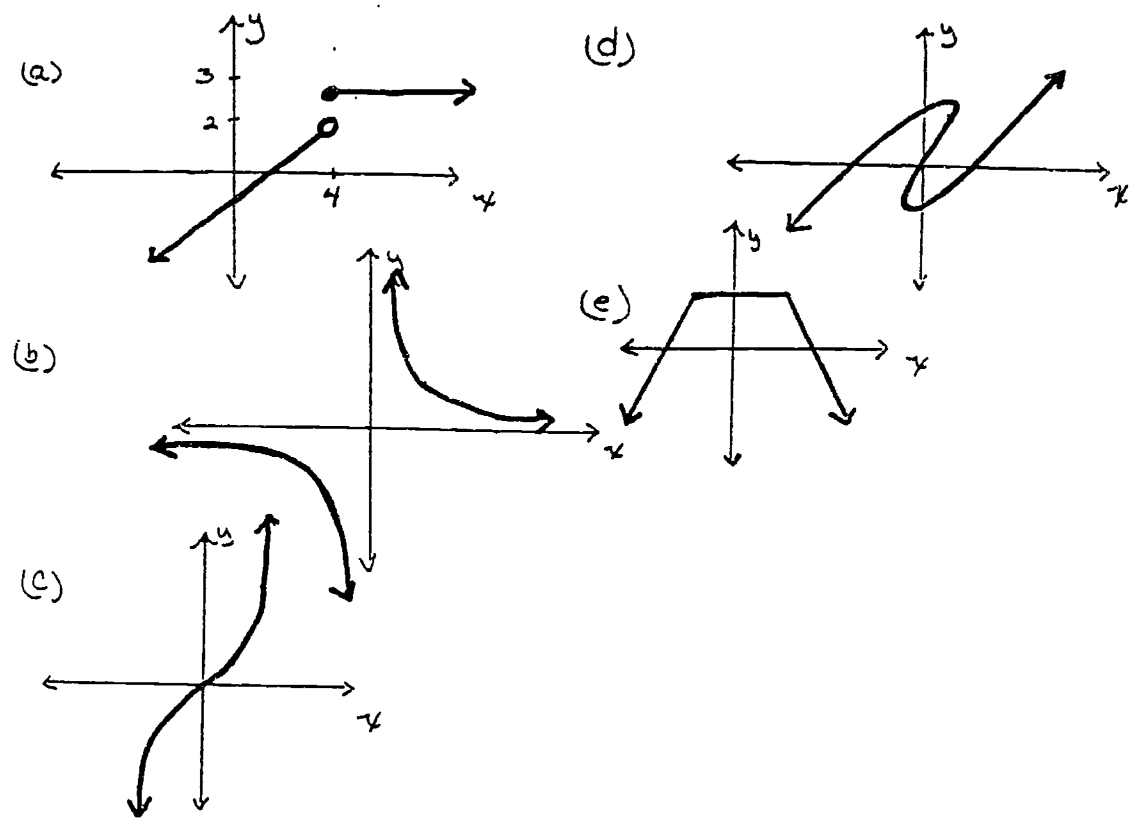
43) I have also had students say to me "You waste too much time in class going over things that you don't ask us on the test." Do you agree or disagree with their view?

	using %	decimals	fractions	equations	factoring	word problems	exponents	graphing	writing proofs	geometric constructions	absolute value & inequalities	
interesting												boring
easy to do/solve												hard to do/solve
applied/real world												theoretical
easiest to learn												most difficult to learn
most useful												least useful
best at												worst at
routine thinking												original thinking
visual												abstract
logical												arbitrary
most liked												least liked
basic												advanced
busy work												thought provoking/challenging
clear												confusing
flexible												rigid

Interview #6 (Function Problem Protocols)

44) Describe what is meant by a function. Give an example of something that is not a function.

45) Which are functions? Which are relations?



46) What does f(x) = 2x + 3 mean?

47) Graph f(x) = 2x + 3. Are straight lines functions?

48) What the x's and y's in the formula: y = mx + b?

49) What is slope?

50) Points (0, 1), (2, 4), and (6, 10) lie on the same line. Compute the slope of the line?

Follow-up:

If you used a different pair of points what would you get? Why?

51) What is meant by the domain and range?

Follow-up:

Referring back to the graphs is #45, ask the student to give the domain and range of the graphs. Or ask the student to write an equation of a function and give it's domain and range.

52) What is the domain and range of $f(x) = \sqrt{x - 4}$?

Interview #7 (Function Problem Protocols)

53) Find the x and y intercepts for the graph of

$y = (x-1)(x-2)(x-3)(x-4)(x-5)$.

54) What does the notation $f(g(x))$ mean?

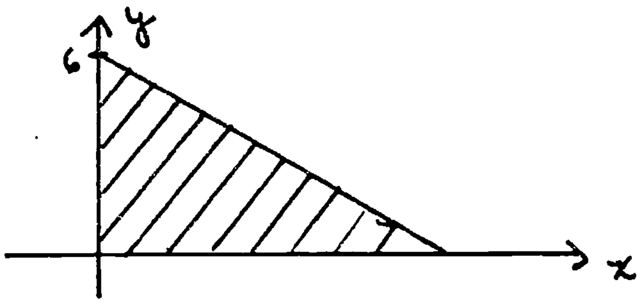
55) $f(x) = 1/x$ and $g(x) = x - 3$.

(a) Find $f(g(2))$, $f(g(0))$, and $(f(g(3))$.

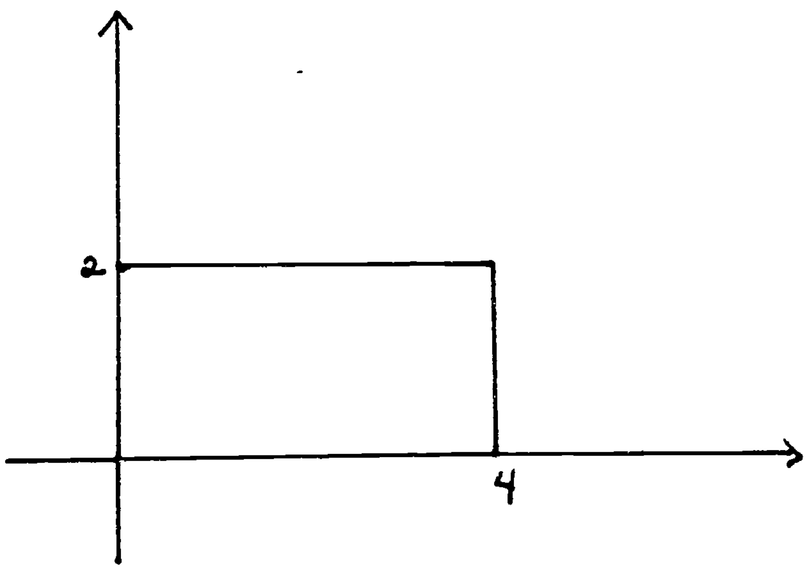
(b) What is the domain of $h(x) = f(g(x))$?

56) Does $f(g(x)) = g(f(x))$?

57) The y-intercept of the line in the figure is 6. Find the slope of the line if the area of the shaded triangle is 72 square units.



58) Prove that the line segment joining the midpoints of the successive sides of a rectangle form a rhombus.



Interview #8 (Function Problem Protocols)

59) During a flu epidemic in a small town, a public health official finds that the total number of people P who have caught the flu after t days is closely approximated by the formula:

$P(t) = 25t - 20 \quad (1 < t < 29)$.

(a) How many have caught the flu after 10 days?

(b) After approximately how many days will 275 have caught the flu?

60) Why did you study functions? What good are they?

61) If (2, a) and (2, b) are points on the graph of function, what can you conclude about a and b?

62) It costs a recording artist $2100 to make a master tape and $1500 for each 1000 tapes produced. The tapes sell for $5 each. How many tapes must be sold before a profit is made?

63) In the Brown family there are these people: Bill, Jane, Sarah, and Tom. In the Jones family there are Allen, Carol, Dave, George, and Patty. Now if I write these peoples names as ordered pairs, that is as (Bill, Brown), (Jane, Brown), (Sarah, Brown) and so on for the Brown family. Also do the same for the Jones family. Does this collection of ordered pairs describe a function? If the order is reversed will it be a function?

Coding Criteria

Conceptual View of Mathematics

Summarizing, a conceptual view of mathematics holds that mathematics is composed of integrated concepts. Students perceive the concepts which underpin procedures as rational, knowable, and vital for their understanding. This view of mathematics is evident by a student's non-reliance on rules and known procedures in problem solving situations, by self-reflection on the selection of procedures and their execution and by the development of a rationale for basic procedures.

Students with a conceptual view:

1) could justify procedures on the basis of first principles or on intuitive number sense,
2) have integrated procedures as opposed to having isolated applications or have multiple ways to approach a problem,
3) use number sense to facilitate approximation or check reasonableness of answers,
4) have the ability to summarize or generalize a process used to solve a problem (not mere repetition of steps--must add some interpretation or perspective to summary), or
5) graded sample test on the basis of validity of process not just answers or familiarity of form.

Procedural View Of Mathematics

Summarizing, a procedural view of mathematics is primarily one that views mathematics as an isolated collection of procedures rules to be memorized. The importance lies in the execution of these procedures and not in the rationale behind them. This view would manifest itself in problem solving situations as an exclusive reliance on formulas or equations to solve these problems. In addition, a procedural view of mathematics would be evident in the student's inability to offer any rationale for basic arithmetic and algebraic procedures.

Students with a procedural view would:

1) execute algorithms without evidencing any ties to other idea or ability to justify procedures in terms of first principles,
2) have the expectation that mathematics can be solved by merely applying a given algorithm or by solving an algebraic equation,
3) used unmonitored trial and error, or
4) graded the sample test on the basis of: answers, not process, familiarity of form and marked problems incorrect on the basis of form.

Demarkations:

strong evidence

sufficient evidence

weak evidence

Uncertain - unable to code problem dialogue

Autonomy

Autonomy is defined here as more than an independence of action. Autonomy is associated with an expectation that mathematics should make internal sense to the individual. By this, it is meant that the student believes she is the primary source of justification for her mathematics. She expects her answers or solutions to internally consistent with her knowledge. A student lacking such an expectation would require an outside authority - teacher or text or answer key to judge the soundness of her solutions. Mathematics for a student who lacked autonomy would represent an external knowledge.

An autonomous student would:

1) show independence from the researcher by (a) challenging the researcher's suggestions or (b) delay or put off responding to researcher's ideas or questions until they have had an opportunity to check a computation or idea for themselves,
2) check answers voluntarily,
3) monitor the reasonableness of answers as problem solution progresses,
4) show an expectation that the problems should make sense,
5) rephrase problems in her own words, or
6) conclude a problem by summarizing idea for herself.

A nonautonomous student would:

1) relay on the researcher to supply answers, hints or clues,
2) express the view that mathematics is memorized rules given by the teacher, or
3) expect others to judge the validity of answers.

Student Rankings

	Beliefs				Functions	
Name	C	P	A	NA	CT	IC
Ann	100%*	0%	100%	0%	85%	15%
Tom	100%	0%	100%	0%	75%	25%
Sue	90%	10%	100%	0%	80%	20%
Steve	75%	25%	75%	25%	70%	30%
Keith	62%	38%	40%	60%	85%	15%
Tara	57%	43%	33%	67%	70%	30%

Note. C = conceptual; P = procedural; A = autonomous; NA = nonautonomous; CT = correct; IC = incorrect.

*Proportion of problems that were codable as conceptual to the total number of problems codable as conceptual or procedural.